SUPPLEMENT X.

TO SECT. II. BOOK. II.

CONCERNING THE RESOLUTION OF DIFFERENTIAL EQUATIONS OF THE THIRD OR HIGHER ORDERS, WHICH INVOLVE ONLY TWO VARIABLES.

1.) Concerned with differential equations of any order, which differentiated anew are able to be integrated.

M. S. exhibited to the Academy on the 8th day of October, 1781.

[E680]

§. l. Let x and y be two variables, between which the differentials of the proposed equation of any order are valid. The differentials may be set out in the usual manner, according to the form :

$$\partial y = p\partial x,\ \partial p = q\partial x,\ \partial q = r\partial x,\ \partial r = s\partial x \text{ etc.}$$

thus so that, with the element ∂x constant, there shall become :

$$p = \frac{\partial y}{\partial x},\ q = \frac{\partial\partial y}{\partial x^2},\ r = \frac{\partial^3 y}{\partial x^3},\ s = \frac{\partial^4 y}{\partial x^4}, \text{ etc.}$$

Again, P and $\mathfrak{P}$ shall be any functions of p; Q and $\mathfrak{Q}$ any functions of q; R and $\mathfrak{R}$ of r; S and $\mathfrak{S}$ of s, etc., which functions not only can be rational, but also irrational, and thus transcending.

§. 2. With these in place, I will show how to integrate two kinds of equations by differentiation, the first of which contains these equations :

$$y - px = \text{P},\ p - qx = \text{Q},\ q - rx = \text{R},\ r - sx = \text{S} \text{ etc.},$$

of which the first can involve any functions of ∂y both rational and irrational, as well as even transcending functions; the second can involve such functions of $\partial\partial y$, the third itself of $\partial^3 y$, the fourth of $\partial^4 y$ and thus henceforth, certainly at this time no one has been able to bring to mind the integration of equations of this kind.

§. 3. Another kind of equation comprises the following equations, which involve any two functions,

$$y + \mathfrak{P}x = \text{P},\ p + \mathfrak{Q}x = \text{Q},\ q + \mathfrak{R}x = \text{R},\ r + \mathfrak{S}x = \text{S} \text{ etc.},$$

the integration of which I will show how to be established by differentiation. But it is evident these preceding equations themselves are to be considered, clearly when there is :

$$\mathfrak{P}=-p,\ \mathfrak{Q}=-q,\ \mathfrak{R}=-r,\ \mathfrak{S}=-s \text{ etc.}$$

Moreover it is apparent, these equations thus can to be complicated, so that certainly no one would have wished to undertake their integration.

Concerning Equations of the First Kind.

Problem I

§.4. *For the proposed differential equation of the first order* $y-px=\mathrm{P}$*, to determine its complete integral.*

Solution.

Since there shall be $\partial y=p\partial x$, if the equation proposed $y-px=\mathrm{P}$ may be differentiated, this equation will be produced, $-x\partial p=\partial\mathrm{P}$, from which, by putting $\partial\mathrm{P}=\mathrm{P}'\partial p$, there is deduced $x=-\mathrm{P}'$. So that if now we may consider *p* as a new variable, we will be able to express that by *x* as well as by *y*. Indeed, since there shall be $y=px+\mathrm{P}$, there will become $y=\mathrm{P}-p\mathrm{P}'$, from which, by eliminating *p,* whenever indeed that calculation may be permitted, it will be possible to establish an equation between *x* and *y*, but which at this point must be regarded as for a particular integral, because it involves no arbitrary constant. But truly, since it will be allowed to divide the equation $-x\partial p=\mathrm{P}'\partial p$ arising from differentiation by ∂p, this same factor equated to zero is agreed to supply the needs of being the complete integral. For on putting $\partial p=0$ there will be $p=\text{const.}=\alpha$, and thus $y=\int p\partial x=\alpha x+\beta$. Indeed this equation is seen to involve two arbitrary constants; but truly the other may be determined from the same proposed equation, since with the substitution made there may become

$\alpha x+\beta-\alpha x=\mathrm{P},$ and thus $\beta=\mathrm{P}=f:\alpha$. [*i.e.* β is a function of α, in an early notation.]

Problem 2.

§.5. *To assign the complete integral for the proposed differential equation of the second order* $p-qx=\mathrm{Q}$.

Solution.

If this equation may be differentiated and there may be written $q\partial x$ in place of ∂p, this equation will be produced, $-x\partial q=\partial\mathrm{Q}$, or, on putting $\partial\mathrm{Q}=\partial\mathrm{Q}'\partial q$, there will become $-x\partial q=\mathrm{Q}'\partial q$. Hence the common factor ∂q equated to zero will give $q=\text{const.}=2\alpha$, from which there becomes

$$p = \int q\partial x = 2\alpha x + \beta, \text{and hence}$$
$$y = \int p\partial x = \alpha xx + \beta x + \gamma,$$

of which three constants $\alpha,\ \beta,\ \gamma$ one may be determined from the proposed equation. Moreover with the division made by ∂q we will have $\mathrm{X} = -\mathrm{Q}'$, from which there is deduced :

$$p = \mathrm{Q} + qx = \mathrm{Q} - q\mathrm{Q}',$$

and hence on account of $\partial x = -\partial\mathrm{Q}' = -\mathrm{Q}''\partial q,$ there will become

$$y = \int p\partial x = \int \mathrm{Q}''\partial q(\mathrm{Q}'q - \mathrm{Q}) + b.$$

Example

§.6. Let $\mathrm{Q} = aq^m$, there will be

$$\mathrm{Q}' = maq^{m-1}$$

and

$$\mathrm{Q}'' = m(m-1)aq^{m-2}.$$

Therefore in this case there will become

$$x = -\mathrm{Q}' = -maq^{m-1},$$

$$y = m(m-1)^2 aa\int q^{2m-2}\partial q + b,$$

or

$$y = \tfrac{m(m-1)^2}{2m-1}aaq^{2m-1} + b.$$

Truly there is $q^{m-1} = -\frac{x}{ma}$, thus so that the value of y itself will be easily expressed by x, with which done the complete integral of this differential of the differential equation will be had :

$$\frac{\partial y}{\partial x} = \frac{x\partial\partial y}{\partial x^2} = \frac{a(\partial\partial y)^m}{\partial x^{2m}}.$$

Problem 3

§.7. *With the differential equation of the third order proposed* $q - rx = \mathrm{R}$*, to investigate its complete integral.*

Solution.

This equation differentiated, on account of $\partial q = r\partial x,$ gives $-x\partial r = \partial\mathrm{R} = \mathrm{R}'\partial r,$ the factor of which equation ∂r equated to zero will provide this equation:

$$y = \alpha x^3 + \beta x^2 + \gamma x + \delta\,,$$

where of the four constants $\alpha,\ \beta,\ \gamma,\ \delta$, one being had determined from the proposed equation. Indeed since hence there shall be

$$p = 3\alpha x^2 + 2\beta x + \gamma,\ q = 6\alpha x + 2\beta,\ r = 6\alpha,$$

on substituting there will be $2\beta = \mathrm{R}$, thus so that only three arbitrary constants may be left in the calculation, as the nature of an equation of this kind postulates. But with the division made by ∂r, this equation $\mathrm{X} = -\mathrm{R}'$ will be satisfied , from which there is deduced $q = \mathrm{R} - r\mathrm{R}'$.
Hence, on account of

$$\partial x = -\partial R' = -R''\partial r\,,$$

there will be found

$$p = \int q\partial x = \int \mathrm{R}''\partial r\left(r\mathrm{R}' - \mathrm{R}\right),$$

and finally $y = \int p\partial x$, where on account of the twofold integration, two arbitrary constants are introduced .

Example

§. 8. Let there be $\mathrm{R} = ar^m$, there will be

$$\mathrm{R}' = mar^{m-1} \text{ and } \mathrm{R}'' = m(m-1)ar^{m-2}\ ,$$

from which there is deduced:

$$p = \tfrac{m(m-1)^2}{2m-1} aar^{2m-1} + b,$$

and on account of

$$\partial x = -\partial\mathrm{R}' = -\mathrm{R}''\partial r = m\left(m-1\right)ar^{m-2}\partial r\,,$$

we obtain :

$$y = \int p\partial x = -\tfrac{m^2(m-1)^3}{(2m-1)(3m-2)} a^3 r^{3m-2} - mabr^{m-1} + c,$$

from which on account of $r^{m-1} = \frac{x}{ma}$ a finite equation may be obtained easily between *x* and *y*, and this will be the complete integral of this differential equation of the third order :

$$\frac{\partial\partial y}{\partial x^2} - \frac{x\partial^3 y}{\partial x^3} = \frac{a\left(\partial^3 y\right)^m}{\partial x^{3m}}.$$

Problem 4

§. 9. *To investigate the complete integral for the proposed differential equation of the fourth order* $r - sx = S$.

Solution.

On account of $\partial r = s\partial x$ the proposed equation on differentiation will become $-x\partial s = \partial S = S'\partial s$, the factor ∂s of which equation gives

$$y = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon \,,$$

where one of the constants may be determined from the proposed equation itself. Again the equation is satisfied $x = -\mathrm{S}'$,from which there is deduced $r = \mathrm{S} - s\mathrm{S}'$, and hence, on account of

$$\partial x = - \,\partial \mathrm{S}' = -\mathrm{S}''\partial s$$

there will be found:

$$q = \int r\partial x,\ p = \int q\partial x \ \text{ et } \ y = \int p\partial x \,,$$

or

$$y = \int \partial x \int \partial x \int r\partial x,$$

where on account of the triple integration there are three arbitrary constants requiring to be added on . It will be allowed to progress to equations of higher orders in a similar manner.

Concerning Equations of the Second Kind.

Problem 5

§. 10. *For the proposed differential equation of the first order of this kind, to investigate its complete integral.*

Solution.

If this same equation $y + \mathfrak{P}x = \mathrm{P}$ may be differentiated, and in place of ∂y there may be written $p\partial x$, this equation will be produced :

$$p\partial x + \mathfrak{P}\partial x + x\partial \mathfrak{P} = \partial \mathrm{P} \,,$$

or on putting $\partial \mathrm{P} = \mathrm{P}'\partial p$, there will be

$$\left(p + \mathfrak{P}\right)\partial x + x\partial \mathfrak{P} = \mathrm{P}'\partial p \,,$$

which divided by $p + \mathfrak{P}$ gives

$$\partial x + x \cdot \partial \cdot l \cdot \left(p + \mathfrak{P}\right) - \tfrac{x\partial p}{p+\mathfrak{P}} = \tfrac{\mathrm{P}'\partial p}{p+\mathfrak{P}}.$$

[Indeed there is $\frac{x\partial\mathfrak{P}}{p+\mathfrak{P}} = x.\partial.l.\left(p+\mathfrak{P}\right) - \frac{x\partial p}{p+\mathfrak{P}}$.]

So that if we now may put

$$\int\frac{\partial p}{p+\mathfrak{P}} = z\,,$$

that equation may be rendered integrable on multiplying by $e^{-z}\left(p+\mathfrak{P}\right)$.
[*i.e.* The integrating factor .] Indeed there will be produced :

$$\left(p+\mathfrak{P}\right)e^{-z}\partial x + (p+\mathfrak{P})x\mathrm{e}^{-z}\partial\cdot l(p+\mathfrak{P}) - x\partial z e^{-z}(p+\mathfrak{P}) = e^{-z}\mathrm{P}'\partial p\ ,$$

the integral of which evidently is

$$xe^{-z}\left(p+\mathfrak{P}\right) = \int e^{-z}\mathrm{P}'\partial p\,,$$

[for $\partial\left(xe^{-z}\left(p+\mathfrak{P}\right)\right) = e^{-z}\left[\partial x\left(p+\mathfrak{P}\right) - x\frac{\partial p}{p+\mathfrak{P}}\left(p+\mathfrak{P}\right) + x\partial.\left(p+\mathfrak{P}\right)\right]$
$= e^{-z}\left[\partial x\left(p+\mathfrak{P}\right) + x\partial\mathfrak{P}\right] = e^{-z}\partial P;$]

from which there is deduced :

$$x = \frac{e^{z}}{p+\mathfrak{P}}\int e^{-z}\mathrm{P}'\partial p = \frac{e^{z}}{p+\mathfrak{P}}\int e^{-z}\partial\mathrm{P},$$

from which there becomes at once:

$$y = \mathrm{P} - \frac{\mathfrak{P}e^{z}}{p+\mathfrak{P}}\int e^{-z}\partial\mathrm{P},$$

where e^{z} is also a function of p, thus so that both the variables *x* and *y* may be expressed by one and the same variable *p*, which expressions now involve an arbitrary constant between themselves, thus so that there shall be no further need for its addition.

Example.

§.11. Let $\mathrm{P} = ap^{m}$ and $\mathfrak{P} = bp^{n}$, thus so that the equation requiring to be integrated shall become $y + bp^{n} = ap^{m}$.
Therefore here there will become:

$$z = \int\frac{\partial p}{p\left(1+bp^{n-1}\right)} = \int\frac{\partial p}{p} - \int\frac{p^{n-2}\partial p}{1+bp^{n-1}}\,,$$

from which there is deduced from the actual integration

$$z = lp - \tfrac{1}{n-1} l\left(1 + bp^{n-1}\right),$$

and from which there becomes:

$$e^{z} = \frac{p}{\left(1+bp^{n-1}\right)^{\frac{1}{n-1}}} \text{ and } e^{-z} = \frac{\left(1+bp^{n-1}\right)^{\frac{1}{n-1}}}{p},$$

on account of which we will have

$$\int e^{-z}\partial \mathrm{P} = am\int p^{m-2}\left(1+bp^{n-1}\right)^{\frac{1}{n-1}}\partial p,$$

in which no transcending quantities are present, thus so that *x* and *y* may be defined easily, and in this manner we obtain the complete integral of this differential equation of the first order :

$$y + bx\frac{\partial y^{n}}{\partial x^{n}} = a\frac{\partial y^{m}}{\partial x^{m}}.$$

Problem 6

§. 12. *For this proposed differential equation of the second order:* $p + \mathfrak{Q}x = \mathrm{Q}$ *, to find its complete integral.*

Solution.

By careful examination it will soon become apparent, this equation has arisen from the preceding, if in place of y, P, $\mathfrak{P}$ the letters p, Q, $\mathfrak{Q}$ may be written since the letters y, p, q, r etc. are progressing by a uniform rule; on account of which done by a substitution from the preceding solution we will have

$$x = \tfrac{e^{z}}{p+\mathfrak{Q}} = \int e^{-z}\partial \mathrm{Q}, \text{ with } z = \int\tfrac{\partial q}{p+\mathfrak{Q}} ;$$

and thus here x will be a function of the quantity q only, from which there becomes

$$\partial x = \tfrac{Q' - x\mathfrak{Q}'}{q+\mathfrak{Q}}\partial q.$$

Then now also p will be defined by a single variable q : for by §.10 there will become :

$$p = \mathrm{Q} - \tfrac{\mathfrak{Q}e^{z}}{q+\mathfrak{Q}}\int e^{-z}\partial \mathrm{Q}.$$

Therefore since there shall be $y = \int p\partial x$, also the quantity y will be expressed by a single function of q , and in this manner a complete solution is agreed on for the solution.

Problem 7

§.13. *From this proposed equation of a third order differential:* $q+\mathfrak{R}x=\mathrm{R}$ *, to assign its complete integral.*

Solution.

This solution can be derived in a similar manner from the first problem of the this second kind (§. 10**)** , while in place of $y,\ \mathrm{P},\ \mathfrak{P}$ there may be written $q,\ \mathrm{R},\ \mathfrak{R}$**,** that which if it were done in the first place for x , would supply this expression :

$$x=\tfrac{e^{z}}{p+\mathfrak{R}}=\int e^{-z}\partial\mathrm{R},\text{ with } z=\int\tfrac{\partial q}{p+\mathfrak{R}},$$

and thus x will be a function of the variable r alone*;* then truly there will be

$$\partial x=\tfrac{\mathrm{R}'-x\mathfrak{R}'}{r+\mathfrak{R}}\partial r.$$

Again here the formula found for pro y and thus transferred will give this expression for q :

$$q=\mathrm{R}-\tfrac{\mathfrak{R}e^{z}}{r+\mathfrak{R}}\int e^{-z}\partial\mathrm{R},$$

which also involves only one variable r and its functions. Therefore because $p=\int q\partial x$ and $y=\int p\partial x,$ there will become $y=\int\partial x\int q\partial x$, and thus also y is expressed by a single variable r .

Problem 8

§. 14. *For the proposed equation of the differential of the fourth order* $r+\mathfrak{S}x=\mathrm{S}$ *, to investigate its integral.*

Solution.

Here there will be

$$x=\tfrac{e^{z}}{s+\mathfrak{S}}x=\int e^{-z}\partial\mathrm{S},\text{ with } z \text{ being } =\int\tfrac{\partial s}{s+\mathfrak{S}}.$$

Again there will be

$$\partial x=\tfrac{\mathrm{S}'-x\mathfrak{S}'}{s+\mathfrak{S}}\partial s,\quad r=\mathrm{S}-\tfrac{\mathfrak{S}e^{z}}{s+\mathfrak{S}}\int e^{-z}\partial\mathrm{S},$$

$$q=\int r\partial x,\ \ p=\int \partial x\int r\partial x,$$

and

$$y=\int p\partial x=\int \partial x\int \partial x\int r\partial x,$$

where all are determined by the single variable s.

§. 15. There is no reason also, why these differential equations, the integrals of which we have shown here, may not be allowed to be adjoined in a certain manner, so that an integration by the same method, which we have used here, may be put in place. In this manner we will obtain innumerable new kinds of differential equations of this kind, which also on being differentiated will be able to lead to the integration, which argument we may investigate in the following problems.

Problem 9

§. 16. *On putting* $p+fq=t$, T *and* $\mathfrak{T}$ *shall be some functions of* t*, either algebraic or transcending, and it shall be proposed to investigate the complete integral of this differential equation of the second order :* $y+fp+\mathfrak{T}x=\mathrm{T}$.

Solution.

There may be put $y+fp=z$, there will become

$$\partial z=\partial x\left(p+fq\right),\ \text{therefore}\ \partial z=t\partial x.$$

Whereby since now the proposed equation shall be $z+\mathfrak{T}x=\mathrm{T}$, on differentiating there arises

$$\partial z+\mathfrak{T}\partial x+x\partial\mathfrak{T}=\partial\mathrm{T},$$

or

$$\left(t+\mathfrak{T}\right)\partial x+x\partial\mathfrak{T}=\partial\mathrm{T},$$

from which this equation is deduced:

$$\partial x+\tfrac{x\partial\mathfrak{T}}{t+\mathfrak{T}}=\tfrac{\partial\mathrm{T}}{t+\mathfrak{T}},$$

towards integrating which, there may be put

$$\int\tfrac{\partial t}{t+\mathfrak{T}}=u,$$

and there will become

$$\int\tfrac{\partial\mathfrak{T}}{t+\mathfrak{T}}=l\left(t+\mathfrak{T}\right)-u,$$

while truly our equation may be rendered integrable, if we may multiply that by $e^{-u}\left(t+\mathfrak{T}\right)$ [*i.e.* the integrating factor]; indeed the integral will be

$$xe^{-u}\left(t+\mathfrak{T}\right)=\int e^{-u}\partial\mathrm{T},$$

from which there is deduced

$$x=\tfrac{e^{u}}{t+\mathfrak{T}}\int e^{-u}\partial\mathrm{T},$$

and thus x may be equal to a certain function of t, which may be able to be found by integration in this manner, and its differential will be

$$\partial x=\tfrac{\partial\mathrm{T}-x\partial\mathfrak{T}}{t+\mathfrak{T}}.$$

Hence therefore there will be $z=\mathrm{T}-\mathfrak{T}x$. Now since there shall be

$$y+fp=z\text{, there will be }y\partial x+f\partial y=z\partial x$$

from which there is deduced

$$\partial y+\tfrac{y\partial x}{f}=\tfrac{z\partial x}{f},$$

which equation multiplied by $e^{\frac{x}{f}}$ gives the integral

$$ye^{\frac{x}{f}}=\tfrac{1}{f}\int e^{\frac{x}{f}}z\partial x,$$

where since both z and x shall be functions of t, y also shall be a function of t only, since there shall be

$$y=\frac{e^{-\frac{x}{f}}}{f}\int e^{\frac{x}{f}}z\partial x.$$

Problem 10

§. 17. *On putting* $p+fq+gr=t$, *if* T *and* $\mathfrak{T}$ *were some functions of* t, *either algebraic or transcending, and this differential equation of the third order were proposed :*

$$y+fp+gq+\mathfrak{T}x=\mathrm{T},$$

to find its complete integral.

Solution.

There may be put

$$y+fp+gq=z,$$

and by differentiating it will become

$$\partial z = \partial x\left(p + fq + gr \right) = t\partial x \, ,$$

and thus our equation requiring to be integrated will be $z + \mathfrak{T}x = \mathrm{T}$, for which there will become as before

$$x = \tfrac{e^u}{t+\mathfrak{T}} \int e^{-u} \partial \mathrm{T}, \quad \text{and} \quad z = \mathrm{T} - \mathfrak{T}x \, ,$$

evidently on putting $\int \frac{\partial t}{t+\mathfrak{T}} = u$. Therefore both these expressions are functions of the variable *t* only, from which also ∂x may be expressed by the same variable. Therefore it yet remains, so that also the other principle variable y may be investigated. But since there shall be $y + fp + gq = z$, in place of the letters p and q the initial values assumed may be written $\frac{\partial y}{\partial x}$ and $\frac{\partial \partial y}{\partial x^2}$, and this equation will be required to be integrated, if the whole equation may be multiplied by ∂x^2:

$$y\partial x^2 + f\partial x\partial y + g\partial\partial y = z\partial x^2 ,$$

in which since both *x* and *z* shall be functions of *t* alone, also *y* will be allowed to be treated as a function of t. But formerly it has been shown by me and others, how such an equation must be treated, therefore such an expansion would be superfluous to repeat here. Indeed it may be sufficient to note, the value of *y* to be designated by the terms of this form $\int \mathrm{e}^{\lambda x} z\partial x$, that therefore will be allowed to be expressed by the variable *t* alone, and thus also *y* will be defined as a of function of *t*.

Problem 11.

§. 18. *By putting* $p + fq + gr + hs = t$, *if* T and $\mathfrak{T}$ *were any functions of t, either algebraic or transcending, and such a differential equation of the fourth order were proposed:*

$$y + fp + gq + hr + \mathfrak{T}x = \mathrm{T} \, ,$$

to search for its complete integral.

Solution.

Let $y + fp + gq + hr = z$, and on differentiating there will be

$$\partial z = \partial x(p + fq + gr + hs) = t\partial x \, ,$$

and the equation requiring to be integrated will become $z+\mathfrak{T}x=\mathrm{T}$, for which again, on assuming

$$\int\frac{\partial t}{t+\mathfrak{T}}=u\,,$$

there will become

$$x=\frac{e^{u}}{t+\mathfrak{T}}\int e^{-u}\partial\mathrm{T},\quad\text{and}\quad z=\mathrm{T}-\mathfrak{T}x\,,$$

thus so that both *x* as well as *z* may be expressed by the single variable *t*. With these found, if in the equation assumed initially, the values of these may be substituted in place of *p, q, r, s,* this equation of the third order will be produced :

$$y\partial x^{3}+f\partial x^{2}\partial y+g\partial x\partial\partial y+h\partial^{3}y=z\partial x^{3},$$

of which the complete integral is allowed to be regarded as known, which are set out by these which are concerned with equations of this kind, thus so that also in this case both the variables *x* and *y* may be expressed by the new variable *t* . Moreover it is readily apparent in this manner for differential equations to be allowed to progress to higher orders. Therefore on this account the calculation of the integral is not to be denied from increased orders. Therefore since here the main problem is concerned with the complete integration of this equation:

$$y+\frac{f\partial y}{\partial x}+\frac{g\partial\partial y}{\partial x^{2}}+\frac{h\partial^{3}y}{\partial x^{3}}+\text{etc.}=z,$$

where *z* is some function of *x,* now here we may apply and briefly show its resolution presented occasionally. This equation may be formed:

$$1+fu+gu^{2}+hu^{3}+iu^{4}+\text{ etc.}=0,$$

the roots *u* of which may be designated by the letters $\alpha,\ \beta,\ \gamma,\ \delta$ etc., with which found there will be as I have shown at some time [see E188, §12 and Euler's *Integral Calculus*, vol. II. §1160.]:

$$y=\frac{e^{\alpha x}\int e^{-\alpha x}z\partial x}{f+2g\alpha+3h\alpha^{2}+4i\alpha^{3}+\text{ etc.}}+\frac{e^{\beta x}\int e^{-\beta x}z\partial x}{f+2g\beta+3h\beta^{2}+4i\beta^{3}+\text{ etc.}}+\text{etc.}$$

Clearly these formulas formed from the individual roots $\alpha,\ \beta,\ \gamma,\ \delta$ etc. and taken together will give the value of *y* and thus the complete integral, since the individual integral formulas involve an arbitrary constant.

2) Example of a differential equation of indefinite order and the integration of those.

M. S. shown to the Academy on the 13th *of December* 1781.

[E681]

§. 19. When differential equations are distinguished following the order of the differentials, the nature of the matter itself is seen to exclude intermediate orders ; since indeed there shall be a need of just as many integrals, of which the number cannot be but an integer. Yet recently I came upon a differential equation of indefinite order, the exponent of which also can be a fractional number, and thus I was able to assign its integral ; because with all due attention it may be observed, the whole analysis, which I have used, I may set out here clearly.

§. 20. From the amazing properties of the fractional powers of the binomial I was considering, which is usually indicated by this character $\left(\frac{p}{q}\right)$, of which the value is produced here :

$$\frac{p}{1}\cdot\frac{p-1}{2}\cdot\frac{p-2}{3}\cdots\cdots\ \frac{p-q+1}{q},$$

the value of this kind of formula $\left(\frac{p}{q}\right)$ came to mind for recalling whole numbers, from which the case also came to mind, in which p and q are not whole numbers, may be able to be assigned. Indeed I observed such a reduction not to succeed directly, from which I have considered its reciprocal value $\frac{1}{\left(\frac{p}{q}\right)}$, of which the value is

$$\frac{1}{p}\cdot\frac{2}{p-1}\cdot\frac{3}{p-2}\cdots\cdots\ \frac{q}{p-q+1}.$$

Hence in the end I put in place

$$\frac{1\cdot2\cdot3\cdots q\times x^{p}}{p(p-1)(p-2)\cdots(p-q+1)}=s,$$

thus so that on putting $x=1$ the desired value of $1:\left(\frac{p}{q}\right)$ may be obtained.

[Note the use of $\left(\frac{p}{q}\right)$ rather than $\binom{p}{q}$ for a binomial coefficient.]

§. 21. Now for the sake of brevity there shall be $1\cdot2\cdot3\cdots q=\mathrm{N}$, so that there may be had

$$s=\frac{\mathrm{N}x^{p}}{p\cdots(p-q+1)},$$

in the denominator of which it is being understood the factors continually decrease by unity. But if now this formula may be differentiated, there will be produced

$$\frac{\partial s}{\partial x}=\frac{\mathrm{N}x^{p-1}}{(p-1)\cdots(p-q+1)},$$

and thus the first factor of the denominator has been removed, and by differentiating anew there will be put in place

$$\frac{\partial\partial s}{\partial x^2}=\frac{\mathrm{N}x^{p-2}}{(p-2)\cdots(p-q+1)}.$$

Therefore by differentiating continually all the factors will be removed from the denominator, and finally it will arrive at this equation :

$$\frac{\partial^q s}{\partial x^q}=\mathrm{N}x^{p-q}.$$

§.22. Therefore, by substituting its value in place of N, we will arrive at this differential equation

$$\frac{\partial^q s}{1\cdots q\partial x^q}=x^{p-q},$$

which therefore will be necessary to integrate just as many times as q contains ones, and the individual integrations thus are required to be put in place, so that on putting $x=0$ the integrals may vanish, and after all the integrations will have been absolute, in place of x there will be written one, and in this way the value of s resulting will give the value of the formula $1:\left(\frac{p}{q}\right)$.

But so that we may set out these integrations more generally, we may write X in place of x^{p-q}, so that we may have this equation requiring to be resolved :

$$\frac{\partial^q s}{1\cdots q\partial x^q}=\mathrm{X}.$$

§. 23. Initially we may multiply this equation by ∂x, and its integral will give

$$\frac{\partial^{q-1} s}{1\cdot 2\cdot 3\cdots q\cdot\partial x^{q-1}}=\int \mathrm{X}\partial x.$$

We may multiply this same equation by $1\cdot\partial x$, and on integrating [again] there will become

$$\frac{\partial^{q-2} s}{2\cdot 3\cdots q\partial x^{q-2}}=\int\partial x\int \mathrm{X}\partial x=x\int \mathrm{X}\partial x-\int \mathrm{X}x\partial x.$$

Indeed repeated integrals of this kind are able to be reduced to simple integrals by known reductions. This equation now multiplied by $2\partial x$ and integrated in the same manner will present

$$\frac{\partial^{q-3}s}{3\cdot 4\cdots q\cdot\partial x^{q-3}}=x^2\int X\partial x-2x\int Xx\partial x+\int Xx^2\partial x.$$

Now on being multiplied by $3\partial x$ and intergraded there will arise

$$\frac{\partial^{q-4}s}{4\cdot 5\cdots q\cdot\partial x^{q-4}}=x^3\int X\partial x-3x^2\int Xx\partial x+3x\int Xx^2\partial x-\int Xx^3\partial x.$$

There will be found in the same manner:

$$\frac{\partial^{q-5}s}{5\cdot 6\cdots q\cdot\partial x^{q-5}}=x^4\int X\partial x-4x^3\int Xx\partial x+6x^2\int Xx^2\partial x-4x\int Xx^3\partial x+\int Xx^4\partial x,$$

and thus in general by calling our characters in the usual manner there will become :

$$\frac{\partial^{q-n}s}{n(n+1)\cdots q\cdot\partial x^{q-n}}=$$
$$x^{n-1}\int X\partial x-\left(\tfrac{n-1}{1}\right)x^{n-2}\int Xx\partial x+\left(\tfrac{n-1}{2}\right)x^{n-3}\int Xx^2\partial x-\left(\tfrac{n-1}{3}\right)x^{n-4}\int Xx^3\partial x+\text{etc.}$$

§. 24. Now we may put $n=q$, and since there shall be $\partial^0 s=s$,this last equation will arise:

$$\frac{s}{q}=x^{q-1}\int X\partial x-\left(\tfrac{q-1}{1}\right)x^{q-2}\int Xx\partial x+\left(\tfrac{q-1}{2}\right)x^{q-3}\int Xx^2\partial x-\text{etc.},$$

the individual terms of which thus must be integrated, so that they may vanish on putting $x=0$, which indeed will always happen, but only if there shall be $q-1>0$, on account of which these integral formulas $\int X\partial x,\ \int Xx\partial x$ etc. must only be integrated without the addition of a constant. Even if now *x* may be increased perhaps in the denominator, by a power of *x*, by which they must be multiplied, again it will be removed.

§. 25. As regards these individual integrals observed, it will now be allowed to put $x=1$ outside the summation signs, certainly which is the case in the question proposed ; and thus [note: $1:q\left(\tfrac{p}{q}\right)=1:\left(\tfrac{p}{q-1}\right)$]

$$1:q\left(\tfrac{p}{q}\right)=\int X\partial x[1-\left(\tfrac{q-1}{1}\right)x+\left(\tfrac{q-1}{2}\right)x^2-\left(\tfrac{q-1}{3}\right)x^3+\text{etc.}],$$

the value of which series clearly is $(1-x)^{q-1}$, thus so that we will have determined this expression :

$$\frac{1}{q\left(\frac{p}{q}\right)}=\int \mathrm{X}\partial x(1-x)^{q-1},$$

of which the value therefore also can be shown by quadrature in the cases, in which q is not a whole number, and thus we have elicited happily the integral of the differential equation $\partial^q s=\mathrm{NX}\partial x^q$ of the indefinite order, and because $\mathrm{X}=x^{p-q}$, all the fractions will be rendered in this manner

$$\left(\frac{p}{q}\right)=\frac{1}{q\int x^{p-q}\partial x(1-x)^{q-1}},$$

and because the exponents of x and of $1-x$ can be interchanged, there will be also

$$\left(\frac{p}{q}\right)=\frac{1}{q\int x^{q-1}\partial x(1-x)^{p-q}},$$

and not long since I have arrived at this formula thus from a very different principle.

[See E421.]

Theorem 1.

§. 26. The value of this character $\left(\frac{p}{q}\right)$ can be reduced to an integral formula, since there shall be

$$\left(\frac{p}{q}\right)=\frac{1}{q\int x^{q-1}\partial x(1-x)^{p-q}},$$

if indeed this integral may be extended from $x=0$ to $x=1$.

Corollary I

§. 27. Therefore on assuming $p=0$ there will become

$$\left(\frac{0}{q}\right)=\frac{1}{q\int x^{q-1}\partial x(1-x)^{-q}}.$$

But I have shown previously to be

$$\int x^{q-1}\partial x(1-x)^{-q}=\frac{\pi}{\sin.\pi q},$$

from which therefore there becomes

$$\left(\frac{0}{q}\right)=\frac{\pi}{\sin.\pi q}.$$

Corollary 2

§. 28. Then by the noted reduction of the integral there is found :

$$\int x^{q-1}\partial x(1-x)^{p-q}=\frac{\pi}{\sin.\pi q}\left(\frac{p-q}{q}\right),$$

of which the value therefore, whenever p is a whole number, on account of which in general there will be

$$\left(\frac{p}{q}\right)=\frac{\sin.\pi q}{\pi}:\left(\frac{p-q}{q}\right).$$

Corollary 3

§. 29. Therefore since in turn there shall be

$$\int x^{q-1}\partial x(1-x)^{p-q}=\frac{1}{q\left(\frac{p}{q}\right)},$$

if here in place of $q-1$ we may write f, and g in place $p-q$, we will have

$$\int x^{r}\partial x(1-x)^{g}=\frac{1}{(1+f)\left(\frac{f+g+1}{f+1}\right)}.$$

Scholium

§. 30. Therefore since we have obtained this formula from an integral equation of indefinite order, we may extend the same investigation wider in the following problem.

Problem 12

§. 31. *For the proposed series, either finite or infinite,*

$$\mathrm{S}=\frac{\mathrm{A}}{\left(\frac{p}{q}\right)}+\frac{\mathrm{B}}{\left(\frac{p+1}{q}\right)}+\frac{\mathrm{C}}{\left(\frac{p+2}{q}\right)}+\frac{\mathrm{D}}{\left(\frac{p+3}{q}\right)}+\text{etc.},$$

to express its value by an integral formula.

Solution.

We may attribute powers of x to the individual terms, and we establish

$$\mathrm{S} = \frac{\mathrm{A}x^{p}}{\left(\frac{p}{q}\right)} + \frac{\mathrm{B}x^{p+1}}{\left(\frac{p+1}{q}\right)} + \frac{\mathrm{C}x^{p+2}}{\left(\frac{p+2}{q}\right)} + \text{etc.},$$

which series therefore, on putting $x = 1$, will present the same proposed series. Where it is required to be observed the letter q keeps the same value in all the terms, truly the other letter p to be increased by one continually, from which the indefinite product $1 \cdot 2 \cdot 3 \cdot \ \cdot \ \cdot q = \mathrm{N}$ will retain the same value in all the terms. Whereby since above from the equation $s = \frac{x^{p}}{\left(\frac{p}{q}\right)}$ we will have deduced this differential equation of indefinite order :

$$\frac{\partial^{q} s}{\partial x^{q}} = \mathrm{N}x^{p-q},$$

the same differential will result from the individual terms of our series, but only if we may increase the exponent p by one, from which therefore we will find

$$\frac{\partial^{q}\mathrm{S}}{\partial x^{q}} = \mathrm{NA}x^{p-q} + \mathrm{NB}x^{p-q+1} + \text{etc.}$$

§. 32. Now we may put

$$\mathrm{A} + \mathrm{B}x + \mathrm{C}x^{2} + \mathrm{D}x^{3} + \text{etc.} \ = \mathrm{V},$$

and there will become

$$\frac{\partial^{q}\mathrm{S}}{\mathrm{N}\partial x^{q}} = x^{p-q}\mathrm{V},$$

on account of which if we may put $x^{p-q}\mathrm{V} \ = \mathrm{X}$, we will have the same equation now examined before

$$\frac{\partial^{q}\mathrm{S}}{1 \cdot 2 \cdot 3 \cdots q \cdot \partial x^{q}} = \mathrm{X},$$

the integration of which repeated q times leads to this expression $S = q\int \mathrm{X}\partial x(1-x)^{q-1}$, from which therefore with the values for X and V being substituted we will obtain the sum sought S, evidently

$$\mathrm{S} = q\int x^{p-q}\partial x\left(\mathrm{A} + \mathrm{B}x + \mathrm{C}x^{2} + \mathrm{D}x^{3} + \text{etc.}\right)(1-x)^{q-1},$$

but only if this integral may be extended from $x = 0$ to $x = 1$, or as we have found before, only if in the integration no constant may be added, then truly there may be taken $x = 1$.

Example

§. 33. Let $\mathrm{V}=(1-x)^n$, thus so that there shall become

$$A=1,\ \mathrm{B}=-\left(\tfrac{n}{1}\right), \mathrm{C}=+\left(\tfrac{n}{N}\right), D=-\left(\tfrac{n}{3}\right) \text{ etc.},$$

and the proposed series will become :

$$\mathrm{S}=\frac{1}{\left(\frac{p}{q}\right)}-\frac{\left(\frac{n}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{n}{2}\right)}{\left(\frac{p+2}{q}\right)}-\frac{\left(\frac{n}{3}\right)}{\left(\frac{p+3}{q}\right)}+\text{etc.},$$

then the sum of this series therefore will be

$$S=q\int x^{p-q}\partial x(1-x)^{q+n-1},$$

or with the exponents of x and $1-x$ interchanged, also there will be

$$S=q\int x^{q+n-1}\partial x(1-x)^{p-q}.$$

But now it is evident this same integral formula can be reduced here to the usual character ; for with the help of § 29 there will be $f=q+n-1$ and $g=p-q$, and hence there will be produced :

$$S=\frac{q}{(q+n)\left(\frac{p+n}{q+n}\right)}.$$

Hence therefore just as we will have this most noteworthy sum of the infinite series from the integral formulas:

$$\frac{1}{\left(\frac{p}{q}\right)}-\frac{\left(\frac{n}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{n}{2}\right)}{\left(\frac{p+2}{q}\right)}-\frac{\left(\frac{n}{3}\right)}{\left(\frac{p+3}{q}\right)}+\frac{\left(\frac{n}{4}\right)}{\left(\frac{p+4}{q}\right)}-\text{etc.}=\frac{q}{(q+n)\left(\frac{p+n}{q+n}\right)}.$$

[Note : These following infinite series may have terms greater than one and are hence diverging. See note in *O.O.* edition for E681.]

Corollary 1

§.34. Therefore if there were $n=0$, evidently this identical equation arises $\frac{1}{\left(\frac{p}{q}\right)}=\frac{1}{\left(\frac{p}{q}\right)}$.

But if $n=1$ there is produced

$$\frac{q}{(q+1)\left(\frac{p+1}{q+1}\right)}=\frac{1}{\left(\frac{p}{q}\right)}-\frac{1}{\left(\frac{p+1}{q}\right)}.$$

If $n = 2$, there becomes

$$\frac{q}{(q+2)\left(\frac{p+2}{q+2}\right)}=\frac{1}{\left(\frac{p}{q}\right)}-\frac{2}{\left(\frac{p+1}{q}\right)}+\frac{1}{\left(\frac{p+2}{q}\right)}.$$

Corollary 2

§. 35. So that the agreement with the truth may become more apparent, we will set out the case to be determined, where $p=3,\ q=2,\ n=4,$ and there will become :

$$\tfrac{q}{q+n}=\tfrac{1}{3},\ \text{and}\ \left(\tfrac{p+n}{q+n}\right)=\left(\tfrac{7}{6}\right)=\left(\tfrac{7}{1}\right)=7.$$

Thence if

$$\left(\tfrac{p}{q}\right)=\left(\tfrac{3}{2}\right)=3,\ \left(\tfrac{p+1}{q}\right)=\left(\tfrac{4}{2}\right)=6,\left(\tfrac{p+2}{q}\right)=\left(\tfrac{5}{2}\right)=10,\ \left(\tfrac{p+3}{q}\right)=\left(\tfrac{6}{2}\right)=15,$$

which is the progression of triangular numbers ; then truly there will become

$$\left(\tfrac{n}{1}\right)=4,\ \left(\tfrac{n}{2}\right)=6,\ \left(\tfrac{n}{3}\right)=4,\ \left(\tfrac{n}{4}\right)=1.$$

With these values substituted there will be

$$\tfrac{1}{3\cdot 7}=\tfrac{1}{3}-\tfrac{4}{6}+\tfrac{6}{10}-\tfrac{4}{15}+\tfrac{1}{21},$$

which agrees exactly.

Example 2

§.36. We may put $\mathrm{V}=(1+x)^{q-1}$, so that there may become :

$$S=q\int x^{p-q}\partial x(1-xx)^{q-1};$$

then truly there will be

$$A=1,\ \mathrm{B}=\left(\tfrac{q-1}{1}\right),\ \mathrm{C}=\left(\tfrac{q-1}{2}\right),\ D=\left(\tfrac{q-1}{3}\right)\ \text{etc.},$$

and thus the proposed series will become

$$\mathrm{S}=\frac{1}{\left(\frac{p}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{p+2}{q}\right)}+\frac{\left(\frac{q-1}{3}\right)}{\left(\frac{p+3}{q}\right)}+\text{etc.},$$

Moreover it is evident, this formula also can be reduced to our characters. For we may put $xx=y$, then there will become

$$S=\tfrac{q}{2}\int y^{p-q-1}\partial y(1-y)^{q-1},$$

or with the exponents interchanged:

$$S=\frac{q}{2}\int y^{q-1}\partial y(1-y)^{\frac{p-q-1}{2}},$$

which compared with § 29 gives $f=q-1,\ g=\frac{p-q-1}{2}$, with which values substituted there is deduced

$$\mathrm{S}=\frac{q}{2q\left(\frac{\frac{p+q-1}{2}}{q}\right)}=\frac{1}{2\left(\frac{\frac{p+q-1}{2}}{q}\right)}=\frac{1}{\left(\frac{p}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{p+2}{q}\right)}+\text{etc.},$$

or if there may be put $\frac{p+q-1}{2}=r$, there will become :

$$\mathrm{S}=\frac{1}{2\left(\frac{r}{q}\right)}=\frac{1}{\left(\frac{p}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{p+2}{q}\right)}+\text{etc.},$$

Corollary I

§.37. Here in the case $q=1$ the sum found is equal the first term. But if we may put $q=2$, there will become

$$\frac{1}{2\left(\frac{\frac{p+1}{2}}{q}\right)}=\frac{1}{\left(\frac{p}{2}\right)}+\frac{1}{\left(\frac{p+1}{2}\right)},$$

that is

$$\frac{4}{pp-1}=\frac{2}{p(p-1)}+\frac{2}{p(p+1)},$$

from which it is apparent the same summation to be agreed to be true, from which indeed no doubt can remain, as long as q is a positive whole number ; on account of which we may consider certain cases, where this is not the case.

Corollary 2

§. 38. But so that the exposition may appear easier, we may consider the case where $r=q$, so that there may become $\left(\frac{r}{q}\right)=1$, but then there will become $p=1+q$, and hence

$$\left(\frac{p}{q}\right)=1+q,\quad \left(\frac{p+1}{q}\right)=\frac{q+1}{1}\cdot\frac{q+2}{2},\quad \left(\frac{p+2}{q}\right)=\frac{q+1}{1}\cdot\frac{q+2}{2}\cdot\frac{q+3}{3},$$

with which substituted this series will arise :

$$\frac{1}{2}=\frac{1}{q+1}+\frac{2(q-1)}{(q+1)(q+2)}+\frac{3(q-1)(q-2)}{(q+1)(q+2)(q+3)}+\frac{4(q-1)(q-2)(q-3)}{(q+1)(q+2)(q+3)(q+4)}+\text{etc.},$$

which series is especially noteworthy, because its sum is $\frac{1}{2}$ always, whatever values may be attributed to q [*i.e.* provided it converges]. For if $q=0$, there will become

$$\frac{1}{2}=1-1+1-1+1-\text{etc.},$$

which series is best know. Now if there shall be $q=-1$, and on account of $q+1=0$ we may multiply all the terms by $q+1$, and this series will be produced

$$0=1-4+9-16+25-\text{etc.},$$

as is readily apparent by taking the differences. We may put $q=\frac{1}{2}$, and this series will be produced :

$$\frac{1}{2}=\frac{2}{3}-\frac{2\cdot 2}{3\cdot 5}+\frac{2\cdot 3}{5\cdot 7}-\frac{2\cdot 4}{7\cdot 9}+\frac{2\cdot 5}{9\cdot 11}-\text{etc.}$$

Therefore since there shall be

$$\frac{2}{3}=1-\frac{1}{3},\ \frac{4}{3\cdot 5}=\frac{2}{3}-\frac{2}{5},\ \frac{6}{5\cdot 7}=\frac{3}{5}-\frac{3}{7},\ \ \frac{8}{7\cdot 9}=\frac{4}{7}-\frac{4}{9},$$

and thus again, with these substituted, this series will be produced :

$$\frac{1}{2}=1-1+1-1+1-\text{etc.},$$

But if we take $q=-\frac{1}{2}$ there will be

$$\frac{1}{2}=2-4+6-8+10-12+\text{etc.},$$

which shall be evident from the differences.

Corollary 3

§. 39. Now we may take $r=0$, so that there may become $p=1-q$. But I have shown to be $\left(\frac{0}{q}\right)=\frac{\sin.q\pi}{q\pi}$, from which there will arise

$$\frac{\pi q}{2\sin.q\pi}=\frac{1}{\left(\frac{1-q}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{2-q}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{3-q}{q}\right)}+\text{etc.},$$

of which there will be a need for the case $q=\frac{1}{2}$ to be established, indeed the left hand member shall become $\frac{\pi}{4}$.
But for the right hand side we well have :

$$\left(\frac{q-1}{1}\right)=-\frac{1}{2},\ \left(\frac{q-1}{2}\right)=\frac{1\cdot 3}{2\cdot 4},\ \left(\frac{q-1}{3}\right)=-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\text{ etc.,}$$

then truly for the denominators :

$$\left(\frac{1-q}{q}\right)=1,\ \left(\frac{2-q}{q}\right)=\frac{3}{2},\ \left(\frac{3-q}{q}\right)=\frac{3\cdot 5}{2\cdot 4},\ \left(\frac{4-q}{q}\right)=\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\text{ etc.,}$$

with which values substituted this series will arise :

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{etc.,}$$

which is a most noteworthy series. Moreover at this point we may put $q=-\frac{1}{2}$, and the left hand member will be $\frac{\pi}{4}$ as before ; but for the right hand side there will be

$$\left(\frac{q-1}{1}\right)=-\frac{3}{2},\ \left(\frac{q-1}{2}\right)=\frac{3\cdot 5}{2\cdot 4},\ \left(\frac{q-1}{3}\right)=\frac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\text{ etc.,}$$

then

$$\left(\frac{1-q}{q}\right)=\frac{1\cdot 3}{2\cdot 4},\ \left(\frac{2-q}{q}\right)=\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6},\ \left(\frac{3-q}{q}\right)=\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\text{ etc.,}$$

hence

$$\frac{\pi}{4}=\frac{2\cdot 4}{1\cdot 3}-\frac{4\cdot 6}{1\cdot 5}+\frac{6\cdot 8}{1\cdot 7}-\frac{8\cdot 10}{1\cdot 9}+\text{etc.,}$$

the truth of which is thus shown. Since there shall be

$$\frac{2\cdot 4}{1\cdot 3}=3-\frac{1}{3},\ \frac{4\cdot 6}{1\cdot 5}=5-\frac{1}{5},\ \frac{6\cdot 8}{1\cdot 7}=7-\frac{1}{7},\ \frac{8\cdot 10}{1\cdot 9}=9-\frac{1}{9}\text{ etc.,}$$

that series will be equal to this:

$$\frac{\pi}{4}=3-\frac{1}{3}-5+\frac{1}{5}+7-\frac{1}{7}-9+\frac{1}{9}+\text{etc.,}$$

which series may be split into these two parts :

$$\frac{\pi}{4}=\begin{cases}\ \ 3-5+7-9+11-13+\text{etc.}\\ -\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\frac{1}{13}-\text{etc.}\end{cases}$$

For the upper it may be noted , the sum by extracting these differences to be :

$$3-5+7-9+11-13+\text{etc.}=1;$$

the lower sum found from the above series, which was

$$\tfrac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\text{etc.},$$

will be

$$-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\text{etc.}=\tfrac{\pi}{4}-1,$$

from which it is evident to become

$$3-\tfrac{1}{3}-5+\tfrac{1}{5}+7-\tfrac{1}{7}-9+\tfrac{1}{9}+\text{etc.}=1+\tfrac{\pi}{4}-1=\tfrac{\pi}{4}.$$

Hence it is therefore apparent, also negative numbers and hence fractions can be taken for q.

General Theorem.

§. 40. If X may denote some function of x, and this differential equation of any order may be proposed :

$$\partial^q y=1\cdot 2\cdot 3\cdots q\mathrm{X}\partial x^q,$$

where the exponent q may denote some number either whole or fractional, either positive or negative, therefore the resolution of this equation will be required by just as many integrations, as if the individual ones may start from $x=0$ and with all performed there may be put $x=1$, then there will be always $y=q\int \mathrm{X}\partial x(1-x)^{q-1}$, clearly with this extension of the integral from $x=0$ to $x=1$.

SUPPLEMENTUM X.

AD SECT. II. TOM. II.
DE
RESOLUTIONE AEQUATIONUM DIFFERENTIALIUM TERTII ALIORUMQUE GRADUUM, QUAE DUAS TANTUM VARIABILES INVOLVUNT. [E680]

1.) De aequationibus differentialibus cujuscunque gradus, quae denuo differentiatae integrari possunt

M. S. Academiae exhibuit die 8. Octobris 1781

§. 1. Sint x et y binae variabiles, inter quas earumque differentialia cuiuscunque gradus aequationes propositae subsistant. Ad formam differentialium tollendam ponatur more solito

$$\partial y = p\partial x,\ \partial p = q\partial x,\ \partial q = r\partial x,\ \partial r = s\partial x \text{ etc.}$$

ita ut, sumto elemento ∂x constante, sit

$$p = \frac{\partial y}{\partial x},\ q = \frac{\partial\partial y}{\partial x^2},\ r = \frac{\partial^3 y}{\partial x^3},\ s = \frac{\partial^4 y}{\partial x^4}, \text{ etc.}$$

Sint porro P et $\mathfrak{P}$ functiones quaecunque ipsius p; Q et $\mathfrak{Q}$ functiones quaecunque ipsius q; R et $\mathfrak{R}$ ipsius r; S et $\mathfrak{S}$ ipsius s etc., quae functiones non solum esse possunt rationales, sed etiam irrationales, atque adeo transcendentes.

§. 2. His positis duo aequationum genera per differentiationem integrare docebo, quarum primum istas continet aequationes:

$$y - px = \mathrm{P},\ p - qx = \mathrm{Q},\ q - rx = \mathrm{R},\ r - sx = \mathrm{S} \text{ etc.},$$

quarum prima involvere potest functiones quascunque ipsius ∂y tam rationales quam irrationales, quin etiam functiones transcendentes; secunda tales functiones ipsius $\partial\partial y$ involvere potest, tertia ipsius $\partial^3 y$ quarta ipsius $\partial^4 y$ et ita porro, cuiusmodi aequationum integratio certe nemini adhuc in mentem venire potuit.

§. 3. Alterum genus aequationum, quarum integrationem per differentiationem expedire docebo, sequentes complectitur aequationes:

$$y + \mathfrak{P}x = \mathrm{P},\ p + \mathfrak{Q}x = \mathrm{Q},\ q + \mathfrak{R}x = \mathrm{R},\ r + \mathfrak{S}x = \mathrm{S} \text{ etc.},$$

quae duas functiones quascunque involvunt. Evidens autem est has aequationes praecedentes in se comprehendere, quando scilicet est

$$\mathfrak{P} = -p,\ \mathfrak{Q} = -q,\ \mathfrak{R} = -r,\ \mathfrak{S} = -s \text{ etc.}$$

Ceterum patet, has aequationes adeo complicatas esse posse, ut nemo certe earum integrationem suscipere voluerit.

De aequationibus prioris generis.

Problema I

§.4. *Proposita aequatione differentiali primi gradus* $y - px = \mathrm{P}$*, eius integrale completum invenire.*

Solutio.

Cum sit $\partial y = p\partial x$, si aequatio proposita differentietur, prodibit haec $-x\partial p = \partial \mathrm{P}$, unde, posito $\partial \mathrm{P} = \mathrm{P}'\partial p$, colligitur $x = -\mathrm{P}'$. Quod si iam *p* tanquam novam variabilem spectemus, per eam tam *x* quam *y* exprimere poterimus. Cum enim sit $y = px + \mathrm{P}$, erit $y = \mathrm{P} - p\mathrm{P}'$, unde, eliminando *p*, quoties quidem calculus id permittet, conflari poterit aequatio inter *x* et *y*, quae autem tantum ut integrale particulare spectari debet, quia nullam involvit constantem arbitrariam. At vero, quoniam aequationem per differentiationem erutam $-x\partial p = \mathrm{P}'\partial p$ per ∂p dividere licuit, iste factor nihilo aequatus integrale completum suppeditare est censendus. Posito enim $\partial p = 0$ erit $p = \text{const.} = \alpha$, ideoque $y = \int p\partial x = \alpha x + \beta$. Haec quidem aequatio duas constantes arbitrarias involvere videtur; at vero altera per ipsam aequationem propositam determinatur, cum facta substitutione fiat

$$\alpha x + \beta - \alpha x = \mathrm{P},\ \text{ideoque } \beta = \mathrm{P} = f : \alpha.$$

Problema 2.

§.5. *Proposita aequatione differentiali secundi gradus* $p - qx = \mathrm{Q}$*, eius integrale completum assignare.*

Solutio.

Si haec aequatio differentietur et loco ∂p scribatur $q\partial x$, prodibit ista $-x\partial q = \partial \mathrm{Q}$, sive, posito $\partial \mathrm{Q} = \partial \mathrm{Q}'\partial q$, erit $-x\partial q = \mathrm{Q}'\partial q$. Hinc factor communis ∂q nihilo aequatus praebet $q = \text{const.} = 2\alpha$, unde fit

$$p = \int q\partial x = 2\alpha x + \beta, \text{hincque}$$

$$y = \int p\partial x = \alpha xx + \beta x + \gamma,$$

quarum trium constantium $\alpha,\ \beta,\ \gamma$ una per aequationem propositam determinatur. Facta autem divisione per ∂q habebimus $\mathrm{X} = -\mathrm{Q}'$, unde colligitur

$$p = \mathrm{Q} + qx = \mathrm{Q} - q\mathrm{Q}',$$

hincque ob $\partial x = -\partial Q' = -Q''\partial q,$ erit

$$y = \int p\partial x = \int Q''\partial q(Q'q - Q) + b..$$

Exemplum

§.6. Sit $Q = aq^m$, erit

$$Q' = maq^{m-1}$$

atque

$$Q'' = m\left(m-1\right)aq^{m-2}.$$

Hoc ergo casu erit

$$x = -Q' = -maq^{m-1},$$

$$y = m(m-1)^2 aa\int q^{2m-2}\partial q + b,$$

sive

$$y = \tfrac{m(m-1)^2}{2m-1} aaq^{2m-1} + b.$$

Est vero $q^{m-1} = -\tfrac{x}{ma}$, ita ut valor ipsius *y* facile per *x* exprimi poterit, quo facto habebitur integrale completum huius aequationis differentio-differentialis

$$\tfrac{\partial y}{\partial x} = \ \tfrac{x\partial\partial y}{\partial x^2} = \tfrac{a\left(\partial\partial y\right)^m}{\partial x^{2m}}.$$

Problema 3

§.7. *Proposita aequatione differentiali tertii gradus* $q - rx = R$ *, eius integrale completum investigare.*

Solutio.

Haec aequatio differentiata, ob $\partial q = r\partial x,$ dat $x\partial r = \partial R = R'\partial r,$ cuius aequationis factor ∂r nihilo aequatus hanc suppeditabit aequationem:

$$y = \alpha x^3 + \beta x^2 + \gamma x + \delta\ ,$$

ubi quatuor constantium $\alpha,\ \beta,\ \gamma,\ \delta$ una ex ipsa aequatione proposita determinata habebitur. Cum enim hinc sit

$$p = 3\alpha x^2 + 2\beta x + \gamma,\ q = 6\alpha x + 2\beta,\ r = 6\alpha,$$

erit substituendo $2\beta = \mathrm{R}$, ita ut tres tantum constantes arbitrariae in calculo relinquantur, uti natura huiusmodi aequationum postulat. Facta autem divisione per ∂r satisfaciet aequatio $\mathrm{X} = -\mathrm{R}'$, unde colligitur $q = \mathrm{R} - r\mathrm{R}'$.
Hinc, ob

$$\partial x = -\partial R' = -R''\partial r\,,$$

reperietur

$$p = \int q\partial x = \int \mathrm{R}''\partial r\left(r\mathrm{R}' - \mathrm{R}\right),$$

ac denique $y = \int p\partial x$, ubi ob duplicem integrationem duae constantes arbitrariae inseruntur.

Exemplum

§. 8. Sit $\mathrm{R} = ar^m$, erit

$$\mathrm{R}' = mar^{m-1} \text{et } \mathrm{R}'' = m(m-1)ar^{m-2}\,,$$

unde colligitur

$$p = \tfrac{m(m-1)^2}{2m-1}aar^{2m-1} + b,$$

atque ob

$$\partial x = -\partial \mathrm{R}' = -\mathrm{R}''\partial r = m\left(m-1\right)ar^{m-2}\partial r\,,$$

nanciscimur

$$y = \int p\partial x = -\tfrac{m^2(m-1)^3}{(2m-1)(3m-2)}a^3r^{3m-2} - mabr^{m-1} + c,$$

unde ob $r^{m-1} = \frac{x}{ma}$ facile obtinetur aequatio finita inter *x* et *y*, haecque *ma* erit integrale completum huius aequationis differentialis tertii gradus:

$$\tfrac{\partial\partial y}{\partial x^2} - \tfrac{x\partial^3 y}{\partial x^3} = \tfrac{a\left(\partial^3 y\right)^m}{\partial x^{3m}}.$$

Problema 4

§. 9. *Proposita aequatione differentiali quarti gradus* $r - sx = S$, *eius integrale completum indagare.*

Solutio.

Ob $\partial r = s\partial x$ fiet, aequationem propositam differentiando, $-x\partial s = \partial S = S'\partial s$, cuius aequationis factor ∂s praebet aequationem finitam

$$y = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon\,,$$

ubi una constantium per ipsam aequationem propositam determinatur. Porro satisfacit aequatio $x = -\mathrm{S}'$, unde colligitur $r = \mathrm{S} - s\mathrm{S}'$, hincque, ob

$$\partial x = -\,\partial S' = -S''\partial s$$

reperitur

$$q = \int r\partial x,\ p = \int q\partial x \ \text{ et } \ y = \int p\partial x\,,$$

sive

$$y = \int \partial x \int \partial x \int r\partial x,$$

ubi ob triplicem integrationem tres adiiciendae sunt constantes arbitrariae.
Simili modo ad aequationes altiorum graduum progredi licet.

De aequationibus secundi generis

Problema 5

§. 10. *Proposita aequatione differentiali primi gradus huiusmodi* $y + \mathfrak{P}x = P$ *,* *eius integrale completum investigare.*

Solutio.

Si ista aequatio $y + \mathfrak{P}x = \mathrm{P}$ differentietur, et loco ∂y *oy* scribatur $p\partial x$, prodit haec:

$$p\partial x + \mathfrak{P}\partial x + x\partial\mathfrak{P} = \partial \mathrm{P}\,,$$

sive posito $\partial \mathrm{P} = \mathrm{P}'\partial p$, erit

$$\left(p + \mathfrak{P}\right)\partial x + x\partial\mathfrak{P} = \mathrm{P}'\partial p\,,$$

quae per $p + \mathfrak{P}$ divisa dat

$$\partial x + x\cdot\partial\cdot l\cdot\left(p + \mathfrak{P}\right) - \tfrac{x\partial p}{p+\mathfrak{P}} = \tfrac{\mathrm{P}'\partial p}{p+\mathfrak{P}}.$$

$$[\text{Est enim } \tfrac{x\partial\mathfrak{P}}{p+\mathfrak{P}} = x.\partial.l.\left(p + \mathfrak{P}\right) - \tfrac{x\partial p}{p+\mathfrak{P}}.]$$

Quod si iam ponamus

$$\int \tfrac{\partial p}{p+\mathfrak{P}} = \mathrm{z}\,,$$

aequatio illa integrabilis reddetur multiplicando per $e^{-\mathrm{z}}\left(p + \mathfrak{P}\right).$ Prodit enim

$$\left(p + \mathfrak{P}\right)e^{-\mathrm{z}}\partial x + (p + \mathfrak{P})x\mathrm{e}^{-\mathrm{z}}\partial{\cdot}l(p + \mathfrak{P}) - x\partial z e^{-\mathrm{z}}(p + \mathfrak{P}) = e^{-\mathrm{z}}\mathrm{P}'\partial p\ \ ,$$

cuius integrale manifesto est

$$xe^{-z}\left(p+\mathfrak{P}\right)=\int e^{-z}\mathrm{P}'\partial p,$$

unde colligitur

$$x=\tfrac{e^{z}}{p+\mathfrak{P}}\int e^{-z}\mathrm{P}'\partial p=\tfrac{e^{z}}{p+\mathfrak{P}}\int e^{-z}\partial\mathrm{P},$$

unde statim fit

$$y=\mathrm{P}-\tfrac{\mathfrak{P}e^{z}}{p+\mathfrak{P}}\int e^{-z}\partial\mathrm{P},$$

ubi e^{z} est etiam functio ipsius *p*, ita ut ambae variabiles *x* et *y* per unam eandemque variabilem *p* exprimantur, quae expressiones iam constantem arbitrariam per se involvunt, ita ut eius adiectione non amplius opus sit.

Exemplum.

§.11. Sit $\mathrm{P}=ap^{m}$ et $\mathfrak{P}=bp^{n}$, ita ut aequatio integranda sit $y+bp^{n}=ap^{m}$. Hic igitur erit

$$z=\int\tfrac{\partial p}{p\left(1+bp^{n-1}\right)}=\int\tfrac{\partial p}{p}-\int\tfrac{p^{n-2}\partial p}{1+bp^{n-1}},$$

unde colligitur actu integrando

$$z=lp-\tfrac{1}{n-1}l\left(1+bp^{n-1}\right),$$

ex quo fit

$$e^{z}=\frac{p}{\left(1+bp^{n-1}\right)^{\frac{1}{n-1}}}\quad\text{et}\quad e^{-z}=\frac{\left(1+bp^{n-1}\right)^{\frac{1}{n-1}}}{p},$$

quamobrem habebimus

$$\int e^{-z}\partial\mathrm{P}=am\int p^{m-2}\left(1+bp^{n-1}\right)^{\frac{1}{n-1}}\partial p,$$

in qua expressione nullae quantitates transcendentes insunt, ita ut *x* et *y* facile definiantur, hocque modo obtinetur integrale completum istius aequationis differentialis primi gradus

$$y+bx\tfrac{\partial y^{n}}{\partial x^{n}}=a\tfrac{\partial y^{m}}{\partial x^{m}}.$$

Problema 6

§. 12. *Proposita hac aequatione differentiali secundi gradus:* $p+\mathfrak{Q}x=\mathrm{Q}$, *eius integrale completum invenire.*

Solutio.

Attendenti mox patebit, hanc aequationem ex praecedente oriri, si loco $y,\ \mathrm{P},\ \mathfrak{P}$ scribantur litterae $p,\ \mathrm{Q},\ \mathfrak{Q}$ quandoquidem litterae *y, p, q, r* etc. uniformi lege progrediuntur; quamobrem facta hac immutatione ex praecedente solutione statim habebimus

$$x = \frac{e^z}{p+\mathfrak{Q}} = \int e^{-z}\partial\mathrm{Q}, \text{ existente } z = \int \frac{\partial q}{p+\mathfrak{Q}};$$

sicque *x* hic erit functio solius quantitatis *q*, ex qua fit

$$\partial x = \frac{Q' - x\mathfrak{Q}'}{q+\mathfrak{Q}}\partial q.$$

Deinde nunc etiam *p* per solam variabilem *q* definietur: erit enim per § 10

$$p = \mathrm{Q} - \frac{\mathfrak{Q}e^z}{q+\mathfrak{Q}}\int e^{-z}\partial\mathrm{Q}.$$

Cum igitur sit $y = \int p\partial x$, etiam quantitas *y* per solam functionem ipsius *q* exprimetur, hocque modo problema perfecte solutum est censendum.

Problema 7

§.13. *Proposita aequatione differentiali tertii gradus hac:* $q + \mathfrak{R}x = \mathrm{R}$ *, eius integrale completum assignare.*

Solutio.

Haec solutio simili modo ex problemate primo huius secundi generis (§ 10**)** derivari potest, dum loco $y,\ \mathrm{P},\ \mathfrak{P}$ scribatur $q,\ \mathrm{R},\ \mathfrak{R}$**,** id quod si primo in aequatione pro *x* fuerit factum, suppeditabit hanc expressionem:

$$x = \frac{e^z}{p+\mathfrak{R}} = \int e^{-z}\partial\mathrm{R}, \text{ existente } z = \int \frac{\partial q}{p+\mathfrak{R}},$$

sicque *x* erit functio solius variabilis *r;* tum vero erit

$$\partial x = \frac{\mathrm{R}' - x\mathfrak{R}'}{r+\mathfrak{R}}\partial r.$$

Formula porro ibi pro *y* inventa et huc translata dabit pro *q* hanc expressionem:

$$q = \mathrm{R} - \frac{\mathfrak{R}e^z}{r+\mathfrak{R}}\int e^{-z}\partial\mathrm{R},$$

quae etiam tantum variabilem *r* eiusque functiones involvit. Quia igitur $p = \int q\partial x$ et $y = \int p\partial x$, erit $y = \int \partial x \int q\partial x$, sicque etiam *y* per solam variabilem

r exprimetur.

Problema 8

§. 14. *Proposita aequatione differentiali quarti gradus* $r + \mathfrak{S}x = \mathrm{S}$ *, eius integrale investigare.*

Solutio.

Hic erit

$$x = \frac{e^z}{s+\mathfrak{S}} = \int e^{-z}\partial \mathrm{S}, \text{ existente } z = \int \frac{\partial s}{s+\mathfrak{S}},$$

Porro erit

$$\partial x = \frac{\mathrm{S}'-x\mathfrak{S}'}{s+\mathfrak{S}}\partial s, \quad r = \mathrm{S} - \frac{\mathfrak{S}e^z}{s+\mathfrak{S}}\int e^{-z}\partial \mathrm{S},$$

$$q = \int r\partial x, \;\; p = \int \partial x \int r\partial x,$$

et

$$y = \int p\partial x = \int \partial x \int \partial x \int r\partial x,$$

ubi omnia per solam variabilem s determinantur.

§. 15. Quin etiam istas aequationes differentiales, quarum integralia hic exhibuimus, certo modo inter se coniungere licet, ut integratio eadem methodo, qua hic usi sumus, institui queat. Hoc modo nanciscemur innumera nova genera huiusmodi aequationum differentialium, quae etiam differentiando ad integrationem perduci poterunt, quod argumentum in sequentibus problematibus pertractemus.

Problema 9

§. 16. *Posito* $p + fq = t$ *, sint T et* $\mathfrak{T}$ *functiones quaecunque ipsius t, sive algebraicae sive transcendentes, ac proposita fuerit haec aequatio differentialis secundi gradus:* $y + fp + \mathfrak{T}x = \mathrm{T}$ *, eius integrale completum investigare.*

Solutio.

Ponatur $y + fp = z$, erit

$$\partial z = \partial x\left(p + fq\right), \text{ ergo } \partial z = t\partial x .$$

Quare cum nunc aequatio proposita sit $z + \mathfrak{T}x = \mathrm{T}$, differentiando prodit

$$\partial z + \mathfrak{T}\partial x + x\partial\mathfrak{T} = \partial \mathrm{T},$$

sive

$$(t+\mathfrak{T})\partial x + x\partial\mathfrak{T} = \partial \mathrm{T},$$

unde colligitur haec aequatio:

$$\partial x + \frac{x\partial\mathfrak{T}}{t+\mathfrak{T}} = \frac{\partial \mathrm{T}}{t+\mathfrak{T}},$$

ad quam integrandam ponatur

$$\int \frac{\partial t}{t+\mathfrak{T}} = u,$$

eritque

$$\int \frac{\partial\mathfrak{T}}{t+\mathfrak{T}} = l(t+\mathfrak{T}) - u,$$

tum vero aequatio nostra integrabilis reddetur, si eam multiplicemus per $e^{-u}(t+\mathfrak{T})$; integrale enim erit

$$xe^{-u}(t+\mathfrak{T}) = \int e^{-u}\partial \mathrm{T},$$

ex quo deducitur

$$x = \frac{e^{u}}{t+\mathfrak{T}}\int e^{-u}\partial \mathrm{T},$$

sicque x aequetur certae functioni ipsius t, quam hoc modo per integrationem invenire licet, eiusque differentiale erit

$$\partial x = \frac{\partial \mathrm{T} - x\partial\mathfrak{T}}{t+\mathfrak{T}}.$$

Hinc igitur prodit $z = \mathrm{T} - \mathfrak{T}x$. Cum nunc sit

$$y + fp = z \text{, erit } y\partial x + f\partial y = z\partial x$$

unde colligitur

$$\partial y + \frac{y\partial x}{f} = \frac{z\partial x}{f},$$

quae aequatio multiplicata per $e^{\frac{x}{f}}$ dat integrale

$$ye^{\frac{x}{f}} = \frac{1}{f}\int e^{\frac{x}{f}} z\partial x,$$

ubi cum tam z quam x sint functiones ipsius t, erit etiam y functio ipsius t tantum, cum sit

$$y = \frac{e^{-\frac{x}{f}}}{f}\int e^{\frac{x}{f}} z\partial x.$$

Problema 10

§. 17. *Posito* $p+fq+qr=t$ *, si fuerint T et* $\mathfrak{T}$ *functiones quaecunque ipsius t, sive algebraicae sive transcendentes, ac proposita fuerit haec aequatio differentialis tertii gradus:*

$$y+fp+gq+\mathfrak{T}x=\mathrm{T}\,,$$

eius integrale completum invenire.

Solutio.

Ponatur

$$y+fp+gq=z,$$

eritque differentiando

$$\partial z=\partial x\left(p+fq+gr\right)=t\partial x\,,$$

sicque nostra aequatio integranda erit $z+\mathfrak{T}x=\mathrm{T}$, pro qua erit ut ante

$$x=\tfrac{e^{u}}{t+\mathfrak{T}}\int e^{-u}\partial\mathrm{T},\quad \text{et}\quad z=\mathrm{T}-\mathfrak{T}x\,,$$

posito scilicet $\int\tfrac{\partial t}{t+\mathfrak{T}}=u$. Ambae igitur illae expressiones functiones erunt solius variabilis *t*, unde etiam ∂x per eandem variabilem exprimetur. Tantum igitur superest, ut etiam altera variabilis principalis *y* indagetur. Cum autem sit $y+fp+gq=z$, loco litterarum *p* et *q* scribantur valores initio assumti $\frac{\partial y}{\partial x}$ et $\frac{\partial\partial y}{\partial x^{2}}$, eritque, si tota aequatio per ∂x^{2} multiplicetur, haec aequatio integranda:

$$y\partial x^{2}+f\partial x\partial y+g\partial\partial y=z\partial x^{2},$$

in qua cum tam *x* quam *z* sint functiones solius *t*, etiam *y* tanquam functionem ipsius *t* tractare licebit. Iam olim autem a me aliisque ostensum est, quomodo talis aequatio tractari debeat, quam ergo evolutionem hic repetere superfluum foret. Sufficiat enim notasse, valorem ipsius *y* per terminos huius formae $\int \mathrm{e}^{\lambda x}z\partial x$ assignari, eum igitur per solam variabilem *t* exprimere licebit, sicque etiam *y* per functionem ipsius *t* definietur.

Problema 11

§. 18. *Posito* $p+fq+gr+hs=t$ *, si fuerint T* et $\mathfrak{T}$ *functiones quaecunque ipsius t, sive algebraicae sive transcendentes, ac proposita fuerit talis aequatio differentialis quarti gradus:*

$$y+fp+gq+hr+\mathfrak{T}x=\mathrm{T}\,,$$

in eius integrale completum inquirere.

Solutio.

Sit $y+fp+gq+hr=z$, eritque differentiando

$$\partial z = \partial x(p + fq + gr + hs) = t\partial x\,,$$

atque aequatio integranda fiet $z + \mathfrak{T}x = \mathrm{T}$, pro qua iterum, sumto

$$\int \tfrac{\partial t}{t+\mathfrak{T}} = u\,,$$

erit

$$x = \tfrac{e^{u}}{t+\mathfrak{T}} \int e^{-u}\partial\mathrm{T}, \quad \text{atque} \quad z = \mathrm{T} - \mathfrak{T}x\,,$$

ita ut tam *x* quam *z* per solam variabilem *t* exprimantur. His inventis, si in aequatione initio assumta loco *p, q, r, s* eorum valores substituantur, prodibit haec aequatio tertii gradus:

$$y\partial x^{3} + f\partial x^{2}\partial y + g\partial x\partial\partial y + h\partial^{3}y = z\partial x^{3},$$

cuius integrale completum per ea, quae circa huiusmodi aequationes sunt prolata, tanquam cognitum spectare licet, ita ut etiam hoc casu ambae variabiles *x* et *y* per novam variabilem *t* exprimantur. Facile autem patet hoc modo ad aequationes differentiates adhuc altiorum graduum progredi licere. Hac igitur ratione calculo integrali haud contemnendum incrementum allatum est censendum. Cum igitur hic praecipuum negotium versetur in integratione completa huiusmodi aequationis:

$$y + \frac{f\partial y}{\partial x} + \frac{g\partial\partial y}{\partial x^{2}} + \frac{h\partial^{3}y}{\partial x^{3}} + \text{etc.} = z,$$

ubi *z* est functio quaecunque ipsius *x*, eius resolutionem iam passim exhibitam huc accommodemus et breviter ostendamus. Formetur haec aequatio:

$$1 + fu + gu^{2} + hu^{3} + iu^{4} + \text{ etc.} = 0,$$

cuius radices *u* designentur litteris $\alpha,\ \beta,\ \gamma,\ \delta$ etc., quibus inventis erit uti iam olim ostendi

$$y = \frac{e^{\alpha x}\int e^{-\alpha x}z\partial x}{f+2g\alpha+3h\alpha^{2}+4i\alpha^{3}+\text{ etc.}} + \frac{e^{\beta x}\int e^{-\beta x}z\partial x}{f+2g\beta+3h\beta^{2}+4i\beta^{3}+\text{ etc.}} + \text{etc.}$$

Hae scilicet formulae ex singulis radicibus $\alpha,\ \beta,\ \gamma,\ \delta$ etc. formatae et iunctim sumtae dabunt valorem ipsius *y* atque adeo integrale completum, quia singulae formulae integrales constantem arbitrariam involvunt.

2) Specimen aequationum differentialium indefiniti gradus earumque integrationis.

M. S. Academiae exhibit die 13. *Decembris* 1781.
[E681]

§. 19. Quando aequationes differentiales secundum gradus differentialium distinguuntur, ipsa rei natura gradus intermedios excludere videtur; cum enim totidem integrationibus opus sit, harum numerus certe non integer esse non potest. Incidi tamen nuper in aequationem differentialem indefiniti gradus, cuius exponens etiam numerus fractus esse potest, atque adeo mihi licuit eius integrale assignare; quod cum omni attentione dignum videatur, totam analysin, qua sum usus, hic dilucide exponam.

§. 20. Cum miras proprietates unciarum potestatum binomii, quas hoc charactere indicare soleo $\left(\frac{p}{q}\right)$, cuius valor est hoc productum:

$$\frac{p}{1}\cdot\frac{p-1}{2}\cdot\frac{p-2}{3}\cdots\cdots\frac{p-q+1}{q},$$

considerassem, in mentem mihi venit valorem huiusmodi formulae $\left(\frac{p}{q}\right)$ ad formulam integralem revocare, unde etiam casus, quibus p et q non sunt numeri integri, assignari queant. Directe quidem talem reductionem non succedere observavi, unde eius valorem reciprocum $\frac{1}{\left(\frac{p}{q}\right)}$ sum contemplatus, cuius valor est

$$\frac{1}{p}\cdot\frac{2}{p-1}\cdot\frac{3}{p-2}\cdots\cdots\frac{q}{p-q+1},$$

Hunc in finem statuo

$$\frac{1\cdot2\cdot3\cdots q\times x^{p}}{p(p-1)(p-2)\cdots(p-q+1)}=s,$$

ita ut posito $x=1$ desideratus valor ipsius $1:\left(\frac{p}{q}\right)$ obtineatur.

§. 21. Sit nunc brevitatis gratia $1\cdot2\cdot3\cdots q=\mathrm{N}$, ut habeatur

$$s=\frac{\mathrm{N}x^{p}}{p\cdots(p-q+1)},$$

in cuius denominatore tenendum est factores continuo unitate decrescere. Quod si iam ista formula differentietur, prodibit

$$\frac{\partial s}{\partial x} = \frac{Nx^{p-1}}{(p-1)\cdots(p-q+1)},$$

sicque primus factor denominatoris est sublatus, ac differentiatione denuo instituta prodibit

$$\frac{\partial\partial s}{\partial x^2} = \frac{Nx^{p-2}}{(p-2)\cdots(p-q+1)}.$$

Hoc igitur modo continuo differentiando omnes factores denominatoris tollentur, ac pervenietur tandem ad hanc aequationem:

$$\frac{\partial^q s}{\partial x^q} = Nx^{p-q}.$$

§.22. Pervenimus igitur, loco N valorem suum substituendo, ad hanc aequationem differentialem

$$\frac{\partial^q s}{1\cdots q\partial x^q} = x^{p-q},$$

quam ergo tot vicibus integrari oporteret, quot q continet unitates, atque singulae integrationes ita sunt instituendae, ut posito $x = 0$ integralia evanescant, et postquam omnes integrationes fuerint absolutae, loco x scribi debebit unitas, hocque modo valor ipsius s resultans dabit valorem formulae $1:\left(\frac{p}{q}\right)$.

Quo autem istas integrationes generalius expediamus, loco x^{p-q} scribamus X, ut habeamus hanc aequationem resolvendam.

$$\frac{\partial^q s}{1\cdots q\partial x^q} = X.$$

§. 23. Hanc aequationem primo multiplicemus per ∂x, eiusque integrale dabit

$$\frac{\partial^{q-1} s}{1\cdot 2\cdot 3\cdots q\cdot\partial x^{q-1}} = \int X\partial x.$$

Istam aequationem ducamus in $1\cdot\partial x$, eritque integrando

$$\frac{\partial^{q-2} s}{2\cdot 3\cdots q\partial x^{q-2}} = \int \partial x\int X\partial x = x\int X\partial x - \int Xx\partial x.$$

Per notas enim reductiones eiusmodi integralia repetita ad simplicia reduci possunt. Haec aequatio iam per $2\partial x$ multiplicata eodemque modo integrata praebebit

$$\frac{\partial^{q-3} s}{3\cdot 4\cdots q\cdot\partial x^{q-3}} = x^2\int X\partial x - 2x\int Xx\partial x + \int Xx^2\partial x.$$

Nunc per $3\partial x$ multiplicando et integrando proveniet

$$\frac{\partial^{q-4}s}{4\cdot5\cdots q\cdot\partial x^{q-4}}=x^3\int X\partial x-3x^2\int Xx\partial x+3x\int Xx^2\partial x-\int Xx^3\partial x.$$

Eodem modo reperietur

$$\frac{\partial^{q-5}s}{5\cdot6\cdots q\cdot\partial x^{q-5}}=x^4\int X\partial x-4x^3\int Xx\partial x+6x^2\int Xx^2\partial x-4x\int Xx^3\partial x+\int Xx^4\partial x,$$

sicque in genere nostros characteres in usum vocando erit

$$\frac{\partial^{q-n}s}{n(n+1)\cdots q\cdot\partial x^{q-n}}=$$
$$x^{n-1}\int X\partial x-\left(\tfrac{n-1}{1}\right)x^{n-2}\int Xx\partial x+\left(\tfrac{n-1}{2}\right)x^{n-3}\int Xx^2\partial x-\left(\tfrac{n-1}{3}\right)x^{n-4}\int Xx^3\partial x+\text{etc.}$$

§. 24. Statuamus nunc $n=q$, et cum sit $\partial^0 s=s$, orietur haec aequatio finita:

$$\frac{s}{q}=x^{q-1}\int X\partial x-\left(\tfrac{q-1}{1}\right)x^{q-2}\int Xx\partial x+\left(\tfrac{q-1}{2}\right)x^{q-3}\int Xx^2\partial x-\text{etc.},$$

cuius singula membra ita integrari debent, ut posito $x=0$ evanescant, quod quidem semper eveniet, si modo sit $q-1>0$, quamobrem ipsae formulae integrales $\int X\partial x,\ \int Xx\partial x$ etc. tantum sine adiectione constantis integrari debent. Etsi enim hoc modo *x* forte in denominatorem ingrediatur, per potestatem ipsius *x*, qua multiplicari debent, iterum tolletur.

§. 25. His circa singula integralia observatis extra signa summatoria iam ponere licebit $x=1$, reperietur quippe qui est casus quaestionis propositae; sicque

$$1:q\left(\tfrac{p}{q}\right)=\int X\partial x[1-\left(\tfrac{q-1}{1}\right)x+\left(\tfrac{q-1}{2}\right)x^2-\left(\tfrac{q-1}{3}\right)x^3+\text{etc.}],$$

cuius seriei valor manifesto est $(1-x)^{q-1}$, ita ut habeamus hanc expressionem determinatam:

$$\frac{1}{q\left(\frac{p}{q}\right)}=\int X\partial x(1-x)^{q-1},$$

cuius ergo valor etiam casibus, quibus *q* non est numerus integer, per quadraturas exhiberi potest, sicque aequationis differentialis indefiniti gradus $\partial^q s = \mathrm{NX}\partial x^q$ integrale feliciter elicuimus, et quia $\mathrm{X} = x^{p-q}$, omnes unciae hoc modo ad formas integrales redigentur

$$\left(\frac{p}{q}\right) = \frac{1}{q\int x^{p-q}\partial x(1-x)^{q-1}},$$

et quia exponentes ipsius x et ipsius $1-x$ permutari possunt, erit etiam

$$\left(\frac{p}{q}\right) = \frac{1}{q\int x^{q-1}\partial x(1-x)^{p-q}},$$

hancque formulam ex principio diversissimo non ita pridem sum adeptus.

Theorema

§. 26. Valor huius characteris $\left(\frac{p}{q}\right)$ reduci potest ad formulam integralem, cum sit

$$\left(\frac{p}{q}\right) = \frac{1}{q\int x^{q-1}\partial x(1-x)^{p-q}},$$

siquidem hoc integrale ab $x=0$ ad $x=1$ extendatur.

Corollarium I

§. 27. Sumto ergo $p=0$ erit

$$\left(\frac{0}{q}\right) = \frac{1}{q\int x^{q-1}\partial x(1-x)^{-q}}.$$

Ostendi autem olim esse

$$\int x^{q-1}\partial x(1-x)^{-q} = \frac{\pi}{\sin.\pi q},$$

unde ergo fiet

$$\left(\frac{0}{q}\right) = \frac{\pi}{\sin.\pi q}.$$

Corollarium 2

§. 28. Deinde per notam integralium reductionem reperitur

$$\int x^{q-1}\partial x(1-x)^{p-q} = \frac{\pi}{\sin.\pi q}\left(\frac{p-q}{q}\right),$$

cuius ergo valor, quoties p est numerus integer, absolute assignari potest, quamobrem in genere erit

$$\left(\frac{p}{q}\right)=\frac{\sin.\pi q}{\pi}:\left(\frac{p-q}{q}\right).$$

Corollarium 3

§. 29. Cum igitur vicissim sit

$$\int x^{q-1}\partial x(1-x)^{p-q}=\frac{1}{q\left(\frac{p}{q}\right)},$$

si hic loco $q-1$ scribamus *f*, et *g* loco $p-q$, habebimus

$$\int x^{r}\partial x(1-x)^{g}=\frac{1}{(1+f)\left(\frac{f+g+1}{f+1}\right)}.$$

Scholion

§. 30. Quoniam igitur hanc formulam integralem nacti sumus ex aequatione integrali indefiniti gradus, eandem investigationem latius extendamus in sequente problemate.

Problema

§. 31. *Proposita serie sive finita sive infinita*

$$\mathrm{S}=\frac{\mathrm{A}}{\left(\frac{p}{q}\right)}+\frac{\mathrm{B}}{\left(\frac{p+1}{q}\right)}+\frac{\mathrm{C}}{\left(\frac{p+2}{q}\right)}+\frac{\mathrm{D}}{\left(\frac{p+3}{q}\right)}+\text{etc.}$$

eius valorem per formulam integralem exprimere.

Solutio

Tribuamus singulis terminis potestates ipsius *x*, ac statuamus

$$\mathrm{S}=\frac{\mathrm{A}x^{p}}{\left(\frac{p}{q}\right)}+\frac{\mathrm{B}x^{p+1}}{\left(\frac{p+1}{q}\right)}+\frac{\mathrm{C}x^{p+2}}{\left(\frac{p+2}{q}\right)}+\text{etc.},$$

quae series ergo, posito $x=1$, praebebit ipsam seriem propositam. Ubi observandum, in omnibus terminis litteram q eundem retinere valorem, alteram vero p continuo unitate augeri, unde productum indefinitum $1\cdot 2\cdot 3\cdot\ \cdot\ \cdot q=\mathrm{N}$ in omnibus terminis eundem retinebit valorem. Quare cum supra ex aequatione $s=\frac{x^{p}}{\left(\frac{p}{q}\right)}$ deduxerimus hanc aequationem differentialem indefiniti gradus:

$$\frac{\partial^q s}{\partial x^q} = \mathrm{N}x^{p-q},$$

ex singulis terminis nostrae seriei idem resultabit differentiale, si modo exponentem p unitate augeamus, unde ergo reperiemus

$$\frac{\partial^q \mathrm{S}}{\partial x^q} = \mathrm{NA}x^{p-q} + \mathrm{NB}x^{p-q+1} + \text{etc.}$$

§. 32. Ponamus nunc

$$\mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 + \text{etc.} = \mathrm{V},$$

eritque

$$\frac{\partial^q \mathrm{S}}{\mathrm{N}\partial x^q} = x^{p-q}\mathrm{V},$$

quamobrem si statuamus $x^{p-q}\mathrm{V} = \mathrm{X}$, habebimus ipsam aequationem iam ante tractatam

$$\frac{\partial^q \mathrm{S}}{1\cdot 2\cdot 3\cdots q\cdot \partial x^q} = \mathrm{X},$$

cuius integratio q vicibus repetita nos perduxit ad hanc expressionem $S = q\int \mathrm{X}\partial x(1-x)^{q-1}$, unde ergo pro X et V valores substituendo nanciscemur summam quaesitam S, scilicet

$$\mathrm{S} = q\int x^{p-q}\partial x\left(\mathrm{A} + \mathrm{B}x + \mathrm{C}x^2 + \mathrm{D}x^3 + \text{etc.}\right)(1-x)^{q-1},$$

si modo hoc integrale ab $x = 0$ ad $x = 1$ extendatur, vel ut ante invenimus, si modo in integratione nulla constans adiiciatur, deinde vero sumatur $x = 1$.

Exemplum

§. 33. Sit $\mathrm{V} = (1-x)^n$, ita ut sit

$$A = 1,\ \mathrm{B} = -\left(\frac{n}{1}\right), \mathrm{C} = +\left(\frac{n}{N}\right), D = -\left(\frac{n}{3}\right) \text{ etc.},$$

et series proposita erit

$$\mathrm{S} = \frac{1}{\left(\frac{p}{q}\right)} - \frac{\left(\frac{n}{1}\right)}{\left(\frac{p+1}{q}\right)} + \frac{\left(\frac{n}{2}\right)}{\left(\frac{p+2}{q}\right)} - \frac{\left(\frac{n}{3}\right)}{\left(\frac{p+3}{q}\right)} + \text{etc.},$$

tum igitur summa huius seriei erit

$$S = q\int x^{p-q}\partial x(1-x)^{q+n-1},$$

sive permutatis exponentibus ipsius x et $1-x$, erit quoque

$$S = q\int x^{q+n-1}\partial x\left(1-x\right)^{p-q}.$$

Nunc autem evidens est hanc ipsam formulam integralem iterum ad characterem hic usitatum reduci posse; ope § 29 erit enim $f = q+n-1$ et $g = p-q$, atque hinc prodibit

$$S = \frac{q}{(q+n)\left(\frac{p+n}{q+n}\right)}.$$

Hinc ergo sive formulis integralibus habebimus hanc summationem seriei infinitae maxime notabilem:

$$\frac{1}{\left(\frac{p}{q}\right)} - \frac{\left(\frac{n}{1}\right)}{\left(\frac{p+1}{q}\right)} + \frac{\left(\frac{n}{2}\right)}{\left(\frac{p+2}{q}\right)} - \frac{\left(\frac{n}{3}\right)}{\left(\frac{p+3}{q}\right)} + \frac{\left(\frac{n}{4}\right)}{\left(\frac{p+4}{q}\right)} - \text{etc.} = \frac{q}{(q+n)\left(\frac{p+n}{q+n}\right)}.$$

Corollarium 1

§.34. Si ergo fuerit $n=0$, oritur aequatio manifeste identica scilicet $\frac{1}{\left(\frac{p}{q}\right)} = \frac{1}{\left(\frac{p}{q}\right)}$.

At si $n=1$ prodit

$$\frac{q}{(q+1)\left(\frac{p+1}{q+1}\right)} = \frac{1}{\left(\frac{p}{q}\right)} - \frac{1}{\left(\frac{p+1}{q}\right)}.$$

Si $n = 2$, fiet

$$\frac{q}{(q+2)\left(\frac{p+2}{q+2}\right)} = \frac{1}{\left(\frac{p}{q}\right)} - \frac{2}{\left(\frac{p+1}{q}\right)} + \frac{1}{\left(\frac{p+2}{q}\right)}.$$

Corollarium 2

§. 35. Quo consensus cum veritate clarius appareat, evolvamus casum determinatum, quo $p=3,\ q=2,\ n=4,$ eritque

$$\frac{q}{q+n} = \frac{1}{3},\ \text{et}\ \left(\frac{p+n}{q+n}\right) = \left(\frac{7}{6}\right) = \left(\frac{7}{1}\right) = 7.$$

Deinde fit

$$\left(\frac{p}{q}\right) = \left(\frac{3}{2}\right) = 3,\ \left(\frac{p+1}{q}\right) = \left(\frac{4}{2}\right) = 6, \left(\frac{p+2}{q}\right) = \left(\frac{5}{2}\right) = 10,\ \left(\frac{p+3}{q}\right) = \left(\frac{6}{2}\right) = 15,$$

quae est progressio numerorum trigonalium; tum vero erit

$$\left(\frac{n}{1}\right) = 4,\ \left(\frac{n}{2}\right) = 6,\ \left(\frac{n}{3}\right) = 4,\ \left(\frac{n}{4}\right) = 1.$$

His igitur valoribus substitutis erit

$$\frac{1}{3\cdot 7}=\frac{1}{3}-\frac{4}{6}+\frac{6}{10}-\frac{4}{15}+\frac{1}{21},$$

quod egregie convenit.

Exemplum 2

§.36. Statuamus $V=(1+x)^{q-1}$, ut fiat

$$S=q\int x^{p-q}\partial x(1-xx)^{q-1};$$

tum vero erit

$$A=1,\ B=\left(\frac{q-1}{1}\right),\ C=\left(\frac{q-1}{2}\right),\ D=\left(\frac{q-1}{3}\right)\text{ etc.,}$$

sicque series proposita erit

$$S=\frac{1}{\left(\frac{p}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{p+2}{q}\right)}+\frac{\left(\frac{q-1}{3}\right)}{\left(\frac{p+3}{q}\right)}+\text{etc.,}$$

Evidens autem est, hanc formulam integralem etiam ad nostros characteres reduci posse. Ponamus enim $xx=y$, erit

$$S=\frac{q}{2}\int y^{p-q-1}\partial y(1-y)^{q-1},$$

sive permutatis exponentibus

$$S=\frac{q}{2}\int y^{q-1}\partial y(1-y)^{\frac{p-q-1}{2}},$$

quae comparata cum § 29 dat $f=q-1,\ g=\frac{p-q-1}{2}$, quibus valoribus substitutis colligitur

$$S=\frac{q}{2q\left(\frac{\frac{p+q-1}{2}}{q}\right)}=\frac{1}{2\left(\frac{\frac{p+q-1}{2}}{q}\right)}=\frac{1}{\left(\frac{p}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{p+2}{q}\right)}+\text{etc.,}$$

vel si ponatur $\frac{p+q-1}{2}=r$, erit

$$S=\frac{1}{2\left(\frac{r}{q}\right)}=\frac{1}{\left(\frac{p}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{p+1}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{p+2}{q}\right)}+\text{etc.,}$$

Corollarium I

§.37. Hic casu $q=1$ summa inventa ipsi termino primo aequatur. Sumamus autem $q=2$, erit

$$\frac{1}{2\left(\frac{\frac{p+1}{2}}{q}\right)}=\frac{1}{\left(\frac{p}{2}\right)}+\frac{1}{\left(\frac{p+1}{2}\right)},$$

hoc est

$$\frac{4}{pp-1}=\frac{2}{p(p-1)}+\frac{2}{p(p+1)},$$

unde patet istam summationem esse veritati consentaneam, de quo quidem nullum superesse potest dubium, quoties q est numerus integer positivus; quamobrem quosdam casus consideremus, ubi non est talis.

Corollarium 2

§. 38. Quo autem evolutio facilior evadat, contemplemur casum quo $r=q$, ut fiat $\left(\frac{r}{q}\right)=1$, tum autem erit $p=1+q$, hincque

$$\left(\frac{p}{q}\right)=1+q,\quad \left(\frac{p+1}{q}\right)=\frac{q+1}{1}\cdot\frac{q+2}{2},\quad \left(\frac{p+2}{q}\right)=\frac{q+1}{1}\cdot\frac{q+2}{2}\cdot\frac{q+3}{3},$$

quibus substitutis orietur haec series:

$$\frac{1}{2}=\frac{1}{q+1}+\frac{2(q-1)}{(q+1)(q+2)}+\frac{3(q-1)(q-2)}{(q+1)(q+2)(q+3)}+\frac{4(q-1)(q-2)(q-3)}{(q+1)(q+2)(q+3)(q+4)}+\text{etc.},$$

quae series notatu maxime est digna, quia eius summa semper est $\frac{1}{2}$, quicunque valores litterae q tribuantur. Si enim sit $q=0$, habebitur

$$\frac{1}{2}=1-1+1-1+1-\text{etc.},$$

quae est series notissima. Sit nunc $q=-1$, et ob $q+1=0$ multiplicemus omnes terminos per $q+1$, prodibitque haec series

$$0=1-4+9-16+25-\text{etc.},$$

uti differentias sumendo facile patet. Ponamus $q=\frac{1}{2}$, et haec series prodibit:

$$\frac{1}{2}=\frac{2}{3}-\frac{2\cdot2}{3\cdot5}+\frac{2\cdot3}{5\cdot7}-\frac{2\cdot4}{7\cdot9}+\frac{2\cdot5}{9\cdot11}-\text{etc.}$$

Cum igitur sit

$$\frac{2}{3}=1-\frac{1}{3},\ \frac{4}{3\cdot5}=\frac{2}{3}-\frac{2}{5},\ \frac{6}{5\cdot7}=\frac{3}{5}-\frac{3}{7},\ \ \frac{8}{7\cdot9}=\frac{4}{7}-\frac{4}{9},$$

et ita porro, his substitutis prodibit haec series:

$$\tfrac{1}{2}=1-1+1-1+1-\text{etc.},$$

At si sumamus $q=-\tfrac{1}{2}$ erit

$$\tfrac{1}{2}=2-4+6-8+10-12+\text{etc.},$$

quod per differentiae fit manifestum.

Corollarium 3

§. 39. Sumamus nunc $r=0$, ut fiat $p=1-q$. Demonstravi autem esse $\left(\frac{0}{q}\right)=\frac{\sin.q\pi}{q\pi}$, unde orietur

$$\frac{\pi q}{2\sin.q\pi}=\frac{1}{\left(\frac{1-q}{q}\right)}+\frac{\left(\frac{q-1}{1}\right)}{\left(\frac{2-q}{q}\right)}+\frac{\left(\frac{q-1}{2}\right)}{\left(\frac{3-q}{q}\right)}+\text{etc.},$$

cuius casum $q=\tfrac{1}{2}$ evolvisse pretium erit, membrum enim sinistrum fit $\tfrac{\pi}{4}$.
Pro parte dextra autem habebimus

$$\left(\frac{q-1}{1}\right)=-\tfrac{1}{2},\ \left(\frac{q-1}{2}\right)=\tfrac{1\cdot 3}{2\cdot 4},\ \left(\frac{q-1}{3}\right)=-\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\ \text{etc.},$$

tum vero pro denominatoribus

$$\left(\frac{1-q}{q}\right)=1,\ \left(\frac{2-q}{q}\right)=\tfrac{3}{2},\ \left(\frac{3-q}{q}\right)=\tfrac{3\cdot 5}{2\cdot 4},\ \left(\frac{4-q}{q}\right)=\tfrac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\ \ \text{etc.},$$

quibus valoribus substitutis orietur haec series:

$$\tfrac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\text{etc.},$$

quae est series notissima. Ponamus autem adhuc $q=-\tfrac{1}{2}$, et membrum sinistrum erit ut ante $\tfrac{\pi}{4}$; pro parte dextra autem erit

$$\left(\frac{q-1}{1}\right)=-\tfrac{3}{2},\ \left(\frac{q-1}{2}\right)=\tfrac{3\cdot 5}{2\cdot 4},\ \left(\frac{q-1}{3}\right)=\tfrac{3\cdot 5\cdot 7}{2\cdot 4\cdot 6}\ \text{etc.},$$

tum

$$\left(\frac{1-q}{q}\right)=\tfrac{1\cdot 3}{2\cdot 4},\ \left(\frac{2-q}{q}\right)=\tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6},\ \left(\frac{3-q}{q}\right)=\tfrac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\ \text{etc.},$$

hinc

$$\frac{\pi}{4} = \frac{2\cdot4}{1\cdot3} - \frac{4\cdot6}{1\cdot5} + \frac{6\cdot8}{1\cdot7} - \frac{8\cdot10}{1\cdot9} + \text{etc.},$$

cuius veritas ita ostenditur. Cum sit

$$\frac{2\cdot4}{1\cdot3} = 3 - \frac{1}{3},\ \ \frac{4\cdot6}{1\cdot5} = 5 - \frac{1}{5},\ \ \frac{6\cdot8}{1\cdot7} = 7 - \frac{1}{7},\ \ \frac{8\cdot10}{1\cdot9} = 9 - \frac{1}{9}\ \text{etc.},$$

erit illa series aequalis huic:

$$\frac{\pi}{4} = 3 - \frac{1}{3} - 5 + \frac{1}{5} + 7 - \frac{1}{7} - 9 + \frac{1}{9} + \text{etc.},$$

quae series in has duas discerpatur:

$$\frac{\pi}{4} = \begin{cases} \quad 3 - 5 + 7 - 9 + 11 - 13 + \text{etc.} \\ -\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.} \end{cases}$$

De superiore notetur, eius summam per differentias erutam esse

$$3 - 5 + 7 - 9 + 11 - 13 + \text{etc.} = 1;$$

inferioris summa ex serie supra inventa, qua erat

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.},$$

erit

$$-\frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \text{etc.} = \frac{\pi}{4} - 1,$$

unde iam manifestum est fore

$$3 - \frac{1}{3} - 5 + \frac{1}{5} + 7 - \frac{1}{7} - 9 + \frac{1}{9} + \text{etc.} = 1 + \frac{\pi}{4} - 1 = \frac{\pi}{4}.$$

Hinc igitur patet, pro q etiam numeros negativos atque adeo fractos accipi posse.

Theorema Generale

§. 40. Si X denotet functionem quamcunque ipsius x, et proposita fuerit haec aequatio differentialis cuiuscunque gradus:

$$\partial^q y = 1\cdot2\cdot3\cdots q\mathrm{X}\partial x^q,$$

ubi exponens q denotet numeros quoscunque sive integros sive fractos sive positivos sive negativos, cuius ergo aequationis resolutio totidem integrationes requirit, quae si singulae ab $x=0$ inchoentur omnibusque peractis statuatur $x=1$, tum semper erit $y=q\int X\partial x(1-x)^{q-1}$, hoc scilicet integrali ab $x=0$ ad $x=1$ extenso.