LEONHARD EULER

FOUNDATIONS OF THE

INTEGRAL CALCULUS

VOLUME FOUR

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SUPPLEMENTS AND ADDITIONS

TO THE
FOUNDATIONS OF INTEGRAL CALCULUS.

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SUPPLEMENT I TO BOOK I, CH. II.

CONCERNING THE INTEGRATION OF IRRATIONAL DIFFERENTIAL FORMULAS.

1.) On the integration of irrational differential formulas. *Proc. Acad. Sc. St.Petersburg. Book. IV. Part I. Pages.* 4-31.

Problem 1.

§.1. *If the function X, besides the variable x, may also involve the irrational formula* $s=\sqrt{(a+bx)}$ *: however thus, so that X shall be a function of the two rational functions x and s, to free the differential formula* $X\partial x$ *from irrationality.*

Solution.

Since so much irrationality shall be present in the formula $s=\sqrt{(a+bx)}$, so that thus just as much shall be required to be removed by a suitable substitution , thus so that the value of x shall not become irrational. Moreover this may be performed, by putting $a+bx=zz$, so that there becomes $s=z$ and $x=\frac{zz-a}{b}$, and hence $\partial x=\frac{2}{b}z\partial z$; with which values substituted, the whole formula of the differential $X\partial x$ becomes rational, in terms of the new variable z .

Example 1.

If there were $\partial y=\frac{\partial x}{\sqrt{(a+bx)}}$, or $\partial y=\frac{\partial x}{s}$, by putting $\sqrt{(a+bx)}=z$, there becomes $\partial y=\frac{2}{b}\partial z$, and on integrating $y=\frac{2z}{b}$, from which with the substitution made, it can be deduced that $y=\frac{2}{b}\sqrt{(a+bx)}+$ C.

Example *2.*

§. 3. If there were $\partial y=\partial x\sqrt{(a+bx)}=s\partial x$, on assuming $\sqrt{(a+bx)}=z$, there becomes $\partial y=z\partial x=\frac{2}{b}zz\partial z$, so that by integrating that becomes $y=\frac{2}{3b}z^3$, and with the substitution made there becomes $y=\frac{2}{3b}(a+bx)^{\frac{3}{2}}+\mathrm{C}$.
Because with the integral, if it may vanish by making $x=0$, there becomes $\mathrm{C}=-\frac{2a\sqrt{a}}{3b}$, and thus $y=\frac{2(a+bx)^{\frac{3}{2}}-2a\sqrt{a}}{3b}$.

Example 3.

§. 4. If there were $\partial y=\frac{x\partial x}{\sqrt{(a+bx)}}$, with the substitution made $\sqrt{(a+bx)}=z$, there will be

$$\partial y=\frac{2(zz-a)\partial z}{bb}=\frac{2zz\partial z-2a\partial z}{bb},$$

from which on integrating there becomes

$$y=\frac{2z^3}{3bb}-\frac{2a}{bb}z+C,$$

and with restitution made :

$$y=\frac{2}{3bb}(a+bx)^{\frac{3}{2}}-\frac{2a}{bb}\sqrt{(a+bx)}+C$$
$$=\frac{2\sqrt{(a+bx)}}{bb}(\tfrac{1}{3}bx-\tfrac{2}{3}a)+C.$$

Example 4.

§.5. If there were $\partial y=\frac{x\partial x}{(a+bx)^{\frac{3}{2}}}$, with the substitution made $\sqrt{(a+bx)}=z$, there becomes $\partial y=\frac{\partial x}{z^3}$; which formula again on account of $\partial x=\frac{2z\partial z}{b}$ becomes $\partial y=\frac{2\partial z}{bzz}$, with which integrated there shall be $y=-\frac{2}{bz}$, or with restitution made, $y=\frac{-2}{b\sqrt{(a+bx)}}+\text{C}$. Where it may be observed, if $\frac{2}{b\sqrt{a}}$ be assumed for C, in which case the integral must vanish by making $x=0$.

Problem 2.

§.6. *If X were some function of the two rational quantities x and s, with* $s=\sqrt[3]{(a+bx)}$ *arising, to free the differential formula* $X\partial x$ *from irrationality.*

Solution.

There may be put $\sqrt[3]{(a+bx)}=z$, so that there becomes $s=z$, there becomes $a+bx=z^3$, and hence $x=\frac{z^3-a}{b}$, and $\partial x=\frac{3z^2}{b}$; with which values substituted the formula becomes rational.

Example 1.

§. 7. If there were

$$\partial y=\frac{\partial x}{\sqrt[3]{(a+bx)}}=\frac{\partial x}{s},$$

on putting $\sqrt[3]{(a+bx)}=z$ and with this value substituted hence arises

$$\partial x = \tfrac{3zz\partial z}{b}, \text{ there will be } \partial y = \tfrac{3z\partial z}{b},$$

thence by integrating there becomes

$$y = \tfrac{3}{2b}zz = \tfrac{s}{2b}\sqrt[3]{(a+bx)^2} + \text{C}.$$

Example 2.

§.8. If there were

$$\partial y = \frac{\partial x}{\sqrt[3]{(a+bx)^2}} = \frac{\partial x}{ss},$$

on putting $\sqrt[3]{(a+bx)} = z$,there becomes, $\partial y = \tfrac{3\partial z}{b}$, hence on integrating

$$y = \tfrac{3}{b}z = \tfrac{3}{b}\sqrt[3]{(a+bx)} + \text{C}.$$

Example 3.

§.9. If there were $\partial y = \partial x\sqrt[3]{(a+bx)} = s\partial x$, with the substitution made there shall be $\partial y = \tfrac{3z^3\partial z}{b}$, hence on integrating

$$y = \tfrac{3z^4}{4b} = \tfrac{3}{4b}(a+bx)\sqrt[3]{(a+bx)} + \text{C}.$$

Problem 3.

§. 10. *If X were a function of the two rational quantities x and s, with* $s = \sqrt[n]{(a+bx)}$ *arising, to free the differential formula* $X\partial x$ *from irrationality.*

Solution.

Putting $\sqrt[n]{(a+bx)} = z$, so that there shall be $s = z$, there becomes $a+bx = z^n$, hence

$$x = \tfrac{z^n - a}{b} \text{ and } \partial x = \tfrac{nz^{n-1}\partial z}{b};$$

with which values substituted the proposed formula $X\partial x$ certainly becomes rational, but only if the number of the exponent *n* were whole.

Example 1.

§. 11. If there were

$$\partial y = \frac{\partial x}{\sqrt[n]{(a+bx)}} = \frac{\partial x}{s},$$

on putting $\sqrt{(a+bx)} = z$, thence on account of the value arising

$$\partial x = \frac{nz^{n-1}\partial z}{b}$$

there will be had

$$\partial y = \frac{nz^{n-2}}{b}\partial z;$$

so that on integrating we deduce

$$y = \frac{n}{b(n-1)} z^{n-1} + \text{C}.$$

or with the values restored

$$y = \frac{n}{b(n-1)}\left(a+bx\right)^{\frac{n-1}{n}} + \text{C} = \frac{n}{b(n-1)} \cdot \frac{a+bx}{\sqrt[n]{(a+bx)}} + \text{C}.$$

Example 2.

§. 12. If there were

$$\partial y = \frac{\partial x}{\sqrt[n]{(a+bx)^{\lambda}}} = \frac{\partial x}{s^{\lambda}},$$

on putting $\sqrt[n]{(a+bx)} = z$, and with the value substituted

$$\partial x = \frac{nz^{n-1}\partial z}{b}, \text{ there becomes}$$

$$\partial y = \frac{nz^{n-1}\partial z}{bz^{\lambda}} = \frac{n}{b} z^{n-\lambda-1}\partial z,$$

with the integral of which gives

$$y = \frac{n}{b(n-\lambda)}\left(a+bx\right)^{\frac{n-\lambda}{n}} + C, \text{ or}$$

$$y = \frac{n}{b(n-\lambda)} \cdot \frac{a+bx}{\sqrt[n]{(a+bx)^{\lambda}}}.$$

Now it is apparent from these examples, that the integration is not to be impeded, even if the exponents n and λ were not whole numbers.

Problem 4.

§. 13. *If X were a function of the two quantities x and s, with there being* $s = \sqrt{[a + b\sqrt{(f + gx)}]}$ *, which formula therefore involves a two-fold irrationality*, *to free the differential formula* $X\partial x$ *from this two-fold irrationality.*

Solution.

Again there may be put $s = \sqrt{[a + b\sqrt{(f + gx)}]} = z$, so that there shall be $s = z$, with the square of that taken there will be $a + b\sqrt{(f + gx)} = zz$, hence:

$$b\sqrt{(f + gx)} = zz - a,$$

and with the square taken again

$$bb(f+gx)=\left(zz-a\right)^2,$$

from which it is deduced

$$x=\frac{(zz-a)^2}{bbg}-\frac{f}{g}, \text{ and hence}$$

equals

$$\partial x=\frac{4z(zz-a)}{bbg}.$$

With which values substituted the whole formula will be returned rational.

Corollary.

§. 14. It can be seen that the irrationality can be removed in the same way, if there were more generally :

$$s=\sqrt[n]{a+b\sqrt[m]{(f+gx)}}.$$

For by putting this formula $=z$, there becomes

$$a+b\sqrt[m]{(f+gx)}=z^n \text{ and } b\sqrt[m]{(f+gx)}=z^n-a.$$

Again $b^m(f+gx)=(z^n-a)^m$, and hence there is deduced :

$$x=\frac{(z^n-a)^m}{b^m g}-\frac{f}{g} \text{ and thus}$$

$$\partial x=\frac{mnz^{n-1}\partial z(z^n-a)^{m-1}}{b^m g}.$$

And thus also in this manner the whole formula emerges rational.

Problem 6.

§. 15. *If X were a function of the two rational quantities s and x, with* $s=\sqrt{\frac{a+bx}{f+gx}}$, *to free the differential formula* $X\partial x$ *from irrationality.*

Solution.

Putting $\sqrt{\frac{a+bx}{f+gx}}=z$, and with the square taken there will be $\frac{a+bx}{f+gx}=zz$, and hence $x=\frac{fzz-a}{b-gzz}$, from which by differentiation it is deduced that

$$\partial x=\frac{2bfz\partial z-2agz\partial z}{(b-gzz)^2}.$$

And with these values substituted the proposed formula $X\partial x$ will be led to rationality.

Example 1.

§. 16. If there were $\partial y = \frac{\partial x}{s} = \frac{\partial x\sqrt{(f+gx)}}{\sqrt{(a+bx)}}$, by putting $\sqrt{\frac{a+bx}{f+gx}} = z$ there will be $\partial y = \frac{dx}{z}$,

and by substituting the value found above in place of ∂x, it is deduced that

$$\partial y = \frac{2(bf-ag)\partial z}{(b-gzz)^2};$$

which formula, as now agreed on well enough, can be reduced according to such $\int\frac{\partial z}{b-gzz}$, the integration of which moreover may be arranged by logarithms or by circular arcs.

Example 2.

§. 17. More specifically, there shall be

$$\partial y = \frac{\partial x\sqrt{(1-x)}}{\sqrt{(1+x)}}, \text{ where } f=1,\ g=-1,\ a=1 \text{ and } b=1, \text{ and thus},$$

$$z = \frac{\sqrt{(1+x)}}{\sqrt{(1-x)}}, \text{ and } \partial x = \frac{4z\partial z}{(1+zz)^2};$$

with which values substituted there becomes $\partial y = \frac{4\partial z}{(1+zz)^2}$. Therefore there may be established

$$\int\frac{4\partial z}{(1+zz)^2} = \frac{\mathrm{A}z}{1+zz} + \mathrm{B}\int\frac{\partial z}{1+zz} = y,$$

from which, with the differentials taken, there becomes

$$\frac{4}{(1+zz)^2} = \frac{\mathrm{A}-\mathrm{A}zz}{(1+zz)^2} + \frac{\mathrm{B}}{1+zz} = \frac{\mathrm{A}+\mathrm{B}+(\mathrm{B}-\mathrm{A})zz}{(1+zz)^2}.$$

Therefore there shall be required to be $\mathrm{A}+\mathrm{B}=4$ and $\mathrm{B}-\mathrm{A}=0$, and thus $\mathrm{A}=2$ and $\mathrm{B}=2$; and because $\int\frac{dz}{1+zz} = \text{Arc.tang.}z$, we arrive at

$$y = \frac{2z}{1+zz} + 2\ \text{Arc.tang.}z;$$

on account of which, with the working restored, since $1+zz = \frac{2}{1-x}$, there will be obtained

$$y = \sqrt{(1-xx)} + 2\ \text{Arc.tang.}\sqrt{\frac{1+x}{1-x}}.$$

Therefore since the tangent of this arc shall be $\sqrt{\frac{1+x}{1-x}}$, its sine $=\sqrt{\frac{1+x}{2}}$ and its cosine $=\sqrt{\frac{1-x}{2}}$; truly the sine of twice the angle will be $\sqrt{(1-xx)}$, and the cosine [of twice the angle] $=-x$, from which there becomes

$$2\ \text{Arc.tang.}\sqrt{\frac{1+x}{1-x}} = \text{Arc.cos}-x = \frac{\pi}{2} + \text{Arc.sin}\,x$$

on account of which for the integral sought, there will be

$$y=\sqrt{(1-xx)}+\tfrac{\pi}{2}+\text{Arc.}\sin x+\text{C},$$

because if thus it may be taken, so that it may vanish on putting $x=0$, there will be

$\text{C}=-1-\frac{\pi}{2}$ and thus,

$$y=\sqrt{(1-xx)}-1+\text{Arc.}\sin x.$$

Therefore then, if it may be assumed that $x=1$, there becomes $y=\frac{\pi}{2}-1$, which value in decimal fractions gives 0,5707963.

Problem 6.

§.15. *If X were a function of the two variables x and s, with* $s=\sqrt[n]{\frac{a+bx}{f+gx}}$, *to direct the formula towards a rational differential* $X\partial x$.

Solution.

Putting $s=\sqrt[n]{\frac{a+bx}{f+gx}}=z$ there will be $\frac{a+bx}{f+gx}=z^n$ and hence $x=\frac{fz^n-a}{b-gz^n}$, consequently $\partial x=\frac{n(bf-ag)fz^{n-1}\partial z}{(b-gz)^{2n}}$; and with these values substituted the whole formula proposed $X\partial x$ will be led to rationality.

Problem 7.

§.19. *If X were a function of the two quantities x and s, with* $s=\sqrt{(a+bxx)}$, *to free the differential formula* $\frac{X\partial x}{x}$ *from irrationality.*

Solution.

We may put $s=\sqrt{(a+bxx)}=z$, there becomes $a+bxx=zz$, hence $xx=\frac{zz-a}{b}$, and because the square *xx* is present in the function X only , therefore even powers of this occur : now with this substitution the function X will emerge rational. Truly with logarithms taken

$$2lx=l(zz-a)-lb,$$

by differentiation there becomes

$$\frac{2\partial x}{x}=\frac{2z\partial z}{zz-a};\ \text{and thus}\ \frac{\partial x}{x}=\frac{z\partial z}{zz-a}.$$

Therefore in this way the proposed formula $X.\frac{\partial x}{x}$ will be returned rational at once.

Example 1.

§. 20. If $\partial y=\frac{x\partial x}{\sqrt{(a+bxx)}}$, there will be $\partial y=\frac{\partial x}{x}\cdot\frac{xx}{\sqrt{(a+bxx)}}=\frac{xx}{s}\cdot\frac{\partial x}{x}$.

Therefore by putting $\sqrt{(a+bxx)}=z$ there becomes $\partial y=\frac{\partial z}{b}$, from which it is deduced by integration $y=\frac{z}{b}=\frac{\sqrt{(a+bxx)}}{b}$.

Example 2.

§. 21. If

$$\partial y=\frac{x^3\partial x}{\sqrt{(a+bxx)}}=\frac{\partial x}{x}\cdot\frac{x^4}{s},$$

by putting $\sqrt{(a+bxx)}=z$, so that there becomes $xx=\frac{zz-a}{b}$ and $\frac{\partial x}{x}=\frac{z\partial z}{zz-a}$, there will be [$\partial y=\frac{x^3\partial x}{\sqrt{(a+bxx)}}=\frac{z\partial z}{zz-a}\cdot\frac{(zz-a)^2}{z}\frac{1}{bb}=\frac{(zz-a)\partial z}{bb}$,] $\partial y=\frac{1}{bb}\partial z\left(zz-a\right)$, and hence by integrating we arrive at $y=\frac{z}{3bb}(zz-3a)$; from which with the working restored, the integral sought will be produced $y=\frac{bxx-2a}{3bb}\sqrt{(a+bxx)}+\text{C}$.

Example 3.

§. 22. If there were

$\partial y=\frac{x^3\partial x}{\sqrt{(a+bxx)^3}}$, there will be $\partial y=\frac{\partial x}{x}\cdot\frac{x^4}{s^3}$; hence on putting $\sqrt{(a+bxx)}=s=z$ there becomes $\partial y=\frac{\partial z}{bb}\left(\frac{zz-a}{zz}\right)$,

so that with the integral taken, $y=\frac{1}{bb}\left(\frac{zz+a}{zz}\right)$, on account of which with the working restored there becomes $y=\frac{2a+bxx}{bb\sqrt{(a+bxx)}}+\text{C}$.

Problem 8.

§. 23. *If X were a rational function of the two quantities x^n and s, with $s=\sqrt[m]{\left(a+bx^n\right)}$, to reduce the differential formula $X\frac{\partial x}{x}$ to rationality.*

Solution.

On putting $s=\sqrt[m]{\left(a+bx^n\right)}=z$, we make $a+bx^n=z^m$ and $x^n=\frac{z^m-a}{b}$. Therefore since only the power x^n occurs in the function X, that will be returned rational, if these values may be substituted. Then indeed with logarithms taken there will be found

$$nlx=l(z^m-a)-lb,$$

and on differentiating,

$$\frac{\partial x}{x}=\frac{mz^{m-1}\partial x}{n\left(z^m-a\right)},$$

and thus the whole formula proposed becomes rational.

Example.

§. 24. Let

$$\partial y = \frac{x^{n-1}\partial x}{\sqrt[m]{\left(a+bx^n\right)}} = \frac{\partial x}{x}\cdot\frac{x^n}{s},$$

and with the factor substituted this equation will arise

$$\partial y = \frac{mz^{m-2}\partial z}{nb},$$

[*i.e.* $\partial y = \frac{mz^{m-1}\partial x}{n\left(z^m-a\right)}\cdot\frac{x^n}{z} = \frac{mz^{m-1}\partial x}{nbx^n}\cdot\frac{x^n}{z},...$] which integrated will give

$$y = \frac{mz^{m-1}}{nb(m-1)} = \frac{m}{nb(m-1)}\sqrt[m]{\left(a+bx^n\right)^{m-1}} + \mathrm{C}, \text{ or}$$

$$y = \frac{m}{nb(m-1)}\cdot\frac{a+bx^n}{\sqrt[m]{a+bx^n}} + \mathrm{C}.$$

Problem 9.

§. 25. *If X were a rational function of the quantities xx and s, with s present and* $s = \sqrt{\frac{a+bxx}{f+gxx}}$, *to free the differential formula* $X\frac{\partial x}{x}$ *irrationality.*

Solution.

On putting $s = \sqrt{\frac{a+bxx}{f+gxx}} = z$, there will become $\frac{a+bxx}{f+gxx} = zz$, hence $xx = \frac{fzz-a}{b-gzz}$, so that the function X shall be completely rational. Again with logarithms taken,

$$2lx = l(fzz-a) - l(b-gzz),$$

the equation may be differentiated, so that there becomes

$$\frac{2\partial x}{x} = \frac{2fz\partial z}{fzz-a} + \frac{2gz\partial z}{b-gzz} = \frac{2(bf-ag)z\partial z}{(fzz-a)(b-gzz)},$$

from which there becomes

$$\frac{\partial x}{x} = \frac{(bf-ag)z\partial z}{(fzz-a)(b-gzz)};$$

and thus the whole differential formula will become rational.

Example.

§. 26. If there were $\partial y = \frac{\partial x}{\sqrt{(f+gxx)}}$, we may represent this formula thus

$$\partial y = \frac{\partial x}{x}\cdot\frac{x}{\sqrt{(f+gxx)}} = \frac{\partial x}{x}\cdot\sqrt{\frac{xx}{f+gxx}}.$$

Therefore here there will be $a=0,\ b=1$, and $z=\frac{x}{\sqrt{f+gxx}}$, thus so that $\partial y=\frac{z\partial z}{x}$; but there will be $\frac{\partial x}{x}=\frac{\partial z}{z(1-gzz)}$, so that there becomes $\partial y=\frac{\partial z}{1-gzz}$, the integration of which formula is expedited by the logarithm, if g were a positive number : but if it were a negative number the integration may be resolved by circular arcs. Therefore there shall be 1°.) $g=+hh$, there will be $\partial y=\frac{\partial z}{1-hhzz}$, and thus $y=\frac{1}{2h}l\frac{1+hz}{1-hz}$; and with the indicated values above restored, there will be

$$y=\frac{1}{2h}l\left(\frac{\sqrt{(f+hhxx)}+hx}{\sqrt{(f+hhxx)}-hx}\right)=\frac{1}{h}l\left(\frac{\sqrt{(f+hhxx)}+hx}{\sqrt{f}}\right).$$

2°.) Let g be a negative quantity, on putting $g=-hh$, there will be

$$\partial y=\frac{\partial z}{1+hhzz}=\frac{1}{h}\cdot\frac{h\partial z}{1+hhzz},$$

from which it is deduced

$$y=\frac{1}{h}\text{Arc.tang.}hz=\frac{1}{h}.\text{Arc.tang.}\frac{hx}{\sqrt{(f-hhxx)}}.$$

Where it is clear, that f must be a positive quantity, because otherwise the differential formula would be imaginary.

Corollary.

§. 27. Therefore from this, if the formula were proposed $\partial y=\sqrt{(1+xx)}$, where $f=1$ and $g=1$, from the first case on account of $h=+1$ there will be

$$\int\frac{\partial x}{\sqrt{(1+xx)}}=l\ [\sqrt{(1+xx)}+x].$$

But if there were

$$\partial y=\frac{\partial x}{\sqrt{(1-xx)}},\text{ where } f=1 \text{ and } g=-1,$$

it is gathered from the second case, that $x=\text{Arc. tang.}\frac{x}{\sqrt{(1-xx)}}$, from which it is concluded

$$\int\frac{\partial x}{\sqrt{(1-xx)}}=\text{Arc. sin.}x=\text{Arc. cos.}\sqrt{(1-xx)}.$$

Problem 10.

§.28. *If X were a rational function of the quantities x^n and s, with there being $s=\sqrt[n]{\left(\frac{a+bx^n}{f+gx^n}\right)}$, to produce a rational differential formula $X\frac{\partial x}{x}$.*

Solution.

There may be put $s=\sqrt[n]{\left(\frac{a+bx^n}{f+gx^n}\right)}=z$, and there will be $\frac{a+bx^n}{f+gx^n}=z^n$, hence $x^n=\frac{fz^n-a}{b-gz^n}$, but moreover with logarithms taken, the equation becomes

$$nlx = l\left(fz^n - a\right) - l\left(b - gz^n\right),$$

and by differentiating

$$\frac{\partial x}{x} = \frac{fz^{n-1}\partial z}{fz^n - a} + \frac{gz^{n-1}}{b - gz^n} = \frac{(bf - ag)z^{n-1}\partial z}{\left(fz^n - a\right)\left(b - gz^n\right)} \;;$$

and with which values substituted the proposed formula shall become rational.

Problem 11.

§.29. *If X were a rational function of the two quantities x^n and s, with $s = \sqrt[m]{\left(\frac{a+bx^n}{f+gx^n}\right)}$, to free the differential formula $X\frac{\partial x}{x}$ from all irrationality.*

Solution.

There may be put $s = \sqrt[m]{\left(\frac{a+bx^n}{f+gx^n}\right)} = z$, and there will become $\frac{a+bx^n}{f+gx^n} = z^m$, from which there becomes $x^n = \frac{fz^m - a}{b - gz^m}$; hence with logarithms taken there the equation becomes

$$nlx = l\left(fz^m - a\right) - l\left(b - gz^m\right),$$

hence by differentiating

$$\frac{n\partial x}{x} = \frac{m(bf - ag)z^{m-1}\partial z}{\left(fz^m - a\right)\left(b - gz^m\right)} \;,$$

and thus

$$\frac{\partial x}{x} = \frac{m(bf - ag)z^{m-1}\partial z}{n\left(fz^m - a\right)\left(b - gz^m\right)},$$

with which values substituted the irrationality of the proposed formula is completely removed.

Problem 12.

§. 30. *If X were a rational function of some two quantities x and s, with $s = \sqrt{(\alpha + \beta x + \gamma xx)}$ being present, to bring the differential formula $X\partial x$ to rationality .*

Solution.

Here it is agreed to distinguish these two cases from each other, as γ were a positive or negative quantity.

I. Let γ be a positive quantity, and there may be put $\gamma = cc$ and $\beta = 2bc$, so that there may be had

$$s=\sqrt{(\alpha+2bcx+ccxx)}=\sqrt{\left[\alpha-bb+(b+cx)^2\right]}$$

where in place of $\alpha-bb$ there may therefore be written e for brevity, so that there shall be

$$s=\sqrt{\left[e+(b+cx)^2\right]}.$$

Now there may be put $s=b+cx+z$, and there will be

$$ss=e+(b+cx)^2=(b+cx)^2+2(b+cx)z+zz,$$

from which it follow, that

$$e-zz=2z(b+cx),\ \text{or}\ b+cx=\tfrac{e-zz}{2z};$$

and hence it is deduced, that

$$x=\tfrac{e-zz}{2cz}-\tfrac{b}{c},\ \text{or}\ x=\tfrac{e-2bz-zz}{2cz}.$$

But the equation differentiated : $b+cx=\frac{e-zz}{2z}$ produces

$$c\partial x=-\tfrac{e\partial z}{2zz}-\tfrac{\partial z}{2}=-\tfrac{e\partial z-zz\partial z}{2zz},$$

from which there is deduced,

$$\partial x=-\frac{\partial z(e+zz)}{2czz},\ \text{but on account of}$$

$$b+cx=\tfrac{e-zz}{2z}\ \text{there becomes}\ s=\tfrac{e+zz}{2z}.$$

Therefore with these values substituted our $X\partial x$ is returned rational. Hence, after the integral of that had been found, in place of z the value found before $\sqrt{\left[e+(b+cx)^2\right]}-b-cx$ will be required to be substituted.

II. But if γ were a negative quantity, on putting

$$\gamma=-cc\ \text{ and }\ \beta=-2bc,$$

so that there may be had

$$s=\sqrt{(\alpha-2bcx-ccxx)}=\sqrt{\left[\alpha+bb-(b+cx)^2\right]},$$

where it is evident, the quantity $\alpha+bb$ by necessity must be positive, so that in general s may escape being imaginary. On account of which ,for the sake of brevity, we put $\alpha+bb=aa$, so that there becomes

$$s=\sqrt{\left[aa-(b+cx)^2\right]},$$

towards making that form rational we may set

$$\sqrt{\left[aa-(b+cx)^2\right]}=a-(b+cx)z,$$

from which with the square taken, there will be

$$aa-(b+cx)^2 = aa-2az(b+cx)+(b+cx)^2 zz$$

which equation is reduced to this :

$$-(b+cx) = -2az+(b+cx)zz,$$

from which there is found

$$(b+cx)=\tfrac{2az}{1+zz}, \text{ and thus}$$

$$x=\tfrac{2az-b-bzz}{c(1+zz)}.$$

But that equation differentiated gives

$$c\partial x = \frac{2a\partial z(1+zz)-4azz\partial z}{(1+zz)^2} = \frac{2a\partial z(1-zz)}{(1+zz)^2};$$

from which there becomes

$$\partial x = \frac{2a\partial z(1-zz)}{c(1+zz)^2}.$$

But again, since there shall be

$$s = a-(b+cx)z, \text{on account of } b+cx=\tfrac{2az}{1+zz}$$

there will be $s=\frac{a(1-zz)}{1+zz}$, on account of which, if in place of x , s and ∂x found, these values may be substituted, the proposed differential formula $X\partial x$ will emerge rational, and will be expressed by the variable z , of which the integral after it had been found, its assumed value may be restored in place of z everywhere $z=\alpha-\sqrt{[aa-(b+cx)^2]}$, and the integral will be obtained expressed by the single variable x .

Example I.

§.31. If there were

$\partial y = \frac{\partial x}{\sqrt{\left[e+(b+cx)^2\right]}}$, which formula pertains to the first case, there will be

$$\partial y = \tfrac{\partial x}{s} = -\tfrac{\partial z}{cz}, \text{because } \partial x = -\tfrac{\partial z(e+zz)}{2ezz} \text{ and } s=\tfrac{e+zz}{2z};$$

the integral of which is $y=-\frac{1}{c}lz$; therefore with the value restored

$$z = l[e+(b+cx)^2]-b-cx, \text{ there will be}$$

$$y=-\tfrac{1}{c}l[\sqrt{[e+(b+cx)^2]}-b-cx]+C,$$

the integral, if it must vanish on putting $x=0$, there becomes

$$C=\tfrac{1}{c}l[\sqrt{(e+bb)}-b].$$

Corollary.

§. 32. If there may be put $b=0$ and $c=1$,or

$\partial y=\frac{\partial x}{\sqrt{(e+xx)}}$, the integral will be

$$y=-\tfrac{1}{c}l[\sqrt{(e+xx)}-x]+l\sqrt{e}=l\frac{\sqrt{e}}{\sqrt{(e+xx)}-x},$$

which formula is reduced to this:

$$y=l\frac{\sqrt{(e+xx)}+x}{\sqrt{e}}.$$

Indeed since again there may be

$\partial.\sqrt{(e+xx)}=\frac{x\partial x}{\sqrt{(e+xx)}}$, there will be

$$\int\frac{x\partial x}{\sqrt{(e+xx)}}=\sqrt{(e+xx)}.$$

Therefore if these two formulas may be combined, this noteworthy equation will be had :

$$\int\frac{A\partial x+Bx\partial x}{\sqrt{(e+xx)}}=\mathrm{A}l\frac{\sqrt{(e+xx)}+x}{\sqrt{e}}+\mathrm{B}\sqrt{(e+xx)}.$$

Example 2.

§.33. Let $\partial y=\frac{\partial x}{\sqrt{\left[aa-(b+cx)^2\right]}}$, which formula referring to the second case, thus so that there shall be $\partial y=\frac{\partial x}{s}$. Therefore since there shall be $\partial x=\frac{2a\partial z(1-zz)}{c(1+zz)^2}$ and $s=\frac{a(1-zz)}{1+zz}$, there will be $\partial y=\frac{\partial x}{s}=\frac{2}{c}\cdot\frac{\partial z}{1+zz}$, from which by integrating there becomes $y=\frac{2}{c}\cdot$ Arc. tang.z. Because therefore there is

$z=\frac{a-\sqrt{[aa-(b+cx)^2]}}{b+cx}$, there will be

$$y=\tfrac{2}{c}\cdot\text{Arc. tang.}\frac{a-\sqrt{[aa-(b+cx)^2]}}{b+cx}+\mathrm{C}.$$

Corollary.

§. 34. Therefore let there be $b=0$ and $c=1$, or the differential formula proposed shall be $\partial y=\frac{\partial x}{\sqrt{[aa-xx]}}$, and there will be found : $y=2.\text{Arc. tang.}\frac{a-\sqrt{[aa-xx]}}{x}+\mathrm{C}.$

Therefore because the tangent of this arc is $\frac{a-\sqrt{[aa-xx]}}{x}$; the tangent of twice the arc will be $=\frac{x}{\sqrt{[aa-xx]}}$, thus so that there shall be $y=\text{Arc. tang.}\frac{x}{\sqrt{[aa-xx]}}$: but the sine of this arc will be $\frac{x}{a}$, and thus the integral sought becomes

$$\int\frac{\partial x}{\sqrt{[aa-xx]}}=\text{Arc. sin.}\tfrac{x}{a}.$$

Because again

$$\partial \cdot \sqrt{[aa-xx]} = -\tfrac{x\partial x}{\sqrt{[aa-xx]}},\text{ there will be}$$

$$\int \tfrac{x\partial x}{\sqrt{[aa-xx]}} = -\sqrt{[aa-xx]}:$$

on account of which this more general integration can be put together

$$\int \tfrac{A\partial x + Bx\partial x}{\sqrt{[aa-xx]}} = \text{A.Arc.sin.}\tfrac{x}{a}. - \text{B}\sqrt{[aa-xx]}.$$

Problem 13.

§. 35. *If V were a rational function of the two quantities* v^n *and* s , *with s given by*

$$s = \sqrt{\left(\alpha + \beta v^n + \gamma v^{2n}\right)},$$

to free the differential formula $Vv^{n-1}\partial v$ *from irrationality.*

Solution.

There may be put $v^n = x$, and there will become

$$s = \sqrt{\left(\alpha + \beta x + \gamma xx\right)} \text{ and } v^{n-1}\partial v = \tfrac{\partial x}{n};$$

therefore now here V will be a function of the two rational quantities *x* and *s,*
with *s* being given by

$$s = \sqrt{\left(\alpha + \beta x + \gamma xx\right)}$$

and the formula will be $\tfrac{V\partial x}{n}$ being free from irrationality; which case clearly agrees with the preceding problem, and thus will have the same solution.

Scholium.

§.36. The preceding cases treated up to this point, can be extended to nearly all differential formulas, which indeed have been examined until now. Yet meanwhile cases of this kind can occur, for which a suitable substitution, necessary to remove the irrationality, is not so easy to observe; but at last such can be found by sharp reasoning, and we set out certain particular examples of kinds in place in published works which cannot be treated by the previous general methods.

Example 1.

§.37. *If this irrational formula were proposed :*

$$\partial P = \frac{\partial x(1+xx)}{(1-xx)\sqrt{(1+xx)^4}},$$

to investigate its integral P.

If anyone wished here to use a substitution of this kind, by which the formula $\sqrt[4]{(1+xx)}$ would be led to rationality, much time and effort would be wasted, yet meanwhile the following single substitution will attend to the matter. There may be put

$$\frac{x\sqrt{2}}{1-xx}=p, \text{ and there will be}$$

$$1+pp=\frac{1+x^4}{(1-xx)^2}$$

hence

$$\sqrt{(1+pp)}=\frac{\sqrt{(1+x^4)}}{1-xx}$$

then indeed on differentiating there will be:

$$\partial p=\frac{\partial x\sqrt{2}(1+xx)}{(1-xx)^2}$$

from which values it is deduced :

$$\frac{\partial p}{\sqrt{(1+pp)}}=\frac{\partial x\sqrt{2}(1+xx)}{(1-xx)\sqrt{(1+x^4)}},$$

which happily agrees with that formula proposed, thus so that there shall be

$$\frac{\partial p}{\sqrt{(1+pp)}}=\partial P\sqrt{2}, \text{ or } \partial P=\frac{1}{\sqrt{2}}\cdot\frac{\partial p}{\sqrt{(1+pp)}};$$

from which it is gathered on integration:

$$P=\frac{1}{\sqrt{2}}l\left[\sqrt{(1+pp)}+p\right].$$

Whereby if in place of p and $\sqrt{(1+pp)}$ given values may be substituted, this very memorable integration will be found :

$$P=\int\frac{\partial x(1+xx)}{(1-xx)\sqrt{(1+x^4)}}=\frac{1}{\sqrt{2}}l\frac{\sqrt{(1+x^4)}+\sqrt{2}}{1+xx}$$

Example 2.

§.38. *If this irrational formula were proposed* $\partial Q=\frac{\partial x(1-xx)}{(1+xx)\sqrt{(1+x^4)}}$, *and with is integral Q to be found*.

According to this excellent method there may be put in place $\frac{x\sqrt{2}}{1+xx}=q$, and there will become

$$\sqrt{(1-qq)}=\frac{\sqrt{(1+x^4)}}{(1+xx)};$$

then truly there will become $\partial q = \frac{\partial x(1-xx)\sqrt{2}}{(1+xx)^2}$, and hence it is deduced that

$$\frac{\partial q}{\sqrt{(1-qq)}} = \frac{\partial x(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}} = \partial Q\sqrt{2}\,,$$

from which there becomes:

$$Q = \tfrac{1}{\sqrt{2}}\int \frac{\partial q}{\sqrt{(1-qq)}} = \tfrac{1}{\sqrt{2}}\,\text{Arc.sin}\,.q.$$

Therefore with the assumed value for q restored, that same integral will be found :

$$Q = \int \frac{\partial x(1-xx)}{(1+xx)\sqrt{(1+x^4)}} = \tfrac{1}{\sqrt{2}}\,\text{Arc.sin}\,.\tfrac{x\sqrt{2}}{1+xx}.$$

Scholium.

§. 39. Since these two formulas

$$\frac{\partial x(1+xx)\sqrt{2}}{(1-xx)\sqrt{(1+x^4)}} \text{ and } \frac{\partial x(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}}$$

shall lead to these simple ones

$$\frac{\partial p}{\sqrt{(1+pp)}} \text{ and } \frac{\partial q}{\sqrt{(1-qq)}},$$

each of which can be made rational easily, these proposed formulas themselves can be freed from irrationality with the aid of a suitable substitution ; from which it is little wonder, that the integrals of these may be able to be shown either by logarithms or circular arcs. For now it has been shown well enough, the integrals of all the rational differential formulas always can be shown by logarithms and circular arcs, or thus algebraically ; which therefore is extended also to these irrational formulas, which with the aid of a certain substitution can be led to rationality. From which in turn several geometers have concluded : if some differential formula in no plane manner may be freed from irrationality, then neither can its integral be able to be expressed by logarithms nor by circular arcs, much less algebraically, but is required to be referred to some other kind of transcendental quantity.
Another combination of the two preceding examples leads us to the following solution.

Example 3.

§.40. *If this differential formula were proposed :*

$$\partial y = \frac{\partial x\sqrt{(1+x^4)}}{1-x^4},$$

to find its integral.
This formula cannot be reduced by either substitution undertaken before: yet with each together the matter can be completed, for indeed its integral may be expedited by

logarithms and circular arcs by the following method. For the formula can be separated into the two following parts, which are

$$\partial y=\frac{\frac{1}{2}\partial x(1+xx)}{(1-xx)\sqrt{\left(1+x^4\right)}}+\frac{\frac{1}{2}\partial x}{(1+xx)\sqrt{\left(1+x^4\right)}},$$

certainly the sum of which produces the same formula of our proposition ; for it produces

$$\partial y=\frac{\frac{1}{2}\partial x(1+xx)^2+\frac{1}{2}\partial x(1-xx)^2}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}}=\frac{\partial x\left(1+x^4\right)}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}}=\frac{\partial x\sqrt{\left(1+x^4\right)}}{1-x^4}.$$

Therefore since if the two preceding examples may be called in to help, clearly there becomes $dy=\frac{1}{2}\partial P+\frac{1}{2}\partial Q$, consequently the integral sought will be $y=\frac{1}{2}P+\frac{1}{2}Q$, which will be able to be expressed in the following manner

$$\int\frac{\partial x\sqrt{\left(1+x^4\right)}}{1-x^4}=\frac{1}{2\sqrt{2}}l\frac{\sqrt{\left(1+x^4\right)}+x\sqrt{2}}{1-xx}+\frac{1}{2\sqrt{2}}\text{Arc.sin.}\frac{x\sqrt{2}}{1+xx}.$$

Example 4.

§. 41. *If this differential formula were proposed* $\partial y=\frac{xx\partial x}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}}$, *to investigate its integral.*

This formula can be treated in a similar manner to the preceding ; for it may be separated into the following two parts :

$$\partial y=\frac{\frac{1}{4}\partial x(1+xx)}{(1-xx)\sqrt{\left(1+x^4\right)}}-\frac{\frac{1}{4}\partial x(1-xx)}{(1+xx)\sqrt{\left(1+x^4\right)}},$$

certainly which taken together produce

$$\partial y=\frac{\frac{1}{4}\partial x(1+xx)^2-\frac{1}{4}\partial x(1-xx)^2}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}}$$

$$=\frac{\frac{1}{4}\partial x.4xx}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}}=\frac{xx\partial x}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}},$$

which since it shall be the proposed formula itself, from the preceding examples it will become $\partial y=\frac{1}{4}\partial \mathrm{P}-\frac{1}{4}\partial Q$, consequently $y=\frac{1}{4}\mathrm{P}-\frac{1}{4}Q$, hence the integral sought will be found expressed thus.

$$\int \frac{xx\partial x}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}} = \frac{1}{4\sqrt{2}} l \frac{\sqrt{\left(1+x^4\right)}+x\sqrt{2}}{1-xx} - \frac{1}{4\sqrt{2}} \operatorname{Arc.sin}.\frac{x\sqrt{2}}{1+xx}.$$

Scholium.

§. 42. These two last examples, if clearly in no manner were able to lead to rationality with the help of some substitutions, they might bear a significant lesson, which conclusion should be mentioned above whenever it should fail : But the matters I have found considered more carefully, all these four examples with the aid of a single substitution at once lead to rationality and thus were able to be integrated ; that which was shown most generally will be worth the effort.

Another resolution of the four last examples.

§. 43. For the first example there may be put

$$v = \frac{x\sqrt{2}}{\sqrt{\left(1+x^4\right)}}, \text{ and there will be } \sqrt{\left(1+vv\right)} = \frac{1+xx}{\sqrt{\left(1+x^4\right)}};$$

then indeed

$$\sqrt{\left(1-vv\right)} = \frac{1-xx}{\sqrt{\left(1+x^4\right)}},$$

from which there becomes

$$\sqrt{\frac{1+vv}{1-vv}} = \frac{1+xx}{1-xx} \text{ and } \sqrt{\left(1-v^4\right)} = \frac{1-x^4}{1+x^4}.$$

But we come to the differentiation :

$$\partial v = \frac{\partial x\left(1-x^4\right)\sqrt{2}}{\left(1+x^4\right)\sqrt{\left(1+x^4\right)}}$$

Since now there shall be $\frac{1-x^4}{1+x^4} = \sqrt{\left(1-v^4\right)}$, there will be

$$\partial v = \frac{\partial x\sqrt{2}.\sqrt{\left(1-x^4\right)}}{\sqrt{\left(1+x^4\right)}}, \text{ or } \frac{\partial v}{\sqrt{\left(1-v^4\right)}} = \frac{\partial x\sqrt{2}}{\sqrt{\left(1+x^4\right)}};$$

which equation is especially noteworthy. Because if now this equation may be multiplied by $\sqrt{\frac{1+vv}{1-vv}} = \frac{1+xx}{1-xx}$, this equation arises :

$$\frac{\partial v}{1-vv} = \frac{\partial x\left(1+xx\right)\sqrt{2}.}{\left(1-xx\right)\sqrt{\left(1+x^4\right)}},$$

and thus there becomes

$$\int \frac{\partial x\left(1+xx\right)}{\left(1-xx\right)\sqrt{\left(1+x^4\right)}} = \frac{1}{2\sqrt{2}}\int \frac{\partial v}{1-vv} = \frac{1}{2\sqrt{2}} l \frac{1+v}{1-v}.$$

From which the equation

$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{\sqrt{(1-v^4)}}=\frac{\partial x}{\sqrt{(1+x^4)}}$ may be multiplied by $\sqrt{\frac{1-vv}{1+vv}}=\frac{1-xx}{1+xx}$,

and the formula of the second example will be produced :

$$\int\frac{\partial x(1-xx)}{(1+xx)\sqrt{(1+x^4)}}=\frac{1}{\sqrt{2}}\int\frac{\partial x}{1+vv}=\frac{1}{\sqrt{2}}\,\text{Arc.}\tan.v\,.$$

Again the same equation

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{\sqrt{(1-v^4)}}=\frac{\partial x}{\sqrt{(1+x^4)}}$$

may be divided by

$$\sqrt{(1-v^4)}=\frac{1-x^4}{1+x^4},$$

and there will be produced :

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{1-v^4}=\frac{\partial x\sqrt{(1+x^4)}}{1-x^4};$$

which is the formula of the third example, thus so that now there shall be

$$\int\frac{\partial x\sqrt{(1+x^4)}}{(1-x^4)}=\frac{1}{\sqrt{2}}\int\frac{\partial v}{1-v^4}=\frac{1}{2\sqrt{2}}\int\frac{\partial v}{1+vv}+\frac{1}{2\sqrt{2}}\int\frac{\partial v}{1-vv},$$

because the integral agrees well with that found before. Finally the last equation found here

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{1-v^4}=\frac{\partial x\sqrt{(1+x^4)}}{(1-x^4)},$$

may be multiplied by $vv=\frac{2xx}{1+x^4}$, so that there may be produced

$$\frac{1}{\sqrt{2}}\cdot\frac{vv\partial v}{1-x^4}=\frac{2xx\partial x\sqrt{(1+x^4)}}{(1-x^4)(1+x^4)}=\frac{2xx\partial x}{(1-x^4)\sqrt{(1+x^4)}},$$

so that for example four there may be deduced

$$\int\frac{xx\partial x}{(1-x^4)\sqrt{(1+x^4)}}=\frac{1}{2\sqrt{2}}\int\frac{vv\partial v}{1-v^4}=-\frac{1}{4\sqrt{2}}\int\frac{\partial v}{1+vv}+\frac{1}{4\sqrt{2}}\int\frac{\partial v}{1-vv},$$

so that since there shall be $v=\frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, there will be

$$\int\frac{\partial v}{1-vv}=\frac{1}{2}l\frac{1+v}{1-v}=\frac{1}{2}l\frac{\sqrt{(1+x^4)}+x\sqrt{2}}{\sqrt{(1+x^4)}-x\sqrt{2}}$$

$$=\frac{1}{2}l\frac{\left[\sqrt{(1+x^4)}+x\sqrt{2}\right]^2}{(1-xx)^2}=l\frac{\sqrt{(1+x^4)}+x\sqrt{2}}{1-xx}.$$

Then truly there is

$$\int\frac{\partial v}{1+vv}=\text{Arc.tang.}v=\text{Arc.sin.}\frac{v}{\sqrt{(1+vv)}}=\text{Arc.sin.}\frac{x\sqrt{2}}{1+xx}.$$

Scholium.

§. 44. But nevertheless it has been permitted to reduce these four equations to rationality, yet the conclusion mentioned above, that all the integral formulas, which in no way were able to be made rational, might belong to another kind of transcendental function, neither shall be able to be expressed by logarithms and the arcs of circles alone, not only remains suspect, but also the falseness of this can be seen clearly. For if the function shall be

$$\mathrm{X}=\frac{a}{\sqrt{(1+xx)}}+\frac{b}{\sqrt[3]{(1+x^3)}}+\frac{c}{\sqrt[4]{(1+x^4)}};$$

then certainly the differential formula $\mathrm{X}\partial x$ in no manner can be led to rationality; yet meanwhile its individual parts

$$\frac{a\partial x}{\sqrt{(1+xx)}},\ \frac{b\partial x}{\sqrt[3]{(1+x^3)}}\ \text{and}\ \frac{c\partial x}{\sqrt[4]{(1+x^4)}}$$

themselves can be effected to become rational and to be integrated by logarithms and circular arcs can be shown. In place of corollaries we have adjoined the following noteworthy problems.

Problem 14.

§.45. *To investigate the values of the integrations of the formulas* $\int\frac{\partial x}{\sqrt{(1+x^4)}}$ and $\int\frac{\partial v}{\sqrt{(1-v^4)}}$ *by series, for the cases in which both* $v=1$ *as well as* $x=1$.

Solution.

Since by putting $v=\frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, as we have done above, it shall be clear, by taking $x=0$ to be also $v=0$, and on taking $x=1$ to be $v=1$, thus so that these two quantities x and v likewise may vanish and likewise may be made equal to one at the same time ; hence we deduce the same differential equation with attention paid to the powers

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{\sqrt{(1-v^4)}}=\frac{\partial x}{\sqrt{(1+x^4)}}$$

which therefore both formulas shall be required to be converted into series ; moreover there will be

$$\frac{1}{\sqrt{(1-v^4)}}=\left(1-v^4\right)^{-\frac{1}{2}}=1+\tfrac{1}{2}v^4+\tfrac{1\cdot3}{2\cdot4}v^8+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}v^{12}+\text{etc., and}$$

$$\frac{1}{\sqrt{(1+x^4)}}=\left(1+x^4\right)^{-\frac{1}{2}}=1-\tfrac{1}{2}x^4+\tfrac{1\cdot3}{2\cdot4}x^8-\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}x^{12}+\text{etc.}$$

Now that multiplied by ∂v and integrated gives

$$\int\frac{\partial v}{\sqrt{(1-v^4)}}=v+\tfrac{1}{2\cdot5}v^5+\tfrac{1\cdot3}{2\cdot4\cdot9}v^9+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot13}v^{13}+\text{etc.}$$

so that on putting $v=1$, the value of this integral will be

$$1+\tfrac{1}{2\cdot5}+\tfrac{1\cdot3}{2\cdot4\cdot9}+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot13}+\tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8\cdot17}+\text{etc.}$$

which series we will indicate by the letter A. In a similar manner the other series multiplied by ∂x and integrated produces

$$\int\frac{\partial x}{\sqrt{(1+x^4)}}=x-\tfrac{1}{2\cdot5}x^5+\tfrac{1\cdot3}{2\cdot4\cdot9}x^9-\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot13}x^{13}+\text{etc.}$$

of which the value on making $x=1$ will be

$$1-\tfrac{1}{2\cdot5}+\tfrac{1\cdot3}{2\cdot4\cdot9}-\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot13}+\tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8\cdot17}-\text{etc.}$$

which we may designate by B, thus so that there shall be $\mathrm{B}=\frac{\mathrm{A}}{\sqrt{2}}$, or $\mathrm{A}=\mathrm{B}\sqrt{2}$;

from which it is apparent, the first series to be had to the second as $\sqrt{2}:1$.

Scholium.

§. 46. The value of the formula of the integral $\int\frac{\partial v}{\sqrt{(1-v^4)}}$ can also be investigated in this manner. Since there shall be

$$\frac{1}{\sqrt{(1-v^4)}}=\frac{(1+vv)^{-\frac{1}{2}}}{\sqrt{(1-vv)}},\ \text{and}$$

$$\left(1+vv\right)^{-\frac{1}{2}}=1-\tfrac{1}{2}vv+\tfrac{1\cdot3}{2\cdot4}v^4-\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}v^6+\text{etc.}$$

it may be observed that $\int\frac{\partial v}{\sqrt{(1-vv)}}=\frac{\pi}{2}$. Then for the integration of the remaining terms there may be put

$$\int\frac{v^{n+2}\partial v}{\sqrt{(1-vv)}}=\mathrm{A}v^{n+1}\sqrt{(1-vv)}+\mathrm{B}\int\frac{v^n\partial v}{\sqrt{(1-vv)}},$$

which equation differentiated gives

$$\frac{v^{n+2}}{\sqrt{(1-vv)}}=\left(n+1\right)\mathrm{A}v^n\sqrt{(1-vv)}-\frac{\mathrm{A}v^{n+2}}{\sqrt{(1-vv)}}+\frac{\mathrm{B}v^n}{\sqrt{(1-vv)}},$$

from which on multiplying by $\sqrt{(1-vv)}$ there is produced

$$v^{n+2} = (n+1)\mathrm{A}v^n - (n+1)\mathrm{A}v^{n+2} - \mathrm{A}^{n+2} + \mathrm{B}v^n .$$

Hence the terms in which v^{n+2} is present , equated to each other produces $1 = -(n+2)\mathrm{A}$, and thus $\mathrm{A} = -\frac{1}{n+2}$; truly the terms containing v^n gives $0 = (n+1)\mathrm{A} + \mathrm{B}$, so that there shall be $\mathrm{B} = \frac{n+1}{n+2}$, thus so that in general there shall be

$$\int \frac{v^{n+2}\partial v}{\sqrt{(1-vv)}} = -\frac{1}{n+2} v^{n+1}\sqrt{(1-vv)} + \frac{n+1}{n+2}\int \frac{v^n\partial v}{\sqrt{(1-vv)}},$$

in order that which integral is required to vanish on putting $v = 0$. Now there may be put $v = 1$, and there becomes

$$\int \frac{v^{n+2}\partial v}{\sqrt{(1-vv)}} = \frac{n+1}{n+2}\int \frac{v^n\partial v}{\sqrt{(1-vv)}};$$

hence therefore by writing the successive values 0 , 2 , 4 , 6 , 8 , etc. for *n,* there becomes

$$\text{I.} \quad \int \frac{vv\partial v}{\sqrt{(1-vv)}} = \frac{1}{2}\cdot\frac{\pi}{2};$$

$$\text{II.} \quad \int \frac{v^4\partial v}{\sqrt{(1-vv)}} = \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2};$$

$$\text{III.} \quad \int \frac{v^6\partial v}{\sqrt{(1-vv)}} = \frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2};$$

etc. etc.

with which values used, in the case $v = 1$ there will be:

$$\int \frac{\partial v}{\sqrt{(1-v^4)}} = \frac{\pi}{2} - \frac{1}{2^2}\cdot\frac{\pi}{2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2}\cdot\frac{\pi}{2} - \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2}\cdot\frac{\pi}{2} + \text{etc.}$$

$$= \frac{\pi}{2}\left(1 - \frac{1^2}{2^2}\cdot\frac{\pi}{2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2} - \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2} + \frac{1^2\cdot 3^2\cdot 5^2\cdot 7^2}{2^2\cdot 4^2\cdot 6^2\cdot 8^2} - \text{etc.}\right)$$

thus, so that from the preceding problem there will be

$$1 - \frac{1}{2.5} + \frac{1\cdot 3}{2.4.9} - \frac{1\cdot 3\cdot 5}{2.4.9.13} + \text{etc.}$$

$$= \frac{\pi}{2}\left(1 - \frac{1^2}{2^2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2} - \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2} + \text{etc.}\right)$$

from which there becomes

$$\frac{\pi}{2} = \frac{1 - \frac{1}{2.5} + \frac{1\cdot 3}{2.4.9} - \frac{1\cdot 3\cdot 5}{2.4.6.13} + \text{etc.}}{1 - \frac{1^2}{2^2} + \frac{1^2\cdot 3^2}{2^2\cdot 4^2} - \frac{1^2\cdot 3^2\cdot 5^2}{2^2\cdot 4^2\cdot 6^2} + \text{etc.}}$$

2) Concerning the integration of the irrational formulas :

$$\int \frac{x^n \partial x}{\sqrt{(aa-2bx+cxx)}}.$$

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Book. VI. Part II. Pages 62 - 67.

Problem 15.

To find the integral of this irrational formula

$$\int \frac{x^n \partial x}{\sqrt{(aa-2bx+cxx)}}.$$

Solution.

§.47. We may begin from the simplest case, where $n=0$, and we seek the integral of the formula $\frac{\partial x}{\sqrt{(aa-2bx+cxx)}}$, which on putting $x=\frac{b+z}{c}$ is transformed into this $\frac{\partial z}{\sqrt{(aacc-bbc+czz)}}$, where it is agreed that two cases are to be distinguished, accordingly as whether c were positive or negative. Therefore in the first place there shall be $c=+ff$, and our formula becomes $\frac{\partial z}{f\sqrt{(aaff-bb+zz)}}$, the integral of which is $\frac{1}{f} l \frac{z+\sqrt{(aaff-bb+zz)}}{C}$ and thus our integral will become

$$\frac{1}{\sqrt{c}} l \frac{cx-b+\sqrt{(aac-2bcx+ccxx)}}{C},$$

therefore taken thus, so that it may vanish on putting $x=0$, there emerges

$$\frac{1}{\sqrt{c}} l \frac{cx-b+\sqrt{c(aa-2bx+cxx)}}{-b+a\sqrt{c}}.$$

But indeed if c were a negative quantity, for example $c=-gg$, the differential formula expressed by z will become $\frac{\partial z}{g\sqrt{(aagg+bb-zz)}}$, the integral of which is $\frac{1}{g}\text{Arc.sin.}\frac{z}{\sqrt{(aagg+bb)}}+\text{C}$; whereby with the integral taken thus, so that it may vanish on putting $x=0$, becomes

$$-\frac{1}{g}\text{Arc.sin.}\frac{cx-b}{\sqrt{(aagg+bb)}}+\frac{1}{g}\text{Arc.sin.}\frac{b}{\sqrt{(aagg+bb)}}.$$

§.48. Now Π may denote the value of the integral of the formula $\int \frac{\partial x}{\sqrt{(aa-2bx+cxx)}}$ thus obtained, so that it may vanish on putting $x=0$, whether c were a positive or negative quantity; and if $c=+ff$ there will be as we have seen :

$$\Pi = \frac{1}{f} l \frac{ffx-b+f\sqrt{(aa-2bx+ffxx)}}{af-b};$$

and indeed in the other case, in which $c=-gg$, there will be

$$\Pi = -\frac{1}{g}\text{Arc.sin}\frac{ggx+b}{\sqrt{(aagg+bb)}} + \frac{1}{g}\text{Arc.sin}\frac{b}{\sqrt{(aagg+bb)}}$$

or with both the arcs taken together we will have

$$\Pi = \frac{1}{g}\text{Arc.sin}\frac{bg\sqrt{(aa-2bx-ggxx)}-abg-ag^3x}{aagg+bb}.$$

Therefore as we will show soon, the integration of the general formula $\int\frac{x^n\partial x}{\sqrt{(aa-2bx+cxx)}}$ can be reduced always to the case $n=0$ but only n were a positive integer, all these integrals can be expressed by the same value Π.

§. 49. Now after the integration of the variable quantity x we may attribute a constant value of this kind, where the irrational formula

$$\sqrt{(aa-2bx+cxx)}$$

may be reduced to zero, that which shall happen, if there may be taken $x=\frac{b\pm\sqrt{(bb-aac)}}{c}$, and thus with two cases. For each case we may put the function Π to be changed into Δ, thus so that for the case $c = ff$ there shall be

$$\Delta = \frac{1}{f} l \frac{\sqrt{(bb-aaff)}}{af-b} = \frac{1}{f} l \sqrt{\frac{b+af}{b-af}};$$

but for the other case, where $c=-gg$

$$\Delta = \frac{1}{g}\text{Arc.sin}\frac{\pm ag\sqrt{(bb+aagg)}}{aagg+bb} = \frac{1}{g}\text{Arc.sin}\frac{ag}{\sqrt{(bb+aagg)}}.$$

But we are going to consider mainly these values of Δ in the following cases, in which the formula of the root $\sqrt{(aa-2bx+cxx)}$ vanishes.

§. 50. Now progressing to the following case, we will consider the formula $s=\sqrt{(aa-2bx+cxx)}-a$, so that clearly it may vanish on making $x=0$, and because there is

$$\partial s = \frac{-b\partial x+cx\partial x}{\sqrt{(aa-2bx+cxx)}}$$

there will be on integrating by parts

$$c\int\frac{cx\partial x}{\sqrt{(aa-2bx+cxx)}} = b\int\frac{\partial x}{\sqrt{(aa-2bx+cxx)}} + s$$

from which we deduce

$$\int \frac{x\partial x}{\sqrt{(aa-2bx+cxx)}} = \frac{b}{c}\Pi + \frac{\sqrt{(aa-2bx+cxx)}-a}{c};$$

whereby, if after the integration we may put $x = \frac{b\pm\sqrt{(bb-aac)}}{c}$, certainly in which cases there shall be $\sqrt{(aa-2bx+cxx)} = 0$ and $\Pi = \Delta$, there will become

$$\int \frac{x\partial x}{\sqrt{(aa-2bx+cxx)}} = \frac{b}{c}\Delta - \frac{a}{c}$$

§.51. Again we may take $s = \sqrt{(aa-2bx+cxx)}$, there becomes $\partial s = \frac{aa\partial x - 3bx\partial x + 2cxx\partial x}{\sqrt{(aa-2bx+cxx)}}$ and on integrating by parts there is deduced

$$2c\int \frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}} = 3b\int \frac{x\partial x}{\sqrt{(aa-2bx+cxx)}} - aa\int \frac{\partial x}{\sqrt{(aa-2bx+cxx)}} + s,$$

from which we deduce at once for the case $\sqrt{(aa-2bx+cxx)} = 0$

$$\int \frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}} = \frac{(3bb-aac)}{2cc}\Delta - \frac{3ab}{2cc}.$$

§. 52. Now towards rising to higher powers we may put $s = xx\sqrt{(aa-2bx+cxx)}$, and hence because there shall be

$$\partial s = \frac{2aax\partial x - 5bxx\partial x + 3cx^3\partial x}{\sqrt{(aa-2bx+cxx)}}, \text{ there will be}$$

$$3c\int \frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}} = 5b\int \frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}} - 2aa\int \frac{x\partial x}{\sqrt{(aa-2bx+cxx)}} + s,$$

and hence again for the case where after the integration there is put

$$x = \frac{b\pm\sqrt{(bb-aac)}}{c}, \text{ there will be found}$$

$$\int \frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}} = \left(\frac{5b^3-3aabc}{2c^3}\right)\Delta - \frac{15abb}{6c^3} + \frac{2a^3}{3cc},$$

$$\text{or } = \left(\frac{5b^3}{2c^3} - \frac{3aab}{2cc}\right)\Delta - \frac{5abb}{2c^3} + \frac{2a^3}{3cc}.$$

§.53. In a similar manner if $s = x^3\sqrt{(aa-2bx+cxx)}$, and hence because there shall be

$$\partial s = \frac{3aaxx\partial x - 7bx^3\partial x + 4cx^4\partial x}{\sqrt{(aa-2bx+cxx)}},$$

on integrating by parts there will be

$$4c\int\frac{x^4\partial x}{\sqrt{(aa-2bx+cxx)}}=7b\int\frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}}$$
$$-3aa\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}+s;$$

since then for the case in which there shall be $\sqrt{(aa-2bx+cxx)}=0$, we will have

$$\int\frac{x^4\partial x}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{35}{8}\frac{b^4}{c^4}-\frac{15aabb}{4c^3}+\frac{3a^4}{8cc}\right)\Delta-\frac{35ab^3}{8c^4}+\frac{55a^3b}{24c^3}.$$

§. 54. But in order that these formulas may be able to be investigated better, we will show the individual cases by factors, just as they rise in order, without any abbreviations, and thus in this manner the integrals of the formulas found will be represented

$$\int\frac{\partial x}{\sqrt{(aa-2bx+cxx)}}=\Delta,$$
$$\int\frac{x\partial x}{\sqrt{(aa-2bx+cxx)}}=\frac{b}{c}\Delta-\frac{a}{c},$$
$$\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{1.3bb}{1.2cc}-\frac{aa}{1.2c}\right)\Delta-\frac{1.3.ab}{1.2.cc},$$
$$\int\frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{1.3.5b^3}{1.2.3c^3}-\frac{1.3.5aab}{1.2.3cc}\right)\Delta-\frac{1.3.5abb}{1.2.3.c^3}+\frac{1.2.2a^3}{1.2.3cc},$$
$$\int\frac{x^4\partial x}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{1.3.5.7b^3}{1.2.2.4c^4}-\frac{1.3.5.6aabb}{1.2.3.4c^4}+\frac{1.3.3a^4}{1.2.3.4cc}\right)\Delta-\frac{1.3.5.7ab^3}{1.2.3.4c^4}+\frac{1.5.11a^3b}{1.2.3.4c^3}.$$

§. 55. Now we will put this general development in place, by assuming $s=x^n\sqrt{(aa-2bx+cxx)}$ and hence because there shall be

$$\partial s=\frac{naax^{n-1}\partial x-(2n+1)bx^n\partial x+(n+1)cx^{n+1}\partial x}{\sqrt{(aa-2bx+cxx)}},$$

thence there is deduced on integrating by parts

$$(n+1)c\int\frac{x^{n+1}\partial x}{\sqrt{(aa-2bx+cxx)}}=(2n+1)b\int\frac{x^n\partial x}{\sqrt{(aa-2bx+cxx)}}$$
$$-naa\int\frac{x^{n-1}\partial x}{\sqrt{(aa-2bx+cxx)}}+x^n\sqrt{(aa-2bx+cxx)}.$$

Because if now truly we may elucidate as before

$$\int\frac{x^{n-1}\partial x}{\sqrt{(aa-2bx+cxx)}}=\mathrm{M}\Delta-\mathfrak{M}\text{ and}$$
$$\int\frac{x^n\partial x}{\sqrt{(aa-2bx+cxx)}}=\mathrm{N}\Delta-\mathfrak{N},$$

thus so that these two formulas shall be known, thus following from these it will be determined that there shall be

$$\int \frac{x^{n+1}\partial x}{\sqrt{(aa-2bx+cxx)}} = \left[\frac{(2n+1)b\mathrm{N}}{(n+1)c} - \frac{naa\mathrm{M}}{(n+1)c}\right]\Delta$$
$$- \frac{(2n+1)b\mathfrak{N}}{(n+1)c} + \frac{naa\mathfrak{M}}{(n+1)c}.$$

Therefore in this way these integrations are allowed to be continued, as long as it pleases, while from any two the following may be formed with the help of this rule, thus so that all these integrals may depend either on logarithms or on circular arcs, according as the coefficient c were either positive or negative. But it is evident that these values themselves cannot be assigned, unless the exponent n were a positive whole number.

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3) Concerning the integration of the formula $\int \frac{\partial x\sqrt{(1+x^4)}}{1-x^4}$, and of others of this kind, by logarithms and circular arcs.

M. S. of the Academy, presented 16 Sept. 1776.

§. 56. Thus since I may have chanced recently, to express the integral of this formula $\int \frac{\partial x\sqrt{(1+x^4)}}{1-x^4}$ by a circular arc and a logarithm, there this integration was considered by me to be more noteworthy, because in no manner could I see, how that could be reduced to rationality, since it is certain that this same formula, which may be seen to be simpler, $\int \partial x\sqrt{(1+x^4)}$, by no means can be made rational, and indeed neither could I see, how the accession of the denominator $1-x^4$ was able to promote this reduction, and hence I concluded to be given irrational differential formulas of this kind, the integrals of which may be permitted to be shown by logarithms and circular arcs, even if they were not able to be freed from irrationality by any substitution : which conclusion indeed appears to prevail everywhere for composite formulas, nevertheless indeed those same formulas

$$\int \frac{\partial x}{\sqrt[3]{(1+x^3)}} \text{ and } \int \frac{\partial x}{\sqrt[4]{(1+x^4)}}$$

are able to be reduced to rationality, yet a formula composed from those

$$\int \partial x\left[\frac{\mathrm{A}}{\sqrt[3]{(1+x^3)}} + \frac{\mathrm{B}}{\sqrt[4]{(1+x^4)}}\right]$$

clearly is unable to be reduced to another rational form by any substitution ; as each part therefore demands its own special substitution.

§. 57. Yet meanwhile with this proposed formula

$$\int \frac{\partial x\sqrt{\left(1+x^4\right)}}{1-x^4} = \mathrm{S}$$

I may be considering more attentively, I have found, that it can be freed from irrationality, evidently with the aid of this particular substitution

$$x = \frac{\sqrt{(1+tt)}+\sqrt{(1-tt)}}{t\sqrt{2}}.$$

Hence indeed there becomes :

$$\partial x = -\frac{\partial t}{tt\sqrt{2(1+tt)}} - \frac{\partial t}{tt\sqrt{2(1-tt)}}$$

which two parts reduced to the same denominator give

$$\partial x = -\frac{\partial t}{tt\sqrt{2\left(1-t^4\right)}}\left[\sqrt{(1-tt)}+\sqrt{(1+tt)}\right].$$

Therefore since there becomes

$$\sqrt{(1+tt)}+\sqrt{(1-tt)} = tx\sqrt{2},$$

with this value substituted there becomes

$$\partial x = -\frac{x\partial t}{t\sqrt{\left(1-t^4\right)}},$$

thus so that there shall be

$$\partial \mathrm{S} = -\frac{x\partial t\sqrt{\left(1+x^4\right)}}{t\left(1-x^4\right)\sqrt{\left(1-t^4\right)}}.$$

§. 58. But again with the square taken there will be

$$xx = \frac{1+\sqrt{\left(1-t^4\right)}}{tt},$$

from which we deduce

$$1+xx = \frac{1+tt+\sqrt{\left(1-t^4\right)}}{tt} = \frac{\sqrt{(1+tt)}}{tt}\left[\sqrt{(1+tt)}+\sqrt{(1-tt)}\right],$$

and thus on account of

$$\sqrt{(1+tt)}+\sqrt{(1-tt)} = tx\sqrt{2}, \text{ there will be}$$

$$1+xx = \frac{x\sqrt{2}(1+tt)}{t}.$$

In a like manner there will be

$$1-xx=-\left(\frac{1-tt+\sqrt{\left(1-t^4\right)}}{tt}\right)$$
$$=-\frac{\sqrt{(1-tt)}}{tt}\left[\sqrt{(1-tt)}+\sqrt{(1+tt)}\right]=-\frac{x\sqrt{2}(1-tt)}{t}.$$

Hence therefore there follows to become
$$1-x^4=-\frac{2xx\sqrt{\left(1-t^4\right)}}{tt},$$
which value substituted into our formula gives
$$\partial S=+\frac{t\partial t\sqrt{\left(1+x^4\right)}}{2x\left(1-t^4\right)}.$$

§. 59. Then with the squares taken we will have

$$(1+xx)^2=\frac{2xx(1+tt)}{tt} \text{ and}$$
$$(1-xx)^2=\frac{2xx(1-tt)}{tt}$$

with which added there will be produced

$$(1+xx)^2+(1-xx)^2=2\left(1+x^4\right)=\frac{4xx}{tt},$$
so that there shall become
$$\sqrt{\left(1+x^4\right)}=\frac{x\sqrt{2}}{t};$$

with which value substituted our formula will change into this :
$$\partial S=\frac{1}{\sqrt{2}}\cdot\frac{\partial t}{1-t^4};$$
which therefore is a rational formula and involves only the variable t.

§. 60. Since again therefore there will be
$$\frac{1}{1-t^4}=\frac{1}{2}\cdot\frac{1}{1+tt}+\frac{1}{2}\cdot\frac{1}{1-tt}$$
then truly by integrating there may be found

$$\int\frac{\partial t}{1+tt}=\text{Arc.tang.}t \text{ and}$$
$$\int\frac{\partial t}{1-tt}=\frac{1}{2}l\frac{1+t}{1-t}=l\frac{1+t}{\sqrt{1-tt}},$$

with which values substituted there will be found :

$$S = \frac{1}{2\sqrt{2}}\text{Arc.tang.}t + \frac{1}{2\sqrt{2}} l \frac{1+t}{\sqrt{1-tt}}.$$

Whereby on returning there shall be $t = \frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, moreover above we find

$$1+x^4 = \frac{2xx}{tt}, \text{ there will be } tt = \frac{2xx}{1+x^4},$$

and hence

$$1-tt = \frac{(1-xx)^2}{1+x^4}, \text{ and thus } \sqrt{1-tt} = \frac{1-xx}{\sqrt{(1+x^4)}},$$

with these values substituted , the integral sought will be expressed by the variable x itself in the following manner :

$$\int \frac{\partial x\sqrt{1+x^4}}{1-x^4} = \frac{1}{2\sqrt{2}}\text{Arc.tang.}\frac{x\sqrt{2}}{\sqrt{1+x^4}} + \frac{1}{2\sqrt{2}} l \frac{x\sqrt{2}+\sqrt{1+x^4}}{1-xx}.$$

§. 61. But this deserves to be asked, by what artifice for that substitution, which first considered may be probed further and may be seen to be developed otherwise ? certainly since no one may have fallen on that, nor also do I remember that, by what reasoning I may have been led to that. Truly after I had considered everything more accurately, I discovered a much simpler method, by which that matter could be resolved without so much mystery, that therefore here it will be agreed to set out clearly.

The proposed integral formula to be treated by a clearer and more natural method.

§. 62. In order that at least it will appear that we may remove the irrationality from the formula $\partial S = \frac{\partial x\sqrt{1+x^4}}{1-x^4}$, we may put $\sqrt{1+x^4} = px$, so that there shall become $\partial S = \frac{px\partial x}{1-x^4}$. Therefore since there becomes $1+x^4 = ppxx$, the root on being extracted will be

$$xx = \tfrac{1}{2}pp + \sqrt{\tfrac{1}{4}p^4 - 1},$$

Here there may be put $\frac{1}{2}pp = q$, so that we may have

$$xx = q + \sqrt{qq-1}, \text{ and}$$

$$2lx = l\left[q + \sqrt{(qq-1)}\right],$$

and hence on differentiation $\frac{2\partial x}{x}=\left[\frac{1}{q+\sqrt{(qq-1)}}\times\left(1+\frac{q}{\sqrt{(qq-1)}}\right)dq\right]=\frac{\partial q}{\sqrt{(qq-1)}}$: therefore with the value $\frac{1}{2}pp$ restored in place of q, there will be $\frac{2\partial x}{x}=\frac{2p\partial p}{\sqrt{\left(p^4-4\right)}}$, and thus there becomes $\partial x=\frac{xp\partial p}{\sqrt{\left(p^4-4\right)}}$, with which value substituted there becomes $\partial S=\frac{p^2x^2\partial p}{\left(1-x^4\right)\sqrt{\left(p^4-4\right)}}$.

§. 63. Now so that hence we may eject the quantity x completely, since we have found

$$xx=\frac{pp+\sqrt{p^4-4}}{2}\text{, there will be}$$

$$x^4=\frac{p^4-2+pp\sqrt{p^4-4}}{2}\text{, and hence}$$

$$1-x^4=\frac{4-p^4-pp\sqrt{p^4-4}}{2}=-\frac{\sqrt{p^4-4}\left[pp+\sqrt{p^4-4}\right]}{2},$$

From which there is deduce to become $\frac{xx}{1-x^4}=-\frac{1}{\sqrt{p^4-4}}$, with which value substituted we obtain the rational differential formula expressed by the new variable p, which is

$$\partial S=-\frac{pp\partial p}{p^4-4},\ \text{with } p \text{ being given by } p=\frac{\sqrt{1+x^4}}{x},$$

from which the same integral is deduced which we produced before. Moreover a similar succession can be made into the formulas of much more general integrals ; just as we will show in the following problem.

Problem l6.

§. 64. *The proposed integral formula* $S=\int\frac{\partial x\sqrt{\left(a+bxx+cx^4\right)}}{a-cx^4}$ *can be freed from all irrationality with the aid of a suitable substitution.*

Solution.

In order that at least a kind of irrationality may be removed, we may put

$$\sqrt{\left(a+bxx+cx^4\right)}=px,$$

so that we may have $S=\int\frac{px\partial x}{a-cx^4}$. Therefore since there shall be

$$p=\frac{\sqrt{\left(a+bxx+cx^4\right)}}{x},\ \text{there will be}$$

$$\partial p = \left[p = \partial\sqrt{\left(\frac{a}{x^2} + b + cx^2\right)}, \right] = \frac{-a\partial x + cx^4\partial x}{xx\sqrt{\left(a+bxx+cx^4\right)}} = \frac{-a\partial x + cx^4\partial x}{px^3},$$

from which there becomes

$$\partial x = -\frac{px^3\partial p}{a-cx^4}$$

with which value substituted there becomes

$$\partial S = -\frac{ppx^4\partial p}{\left(a-cx^4\right)^2}.$$

§. 65. Then since there shall be

$$a + cx^4 = (pp-b)xx,$$

and hence again

$$\left(a+cx^4\right)^2 = (pp-b)^2 x^4,$$

$4acx^4$ may be taken away, and there will remain

$$\left(a-cx^4\right)^2 = \left[(pp-b)^2 - 4ac\right]x^4,$$

with which substituted our formula becomes

$$\partial S = -\frac{pp\partial p}{(pp-b)^2 - 4ac},$$

And thus the variable quantity *x* has been completely removed from the calculation, and we have deduced the differential formula in short to a rational form, of which therefore the integration may be worked out by logarithms and circular arcs without more difficulty. Indeed why also may formulas hitherto more general not be treated by the same method.

Problem 17.

§. 66. *To free this proposed integral*

$$S = \int \frac{x^{n-2}\partial x\sqrt[n]{\left(a+bx^n+cx^{2n}\right)}}{a-cx^{2n}}$$

from all irrationality with the help of a suitable substitution.

Solution.

Therefore we may use

$$\sqrt[n]{\left(a+bx^n+cx^{2n}\right)} = px,$$

so that by this substitution the proposed formula may adopt this form

$$\partial S = \frac{px^{n-1}\partial x}{a-cx^{2n}};$$

$$p^n = \frac{a+bx^n+cx^{2n}}{x^n},$$

by differentiation there will become

$$p^{n-1}\partial p = -\frac{\partial x\left(a-cx^{2n}\right)}{x^{n+1}},$$

from which there becomes

$$\partial x = -\frac{p^{n-1}x^{n+1}\partial p}{a-cx^{2n}},$$

with which value substituted our formula will adopt this form

$$\partial S = -\frac{p^n x^{2n}\partial p}{\left(a-cx^{2n}\right)^2}.$$

§. 67. Then since there shall be

$$a+cx^{2n} = \left(p^n-b\right)x^n, \text{ there will be}$$

$$\left(a+cx^{2n}\right)^2 = \left(p^n-b\right)^2 x^{2n};$$

hence $4acx^{2n}$ may be taken away, and there will remain

$$\left(a-cx^{2n}\right)^2 = \left[\left(p^n-b\right)^2 - 4ac\right]x^{2n},$$

with this value substituted there becomes

$$\partial S = -\frac{p^n\partial p}{\left(p^n-b\right)^2-4ac},$$

which therefore is entirely rational, and thus the integration by logarithms and circular arcs is easily performed.

Problem l8.

§. 68. *To find still more general integral formulas, which with the aid of the substitution*

$$\sqrt[n]{\left(a+bx^n+cx^{2n}\right)} = px$$

may be able to become rational.

Solution.

Because we have found in the preceding problem, with the aid of the substitution of this differential formula

$$\frac{x^{n-2}\partial x\sqrt[n]{\left(a+bx^n+cx^{2n}\right)}}{a-cx^{2n}}$$

to be reduced to that rational formula

$$-\frac{p^n\partial p}{\left(p^n-b\right)^2-4ac}, \text{ there will be}$$

$$\frac{\mathrm{P}x^{n-2}\partial x\sqrt[n]{\left(a+bx^n+cx^{2n}\right)}}{a-cx^{2n}}=-\frac{\mathrm{P}p^n\partial p}{\left(p^n-b\right)^2-4ac}$$

where in place of P some functions of x of the same kind can be accepted, so that with the same substitution made they may present rational functions of p, since that can be done in an infinite number of ways, the particulars of which we shall run through here.

§. 69. Since by the strength of the substitution

$$\frac{\sqrt[n]{\left(a+bx^n+cx^{2n}\right)}}{x}=p$$

in place of P any power of p can be assumed, which shall be p^λ.
Therefore $\mathrm{P}=p^\lambda\mathrm{Q}$ may be assumed, and also there will be

$$\mathrm{P}=\frac{Q\sqrt[n]{\left(a+bx^n+cx^{2n}\right)^\lambda}}{x^\lambda};$$

with which values substituted this equation will be produced

$$\mathrm{P}=\frac{Qx^{n-\lambda-2}\partial x\sqrt[n]{\left(a+bx^n+cx^{2n}\right)^{\lambda+1}}}{a-cx^{2n}}=-\frac{Qp^{n+\lambda}\partial p}{\left(p^n-b\right)^2-4ac}$$

which latter is rational again.

§. 70. Just as in the preceding problem, we found also there was

$$\frac{\left(a-cx^{2n}\right)^2}{x^{2n}}=\left(p^n-b\right)^2-4ac$$

as on account of the formula for Q we have supposed a power of the exponent i of these quantities, or rather of the reciprocal of these quantities, evidently there may be taken

$$\mathrm{Q}=\frac{x^{2in}}{\left(a-cx^{2n}\right)^{2i}}=\frac{1}{\left[\left(p^n-b\right)^2-4ac\right]^i}$$

With which values substituted we will obtain this formula of the greatest extent

$$\frac{x^{(2i+1)n-\lambda-2}\partial x\sqrt[n]{\left(a+bx^n+cx^{2n}\right)^{\lambda+1}}}{\left(a-cx^{2n}\right)^{2i+1}}=-\frac{p^{n+\lambda}\partial p}{\left[\left(p^n-b\right)^2-4ac\right]^{i+1}};$$

where for the letters λ and *i* some whole numbers either positive or negative are allowed to be taken, for the differential formula expressed by *p* will remain rational always.

§. 71. Since there is no reason why this reduction cannot be rendered more generally, on account of which it is not necessary that λ shall be a whole number : For whatever fraction may be assumed for λ, the formula expressed by *p* will be always readily reduced to rationality. If indeed we may put $\lambda=\frac{\mu}{\nu}$, the right hand side becomes

$$-\frac{p^{\frac{\nu n+\mu}{\nu}}\partial p}{\left[\left(p^n-b\right)^2-4ac\right]^{i+1}},$$

which is rendered rational by putting $p=q^{\nu}$, since there will be $\partial p=\nu q^{\nu-1}\partial q$, and thus this side becomes

$$-\frac{\nu p^{\mu+\nu n+\nu-1}\partial q}{\left[\left(p^{\nu n}-b\right)^2-4ac\right]^{i+1}}.$$

But now it will be required to use this substitution

$$\sqrt[n]{\left(a+bx^n+cx^{2n}\right)}=q^{\nu}x,$$

and this reduction will be had

$$\frac{x^{(2i+1)n-\frac{\mu}{\nu}-2}\partial x\sqrt[n]{\left(a+bx^n+cx^{2n}\right)^{\frac{\mu+\nu}{\nu}}}}{\left(a-cx^{2n}\right)^{2i+1}}$$
$$=-\frac{\nu q^{\mu+\nu n+\nu-1}\partial q}{\left[\left(q^{\nu n}-b\right)^2-4ac\right]^{i+1}},$$

which latter formula certainly is rational.

§. 72. So that we may remove the fractional exponent of *x* itself on the left hand side, we may put $x=y^{\nu}$, and there will become

$$\frac{y^{(2i+1)nv-\mu-v-1}\partial y\sqrt[n]{\left(a+by^{nv}+cy^{2nv}\right)^{\mu+v}}}{\left(a-cy^{2nv}\right)^{2i+1}}$$
$$=-\frac{q^{\mu+vn+v-1}\partial q}{\left[\left(q^{nv}-b\right)^{2}-4ac\right]^{i+1}},$$

but which expression may seem to be much more general, than it actually is.
For if in place of *nv* we may write *n* everywhere this equation emerges

$$\frac{y^{(2i+1)n-\mu-v-1}\partial y\sqrt[n]{\left(a+by^{n}+cy^{2n}\right)^{\mu+v}}}{\left(a-cy^{2n}\right)^{2i+1}}$$
$$=-\frac{q^{\mu+v+n-1}\partial q}{\left[\left(q^{n}-b\right)^{2}-4ac\right]^{i+1}};$$

moreover this equation clearly does not disagree with that other §. 70 ; indeed here if we write λ in place of $\mu+v-1$ and *q* in place of y so that *x* and *p* as before, the same preceding equation is recovered , and thus it will suffice to assume whole numbers in place of λ .

Corollary 1.

§. 73. So that the nature of these formulas may be seen clearer, we shall take $n=2$, and the differential formula involving the variable *x* will be

$$\frac{x^{4i-\lambda}\partial x\sqrt{\left(a+bxx+cx^{4}\right)^{\lambda+1}}}{\left(a-cx^{4}\right)^{2i+1}},$$
$$=-\frac{q^{\mu+v+n-1}\partial q}{\left[\left(q^{n}-b\right)^{2}-4ac\right]^{i+1}};$$

which, with the substitution made $\left(a+bxx+cx^{4}\right)=px,$ is transformed into this ratio

$$-\frac{p^{\lambda+2}\partial p}{\left[(pp-b)^{2}-4ac\right]^{i-1}},$$

from which on taking $\lambda=4i$ this equation will result

$$\frac{\partial x\sqrt{\left(a+bxx+cx^{4}\right)^{4i+1}}}{\left(a-cx^{4}\right)^{ci+1}}=-\frac{p^{4i+2}\partial p}{\left[(pp-b)^{2}-4ac\right]^{i+1}},$$

in which again if there may be put $i=0$, there becomes

$$\frac{\partial x\sqrt{\left(a+bxx+cx^4\right)}}{\left(a-cx^4\right)}=-\frac{pp\partial p}{(pp-b)^2-4ac};$$

which, if in addition there may be put $a=1,\ b=0$ and $c=1$, becomes

$$\frac{\partial x\sqrt{\left(1+x^4\right)}}{1-x^4}=-\frac{pp\partial p}{p^4-4},$$

which is that same reduction, which was found above in §. 63.

Corollary 2.

§. 74. If we take $n=3$, this general reduction will be produced

$$\frac{x^{6i-\lambda+1}\partial x\sqrt[3]{\left(a+bx^3+cx^6\right)^{\lambda+1}}}{\left(a-cx^6\right)^{2i+1}}=-\frac{p^{\lambda+3}\partial p}{\left[\left(p^3-b\right)^2-4ac\right]^{i+1}};$$

which on putting $i=0$ changes into this

$$\frac{x^{-\lambda+1}\partial x\sqrt[3]{\left(a+bx^3+cx^6\right)^{\lambda+1}}}{a-cx^6}=-\frac{p^{\lambda+3}\partial p}{\left(p^3-b\right)^2-4ac};$$

indeed on putting $b=0$, produces this neater formula

$$\frac{x^{-\lambda+1}\partial x\sqrt[3]{\left(a+cx^6\right)^{\lambda+1}}}{a-cx^6}=-\frac{p^{\lambda+3}\partial p}{p^6-4ac};$$

of which it will be a pleasure to set out these two cases.

I. Let $\lambda=0$, and there will be

$$\frac{x\partial x\sqrt[3]{\left(a+cx^6\right)}}{a-cx^6}=-\frac{p^3\partial p}{p^6-4ac};$$

which will be rendered neater by putting $xx=y$, indeed there will be found

$$\frac{\partial y\sqrt[3]{\left(a+cy^3\right)}}{a-cy^3}=-\frac{2p^3\partial p}{p^6-4ac}.$$

II. Moreover on taking $\lambda = 1$, this expression will be produced

$$\frac{\partial x \sqrt[3]{\left(a+cx^6\right)^2}}{a-cx^6} = -\frac{p^4 \partial p}{p^6 - 4ac}.$$

Scholium.

§. 75. From these examples it is understood well enough, that outstanding reductions are able to be deduced from our general formulas, the resolution of which, provided our method may be adhered to, may be seen to be superior to all the strengths of analysis.

4.) A memorable kind of differential formulas of the greatest irrationality, which still can be reduced to rationality.

M. S. of the Academy, exhibited on 15th of May 1777.

§. 76. Since I have examined this differential formula recently

$$\frac{\partial x}{(1-xx)\sqrt[4]{(2xx-1)}}$$

and as I had been led to its rationality in an unusual way, soon I saw the same method to succeed in this more general formula

$$\frac{\partial x}{(a+bxx)\sqrt[4]{(a+2bxx)}},$$

and thus in this much more general case

$$\frac{\partial x}{\left(a+bx^n\right)\sqrt[2n]{\left(a+2bx^n\right)}},$$

where the irrationality can rise to however great height, the resolution of which can be established in the following manner.

§. 77. Clearly I make use of this substitution $\frac{x}{\sqrt[2n]{\left(a+2bx^n\right)}} = Z$, so that our formula requiring to be integrated, as we indicate by $\partial \mathrm{V}$, becomes $\partial \mathrm{V} = \frac{\partial x}{x} \cdot \frac{Z}{a+bx^n}$, therefore with the logarithms taken will be

$$lZ = lx - \tfrac{1}{2n} l\left(a + 2bx^n\right),$$

from which by differentiation there will be

$$\frac{\partial Z}{Z}=\frac{\partial x}{x}-\frac{bx^{n-1}\partial x}{a+2bx^{n}}=\frac{\partial x\left(a+bx^{n}\right)}{x\left(a+2bx^{n}\right)},$$

therefore there will be

$$\frac{\partial x}{x}=\frac{\partial Z\left(a+2bx^{n}\right)}{Z\left(a+bx^{n}\right)},$$

hence our formula therefore will be

$$\partial \mathrm{V}=\frac{\partial Z\left(a+2bx^{n}\right)}{\left(a+bx^{n}\right)^{2}}.$$

Therefore since there will be

$$Z^{2n}=\frac{x^{2n}}{a+2bx^{n}},\ \text{there will be}\ a+2bx^{n}=\frac{x^{2n}}{Z^{2n}},$$

and thus

$$\partial \mathrm{V}=\frac{x^{2n}\partial Z}{Z^{2n}\left(a+bx^{n}\right)^{2}}.$$

Again since there shall be $aa+2abx^{n}=\frac{ax^{2n}}{Z^{2n}}$, bbx^{2n} is added to each side, and there will be produced

$$\left(a+bx^{n}\right)^{2}=\frac{ax^{2n}}{Z^{2n}}+bbx^{2n}=\frac{x^{2n}\left(a+bbZ^{2n}\right)}{Z^{2n}},$$

with which value substituted, our formula will be come upon

$$\partial \mathrm{V}=\frac{\partial Z}{a+bbZ^{2n}},$$

therefore which formula is rational, and thus can be integrated by logarithms and circular arcs.

§.78. Again I have observed, since here after the root sign only a binomial is involved, in place of this it is possible to introduce some trinomial, and thus polynomials too. Moreover for the trinomial the differential formula will have such a form

$$\partial \mathrm{V}=\frac{\partial x}{\left(a+bx^{n}\right)\sqrt[3n]{\left(aa+3abx^{n}+3bbx^{2n}\right)}},$$

where therefore irrationalities may rise to a much higher order. Truly nothing less than this formula also will be able to be freed from irrationality with the aid of a similar substitution

$$Z=\frac{x}{\sqrt[3n]{\left(aa+3abx^{n}+3bbx^{2n}\right)}};$$

hence indeed with the logarithms taken by differentiation we will obtain :

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{abx^{n-1}\partial x - 2bbx^{2n-1}\partial x}{aa+3abx^{n}+3bbx^{2n}},$$

or

$$\frac{\partial Z}{Z} = \frac{\partial x\left(a+bx^{n}\right)^{2}}{x\left(aa+3abx^{n}+3bbx^{2n}\right)},$$

and thus

$$\frac{\partial x}{x} = \frac{\partial Z}{Z}\cdot\frac{aa+3abx^{n}+3bbx^{2n}}{\left(a+bx^{n}\right)^{2}}.$$

Therefore since now our formula shall be $\partial \mathrm{V} = \frac{\partial x}{x}\cdot\frac{Z}{a+bx^{n}}$, with the element $\partial \mathrm{Z}$ introduced, we will obtain

$$\partial V = \frac{\partial Z\left(aa+3abx^{n}+3bbx^{2n}\right)}{\left(a+bx^{n}\right)^{2}}.$$

§. 79. Therefore since by the strength of the substitution there shall be

$$\left(aa+3abx^{n}+3bbx^{2n}\right) = \frac{x}{Z},$$

there will be

$$aa+3abx^{n}+3bbx^{2n} = \frac{x^{3n}}{Z^{3n}}.$$

Both sides may be multiplied by *a*, and $b^{3}x^{3n}$ may be added to both sides, and there becomes

$$\left(a+bx^{n}\right)^{3} = \frac{x^{3n}\left(a+b^{2}Z^{3n}\right)}{Z^{3n}}:$$

therefore with this value substituted from our formula the letter *x* will be completely excluded, and there will be produced $\partial \mathrm{V} = \frac{\partial Z}{a+b^{3}Z^{n}}$. Therefore the integral of this will always be allowed to be found by logarithms and circular arcs.

§. 80. But for quadrinomials for the sake of brevity we may put

$$\sqrt[4n]{\left(a^{3}+4aabx^{n}+6abbx^{2n}+4b^{3}x^{3n}\right)} = \mathrm{S},$$

and this formula is proposed to be reduced to rationality

$$\partial \mathrm{V} = \frac{\partial x}{\left(a+bx^n\right)\mathrm{S}},$$

as that will succeed in a similar manner with the aid of this substitution $\frac{x}{\mathrm{S}} = \mathrm{Z}$, from which our formula will become $\partial \mathrm{V} = \frac{\partial x}{x} \cdot \frac{Z}{a+bx^n}$. Now since there shall be

$$\frac{\partial S}{S} = \frac{aabx^{n-1}\partial x + 3aabx^{2n-1}\partial x + 3b^3x^{3n-1}\partial x}{a^3+4aabx^n+6abbx^{2n}+4b^3x^{3n}},$$

or

$$\frac{\partial S}{S} = \frac{\partial x}{x} \cdot \frac{bx^n\left(aa+3abx^n+3bbx^{2n}\right)}{S^{4n}},$$

there will be $\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S}$; consequently

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} \cdot \frac{\left(a+bx^n\right)^4}{\mathrm{S}^{4n}}, \text{ and hence } \frac{\partial x}{x} = \frac{\mathrm{S}^{4n}\partial Z}{Z\left(a+bx^n\right)^4},$$

with which value substituted our formula will become

$$\partial V = \frac{\mathrm{S}^{4n}\partial Z}{\left(a+bx^n\right)^4}.$$

§. 81. But since there shall be

$$S^{4n} = a^3 + 4aabx^n + 6abbx^{2n} + 4b^3x^{3n}, \text{ there becomes}$$

$$aS^{4n} + b^4x^{4n} = \left(a+bx^n\right)^4,$$

with which value substituted there will be

$$\partial V = \frac{\mathrm{S}^{4n}\partial Z}{a\mathrm{S}^{4n}+b^4x^{4n}}:$$

therefore because we have put $\mathrm{Z} = \frac{x}{s}$, there will be $\mathrm{S} = \frac{x}{Z}$, and thus $\mathrm{S}^{4n} = \frac{x^{4n}}{Z^{4n}}$, which value replaced will give

$$\partial V = \frac{\partial Z}{a+b^4Z^{4n}},$$

and thus likewise has been transformed to rationality.

§. 82. Hence now it is understood easily, how the differential formulas for all the polynomials must be prepared together, so that by such a substitution they may be made rational, which we will prepare in the following problem.

Problem 19

§. 83. *If this differential formula were proposed*

$$\partial \mathrm{V} = \frac{\partial x}{\left(a+bx^n\right)\sqrt[\lambda n]{\left(\left(a+bx^n\right)^{\lambda}-b^{\lambda}x^{\lambda n}\right)}},$$

to reduce that to rationality, however the magnitude may be taken for n and λ.

Solution.

Also we put here for the sake of brevity

$$\sqrt[\lambda n]{\left(\left(a+bx^n\right)^{\lambda}-b^{\lambda}x^{\lambda n}\right)} = \mathrm{S},$$

so that the formula becomes

$$\partial \mathrm{V} = \frac{\partial x}{\left(a+bx^n\right)\mathrm{S}},$$

and in addition there is made $\frac{x}{\mathrm{S}} = \mathrm{Z}$, so that we may have

$$\partial \mathrm{V} = \frac{\partial x}{x}\cdot\frac{Z}{a+bx^n}.$$

Now by differentiating the logarithm there will be found

$$\frac{\partial S}{S} = \frac{bx^{n-1}\partial x\left(a+bx^n\right)^{\lambda-1}-b^{\lambda}x^{\lambda n-1}\partial x}{\mathrm{S}^{\lambda n}}, \text{ or}$$

$$\frac{\partial \mathrm{S}}{\mathrm{S}} = \frac{\partial x}{x}\cdot\frac{bx^{n}\left(a+bx^n\right)^{\lambda-1}-b^{\lambda}x^{\lambda n}}{\mathrm{S}^{\lambda n}}.$$

Therefore since there shall be $\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S}$, with this value substituted there will be

$$\frac{\partial \mathrm{Z}}{\mathrm{Z}} = \frac{\partial x}{x}\cdot\frac{a\left(a+bx^n\right)^{\lambda-1}}{\mathrm{S}^{\lambda n}},$$

and hence in turn there will become

$$\frac{\partial x}{x} = \frac{\mathrm{S}^{\lambda n}\partial \mathrm{Z}}{a\mathrm{Z}\left(a+bx^n\right)^{\lambda-1}},$$

with which value substituted we obtain

$$\partial \mathrm{V} = \frac{\mathrm{S}^{\lambda n}\partial Z}{a\left(a+bx^n\right)^{\lambda}},$$

because now there becomes $\left(a+bx^n\right)^{\lambda} = \mathrm{S}^{\lambda n} + b^{\lambda}x^{\lambda n}$, there will be

$$\partial V = \frac{S^{\lambda n}\partial Z}{a\left(S^{\lambda n}+b^{\lambda}x^{\lambda n}\right)}.$$

Finally on account of $S=\frac{x}{Z}$, thus $S^{\lambda n}=\frac{x^{\lambda n}}{Z^{\lambda n}}$, with this value substituted there will be obtained

$$\partial V = \frac{\partial Z}{a\left(1+b^{\lambda}Z^{\lambda n}\right)},$$

which is rational involving the single variable Z , of which the integral thus can be assigned by logarithms and circular arcs.

Corollary 1.

§. 84. The same solution also has a place, if fractional numbers may be taken for λ, by which account they are involved again after the root sign : thus if there were $\lambda=\frac{2}{n}$, the formula of the root will be

$$S=\sqrt{\left(a+bx^{n}\right)^{\frac{2}{\lambda}}-b^{\frac{2}{\lambda}}xx},$$

and the integral of our formula

$$\partial V = \frac{\partial x}{a\left(1+bx^{n}\right)S}$$

will be

$$V=\frac{1}{a}\int\frac{\partial Z}{1+b^{\frac{2}{n}}ZZ}=\frac{1}{ab^{\frac{1}{n}}}\text{Arc.tang.}b^{\frac{1}{n}}Z.$$

Corollary 2.

§. 85. So that this may be made clearer, we may take $a=i,\ b=i,$ and $n=4,$ in order that for the last case there shall be

$$S=\sqrt{\left(1+x^{4}\right)^{\frac{1}{2}}-xx},\ \text{and}\ \partial V=\frac{\partial x}{\left(1+x^{4}\right)\sqrt{\left(1+x^{4}\right)^{\frac{1}{2}}-xx}},$$

with the integral of this put in place,

$$Z=\frac{x}{\sqrt{\left(1+x^{4}\right)^{\frac{1}{2}}-xx}},\ \text{there will be}$$

$$V=\text{Arc.tang. } Z,\ \text{or}\ V=\frac{x}{\sqrt{\left(1+x^{4}\right)^{\frac{1}{2}}-xx}}.$$

But if with $n=4$ in place and $a=1,$ there was $b=-1$, and thus

$$S=\sqrt{\left(1-x^{4}\right)^{\frac{1}{2}}-xx\sqrt{-1}},$$

this formula appears imaginary.

Corollary 3.

§. 86. For the same case, and there will be $\lambda = \frac{2}{n}$, there shall be $n = 6$, $a = 1$ and $b = 1$, and there becomes

$$\mathrm{S} = \sqrt{\left(1+x^{6}\right)^{\frac{1}{3}} - xx}, \text{ and thus}$$

$$\partial \mathrm{V} = \frac{\partial x}{\left(1+x^{6}\right)\sqrt{\left(1+x^{6}\right)^{\frac{1}{3}} - xx}},$$

The integral of this, on putting $\frac{x}{\mathrm{S}} = \mathrm{Z}$, will be

$$\mathrm{V} = \text{Arc.tang.}\mathrm{Z} = \text{Arc.tang.}\frac{x}{\sqrt{\left(1+x^{6}\right)^{\frac{1}{3}} - xx}}.$$

And in a similar manner other examples of this kind can be formed as desired; truly whatsoever formula of the problem is certainly general, yet it can still become much more general, as we are about to show in the following problem.

Problem 20.

§. 87. *If this much more general differential formula may be proposed, certainly in which three exponents of the unknowns λ, n, and m occur*

$$\partial \mathrm{V} = \frac{x^{m-1}\partial x}{\left(a+bx^{n}\right)\left(\sqrt[\lambda n]{\left(\left(a+bx^{n}\right)^{\lambda} - b^{\lambda}x^{\lambda n}\right)}\right)^{m}},$$

to free that from irrationality.

Solution.

Again for the sake of brevity there is put

$$\sqrt[\lambda n]{\left(\left(a+bx^{n}\right)^{\lambda} - b^{\lambda}x^{\lambda n}\right)} = \mathrm{S},$$

so that the formula proposed to be integrated becomes

$$\partial \mathrm{V} = \frac{x^{m-1}\partial x}{\left(a+bx^{n}\right)S^{m}} = \frac{\partial x}{x}\cdot\frac{x^{m}}{\left(a+bx^{n}\right)S^{m}},$$

which therefore if again as before we may put $\frac{x}{\mathrm{S}} = \mathrm{Z}$, there becomes

$$\partial \mathrm{V} = \frac{\partial x}{x}\cdot\frac{Z^{m}}{a+bx^{n}},$$

from which it is required to eliminate the variable *x* completely. Now since both the letters S and Z have the same values, as in the preceding problem and thus that same formula $\partial \mathrm{V}$ may arise, if the preceding may be multiplied by Z^{m-1}, also we will obtain the integral sought, provided that we will multiply the above integral by Z^{m-1}, with which done the integral sought will be

$$\mathrm{V} = \frac{1}{a}\int \frac{Z^{m-1}\partial Z}{1+b^{\lambda}Z^{\lambda n}}.$$

Corollary 1.

§. 88. If we may take a negative exponent *m*, the irrationality will be moved to the numerator, thus on putting $m=-1$ we will have

$$\partial \mathrm{V} = \frac{\partial x \sqrt[\lambda n]{\left(\left(a+bx^{n}\right)^{\lambda} - b^{\lambda}x^{\lambda n}\right)}}{xx\left(a+bx^{n}\right)},$$

of which the integral therefore expressed by Z will be

$$\mathrm{V} = \frac{1}{a}\int \frac{\partial Z}{ZZ\left(a+b^{\lambda}Z^{\lambda n}\right)}.$$

Indeed the irrationality also is able to be expressed more simply by this exponent *m*, just as if we may take $m=\lambda$, there will be

$$\partial \mathrm{V} = \frac{x^{\lambda-1}\partial x}{\left(a+bx^{n}\right)\sqrt[n]{\left(\left(a+bx^{n}\right)^{\lambda} - b^{\lambda}x^{\lambda n}\right)}}$$

Of which the integral on putting $\mathrm{Z} = \frac{x}{\mathrm{S}}$, with S retaining the above value will be

$$\mathrm{V} = \frac{1}{a}\int \frac{Z^{\lambda-1}\partial Z}{a+b^{\lambda}Z^{n\lambda}}.$$

Corollary 2.

§. 89. Then indeed also if we may assume a fraction for *m*, the irrationality will be more complicated according to this, just as if we take $m=\frac{1}{2}$, the differential formula now will become :

$$\partial \mathrm{V} = \frac{\partial x}{\left(a+bx^{n}\right)\sqrt[2\lambda n]{x^{\lambda n}\left[\left(a+bx^{n}\right)^{\lambda} - b^{\lambda}x^{\lambda n}\right]}}.$$

Truly this case may be easily recalled to the first problem by putting $x=vv$, thus so that there shall be

$$\partial \mathrm{V} = \frac{2\partial v}{\left(a+bv^{2n}\right)\sqrt[2\lambda n]{\left[\left(a+bv^{2n}\right)^{\lambda} - b^{\lambda}v^{2\lambda n}\right]}},$$

which formula does not disagree from the first problem otherwise except that here the exponent *n* shall be twice as big.

Scholium.

§. 90. Although the two letters *a* and *b* can be taken negative as well as positive, as it pleases, yet cases occur, which cannot be used according to this general form: just as if this formula may be proposed $\frac{\partial x}{\left(1-xx\sqrt[4]{(2xx-1)}\right)}$, this will not be contained in the first problem, because there must be $aa=-1$ [*cf* §. 78], since that in general may eventuate, even in this case we may subjugate the general problem to accommodate this case.

Problem 21.

§. 91. *If this differential formula may be extended the furthest to include three indeterminate exponents*

$$\partial \mathrm{V}=\frac{x^{m-1}\partial x}{\left(fx^{n}-g\right)\sqrt[\lambda n]{\left(f^{\lambda}x^{n\lambda}-\left(fx^{n}-g\right)^{\lambda}\right)^{m}}},$$

to free that from all irrationality.

Solution.

As previously, for the sake of brevity we may put

$$\sqrt[\lambda n]{\left(f^{\lambda}x^{\lambda n}-\left(fx^{n}-g\right)^{\lambda}\right)}=\mathrm{S},$$

then truly $\mathrm{Z}=\frac{x}{\mathrm{S}}$, so that the differential formula becomes

$$\partial \mathrm{V}=\frac{\partial x}{x}\cdot\frac{Z^{m}}{fx^{n}-g}.$$

But now by taking the logarithmic differential there is :

$$\frac{\partial \mathrm{S}}{S}=\frac{\partial x}{x}\cdot\frac{f^{\lambda}x^{\lambda n}-fx^{n}\left(fx^{n}-g\right)^{\lambda-1}}{S^{\lambda n}},$$

and hence there is gathered to be:

$$\frac{\partial \mathrm{Z}}{Z}=\frac{\partial x}{x}-\frac{\partial \mathrm{S}}{S}=\frac{\partial x}{x}\cdot\frac{g\left(fx^{n}-g\right)^{\lambda-1}}{S^{\lambda n}},$$

and thus there will be had:

$$\frac{\partial x}{x}=\frac{\partial Z}{Z}\cdot\frac{S^{\lambda n}}{g\left(fx^{n}-g\right)^{\lambda-1}},$$

with which value substituted we arrive at

$$\partial \mathrm{V} = \frac{Z^{m-1}\partial Z S^{\lambda n}}{g\left(fx^n - g\right)^{\lambda}}.$$

But it is clear that $\left(fx^n - g\right)^{\lambda} = f^{\lambda}x^{\lambda n} - S^{\lambda n}$, and thus

$$\partial \mathrm{V} = \frac{Z^{m-1}S^{\lambda n}\partial Z}{g\left(f^{\lambda}x^{\lambda n} + S^{\lambda n}\right)};$$

from which finally on account of $S = \frac{x}{Z}$ this form may be adopted :

$$\partial \mathrm{V} = \frac{Z^{m-1}\partial Z}{g\left(f^{\lambda}Z^{\lambda n} - 1\right)},$$

which formula disagrees with the preceding by sign only.

Vol. IV.

SUPPLEMENTUM I.

AD TOM. I. CAP. II.

DE
INTEGRATIONE FORMULARUM DIFFERENTIALIUM
IRRATIONALIUM.

1.) De integratione formularum differentialium irrationalium. *Acta Academiae Scientiar. Petropolitanae. Tom.IV. Pars I. Pag.* 4-31.

Problema 1.

§. 1. *Si functio X praeter ipsam variabilem x etiam formulam irrationalem* $s=\sqrt{(a+bx)}$ *involvat : ita tamen, ut X sit functio rationalis binarum quantitatum x et s, formulam differentialem* $X\partial x$ *ab irrationalitate liberare.*

Solutio.

Cum irrationalitas tantum in formula $s=\sqrt{(a+bx)}$ insit, hanc tantum ita per idoneam substitutionem tolli oportet , ut inde valor ipsius x non fiat irrationalis. Hoc autem praestabitur, ponendo $a+bx=zz$, ut fiat $s=z$ et $x=\frac{zz-a}{b}$, hincque $\partial x=\frac{2}{b}z\partial z$; quibus valoribus substitutis, tota formula differentialis $X\partial x$ ad rationalem, novam variabilem z complectens, perducitur.

Exemplumt 1.

Si fuerit $\partial y=\frac{\partial x}{\sqrt{(a+bx)}}$, seu $\partial y=\frac{\partial x}{s}$, posito $\sqrt{(a+bx)}=z$, fiet $\partial y=\frac{2}{b}\partial z$, et integrando $y=\frac{2z}{b}$, unde facta substitutione colligitur $y=\frac{2}{b}\sqrt{(a+bx)}+$ C.

Exemplum *2.*

§. 3. Si fuerit $\partial y=\partial x\sqrt{(a+bx)}=s\partial x$, sumto $\sqrt{(a+bx)}=z$, erit $\partial y=z\partial x=\frac{2}{b}zz\partial z$, unde integrando fit $y=\frac{2}{3b}z^3$, et facta substitutione prodit

$$y=\frac{2}{3b}(a+bx)^{\frac{3}{2}}+\text{C}.$$

Quod integrale si debeat evanescere facto $x=0$, fiet $\text{C}=-\frac{2a\sqrt{a}}{3b}$,
ideoque

$$y = \frac{2(a+bx)^{\frac{3}{2}} - 2a\sqrt{a}}{3b}.$$

Exemplum 3.

§. 4. Si fueret $\partial y = \frac{x\partial x}{\sqrt{(a+bx)}}$, facta substitutione $\sqrt{(a+bx)} = z$, erit

$$\partial y = \frac{2(zz-a)\partial z}{bb} = \frac{2zz\partial z - 2a\partial z}{bb},$$

unde fit integrando

$$y = \frac{2z^3}{3bb} - \frac{2a}{bb}z + C,$$

et facta restitutione

$$y = \frac{2}{3bb}(a+bx)^{\frac{3}{2}} - \frac{2a}{bb}\sqrt{(a+bx)} + C$$
$$= \frac{2\sqrt{(a+bx)}}{bb}\left(\tfrac{1}{3}bx - \tfrac{2}{3}a\right) + C.$$

Exemplum 4.

§.5. Si fueret $\partial y = \frac{x\partial x}{(a+bx)^{\frac{3}{2}}}$, facta substitutione $\sqrt{(a+bx)} = z$, erit $\partial y = \frac{\partial x}{z^3}$; quae formula porro ob $\partial x = \frac{2z\partial z}{b}$ abit in $\partial y = \frac{2\partial z}{bzz}$, qua integrata sit $y = -\frac{2}{bz}$, seu facta restitutione, $y = \frac{-2}{b\sqrt{(a+bx)}} + \mathrm{C}$. Ubi notetur, pro C sumi debere $\frac{2}{b\sqrt{a}}$, casu quo integrale evanescere debeat facto $x = 0$.

Problema 2.

§.6. *Si fuerit X functio quaecunque rationalis binarum quantitatum x et s, existente* $s = \sqrt[3]{(a+bx)}$, *formulam dijferentialem* $X\partial x$ *ab irrationalitate liberare.*

Solutio.

Ponatur $\sqrt[3]{(a+bx)} = z$, ut sit $s = z$, erit $a + bx = z^3$, hinque $x = \frac{z^3-a}{b}$, et $\partial x = \frac{3z^2}{b}$; quibus valoribus substitutis tota formula fiet rationalis.

Exemplum 1.

§. 7. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt[3]{(a+bx)}} = \frac{\partial x}{s},$$

posito $\sqrt[3]{(a+bx)} = z$ et substituto valore hinc nato

$$\partial x = \frac{3zz\partial z}{b}, \text{ erit } \partial y = \frac{3z\partial z}{b},$$

unde integrando sit

$$y = \frac{3}{2b} zz = \frac{s}{2b} \sqrt[3]{(a+bx)^2} + \text{C}.$$

Exemplum 2.

§.8. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt[3]{(a+bx)^2}} = \frac{\partial x}{ss},$$

posito $\sqrt[3]{(a+bx)} = z$ fiet $\partial y = \frac{3\partial z}{b}$, hinc integrando

$$y = \frac{3}{b} z = \frac{3}{b} \sqrt[3]{(a+bx)} + \text{C}.$$

Exemplum 3.

§.9. Si fuerit $\partial y = \partial x \sqrt[3]{(a+bx)} = s\partial x$, facta substitute sit $\partial y = \frac{3z^3 \partial z}{b}$, hinc integrando $y = \frac{3z^4}{4b} = \frac{3}{4b}(a+bx)\sqrt[3]{(a+bx)} + \text{C}$.

Problema 3.

§. 10. *Si fuerit X functio rationalis binarum quantitatum x et s, existente* $s = \sqrt[n]{(a+bx)}$, *formulam differentialem* $X\partial x$ *ab irrationalitate liberare.*

Solutio.

Ponatur $\sqrt[n]{(a+bx)} = z$, ut sit $s = z$, erit $a + bx = z^n$, hinc

$$x = \frac{z^n - a}{b} \text{ et } \partial x = \frac{nz^{n-1}\partial z}{b};$$

quibus valoribus substitutis formula proposita $X\partial x$ certe fiet rationalis, si modo numerus exponentialis n fuerit integer.

Exemplum 1.

§. 11. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt[n]{(a+bx)}} = \frac{\partial x}{s},$$

posito $\sqrt{(a+bx)} = z$,ob valorem inde natum

$$\partial x = \frac{nz^{n-1}\partial z}{b}$$

habebitur

$$\partial y = \frac{nz^{n-2}}{b}\partial z;$$

unde integrando colligimus

$$y = \frac{n}{b(n-1)} z^{n-1} + \mathrm{C}.$$

sive restitutis valoribus

$$y = \frac{n}{b(n-1)}\left(a+bx\right)^{\frac{n-1}{n}} + \mathrm{C} = \frac{n}{b(n-1)} \cdot \frac{a+bx}{\sqrt[n]{(a+bx)}} + \mathrm{C}.$$

Exemplum 2.

§. 12. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt[n]{(a+bx)^{\lambda}}} = \frac{\partial x}{s^{\lambda}},$$

posito $\sqrt[n]{\left(a+bx\right)} = z$, et substituto valore

$$\partial x = \frac{nz^{n-1}\partial z}{b}, \text{ fiet}$$

$$\partial y = \frac{nz^{n-1}\partial z}{bz^{\lambda}} = \frac{n}{b} z^{n-\lambda-1}\partial z,$$

cujus integrale dat

$$y = \frac{n}{b(n-\lambda)}\left(a+bx\right)^{\frac{n-\lambda}{n}} + C, \text{sive}$$

$$y = \frac{n}{b(n-\lambda)} \cdot \frac{a+bx}{\sqrt[n]{(a+bx)^{\lambda}}}.$$

Ex his autem exemplis jam apparet, integrationem non impediri, etiamsi exponentes *n* et λ non fuerint numeri integri.

Problema 4.

§. 13. *Si fuerit X functio rationalis binarum quantitatum x et s, existente* $s = \sqrt{\ [a + b\sqrt{(f+gx)}]}$ *, quae formula ergo duplicem irrationalitatem involvit, formulam differentialem* $X\partial x$ *ab hac duplici irrationalitate liberare.*

Solutio.

Ponatur iterum $s = \sqrt{\ [a + b\sqrt{(f+gx)}]} = z$, ut sit $s = z$, erit sumtis quadratis $a + b\sqrt{(f+gx)} = zz$, hinc:

$$b\sqrt{(f+gx)} = zz - a,$$

ac sumtis denuo quadratis

$$bb(f+gx)=\left(zz-a\right)^2,$$

unde colligitur

$$x=\tfrac{\left(zz-a\right)^2}{bbg}-\tfrac{f}{g}, \text{ hincque}$$

eque

$$\partial x=\tfrac{4z\left(zz-a\right)}{bbg}.$$

Quibus valoribus substitutis tota formula reddetur rationalis.

Corollarium.

§. 14. Perspicuum est, eodem modo irrationalitatem tolli posse, si fuerit multo generalius

$$s=\sqrt[n]{a+b\sqrt[m]{(f+gx)}}.$$

Posita enim hac formula $=z$, fiet

$$a+b\sqrt[m]{(f+gx)}=z^n \text{ et } b\sqrt[m]{(f+gx)}=z^n-a.$$

Porro $b^m(f+gx)=(z^n-a)^m$, et hinc colligitur

$$x=\tfrac{(z^n-a)^m}{b^m g}-\tfrac{f}{g} \text{ ideoque}$$

$$\partial x=\tfrac{mnz^{n-1}\partial z(z^n-a)^{m-1}}{b^m g}$$

Sicque etiam hoc modo tota formula rationalis evadet.

Problema 6.

§. 15. *Si fuerit X functio rationalis binarum quantitatum s et x, existente* $s=\sqrt{\tfrac{a+bx}{f+gx}}$, *formulam differentialem* $X\partial x$ *ab irrationalitate liberare.*

Solutio,

Ponatur $\sqrt{\tfrac{a+bx}{f+gx}}=z$, et sumtis quadratis erit $\tfrac{a+bx}{f+gx}=zz$, hincque $x=\tfrac{fzz-a}{b-gzz}$,
unde differentiando colligitur

$$\partial x=\tfrac{2bfz\partial z-2agz\partial z}{\left(b-gzz\right)^2}.$$

Hisque valoribus substitutis formula proposita $X\partial x$ ad rationalitatem erit perducta.

Exemplum 1.

§. 16. Si fuerit $\partial y = \frac{\partial x}{s} = \frac{\partial x\sqrt{(f+gx)}}{\sqrt{(a+bx)}}$, posito

$$\sqrt{\frac{a+bx}{f+gx}} = z \text{ erit } \partial y = \frac{dx}{z},$$

et substituto loco ∂x valore supra invente colligitur

$$\partial y = \frac{2(bf-ag)\partial z}{(b-gzz)^2};$$

quae formula, uti jam satis constat, reduci potest ad talem $\int\frac{\partial z}{b-gzz}$, cujus autem integratio vel per logarithmos vel per arcus circulares expedietur.

Exemplum 2.

§. 17. Sit specialius $\partial y = \frac{\partial x\sqrt{(1-x)}}{\sqrt{(1+x)}}$, ubi $f=1,\ g=-1,\ a=1$ et $b=1$, ideoque,

$$z = \frac{\sqrt{(1+x)}}{\sqrt{(1-x)}}, \text{ et } \partial x = \frac{4z\partial z}{(1+zz)^2};$$

quibus valoribus substitutis fiet $\partial y = \frac{4\partial z}{(1+zz)^2}$. Statuatur ergo

$$\int\frac{4\partial z}{(1+zz)^2} = \frac{\mathrm{A}z}{1+zz} + \mathrm{B}\int\frac{\partial z}{1+zz} = y,$$

unde sumtis differentialibus fiet

$$\frac{4}{(1+zz)^2} = \frac{\mathrm{A}-\mathrm{A}zz}{(1+zz)^2} + \frac{\mathrm{B}}{1+zz} = \frac{\mathrm{A}+\mathrm{B}+(\mathrm{B}-\mathrm{A})zz}{(1+zz)^2}.$$

Oportet igitur sit $\mathrm{A}+\mathrm{B}=4$ et $\mathrm{B}-\mathrm{A}=0$, ideoque $\mathrm{A}=2$ et $\mathrm{B}=2$; et quia $\int\frac{dz}{1+zz} = \text{Arc.tang.}z$, adipiscimur

$$y = \frac{2z}{1+zz} + 2\ \text{Arc.tang.}z;$$

quocirca facta restitutione, ob $1+zz = \frac{2}{1-x}$, obtinebitur

$$y = \sqrt{(1-xx)} + 2\ \text{Arc.tang.}\sqrt{\frac{1+x}{1-x}}.$$

Cum igitur hujus arcus tangens sit $\sqrt{\frac{1+x}{1-x}}$, erit ejus sinus $=\sqrt{\frac{1+x}{2}}$ et cosinus $=\sqrt{\frac{1-x}{2}}$; anguli vero dupli sinus erit $\sqrt{(1-xx)}$ et cosinus $=-x$, unde fiet

$$2\ \text{Arc.tang.}\sqrt{\frac{1+x}{1-x}} = \text{Arc.cos}-x = \frac{\pi}{2} + \text{Arc.sin}\,x$$

quocirca integrale quaesitum erit

$$y = \sqrt{(1-xx)} + \frac{\pi}{2} + \text{Arc.sin}\,x + \mathrm{C},$$

quod si ita capi debeat, ut evanescat posito $x=0$, erit

$C=-1-\frac{\pi}{2}$ ideoque,

$$y=\sqrt{(1-xx)}-1+\text{Arc.}\sin x .$$

Tum igitur, si sumatur $x=1$, fiet $y=\frac{\pi}{2}-1$, qui valor in fractionibus decimalibus dat 0,5707963.

Problema 6.

§.15. *Si fuerit X functio rationalis binarum variabilium x et s, existente* $s=\sqrt[n]{\frac{a+bx}{f+gx}}$, *formulam differentialem* $X\partial x$ *ad rationalitatem perducere.*

Solutio.

Posito $s=\sqrt[n]{\frac{a+bx}{f+gx}}=z$ erit $\frac{a+bx}{f+gx}=z^n$ hincque $x=\frac{fz^n-a}{b-gz^n}$, consequenter $\partial x=\frac{n(bf-ag)fz^{n-1}\partial z}{(b-gz)^{2n}}$; hisque valoribus substitutis tota formula. proposita $X\partial x$ ad rationalitatem erit perducta.

Problema 7.

§.19. *Si fuerit X functio binarum quantitatum x et s, existente* $s=\sqrt{(a+bxx)}$, *formulam differentialem* $\frac{X\partial x}{x}$; *ab irrationalitate liberare.*

Solutio.

Ponamus $s=\sqrt{(a+bxx)}=z$, erit $a+bxx=zz$, hinc $xx=\frac{zz-a}{b}$, et quia in functione X tantum quadratum *xx*, ejusque ergo potestates pares occurrunt : hac substitutione jam functio X evadet rationalis. Sumtis vero logarithmis

$$2lx=l(zz-a)-lb,$$

differentiando fit

$$\frac{2\partial x}{x}=\frac{2z\partial z}{zz-a}; \text{ ideoque } \frac{\partial x}{x}=\frac{z\partial z}{zz-a}.$$

Hoc ergo modo formula proposito $X.\frac{\partial x}{x}$ prorsus reddetur rationalis.

Exemplum 1.

§. 20. Si fuerit

$$\partial y=\frac{x\partial x}{\sqrt{(a+bxx)}}, \text{ erit } \partial y=\frac{\partial x}{x}\cdot\frac{xx}{\sqrt{(a+bxx)}}=\frac{xx}{s}\cdot\frac{\partial x}{x}.$$

Posito ergo $\sqrt{(a+bxx)}=z$ erit $\partial y=\frac{\partial z}{b}$, unde colligitur integrando $y=\frac{z}{b}=\frac{\sqrt{(a+bxx)}}{b}$.

Exemplum 2.

§. 21. Si fuerit

$$\partial y = \frac{x^3 \partial x}{\sqrt{(a+bxx)}} = \frac{\partial x}{x} \cdot \frac{x^4}{s},$$

ponendo $\sqrt{(a+bxx)} = z$, ut sit $xx = \frac{zz-a}{b}$ et $\frac{\partial x}{x} = \frac{z\partial z}{zz-a}$, erit $\partial y = \frac{1}{bb}\partial z\left(zz-a\right)$, hincque integrando adipiscimur $y = \frac{z}{3bb}(zz-3a)$; unde facta restitutione prodibit integrale quaesitum $y = \frac{bxx-2a}{3bb}\sqrt{(a+bxx)} + \text{C}$.

Exemplum 3.

§. 22. Si fuerit

$$\partial y = \frac{x^3 \partial x}{\sqrt{(a+bxx)^3}}, \text{erit } \partial y = \frac{\partial x}{x} \cdot \frac{x^4}{s^3};$$

hinc posito
$\sqrt{(a+bxx)} = s = z$ fiet $\partial y = \frac{\partial z}{bb}\left(\frac{zz-a}{zz}\right)$, unde sumto integrali fiet $y = \frac{1}{bb}\left(\frac{zz+a}{zz}\right)$, quocirca facta restitutione resultat $y = \frac{2a+bxx}{bb\sqrt{(a+bxx)}} + \text{C}$.

Problema 8.

§. 23. *Si fuerit X functio rationalis binarum quantitatum x^n et s, existente $s = \sqrt[m]{\left(a+bx^n\right)}$, formulam differentialem $X\frac{\partial x}{x}$ ad rationalitatem perducere.*

Solutio.

Posito $s = \sqrt[m]{\left(a+bx^n\right)} = z$, fiet $a+bx^n = z^m$ et $x^n = \frac{z^m-a}{b}$. Quia igitur in functione X tantum potestas x^n occurrit, ea rationalis reddetur, si hi valores substituantur. Tum vero sumtis logarithmis habebitur

$$nlx = l(z^m - a) - lb,$$

et differentiando

$$\frac{\partial x}{x} = \frac{mz^{m-1}\partial x}{n\left(z^m-a\right)},$$

sicque tota formula proposita fiet rationalis.

Exemplum.

§. 24. Sit

$$\partial y = \frac{x^{n-1}\partial x}{\sqrt[m]{\left(a+bx^n\right)}} = \frac{\partial x}{x} \cdot \frac{x^n}{s},$$

factaque substitutione orietur haec aequatio

$$\partial y = \frac{mz^{m-2}\partial z}{nb},$$

qua integrata prodibit

$$y=\frac{mz^{m-1}\partial z}{nb(m-1)}=\frac{m}{nb(m-1)}\sqrt[m]{\left(a+bx^{n}\right)^{m-1}}+\mathrm{C},\ \text{sive}$$

$$y=\frac{m}{nb(m-1)}\cdot\frac{a+bx^{n}}{\sqrt[m]{a+bx^{n}}}+\mathrm{C}.$$

Problema 9.

§. 25. *Si fuerit X functio rationalis quantitatum xx et s, existente* $s=\sqrt{\frac{a+bxx}{f+gxx}}$, *formulam differentialem* $X\frac{\partial x}{x}$ *ab irrationalitate liberare.*

Solutio.

Ponatur $s=\sqrt{\frac{a+bxx}{f+gxx}}=z$, eritque $\frac{a+bxx}{f+gxx}=zz$, hinc $xx=\frac{fzz-a}{b-gzz}$, unde functio X penitus sit rationalis. Porro sumtis logarithmis

$$2lx=l(fzz-a)-l(b-gzz),$$

differentietur, ut prodeat

$$\frac{2\partial x}{x}=\frac{2fz\partial z}{fzz-a}+\frac{2gz\partial z}{b-gzz}=\frac{2(bf-ag)z\partial z}{(fzz-a)(b-gzz)},$$

unde fit

$$\frac{\partial x}{x}=\frac{(bf-ag)z\partial z}{(fzz-a)(b-gzz)};$$

sicque tota formula differentialis fiet rationalis.

Exemplum.

§. 26. Si fuerit $\partial y=\frac{\partial x}{\sqrt{(f+gxx)}}$, repræsentemus hanc formulam ita

$$\partial y=\frac{\partial x}{x}\cdot\frac{x}{\sqrt{(f+gxx)}}=\frac{\partial x}{x}\cdot\sqrt{\frac{xx}{f+gxx}}.$$

Hic ergo erit $a=0,\ b=1$, et $z=\frac{x}{\sqrt{f+gxx}}$, ita ut $\partial y=\frac{z\partial z}{x}$; erit autem

$$\frac{\partial x}{x}=\frac{\partial z}{z(1-gzz)},$$

unde fit $\partial y=\frac{\partial z}{1-gzz}$, cujus formulae integratio per logarithmo expedietur, si fuerit g fuerit numerus positivus : sin autem fuerit negativus per arcus circulares absolvetur. Sit igitur 1°.) $g=+hh$, erit $\partial y=\frac{\partial z}{1-hhzz}$, ideoque $y=\frac{1}{2h}l\frac{1+hz}{1-hz}$; et restitutis valoribus supra indicatis , erit

$$y=\frac{1}{2h}l\left(\frac{\sqrt{(f+hhxx)}+hx}{\sqrt{(f+hhxx)}-hx}\right)=\frac{1}{h}l\left(\frac{\sqrt{(f+hhxx)}+hx}{\sqrt{f}}\right).$$

Sit 2°.) g quantitas negativa , puta $g=-hh$, erit

$$\partial y=\frac{\partial z}{1+hhzz}=\frac{1}{h}\cdot\frac{h\partial z}{1+hhzz},$$

unde colligitur

$$y=\frac{1}{h}\text{Arc.tang.}hz=\frac{1}{h}.\text{Arc.tang.}\frac{hx}{\sqrt{(f-hhxx)}}.$$

Ubi manifestum est, f esse debere quantitatem positivam, quia alioquin formula differentialis esset imaginaria.

Corollarium.

§. 27. Hinc ergo si proponatur formula
$\partial y=\sqrt{(1+xx)}$, ubi $f=1$ et $g=1$, ex casu priore ob $h=+1$ erit

$$\int\frac{\partial x}{\sqrt{(1+xx)}}=l\,[\sqrt{(1+xx)}+x].$$

At si fuerit

$$\partial y=\frac{\partial x}{\sqrt{(1-xx)}},\ \text{ubi}\ f=1\ \text{et}\ g=-1,$$

colligitur ex casu posteriore $x=\text{Arc. tang.}\frac{x}{\sqrt{(1-xx)}}$, unde concluditur

$$\int\frac{\partial x}{\sqrt{(1-xx)}}=\text{Arc. sin.}x=\text{Arc. cos.}\sqrt{(1-xx)}.$$

Problema 10.

§.28. *Si fuerit X functio rationalis quantitatum* x^n *et s , existente* $s=\sqrt[n]{\left(\frac{a+bx^n}{f+gx^n}\right)}$, *formulam differentialem* $X\frac{\partial x}{x}$ *rationalem efficere.*

Solutio.

Ponatur $s=\sqrt[n]{\left(\frac{a+bx^n}{f+gx^n}\right)}=z$, eritque $\frac{a+bx^n}{f+gx^n}=z^n$, hinc $x^n=\frac{fz^n-a}{b-gz^n}$, tum autem sumtis logarithmis, erit

$$nlx=l\left(fz^n-a\right)-l\left(b-gz^n\right),$$

et differentiando

$$\frac{\partial x}{x}=\frac{fz^{n-1}\partial z}{fz^n-a}+\frac{gz^{n-1}}{b-gz^n}=\frac{(bf-ag)z^{n-1}\partial z}{\left(fz^n-a\right)\left(b-gz^n\right)}\ ;$$

quibus valoribus substitutis formula proposita fit rationalis.

Problema 11.

§.29. *Si fuerit X functio rationalis binarum quantitatum x^n et s, existente* $s=\sqrt[m]{\left(\frac{a+bx^n}{f+gx^n}\right)}$, *formulam differentialem* $X\frac{\partial x}{x}$ *ab omni irrationalitate liberare.*

Solutio.

Statatur $s=\sqrt[m]{\left(\frac{a+bx^n}{f+gx^n}\right)}=z$, eritque $\frac{a+bx^n}{f+gx^n}=z^m$, unde fit $x^n=\frac{fz^m-a}{b-gz^m}$; hinc sumtis logarithmis erit

$$nlx=l\left(fz^m-a\right)-l\left(b-gz^m\right),$$

hinc differentiando

$$\frac{n\partial x}{x}=\frac{m(bf-ag)z^{m-1}\partial z}{\left(fz^m-a\right)\left(b-gz^m\right)},$$

ideoque

$$\frac{\partial x}{x}=\frac{m(bf-ag)z^{m-1}\partial z}{n\left(fz^m-a\right)\left(b-gz^m\right)},$$

quibus valoribus substitutis irrationalitas formulae propositae penitus tollitur.

Problema 12.

§. 30. *Si fuerit X functio rationalis quaecunque binarum quantitatum x et s, existente* $s=\sqrt{(\alpha+\beta x+\gamma xx)}$, *formulam differentialem* $X\partial x$ *ad rationalitatem perducere.*

Solutio.

Hic duos casus a se invicem distingui convenit, prout γ fuerit vel quantitas positiva vel negativa.

I. Sit γ quantitas positiva, ac ponatur $\gamma=cc$ et $\beta=2bc$, ut habeatur

$$s=\sqrt{(\alpha+2bcx+ccxx)}=\sqrt{\left[\alpha-bb+\left(b+cx\right)^2\right]}$$

ubi loco $\alpha-bb$ brevitatis ergo scribatur e, ut sit

$$s=\sqrt{\left[e+\left(b+cx\right)^2\right]}.$$

Jam statuatur $s=b+cx+z$, eritque

$$ss=e+\left(b+cx\right)^2=\left(b+cx\right)^2+2(b+cx)z+zz,$$

unde sequitur

$$e-zz=2z(b+cx),\ \text{sive}\ \ b+cx=\tfrac{e-zz}{2z};$$

hincque colligitur

$$x=\tfrac{e-zz}{2cz}-\tfrac{b}{c},\ \text{seu}\ x=\tfrac{e-2bz-zz}{2cz}.$$

Aequatio autem $b+cx=\tfrac{e-zz}{2z}$ differentiata praebet

$$c\partial x=-\tfrac{e\partial z}{2zz}-\tfrac{\partial z}{2}=-\tfrac{e\partial z-zz\partial z}{2zz},$$

unde deducitur

$$\partial x=-\tfrac{\partial z(e+zz)}{2czz},\ \text{at ob}$$

$$b+cx=\tfrac{e-zz}{2z}\ \ \text{fiet}\ \ s=\tfrac{e+zz}{2z}.$$

His ergo valoribus substitutis formula nostra $X\partial x$ reddetur rationalis. Postquam igitur ejus integrale fuerit inventum, loco z valor ante inventus $\sqrt{\left[e+\left(b+cx\right)^2\right]}-b-cx$ erit substituendus.

II. Sin autem γ fuerit quantitas negativa , ponatur

$$\gamma=-cc\ \ \text{et}\ \ \beta=-\ 2bc,$$

ut habeatur

$$s=\sqrt{\left(\alpha-2bcx-ccxx\right)}=\sqrt{\left[\alpha+bb-(b+cx)^2\right]},$$

ubi evidens est, quantitatem $\alpha+bb$ necessario esse debere positivam , quia alioquin s evaderet imaginarium. Quamobrem ponamus brevitatis gratia $\alpha+bb=aa$, ut fiat

$$s=\sqrt{\left[aa-(b+cx)^2\right]}$$

ad quam formam rationalem efficiendam statuamus

$$\sqrt{\left[aa-(b+cx)^2\right]}=a-(b+cx)z,$$

unde sumtis quadratis erit

$$aa-\left(b+cx\right)^2=aa-2az\left(b+cx\right)+(b+cx)^2zz$$

quae aequatio reducitur ad hanc :

$$-(b+cx)=-2az+\left(b+cx\right)zz,$$

unde reperitur

$$(b+cx)=\tfrac{2az}{1+zz},\ \ \text{ideoque}$$

$$x=\tfrac{2az-b-bzz}{c(1+zz)}.$$

Illa autem aequatio differentiata dat

$$c\partial x=\frac{2a\partial z(1+zz)-4azz\partial z}{(1+zz)^2}=\frac{2a\partial z(1-zz)}{(1+zz)^2};$$

unde fit

$$\partial x = \frac{2a\partial z(1-zz)}{c(1+zz)^2}.$$

Porro autem, cum sit

$$s = a-(b+cx)z, \text{ob } b+cx = \tfrac{2az}{1+zz}$$

erit $s = \frac{a(1-zz)}{1+zz}$, quocirca, si loco x , s et ∂x inventi hi valores substituantur, formula proposita differentialis $X\partial x$ evadet rationalis, et per variabilem z exprimetur , cujus integrale postquam fuerit inventum, loco z ubique ejus restituatur valor assumtus $z = \alpha - \sqrt{[aa-(b+cx)^2]}$, et integrale obtinebitur per solam variabilem x expressum.

Exemplum I.

§.31. Si fuerit

$$\partial y = \frac{\partial x}{\sqrt{\left[e+(b+cx)^2\right]}},$$ quae formula ad casum priorem pertinet , erit

$$\partial y = \tfrac{\partial x}{s} = -\tfrac{\partial z}{cz}, \text{ob } \partial x = -\frac{\partial z(e+zz)}{2ezz} \text{ et } s = \tfrac{e+zz}{2z};$$

cujus integrale est $y = -\frac{1}{c}lz$; restitututo ergo valore

$$z = l[e+(b+cx)^2]-b-cx, \text{ erit}$$

$$y = -\tfrac{1}{c}l[\sqrt{[e+(b+cx)^2]}-b-cx]+C,$$

integrale si evanescere debeat posito $x = 0$, fiet

$$C = \tfrac{1}{c}l[\sqrt{(e+bb)}-b].$$

Corollarium.

§. 32. Si ponatur $b = 0$ et $c = 1$, sive

$$\partial y = \frac{\partial x}{\sqrt{(e+xx)}}, \text{ erit integrale}$$

$$y = -\tfrac{1}{c}l[\sqrt{(e+xx)}-x]+l\sqrt{e} = l\frac{\sqrt{e}}{\sqrt{(e+xx)}-x},$$

quae formula reducitur ad hanc

$$y = l\frac{\sqrt{(e+xx)}+x}{\sqrt{e}}.$$

Cum vero porro sit

$$\partial.\sqrt{(e+xx)} = \frac{x\partial x}{\sqrt{(e+xx)}}, \text{ erit}$$

$$\int \frac{x\partial x}{\sqrt{(e+xx)}} = \sqrt{(e+xx)}.$$

Si igitur hae duae formulae combinentur, habebitur ista integratio notatu digna

$$\int \frac{A\partial x + Bx\partial x}{\sqrt{(e+xx)}} = \mathrm{A}l\,\frac{\sqrt{(e+xx)}+x}{\sqrt{e}} + \mathrm{B}\sqrt{(e+xx)}.$$

Exemplum 2.

§.33. Sit $\partial y = \frac{\partial x}{\sqrt{\left[aa-(b+cx)^2\right]}}$, quae formula ad casum secundum est referenda, ita ut sit $\partial y = \frac{\partial x}{s}$. Cum igitur sit $\partial x = \frac{2a\partial z(1-zz)}{c(1+zz)^2}$ et $s = \frac{a(1-zz)}{1+zz}$, erit $\partial y = \frac{\partial x}{s} = \frac{2}{c}\cdot\frac{\partial z}{1+zz}$, unde fit integrando $y = \frac{2}{c}\cdot$ Arc. tang.z. Quia igitur est

$$z = \frac{a-\sqrt{[aa-(b+cx)^2]}}{b+cx}, \text{ erit}$$

$$y = \frac{2}{c}\cdot \text{Arc. tang.}\frac{a-\sqrt{[aa-(b+cx)^2]}}{b+cx} + \mathrm{C}.$$

Corollarium.

§. 34. Sit igitur $b = 0$ et $c = 1$, seu formula differentialis proposita $\partial y = \frac{\partial x}{\sqrt{[aa-xx]}}$, reperieturque $y = 2.\text{Arc. tang.}\frac{a-\sqrt{[aa-xx]}}{x} + \mathrm{C}.$

Quia igitur tangens hujus arcus est $\frac{a-\sqrt{[aa-xx]}}{x}$; tangens dupli arcus erit $= \frac{x}{\sqrt{[aa-xx]}}$ ita ut sit

$y =$ Arc. tang.$\frac{x}{\sqrt{[aa-xx]}}$: hujus autem arcus sinus erit $\frac{x}{a}$, sicque integrale quaesitum

$$\int \frac{\partial x}{\sqrt{[aa-xx]}} = \text{Arc. sin.}\frac{x}{a}.$$

Quia porro

$$\partial\cdot\sqrt{[aa-xx]} = -\frac{x\partial x}{\sqrt{[aa-xx]}}, \text{ erit}$$

$$\int \frac{x\partial x}{\sqrt{[aa-xx]}} = -\sqrt{[aa-xx]}:$$

quocirca ista generalior conficitur integratio

$$\int \frac{A\partial x + Bx\partial x}{\sqrt{[aa-xx]}} = \mathrm{A}.\text{Arc.sin.}\frac{x}{a}. - \mathrm{B}\sqrt{[aa-xx]}.$$

Problema 13.

§. 35. *Si fuerit V functio rationalis binarum quantitatum* v^n *et* s , *existente*

$$s = \sqrt{\left(\alpha + \beta v^n + \gamma v^{2n}\right)},$$

formulam differentialem $Vv^{n-1}\partial v$ ab irrationalitate liberare.

Solutio.

Ponatur $v^n = x$, erit

$$s = \sqrt{(\alpha + \beta x + \gamma xx)} \text{ et } v^{n-1}\partial v = \frac{\partial x}{n};$$

hic ergo jam erit V functio rationalis binarum quantitatum *x* et *s,*
existente

$$s = \sqrt{(\alpha + \beta x + \gamma xx)}$$

et formula ab irrationalitate liberanda erit $\frac{V\partial x}{n}$; qui casus prorsus convenit cum problemate praecedente, ideoque eandem habebit solutionem.

Scholion.

§.36. Praecepta hactenus tradita ad omnes fere formulas differentiales, quae quidem adhuc tractari potuerunt, extenduntur. Interim tamen ejusmodi casus occurrere possunt, quibus idonea substitutio, ad irrationalitatem tollendam necessaria, non tam facile perspicitur, sed acri judicio demum investigare licet, in quo negotio cum praecepta generalia tradere nondum liceat, exempla quaedam particularia speciminis loco in medium afferamus.

Exemplum 1.

§.37. *Si propositat fuerit haec formula irrationalis*

$$\partial P = \frac{\partial x(1+xx)}{(1-xx)\sqrt{(1+xx)^4}},$$

ejus integrale P investigare.

Si quis hic ejusmodi uti vellet substitutione, qua formula $\sqrt{(1+xx)^4}$ ad rationalitatem perduceretur, oleum et operam esset perditurus, interim tamen singulari artificio sequens substitutio negotium conficere poterit. Statuatur

$$\frac{x\sqrt{2}}{1-xx} = p, \text{ eritque}$$

$$1 + pp = \frac{1+x^4}{(1-xx)^2}$$

hinc

$$\sqrt{(1+pp)} = \frac{\sqrt{(1+x^4)}}{1-xx}$$

tum vero erit differentiando

$$\partial p = \frac{\partial x\sqrt{2}(1+xx)}{(1-xx)^2}$$

ex quibus valoribus colligitur

$$\frac{\partial p}{\sqrt{(1+pp)}} = \frac{\partial x\sqrt{2}(1+xx)}{(1-xx)\sqrt{(1+x^4)}},$$

quae feliciter cum formula ipsa proposita convenit, ita ut sit

$$\frac{\partial p}{\sqrt{(1+pp)}} = \partial P\sqrt{2}, \text{ sive } \partial P = \frac{1}{\sqrt{2}} \cdot \frac{\partial p}{\sqrt{(1+pp)}};$$

unde colligitur integrando

$$P = \frac{1}{\sqrt{2}} l\left[\sqrt{(1+pp)} + p\right].$$

Quare si loco p et $\sqrt{(1+pp)}$ valores dati substituantur, haec obtinetur integratio satis memorabilis

$$P = \int \frac{\partial x(1+xx)}{(1-xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} l \frac{\sqrt{(1+x^4)}+\sqrt{2}}{1+xx}$$

Exemplum 2.

§.38. *Si proposita fuerit haec formula irrationalis* $\partial Q = \frac{\partial x(1-xx)}{(1+xx)\sqrt{(1+x^4)}}$, *ejus integrale Q investigare.*

Ad hoc praestandum fiat $\frac{x\sqrt{2}}{1+xx} = q$, eritque

$$\sqrt{(1-qq)} = \frac{\sqrt{(1+x^4)}}{(1+xx)};$$

tum vero erit $\partial q = \frac{\partial x(1-xx)\sqrt{2}}{(1+xx)^2}$, atque hinc colligitur

$$\frac{\partial q}{\sqrt{(1-qq)}} = \frac{\partial x(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}} = \partial Q\sqrt{2},$$

unde fit

$$Q = \frac{1}{\sqrt{2}}\int \frac{\partial q}{\sqrt{(1-qq)}} = \frac{1}{\sqrt{2}} \text{Arc.sin}.q.$$

Restituto ergo pro q valore assumto , ista obtinebitur integratio

$$Q = \int \frac{\partial x(1-xx)}{(1+xx)\sqrt{(1+x^4)}} = \frac{1}{\sqrt{2}} \text{Arc.sin}.\frac{x\sqrt{2}}{1+xx}.$$

Scholion.

§. 39. Cum istae duae formulae

$$\frac{\partial x(1+xx)\sqrt{2}}{(1-xx)\sqrt{(1+x^4)}} \text{ et } \frac{\partial x(1-xx)\sqrt{2}}{(1+xx)\sqrt{(1+x^4)}}$$

perductae sint ad has simplices

$$\frac{\partial p}{\sqrt{(1+pp)}} \text{ et } \frac{\partial q}{\sqrt{(1-qq)}},$$

quarum utraque facile ab irrationalitate liberatur, istae ipsae formulae propositae ope idoneae substitutionis ab irrationalitate liberari possunt ; unde mirum non est, earum integrali a sive per logarithmum sive per arcum circularem exhiberi potuisse. Satis enim jam est ostensum , omnium formularum differentialium rationalium integralia semper vel per logarithmos et arcus circulares, vel adeo algebraice exhiberi posse ; quod igitur etiam de illis formulis irrationalibus est tenendum, quas certae substitutionis ope ad rationalitatem perducere licet. Unde vicissim plures Geometrae concluserunt:
si quae formula differentialis nullo plane modo ab irrationalitate liberari queat , tum ejus integrale etiam neque per logarithmos nec arcus circulares, multo minus algebraice exprimi posse, sed ad aliud genus quantitatum transcendentium referri oportere.
Caeterum combinatio duorum praecedentium exemplorum manuducit ad solutionem sequentium.

Exemplum 3.

§.40. *Si proposita fuerit haec formula differentialis*

$$\partial y=\frac{\partial x\sqrt{\left(1+x^4\right)}}{1-x^4},$$

ejus integrale invenire.
Hanc formulam per neutram substitutionem ante usurpatam rationalem reddere licet: utraque tamen juncta negotium confici poterit, namque ejus integrale per logarithmos et arcus circulares sequenti artificio expedietur. Formula enim proposita in binas sequentes partes discerpi potest, quae sunt

$$\partial y=\frac{\frac{1}{2}\partial x(1+xx)}{(1-xx)\sqrt{\left(1+x^4\right)}}+\frac{\frac{1}{2}\partial x}{(1+xx)\sqrt{\left(1+x^4\right)}},$$

quippe quarum summa ipsam formulam nostram propositam producit ; prodit enim

$$\partial y=\frac{\frac{1}{2}\partial x(1+xx)^2+\frac{1}{2}\partial x(1-xx)^2}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}}=\frac{\partial x\left(1+x^4\right)}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}}=\frac{\partial x\sqrt{\left(1+x^4\right)}}{1-x^4}.$$

Quod si ergo duo praecedentia. exempla in subsidium vocentur, manifesto fiet $dy=\frac{1}{2}\partial P+\frac{1}{2}\partial Q$, consequenter integrale quaesitum erit $y=\frac{1}{2}P+\frac{1}{2}Q$, quod sequenti modo exprimere licebit

$$\int\frac{\partial x\sqrt{\left(1+x^4\right)}}{1-x^4}=\frac{1}{2\sqrt{2}}l\frac{\sqrt{\left(1+x^4\right)}+x\sqrt{2}}{1-xx}+\frac{1}{2\sqrt{2}}\text{Arc.sin.}\frac{x\sqrt{2}}{1+xx}.$$

Exemplum 4.

§. 41. *Si proposita fuerit haec formula differentialis* $\partial y=\frac{xx\partial x}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}}$, *ejus integrale investigare.*

Haec formula simili modo ac praecedens tractari potest ; discerpatur enim in sequentes duas partes:

$$\partial y=\frac{\frac{1}{4}\partial x(1+xx)}{(1-xx)\sqrt{\left(1+x^4\right)}}-\frac{\frac{1}{4}\partial x(1-xx)}{(1+xx)\sqrt{\left(1+x^4\right)}},$$

quippe quae conjunctae producunt

$$\partial y=\frac{\frac{1}{4}\partial x(1+xx)^2-\frac{1}{4}\partial x(1-xx)^2}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}}$$

$$=\frac{\frac{1}{4}\partial x.4xx}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}}=\frac{xx\partial x}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}},$$

quae cum sit ipsa formula proposita, erit ex praecedentibus exemplis $\partial y=\frac{1}{4}\partial \mathrm{P}-\frac{1}{4}\partial Q$, consequenter $y=\frac{1}{4}\mathrm{P}-\frac{1}{4}Q$, hinc integrale quaesitum ita reperietur expressum.

$$\int\frac{xx\partial x}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}}=\frac{1}{4\sqrt{2}}l\frac{\sqrt{\left(1+x^4\right)}+x\sqrt{2}}{1-xx}-\frac{1}{4\sqrt{2}}\mathrm{Arc.sin}.\frac{x\sqrt{2}}{1+xx}.$$

Scholion.

§. 42. Haec duo postrema exempla si nullo plane modo ope cujuspiam substitutionis ad rationalitatem perduci possent, insigne praeberent documentum, quod conclusio supra memorata quandoque fallere possit : Re autem attentius perpensa inveni, omnia haec quatuor exempia ope unicae substitutionis immediate ad rationalitatem perduci ideoque integrari posse ; id quod ostendisse utique operae erit pretium.

Alia resolutio
quatuor postremorum exemplorum.

§. 43. Statuatur pro primo exemplo

$$v=\frac{x\sqrt{2}}{\sqrt{\left(1+x^4\right)}},\ \text{eritque}\ \sqrt{(1+vv)}=\frac{1+xx}{\sqrt{\left(1+x^4\right)}};$$

tum vero

$$\sqrt{(1-vv)}=\frac{1-xx}{\sqrt{\left(1+x^4\right)}},$$

unde fit

$$\sqrt{\frac{1+vv}{1-vv}}=\frac{1+xx}{1-xx}\ \text{et}\ \sqrt{\left(1-v^4\right)}=\frac{1-x^4}{1+x^4}.$$

At differentiando adipiscimur

$$\partial v=\frac{\partial x\left(1-x^4\right)\sqrt{2}}{\left(1+x^4\right)\sqrt{\left(1+x^4\right)}}$$

Cum nunc sit $\frac{1-x^4}{1+x^4}=\sqrt{\left(1-v^4\right)}$, erit

$$\partial v=\frac{\partial x\sqrt{2}.\sqrt{\left(1-x^4\right)}}{\sqrt{\left(1+x^4\right)}},\ \text{sive}\ \frac{\partial v}{\sqrt{\left(1-v^4\right)}}=\frac{\partial x\sqrt{2}}{\sqrt{\left(1+x^4\right)}};$$

quae aequalitas maxime est notatu digna. Quod si jam haec aequatio multiplicetur per $\sqrt{\frac{1+vv}{1-vv}}=\frac{1+xx}{1-xx}$, nascetur haec aequatio

$$\frac{\partial v}{1-vv}=\frac{\partial x(1+xx)\sqrt{2}.}{(1-xx)\sqrt{\left(1+x^4\right)}},$$

sicque erit

$$\int\frac{\partial x(1+xx)}{(1-xx)\sqrt{\left(1+x^4\right)}}=\frac{1}{2\sqrt{2}}\int\frac{\partial v}{1-vv}=\frac{1}{2\sqrt{2}}\,l\,\frac{1+v}{1-v}.$$

Deinde aequatio

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{\sqrt{\left(1-v^4\right)}}=\frac{\partial x}{\sqrt{\left(1+x^4\right)}}\ \text{multiplicetur per}\ \sqrt{\frac{1-vv}{1+vv}}=\frac{1-xx}{1+xx},$$

ac prodibit formula exempli secundi

$$\int\frac{\partial x(1-xx)}{(1+xx)\sqrt{\left(1+x^4\right)}}=\frac{1}{\sqrt{2}}\int\frac{\partial x}{1+vv}=\frac{1}{\sqrt{2}}\,\text{Arc.}\tan.v\,.$$

Porro eadem aequatio

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{\sqrt{\left(1-v^4\right)}}=\frac{\partial x}{\sqrt{\left(1+x^4\right)}}$$

dividatur per

$$\sqrt{\left(1-v^4\right)}=\frac{1-x^4}{1+x^4},$$

et prodibit

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{1-v^4}=\frac{\partial x\sqrt{\left(1+x^4\right)}}{1-x^4};$$

quae est ipsa formula exempli tertii , ita ut jam sit

$$\int\frac{\partial x\sqrt{\left(1+x^4\right)}}{\left(1-x^4\right)}=\frac{1}{\sqrt{2}}\int\frac{\partial v}{1-v^4}=\frac{1}{2\sqrt{2}}\int\frac{\partial v}{1+vv}+\frac{1}{2\sqrt{2}}\int\frac{\partial v}{1-vv},$$

quod integrale cum ante invento egregie convenit. Tandem postrema aequatio hic inventa

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{1-v^4}=\frac{\partial x\sqrt{\left(1+x^4\right)}}{\left(1-x^4\right)},$$

ducatur in $vv=\frac{2xx}{1+x^4}$, ut prodeat

$$\frac{1}{\sqrt{2}}\cdot\frac{vv\partial v}{1-x^4}=\frac{2xx\partial x\sqrt{\left(1+x^4\right)}}{\left(1-x^4\right)\left(1+x^4\right)}=\frac{2xx\partial x}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}},$$

unde pro exemplo quarto colligitur

$$\int\frac{xx\partial x}{\left(1-x^4\right)\sqrt{\left(1+x^4\right)}}=\frac{1}{2\sqrt{2}}\int\frac{vv\partial v}{1-v^4}=-\frac{1}{4\sqrt{2}}\int\frac{\partial v}{1+vv}+\frac{1}{4\sqrt{2}}\int\frac{\partial v}{1-vv},$$

unde cum sit $v=\frac{x\sqrt{2}}{\sqrt{\left(1+x^4\right)}}$, erit

$$\int\frac{\partial v}{1-vv}=\frac{1}{2}l\frac{1+v}{1-v}=\frac{1}{2}l\frac{\sqrt{\left(1+x^4\right)}+x\sqrt{2}}{\sqrt{\left(1+x^4\right)}-x\sqrt{2}}$$

$$=\frac{1}{2}l\frac{\left[\sqrt{\left(1+x^4\right)}+x\sqrt{2}\right]^2}{\left(1-xx\right)^2}=l\frac{\sqrt{\left(1+x^4\right)}+x\sqrt{2}}{1-xx}.$$

Deinde vero est

$$\int\frac{\partial v}{1+vv}=\text{Arc.tang.}v=\text{Arc.sin.}\frac{v}{\sqrt{(1+vv)}}=\text{Arc.sin.}\frac{x\sqrt{2}}{1+xx}.$$

Scholion.

§. 44. Quanquam autem haec quatuor exempla ad rationalitatem reducere licuit, tamen conclusio supra memorata, quod omnes formulae integrales , quae nullo modo rationales effici queant, ad aliud pertineant transcendentium genus , neque per solos logarithmos et arcus circulares expediri possint, non solum manet suspecta, sed etiam falsitas ejus evidenter ob oculos poni potest. Sit enim functio

$$\mathrm{X}=\frac{a}{\sqrt{(1+xx)}}+\frac{b}{\sqrt[3]{\left(1+x^3\right)}}+\frac{c}{\sqrt[4]{\left(1+x^4\right)}};$$

tum certe formula differentialis $\mathrm{X}\partial x$ nullo modo ad rationalitatem perduci poterit ; interim tamen singulos ejus partes

$$\frac{a\partial x}{\sqrt{(1+xx)}},\ \frac{b\partial x}{\sqrt[3]{\left(1+x^3\right)}}\ \text{et}\ \frac{c\partial x}{\sqrt[4]{\left(1+x^4\right)}}$$

seorsim rationales effici et integralia per logarithmos et arcus circulares exhiberi possunt. Corollarinis loco hic sequens problema notatu dignum adjungamus.

Problema 14.

§.45. *Formularum integralium* $\int\frac{\partial x}{\sqrt{\left(1+x^4\right)}}$ et $\int\frac{\partial v}{\sqrt{\left(1-v^4\right)}}$ *valores per series investigare*, *pro casibus*, *quibus ponitur tam* $v=1$ *quam* $x=1$.

Solutio.

Cum posito $v=\frac{x\sqrt{2}}{\sqrt{(1+x^4)}}$, ut supra fecimus, evidens sit, sumto $x=0$ fore etiam $v=0$, et sumto $x=1$ fore $v=1$, ita ut hae duae quantitates x et v simul evanescant et simul unitati aequentur ; hinc deducimus istam aequationem differentialem attentione dignissimam

$$\frac{1}{\sqrt{2}}\cdot\frac{\partial v}{\sqrt{(1-v^4)}}=\frac{\partial x}{\sqrt{(1+x^4)}}$$

quas ergo ambas formulas in series converti oportet; erit autem

$$\frac{1}{\sqrt{(1-v^4)}}=\left(1-v^4\right)^{-\frac{1}{2}}=1+\frac{1}{2}v^4+\frac{1\cdot3}{2\cdot4}v^8+\frac{1\cdot3\cdot5}{2\cdot4\cdot6}v^{12}+\text{etc. et}$$

$$\frac{1}{\sqrt{(1+x^4)}}=\left(1+x^4\right)^{-\frac{1}{2}}=1-\frac{1}{2}x^4+\frac{1\cdot3}{2\cdot4}x^8-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}x^{12}+\text{etc.}$$

Illa jam per ∂v multiplicata et integrata praebet

$$\int\frac{\partial v}{\sqrt{(1-v^4)}}=v+\frac{1}{2\cdot5}v^5+\frac{1\cdot3}{2\cdot4\cdot9}v^9+\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot13}v^{13}+\text{etc.}$$

unde posito $v=1$, valor hujus integralis erit

$$1+\frac{1}{2\cdot5}+\frac{1\cdot3}{2\cdot4\cdot9}+\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot13}+\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8\cdot17}+\text{etc.}$$

quam seriem littera A indicemus. Simili modo altera series in ∂x ducta et integrata producit

$$\int\frac{\partial x}{\sqrt{(1+x^4)}}=x-\frac{1}{2\cdot5}x^5+\frac{1\cdot3}{2\cdot4\cdot9}x^9-\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot13}x^{13}+\text{etc.}$$

cujus valor facto $x=1$ erit

$$1-\frac{1}{2\cdot5}+\frac{1\cdot3}{2\cdot4\cdot9}-\frac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot13}+\frac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8\cdot17}-\text{etc.}$$

quem littera B designemus, ita ut sit $\mathrm{B}=\frac{\mathrm{A}}{\sqrt{2}}$, sive $\mathrm{A}=\mathrm{B}\sqrt{2}$;
unde patet, priorem seriem se habere ad posteriorem ut $\sqrt{2}:1$.

Scholion.

§. 46. Valor formulae integralis $\int\frac{\partial v}{\sqrt{(1-v^4)}}$ etiam hoc modo per seriem investigari potest.

Cum sit

$$\frac{1}{\sqrt{(1-v^4)}}=\frac{(1+vv)^{-\frac{1}{2}}}{\sqrt{(1-vv)}},\text{ et}$$

$$\left(1+vv\right)^{-\frac{1}{2}}=1-\frac{1}{2}vv+\frac{1\cdot3}{2\cdot4}v^4-\frac{1\cdot3\cdot5}{2\cdot4\cdot6}v^6+\text{etc.}$$

notetur esse $\int\frac{\partial v}{\sqrt{(1-vv)}}=\frac{\pi}{2}$. Deinde pro integratione reliquorum terminorum ponatur

$$\int\frac{v^{n+2}\partial v}{\sqrt{(1-vv)}}=-\frac{1}{n+2}v^{n+1}\sqrt{(1-vv)}+\frac{n+1}{n+2}\int\frac{v^{n}\partial v}{\sqrt{(1-vv)}},$$

quae aequatio differentiata dat

$$\frac{v^{n+2}}{\sqrt{(1-vv)}}=(n+1)\mathrm{A}v^{n}\sqrt{(1-vv)}-\frac{\mathrm{A}v^{n+2}}{\sqrt{(1-vv)}}+\mathrm{B}\frac{v^{n}}{\sqrt{(1-vv)}},$$

unde per $\sqrt{(1-vv)}$ multiplicando prodit

$$v^{n+2}=(n+1)\mathrm{A}v^{n}-(n+1)\mathrm{A}v^{n+2}-\mathrm{A}^{n+2}+\mathrm{B}v^{n}.$$

Hinc termini in quibus inest v^{n+2} , inter se aequati praebent $1=-(n+2)\mathrm{A}$, ideoque $\mathrm{A}=-\frac{1}{n+2}$; termini vero v^{n} continentes praebent $0=(n+1)\mathrm{A}+\mathrm{B}$, unde sit $\mathrm{B}=\frac{n+1}{n+2}$, ita ut in genere sit

$$\int\frac{v^{n+2}\partial v}{\sqrt{(1-vv)}}=-\frac{1}{n+2}v^{n+1}\sqrt{(1-vv)}+\frac{n+1}{n+2}\int\frac{v^{n}\partial v}{\sqrt{(1-vv)}},$$

quod integrale uti requiritur evanescit posito $v=0$. Ponatur nunc $v=1$, eritque

$$\int\frac{v^{n+2}\partial v}{\sqrt{(1-vv)}}=\frac{n+1}{n+2}\int\frac{v^{n}\partial v}{\sqrt{(1-vv)}};$$

hinc ergo pro *n* scribendo successive valores 0 , 2 , 4 , 6 , 8 , etc. erit

$$\text{I.}\quad\int\frac{vv\partial v}{\sqrt{(1-vv)}}=\frac{1}{2}\cdot\frac{\pi}{2};$$

$$\text{II.}\quad\int\frac{v^{4}\partial v}{\sqrt{(1-vv)}}=\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2};$$

$$\text{III.}\quad\int\frac{v^{6}\partial v}{\sqrt{(1-vv)}}=\frac{5}{6}\cdot\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2};$$

etc. etc.

quibus valoribus adhibitis , erit casu $v=1$

$$\int\frac{\partial v}{\sqrt{(1-v^{4})}}=\frac{\pi}{2}-\frac{1}{2^{2}}\cdot\frac{\pi}{2}+\frac{1^{2}\cdot3^{2}}{2^{2}\cdot4^{2}}\cdot\frac{\pi}{2}-\frac{1^{2}\cdot3^{2}\cdot5^{2}}{2^{2}\cdot4^{2}\cdot6^{2}}\cdot\frac{\pi}{2}+\text{etc.}$$

$$=\frac{\pi}{2}\left(1-\frac{1^{2}}{2^{2}}\cdot\frac{\pi}{2}+\frac{1^{2}\cdot3^{2}}{2^{2}\cdot4^{2}}-\frac{1^{2}\cdot3^{2}\cdot5^{2}}{2^{2}\cdot4^{2}\cdot6^{2}}+\frac{1^{2}\cdot3^{2}\cdot5^{2}\cdot7^{2}}{2^{2}\cdot4^{2}\cdot6^{2}\cdot8^{2}}-\text{etc.}\right)$$

ita ut sit ex problemate praecedente

$$1-\frac{1}{2.5}+\frac{1\cdot3}{2.4.9}-\frac{1\cdot3\cdot5}{2.4.9.13}+\text{etc.}$$

$$=\frac{\pi}{2}\left(1-\frac{1^{2}}{2^{2}}+\frac{1^{2}\cdot3^{2}}{2^{2}\cdot4^{2}}-\frac{1^{2}\cdot3^{2}\cdot5^{2}}{2^{2}\cdot4^{2}\cdot6^{2}}+\text{etc.}\right)$$

unde fit

$$\frac{\pi}{2}=\frac{1-\frac{1}{2.5}+\frac{1\cdot3}{2.4.9}-\frac{1\cdot3\cdot5}{2.4.6.13}+\text{etc.}}{1-\frac{1^{2}}{2^{2}}+\frac{1^{2}\cdot3^{2}}{2^{2}\cdot4^{2}}-\frac{1^{2}\cdot3^{2}\cdot5^{2}}{2^{2}\cdot4^{2}\cdot6^{2}}+\text{etc.}}$$

2) De integratione formulae irrationalis

$$\int \frac{x^n \partial x}{\sqrt{(aa-2bx+cxx)}}.$$

Acta Academiae Scientiar. Petropolitanae.
Tom. VI. Pars II. Pag. 62 - 67.

Problema 15.

Invenire integrale hujus formulae irrationalis

$$\int \frac{x^n \partial x}{\sqrt{(aa-2bx+cxx)}}.$$

Solutio.

§.47. Incipiamus a casu simplicissimo, quo $n=0$, quaeramus integrale formulae $\frac{\partial x}{\sqrt{(aa-2bx+cxx)}}$, quae posito $x=\frac{b+z}{c}$ transit in hanc $\frac{\partial z}{\sqrt{(aacc-bbc+czz)}}$, ubi duo casus distingui convenit, prout c fuerit vel quantitas positiva vel negativa. Sit igitur primo $c=+ff$, et formula nostra fiet $\frac{\partial z}{f\sqrt{(aaff-bb+zz)}}$, cujus integrale est $\frac{1}{f}l\frac{z+\sqrt{(aaff-bb+zz)}}{C}$ ideoque erit nostrum integrale

$$\frac{1}{\sqrt{c}}l\frac{cx-b+\sqrt{(aac-2bcx+ccxx)}}{C},$$

quod ergo ita sumtum, ut evanescat posito $x=0$, evadet

$$\frac{1}{\sqrt{c}}l\frac{cx-b+\sqrt{c(aa-2bx+cxx)}}{-b+a\sqrt{c}}.$$

At vero si c fuerit quantitas negativa, puta $c=-gg$, formula differentialis per z expressa erit $\frac{\partial z}{g\sqrt{(aagg+bb-zz)}}$, cujus integrale est

$\frac{1}{g}\text{Arc.sin.}\frac{z}{\sqrt{(aagg+bb)}}+\text{C}$; quare integrale ita sumtum , ut evanescat posito $x=0$, fiet

$$-\frac{1}{g}\text{Arc.sin.}\frac{cx-b}{\sqrt{(aagg+bb)}}+\frac{1}{g}\text{Arc.sin.}\frac{b}{\sqrt{(aagg+bb)}}.$$

§.48. Denotet nunc Π valorem formulae integralis $\int\frac{\partial x}{\sqrt{(aa-2bx+cxx)}}$ ita sumtum, ut evanescat posito $x=0$, sive c fuerit quantitas positiva sive negativa; ac si sit $c=+ff$ erit uti vidimus

$$\Pi=\frac{1}{f}l\frac{ffx-b+f\sqrt{(aa-2bx+ffxx)}}{af-b};$$

altero vero casu, quo $c=-gg$, erit

$$\Pi=-\frac{1}{g}\text{Arc.sin}\frac{ggx+b}{\sqrt{(aagg+bb)}}+\frac{1}{g}\text{Arc.sin}\frac{b}{\sqrt{(aagg+bb)}}$$

sive ambobus arcubus contractis habebimus

$$\Pi = \frac{1}{g}\text{Arc.sin}\frac{bg\sqrt{(aa-2bx-ggxx)}-abg-ag^3x}{aagg+bb}.$$

Quoniam igitur mox ostendemus, integrationem formulae generalis $\int\frac{x^n\partial x}{\sqrt{(aa-2bx+cxx)}}$ semper reduci posse ad casum $n=0$ si modo fuerit n numerus integer positivus, omnia haec integralia per istum valorem Π exprimi poterunt

§. 49. Jam post integrationem quantitati variabili x ejusmodi valorem constautem tribuamus, quo formula irrationalis

$$\sqrt{(aa-2bx+cxx)}$$

ad nihilum redigatur, id quod fit, si sumatur $x=\frac{b\pm\sqrt{(bb-aac)}}{c}$, ideoque duobus casibus. Ponamus pro utroque casu functionem Π abire in Δ, ita ut casu $c=ff$ sit

$$\Delta = \frac{1}{f}l\frac{\sqrt{(bb-aaff)}}{af-b} = \frac{1}{f}l\sqrt{\frac{b+af}{b-af}};$$

pro altero autem casu , quo $c=-gg$

$$\Delta = \frac{1}{g}\text{Arc.sin}\frac{\pm ag\sqrt{(bb+aagg)}}{aagg+bb} = \frac{1}{g}\text{Arc.sin}\frac{ag}{\sqrt{(bb+aagg)}}.$$

Hos autem valores Δ in sequentibus casibus , quibus ipsa formula radicalis $\sqrt{(aa-2bx+cxx)}$ evanescit, potissimum sumus contemplaturi.

§. 50. Nunc ad sequentem casum progressuri, consideremus formulam $s=\sqrt{(aa-2bx+cxx)}-a$, ut scilicet evanescat facto $x=0$, et quoniam est

$$\partial s = \frac{-b\partial x+cx\partial x}{\sqrt{(aa-2bx+cxx)}}$$

erit vicissim integrando

$$c\int\frac{cx\partial x}{\sqrt{(aa-2bx+cxx)}} = b\int\frac{\partial x}{\sqrt{(aa-2bx+cxx)}}+s$$

unde colligimus

$$\int\frac{x\partial x}{\sqrt{(aa-2bx+cxx)}} = \frac{b}{c}\Pi+\frac{\sqrt{(aa-2bx+cxx)}-a}{c};$$

quare si post integrationem statuamus $x=\frac{b\pm\sqrt{(bb-aac)}}{c}$, quippe quibus casibus fit $\sqrt{(aa-2bx+cxx)}=0$ et $\Pi=\Delta$, fiet

$$\int\frac{x\partial x}{\sqrt{(aa-2bx+cxx)}} = \frac{b}{c}\Delta-\frac{a}{c}$$

§.51. Sumamus porro $s=\sqrt{(aa-2bx+cxx)}$, fiet $\partial s=\frac{aa\partial x-3bx\partial x+2cxx\partial x}{\sqrt{(aa-2bx+cxx)}}$ vicissim integrando colligitur

$$2c\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}=3b\int\frac{x\partial x}{\sqrt{(aa-2bx+cxx)}}-aa\int\frac{\partial x}{\sqrt{(aa-2bx+cxx)}}+s,$$

unde statim pro casu $\sqrt{(aa-2bx+cxx)}=0$ deducimus

$$\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}=\frac{(3bb-aac)}{2cc}\Delta-\frac{3ab}{2cc}.$$

§. 52. Jam ad altiores potestates ascensuri statuamus $s=xx\sqrt{(aa-2bx+cxx)}$, et quia hinc fit

$$\partial s=\frac{2aax\partial x-5bxx\partial x+3cx^3\partial x}{\sqrt{(aa-2bx+cxx)}}\text{, erit}$$

$$3c\int\frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}}=5b\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}-2aa\int\frac{x\partial x}{\sqrt{(aa-2bx+cxx)}}+s,$$

hincque porro pro casu quo post integrationem statuitur

$$x=\frac{b\pm\sqrt{(bb-aac)}}{c}\text{, habebitur}$$

$$\int\frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{5b^3-3aabc}{2c^3}\right)\Delta-\frac{15abb}{6c^3}+\frac{2a^3}{3cc},$$

$$\text{vel}=\left(\frac{5b^3}{2c^3}-\frac{3aab}{2cc}\right)\Delta-\frac{5abb}{2c^3}+\frac{2a^3}{3cc}.$$

§.53. Simili modo si $s=x^3\sqrt{(aa-2bx+cxx)}$, et quia hinc fit

$$\partial s=\frac{3aaxx\partial x-7bx^3\partial x+4cx^4\partial x}{\sqrt{(aa-2bx+cxx)}},$$

erit vicissim integrando

$$4c\int\frac{x^4\partial x}{\sqrt{(aa-2bx+cxx)}}=7b\int\frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}}$$
$$-3aa\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}+s;$$

tum igitur pro casu quo fit $\sqrt{(aa-2bx+cxx)}=0$, habebimus

$$\int\frac{x^4\partial x}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{35}{8}\frac{b^4}{c^4}-\frac{15aabb}{4c^3}+\frac{3a^4}{8cc}\right)\Delta-\frac{35ab^3}{8c^4}+\frac{55a^3b}{24c^3}.$$

§. 54. Quo autem ordo in his formulis melius explorari possit, singulas exhibeamus per factores, quemadmodum ordine oriuntur, sine ulla abbreviatione, atque hoc modo formulae integrales inventae ita repraesententur

$$\int\frac{\partial x}{\sqrt{(aa-2bx+cxx)}}=\Delta,$$

$$\int\frac{x\partial x}{\sqrt{(aa-2bx+cxx)}}=\frac{b}{c}\Delta-\frac{a}{c},$$

$$\int\frac{xx\partial x}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{1.3bb}{1.2cc}-\frac{aa}{1.2c}\right)\Delta-\frac{1.3.ab}{1.2.cc},$$

$$\int\frac{x^3\partial x}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{1.3.5b^3}{1.2.3c^3}-\frac{1.3.5aab}{1.2.3cc}\right)\Delta-\frac{1.3.5abb}{1.2.3.c^3}+\frac{1.2.2a^3}{1.2.3cc},$$

$$\int\frac{x^4\partial x}{\sqrt{(aa-2bx+cxx)}}=\left(\frac{1.3.5.7b^3}{1.2.2.4c^4}-\frac{1.3.5.6aabb}{1.2.3.4c^4}+\frac{1.3.3a^4}{1.2.3.4cc}\right)\Delta-\frac{1.3.5.7ab^3}{1.2.3.4c^4}+\frac{1.5.11a^3b}{1.2.3.4c^3}.$$

§. 55. Instituamus nunc in genere istam evolutionem , sumendo $s=x^n\sqrt{(aa-2bx+cxx)}$ et quia hinc sit

$$\partial s=\frac{naax^{n-1}\partial x-(2n+1)bx^n\partial x+(n+1)cx^{n+1}\partial x}{\sqrt{(aa-2bx+cxx)}},$$

inde vicissim integrando colligitur

$$(n+1)c\int\frac{x^{n+1}\partial x}{\sqrt{(aa-2bx+cxx)}}=(2n+1)b\int\frac{x^n\partial x}{\sqrt{(aa-2bx+cxx)}}$$
$$-naa\int\frac{x^{n-1}\partial x}{\sqrt{(aa-2bx+cxx)}}+x^n\sqrt{(aa-2bx+cxx)}.$$

Quod si vero jam ante elicuerimus

$$\int\frac{x^{n-1}\partial x}{\sqrt{(aa-2bx+cxx)}}=\mathrm{M}\Delta-\mathfrak{M}\text{ et}$$
$$\int\frac{x^n\partial x}{\sqrt{(aa-2bx+cxx)}}=\mathrm{N}\Delta-\mathfrak{N},$$

ita ut hae duae formulae sint cognitae, sequens ex iis ita determinabitur, ut sit

$$\int\frac{x^{n+1}\partial x}{\sqrt{(aa-2bx+cxx)}}=\left[\frac{(2n+1)b\mathrm{N}}{(n+1)c}-\frac{naa\mathrm{M}}{(n+1)c}\right]\Delta$$
$$-\frac{(2n+1)b\mathfrak{N}}{(n+1)c}+\frac{naa\mathfrak{M}}{(n+1)c}.$$

Hoc igitur modo has integrationes, quousque libuerit, continuare licet, dum ex binis quibusque sequens ope hujus regulae formatur, ita ut omnia haec integralia vel a

logarithmis vel ab arcubus circularibus pendeant, prouti coefficiens c fuerit vel positivus vel negativus.
Manifestum autem est istos valores assignari non posse, nisi exponens n fuerit numerus integer positivus.

3) De integratione formulae $\int\frac{\partial x\sqrt{(1+x^4)}}{1-x^4}$, aliarumque ejusdem generis, per logarithmos et arcus circulares.

M. S. Academiae exhib. die 16 Sept. 1776.

§. 56. Cum mihi non ita pridem contigisset, integrale hujus formulae $\int\frac{\partial x\sqrt{(1+x^4)}}{1-x^4}$ per arcum circularem et logarthmum exprimere, haec integratio eo magis mihi visa est notatu digna, quod nullo modo perspiciebam, eam ad rationalitatem perduci posse , quandoquidem certum est, istam formulam, quae simplicior videatur, $\int\partial x\sqrt{(1+x^4)}$, neutiquam ad rationalitatem revocari posse, neque enim videbam, accessionem denominatoris $1-x^4$ hanc reductionem promovere posse, hincque concludebam dari ejusmodi formulas differentiales irrationales, quarum integralia per logarithmos et arcus circulares exhibere liceat, etiamsi nulla substitutione ab irrationalitate liberari queant: quaequidem conclusio utique valet pro formulis compositis, quanquam enim istae formulae

$$\int\frac{\partial x}{\sqrt[3]{(1+x^3)}} \text{ et } \int\frac{\partial x}{\sqrt[4]{(1+x^4)}}$$

ad rationalitatem reduci possunt, tamen formula ex iis composita

$$\int\partial x\left[\frac{A}{\sqrt[3]{(1+x^3)}}+\frac{B}{\sqrt[4]{(1+x^4)}}\right]$$

per nullam plane substitutionem ad aliam formulam rationalem reduci potest ; propterea quod utraque pars peculiarem substitutionem postulat.

§. 57. Interim tamen cum formulam propositam

$$\int\frac{\partial x\sqrt{(1+x^4)}}{1-x^4}=S$$

attentius essem contemplatus, inveni, eam ab irrationalitate liberari posse, ope hujus substitutionis prorsus singularis

$$x=\frac{\sqrt{(1+tt)}+\sqrt{(1-tt)}}{t\sqrt{2}}.$$

Hinc enim fit

$$\partial x=-\frac{\partial t}{tt\sqrt{2(1+tt)}}-\frac{\partial t}{tt\sqrt{2(1-tt)}}$$

quae duae partes ad eundem denominatorem reductae dant

$$\partial x=-\frac{\partial t}{tt\sqrt{2\left(1-t^4\right)}}\left[\sqrt{\left(1-tt\right)}+\sqrt{\left(1+tt\right)}\right].$$

Cum igitur sit

$$\sqrt{\left(1+tt\right)}+\sqrt{\left(1-tt\right)}=tx\sqrt{2},$$

hoc valore substituto fiet

$$\partial x=-\frac{x\partial t}{t\sqrt{\left(1-t^4\right)}},$$

ita ut sit

$$\partial \mathrm{S}=-\frac{x\partial t\sqrt{\left(1+x^4\right)}}{t\left(1-x^4\right)\sqrt{\left(1-t^4\right)}}.$$

§. 58. Porro autem sumtis quadratis erit

$$xx=\frac{1+\sqrt{\left(1-t^4\right)}}{tt},$$

unde colligimus

$$1+xx=\frac{1+tt+\sqrt{\left(1-t^4\right)}}{tt}=\frac{\sqrt{(1+tt)}}{tt}\left[\sqrt{\left(1+tt\right)}+\sqrt{\left(1-tt\right)}\right],$$

sicque ob

$$\sqrt{\left(1+tt\right)}+\sqrt{\left(1-tt\right)}=tx\sqrt{2},\ \text{erit}$$

$$1+xx=\frac{x\sqrt{2}(1+tt)}{t}.$$

Simili modo erit

$$1-xx=-\left(\frac{1-tt+\sqrt{\left(1-t^4\right)}}{tt}\right)$$

$$=-\frac{\sqrt{(1-tt)}}{tt}\left[\sqrt{\left(1-tt\right)}+\sqrt{\left(1+tt\right)}\right]=-\frac{x\sqrt{2}(1-tt)}{t}.$$

Hinc igitur sequitur fore

$$1-x^4=-\frac{2xx\sqrt{\left(1-t^4\right)}}{tt},$$

qui valor in nostra formula substitutus praebet

$$\partial \mathrm{S}=+\frac{t\partial t\sqrt{\left(1+x^4\right)}}{2x\left(1-t^4\right)}.$$

§. 59. Deinde sumtis quadratis habebimus

$$(1+xx)^2 = \frac{2xx(1+tt)}{tt} \text{ et}$$
$$(1-xx)^2 = \frac{2xx(1-tt)}{tt}$$

quibus additis prodibit

$$(1+xx)^2 + (1-xx)^2 = 2\left(1+x^4\right) = \frac{4xx}{tt},$$

unde fit

$$\sqrt{\left(1+x^4\right)} = \frac{x\sqrt{2}}{t};$$

quo valore substituto nostra formula abit in hanc:

$$\partial S = \frac{1}{\sqrt{2}} \cdot \frac{\partial t}{1-t^4};$$

quae ergo formula est rationalis et solam variabilem t complectitur.

§. 60. Cum igitur porro sit

$$\frac{1}{1-t^4} = \frac{1}{2} \cdot \frac{1}{1+tt} + \frac{1}{2} \cdot \frac{1}{1-tt}$$

tum vero integrando reperiatur

$$\int \frac{\partial t}{1+tt} = \text{Arc.tang.}t \text{ et}$$
$$\int \frac{\partial t}{1-tt} = \frac{1}{2} l \frac{1+t}{1-t} = l \frac{1+t}{\sqrt{1-tt}},$$

quibus valoribus substitutis reperietur

$$\mathrm{S} = \frac{1}{2\sqrt{2}} \text{Arc.tang.}t + \frac{1}{2\sqrt{2}} l \frac{1+t}{\sqrt{1-tt}}.$$

Quare cum regrediendo sit $t = \frac{x\sqrt{2}}{\sqrt{\left(1+x^4\right)}}$, supra autem invenerimus

$$1+x^4 = \frac{2xx}{tt}, \text{ erit } tt = \frac{2xx}{1+x^4},$$

hinque

$$1-tt = \frac{(1-xx)^2}{1+x^4}, \text{ ideoque } \sqrt{1-tt} = \frac{1-xx}{\sqrt{\left(1+x^4\right)}},$$

his valoribus substitutis , integrale quaesitum per ipsam variabilem x sequenti modo exprimetur

$$\int \frac{\partial x\sqrt{1+x^4}}{1-x^4} = \frac{1}{2\sqrt{2}} \text{Arc.tang.} \frac{x\sqrt{2}}{\sqrt{1+x^4}} + \frac{1}{2\sqrt{2}} l \frac{x\sqrt{2}+\sqrt{1+x^4}}{1-xx}.$$

§. 61. Hic autem merito quaeretur, quonam artificio ad substitutionem illam, quae primo intuitu a scopo prorsus aliena videtur pertigerim [pertingerem] ? quandoquidem nemo certe in eam incidisset, neque etiam ipse memini, quanam ratione ad eam sim perductus. Verum postquam omnia momenta accuratius perpendissem , methodum multo planiorem detexi, qua istud negotium sine tot ambagibus absolvi potest, quam igitur hic perspicue proponi conveniet.

Methodus planior et magis naturalis, formulam integralem propositam tractandi.

§. 62. Quo ex formula $\partial S = \frac{\partial x\sqrt{1+x^4}}{1-x^4}$ irrationalitatam saltem apparenter tollamur, ponamus $\sqrt{1+x^4} = px$, ut fiat $\partial S = \frac{px\partial x}{1-x^4}$. Cum igitur sit $1+x^4 = ppxx$, erit radicem extrahendo

$$xx = \tfrac{1}{2}pp + \sqrt{\tfrac{1}{4}p^4 - 1} ,$$

Ponatur hic $\frac{1}{2}pp = q$, ut habeamus

$$xx = q + \sqrt{qq-1}, \text{ et}$$
$$2lx = l\left[q + \sqrt{(qq-1)}\right],$$

hincque differentiando $\frac{2\partial x}{x} = \frac{\partial q}{\sqrt{(qq-1)}}$: ergo loco q restitio valore $\frac{1}{2}pp$, erit $\frac{2\partial x}{x} = \frac{2p\partial p}{\sqrt{(p^4-4)}}$, sicque fiet $\partial x = \frac{xp\partial p}{\sqrt{(p^4-4)}}$, quo valore substituto fit $\partial S = \frac{p^2x^2\partial p}{(1-x^4)\sqrt{(p^4-4)}}$.

§. 63. Ut nunc hinc quantitatem x penitus ejiciamus, quoniam invenimus

$$xx = \frac{pp+\sqrt{p^4-4}}{2}, \text{ erit}$$
$$x^4 = \frac{p^4-2+pp\sqrt{p^4-4}}{2}, \text{hincque}$$
$$1-x^4 = \frac{4-p^4-pp\sqrt{p^4-4}}{2} = -\frac{\sqrt{p^4-4}\left[pp+\sqrt{p^4-4}\right]}{2},$$

Unde colligitur fore $\frac{xx}{1-x^4} = -\frac{1}{\sqrt{p^4-4}}$, quo valore substituto impetramus formulam differentialem rationalem per novam variabilem p expressam , quae est

$$\partial S = -\frac{pp\partial p}{p^4-4}, \text{ existente } p = \frac{\sqrt{1+x^4}}{x},$$

unde idem integrale, quod ante nacti sumus, deducitur. Similis autem substitutio cum successu adbiheri potest in formulis integralibus multo magis generalibus; veluti in sequente problemate ostendemus.

Problema 16.

§. 64. *Propositam formulam integralem* $S=\int\frac{\partial x\sqrt{\left(a+bxx+cx^4\right)}}{a-cx^4}$ *ope idoneae substitutionis ab omni irrationalitate liberare.*

Solutio.

Ad speciem saltem irrationalitatis tollendam, ponamus

$$\sqrt{\left(a+bxx+cx^4\right)}=px,$$

ut habeamus $S=\int\frac{px\partial x}{a-cx^4}$. Cum igitur sit

$$p=\frac{\sqrt{\left(a+bxx+cx^4\right)}}{x},\ \text{erit}$$

$$\partial p=-\frac{a\partial x+cx^4\partial x}{xx\sqrt{\left(a+bxx+cx^4\right)}}=-\frac{a\partial x+cx^4\partial x}{px^3},$$

unde erit

$$\partial x=\frac{px^3\partial p}{a-cx^4}$$

quo valore substituto fiet

$$\partial S=-\frac{ppx^4\partial p}{\left(a-cx^4\right)^2}.$$

§. 65. Deinde cum sit

$$a+cx^4=\left(pp-b\right)xx,$$

hincque porro

$$\left(a+cx^4\right)^2=\left(pp-b\right)^2x^4,$$

aufferatur $4acx^4$, ac remanebit

$$\left(a-cx^4\right)^2=\left[\left(pp-b\right)^2-4ac\right]x^4,$$

quo substituto formula nostra fiet

$$\partial S=-\frac{pp\partial p}{\left(pp-b\right)^2-4ac},$$

Sicque quantitas variabilis *x* penitus e calculo est extrusa , ac deducti sumus ad formulam differentialem prorsus rationalem, cujus ergo integratio per logarithmos et arcus circulares nulla amplius laborat difficultate. Quin etiam formulae adhuc generaliores eodem modo feliciter tracturi poterunt.

Problema l7.

§. 66. *Propositam hanc formulam integralem*

$$S=\int\frac{x^{n-2}\partial x\sqrt[n]{\left(a+bx^{n}+cx^{2n}\right)}}{a-cx^{2n}}$$

ope idoneae substitutionis ab omni irrationalitate liberare.

Solutio.

Utamur igitur hac substitutione

$$\sqrt[n]{\left(a+bx^{n}+cx^{2n}\right)}=px\,,$$

ut formula proposita hanc induat formam

$$\partial S=\frac{px^{n-1}\partial x}{a-cx^{2n}};$$

tum vero cum sit

$$p^{n}=\frac{a+bx^{n}+cx^{2n}}{x^{n}},$$

erit differentiando

$$p^{n-1}\partial p=-\frac{\partial x\left(a-cx^{2n}\right)}{x^{n+1}},$$

unde fit

$$\partial x=-\frac{p^{n-1}x^{n+1}\partial p}{a-cx^{2n}},$$

quo valore substituto formula nostra induet hanc formam

$$\partial S=-\frac{p^{n}x^{2n}\partial p}{\left(a-cx^{2n}\right)^{2}}.$$

§. 67. Deinde cum sit

$$a+cx^{2n}=\left(p^{n}-b\right)x^{n},\ \text{erit}$$

$$\left(a+cx^{2n}\right)^{2}=\left(p^{n}-b\right)^{2}x^{2n};$$

hinc subtrahatur $4acx^{2n}$, et remanebit

$$\left(a-cx^{2n}\right)^{2}=\left[\left(p^{n}-b\right)^{2}-4ac\right]x^{2n},$$

substituto igitur hoc valore fiet

$$\partial S=-\frac{p^{n}\partial p}{\left(p^{n}-b\right)^{2}-4ac},$$

quae ergo omnino est rationalis , atque adeo integratio per logarithmos et arcus circulares facile expeditur.

Problema l8.

§. 68. *lnvenire formulas integrales adhuc generaliores, quae ope substitutionis*

$$\sqrt[n]{\left(a+bx^n+cx^{2n}\right)}=px$$

ad rationalitatem perduci queant.

Solutio.

Quoniam in praecedente problemate invenimus , hanc formulam differentialem

$$\frac{x^{n-2}\partial x\sqrt[n]{\left(a+bx^n+cx^{2n}\right)}}{a-cx^{2n}}$$

ope hujus substitutionis reduci ad istam formulam rationalem

$$-\frac{p^n\partial p}{\left(p^n-b\right)^2-4ac}, \text{ erit}$$

$$\frac{\mathrm{P}x^{n-2}\partial x\sqrt[n]{\left(a+bx^n+cx^{2n}\right)}}{a-cx^{2n}}=-\frac{\mathrm{P}p^n\partial p}{\left(p^n-b\right)^2-4ac}$$

ubi loco P functiones quaecunque ipsius x accipi possunt ejusmodi, ut facta substitutione praebeant functiones rationales ipsius p, id quod infinitis modis fieri poterit, quorum praecipuos hic percurramus.

§. 69. Cum vi substitutionis sit

$$\frac{\sqrt[n]{\left(a+bx^n+cx^{2n}\right)}}{x}=p$$

loco P potestas quaecunque ipsius p assumi poterit, quae sit p^λ.

Sumatur igitur $\mathrm{P}=p^\lambda\mathrm{Q}$, eritque etiam

$$\mathrm{P}=\frac{Q\sqrt[n]{\left(a+bx^n+cx^{2n}\right)^\lambda}}{x^\lambda};$$

quibus valoribus substitutis prodibit ista aequatio

$$\mathrm{P}=\frac{Qx^{n-\lambda-2}\partial x\sqrt[n]{\left(a+bx^n+cx^{2n}\right)^{\lambda+1}}}{a-cx^{2n}}=-\frac{Qp^{n+\lambda}\partial p}{\left(p^n-b\right)^2-4ac}$$

quae posterior formula denuo est rationalis.

§. 70. Deinde in praecedentc problemate quoque invenimus esse

$$\frac{\left(a-cx^{2n}\right)^2}{x^{2n}}=\left(p^n-b\right)^2-4ac$$

quam ob rem pro Q sumamus potestatem exponentis *i* harum quantitatum, vel potius harum qùantitatum reciprocam, scilicet capiatur

$$\mathrm{Q}=\frac{x^{2in}}{\left(a-cx^{2n}\right)^{2i}}=\frac{1}{\left[\left(p^n-b\right)^2-4ac\right]^i}$$

Quibus valoribus substitutis obtinebimus formulam latissime patentem hanc

$$\frac{x^{(2i+1)n-\lambda-2}\partial x\sqrt[n]{\left(a+bx^n+cx^{2n}\right)^{\lambda+1}}}{\left(a-cx^{2n}\right)^{2i+1}}=-\frac{p^{n+\lambda}\partial p}{\left[\left(p^n-b\right)^2-4ac\right]^{i+1}};$$

ubi pro litteris λ et *i* numeros quoscunque integros sive positivos sive negativos accipere licet, perpetuo enim formula differentialis per *p* expressa manebit rationalis.

§. 71. Quin etiam haec reductio multo generalior reddi potest, propterea quod necessum non est ut λ sit numerus integer : Quaecunque enim fractio pro λ assumatur, formula per *p* expressa semper facile ad rationalitatem reduci poterit. Si enim ponamus $\lambda=\frac{\mu}{\nu}$, membrum dextrum fiet

$$-\frac{p^{\frac{\nu n+\mu}{\nu}}\partial p}{\left[\left(p^n-b\right)^2-4ac\right]^{i+1}},$$

quae rationalis redditur ponendo $p=q^\nu$, erit enim $\partial p=\nu q^{\nu-1}\partial q$, ideoque hoc membrum

$$-\frac{\nu p^{\mu+\nu n+\nu-1}\partial q}{\left[\left(p^{\nu n}-b\right)^2-4ac\right]^{i+1}}.$$

Nunc autem uti oportebit hac substitutione

$$\sqrt[n]{\left(a+bx^n+cx^{2n}\right)}=q^\nu x,$$

atque habebitur ista reductio

$$\frac{x^{(2i+1)n-\frac{\mu}{\nu}-2}\partial x\sqrt[n]{\left(a+bx^n+cx^{2n}\right)^{\frac{\mu+\nu}{\nu}}}}{\left(a-cx^{2n}\right)^{2i+1}}$$

$$=-\frac{\nu q^{\mu+\nu n+\nu-1}\partial q}{\left[\left(q^{\nu n}-b\right)^2-4ac\right]^{i+1}},$$

quae postrema formula utique est rationalis.

§. 72. Ut etiam in membro sinistro exponentes fractos ipsius *x* tollamus , ponamus $x = y^{\nu}$, eritque

$$\frac{y^{(2i+1)n\nu-\mu-\nu-1}\partial y\sqrt[n]{\left(a+by^{n\nu}+cy^{2n\nu}\right)^{\mu+\nu}}}{\left(a-cy^{2n\nu}\right)^{2i+1}} = -\frac{q^{\mu+\nu n+\nu-1}\partial q}{\left[\left(q^{n\nu}-b\right)^2-4ac\right]^{i+1}},$$

quae expressio autem multo generalior videtur , quam revera est.
Si enim loco *nν* ubique scribamus *n* resultat ista aequatio

$$\frac{y^{(2i+1)n-\mu-\nu-1}\partial y\sqrt[n]{\left(a+by^{n}+cy^{2n}\right)^{\mu+\nu}}}{\left(a-cy^{2n}\right)^{2i+1}} = -\frac{q^{\mu+\nu+n-1}\partial q}{\left[\left(q^{n}-b\right)^2-4ac\right]^{i+1}};$$

haec autem aequatio manifesto non discrepat ab illa §. 70. allata ; si enim hic loco $\mu+\nu-1$ scribamus λ et loco *y* et *q* ut ante *x* et *p*, ipsa praecedens aequatio reperitur , sicque sufficiet loco λ numeros integros assumere.

Corollarium 1.

§. 73. Quo clarius indoles harum formularum perspiciatur, sumamus $n = 2$, et formula differentialis variabilem *x* involvens erit

$$\frac{x^{4i-\lambda}\partial x\sqrt{\left(a+bxx+cx^4\right)^{\lambda+1}}}{\left(a-cx^4\right)^{2i+1}}, = -\frac{q^{\mu+\nu+n-1}\partial q}{\left[\left(q^{n}-b\right)^2-4ac\right]^{i+1}};$$

quae facta substitutione $\left(a+bxx+cx^4\right) = px,$ transmutatur in hanc rationalem

$$-\frac{p^{\lambda+2}\partial p}{\left[(pp-b)^2-4ac\right]^{i-1}},$$

unde sumendo $\lambda = 4i$ resultat ista aequatio

$$\frac{\partial x\sqrt{\left(a+bxx+cx^4\right)^{4i+1}}}{\left(a-cx^4\right)^{ci+1}}=-\frac{p^{4i+2}\partial p}{\left[(pp-b)^2-4ac\right]^{i+1}},$$

in qua si porro ponatur $i=0$, fiet

$$\frac{\partial x\sqrt{\left(a+bxx+cx^4\right)}}{\left(a-cx^4\right)}=-\frac{pp\partial p}{(pp-b)^2-4ac};$$

quae si insuper ponatur $a=1,\ b=0$ et $c=1$, praebet

$$\frac{\partial x\sqrt{\left(1+x^4\right)}}{1-x^4}=-\frac{pp\partial p}{p^4-4},$$

quae est ipsa reductio, quae supra §. 63. fuerat inventa.

Corollarium 2.

§. 74. Si sumamus $n=3$, prodibit ista reductio generalis

$$\frac{x^{6i-\lambda+1}\partial x\sqrt[3]{\left(a+bx^3+cx^6\right)^{\lambda+1}}}{\left(a-cx^6\right)^{2i+1}}=-\frac{p^{\lambda+3}\partial p}{\left[\left(p^3-b\right)^2-4ac\right]^{i+1}};$$

quae ponendo $i=0$ migrat in hanc

$$\frac{x^{-\lambda+1}\partial x\sqrt[3]{\left(a+bx^3+cx^6\right)^{\lambda+1}}}{a-cx^6}=-\frac{p^{\lambda+3}\partial p}{\left(p^3-b\right)^2-4ac};$$

posito vero $b=0$, haec prodit formula concinnior

$$\frac{x^{-\lambda+1}\partial x\sqrt[3]{\left(a+cx^6\right)^{\lambda+1}}}{a-cx^6}=-\frac{p^{\lambda+3}\partial p}{p^6-4ac};$$

cujus duos casus evolvisse juvabit.

I. Sit $\lambda=0$, eritque

$$\frac{x\partial x\sqrt[3]{\left(a+cx^6\right)}}{a-cx^6}=-\frac{p^3\partial p}{p^6-4ac};$$

quae concinnior redditur ponendo $xx=y$, reperietur enim

$$\frac{\partial y\sqrt[3]{\left(a+cy^3\right)}}{a-cy^3}=-\frac{2p^3\partial p}{p^6-4ac}.$$

II. Sumto autem $\lambda = 1$, ista prodit expressio

$$\frac{\partial x\sqrt[3]{\left(a+cx^6\right)^2}}{a-cx^6} = -\frac{p^4\partial p}{p^6-4ac}.$$

Scholion.

§. 75. Ex his exemplis satis intelligitur, quam egregie reductiones ex nostris formulis generalibus deduci queant, quarum resolutio, nisi methodus nostra adhibeatur, omnes vires analyseos superare videatur.

4.) Memorabile genus formularum differentialium maxime irrationalium, quas tamen ad rationalitatem perducere licet.

M. S. Academiae exhib. d. 15. Maii 1777.

§. 76. Cum nuper hanc formulam differentialem

$$\frac{\partial x}{(1-xx)\sqrt[4]{(2xx-1)}}$$

tractassem eamque singulari modo ad rationalitatem perduxissem, mox vidi eandem methodum succedere in hac generaliori

$$\frac{\partial x}{(a+bxx)\sqrt[4]{(a+2bxx)}},$$

atque adeo in hac multo generaliori

$$\frac{\partial x}{\left(a+bx^n\right)\sqrt[2n]{\left(a+2bx^n\right)}},$$

ubi irrationalitas ad ordinem quantumvis altum assurgere potest,
cujus resolutio sequenti modo instituitur.

§. 77. Utor scilicet hac substitutione $\frac{x}{\sqrt[2n]{\left(a+2bx^n\right)}} = Z$, ut formula nostra integranda , quam per ∂V indicemus, fiat $\partial V = \frac{\partial x}{x}\cdot\frac{Z}{a+bx^n}$, sumtis ergo logarithmis erit

$$lZ = lx - \tfrac{1}{2n}l\left(a+2bx^n\right),$$

unde differentiando fit

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{bx^{n-1}\partial x}{a+2bx^n} = \frac{\partial x\left(a+bx^n\right)}{x\left(a+2bx^n\right)},$$

erit ergo

$$\frac{\partial x}{x} = \frac{\partial Z\left(a+2bx^n\right)}{Z\left(a+bx^n\right)},$$

hinc ergo nostra formula erit

$$\partial V = \frac{\partial Z\left(a+2bx^n\right)}{\left(a+bx^n\right)^2}.$$

Cum igitur sit

$$Z^{2n} = \frac{x^{2n}}{a+2bx^n}, \text{ erit } a+2bx^n = \frac{x^{2n}}{Z^{2n}},$$

ideoque

$$\partial V = \frac{x^{2n}\partial Z}{Z^{2n}\left(a+bx^n\right)^2}.$$

Cum porro sit $aa+2abx^n = \frac{ax^{2n}}{Z^{2n}}$, addatur utrinque bbx^{2n},
et prodibit

$$\left(a+bx^n\right)^2 = \frac{ax^{2n}}{Z^{2n}} + bbx^{2n} = \frac{x^{2n}\left(a+bbZ^{2n}\right)}{Z^{2n}},$$

quo valore substituto nostra formula evadet

$$\partial V = \frac{\partial Z}{a+bbZ^{2n}},$$

quae ergo formula est rationalis, ideoque per logarithmos et arcus circulares integrari poterit.

§.78. Observavi porro, cum hic post signum radicale tantum binomium involvatur , ejus loco quoque trinomia, atque adeo polynomia introduci posse. Pro trinomiis autem formula differentialis talem habebit formam

$$\partial V = \frac{\partial x}{\left(a+bx^n\right)\sqrt[3n]{\left(aa+3abx^n+3bbx^{2n}\right)}},$$

ubi ergo irrationalitas ad ordinem multo altiorem ascendit. Nihilo vero minus etiam ista formula ab irrationalitate liberari poterit ope similis substitutionis

$$Z = \frac{x}{\sqrt[3n]{\left(aa+3abx^n+3bbx^{2n}\right)}};$$

hinc enim sumtis logarithmis per differentiationem nanciscemur

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{abx^{n-1}\partial x - 2bbx^{2n-1}\partial x}{aa+3\ abx^n+3bbx^{2n}}, \text{ seu}$$

$$\frac{\partial Z}{Z} = \frac{\partial x\left(a+bx^n\right)^2}{x\left(aa+3\ abx^n+3bbx^{2n}\right)},$$

ideoque

$$\frac{\partial x}{x} = \frac{\partial Z}{Z} \cdot \frac{aa+3abx^n+3bbx^{2n}}{\left(a+bx^n\right)^2}.$$

Cum igitur nostra formula jam sit $\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a+bx^n}$, introducto elemento ∂Z, obtinebimus

$$\partial V = \frac{\partial Z\left(aa+3abx^n+3bbx^{2n}\right)}{\left(a+bx^n\right)^2}.$$

§. 79. Cum igitur vi substitutionis sit

$$\left(aa+3abx^n+3bbx^{2n}\right) = \frac{x}{Z}, \text{ erit}$$

$$aa+3abx^n+3bbx^{2n} = \frac{x^{3n}}{Z^{3n}}.$$

Multiplicetur utrinque per a, et addatur utrinque b^3x^{3n}, eritque

$$\left(a+bx^n\right)^3 = \frac{x^{3n}\left(a+b^2Z^{3n}\right)}{Z^{3n}}:$$

hoc igitur valore substituto ex formula nostra littera x penitus excludetur, prodibitque $\partial V = \frac{\partial Z}{a+b^3Z^n}$. Cujus ergo integrale semper per logarithmos et arcus circulares reperire licebit.

§. 80. Pro quadrinomiis autem ponamus brevitatis gratia

$$\sqrt[4n]{\left(a^3+4aabx^n+6abbx^{2n}+4b^3x^{3n}\right)} = S,$$

ac formula ad rationalitatem reducenda proponatur haec

$$\partial V = \frac{\partial x}{\left(a+bx^n\right)S},$$

id quod simili modo succedet ope hujus substitutionis $\frac{x}{S} = Z$, unde formula nostra erit $\partial V = \frac{\partial x}{x} \cdot \frac{Z}{a+bx^n}$. Cum nunc sit

$$\frac{\partial S}{S} = \frac{aabx^{n-1}\partial x+3aabx^{2n-1}\partial x+3b^3x^{3n-1}\partial x}{a^3+4aabx^n+6abbx^{2n}+4b^3x^{3n}},$$

sive

$$\frac{\partial S}{S} = \frac{\partial x}{x} \cdot \frac{bx^n\left(aa+3abx^n+3bbx^{2n}\right)}{S^{4n}},$$

erit $\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S}$; consequenter

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} \cdot \frac{\left(a+bx^n\right)^4}{S^{4n}}, \text{ hincque } \frac{\partial x}{x} = \frac{S^{4n}\partial Z}{Z\left(a+bx^n\right)^4},$$

quo valore substituto formula nostra erit

$$\partial V=\frac{S^{4n}\partial Z}{\left(a+bx^n\right)^4}.$$

§. 81. Cum autem sit

$$S^{4n}=a^3+4aabx^n+6abbx^{2n}+4b^3x^{3n}, \text{ erit}$$

$$aS^{4n}+b^4x^{4n}=\left(a+bx^n\right)^4,$$

quo valore substituto erit

$$\partial V=\frac{S^{4n}\partial Z}{aS^{4n}+b^4x^{4n}}:$$

quia igitur posuimus $Z=\frac{x}{s}$, erit $S=\frac{x}{Z}$, ideoque $S^{4n}=\frac{x^{4n}}{Z^{4n}}$, qui valor surrogatus dabit

$$\partial V=\frac{\partial Z}{a+b^4Z^{4n}},$$

sicque itidem ad rationalitatem est perducta.

§. 82. Hinc jam facile intelligitur, quo modo pro omnibus polynomiis formulae differentiales comparatae esse debeant, ut tali substitutione ad rationalitatem perduci queant, id quod in sequente problemate expediamus

Problema 19

§. 83. *Si proposita fuerit haec formula differentialis*

$$\partial V=\frac{\partial x}{\left(a+bx^n\right)\sqrt[\lambda n]{\left(\left(a+bx^n\right)^\lambda-b^\lambda x^{\lambda n}\right)}},$$

eam ad rationalitatem reducere , quantumvis magni numeri pro n et λ accipiantur.

Solutio.

Ponamus etiam hic brevitatis gratia

$$\sqrt[\lambda n]{\left(\left(a+bx^n\right)^\lambda-b^\lambda x^{\lambda n}\right)}=S,$$

ut formula fiat

$$\partial V=\frac{\partial x}{\left(a+bx^n\right)S},$$

fiatque insuper $\frac{x}{S}=Z$, ut habeamus

$$\partial V=\frac{\partial x}{x}\cdot\frac{Z}{a+bx^n}.$$

Jam logarithmos differentiando reperietur

$$\frac{\partial S}{S} = \frac{bx^{n-1}\partial x\left(a+bx^n\right)^{\lambda-1} - b^\lambda x^{\lambda n-1}\partial x}{\mathrm{S}^{\lambda n}}, \text{ sive}$$

$$\frac{\partial \mathrm{S}}{\mathrm{S}} = \frac{\partial x}{x}\cdot\frac{bx^n\left(a+bx^n\right)^{\lambda-1} - b^\lambda x^{\lambda n}}{\mathrm{S}^{\lambda n}}.$$

Cum igitur sit $\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S}$, hoc valore substituto erit

$$\frac{\partial \mathrm{Z}}{\mathrm{Z}} = \frac{\partial x}{x}\cdot\frac{a\left(a+bx^n\right)^{\lambda-1}}{\mathrm{S}^{\lambda n}},$$

hincque vicissim erit

$$\frac{\partial x}{x} = \frac{\mathrm{S}^{\lambda n}\partial \mathrm{Z}}{a\mathrm{Z}\left(a+bx^n\right)^{\lambda-1}},$$

quo valore substituto impetramus

$$\partial \mathrm{V} = \frac{\mathrm{S}^{\lambda n}\partial Z}{a\left(a+bx^n\right)^{\lambda}},$$

quia nunc est $\left(a+bx^n\right)^{\lambda} = \mathrm{S}^{\lambda n} + b^\lambda x^{\lambda n}$, erit

$$\partial \mathrm{V} = \frac{\mathrm{S}^{\lambda n}\partial Z}{a\left(\mathrm{S}^{\lambda n} + b^\lambda x^{\lambda n}\right)}.$$

Denique ob $\mathrm{S} = \frac{x}{\mathrm{Z}}$, ideoque $\mathrm{S}^{\lambda n} = \frac{x^{\lambda n}}{\mathrm{Z}^{\lambda n}}$, hoc valore substituto obtinebitur

$$\partial \mathrm{V} = \frac{\partial Z}{a\left(1+b^\lambda Z^{\lambda n}\right)},$$

quae est rationalis unicam variabilem Z involvens, cujus adeo integrale per logarithmos et arcus circulares assignari poterit.

Corollarium 1.

§. 84. Eadem solutio etiam locum habet, si pro λ numeri fracti accipiantur, qua ratione post signum radicale denuo radicalia involvuntur : ita si fuerit $\lambda = \frac{2}{n}$, erit formula radicalis

$$\mathrm{S} = \sqrt{\left(a+bx^n\right)^{\frac{2}{\lambda}} - b^{\frac{2}{\lambda}}xx},$$

et formulae nostrae

$$\partial \mathrm{V} = \frac{\partial x}{a\left(1+bx^n\right)\mathrm{S}}$$

integrale erit

$$\mathrm{V}=\frac{1}{a}\int\frac{\partial Z}{1+b^{\frac{2}{n}}ZZ}=\frac{1}{ab^{\frac{1}{n}}}\,\text{Arc.tang.}\,b^{\frac{1}{n}}Z.$$

Corollarium 2.

§. 85. Quo haec clariora reddantur, capiamus $a=i,\ b=i,$ et $n=4,$ ut pro postremo casu sit

$$\mathrm{S}=\sqrt{\left(1+x^4\right)^{\frac{1}{2}}-xx},\ \text{et}\ \partial V=\frac{\partial x}{\left(1+x^4\right)\sqrt{\left(1+x^4\right)^{\frac{1}{2}}-xx}},$$

cujus integrale posito

$$Z=\frac{x}{\sqrt{\left(1+x^4\right)^{\frac{1}{2}}-xx}},\ \text{erit}$$

$$\mathrm{V}=\text{Arc.tang. Z, sive V}=\frac{x}{\sqrt{\left(1+x^4\right)^{\frac{1}{2}}-xx}}.$$

Sin autem manente $n=4$ et $a=1,$ fuerit $b=-1$, ideoque

$$\mathrm{S}=\sqrt{\left(1-x^4\right)^{\frac{1}{2}}-xx\sqrt{-1}},$$

ipsa formula prodiret imaginaria.

Corollarium 3.

§. 86. Pro eodem casu , eritque $\lambda=\frac{2}{n},$ sit $n=6$, et $a=1$ et $b=1$, eritque

$$\mathrm{S}=\sqrt{\left(1+x^6\right)^{\frac{1}{3}}-xx},\quad \text{ideoque}$$

$$\partial\mathrm{V}=\frac{\partial x}{\left(1+x^6\right)\sqrt{\left(1+x^6\right)^{\frac{1}{3}}-xx}},$$

Cujus integrale posito $\frac{x}{\mathrm{S}}=\mathrm{Z}$, erit

$$\mathrm{V}=\text{Arc.tang.Z}=\text{Arc.tang.}\frac{x}{\sqrt{\left(1+x^6\right)^{\frac{1}{3}}-xx}}.$$

Similique modo alia hujus generis exempla pro lubitu formari possunt; verum quamquam formula problematis admodum est generalis , tamen adhuc multo magis generalior fieri potest , uti in sequente problemate sumus ostensuri.

Problema 20.

§. 87. *Si proponatur ista formula differentialis multo generalior, quippe in qua tres occurrunt exponentes indeterminati λ, n, et m,*

$$\partial V = \frac{x^{m-1}\partial x}{\left(a+bx^n\right)\left(\sqrt[\lambda n]{\left(\left(a+bx^n\right)^{\lambda}-b^{\lambda}x^{\lambda n}\right)}\right)^m},$$

eam ab irrationalitate liberare.

Solutio.

Ponatur iterum brevitatis gratia

$$\sqrt[\lambda n]{\left(\left(a+bx^n\right)^{\lambda}-b^{\lambda}x^{\lambda n}\right)} = S,$$

ut formula integranda proposita fiat

$$\partial V = \frac{x^{m-1}\partial x}{\left(a+bx^n\right)S^m} = \frac{\partial x}{x}\cdot\frac{x^m}{\left(a+bx^n\right)S^m},$$

quae ergo si porro ut ante statuamus $\frac{x}{S} = Z$, fiet

$$\partial V = \frac{\partial x}{x}\cdot\frac{Z^m}{a+bx^n},$$

unde variabilem x penitus eliminari oportet. Quoniam nunc ambae litterae S et Z eosdem habent valores, ut in problemate praecedente atque adeo ipsa formula ∂V oriatur , si praecedens per Z^{m-1} multiplicetur, etiam integrale quaesitum obtinebimus, dum superius integrale per Z^{m-1} multiplicabimus, quo facto erit integrale quaesitum

$$V = \frac{1}{a}\int\frac{Z^{m-1}\partial Z}{1+b^{\lambda}Z^{\lambda n}}.$$

Corollarium 1.

§. 88. Si exponentem m negativum capiamus, irrationalitas in numeratorem transferetur, ita posita $m=-1$ habebimus

$$\partial V = \frac{\partial x\sqrt[\lambda n]{\left(\left(a+bx^n\right)^{\lambda}-b^{\lambda}x^{\lambda n}\right)}}{xx\left(a+bx^n\right)},$$

cujus ergo integrale per Z expressum erit

$$V = \frac{1}{a}\int\frac{\partial Z}{ZZ\left(a+b^{\lambda}Z^{\lambda n}\right)}.$$

Quin etiam per hunc exponentem m irrationalitas simplicior reddi poterit, veluti si sumamus $m=\lambda$, erit

$$\partial V = \frac{x^{\lambda-1}\partial x}{\left(a+bx^n\right)\sqrt[n]{\left(\left(a+bx^n\right)^{\lambda}-b^{\lambda}x^{\lambda n}\right)}}$$

Cujus integrale posito $Z = \frac{x}{S}$, retinente S superiorem valorem erit

$$V = \frac{1}{a}\int \frac{Z^{\lambda-1}\partial Z}{a+b^{\lambda}Z^{n\lambda}}.$$

Corollarium 2.

§. 89. Deinde vero etiam si pro m fractionem assumamus, irrationalitas adhuc magis complicabitur, veluti si sumamus $m = \frac{1}{2}$, formula differentialis jam erit

$$\partial V = \frac{\partial x}{\left(a+bx^{n}\right)\sqrt[2\lambda n]{x^{\lambda n}\left[\left(a+bx^{n}\right)^{\lambda} - b^{\lambda}x^{\lambda n}\right]}}.$$

Verum hic casus facile ad primum problema revocatur statuendo
$x = vv$, ita ut sit

$$\partial V = \frac{2\partial v}{\left(a+bv^{2n}\right)\sqrt[2\lambda n]{\left[\left(a+bv^{2n}\right)^{\lambda} - b^{\lambda}v^{2\lambda n}\right]}},$$

quae formula a primo problemate aliter non discrepat nisi quod hic exponens n duplo sit major.

Scholion.

§. 90. Quamquam binae litterae a et b pro lubitu tam negative, quam positive accipi possunt, tamen occurrunt casus, qui sub hac generali forma non comprehenduntur: veluti si proponatur haec formula $\frac{\partial x}{\left(1-xx\sqrt[4]{(2xx-1)}\right)}$, haec in problemate primo non continetur, quia fieri deberet $aa = -1$, quod cum in genere evenire posset, etiam problema generale ad hunc casum accommodatum subjungamus.

Problema 21.

§. 91. *Si ponatur ista formula differentialis latissime patens tres exponentes indeterminatos involvens*

$$\partial V = \frac{x^{m-1}\partial x}{\left(fx^{n}-g\right)\sqrt[\lambda n]{\left(f^{\lambda}x^{n\lambda} - \left(fx^{n}-g\right)^{\lambda}\right)^{m}}},$$

eam ab omni irrationalitate liberare.

Solutio.

Statuamus ut ante brevitatis gratia

$$\sqrt[\lambda n]{\left(f^{\lambda}x^{\lambda n} - \left(fx^{n}-g\right)^{\lambda}\right)} = S,$$

tum vero $Z = \frac{x}{S}$, ut formula differentialis fiat

$$\partial V = \frac{\partial x}{x} \cdot \frac{Z^m}{fx^n - g}.$$

Nunc autem sumendo differentialia logarithmica est

$$\frac{\partial S}{S} = \frac{\partial x}{x} \cdot \frac{f^{\lambda} x^{\lambda n} - fx^n \left(fx^n - g\right)^{\lambda - 1}}{S^{\lambda n}},$$

atque hinc colligitur fore

$$\frac{\partial Z}{Z} = \frac{\partial x}{x} - \frac{\partial S}{S} = \frac{\partial x}{x} \cdot \frac{g\left(fx^n - g\right)^{\lambda - 1}}{S^{\lambda n}},$$

sicque habebitur

$$\frac{\partial x}{x} = \frac{\partial Z}{Z} \cdot \frac{S^{\lambda n}}{g\left(fx^n - g\right)^{\lambda - 1}},$$

quo valore substituto nanciscimur

$$\partial V = \frac{Z^{m-1} \partial Z S^{\lambda n}}{g\left(fx^n - g\right)^{\lambda}}.$$

Manifesto autem est $\left(fx^n - g\right)^{\lambda} = f^{\lambda} x^{\lambda n} - S^{\lambda n}$, ideoque

$$\partial V = \frac{Z^{m-1} S^{\lambda n} \partial Z}{g\left(f^{\lambda} x^{\lambda n} + S^{\lambda n}\right)};$$

unde postremo ob $S = \frac{x}{Z}$ concluditur haec forma.

$$\partial V = \frac{Z^{m-1} \partial Z}{g\left(f^{\lambda} Z^{\lambda n} - 1\right)},$$

quae formula a praecedentibus tantum signis discrepat.