CHAPTER IV

CONCERNING THE RESOLUTION OF HOMOGENEOUS DIFFERENTIAL EQUATIONS

PROBLEM 85

492. *If v is equal to some function of the two quantities t and u determined thus by the three variables x, y and z, so that there shall be*

$$t = \alpha x + \beta z \text{ and } u = \gamma y + \delta z,$$

thence to define the differential formulas of all orders of this.

SOLUTION

Since v shall be a function of the quantities

$$t = \alpha x + \beta z \text{ and } u = \gamma y + \delta z,$$

the differential formulas of this will become known arising from these two variables, evidently

$$\left(\frac{dv}{dt}\right),\ \left(\frac{dv}{du}\right),\ \left(\frac{ddv}{dt^2}\right),\ \left(\frac{ddv}{dtdu}\right),\ \left(\frac{ddv}{du^2}\right) \text{ etc.};$$

moreover hence we can deduce at once

$$\left(\frac{dv}{dx}\right) = \alpha\left(\frac{dv}{dt}\right),\quad \left(\frac{dv}{dy}\right) = \gamma\left(\frac{dv}{du}\right),\quad \left(\frac{dv}{dz}\right) = \beta\left(\frac{dv}{dt}\right) + \delta\left(\frac{dv}{du}\right),$$

clearly differential formulas of the first order. But for differential formulas of the second order we arrive at

$$\left(\frac{ddv}{dx^2}\right) = \alpha\alpha\left(\frac{ddv}{dt^2}\right),\ \left(\frac{ddv}{dy^2}\right) = \gamma\gamma\left(\frac{ddv}{du^2}\right),$$
$$\left(\frac{ddv}{dz^2}\right) = \beta\beta\left(\frac{ddv}{dt^2}\right) + 2\beta\delta\left(\frac{ddv}{dtdu}\right) + \delta\delta\left(\frac{ddv}{du^2}\right),$$
$$\left(\frac{ddv}{dxdy}\right) = \alpha\gamma\left(\frac{ddv}{dtdu}\right),\quad \left(\frac{ddv}{dxdz}\right) = \alpha\beta\left(\frac{ddv}{dt^2}\right) + \alpha\delta\left(\frac{ddv}{dtdu}\right)$$

and

$$\left(\frac{ddv}{dydz}\right)=\beta\gamma\left(\frac{ddv}{dtdu}\right)+\gamma\delta\left(\frac{ddv}{du^2}\right).$$

In a similar manner we ascend to the third order

$$\left(\frac{d^3v}{dx^3}\right)=\alpha^3\left(\frac{d^3v}{dt^3}\right),\ \left(\frac{d^3v}{dy^3}\right)=\gamma^3\left(\frac{d^3v}{du^3}\right),$$

$$\left(\frac{d^3v}{dz^3}\right)=\beta^3\left(\frac{d^3v}{dt^3}\right)+3\beta^2\delta\left(\frac{d^3v}{dt^2du}\right)+3\beta\delta^2\left(\frac{d^3v}{dtdu^2}\right)+\delta^3\left(\frac{d^3v}{du^3}\right),$$

$$\left(\frac{d^3v}{dx^2dy}\right)=\alpha\alpha\gamma\left(\frac{d^3v}{dt^2du}\right),\quad \left(\frac{d^3v}{dxdy^2}\right)=\alpha\gamma\gamma\left(\frac{d^3v}{dtdu^2}\right),$$

$$\left(\frac{d^3v}{dx^2dy}\right)=\alpha\alpha\beta\left(\frac{d^3v}{dt^3}\right)+\alpha\alpha\delta\left(\frac{d^3v}{dt^2du}\right),\quad \left(\frac{d^3v}{dy^2dz}\right)=\beta\gamma\gamma\left(\frac{d^3v}{dtdu^2}\right)+\gamma\gamma\delta\left(\frac{d^3v}{du^3}\right),$$

$$\left(\frac{d^3v}{dxdz^2}\right)=\alpha\beta\beta\left(\frac{d^3v}{dt^3}\right)+2\alpha\beta\delta\left(\frac{d^3v}{dt^2du}\right)+\alpha\delta\delta\left(\frac{d^3v}{dtdu^2}\right),$$

$$\left(\frac{d^3v}{dydz^2}\right)=\beta\beta\gamma\left(\frac{d^3v}{dt^2du}\right)+2\beta\gamma\delta\left(\frac{d^3v}{dtdu^2}\right)+\gamma\delta\delta\left(\frac{d^3v}{du^3}\right),$$

$$\left(\frac{d^3v}{dxdydz}\right)=\alpha\beta\gamma\left(\frac{d^3v}{dt^2du}\right)+\alpha\gamma\delta\left(\frac{d^3v}{dtdu^2}\right),$$

from which it is seen easily, how these differential formulas could be continued to higher orders.

SCHOLIUM 1

493. It will be seen that perhaps this problem ought to be taken more generally thus, by defining the quantities *t* and *u* by the three variables x, y, z, so that there shall be

$$t=\alpha x+\beta y+\gamma z \quad\text{and}\quad u=\delta x+\varepsilon y+\zeta z;$$

truly since this hypothesis shall be made to that end, that *v* will be made a function of the two variables *t* and *u*, then clearly also *v* can be seen as a function of the two quantities $\varepsilon t-\beta u$ and $\delta t-\alpha u$, of which the first one shall be free from *y*, and the latter one truly from *x*. On which account the hypothesis assumed is to be considered the widest allowed ; yet perhaps it will be considered with this exception, if there should be $t=x+z$ and $u=x-z$, because here the value of *u* may not be retained [as β and ε are both zero]; truly also in this case the quantity *v* considered as a function of $t+u$ and $t-u$ becomes a function of *x* and *z*, which case certainly is contained in the hypothesis on taking $\beta=0$ and $\gamma=0$.

SCHOLIUM 2

494. Thus I have proposed this problem, because I do not undertake to treat other differential equations here, in which unless a value of this kind is satisfied, that v is equal to a function of some kind of the two new variables t and u, which thus depend on the principle variables $x, y, z,$ so that, just as I have assumed,

$$t=\alpha x+\beta z \text{ and } u=\gamma y+\delta z .$$

But equations of this kind, which can be satisfied in this way, may be apparent readily to be homogeneous, thus so that the equation cannot be resolved unless it depends on the individual differential of the same order multiplied by constant quantities and added among themselves, which I have used in the preceding part [§416- §428, Section III of this volume] under the title of homogeneous equations.

Therefore in the proposed homogeneous equation of this kind, in place of the individual differential formulas formed by the elements dx, dy, dz, here values are substituted formed from the elements dt and du and then the individual members, as far as a certain differential formula arising from the elements dt et du are included, may be reduced to zero separately and thus the ratios $\frac{\beta}{\alpha}$ and $\frac{\delta}{\gamma}$ may be determined, since the question does not depend only on these quantities but also on the ratios of these. Therefore since only two things are left to be investigated, if with more equations requiring to be satisfied, homogeneous equations of this kind cannot be resolved in this manner, unless these extra equations can be recalled to only two equations, and that which will be explained more clearly in the following.

PROBLEM 86

495. *With the proposed homogeneous equation of the first order*

$$A\left(\tfrac{dv}{dx}\right)+B\left(\tfrac{dv}{dy}\right)+C\left(\tfrac{dv}{dz}\right)=0$$

to investigate the nature of the function v of the three variables x, y and z.

SOLUTION

There is devised $v=\Gamma:(t \text{ and } u)$ with

$$t=\alpha x+\beta z \text{ and } u=\gamma y+\delta z$$

present, and with the substitution made from the previous problem our equation may be divided into two parts

$$\left(\tfrac{dv}{dt}\right)(A\alpha+C\beta)+\left(\tfrac{dv}{du}\right)(B\gamma+C\delta)=0,$$

of which each separately reduced to zero gives

$$\frac{\beta}{\alpha}=\frac{-A}{C} \quad \text{and} \quad \frac{\delta}{\beta}=\frac{-B}{C}$$

from which there arises

$$t=Cx-Az \quad \text{and} \quad u=Cy-Bz\,.$$

Whereby the integral of the proposed equation will be

$$v=\Gamma:\left(\overline{Cx-Az} \quad \text{and} \quad \overline{Cy-Bz}\right),$$

which also can be shown more neatly

$$v=\Gamma:\left(\tfrac{x}{A}-\tfrac{z}{C} \quad \text{and} \quad \tfrac{y}{B}-\tfrac{z}{C}\right)$$

COROLLARY 1

496. With the variables interchanged it is evident that this integral also can be expressed thus :

$$v=\Gamma:\left(\tfrac{x}{A}-\tfrac{y}{B} \quad \text{and} \quad \tfrac{y}{B}-\tfrac{z}{C}\right) \text{ or } v=\Gamma:\left(\tfrac{x}{A}-\tfrac{y}{B} \quad \text{and} \quad \tfrac{x}{A}-\tfrac{z}{C}\right),$$

because there shall be

$$\tfrac{x}{A}-\tfrac{y}{B}=\left(\tfrac{x}{A}-\tfrac{z}{C}\right)-\left(\tfrac{y}{B}-\tfrac{z}{C}\right).$$

COROLLARY 2

497. So that also with these three formulas set up from the proposed equation

$$\tfrac{x}{A}-\tfrac{y}{B}, \quad \tfrac{x}{A}-\tfrac{z}{C}, \quad \tfrac{y}{B}-\tfrac{z}{C}$$

some function from these in some manner will suffice to put together a suitable value for v. Because indeed each one of these two formulas is the difference of the remaining two, such a function is agreed to include only two variables.

COROLLARY 3

498. Likewise it is the case, with whichever of these three forms of the integral we may use ; but when the two new variables t and u should be equal to each other, then another is to be used. Just as if there should be $C=0$, then the first form $v=\Gamma:(z \text{ and } z)$, as a function of z only, would be useless and the complete integral would become either

$$v=\Gamma:\left(\tfrac{x}{A}-\tfrac{y}{B} \quad \text{and} \quad z\right) \quad \text{or} \quad v=\Gamma:\left(\overline{Bx-Ay} \quad \text{and} \quad z\right)$$

PROBLEM 87

499. *With the proposed homogeneous equation of the second order*

$$A\left(\frac{ddv}{dx^2}\right)+B\left(\frac{ddv}{dy^2}\right)+C\left(\frac{ddv}{dz^2}\right)+2D\left(\frac{ddv}{dxdy}\right)+2E\left(\frac{ddv}{dxdz}\right)+2F\left(\frac{ddv}{dydz}\right)=0$$

to investigate the case, in which the integral of this can be expressed in this form $\Gamma:(t \text{ and } u)$ *with* $t=\alpha x+\beta z$ *and* $u=\gamma y+\delta z$ *arising.*

SOLUTION

With the substitution made following the formulas treated in Problem 85 the equation proposed may be resolved into the three following members

$$\left.\begin{array}{l}\left(\frac{ddv}{dt^2}\right)(A\alpha\alpha+C\beta\beta+2E\alpha\beta)\\ +\left(\frac{ddv}{dtdu}\right)(2C\beta\delta+2D\alpha\gamma+2E\alpha\delta+2F\beta\gamma)\\ +\left(\frac{ddv}{du^2}\right)(B\gamma\gamma+C\delta\delta+2F\gamma\delta)\end{array}\right\}=0,$$

each of which separately must be equated to zero. But the first gives

$$\frac{\beta}{\alpha}=\frac{-E+\sqrt{(EE-AC)}}{C},$$

the last truly

$$\frac{\delta}{\gamma}=\frac{-F+\sqrt{(FF-BC)}}{C}$$

which values substituted into the middle equation, which thus may be returned :

$$\frac{C\beta\delta}{\alpha\gamma}+D+\frac{E\delta}{\gamma}+\frac{F\beta}{\alpha}=0,$$

gives this equation

$$EF-CD=\sqrt{(EE-AC)}(FF-BC),$$

from which equation a condition is satisfied between the coefficients *A, B, C, D, E, F,* so that here there is a place for an applicable solution. But since the equation expanded out gives

$$CCDD - 2CDEF + BCEE + ACFF - ABCC = 0,$$

from which there is produced

$$C = \frac{2DEF - BEE - AFF}{DD - AB},$$

because a common multiplying factor C has entered. But just as often as this condition has a place, so that it becomes

$$AFF + BEE + CDD = ABC + 2DEF,$$

also this algebraic equation

$$Axx + Byy + Czz + 2Dxy + 2Exz + 2Fyz$$

formed from the proposed equation can be resolved into two factors, nor therefore in other cases is there room for a solution to be given.

Therefore so that duly we may explain these cases allowing a solution, we may put the factors of this form to be

$$(ax + by + cz)(fx + gy + hz),$$

because hence it comes about, if there should be

$$A = af, \qquad B = bg, \qquad C = ch,$$
$$2D = ag + bf, \ 2E = ah + cf, \ 2F = bk + cg,$$

from which certainly there arises

$$AFF + BEE + CDD = ABC + 2DEF.$$

Moreover for the solution, it is deduced hence that either

$$\frac{\beta}{\alpha} = \frac{-a}{c} \text{ or } \frac{\beta}{\alpha} = \frac{-f}{h}$$

and either

$$\frac{\delta}{\gamma} = \frac{-b}{c} \text{ or } \frac{\delta}{\gamma} = \frac{-g}{h},$$

where it is required to note for the fractions $\frac{\beta}{\alpha}$ and $\frac{\delta}{\gamma}$ that it is necessary to take together the values written below, so that there shall be

$$\text{either } t = cx - az \quad \text{and} \quad u = cy - bz,$$
$$\text{or } \quad t = hx - fz \quad \text{and} \quad u = hy - gz.$$

On account of which the complete integral from these cases admitting a solution will be

$$v = \Gamma : \left(\overline{cx - az} \text{ and } \overline{cy - bz}\right) + \Delta : \left(\overline{hx - fz} \text{ and } \overline{hy - gz}\right)$$

or

$$v = \Gamma : \left(\tfrac{x}{a} - \tfrac{z}{c} \text{ and } \tfrac{y}{b} - \tfrac{z}{c}\right) + \Delta : \left(\tfrac{x}{f} - \tfrac{z}{h} \text{ and } \tfrac{y}{g} - \tfrac{z}{h}\right).$$

COROLLARY 1

500. Therefore other homogeneous equations of the second order are unable to be resolved in this manner, unless they may be contained in this form

$$af\left(\tfrac{ddv}{dx^2}\right) + bg\left(\tfrac{ddv}{dy^2}\right) + ch\left(\tfrac{ddv}{dz^2}\right)$$
$$+ (ag + bf)\left(\tfrac{ddv}{dxdy}\right)(ah + cf)\left(\tfrac{ddv}{dxdz}\right) + (bh + cg)\left(\tfrac{ddv}{dydz}\right) = 0;$$

then truly the complete integral will be

$$v = \Gamma : \left(\tfrac{x}{a} - \tfrac{z}{c} \text{ and } \tfrac{y}{b} - \tfrac{z}{c}\right) + \Delta : \left(\tfrac{x}{f} - \tfrac{z}{h} \text{ and } \tfrac{y}{g} - \tfrac{z}{h}\right).$$

COROLLARY 2

501. From which moreover it may be more easily discerned, whether or not it may be possible to reduce some proposed equation

$$A\left(\tfrac{ddv}{dx^2}\right) + B\left(\tfrac{ddv}{dy^2}\right) + C\left(\tfrac{ddv}{dz^2}\right) + 2D\left(\tfrac{ddv}{dxdy}\right) + 2E\left(\tfrac{ddv}{dxdz}\right) + 2F\left(\tfrac{ddv}{dydz}\right) = 0$$

to that, and thence this algebraic form may be formed

$$Axx + Byy + Czz + 2Dxy + 2Exz + 2Fyz;$$

which if it is allowed to be resolved into two rational factors

$$(ax + by + cz)(fx + gy + hz),$$

the complete integral of this can be shown at once.

COROLLARY 3

502. Yet the single case, in which those two factors are made equal to each other, demands an exception, because then the two functions found merge into one. Truly it is deduced from above [§ 416], if this comes about, so that there shall be $f=a$, $b=g$ and $h=c$, the complete integral can thus be expressed

$$z=x\Gamma:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ and }\tfrac{y}{b}-\tfrac{z}{c}\right)+\Delta:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ and }\tfrac{y}{b}-\tfrac{z}{c}\right)$$

SCHOLIUM 1

503. Therefore in which cases a homogeneous equation of the second order admits a resolution, with these also are included in themselves two homogeneous equations of the first order

$$a\left(\tfrac{dv}{dx}\right)+b\left(\tfrac{dv}{dy}\right)+c\left(\tfrac{dv}{dz}\right)=0$$

and

$$f\left(\tfrac{dv}{dx}\right)+g\left(\tfrac{dv}{dy}\right)+h\left(\tfrac{dv}{dz}\right)=0,$$

clearly each of which satisfies that equation, and the complete integrals of these taken jointly suffices for the complete integral of this.

Hence another way is uncovered of finding the integrals of homogeneous equations of the second order, by constructing an equation of the first order with themselves satisfying

$$a\left(\tfrac{dv}{dx}\right)+b\left(\tfrac{dv}{dy}\right)+c\left(\tfrac{dv}{dz}\right)=0\ ;$$

then from this by a threefold differentiation three new equations may be formed :

$$a\left(\tfrac{ddv}{dx^2}\right)+b\left(\tfrac{ddv}{dxdy}\right)+c\left(\tfrac{ddv}{dxdz}\right)=0,$$
$$a\left(\tfrac{ddv}{dxdy}\right)+b\left(\tfrac{ddv}{dy^2}\right)+c\left(\tfrac{ddv}{dydz}\right)=0,$$
$$a\left(\tfrac{ddv}{dxdz}\right)+b\left(\tfrac{ddv}{dydz}\right)+c\left(\tfrac{ddv}{dz^2}\right)=0,$$

of which the first multiplied by *f*, the second by *g* and the third by *h* and gathered together into one sum produce one general equation itself, the integral of which we have shown above. Therefore that can be considered as if produced from two homogeneous equations of the first order, from which taken together the complete integral is formed.

SCHOLUM 2

504. Therefore an infinite number of homogeneous second order equations here are excluded, which reject the integration or are unable in this manner to be reduced to equations of the first order; all which excluded cases are recognised from this criterion, if there should not be

$$AFF + BEE + CDD = ABC + 2DEF\ .$$

This equation $\left(\frac{ddv}{dxdy}\right) = \left(\frac{ddv}{dz^2}\right)$ is of this kind, which therefore with such an integral, of the kind we have assumed here, is not allowed ; nor also is another way apparent of finding the complete integral of this. But innumerable particular integrals are able to be shown easily and which of course involve arbitrary functions, but still of one variable quantity, which on being set up in the present circumstances cannot be considered otherwise than particular integrals. For if there is put

$$v = \Gamma:(\alpha x + \beta y + \gamma z)\,,$$

with the substitution made there must become $\alpha\beta = \gamma\gamma$ or on assuming $\gamma = 1$ there must be $\alpha\beta = 1$; whereby accordingly innumerable formulas of this kind taken together give satisfaction, so that there shall be

$$v = \Gamma:\left(\tfrac{\alpha}{\beta}x + \tfrac{\beta}{\alpha}y + z\right) + \Delta:\left(\tfrac{\gamma}{\delta}x + \tfrac{\delta}{\gamma}y + z\right) + \Sigma:\left(\tfrac{\varepsilon}{\zeta}x + \tfrac{\zeta}{\varepsilon}y + z\right) + \text{etc.},$$

where for $\alpha, \beta, \gamma, \delta$ etc. it is permitted to take any numbers. But however many of an indefinite number of different formulas of this kind are taken together, still the integral can only be considered as particular. From which it is understood that the complete integral of this equation $\left(\frac{ddv}{dxdy}\right) = \left(\frac{ddv}{dz^2}\right)$ is at present of the greatest concern, and a method for arriving at that would extend the limits of analysis considerably.

Moreover homogeneous equations of the third order demand much greater restriction, so that the complete integration may succeed in this manner, as will be shown in the following problem.

PROBLEM 88

505. *To define those cases of homogeneous equations of the third order, in which the complete integral can be shown by the assumed form or that can be reduced to the form of homogeneous equations of the first order.*

SOLUTION

This equation of the first order is devised to be contained in a homogeneous equation of the third order :

$$a\left(\frac{dv}{dx}\right)+b\left(\frac{dv}{dy}\right)+c\left(\frac{dv}{dz}\right)=0$$

which so that it satisfies an equation of the third order :

$$A\left(\frac{d^3v}{dx^3}\right)+B\left(\frac{d^3v}{dy^3}\right)+C\left(\frac{d^3v}{dz^3}\right)$$
$$+D\left(\frac{d^3v}{dx^2dy}\right)+E\left(\frac{d^3v}{dxdy^2}\right)+F\left(\frac{d^3v}{dx^2dz}\right)+G\left(\frac{d^3v}{dxdz^2}\right)+H\left(\frac{d^3v}{dy^2dz}\right)+I\left(\frac{d^3v}{dydz^2}\right)$$
$$+K\left(\frac{d^3v}{dxdydz}\right)=0,$$

it is necessary that this algebraic expression

$$Ax^3+By^3+Cz^3+Dxxy+Exyy+Fxxz+Gxzz+Hyyz+Iyzz+Kxyz$$

should have a factor $ax+by+cz$; but unless the other factor again can be resolved into two simple factors, in accordance with being referred to a homogeneous equation of the second order, which solution will be rejected. Whereby so that the complete integration shall succeed, it is necessary that the expression be constructed from three simple factors, which shall be

$$(ax+by+cz)(fx+gy+hz)(kx+my+nz),$$

and hence the coefficients of the general equation thus themselves will be considered :

$$A=afk,\ B=bgm,\ C=chn,$$
$$D=afm+agk+bfk,\ \ E=agm+bfm+bgk,\ \ F=afn+ahk+cfk,$$
$$G=ahn+cfn+chk,\ \ H=bgn+bhm+cgm,\ I=bhn+cgn+chm,$$
$$K=agn+ahm+bfn+bhk+cfm+cgk$$

and then the complete integral will be

$$v=\Gamma:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ and }\tfrac{y}{b}-\tfrac{z}{c}\right)+\Delta:\left(\tfrac{x}{f}-\tfrac{z}{h}\text{ and }\tfrac{y}{g}-\tfrac{z}{h}\right)+\Sigma:\left(\tfrac{x}{k}-\tfrac{z}{n}\text{ and }\tfrac{y}{m}-\tfrac{z}{n}\right),$$

evidently whichever simple factor gives a arbitrary function of the two variables.

COROLLARY 1

506. In whichever of these functions, the variables *x, y, z* are allowed to be permuted among themselves; so that also whichever as if put together from the three variables, clearly the first [will be formed] from these

$$\tfrac{x}{a}-\tfrac{y}{b},\quad \tfrac{y}{b}-\tfrac{z}{c}\quad\text{and}\quad \tfrac{z}{c}-\tfrac{x}{a}$$

and in a like manner for the rest.

COROLLARY 2

507. If two factors were equal, $f=a,\ g=b,\ h=c$, in which case the two first functions merge into one function, in place of these there must be written

$$x\Gamma:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ and }\tfrac{y}{b}-\tfrac{z}{c}\right)+\Delta:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ and }\tfrac{y}{b}-\tfrac{z}{c}\right),$$

but if all three were equal, so that above there shall be $k=a,\ m=b,\ n=c$, then the complete integral will be

$$v=xx\Gamma:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ and }\tfrac{y}{b}-\tfrac{z}{c}\right)+x\Delta:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ and }\tfrac{y}{b}-\tfrac{z}{c}\right)+\Sigma:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ and }\tfrac{y}{b}-\tfrac{z}{c}\right).$$

COROLLARY 3

508. Just as here we have multiplied the first two parts by *xx* and *x*, thus also we may multiply by *yy* and *y*, likewise by *zz* and *z* ; likewise indeed it is, which whichever variables we have used, provided that it shall not be that which perhaps occurs alone after the sign of the function ; clearly if there should be $a=0$ and functions of the quantities *x* and $\tfrac{y}{b}-\tfrac{z}{c}$ must be taken, then the multipliers *xx* and *x* must be excluded.

SCHOLIUM 1

509. In a like manner it is apparent that homogeneous equations of the fourth order cannot be resolved by this method, unless it can be resolved into four simple equations and as if the products of these are able to be considered. Although here actually no resolution into factors may be possible, still from examples produced clearly it is observed, in what manner an expression may be formed involving the three variables *x, y, z* must be formed from a homogeneous differential equation of any order ; if which it is able to be resolved into simple factors of the form

$ax + by + cz$, thence likewise the complete integral of the differential equation will be shown easily, since any factor of the two variables may suffice establishing a function part of the integral, thus so that also this part of the differential equation taken separately may be satisfactory and will be possible to consider as a particular integral. But if that algebraic expression thus were prepared, so that it had certain simple factors, but not so many as the dimensions, certain individual particular integrals will be prepared, but which taken together will not suffice for a complete integral. Just as if this differential equation of the third order were proposed :

$$a\left(\frac{d^3v}{dx^2dy}\right)+b\left(\frac{d^3v}{dxdy^2}\right)-a\left(\frac{d^3v}{dxdz^2}\right)-b\left(\frac{d^3v}{dydz^2}\right)=0,$$

because the algebraic form

$$axxy + bxyy - axzz - byzz$$

may have the simple factor $ax + by$, for that value will certainly be satisfactory

$$v = \Gamma:\left(\tfrac{x}{a}-\tfrac{y}{b}\text{ et } z\right)\ ;$$

but two arbitrary functions are still missing containing the complete integral of this equation

$$\left(\frac{ddv}{dxdy}\right)-\left(\frac{ddv}{dz^2}\right)=0\,,$$

from which evidently the other factor $xy - zz$ of that expression arises. Therefore as often as these algebraic expressions formed from homogeneous differential equations of higher orders admit to a resolution into factors, even if not simple ones, hence at any rate we learn how the integration of these are able to be recalled as equations of lower order, since without doubt there is great interest in hard investigations of this kind,.

SCHOLIUM 2

510. These [parts] exist, which I have been able to advance concerning functions of three variables, certainly only because in which the first principles of this science are included, that require to be investigated from some given relation of the differentials, concerning which the further development is to be recommended to the sagacity of the geometers with the greatest enthusiasm. Indeed so much is lacking which are required at this point in the theory of motion of fluids, to that analysis the higher parts may be referred, so that these speculations may be considered as sterile or as rather generally, of which therefore the usefulness with the first part of the integral calculus can by no means be seen to be neglected.

But on that account these latter parts deserve to be developed more, because the theory of fluids certainly may be concerned with functions of four variables, the nature of which, is required to be investigated from differential equations of the second order, as indeed I do not wished to touch on a part of this on account of the lack of space. But in this theory the resolution of this equation

$$\left(\frac{ddv}{dt^2}\right)=\left(\frac{ddv}{dx^2}\right)+\left(\frac{ddv}{dy^2}\right)+\left(\frac{ddv}{dz^2}\right)$$

is of the greatest interest, where the three x, y, z letters express the coordinates, truly t the elapsed time, and of these four variables a function is sought, which substituted in place of v satisfies that equation. [See Euler's extensive researches on the theory of motion of fluids, written in French mainly, and reviewed in English in Vol. 12 of the Opera Omnia edition by C. Truesdell. This is a volume most worthy of attention by those lacking in language skills, and who have a desire to find out the extent of Euler's achievements in the area of fluid dynamics.]
But so far from the reports the complete integral of this equation is deduced easily to include two arbitrary functions, each of which shall be a function of the three variables, and all the other solutions appearing less widely are to be considered as incomplete. But the situation can show innumerable particular solutions; just as if we put

$$v=\Gamma:(\alpha x+\beta y+\gamma z+\delta t),$$

there is found

$$\delta\delta=\alpha\alpha+\beta\beta+\gamma\gamma;$$

which since it can happen in boundless ways, an infinity of functions of this kind added together will show a suitable value for v . Then indeed these values are satisfactory

$$v=\frac{\Gamma:\left(t\pm\sqrt{(xx+yy+zz)}\right)}{\sqrt{(xx+yy+zz)}},\quad v=\frac{\Gamma:\left(y\pm\sqrt{(tt-yy-zz)}\right)}{\sqrt{(tt-yy-zz)}},$$
$$v=\frac{\Gamma:\left(x\pm\sqrt{(tt-yy-zz)}\right)}{\sqrt{(tt-yy-zz)}},\quad v=\frac{\Gamma:\left(z\pm\sqrt{(tt-xx-yy)}\right)}{\sqrt{(tt-xx-yy)}},$$

of which the investigation now is more difficult. But since these functions are of one variable only, they show particular integrals mainly, which accordingly also even now will be particular, if for v they may be considered functions of two variables, but which indeed it is not permitted to suppose. Whereby since the complete integral must include thus two arbitrary functions of three variables, it is easily understood, how far at this stage we may be from the goal.

CAPUT IV

DE AEQUATIONUM DIFFERENTIALIUM HOMOGENEARUM RESOLUTIONE

PROBLEMA 85

492. *Si v aequetur functioni cuicunque binarum quantitatum t et u ita per tres variabiles x, y et z determinatarum, ut sit*

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z,$$

eius formulas differentiales omnium graduum inde definire.

SOLUTIO

Cum v sit functio quantitatum

$$t = \alpha x + \beta z \text{ et } u = \gamma y + \delta z,$$

eius formulae differentiales ex his duabus variabilibus natae innotescent, scilicet

$$\left(\frac{dv}{dt}\right), \left(\frac{dv}{du}\right), \left(\frac{ddv}{dt^2}\right), \left(\frac{ddv}{dtdu}\right), \left(\frac{ddv}{du^2}\right) \text{ etc.};$$

hinc autem statim colligimus

$$\left(\frac{dv}{dx}\right) = \alpha\left(\frac{dv}{dt}\right), \quad \left(\frac{dv}{dy}\right) = \gamma\left(\frac{dv}{du}\right), \quad \left(\frac{dv}{dz}\right) = \beta\left(\frac{dv}{dt}\right) + \delta\left(\frac{dv}{du}\right),$$

formulas scilicet differentiales primi gradus. Pro formulis autem differentialibus secundi gradus adipiscimur

$$\left(\frac{ddv}{dx^2}\right) = \alpha\alpha\left(\frac{ddv}{dt^2}\right), \left(\frac{ddv}{dy^2}\right) = \gamma\gamma\left(\frac{ddv}{du^2}\right),$$
$$\left(\frac{ddv}{dz^2}\right) = \beta\beta\left(\frac{ddv}{dt^2}\right) + 2\beta\delta\left(\frac{ddv}{dtdu}\right) + \delta\delta\left(\frac{ddv}{du^2}\right),$$
$$\left(\frac{ddv}{dxdy}\right) = \alpha\gamma\left(\frac{ddv}{dtdu}\right), \quad \left(\frac{ddv}{dxdz}\right) = \alpha\beta\left(\frac{ddv}{dt^2}\right) + \alpha\delta\left(\frac{ddv}{dtdu}\right)$$

et

$$\left(\frac{ddv}{dydz}\right) = \beta\gamma\left(\frac{ddv}{dtdu}\right) + \gamma\delta\left(\frac{ddv}{du^2}\right).$$

Simili modo ad tertium gradum ascendimus

$$\left(\frac{d^3v}{dx^3}\right)=\alpha^3\left(\frac{d^3v}{dt^3}\right),\ \left(\frac{d^3v}{dy^3}\right)=\gamma^3\left(\frac{d^3v}{du^3}\right),$$
$$\left(\frac{d^3v}{dz^3}\right)=\beta^3\left(\frac{d^3v}{dt^3}\right)+3\beta^2\delta\left(\frac{d^3v}{dt^2du}\right)+3\beta\delta^2\left(\frac{d^3v}{dtdu^2}\right)+\delta^3\left(\frac{d^3v}{du^3}\right),$$
$$\left(\frac{d^3v}{dx^2dy}\right)=\alpha\alpha\gamma\left(\frac{d^3v}{dt^2du}\right),\quad \left(\frac{d^3v}{dxdy^2}\right)=\alpha\gamma\gamma\left(\frac{d^3v}{dtdu^2}\right),$$
$$\left(\frac{d^3v}{dx^2dy}\right)=\alpha\alpha\beta\left(\frac{d^3v}{dt^3}\right)+\alpha\alpha\delta\left(\frac{d^3v}{dt^2du}\right),\quad \left(\frac{d^3v}{dy^2dz}\right)=\beta\gamma\gamma\left(\frac{d^3v}{dtdu^2}\right)+\gamma\gamma\delta\left(\frac{d^3v}{du^3}\right),$$
$$\left(\frac{d^3v}{dxdz^2}\right)=\alpha\beta\beta\left(\frac{d^3v}{dt^3}\right)+2\alpha\beta\delta\left(\frac{d^3v}{dt^2du}\right)+\alpha\delta\delta\left(\frac{d^3v}{dtdu^2}\right),$$
$$\left(\frac{d^3v}{dydz^2}\right)=\beta\beta\gamma\left(\frac{d^3v}{dt^2du}\right)+2\beta\gamma\delta\left(\frac{d^3v}{dtdu^2}\right)+\gamma\delta\delta\left(\frac{d^3v}{du^3}\right),$$
$$\left(\frac{d^3v}{dxdydz}\right)=\alpha\beta\gamma\left(\frac{d^3v}{dt^2du}\right)+\alpha\gamma\delta\left(\frac{d^3v}{dtdu^2}\right),$$

unde facile patet, quomodo has formulas differentiales ad altiores gradus continuari oporteat.

SCHOLION 1

493. Hoc problema fortasse generalius concipi debuisse videbitur quantitates *t* et *u* ita per tres variabiles *x*, *y*, *z* definiendo, ut esset

$$t=\alpha x+\beta y+\gamma z \quad \text{et} \quad u=\delta x+\varepsilon y+\zeta z;$$

verum cum haec hypothesis in eum tantum finem sit facta, ut *v* fieret functio ipsarum *t* et *u*, evidens tum quoque *v* spectari posse ut functionem harum duarum quantitatum $\varepsilon t-\beta u$ et $\delta t-\alpha u$, quarum illa ab *y*, haec vero ab *x* erit libera. Quocirca hypothesis assumta latissime patere est censenda; exceptio tamen forte hinc admittenda videbitur, si fuerit $t=x+z$ et $u=x-z$, quia hic ipsius *u* valor non continetur; verum etiam hoc casu quantitas *v* ut functio ipsarum $t+u$ et $t-u$ spectata fiet functio ipsarum *x* et *z*, qui casus utique in hypothesi continetur sumtis $\beta=0$ et $\gamma=0$.

SCHOLION 2

494. Hoc problema ideo praemisi, quia alias aequationes differentiales tractare hic non sustineo, nisi quibus eiusmodi valor satisfacit, ut *v* aequetur functioni cuicunque binarum novarum variabilium *t* et *u*, quae ab principalibus *x*, *y*, *z* ita pendeant, ut sit, quemadmodum assumsi,

$$t=\alpha x+\beta z \text{ et } u=\gamma y+\delta z.$$

Huiusmodi autem aequationes, quibus hoc modo satisfieri potest, esse homogeneas facile patet, ita ut aequatio resolvenda constet nonnisi formulis differentialibus eiusdem gradus singulis per constantes quantitates multiplicatis et inter se additis, qua appellatione aequationum homogenearum iam in parte praecedente [§ 416-428] sum usus.

Proposita ergo huiusmodi aequatione homogenea loco singularum formularum differentialium per elementa dx, dy, dz formatarum substituantur valores hic inventi per elementa dt et du formati et tum singula membra, quatenus certam formulam differentialem ex elementis dt et du natam complectuntur, seorsim ad nihilum redigantur indeque rationes $\frac{\beta}{\alpha}$ et $\frac{\delta}{\gamma}$ determinentur, quandoquidem quaestio non tam circa has ipsas quantitates quam earum rationes versatur. Quoniam igitur duae tantum res investigationi relinquuntur, si pluribus aequationibus fuerit satisfaciendum, eiusmodi aequationes homogeneae hac ratione resolvi nequeunt, nisi casu plures illae aequationes ad duas tantum revocentur, id quod in sequentibus clarius explicabitur.

PROBLEMA 86

495. *Proposita aequatione homogenea primi gradus*

$$A\left(\frac{dv}{dx}\right)+B\left(\frac{dv}{dy}\right)+C\left(\frac{dv}{dz}\right)=0$$

investigare naturam functionis v trium variabilium x, y et z.

SOLUTIO

Fingatur $v=\Gamma:(t \text{ et } u)$ existente

$$t=\alpha x+\beta z \text{ et } u=\gamma y+\delta z$$

et facta substitutione ex problemate praecedente aequatio nostra in duas partes dividetur

$$\left(\frac{dv}{dt}\right)(A\alpha+C\beta)+\left(\frac{dv}{du}\right)(B\gamma+C\delta)=0,$$

quarum utraque seorsim ad nihilum reducta praebet

$$\frac{\beta}{\alpha}=\frac{-A}{C} \quad \text{et} \quad \frac{\delta}{\beta}=\frac{-B}{C}$$

unde fit

$$t=Cx-Az \quad \text{et} \quad u=Cy-Bz.$$

Quare aequationis propositae integrale completum erit

$$v=\Gamma:\left(\overline{Cx-Az} \quad \text{et} \quad \overline{Cy-Bz}\right),$$

quod etiam concinnius ita exhiberi potest

$$v = \Gamma : \left(\tfrac{x}{A} - \tfrac{z}{C} \quad \text{et} \quad \tfrac{y}{B} - \tfrac{z}{C} \right)$$

COROLLARIUM 1

496. Permutandis variabilibus hoc integrale etiam ita exprimi posse evidens est

$$v = \Gamma : \left(\tfrac{x}{A} - \tfrac{y}{B} \quad \text{et} \quad \tfrac{y}{B} - \tfrac{z}{C} \right) \text{ vel } v = \Gamma : \left(\tfrac{x}{A} - \tfrac{y}{B} \quad \text{et} \quad \tfrac{x}{A} - \tfrac{z}{C} \right),$$

quoniam est

$$\tfrac{x}{A} - \tfrac{y}{B} = \left(\tfrac{x}{A} - \tfrac{z}{C} \right) - \left(\tfrac{y}{B} - \tfrac{z}{C} \right).$$

COROLLARIUM 2

497. Quin etiam constitutis ex aequatione proposita his tribus formulis

$$\tfrac{x}{A} - \tfrac{y}{B}, \quad \tfrac{x}{A} - \tfrac{z}{C}, \quad \tfrac{y}{B} - \tfrac{z}{C}$$

functio quaecunque ex iis utcunque conflata valorem idoneum pro v suppeditabit. Quoniam enim harum binarum formularum unaquaeque est differentia binarum reliquarum, talis functio duas tantum variabiles complecti est censenda.

COROLLARIUM 3

498. Perinde est, quanam harum trium formarum integralium utamur; quando autem binae novae variabiles t et u inter se fuerint aequales, tum alia est utendum. Veluti si esset $C = 0$, prima forma $v = \Gamma : (z \text{ et } z)$, utpote functio solius z, foret inutilis et integrale completum esset futurum seu $v = \Gamma : \left(\tfrac{x}{A} - \tfrac{y}{B} \quad \text{et} \quad z \right)$ seu $v = \Gamma : \left(\overline{Bx - Ay} \quad \text{et} \quad z \right)$

PROBLEMA 87

499. *Proposita aequatione homogenea secundi gradus*

$$A\left(\frac{ddv}{dx^2}\right)+B\left(\frac{ddv}{dy^2}\right)+C\left(\frac{ddv}{dz^2}\right)+2D\left(\frac{ddv}{dxdy}\right)+2E\left(\frac{ddv}{dxdz}\right)+2F\left(\frac{ddv}{dydz}\right)=0$$

casus investigare, quibus eius integrale hac forma $\Gamma:(t \text{ et } u)$ *exprimi potest existente*

$$t=\alpha x+\beta z \text{ et } u=\gamma y+\delta z.$$

SOLUTIO

Facta substitutione secundum formulas in Problemate 85 traditas aequatio proposita in tria membra sequentia resolvetur

$$\left.\begin{aligned}&\left(\frac{ddv}{dt^2}\right)(A\alpha\alpha+C\beta\beta+2E\alpha\beta)\\ &+\left(\frac{ddv}{dtdu}\right)(2C\beta\delta+2D\alpha\gamma+2E\alpha\delta+2F\beta\gamma)\\ &+\left(\frac{ddv}{du^2}\right)(B\gamma\gamma+C\delta\delta+2F\gamma\delta)\end{aligned}\right\}=0,$$

quorum singula seorsim nihilo debent aequari. At primum praebet

$$\frac{\beta}{\alpha}=\frac{-E+\sqrt{(EE-AC)}}{C},$$

ultimum vero

$$\frac{\delta}{\gamma}=\frac{-F+\sqrt{(FF-BC)}}{C}$$

qui valores in media, quae ita referatur

$$\frac{C\beta\delta}{\alpha\gamma}+D+\frac{E\delta}{\gamma}+\frac{F\beta}{\alpha}=0,$$

substituti suppeditant hanc aequationem

$$EF-CD=\sqrt{(EE-AC)(FF-BC)},$$

qua aequatione conditio inter coefficientes *A, B, C, D, E, F* continetur, ut solutio hic applicata locum invenire possit. Haec autem aequatio evoluta dat

$$CCDD-2CDEF+BCEE+ACFF-ABCC=0,$$

unde fit

$$C=\frac{2DEF-BEE-AFF}{DD-AB},$$

quia factor C per multiplicationem est ingressus. Quoties autem haec conditio habet locum, ut sit

$$AFF+BEE+CDD=ABC+2DEF,$$

toties haec expressio algebraica ex aequatione proposita formanda

$$Axx+Byy+Czz+2Dxy+2Exz+2Fyz$$

in duos factores potest resolvi neque ergo aliis casibus solutio hic adhibita locum habere potest.

Quo ergo hos casus solutionem admittentes rite evolvamus, ponamus huius formae factores esse

$$(ax+by+cz)(fx+gy+hz),$$

quod ergo eveniet, si fuerit

$$A=af,\quad B=bg,\quad C=ch,$$
$$2D=ag+bf,\ 2E=ah+cf,\ 2F=bk+cg,$$

unde utique fit

$$AFF+BEE+CDD=ABC+2DEF.$$

Hinc autem pro solutione colligitur

$$\text{vel}\ \frac{\beta}{\alpha}=\frac{-a}{c}\ \text{vel}\ \frac{\beta}{\alpha}=\frac{-f}{h}$$

et

$$\text{vel}\ \frac{\delta}{\gamma}=\frac{-b}{c}\ \text{vel}\ \frac{\delta}{\gamma}=\frac{-g}{h},$$

ubi observari oportet pro fractionibus $\frac{\beta}{\alpha}$ et $\frac{\delta}{\gamma}$ valores sibi subscriptos coniungi oportere, ita ut sit

$$\text{vel}\ t=cx-az\ \text{et}\ u=cy-bz,$$
$$\text{vel}\ t=hx-fz\ \text{et}\ u=hy-gz.$$

Quocirca pro his casibus solutionem admittentibus integrale completum erit

$$v=\Gamma:\left(\overline{cx-az}\text{ et }\overline{cy-bz}\right)+\Delta:\left(\overline{hx-fz}\text{ et }\overline{hy-gz}\right)$$

seu

$$v=\Gamma:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ et }\tfrac{y}{b}-\tfrac{z}{c}\right)+\Delta:\left(\tfrac{x}{f}-\tfrac{z}{h}\text{ et }\tfrac{y}{g}-\tfrac{z}{h}\right).$$

COROLLARIUM 1

500. Hoc ergo modo aliae aequationes homogeneae secundi gradus resolvi nequeunt, nisi quae in hac forma continentur

$$af\left(\tfrac{ddv}{dx^2}\right)+bg\left(\tfrac{ddv}{dy^2}\right)+ch\left(\tfrac{ddv}{dz^2}\right)$$
$$+(ag+bf)\left(\tfrac{ddv}{dxdy}\right)(ah+cf)\left(\tfrac{ddv}{dxdz}\right)+(bh+cg)\left(\tfrac{ddv}{dydz}\right)=0;$$

tum vero integrale completum erit

$$v=\Gamma:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ et }\tfrac{y}{b}-\tfrac{z}{c}\right)+\Delta:\left(\tfrac{x}{f}-\tfrac{z}{h}\text{ et }\tfrac{y}{g}-\tfrac{z}{h}\right).$$

COROLLARIUM 2

501. Quo autem facilius dignoscatur, utrum aequatio quaepiam proposita

$$A\left(\tfrac{ddv}{dx^2}\right)+B\left(\tfrac{ddv}{dy^2}\right)+C\left(\tfrac{ddv}{dz^2}\right)+2D\left(\tfrac{ddv}{dxdy}\right)+2E\left(\tfrac{ddv}{dxdz}\right)+2F\left(\tfrac{ddv}{dydz}\right)=0$$

eo reduci possit necne, formetur inde haec forma algebraica

$$Axx+Byy+Czz+2Dxy+2Exz+2Fyz;$$

quae si resolvi patiatur in duos factores rationales

$$(ax+by+cz)(fx+gy+hz),$$

eius integrale completum hinc statim exhiberi potest.

COROLLARIUM 3

502. Unicus tantum casus, quo duo isti factores inter se fiunt aequales, exceptionem postulat, quoniam tum binae functiones inventae in unam coalescerent. Verum ex superioribus [§ 416] colligitur, si hoc eveniat, ut sit $f = a$, $b = g$ et $h = c$, integrale completum ita exprimi

$$z = x\Gamma : \left(\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} - \tfrac{z}{c}\right) + \Delta : \left(\tfrac{x}{a} - \tfrac{z}{c} \text{ et } \tfrac{y}{b} - \tfrac{z}{c}\right)$$

SCHOLION 1

503. Quibus ergo casibus aequatio homogenea secundi gradus resolutionem admittit, iis quoque in se complectitur duas aequationes homogeneas primus gradus

$$a\left(\tfrac{dv}{dx}\right) + b\left(\tfrac{dv}{dy}\right) + c\left(\tfrac{dv}{dz}\right) = 0$$

et

$$f\left(\tfrac{dv}{dx}\right) + g\left(\tfrac{dv}{dy}\right) + h\left(\tfrac{dv}{dz}\right) = 0,$$

quippe quarum utraque illi satisfacit, et harum integralia completa iunctim sumta illius integrale completum suppeditant.

Hinc alia via aperitur aequationum homogenearum secundi gradus integralia inveniendi fingendo aequationem primi gradus ipsis satisfacientem

$$a\left(\tfrac{dv}{dx}\right) + b\left(\tfrac{dv}{dy}\right) + c\left(\tfrac{dv}{dz}\right) = 0 \;;$$

tum ex hac per triplicem differentiationem tres novae formentur

$$a\left(\tfrac{ddv}{dx^2}\right) + b\left(\tfrac{ddv}{dxdy}\right) + c\left(\tfrac{ddv}{dxdz}\right) = 0,$$
$$a\left(\tfrac{ddv}{dxdy}\right) + b\left(\tfrac{ddv}{dy^2}\right) + c\left(\tfrac{ddv}{dydz}\right) = 0,$$
$$a\left(\tfrac{ddv}{dxdz}\right) + b\left(\tfrac{ddv}{dydz}\right) + c\left(\tfrac{ddv}{dz^2}\right) = 0,$$

quarum prima per *f*, secunda per *g* et tertia per *h* multiplicatae et in unam summam collectae ipsam illam aequatione generalem producunt, cuius integrale supra exhibuimus. Ea ergo quasi productum ex binis aequationibus homogeneis primi gradus spectari poterit, ex quibus coniunctis integrale completum formatur.

SCHOLION 2

504. Infinitae ergo aequationes homogeneae secundi gradus hic excluduntur, quae hoc modo integrationem respuunt seu ad aequationes primi gradus reduci nequeunt; qui casus exclusi omnes ex hoc criterio agnoscuntur, si non fuerit

$$AFF + BEE + CDD = ABC + 2DEF.$$

Huius generis est ista aequatio $\left(\frac{ddv}{dxdy}\right) = \left(\frac{ddv}{dz^2}\right)$, quae ergo tale integrale, cuiusmodi hic assumsimus, non admittit; neque etiam alia patet via eius integrale completum investigandi. Integralia autem particularia facile innumera exhiberi possunt et quae adeo functiones arbitrarias complectuntur, sed tantum unius quantitatis variabilis, quae in praesenti instituto nonnisi integralia particularia constituere sunt censendae. Si enim ponatur

$$v = \Gamma:(\alpha x + \beta y + \gamma z),$$

facta substitutione fieri debet $\alpha\beta = \gamma\gamma$ seu sumto $\gamma = 1$ debet esse $\alpha\beta = 1$; quare innumerabiles adeo huiusmodi formulae coniunctae satisfaciunt, ut sit

$$v = \Gamma:\left(\frac{\alpha}{\beta}x + \frac{\beta}{\alpha}y + z\right) + \Delta:\left(\frac{\gamma}{\delta}x + \frac{\delta}{\gamma}y + z\right) + \Sigma:\left(\frac{\varepsilon}{\zeta}x + \frac{\zeta}{\varepsilon}y + z\right) + \text{etc.},$$

ubi pro $\alpha, \beta, \gamma, \delta$ etc. numeros quoscunque accipere licet. Quamvis autem infinitae huiusmodi formulae diversae coniungantur, tamen integrale nonnisi pro particulari haberi potest. Ex quo intelligitur integrationem completam istius aequationis $\left(\frac{ddv}{dxdy}\right) = \left(\frac{ddv}{dz^2}\right)$ maximi esse momenti methodumque eo perveniendi fines Analyseos non mediocriter esse prolaturam.

Aequationes autem homogeneae tertii gradus multo maiorem restrictionem exigunt, ut integratio completa hoc modo succedat, uti sequenti problemate ostendetur.

PROBLEMA 88

505. *Aequationum homogenearum tertii gradus eos casus definire, quibus integrale completum per formam assumtam exhiberi seu ad formam aequationum homogenearum primi gradus reduci potest.*

SOLUTIO

In aequatione homogenea tertii gradus fingatur contineri haec primi gradus

$$a\left(\frac{dv}{dx}\right) + b\left(\frac{dv}{dy}\right) + c\left(\frac{dv}{dz}\right) = 0$$

quae ut satisfaciat aequationi tertii gradus

$$A\left(\frac{d^3v}{dx^3}\right)+B\left(\frac{d^3v}{dy^3}\right)+C\left(\frac{d^3v}{dz^3}\right)$$
$$+D\left(\frac{d^3v}{dx^2dy}\right)+E\left(\frac{d^3v}{dxdy^2}\right)+F\left(\frac{d^3v}{dx^2dz}\right)+G\left(\frac{d^3v}{dxdz^2}\right)+H\left(\frac{d^3v}{dy^2dz}\right)+I\left(\frac{d^3v}{dydz^2}\right)$$
$$+K\left(\frac{d^3v}{dxdydz}\right)=0,$$

necesse est, ut expressio haec algebraica

$$Ax^3+By^3+Cz^3+Dxxy+Exyy+Fxxz+Gxzz+Hyyz+Iyzz+Kxyz$$

factorem habeat $ax+by+cz$; nisi autem alter factor denuo in duos simplices sit resolubilis, ad aequationem homogeneam secundi gradus referetur, quae solutionem respuit. Quare ut integratio completa succedat, necesse est istam expressionem tribus constare factoribus simplicibus, qui sint

$$(ax+by+cz)(fx+gy+hz)(kx+my+nz),$$

hincque aequationis generalis coefficientes ita se habebunt

$$A=afk,\ B=bgm,\ C=chn,$$
$$D=afm+agk+bfk,\ \ E=agm+bfm+bgk,\ \ F=afn+ahk+cfk,$$
$$G=ahn+cfn+chk,\ \ H=bgn+bhm+cgm,\ I=bhn+cgn+chm,$$
$$K=agn+ahm+bfn+bhk+cfm+cgk$$

ac tum integrale completum erit

$$v=\Gamma:\left(\frac{x}{a}-\frac{z}{c}\text{ et }\frac{y}{b}-\frac{z}{c}\right)+\Delta:\left(\frac{x}{f}-\frac{z}{h}\text{ et }\frac{y}{g}-\frac{z}{h}\right)+\Sigma:\left(\frac{x}{k}-\frac{z}{n}\text{ et }\frac{y}{m}-\frac{z}{n}\right),$$

quilibet scilicet factor simplex praebet functionem arbitrariam duarum variabilium.

COROLLARIUM 1

506. In qualibet harum functionum variabiles *x, y, z* inter se permutare licet; quin etiam quaelibet quasi ex tribus variabilibus conflata spectari potest, prima nempe ex his

$$\frac{x}{a}-\frac{y}{b},\quad \frac{y}{b}-\frac{z}{c}\quad\text{et}\quad \frac{z}{c}-\frac{x}{a}$$

similique modo de ceteris.

COROLLARIUM 2

507. Si duo factores fuerint aequales, $f=a, g=b, h=c$, quo casu duae priores functiones in unam coalescerent, earum loco scribi debet

$$x\Gamma:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ et }\tfrac{y}{b}-\tfrac{z}{c}\right)+\Delta:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ et }\tfrac{y}{b}-\tfrac{z}{c}\right),$$

at si omnes tres fuerint aequales, ut insuper sit $k=a, m=b, n=c$, integrale completum erit

$$v=xx\Gamma:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ et }\tfrac{y}{b}-\tfrac{z}{c}\right)+x\Delta:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ et }\tfrac{y}{b}-\tfrac{z}{c}\right)+\Sigma:\left(\tfrac{x}{a}-\tfrac{z}{c}\text{ et }\tfrac{y}{b}-\tfrac{z}{c}\right).$$

COROLLARIUM 3

508. Quemadmodum hic duas priores partes per *xx* et *x* multiplicavimus, ita eas quoque per *yy* et *y*, item *zz* et *z* multiplicare possemus; perinde enim est, quanam variabili hic utamur, dum ne sit ea, quae forte sola post signum functionis occurrit; scilicet si esset $a=0$ et functiones quantitatum x et $\frac{y}{b}-\frac{z}{c}$ capi debeant, tum multiplicatores *xx* et *x* excludi deberent.

SCHOLION 1

509. Simili modo patet aequationes homogeneas quarti gradus hac methodo resolvi non posse, nisi in quatuor eiusmodi aequationes simplices resolvi et quasi earum producta spectari queant. Etsi enim hic revera nulla resolutio in factores locum habeat, tamen ex allatis exemplis clare perspicitur, quemadmodum ex aequatione differentiali homogenea cuiuscunque gradus expressio algebraica eiusdem gradus ternas variabiles *x, y, z* involvens debeat formari; quae si in factores simplices formae $ax+by+cz$ resolvi queat, simul inde aequationis differentialis integrale completum facile exhibebitur, cum quilibet factor functionem duarum variabilium suppeditet integralis partem constituentem, ita ut etiam haec pars seorsim sumta aequationi differentiali satisfaciat et pro integrali particulari haberi possit. At si illa expressio algebraica ita fuerit comparata, ut factores quidem habeat simplices, sed non tot, quot dimensiones, singuli quidem integralia particularia praebebunt, quae autem iunctim sumta non integrale completum suppeditabunt. Veluti si proponatur haec aequatio differentialis tertii gradus

$$a\left(\frac{d^3v}{dx^2dy}\right)+b\left(\frac{d^3v}{dxdy^2}\right)-a\left(\frac{d^3v}{dxdz^2}\right)-b\left(\frac{d^3v}{dydz^2}\right)=0,$$

quia forma algebraica

$$axxy+bxyy-axzz-byzz$$

factorem habet simplicem $ax+by$, illi utique satisfaciet valor

$$v = \Gamma:\left(\tfrac{x}{a} - \tfrac{y}{b} \text{ et } z\right)\ ;$$

pro integrali autem completo adhuc desunt duae functiones arbitrariae integrale completum huius aequationis

$$\left(\frac{ddv}{dxdy}\right) - \left(\frac{ddv}{dz^2}\right) = 0$$

continentes, ex qua quippe alter factor $xy - zz$ illius expressionis nascitur. Quoties ergo hae expressiones algebraicae ex aequationibus differentialibus homogeneis altiorum graduum formatae resolutionem in factores, etsi non simplices, admittant, hinc saltem discimus, quomodo earum integratio ad aequationes inferiorum graduum revocari possit, quod in huiusmodi arduis investigationibus sine dubio rnaximi est momenti.

SCHOLION 2

510. Haec sunt, quae de functionibus trium variabilium ex data quadam differentialium relatione investigandis proferre potui, in quibus utique nonnisi prima elementa huius scientiae continentur, quorum ulterior evolutio sagacitati Geometrarum summo studio est commendanda. Tantum enim abest, ut hae speculationes pro sterilibus sint habendae, ut potius pleraque, quae adhuc in theoria motus fluidorum desiderantur, ad has Analyseos partes sublimiores sint referenda, quarum propterea utilitas neutiquam parti priori calculi integralis postponenda videtur.

Eo magis autem hae partes posteriores excoli merentur, quod theoria fluidorum adeo circa functiones quatuor variabilium versetur, quarum naturam ex aequationibus differentialibus secundi gradus investigari oportet, quam partem ob penuriam materiae ne attingere quidem volui. In hac autem theoria resolutio huius aequationis

$$\left(\frac{ddv}{dt^2}\right) = \left(\frac{ddv}{dx^2}\right) + \left(\frac{ddv}{dy^2}\right) + \left(\frac{ddv}{dz^2}\right)$$

maximi est momenti, ubi litterae $x, y,$ z ternas coordinatas, t vero tempus elapsum exprimunt harumque quatuor variabilium functio quaeritur, quae loco v substituta illi aequationi satisfaciat. Ex hactenus autem allatis facile colligitur integrale completum huius aequationis duas complecti debere functiones arbitrarias, quarum utraque sit functio trium variabilium, aliasque solutiones omnes minus late patentes pro incompletis esse habendas. Facili autem negotio innumeras solutiones partieulares exhibere licet; veluti si ponamus

$$v = \Gamma:\left(ax + \beta y + \gamma z + \delta t\right),$$

reperitur

$$\delta\delta = \alpha\alpha + \beta\beta + \gamma\gamma\ ;$$

quod cum infinitis modis fieri possit, infinitae huiusmodi functiones additae valorem idoneum pro v exhibebunt. Deinde etiam satisfaciunt isti valores

$$v=\frac{\Gamma:\left(t\pm\sqrt{(xx+yy+zz)}\right)}{\sqrt{(xx+yy+zz)}},\quad v=\frac{\Gamma:\left(y\pm\sqrt{(tt-yy-zz)}\right)}{\sqrt{(tt-yy-zz)}},$$

$$v=\frac{\Gamma:\left(x\pm\sqrt{(tt-yy-zz)}\right)}{\sqrt{(tt-yy-zz)}},\quad v=\frac{\Gamma:\left(z\pm\sqrt{(tt-xx-yy)}\right)}{\sqrt{(tt-xx-yy)}},$$

quorum quidem investigatio iam est difficilior. Cum autem hae functiones tantum sint unius variabilis, integralia maxime particularia exhibent, quae adeo etiamnum forent particularia, si pro v functiones binarum variabilium haberentur, quales autem ne suspicari quidem licet. Quare cum integrale completum duas adeo functiones arbitrarias trium variabilium complecti debeat, facile intelligitur, quantopere adhue ab hoc scopo simus remoti.