INTEGRAL CALCULUS

FINAL BOOK.

THE FIRST PART

OR

THE INVESTIGATION OF FUNCTIONS OF TWO VARIABLES FROM A GIVEN RELATION OF THE DIFFERENTIAL OF ANY ORDER.

SECTION THREE

THE INVESTIGATION OF FUNCTIONS OF TWO VARIABLES FROM A GIVEN RELATION OF THE DIFFERENTIALS OF THE THIRD AND OF HIGHER ORDERS

CHAPTER I

CONCERNING THE RESOLUTION OF THE MOST SIMPLE EQUATIONS INVOLVING A SINGLE DIFFERENTIAL FORMULA

PROBLEM 61

379. *To investigate the nature of functions of two variables x and y, if some formula of the differentials of the third order should vanish.*

SOLUTION

Let z be that function sought, and since there shall be four differential formulas of this of the third order

$$\left(\frac{d^3z}{dx^3}\right),\quad \left(\frac{d^3z}{dx^2dy}\right),\quad \left(\frac{d^3z}{dxdy^2}\right),\quad \text{and}\quad \left(\frac{d^3z}{dy^3}\right),$$

provided any of these is put in place equal to zero, we will have just as many cases to be explained.

I. Therefore in the first place let there be $\left(\frac{d^3z}{dx^3}\right)=0$ and on taking y constant the first integration gives

$$\left(\frac{d^2z}{dx^2}\right)=\Gamma{:}y\ ;$$

then in a like manner the second integration gives

$$\left(\frac{dz}{dx}\right)=x\Gamma{:}y+\Delta{:}y\,,$$

from which finally there becomes

$$z=\tfrac{1}{2}xx\Gamma{:}y+x\Delta{:}y+\Sigma y\,,$$

where $\Gamma{:}y$, $\Delta{:}y$ and $\Sigma{:}y$ denote any functions of y, thus so that on account of the threefold integration there shall be three arbitrary functions entered into the calculation, as the nature of the problem demands.

II. Let there be $\left(\frac{d^3z}{dx^2dy}\right)=0$ and with the first two integrated by the variability of x only so that there is found as before

$$\left(\frac{dz}{dy}\right)=x\Gamma'{:}y+\Delta'{:}y$$

but now with only y considered for variation we come upon

$$z = x\Gamma{:}y + \Delta{:}y + \Sigma{:}x,$$

since the letters with the signs of the inscribed functions here always will have this meaning, so that there shall be

$$\int dy\Gamma'{:}y = \Gamma{:}y \quad \text{and} \quad \int dy\Delta'{:}y = \Delta{:}y.$$

III. Let there be $\left(\frac{d^3z}{dxdy^2}\right) = 0$, and because here the case does not differ from the preceding, except that the two variables x and y shall be interchanged among themselves, the integral sought is

$$z = y\Gamma{:}x + \Delta{:}x + \Sigma{:}y$$

IV. Let there be $\left(\frac{d^3z}{dy^3}\right) = 0$ and because of the similar change from the first case there is understood to become

$$z = \tfrac{1}{2}yy\Gamma{:}x + y\Delta{:}x + \Sigma x.$$

COROLLARY 1

380. Three arbitrary functions either of x or of y only have been introduced here by the threefold integration; all three are of y only in the first case $\left(\frac{d^3z}{dx^3}\right) = 0$, truly only of x in the fourth case $\left(\frac{d^3z}{dy^3}\right) = 0$; truly two are functions of y and one of x in the second case $\left(\frac{d^3z}{dx^2dy}\right) = 0$; but on the other hand two are of x and one of y in the third case $\left(\frac{d^3z}{dxdy^2}\right) = 0$.

COROLLARY 2

381. Again it will be helpful to observe, if two or more arbitrary functions occur of the same variable, for example x, one indeed to be put in place absolutely, the other to be multiplied by y, truly the third, if present, by $\tfrac{1}{2}yy$ or, because it returns the same thing, to be agreed to be multiplied by yy.

COROLLARY 3

382. But it is to be understood that these functions thus are to be left by our choice, so that also discontinuous functions or outside the law of continuity are not to be excluded. Clearly if some line is described freely by hand, the applied line [*i.e.* the y-coordinate] corresponding to the abscissa x will refer to a function $\Gamma{:}x$ of this kind.

SCHOLIUM 1

383. Here I decide to spend less time over the transformation of differential formulas of higher order, while some others are introduced into the calculation in place of the two variables *x* and *y*, because in general the expressions become exceedingly complicated and will have scarcely any use, then indeed especially, since the method for finding these transformations has been treated well enough above (§ 229). Yet a simpler case, in which two new variables *t* and *u* are to be introduced in place of *x* and *y* thus is are taken, so that there shall be

$$t=\alpha x+\beta y \quad \text{and} \quad u=\gamma x+\delta y\,,$$

this also I will adapt to higher differential formulas. Therefore since we have seen [§ 233] with formulas of the first order

$$\left(\frac{dz}{dx}\right)=\alpha\left(\frac{dz}{dt}\right)+\gamma\left(\frac{dz}{du}\right),\quad \left(\frac{dz}{dy}\right)=\beta\left(\frac{dz}{dt}\right)+\delta\left(\frac{dz}{du}\right)$$

and with formulas of the second order

$$\left(\frac{ddz}{dx^2}\right)=\alpha^2\left(\frac{ddz}{dt^2}\right)+2\alpha\gamma\left(\frac{ddz}{dtdu}\right)+\gamma^2\left(\frac{ddz}{du^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right)=\alpha\beta\left(\frac{ddz}{dt^2}\right)+\left(\alpha\delta+\beta\gamma\right)\left(\frac{ddz}{dtdu}\right)+\gamma\delta\left(\frac{ddz}{du^2}\right),$$
$$\left(\frac{ddz}{dy^2}\right)=\beta^2\left(\frac{ddz}{dt^2}\right)+2\beta\delta\left(\frac{ddz}{dtdu}\right)+\delta^2\left(\frac{ddz}{du^2}\right),$$

with formulas of the third order there will be

$$\left(\frac{d^3z}{dx^3}\right)=\alpha^3\left(\frac{ddz}{dt^2}\right)+3\alpha^2\gamma\left(\frac{d^3z}{dt^2du}\right)+3\alpha\gamma^2\left(\frac{d^3z}{dtdu^2}\right)+\gamma^3\left(\frac{d^3z}{du^3}\right),$$
$$\left(\frac{d^3z}{dx^2dy}\right)=\alpha^2\beta\left(\frac{d^3z}{dt^3}\right)+\left(\alpha^2\delta+2\alpha\beta\gamma\right)\left(\frac{d^3z}{dt^2du}\right)+\left(\beta\gamma^2+2\alpha\gamma\delta\right)\left(\frac{d^3z}{dtdu^2}\right)+\gamma^2\delta\left(\frac{d^3z}{du^3}\right),$$
$$\left(\frac{d^3z}{dxdy^2}\right)=\alpha\beta^2\left(\frac{d^3z}{dt^3}\right)+\left(\beta^2\gamma+2\alpha\beta\gamma\right)\left(\frac{d^3z}{dt^2du}\right)+\left(\alpha\delta^2+2\beta\gamma\delta\right)\left(\frac{d^3z}{dtdu^2}\right)+\gamma^2\delta\left(\frac{d^3z}{du^3}\right),$$
$$\left(\frac{d^3z}{dy^3}\right)=\beta^3\left(\frac{ddz}{dt^2}\right)+3\beta^2\delta\left(\frac{d^3z}{dt^2du}\right)+3\beta\delta^2\left(\frac{d^3z}{dtdu^2}\right)+\delta^3\left(\frac{d^3z}{du^3}\right),$$

and with formulas of the fourth order :

	$\left(\frac{d^4z}{dt^4}\right)$	$\left(\frac{d^4z}{dt^3du}\right)$	$\left(\frac{d^4z}{dt^2du^2}\right)$	$\left(\frac{d^4z}{dtdu^3}\right)$	$\left(\frac{d^4z}{du^4}\right)$
$\left(\frac{d^4z}{dx^4}\right)=$	α^4	$4\alpha^3\gamma$	$6\alpha^2\gamma^2$	$4\alpha\gamma^3$	γ^4
$\left(\frac{d^4z}{dx^3dy}\right)=$	$\alpha^3\beta$	$\alpha^3\delta+3\alpha^2\beta\gamma$	$3\alpha^2\gamma\delta+3\alpha\beta\gamma^2$	$3\alpha\gamma^2\delta+\beta\gamma^3$	$\gamma^3\delta$
$\left(\frac{d^4z}{dx^2dy^2}\right)=$	$\alpha^2\beta^2$	$2\alpha^2\beta\delta+3\alpha\beta^2\gamma$	$\alpha^2\delta^2+4\alpha\beta\gamma\delta+\beta^2\gamma^2$	$2\alpha\gamma\delta^2+2\beta\gamma^2\delta$	$\gamma^2\delta^2$
$\left(\frac{d^4z}{dxdy^3}\right)=$	$\alpha\beta^3$	$3\alpha\beta^2\delta+\beta^3\gamma$	$3\alpha\beta\delta^2+3\beta^2\gamma\delta$	$\alpha\delta^3+3\beta\gamma\delta^2$	$\gamma\delta^3$
$\left(\frac{d^4z}{dy^4}\right)=$	β^4	$4\beta^3\delta$	$6\beta^2\delta^2$	$4\beta\delta^3$	δ^4

from which likewise the law for higher orders can be elucidated; evidently for the general form $\left(\frac{d^{m+n}z}{dx^m dy^n}\right)$, these coefficients are the same as arise from the workings of this formula $(\alpha+\gamma v)^m(\beta+\delta v)^n$, if indeed the terms of the powers of v are set aside.

SCHOLIUM 2

384. I observe that the evolution of this formula is connected to the principles established previously, shown with great care [§201 of the *Differential Calculus*].
Therefore let there be

$$s=(\alpha+\gamma v)^m(\beta+\delta v)^n$$

and there may be put

$$s=A+Bv+Cv^2+Dv^3+Ev^4+Fv^5+\text{etc.},$$

where indeed in the first place it is apparent that $A=\alpha^m\beta^n$; truly with the remaining coefficients found from the differentials of the logarithms we will have

$$\frac{ds}{sdv}=\frac{m\gamma}{\alpha+\gamma v}+\frac{n\delta}{\beta+\delta v}$$

and thus

$$\frac{ds}{dv}\left(\alpha\beta+(\alpha\delta+\beta\gamma)v+\gamma\delta vv\right)-s\left(m\beta\gamma+n\alpha\delta+(m+n)\gamma\delta v\right)=0;$$

where if in place of *s*, the series assumed is substituted, this equation will arise

$$\begin{aligned}
0 = a\beta B \quad &+ 2\alpha\beta Cv \quad + 3\alpha\beta Dv^2 \quad + 4\alpha\beta Ev^3 \quad + 5\alpha\beta Fv^4 + \text{ etc.}\\
&+ \alpha\delta B \quad + 2\alpha\delta C \quad + 3\alpha\delta D \quad + 4\alpha\delta E\\
&+ \beta\gamma B \quad + 2\beta\gamma C \quad + 3\beta\gamma D \quad + 4\beta\gamma E\\
&\qquad\quad + \gamma\delta B \quad + 2\gamma\delta C \quad + 3\gamma\delta D\\
-m\beta\gamma A \quad &- m\beta\gamma B \quad - m\beta\gamma C \quad - m\beta\gamma D \quad - m\beta\gamma E\\
-n\alpha\delta A \quad &- n\alpha\delta B \quad - n\alpha\delta C \quad - n\alpha\delta D \quad - n\alpha\delta E\\
&-(m+n)\gamma\delta A-(m+n)\gamma\delta B-(m+n)\gamma\delta C-(m+n)\gamma\delta D,
\end{aligned}$$

from which any coefficient from the preceding thus may be defined

$$\begin{aligned}
A &= \alpha^m\beta^n,\\
B &= \tfrac{m\beta\gamma+n\alpha\delta}{\alpha\beta},\\
C &= \tfrac{(m-1)\beta\gamma+(n-1)\alpha\delta}{2\alpha\beta}B+\tfrac{(m+n)\gamma\delta}{2\alpha\beta}A,\\
D &= \tfrac{(m-2)\beta\gamma+(n-2)\alpha\delta}{3\alpha\beta}C+\tfrac{(m+n-1)\gamma\delta}{3\alpha\beta}B,\\
E &= \tfrac{(m-3)\beta\gamma+(n-3)\alpha\delta}{4\alpha\beta}D+\tfrac{(m+n-2)\gamma\delta}{4\alpha\beta}C\\
&\text{etc.}
\end{aligned}$$

Therefore with these coefficients found if there is put

$$t = \alpha x + \beta y \quad \text{and} \quad u = \gamma x + \delta y,$$

the transformation of any differential formulas thus will be had, so that there becomes

$$\left(\tfrac{d^{m+n}z}{dx^m dy^n}\right) = A\left(\tfrac{d^{m+n}z}{dt^{m+n}}\right)+B\left(\tfrac{d^{m+n}z}{dt^{m+n-1}du}\right)+C\left(\tfrac{d^{m+n}z}{dt^{m+n-2}du^2}\right)+\text{etc.}$$

PROBLEM 62

385. *To investigate the nature of functions of two variables x and y, if the formula for the differentials of any order vanishes.*

SOLUTION

From these, which we have shown concerning the differential formulas of the third order equal to zero in the preceding problem, it is clear enough how the solution of this problem for differential formulas of the fourth order thus may be had.

I. If there shall be $\left(\frac{d^4z}{dx^4}\right)=0$, then there will be $z=x^3\Gamma:y+x^2\Delta:y+x\Sigma:y+\Theta:y$.

II. If there shall be $\left(\frac{d^4z}{dx^3dy}\right)=0$, then there will be $z=x^2\Gamma:y+x\Delta:y+\Sigma:y+\Theta:x$.

III. If there shall be $\left(\frac{d^4z}{dx^2dy^2}\right)=0$, then there will be $z=x\Gamma:y+\Delta:y+y\Sigma:x+\Theta:x$.

IV. If there shall be $\left(\frac{d^4z}{dxdy^3}\right)=0$, then there will be $z=\Gamma:y+y^2\Delta:y+y\Sigma:x+\Theta:x$.

V. If there shall be $\left(\frac{d^4z}{dy^4}\right)=0$, then there will be $z=y^3\Gamma:x+y^2\Delta:x+y\Sigma:x+\Theta:x$;

from which likewise the progression to higher orders is evident.

COROLLARIUM 1

386. Since here four arbitrary functions occur, evidently it is required to put in place just as many integrations, that may be present according to the criterion of the complete integral itself.

COROLLARY 2

387. Then it is easy in turn to show the forms of the equation found to satisfy the proposed equation. Since thus we have found for the third case

$$z=x\Gamma:y+\Delta:\mathrm{y}+y\Sigma:x+\Theta:x,$$

hence on differentiation we deduce

in the first place $\left(\frac{dz}{dx}\right)=\Gamma:y+y\Sigma':x+\Theta':x$,

then $\left(\frac{ddz}{dx^2}\right)=y\Sigma'':x+\Theta'':x$, in the third place $\left(\frac{ddz}{dx^2dy}\right) = \Sigma'':x,$

and

in the fourth place $\left(\frac{ddz}{dx^2dy^2}\right) = 0$

and the same is arrived at, in whatever order the differentiations are put in place, either with the variable x or y taken alone.

SCHOLIUM 1

388. So far we have assumed one differential formula to be equal to zero ; but the calculation succeeds likewise, if a formula equal to any function of this kind of x and y is put in place, just as I am going to assume in the following problems. Yet I consider this needs to be impressed on us, if V were some function of the two variables x et y, then $\int Vdx$ denotes that integral, which may be obtained, if x alone may be considered to be variable, in this truly the formula $\int Vdy$ only has y for a variable ; which likewise is required to be held for repeated integrals such as $\int dx\int Vdx$, where in each only x is taken to be variable, in this truly $\int dy\int Vdx$, after the integration $\int Vdx$ the variability of x only should have been removed, then in the other integration $\int dy\int Vdx$ the variability is to be taken from y. And since likewise, whichever integration is first put in place, also this distinction by means of the integral signs can be removed and this twin integral $\iint Vdxdy$ thus can be shown; and hence it is understood, how it is required to interpret these formulas

$$\iiint Vdxdy \quad \text{or} \quad \int^3 Vdx^2dy \quad \text{and} \quad \int^{m+n} Vdx^m dy^n\ ;$$

it is evident here that we fix indices above the integration sign $\int$, in short just as the formulas are accustomed to be affixed with the differentiation sign d, which indicate obviously, how many times the integration is to be repeated.

SCHOLIUM 2

389. Here we take these repeated integrals thus to be put in place here, so that no relation between the two variables x and y may be called upon to help ; which circumstances therefore are to be attended to with great care, since generally, when there is a need for such integrations, the calculation must be put in place directly in a different manner. For if indeed for some proposed body the volume or surface shall be investigated by geometry, the formula must be presented by a two fold integration of this kind $\iint Vdxdy$, with some function V of x and y arising; where indeed in the first place the integral $\int Vdy$ is sought with x regarded as constant, but it is required to consider the prescribed limits of the integral in the completed integration, provided of course it is prescribed for the one limit, that here the integral $\int Vdy$ should vanish on putting $y = 0$, truly with the other, that for it to be extended as far as to make y equal to some given function of x. Then truly, after this [has been set up] the integral $\int Vdy$ can be found in that manner, finally the other integration is undertaken for the formula $dx\int Vdy$, in which the quantity y is no longer present, then in place of this a certain function of x has been substituted and that formula has returned now to include a single variable x. Therefore here it is to be agreed that in the first complete integration of the variable y is changed into a function of x, which therefore in the other integration, where x is variable, will be allowed to regard the smallest value as the constant. From which it is apparent that this case with the whole space [or volume divided up] is different from these repeating integrals which we consider here ; accordingly here therefore we consider that less, with that special account of this only able to have a place in the formula $\iint Vdxdy$, truly with the rest, the attention is directed to where either the differential dx or dy is repeated more often. As on account of this reason a whole relation, which perhaps can be put in place by one integration completed between the two variables x and y, we may deservedly remove.

PROBLEM 63

390. *If a certain formula of the third or of higher order is equal to some function of the two variables x and y, to define the nature of the function z.*

SOLUTION

Let V be some function of the two variables x and y and beginning from the formulas of the third order there shall be in the first place $\left(\frac{d^3z}{dx^3}\right) = V$ and on putting x to be variable there will be

$$\left(\frac{ddz}{dx^2}\right) = \int Vdx + \Gamma{:}y\,;$$

then again truly,

$$\left(\frac{dz}{dx}\right) = \int dx\int Vdx + x\Gamma{:}y + \Delta{:}y = \iint Vdx^2 + x\Gamma{:}y + \Delta{:}y$$

and finally

$$z = \int^3 Vdx^3 + \tfrac{1}{2}x^2\Gamma{:}y + x\Delta{:}y + \Sigma{:}y\,.$$

In a similar manner it is apparent, if there should be $\left(\frac{d^3z}{dx^2dy}\right)=V$, to become

$$z=\int^3 Vdx^2dy+x\Gamma:y+\Delta:y+\Sigma:x\,,$$

and if there shall be $\left(\frac{d^3z}{dxdy^2}\right)=V$, there will be

$$z=\int^3 Vdxdy^2+\Gamma:y+y\Delta:x+\Sigma:x\ ;$$

if there shall be $\left(\frac{d^3z}{dy^3}\right)=V$, then there becomes

$$z=\int^3 Vdy^3+y^2\Gamma:x+y\Delta:x+\Sigma:x.$$

In the same manner we may find on progressing to formulas of higher grades, as follows; if there shall be $\left(\frac{d^4z}{dx^4}\right)=V$, to become

$$z=\int^4 Vdx^4+x^3\Gamma:y+x^2\Delta:y+x\Sigma:y+\Theta:y\ ;$$

if there shall be $\left(\frac{d^4z}{dx^3dy}\right)=V$, to become

$$z=\int^4 Vdx^3dy+x^2\Gamma:y+x\Delta:y+\Sigma:y+\Theta:x$$

if there shall be $\left(\frac{d^4z}{dx^2dy^2}\right)=V$, to become

$$z=\int^4 Vdx^2dy^2+x\Gamma:y+\Delta:y+y\Sigma:x+\Theta:x\ ;$$

if there shall be $\left(\frac{d^4z}{dxdy^3}\right)=V$, to become

$$z=\int^4 Vdxdy^3+\Gamma:y+y^2\Delta:x+y\Sigma:x+\Theta:x\ ;$$

if there shall be $\left(\frac{d^4z}{dy^4}\right)=V$, to become

$$z=\int^4 Vdy^4+y^3\Gamma:x+y^2\Delta:x+y\Sigma:x+\Theta:x\ ;$$

nor is there any need for further explanations for higher orders.

COROLLARY 1

391. Just as the usual integral sign in the first book now accordingly involves a constant introduced by the integration, thus here also arbitrary functions are agreed to be introduced now into the formula involving the integral, thus so that there is no need to express these.

COROLLARY 2

392. Therefore it is sufficient for the equation $\left(\frac{d^3z}{dx^3}\right)=V$ with the triple integral to be given in this manner $z=\int^3 Vdx^3$, which form now with the ability to include the above parts added

$$xx\Gamma{:}y+x\Delta{:}y+\Sigma{:}y\ ;$$

which likewise is understood for the rest.

COROLLARY 3

393. If therefore in general this equation is considered

$$\left(\frac{d^{m+n}z}{dx^m dy^n}\right)=V$$

the integral of this may be shown at once in this way

$$z=\int^{m+n} Vdx^m dy^n\ ,$$

which now has the ability to include all these arbitrary functions put together by the same number of integrations from the number $m+n$.

SCHOLIUM

394. These cases certainly are the most simple, which may be seen to be referred to according to this chapter, but with the difficulty for greater complexity it is necessary to establish certain precepts, since at this stage this part of the integral calculus has scarcely began to be worked on. Yet meanwhile it is understood, if more complicated equations are able to be recalled as simpler equations by the aid of certain transformations, also the integration of these will be able to be brought out; because here indeed the work may be considered to be pursued without much reward. Therefore I proceed to more abstruse cases and these have been prepared thus, so that they are able to be solved with the help of equations of lesser orders, from which indeed a conspicuous method will be deduced extending to greater use, which it will be possible to use more often successfully. Nor yet in this development is it agreed to be particularly abstract, however it suffices that certain known resources are indeed brought to light at this stage.

CALCULI INTEGRALIS
LIBER POSTERIOR.
PARS PRIMA
SEU
INVESTIGATIO FUNCTIONUM DUARUM
VARIABILIUM EX DATA DIFFERENTIALIUM
CUIUSVIS GRADUS RELATIONE.

SECTIO TERTIA

INVESTIGATIO DUARUM VARIABILIUM FUNCTIONUM EX DATA DIFFERENTIALIUM TERTII ALTIORUMQUE GRADUUM RELATIONE

CAPUT I

DE RESOLUTIONE AEQUATIONUM SIMPLICISSIMARUM UNICAM FORMULAM DIFFERENTIALEM INVOLVENTIUM

PROBLEMA 61

379. *Indolem functionis binarum variabilium x et y indagare, si eius quaepiam formula differentialis tertii gradus evanescat.*

SOLUTIO

Sit z functio illa quaesita, et cum eius sint quatuor formulae differentiales tertii gradus

$$\left(\frac{d^3z}{dx^3}\right), \quad \left(\frac{d^3z}{dx^2dy}\right), \quad \left(\frac{d^3z}{dxdy^2}\right), \quad \text{et} \quad \left(\frac{d^3z}{dy^3}\right),$$

prout quaelibet harum nihilo aequalis statuitur, totidem habemus casus evolvendos.

I. Sit igitur primo $\left(\frac{d^3z}{dx^3}\right)=0$ et sumta y constante prima integratio praebet

$$\left(\frac{d^2z}{dx^2}\right)=\Gamma{:}y\ ;$$

tum simili modo secunda integratio dat

$$\left(\frac{dz}{dx}\right)=x\Gamma{:}y+\Delta{:}y\,,$$

unde tandem fit

$$z=\tfrac{1}{2}xx\Gamma{:}y+x\Delta{:}y+\Sigma y\,,$$

ubi $\Gamma{:}y$, $\Delta{:}y$ et $\Sigma{:}y$ denotant functiones quascunque ipsius y, ita ut ob triplicem integrationem tres functiones arbitrariae in calculum sint ingressae, ut rei natura postulat.

II. Sit $\left(\frac{d^3z}{dx^2dy}\right)=0$ ac primo bis integrando per solius x variabilitatem reperitur ut ante

$$\left(\frac{dz}{dy}\right)=x\Gamma'{:}y+\Delta'{:}y$$

nunc autem sola y pro variabili habita adipiscimur

$$z=x\Gamma{:}y+\Delta{:}y+\Sigma{:}x\,,$$

quandoquidem apices signis functionum inscripti hic semper hunc habent significatum, ut sit

$$\int dy\Gamma':y = \Gamma:y \quad \text{et} \quad \int dy\Delta':y = \Delta:y\,.$$

III. Sit $\left(\frac{d^3z}{dxdy^2}\right) = 0$, et quia hic casus a praecedente non differt, nisi quod binae variabiles *x* et *y* inter se sint permutatae, integrale quaesitum est

$$z = y\Gamma:x + \Delta:x + \Sigma:y$$

IV. Sit $\left(\frac{d^3z}{dy^3}\right) = 0$ et ob similem permutationem ex casu primo intelligitur fore

$$z = \tfrac{1}{2}yy\Gamma:x + y\Delta:x + \Sigma x\,.$$

COROLLARIUM 1

380. Tres functiones arbitrariae hic per triplicem integrationem ingressae sunt vel ipsius *x* vel ipsius *y* tantum; omnes tres sunt ipsius *y* tantum casu primo $\left(\frac{d^3z}{dx^3}\right) = 0$, ipsius *x* vero tantum casu quarto $\left(\frac{d^3z}{dy^3}\right) = 0$; duae vero sunt ipsius *y* et una ipsius *x* casu secundo $\left(\frac{d^3z}{dx^2dy}\right) = 0$; contra autem duae ipsius *x* et una ipsius *y* casu tertio $\left(\frac{d^3z}{dxdy^2}\right) = 0$.

COROLLARIUM 2

381. Porro observasse iuvabit, si eiusdem variabilis, puta *x*, duae pluresve occurrant functiones arbitrariae, unam quidem absolute poni, alteram per *y* multiplicari, tertiam vero, si adsit, per $\frac{1}{2}yy$ seu, quod eodem redit, per *yy* multiplicatam accedere.

COROLLARIUM 3

382. Perpetuo autem tenendum est has functiones ita arbitrio nostro relinqui, ut etiam functiones discontinuae seu nulla continuitatis lege contentae non excludantur. Scilicet si libero manus tractu linea quaecunque describatur, applicata respondens abscissae *x* huiusmodi functionem $\Gamma:x$ referet.

SCHOLION 1

383. Minus hic immorandum arbitror transformationi formularum differentialium altioris gradus, dum loco binarum variabilium *x* et *y* aliae quaecunque in calculum introducuntur, quoniam in genere expressiones nimis fierent complicatae vixque ullum usum habiturae, tum vero imprimis, quod methodus has transformationes inveniendi iam supra (§ 229) satis luculenter est tradita. Casum tantum simpliciorem, quo binae novae variabiles *t* et *u* loco *x* et *y* introducendae ita accipiuntur, ut sit

$$t=\alpha x+\beta y \quad \text{et} \quad u=\gamma x+\delta y,$$

hic quoque ad formulas differentiales altiores accommodabo. Cum igitur viderimus esse [§ 233] pro formulis primi gradus

$$\left(\tfrac{dz}{dx}\right)=\alpha\left(\tfrac{dz}{dt}\right)+\gamma\left(\tfrac{dz}{du}\right),\quad \left(\tfrac{dz}{dy}\right)=\beta\left(\tfrac{dz}{dt}\right)+\delta\left(\tfrac{dz}{du}\right)$$

et pro formulis secundi gradus

$$\left(\tfrac{ddz}{dx^2}\right)=\alpha^2\left(\tfrac{ddz}{dt^2}\right)+2\alpha\gamma\left(\tfrac{ddz}{dtdu}\right)+\gamma^2\left(\tfrac{ddz}{du^2}\right),$$
$$\left(\tfrac{ddz}{dxdy}\right)=\alpha\beta\left(\tfrac{ddz}{dt^2}\right)+\left(\alpha\delta+\beta\gamma\right)\left(\tfrac{ddz}{dtdu}\right)+\gamma\delta\left(\tfrac{ddz}{du^2}\right),$$
$$\left(\tfrac{ddz}{dy^2}\right)=\beta^2\left(\tfrac{ddz}{dt^2}\right)+2\beta\delta\left(\tfrac{ddz}{dtdu}\right)+\delta^2\left(\tfrac{ddz}{du^2}\right),$$

erit pro formulis tertii gradus

$$\left(\tfrac{d^3z}{dx^3}\right)=\alpha^3\left(\tfrac{ddz}{dt^2}\right)+3\alpha^2\gamma\left(\tfrac{d^3z}{dt^2du}\right)+3\alpha\gamma^2\left(\tfrac{d^3z}{dtdu^2}\right)+\gamma^3\left(\tfrac{d^3z}{du^3}\right),$$
$$\left(\tfrac{d^3z}{dx^2dy}\right)=\alpha^2\beta\left(\tfrac{d^3z}{dt^3}\right)+\left(\alpha^2\delta+2\alpha\beta\gamma\right)\left(\tfrac{d^3z}{dt^2du}\right)+\left(\beta\gamma^2+2\alpha\gamma\delta\right)\left(\tfrac{d^3z}{dtdu^2}\right)+\gamma^2\delta\left(\tfrac{d^3z}{du^3}\right),$$
$$\left(\tfrac{d^3z}{dxdy^2}\right)=\alpha\beta^2\left(\tfrac{d^3z}{dt^3}\right)+\left(\beta^2\gamma+2\alpha\beta\gamma\right)\left(\tfrac{d^3z}{dt^2du}\right)+\left(\alpha\delta^2+2\beta\gamma\delta\right)\left(\tfrac{d^3z}{dtdu^2}\right)+\gamma^2\delta\left(\tfrac{d^3z}{du^3}\right),$$
$$\left(\tfrac{d^3z}{dy^3}\right)=\beta^3\left(\tfrac{ddz}{dt^2}\right)+3\beta^2\delta\left(\tfrac{d^3z}{dt^2du}\right)+3\beta\delta^2\left(\tfrac{d^3z}{dtdu^2}\right)+\delta^3\left(\tfrac{d^3z}{du^3}\right)$$

et pro formulis quarti gradus

$\left(\tfrac{d^4z}{dt^4}\right)$	$\left(\tfrac{d^4z}{dt^3du}\right)$	$\left(\tfrac{d^4z}{dt^2du^2}\right)$	$\left(\tfrac{d^4z}{dtdu^3}\right)$	$\left(\tfrac{d^4z}{du^4}\right)$
$\left(\tfrac{d^4z}{dx^4}\right)=\alpha^4$	$4\alpha^3\gamma$	$6\alpha^2\gamma^2$	$4\alpha\gamma^3$	γ^4
$\left(\tfrac{d^4z}{dx^3dy}\right)=\alpha^3\beta$	$\alpha^3\delta+3\alpha^2\beta\gamma$	$3\alpha^2\gamma\delta+3\alpha\beta\gamma^2$	$3\alpha\gamma^2\delta+\beta\gamma^3$	$\gamma^3\delta$
$\left(\tfrac{d^4z}{dx^2dy^2}\right)=\alpha^2\beta^2$	$2\alpha^2\beta\delta+3\alpha\beta^2\gamma$	$\alpha^2\delta^2+4\alpha\beta\gamma\delta+\beta^2\gamma^2$	$2\alpha\gamma\delta^2+2\beta\gamma^2\delta$	$\gamma^2\delta^2$
$\left(\tfrac{d^4z}{dxdy^3}\right)=\alpha\beta^3$	$3\alpha\beta^2\delta+\beta^3\gamma$	$3\alpha\beta\delta^2+3\beta^2\gamma\delta$	$\alpha\delta^3+3\beta\gamma\delta^2$	$\gamma\delta^3$
$\left(\tfrac{d^4z}{dy^4}\right)=\beta^4$	$4\beta^3\delta$	$6\beta^2\delta^2$	$4\beta\delta^3$	δ^4

unde simul lex pro altioribus gradibus elucet; pro formula scilicet generali $\left(\frac{d^{m+n}z}{dx^m dy^n}\right)$ hi coefficientes iidem sunt, qui oriuntur ex evolutione huius formae $(\alpha+\gamma v)^m(\beta+\delta v)^n$, siquidem termini secundum potestates ipsius v disponantur.

SCHOLION 2

384. Haud alienum fore arbitror evolutionem istius formulae ex principus ante stabilitis accuratius docere.
Sit igitur

$$s=(\alpha+\gamma v)^m(\beta+\delta v)^n$$

ac ponatur

$$s=A+Bv+Cv^2+Dv^3+Ev^4+Fv^5+\text{etc.},$$

ubi quidem primo patet esse $A=\alpha^m\beta^n$; pro reliquis vero coefficientibus inveniendis sumtis differentialibus logarithmorum habebimus

$$\frac{ds}{sdv}=\frac{m\gamma}{\alpha+\gamma v}+\frac{n\delta}{\beta+\delta v}$$

ideoque

$$\frac{ds}{dv}\left(\alpha\beta+(\alpha\delta+\beta\gamma)+\gamma\delta vv\right)-s\left(m\beta\gamma+n\alpha\delta+(m+n)\gamma\delta v\right)=0\,;$$

ubi si loco s series assumta substituatur, orietur haec aequatio

$$\begin{array}{llllll}
0=a\beta B & +2\alpha\beta Cv & +\ 3\alpha\beta Dv^2 & +4\alpha\beta Ev^3 & +\ 5\alpha\beta Fv^4 & +\ \text{etc.}\\
 & +\ \alpha\delta B & +\ 2\alpha\delta C & +3\alpha\delta D & +\ 4\alpha\delta E & \\
 & +\ \beta\gamma B & +2\beta\gamma C & +3\beta\gamma D & +\ 4\beta\gamma E & \\
 & & +\ \gamma\delta B & +2\gamma\delta C & +\ 3\gamma\delta D & \\
-m\beta\gamma A & -m\beta\gamma B & -m\beta\gamma C & -m\beta\gamma D & -m\beta\gamma E & \\
-n\alpha\delta A & -\ n\alpha\delta B & -\ n\alpha\delta C & -\ n\alpha\delta D & -n\alpha\delta E & \\
 & -(m+n)\gamma\delta A & -(m+n)\gamma\delta B & -(m+n)\gamma\delta C & -(m+n)\gamma\delta D &
\end{array}$$

unde quilibet coefficiens ex praecedentibus ita definitur

$$A = \alpha^m \beta^n,$$
$$B = \frac{m\beta\gamma + n\alpha\delta}{\alpha\beta},$$
$$C = \frac{(m-1)\beta\gamma + (n-1)\alpha\delta}{2\alpha\beta} B + \frac{(m+n)\gamma\delta}{2\alpha\beta} A,$$
$$D = \frac{(m-2)\beta\gamma + (n-2)\alpha\delta}{3\alpha\beta} C + \frac{(m+n-1)\gamma\delta}{3\alpha\beta} B,$$
$$E = \frac{(m-3)\beta\gamma + (n-3)\alpha\delta}{4\alpha\beta} D + \frac{(m+n-2)\gamma\delta}{4\alpha\beta} C$$
etc.

His igitur coefficientibus inventis si ponatur

$$t = \alpha x + \beta y \quad \text{et} \quad u = \gamma x + \delta y,$$

transformatio formulae differentialis cuiuscunque ita se habebit, ut sit

$$\left(\frac{d^{m+n}z}{dx^m dy^n}\right) = A\left(\frac{d^{m+n}z}{dt^{m+n}}\right) + B\left(\frac{d^{m+n}z}{dt^{m+n-1}du}\right) + C\left(\frac{d^{m+n}z}{dt^{m+n-2}du^2}\right) + \text{etc.}$$

PROBLEMA 62

385. *Indolem functionis binarum variabilium x et y investigare, si eius formula differentialis cuiuscunque gradus evanescat.*

SOLUTIO

Ex iis, quae de formulis differentialibus tertii gradus nihilo aequatis ostendimus in praecedente problemate, satis perspicunm est solutionem huius problematis pro formulis differentialibus quarti gradus ita se habere.

I. Si sit $\left(\frac{d^4z}{dx^4}\right) = 0$, erit

$$z = x^3\Gamma{:}y + x^2\Delta{:}y + x\Sigma{:}y + \Theta{:}y.$$

II. Si sit $\left(\frac{d^4z}{dx^3dy}\right) = 0$, erit

$$z = x^2\Gamma{:}y + x\Delta{:}y + \Sigma{:}y + \Theta{:}x.$$

III. Si sit $\left(\frac{d^4z}{dx^2dy^2}\right) = 0$, erit

$$z = x\Gamma{:}y + \Delta{:}y + y\Sigma{:}x + \Theta{:}x.$$

IV. Si sit $\left(\frac{d^4z}{dxdy^3}\right) = 0$, erit

$$z = \Gamma{:}y + y^2\Delta{:}y + y\Sigma{:}x + \Theta{:}x.$$

V. Si sit $\left(\frac{d^4z}{dy^4}\right)=0$, erit

$$z = y^3\Gamma{:}x + y^2\Delta{:}x + y\Sigma{:}x + \Theta{:}x;$$

unde simul progressus ad altiores gradus est manifestus.

COROLLARIUM 1

386. Cum hic quatuor functiones arbitrariae occurrant, totidem scilicet, quot integrationes institui oportet, in hoc ipso criterium integrationis completae continetur.

COROLLARIUM 2

387. Quin etiam vicissim facile ostenditur formas inventas aequationi propositae satisfacere. Sic cum pro casu tertio invenerimus

$$z = x\Gamma{:}y + \Delta{:}\mathrm{y} + y\Sigma{:}x + \Theta{:}x,$$

differentiando hinc colligimus

$$\text{primo} \quad \left(\frac{dz}{dx}\right) = \Gamma{:}y + y\Sigma'{:}x + \Theta'{:}x,$$

$$\text{deinde} \quad \left(\frac{ddz}{dx^2}\right) = y\Sigma''{:}x + \Theta''{:}x,$$

$$\text{tertio} \quad \left(\frac{ddz}{dx^2dy}\right) = \Sigma''{:}x,$$

et

$$\text{quarto} \quad \left(\frac{ddz}{dx^2dy^2}\right) = 0$$

eodemque pervenitur, quocunque ordine differentiationes vel solam *x* vel solam *y* variabilem sumendo instituantur.

SCHOLION 1

388. Hactenus unam formulam differentialem nihilo esse aequalem assumsimus; calculus autem perinde succedit, si huius modi formula functioni cuicunque ipsarum *x* et *y* aequalis statuatur, quemadmodum in sequentibus problematibus sum ostensurus. Hoc tantum inculcandum censeo, si V fuerit functio quaecunque binarum variabilium *x* et *y*, tum $\int Vdx$ id denotare integrale, quod obtinetur, si sola *x* pro variabili habeatur, in hac vero formula $\int Vdy$ solam *y* pro variabili haberi; quod idem tenendum est de integrationibus. repetitis veluti $\int dx\int Vdx$, ubi in utraque sola *x* variabilis assumitur, in hac vero $\int dy\int Vdx$, postquam integrale $\int Vdx$ ex sola ipsius *x* variabilitate fuerit erutum, tum in altera integratione $\int dy \int V dx$ solam *y* variabilem accipiendam esse. Et cum perinde, utra integratio prior instituatur, etiam hoc discrimen e modo signandi tolli potest hocque integrale geminatum ita $\iint V dx dy$ exhiberi; hincque intelligitur, quomodo has formulas

$$\iiint V dx dy \quad \text{seu} \quad \int^3 V dx^2 dy \quad \text{et} \quad \int^{m+n} V dx^m dy^n$$

interpretari oporteat; hic scilicet signo integrationis $\int$ indices suffigimus, prorsus uti signo differentiationis *d* suffigi solent, quippe qui indicant, quoties integratio sit repetenda.

SCHOLION 2

389. Singulas has integrationes repetendas ita institui hic assumimus, ut nulla relatio inter binas variabiles *x* et *y* in subsidium vocetur; quae circumstantia eo diligentius est animadvertenda, cum vulgo, ubi talibus integrationibus opus est, calculus prorsus diverso modo institui debeat. Quodsi enim proposito quopiam corpore geometrico eius soliditas seu superficies sit investiganda, per duplicem integrationem huiusmodi formula $\iint V dx dy$ evolvi debet existente *V* certa functione ipsarum *x* et *y;* ubi quidem primo quaeritur integrale $\int V dy$ spectata *x* ut constante, at absoluta integratione ad terminos integrationi praescriptos respici oportet, dum scilicet altero praescribitur, ut hoc integrale $\int V dy$ evanescat posito $y = 0$, altero vero id eo usque extendendum est, donec *y* datae cuipiam functioni ipsius *x* aequetur. Tum vero, postquam hoc integrale $\int V dy$ isto modo fuerit determinatum, altera demum integratio formulae $dx \int V dy$ suscipitur, in qua quantitas *y* non amplius inest, dum eius loco certa quaepiam functio ipsius *x* est substituta eaque formula iam revera unicam variabilem *x* complectitur. Hic ergo prima integratione absoluta variabilis *y* in functionem ipsius *x* abire est censenda, quam propterea in altera integratione, ubi *x* est variabilis, minime ut constantem spectare licebit. Ex quo patet hunc casum toto coelo esse diversum ab iis integrationibus repetendis, quas hic contemplamur; ad quem propterea hic eo minus respicimus, cuius ista peculiaris ratio tantum in formula $\iint V dx dy$ locum habere possit, reliquis vero, ubi alterum differentiale *dx* vel *dy* saepius repetitur, adeo adversetur. Quam ob causam hinc omnem relationem, quae forte peracta una integratione inter binas variabiles *x* et *y* statui posset, merito removemus.

PROBLEMA 63

390. *Si formula quaepiam differentialis tertii altiorisve gradus aequetur functioni cuicunque binarum variabilium x et y, indolem functionis z definire.*

SOLUTIO

Sit V functio quaecunque binarum variabilium x et y et incipientes a formulis tertii ordinis sit primo $\left(\frac{d^3z}{dx^3}\right)=V$ et posita sola x variabili erit

$$\left(\frac{ddz}{dx^2}\right)=\int Vdx+\Gamma{:}y\,;$$

tum vero porro

$$\left(\frac{dz}{dx}\right)=\int dx\int Vdx+x\Gamma{:}y+\Delta{:}y=\iint Vdx^2+x\Gamma{:}y+\Delta{:}y$$

ac denique

$$z=\int^3 Vdx^3+\tfrac{1}{2}x^2\Gamma{:}y+x\Delta{:}y+\Sigma{:}y\,.$$

Simili modo patet, si fuerit $\left(\frac{d^3z}{dx^2dy}\right)=V$, fore

$$z=\int^3 Vdx^2dy+x\Gamma{:}y+\Delta{:}y+\Sigma{:}x\,,$$

ac si sit $\left(\frac{d^3z}{dxdy^2}\right)=V$, erit

$$z=\int^3 Vdxdy^2+\Gamma{:}y+y\Delta{:}x+\Sigma{:}x\ ;$$

si sit $\left(\frac{d^3z}{dy^3}\right)=V$, erit

$$z=\int^3 Vdy^3+y^2\Gamma{:}x+y\Delta{:}x+\Sigma{:}x.$$

Eodem modo ad formulas altiorum graduum progredientes reperiemus, ut sequitur; si sit $\left(\frac{d^4z}{dx^4}\right)=V$, fore

$$z=\int^4 Vdx^4+x^3\Gamma{:}y+x^2\Delta{:}y+x\Sigma{:}y+\Theta{:}y$$

si sit $\left(\frac{d^4z}{dx^3dy}\right)=V$, fore

$$z = \int^4 Vdx^3dy + x^2\Gamma{:}y + x\Delta{:}y + \Sigma{:}y + \Theta{:}x$$

si sit $\left(\frac{d^4z}{dx^2dy^2}\right) = V$, fore

$$z = \int^4 Vdx^2dy^2 + x\Gamma{:}y + \Delta{:}y + y\Sigma{:}x + \Theta{:}x\,;$$

Si sit $\left(\frac{d^4z}{dxdy^3}\right) = V$, fore

$$z = \int^4 Vdxdy^3 + \Gamma{:}y + y^2\Delta{:}x + y\Sigma{:}x + \Theta{:}x\,;$$

si sit $\left(\frac{d^4z}{dy^4}\right) = V$, fore

$$z = \int^4 Vdy^4 + y^3\Gamma{:}x + y^2\Delta{:}x + y\Sigma{:}x + \Theta{:}x\,;$$

neque pro altioribus gradibus res eget ulteriori explicatione.

COROLLARIUM 1

391. Quemadmodum signum integrationis in primo libro usitatum iam per se involvit constantem per integrationem ingredientem, ita quoque hic functiones arbitrariae per integrationem ingressae iam in formula integrali involvi sunt censendae, ita ut non sit opus eas exprimere.

COROLLARIUM 2

392. Sufficit ergo pro aequatione $\left(\frac{d^3z}{dx^3}\right) = V$ integrale triplicatum hoc modo dedisse $z = \int^3 Vdx^3$, quae forma iam potestate complectitur partes supra adiectas

$$xx\Gamma{:}y + x\Delta{:}y + \Sigma{:}y\,;$$

quod idem de reliquis est tenendum.

COROLLARIUM 3

393. Si ergo in genere haec habeatur aequatio

$$\left(\frac{d^{m+n}z}{dx^m dy^n}\right) = V$$

eius integrale statim hoc modo exhibetur

$$z = \int^{m+n} Vdx^m dy^n\,,$$

quae potestate iam involvit omnes illas functiones arbitrarias numero $m+n$ per totidem integrationes invectas.

SCHOLION

394. Hi casus utique sunt simplicissimi, qui ad hoc caput referendi videntur, pro magis autem complicatis vix certa praecepta tradere licet, cum ista calculi integralis pars vix adhuc coli sit coepta. Interim tamen iam intelligitur, si aequationes magis complicatas ope cuiusdam transformationis ad has simplicissimas revocare liceat, etiam earum integrationem in promtu esse futuram; quod quidem negotium hic non copiosius persequendum videtur.
Progredior igitur ad casus magis reconditos eosque ita comparatos, ut ope aequationum inferiorum ordinum expediri queant, unde quidem insignis methodus satis late patens colligi poterit, qua saepius haud sine successu uti licebit. Neque tamen in hac pertractatione nimis diffusum esse convenit, sed sufficiet praecipuos fontes adhuc quidem cognitos patefecisse.