CHAPTER V

A PARTICULAR TRANSFORMATION OF THE SAME EQUATIONS

PROBLEM 56

349. *With this proposed equation*

$$\left(\frac{ddz}{dy^2}\right)=P\left(\frac{ddz}{dx^2}\right)+Q\left(\frac{dz}{dx}\right)+Rz\,,$$

in which P, Q, R shall be functions of x only, that with the aid of the substitution

$$z=M\left(\frac{dv}{dx}\right)+Nv\,,$$

where M and N also shall be functions of x only, may be changed into another equation of the same form, so that there may be produced

$$\left(\frac{ddv}{dy^2}\right)=F\left(\frac{ddv}{dx^2}\right)+G\left(\frac{dv}{dx}\right)+Hv$$

with functions F, G, H present of x only.

SOLUTION

Because the quantities *M* and *N* are free of *y*, there becomes

$$\left(\frac{ddz}{dy^2}\right)=M\left(\frac{d^3v}{dxdy^2}\right)+N\left(\frac{ddv}{dy^2}\right),[*]$$

which form by the equation, that we may assume to result finally, will turn into this [*i.e.* on substituting the differentials $\left(\frac{d^3v}{dxdy^2}\right)$ and $\left(\frac{ddv}{dy^2}\right)$ derived from the final equation into [*] :]

$$\left(\frac{ddz}{dy^2}\right)=MF\left(\frac{d^3v}{dx^3}\right)+\begin{matrix}\frac{MdF}{dx}\\+MG\\+NF\end{matrix}\left(\frac{ddv}{dx^2}\right)+\begin{matrix}\frac{MdG}{dx}\\+MH\\+NG\end{matrix}\left(\frac{dv}{dx}\right)+\begin{matrix}\frac{MdH}{dx}\\+NH\\ \end{matrix}v$$

[Note that only the coefficients are displayed on the second and third lines, a common practice with Euler.]
Then truly for the other member of the proposed equation our substitution gives

$$\left(\frac{dz}{dx}\right) = M\left(\frac{ddv}{dx^2}\right) + \frac{dM}{dx}\left(\frac{dv}{dx}\right) + \frac{dN}{dx}v$$
$$+N \quad \vdots$$

and hence again

$$\left(\frac{ddz}{dx^2}\right) = M\left(\frac{d^3v}{dx^3}\right) + \left(\frac{2dM}{dx} + N\right)\left(\frac{ddv}{dx^2}\right) + \left(\frac{ddM}{dx^2} + \frac{2dN}{dx}\right)\left(\frac{dv}{dx}\right) + \frac{ddN}{dx^2}v.$$

Since now there shall be by hypothesis

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{d^2z}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz\,,$$

if here the values just found are substituted, and the individual members $\left(\frac{d^3v}{d^3x}\right)$, $\left(\frac{ddv}{dx^2}\right)$, $\left(\frac{dv}{dx}\right)$ and v themselves are reduced to zero, the four following equations arise, evidently

from the equation	there is deduced
$\left(\frac{d^3v}{dx^3}\right)$	$MF = MP\,,$
$\left(\frac{ddv}{dx^2}\right)$	$\frac{MdF}{dx} + MG + NF = \left(\frac{2dM}{dx} + N\right)P + MQ,$
$\left(\frac{dv}{dx}\right)$	$\frac{MdG}{dx} + MH + NG = \left(\frac{ddM}{dx^2} + \frac{2dN}{dx}\right)P + \left(\frac{dM}{dx} + N\right)Q + MR,$
v	$\frac{MdH}{dx} + NH = \frac{ddN}{dx^2}P + \frac{dN}{dx}Q + NR,$

From which in the first place P, Q and R are to be sought most conveniently.
Now the first gives at once $P = F$, from which the second equation becomes

$$\frac{MdF - 2FdM}{Mdx} + G = Q\,.$$

Moreover from the two final equations there is deduced, on eliminating R,

$$\frac{M(NdG - MdH)}{dx} + NNG = \left(\frac{NddM - MddN}{dx^2} + \frac{2NdN}{dx}\right)F + \left(\frac{NdM - MdN}{dx} + \frac{NdN}{dx}\right)Q$$

and on substituting that value found for Q

$$0 = \frac{MMdH}{dx} - \frac{MNdG}{dx} + \frac{NddM - MddN}{dx^2}F + \frac{2NFdN}{dx} + \frac{NdM - MdN}{dx}G$$
$$+ \frac{NdM - MdN}{dx^2}dF + \frac{NNdF}{dx} - \frac{2FdM(NdM - MdN)}{Mdx^2} - \frac{2NNFdM}{Mdx},$$

which equation multiplied by $\frac{dx}{MM}$ conveniently is returned integrable, and the integral is found to be :

$$C = H - \frac{N}{M}G + \frac{NdM - MdN}{MMdx}F + \frac{NNF}{MM}.$$

Therefore if for the sake of brevity we put $N = Ms$, there will be hence

$$C = H - Gs - \frac{Fds}{dx} + Fss$$

or

$$ds + \frac{G}{F}sdx - ssdx + \frac{(C-H)dx}{F} = 0.$$

Now hence either the quantity $s = \frac{N}{M}$ can be defined, or from one of the functions F, G and H, the letters P, Q and R thus will be determined from that equation proposed, so that there shall be

$$\text{I. } P = F,$$
$$\text{II. } Q = G + \frac{dF}{dx} - \frac{2FdM}{Mdx},$$

and from the final equation there is determined

$$R = H + \frac{MdH}{Ndx} - \frac{FddN}{Ndx^2} - \frac{dN}{Ndx}\left(G + \frac{dF}{dx} - \frac{2FdM}{Mdx}\right),$$

which value emerges, on account of $N = Ms$,

$$R = H + \frac{dH}{sdx} - \frac{Gds}{sdx} - \frac{GdM}{Mdx} - \frac{Fdds}{sdx^2} - \frac{FddM}{Mdx^2} + \frac{2FdM^2}{MMdx^2} - \frac{dFds}{sdx^2} - \frac{dFdM}{Mdx^2},$$

and since the equation found, if it should be differentiated, will give

$$0 = dH - Gds - sdG - \frac{Fdds}{dx} - \frac{dFds}{dx} + 2Fsds + ssdF,$$

we will obtain

$$\text{III. } R = H - \frac{GdM}{Mdx} + \frac{dG}{dx} - \frac{FddM}{Mdx^2} - \frac{2Fds}{dx} + \frac{2FdM^2}{MMdx^2} - \frac{sdF}{dx} - \frac{dFdM}{Mdx^2};$$

from which, if the equation

$$\left(\frac{ddv}{dy^2}\right)=F\left(\frac{ddv}{dx^2}\right)+G\left(\frac{dv}{dx}\right)+Hv$$

allows resolution, also the resolution of this equation will succeed

$$\left(\frac{ddz}{dy^2}\right)=P\left(\frac{ddz}{dx^2}\right)+Q\left(\frac{dz}{dx}\right)+Rz,$$

since there shall be

$$z=M\left(\frac{dv}{dx}\right)+Nv=M\left(sv+\left(\frac{dv}{dx}\right)\right).$$

COROLLARY 1

350. If there is put $M=1$, so that there becomes $z=sv+\left(\frac{dv}{dx}\right)$, then there will be

$$P=F,\ Q=G+\frac{dF}{dx}\ \text{ and }\ R=H+\frac{dG}{dx}-\frac{2Fds+sdF}{dx}$$

nor is the use of this reduction restricted in this way, because, if hence in place of z there is put Mz, also the resolution of the equation hence arising becomes easy.

COROLLARY 2

351. Therefore as often as the resolution of the equation

$$\left(\frac{ddv}{dy^2}\right)=F\left(\frac{ddV}{dx^2}\right)+G\left(\frac{dv}{dx}\right)+Hv$$

is possible, so also the resolution of this equation

$$\left(\frac{ddz}{dy^2}\right)=F\left(\frac{ddz}{dx^2}\right)+\left(G+\frac{dF}{dx}\right)\left(\frac{dz}{dx}\right)+\left(H+\frac{dG}{dx}-\frac{2Fds+sdF}{dx}\right)z$$

will succeed, but only if s is taken from this equation

$$Fds+Gsdx-Fssdx+\left(C-H\right)dx=0;$$

for then there will be $z=sv+\left(\frac{dv}{dx}\right)$. Moreover the letters F, G, H are functions of x only.

SCHOLIUM

352. This reduction may be seen especially to supply a natural method for bringing about integrations of this kind, which also involve the differentials of functions. Indeed if the integral of an equation for a given *v* shall be $v=\varphi:t$, with *t* being some function arising of *x* and *y*, on account of $dv=dt\varphi':t$ there will be $\left(\frac{dv}{dx}\right)=\left(\frac{dt}{dx}\right)\varphi':t$ and thence we will have for *z*, the integral of the differential equation, [assuming the solution in terms of *v*, $z=sv+\left(\frac{dv}{dx}\right)$ derived above for differential equations of this kind],

$$z=s\varphi:t+\left(\frac{dt}{dx}\right)\varphi':t.$$

Then if there should be more generally, $v=u\varphi:t$, there becomes

$$z=su\varphi:t+\left(\frac{du}{dx}\right)\varphi:t+u\left(\frac{dt}{dx}\right)\varphi':t,$$

from which account it is seen how equations of this kind may be come upon, the integrals of which besides being a function $\varphi:t$ also are functions arising from the differentiation of this $\varphi':t$ and thus also the following $\varphi'':t,\varphi''':t$ etc. may be involved. On account of which there will be a need to establish this reduction carefully.

PROBLEM 57

353. *With the resolution of this equation granted*

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{m}{x}\left(\frac{dv}{dx}\right)+\frac{n}{xx}v$$

to find another equation of this form

$$\left(\frac{ddz}{dy^2}\right)=P\left(\frac{ddz}{dx^2}\right)+Q\left(\frac{dz}{dx}\right)+Rz$$

for which there shall be

$$z=sv+\left(\frac{dv}{dx}\right).$$

SOLUTION

From a comparison made with the preceding problem we have

$$F=1,\quad G=\frac{m}{x}\quad\text{and}\quad H=\frac{n}{xx},$$

from which it is required to define the quantity s from this equation

$$ds+\frac{msdx}{x}-ssdx+\left(f-\frac{n}{xx}\right)dx=0,$$

[The constant C above is now called f.]

with which found on account of $\frac{dG}{dx}=-\frac{m}{xx}$ the equation sought will be, [because $P=F$, $Q=G+\frac{dF}{dx}$ and $R=H+\frac{dG}{dx}-\frac{2Fds+sdF}{dx}$],

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\left(\frac{n-m}{xx}-\frac{2ds}{dx}\right)z$$

or in place of ds, with the value thence substituted

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\left(2f-\frac{n+m}{xx}+\frac{2ms}{x}-2ss\right)z$$

for which there is

$$z=sv+\left(\frac{dv}{dx}\right).$$

I. In the first place we may put the constant quantity $f=0$, so that there shall be

$$ds+\frac{msdx}{x}-ssdx-\frac{ndx}{xx}=0,$$

a particular integral of which is $s=\frac{\alpha}{x}$ with

$$-\alpha+m\alpha-\alpha\alpha-n=0 \quad \text{arising, or} \quad \alpha\alpha-(m-1)\alpha+n=0,$$

from which on account of $\frac{ds}{dx}=\frac{-\alpha}{xx}$ this equation arises

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\frac{2\alpha-m+n}{xx}z$$

for which there is

$$z=\frac{\alpha}{x}v+\left(\frac{dv}{dx}\right),$$

or with an exclusive value $n=\alpha(m-1-\alpha)$, if the resolution of this equation shall agree

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{m}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(m-1-\alpha)}{xx}v\,,$$

then for this case there will be

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\frac{(\alpha-1)(m-\alpha)}{xx}z\,,$$

and there will be

$$z=\frac{\alpha}{x}v+\left(\frac{dv}{dx}\right).$$

II. Keeping $f=0$, we seek a complete value for *s* by putting

$s=\frac{\alpha}{x}+\frac{1}{t}$ and there becomes, on putting $n=(m-1)\alpha-\alpha\alpha$,

$$dt+\frac{(2a-m)tdx}{x}+dx=0\,,$$

which multiplied by $x^{2\alpha-m}$ and integrated gives

$$t=\frac{cx^{m-2\alpha}}{2\alpha-m+1}-\frac{x}{2\alpha-m+1}$$

and

$$s=\frac{\alpha cx^{m-2\alpha-1}+\alpha-m+1}{x\left(cx^{m-2\alpha-1}-1\right)}=\frac{\alpha}{x}+\frac{2\alpha-m+1}{x\left(cx^{m-2\alpha-1}-1\right)}\,,$$

from which there becomes

$$\frac{ds}{dx}=\frac{-\alpha}{xx}+\frac{(m-2\alpha-1)(m-2\alpha)}{xx\left(cx^{m-2\alpha-1}-1\right)}+\frac{(m-2\alpha-1)^2}{xx\left(cx^{m-2\alpha-1}-1\right)^2}\,.$$

Here the case $c=0$ may be noted especially, in which there becomes

$$s=\frac{m-\alpha-1}{x}\quad\text{and}\quad\frac{ds}{dx}=\frac{-m+\alpha+1}{xx}$$

thus so that from the given equation

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{m}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(m-1-\alpha)}{xx}v$$

for this equation

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\frac{(\alpha+1)(m-2-\alpha)}{xx}z$$

there shall become

$$z=\frac{m-\alpha-1}{x}v+\left(\frac{dv}{dx}\right).$$

But for the general value let $m-2\alpha-1=\beta$, so that there is had

$$s=\frac{\alpha}{x}-\frac{\beta}{x\left(cx^{\beta}-1\right)}\quad\text{and}\quad\frac{ds}{dx}=\frac{-\alpha}{xx}+\frac{\beta(\beta+1)}{xx\left(cx^{\beta}-1\right)}+\frac{\beta\beta}{xx\left(cx^{\beta}-1\right)^2},$$

from which, if this equation is given

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{2\alpha+\beta+1}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(\alpha+\beta)}{xx}v,$$

with the help of which this is resolved

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{2\alpha+\beta+1}{x}\left(\frac{dz}{dx}\right)+\left((\alpha-1)(\alpha+\beta+1)-\frac{2\beta(\beta+1)}{cx^{\beta}-1}-\frac{2\beta\beta}{\left(cx^{\beta}-1\right)^2}\right)\frac{z}{xx},$$

since there shall be

$$z=\left(\alpha-\frac{\beta}{cx^{\beta}-1}\right)\frac{v}{x}+\left(\frac{dv}{dx}\right)$$

III. Also we may have some ratio of the constant f and we may put $f=\frac{1}{aa}$, so that on making $n=\alpha(m-1-a)$ we have

$$ds+\frac{msdx}{x}-ssdx-\frac{\alpha(m-1-\alpha)dx}{xx}+\frac{dx}{aa}=0,$$

which on putting $s=\frac{\alpha}{x}+\frac{1}{t}$ changes into

$$dt-\frac{(m-2\alpha)tdx}{x}+dx=\frac{tt}{aa}dx.$$

Let there be $m-2\alpha=\gamma$, so that the given equation shall be

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) + \frac{2\alpha+\gamma}{x}\left(\frac{dv}{dx}\right) + \frac{\alpha(\alpha+\gamma-1)}{xx}v$$

and with the quantity s found this equation arises

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{2\alpha+\gamma}{x}\left(\frac{dz}{dx}\right) + \left(\frac{\alpha\alpha-3\alpha+\alpha\gamma-\gamma}{xx} - \frac{2ds}{dx}\right)z$$

or

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{2\alpha+\gamma}{x}\left(\frac{dz}{dx}\right) + \left(\frac{(\alpha-1)(\alpha+\gamma)}{xx} + \frac{2dt}{ttdx}\right)z\,,$$

for which there is [the solution]

$$z = \left(\frac{\alpha}{x} + \frac{1}{t}\right)v + \left(\frac{dv}{dx}\right),$$

where the whole business of finding the quantity t returns from the equation

$$dt - \frac{\gamma tdx}{x} + dx = \frac{tt}{aa}dx.$$

In the end this [substitution] is put in place $t = a - \frac{aadu}{udx}$ and there is found

$$\frac{ddu}{dx^2} - \frac{\gamma du}{xdx} - \frac{2du}{adx} + \frac{\gamma u}{ax} = 0\,,$$

a twofold resolution of which is given, in the one case on putting

$$u = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{ etc.},$$

with arising

$$B = \frac{\gamma}{\gamma a}A,\quad C = \frac{\gamma-2}{2(\gamma-1)a}B,\quad D = \frac{\gamma-4}{3(\gamma-2)a}C,\quad E = \frac{\gamma-6}{4(\gamma-3)a}D \quad \text{etc.},$$

with the other case on putting

$$u = Ax^{\gamma+1} + Bx^{\gamma+2} + Cx^{\gamma+3} + Dx^{\gamma+4} + Ex^4 + \text{etc.},$$

where

$$B = \frac{\gamma+2}{(\gamma+2)a}A,\quad C = \frac{\gamma+4}{2(\gamma+3)a}B,\quad D = \frac{\gamma+6}{3(\gamma+4)a}C,\quad E = \frac{\gamma+8}{4(\gamma+5)a}D \quad \text{etc.},$$

the first series of which is terminated, if there shall be a whole positive even number γ, and in the other, indeed, if negative. Which even if they are particular values, yet above we have shown now [see §837 and §967 of Book II], how from these the complete values shall be elicited.

COROLLARIUM 1

354. Moreover above we have seen (§ 333) that this equation

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{2m}{x}\left(\frac{dv}{dx}\right)+\frac{(m+i)(m-i-1)}{xx}v$$

is integrable, if *i* should be some whole number, from which we deduce that this equation

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{m}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(m-i-\alpha)}{xx}v$$

is allowed to be integrated, as often as either $\alpha=\frac{1}{2}m+i$ or $\alpha=\frac{1}{2}m-i-1$ or $m-2\alpha$ shall be either a positive or negative whole number, which cases, on account of $m-2\alpha=\gamma$, agree with the cases of integrability for the general value of *s* being found.

COROLLARY 2

355. But when from this equation it is possible to define the function *v*, then also these two following equations [§ 353] similar to that one are able to be resolved

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\frac{(\alpha-1)(m-\alpha)}{xx}z$$

and

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\frac{(\alpha+1)(m-\alpha-2)}{xx}z,$$

since for the one there shall be

$$z=\frac{\alpha}{x}v+\left(\frac{dv}{dx}\right)$$

for the other

$$z=\frac{m-\alpha-1}{x}v+\left(\frac{dv}{dx}\right).$$

COROLLARY 3

356. Truly in addition also equations of other kinds, where the final term is not of the form $\frac{n}{xx}z$, are able to be resolved, and which are found if a more general value of the quantity *s* and thus the ratio of the constant *f* may be considered.

EXAMPLE 1

357. *With the proposed equation* $\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)$, *for which there is*

$$v=\ \pi:(x+y)+\varphi:(x-y),$$

to find more complicated equations, which are able to be integrated with the aid of this.

Since here there shall be $F=1$, $G=0$ and $H=0$, this equation may be resolved [from § 349] :

$$ds-ssdx+Cdx=0$$

and the integral of this equation

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{2ds}{dx}z$$

will be

$$z=sv+\left(\frac{dv}{dx}\right)$$

But initially taking the constant $C=0$ makes $\frac{ds}{ss}=dx$ and $\frac{1}{s}=c-x$ or $s=\frac{1}{c-x}$ and $\frac{ds}{dx}=\frac{1}{(c-x)^2}$, where indeed without any restriction there can be put $c=0$, so that the integral of this equation

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{2}{xx}z$$

shall be

$$z=-\tfrac{1}{x}\big(\pi:(x+y)+\varphi:(x-y)\big)+\pi':(x+y)+\varphi':(x-y).$$

Then there shall be $C=aa$ and on account of $ds=dx(ss-aa)$ there will be made $x=\frac{1}{2a}\frac{s-a}{s+a}-\frac{1}{2a}lA$ and hence

$$\frac{s-a}{s+a}=Ae^{2ax}\quad\text{and}\quad s=\frac{a\left(1+Ae^{2ax}\right)}{1-Ae^{2ax}}$$

from which

$$\frac{ds}{dx}=\frac{4Aaae^{2ax}}{\left(1-Ae^{2ax}\right)^2}$$

and the integral of the equation

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{8Aaae^{2ax}}{\left(1-Ae^{2ax}\right)^2}z$$

is

$$z=\frac{a\left(1+Ae^{2ax}\right)}{1-Ae^{2ax}}v+\left(\frac{dv}{dx}\right).$$

Finally there shall be
$C=-aa$ and on account of $ds=dx\left(aa+ss\right)$ there becomes $ax+b=\text{Ang.tang.}\frac{s}{a}$
and hence

$$s=a\,\text{tang.}\left(ax+b\right) \text{ and } \frac{ds}{dx}=\frac{aa}{\cos.^{2}\left(ax+b\right)},$$

on account of which the integral of this equation

$$\left(\frac{ddz}{dy^{2}}\right)=\left(\frac{ddz}{dx^{2}}\right)-\frac{aa}{\cos.^{2}\left(ax+b\right)}z$$

shall be

$$z=\frac{a\sin.\left(ax+b\right)}{\cos.\left(ax+b\right)}v+\left(\frac{dv}{dx}\right)$$

EXAMPLE 2

358. *With the proposed equation*

$$\left(\frac{ddv}{dy^{2}}\right)=\left(\frac{ddv}{dx^{2}}\right)-\frac{2}{xx}v,$$

the integral of which is agreed upon [§ 357], *to find others integrable with the aid of this.*

For in this case we have

$$ds-ssdx+\left(C+\frac{2}{xx}\right)dx=0,$$

with which resolved, the integral of this equation

$$\left(\frac{ddz}{dy^{2}}\right)=\left(\frac{ddz}{dx^{2}}\right)-2\left(\frac{1}{xx}+\frac{ds}{dx}\right)z$$

will be

$$z=sv+\left(\frac{dv}{dx}\right).$$

1. Initially let $C=0$ and from the equation $ds-ssdx+\frac{2dx}{xx}=0$ there becomes in particular $s=\frac{1}{x}$ or $s=-\frac{2}{x}$.
Therefore there is put $s=\frac{1}{x}+\frac{1}{t}$ and there will be

$$dt + \frac{2tdx}{x} + dx = 0\,,$$

[For $dt + \frac{2tdx}{x} + dx = 0$ gives $x^2dt + 2txdx + x^2dx = 0,$ or $d\left(x^2t\right) + x^2dx = 0,$]

hence $txx + \frac{1}{3}x^3 = \frac{1}{3}a^3$. Hence

$t = \frac{a^3-x^3}{3xx}$ and $s = \frac{a^3+2x^3}{x\left(a^3-x^3\right)}$, and hence $\frac{ds}{dx} + \frac{1}{xx} = \frac{3x\left(2a^3+x^3\right)}{\left(a^3-x^3\right)^2}$,

from which the integral of this equation

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{6x\left(2a^3+x^3\right)}{\left(a^3-x^3\right)^2}z$$

is

$$z = \frac{a^3+2x^3}{x\left(a^3-x^3\right)}v + \left(\frac{dv}{dx}\right).$$

II. Let there be $C = \frac{1}{cc}$ and on putting $s = \frac{1}{x} + \frac{1}{t}$ there is made

$$dt + \frac{2tdx}{x} + dx = \frac{ttdx}{cc}\,,$$

to which particularly there corresponds $t = c + \frac{cc}{x}$, so that there shall be

$$s = \frac{cc+cx+xx}{cx(c+x)} \quad \text{and} \quad \frac{ds}{dx} + \frac{1}{xx} = \frac{1}{(c+x)^2}$$

and the integral of this equation

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{2}{(c+x)^2}z$$

shall be

$$z = \frac{cc+cx+xx}{cx(c+x)}v + \left(\frac{dv}{dx}\right).$$

But in accordance with finding the complete integral for *t* there is put in place

$$t = c + \frac{cc}{x} + \frac{1}{u}$$

and there becomes

$$du + \frac{2udx}{c} + \frac{dx}{cc} = 0 \quad \text{or} \quad dx = \frac{-ccdu}{1+2cu}\,,$$

hence

$$x = b - 2l(1+2cu)\,,$$

therefore

$$u = \frac{e^{\frac{2(b-x)}{c}} - 1}{2c},$$

from which

$$t = c + \frac{cc}{x} + \frac{2c}{e^{\frac{2(b-x)}{c}} - 1} \quad \text{and} \quad s = \frac{1}{x} + \frac{x\left(e^{\frac{2(b-x)}{c}} - 1\right)}{c\left((c+x)e^{\frac{2(b-x)}{c}} + x - c\right)},$$

and

$$\frac{ds}{dx} + \frac{1}{xx} = \frac{-dt}{ttdx} = \frac{1}{tt}\left(1 + \frac{2t}{x} - \frac{tt}{cc}\right) = \frac{1}{tt}\left(\frac{tt}{cc} - \frac{4e^{\frac{2(b-x)}{c}}}{\left(e^{\frac{2(b-x)}{c}} - 1\right)^2}\right)$$

SCHOLIUM

359. Since above [§ 333] we have found that this equation

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) - \frac{i(i+1)}{xx}v$$

admits to integration, clearly which case arises from the general form (§ 354) on taking $m = 0$, there will be with this problem translated here

$$ds - ssdx + \left(f + \frac{i(i+1)}{xx}\right)dx = 0$$

and hence with the quantity s found , the integral of this equation

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \left(2f + \frac{i(i+1)}{xx} - 2ss\right)z$$

will be

$$z = sv + \left(\frac{dv}{dx}\right)$$

I. Because if now we should take $f = 0$, there will be particularly $s = \frac{i}{x}$ or $s = \frac{-i-1}{x}$ from which indeed the form of the integrable equation is not changed. But on making $s = \frac{i}{x} + \frac{1}{t}$ there arises

$$dt + \frac{2itdx}{x} + dx = 0,$$

the integral of which is

$$x^{2i}t+\frac{1}{2i+1}x^{2i+1}=\frac{g}{2i+1}$$

and thus

$$s=\frac{ig+(i+1)x^{2i+1}}{x\left(g-x^{2i+1}\right)}$$

and the integrable equation becomes

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{i(i-1)gg+6i(i+1)gx^{2i+1}+(i+1)(i+2)x^{4i+2}}{xx\left(g-x^{2i+1}\right)^2}z$$

II. But on not rejecting f there shall be $s=\frac{i}{x}+u$ and there becomes

$$-du+\frac{2iudx}{x}+\frac{uudx}{x}=fdx;$$

which as it may be converted into differential equation of the second order, it can be resolved easily by a series, and there is put :
[This corresponds to a special case of the equation treated in §967 of volume II, as indicated by the editor of the *O.O.* edition.]

$$u=\sqrt{f}-\frac{i}{x}-\frac{dr}{rdx}$$

and there emerges

$$\frac{ddr}{dx^2}-\frac{2dr\sqrt{f}}{dx}-\frac{i(i+1)r}{xx}=0\,.$$

Let there be $\sqrt{f}=a$ and there is put in place

$$r=Ax^{i+1}+Bx^{i+2}+Cx^{i+3}+Dx^{i+4}+\text{etc.}$$

and there is found

$$B=\frac{2(i+1)a}{1(2i+2)}A,\quad C=\frac{2(i+2)a}{2(2i+3)}B,\quad D=\frac{2(i+3)a}{3(2i+4)}C,\quad E=\frac{2(i+4)a}{4(2i+5)}D,\quad \text{etc.},$$

which is terminated, whenever *i* is a negative whole number.

But if on the other hand there is put in place

$$r=Ax^{-i}+Bx^{1-i}+Cx^{2-i}+Dx^{3-i}+\text{etc.},$$

the following relation is produced

$$B=\frac{2ia}{2i}A,\quad C=\frac{2(i-1)a}{2(2i-1)}B,\quad D=\frac{2(i-2)a}{3(2i-2)}C,\quad E=\frac{2(i-3)a}{4(2i-3)}D,\quad \text{etc.},$$

which is terminated, whenever i is a positive whole number.

PROBLEM 58

360. *With the proposed equation*

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)-\frac{2aa}{\cos.^2(ax+b)}v\,,$$

the integral of which is [See § 357.] :

$$v=a\text{tang.}(ax+b).\left(\pi:(x+y)+\varphi:(x-y)\right)+\pi':(x+y)+\varphi':(x-y)\,,$$

by the transformation related here, to find other integral equations with the aid of that.

SOLUTION

For the sake of brevity we may put $ax+b=\omega$, so that there shall be $d\omega=adx$, and from § 351, since there shall be $F=1,\quad G=0,\quad H=\frac{-2aa}{\cos.^2\omega}$, the quantity s is sought from this equation

$$ds-ssdx+\left(C+\frac{2aa}{\cos.^2\omega}\right)dx=0$$

and the integral of this equation

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-\left(\frac{2aa}{\cos.^2\omega}+\frac{2ds}{dx}\right)z$$

will be $z=sv+\left(\frac{dv}{dx}\right)$ or

$$\begin{aligned}z&=as\text{tang.}\omega\cdot\left(\pi:(x+y)+\varphi:(x-y)\right)+s\left(\pi':(x+y)+\varphi':(x-y)\right)\\&+\frac{aa}{\cos.^2\omega}\left(\pi:(x+y)+\varphi:(x-y)\right)+a\text{tang.}\omega\cdot\left(\pi':(x+y)+\varphi':(x-y)\right)+\left(\pi'':(x+y)+\varphi'':(x-y)\right).\end{aligned}$$

Therefore the whole business is reduced to finding the quantity s, towards which end we may put

$$s=\alpha\text{tang.}\omega-\frac{du}{udx}\,,$$

and there becomes

$$\frac{ds}{dx}=\frac{\alpha a}{\cos.^2\omega}-\frac{ddu}{udx^2}+\frac{du^2}{uudx^2}$$

and with the substitution made there becomes

$$\frac{\alpha a}{\cos.^2\omega}-\frac{\alpha\alpha\sin.^2\omega}{\cos.^2\omega}+C+\frac{2aa}{\cos.^2\omega}-\frac{ddu}{udx^2}+\frac{2\alpha du}{udx}\text{tang.}\omega=0.$$

Now on account of

$$-\frac{\alpha\alpha\sin.^2\omega}{\cos.^2\omega} = -\frac{\alpha\alpha}{\cos.^2\omega} + \alpha\alpha$$

α is assumed thus, so that there becomes

$$-\ \alpha\alpha + a\alpha + 2aa = 0.$$

Therefore there is taken $\alpha = -a$, so that there shall be

$$s = -a\text{tang.}\omega - \frac{du}{udx},$$

and this equation can be considered for finding the quantity u

$$\frac{ddu}{udx^2} + \frac{2adu}{udx}\text{tang.}\omega + naau = 0$$

on putting $C = -aa - naa$ or

$$\frac{ddu}{d\omega^2} + \frac{2du}{d\omega}\text{tang.}\omega + nu = 0$$

on account of $dx = \frac{d\omega}{a}$; the resolution of which is seen to be not a little difficult, but between many ways that to be treated here is considered to be the most suitable to be established.

There is devised

$$u = A\cos.\lambda\omega + B\cos.(\lambda+2)\omega + C\cos.(\lambda+4)\omega + \text{etc.}$$

and there will be

$$\frac{du}{d\omega} = -\lambda A\sin.\lambda\omega - (\lambda+2)B\sin.(\lambda+2)\omega - (\lambda+4)C\sin.(\lambda+4)\omega - \text{etc.}$$

$$\frac{ddu}{d\omega^2} = -\lambda\lambda A\cos.\lambda\omega - (\lambda+2)^2 B\cos.(\lambda+2)\omega - (\lambda+4)^2 C\cos.(\lambda+4)\omega - \text{etc.}$$

and the equation represented by this form

$$\frac{ddu}{d\omega^2}\cos.\omega + \frac{4du}{d\omega}\sin.\omega + 2nu\cos.\omega = 0$$

will give

$$0=-\lambda\lambda A\cos.(\lambda-1)\omega-(\lambda+2)^2 B\cos.(\lambda+1)\omega-(\lambda+4)^2 C\cos.(\lambda+3)\omega-\text{etc.}$$

$$\begin{array}{lll}
 & -\lambda\lambda A & -(\lambda+2)^2 B \\
-2\lambda A & -2(\lambda+2)B & -2(\lambda+4)C \\
 & +2\lambda A & +2(\lambda+2)B \\
+nA & +nB & +nC \\
 & +nA & +nB
\end{array}$$

from which λ thus is required to be taken, so that there shall be

$$\lambda\lambda+2\lambda=n \quad \text{or} \quad \lambda=-1\pm\sqrt{(n+1)}$$

and a twofold value for λ may be considered. Now in addition the second term on account of $n=\lambda\lambda+2\lambda$ gives $B=\frac{\lambda}{\lambda+2}A$, the third conveniently gives $C=0$, and from which all the following vanish.

We assume $n=mm-1$, so that there shall be

$$\lambda=-1\pm m \text{ and } B=\frac{-1\pm m}{1\pm m}A,$$

and the complete integral is seen to be concluded

$$u=A\left(\cos.(m-1)\omega+\tfrac{m-1}{m+1}\cos.(m+1)\omega\right)+$$
$$\mathfrak{A}\left(\cos.(m+1)\omega+\tfrac{m+1}{m-1}\cos.(m-1)\omega\right).$$

Let there be

$$A=(m+1)B \quad \text{and} \quad \mathfrak{A}=(m-1)\mathfrak{B};$$

then there becomes

$$u=(m+1)(B+\mathfrak{B})\cos.(m-1)\omega+(m-1)(B+\mathfrak{B})\cos.(\mathrm{m}+1)\omega;$$

since two constants may coalesce into one [*i. e.* 1], this is a particular integral only, but from which henceforth the complete integral can be elicited [§ 361, §§ 362]. Therefore since there will be

$$\frac{du}{ud\omega}=\frac{-(mm-1)\sin.(m-1)\omega-(mm-1)\sin.(m+1)\omega}{(m+1)\cos.(m-1)\omega+(m-1)\cos.(m+1)\omega}$$

there is, [recalling that $s=\alpha\text{tang.}\omega-\frac{du}{udx}$, $\alpha=-a$, and $d\omega=adx$,]

$$\frac{s}{a}=-\text{tang.}\omega+\frac{(mm-1)(\sin.(m-1)\omega+\sin.(m+1)\omega)}{(m+1)\cos.(m-1)\omega+(m-1)\cos.(m+1)\omega}$$

for the equation

$$\frac{ds}{ad\omega}-\frac{ss}{aa}-mm+\frac{2}{\cos.^2\omega}=0$$

on account of $C=-(n+1)aa=-mmaa$.

[Recalling that initially $ds-ssdx+\left(C+\frac{2aa}{\cos.^2\omega}\right)dx=0$ and $d\omega=adx$.]

But that equation found is reduced to this form

$$\frac{s}{a}=-\text{tang.}w+\frac{(mm-1)\text{tang.}m\omega}{m+\text{tang.}m\omega\ \text{tang.}\omega},$$

which expression substituted it taken to satisfy that very well indeed. We will write Θ in place of this, and we put $\frac{s}{a}=\Theta+\frac{1}{t}$ for the complete integral to be elicited and there will be produced

$$-\frac{dt}{ttd\omega}-\frac{2\Theta}{t}-\frac{1}{tt}=0 \text{ or } dt+2\Theta td\omega+d\omega=0.$$

But just as there was before

$$\Theta=\frac{s}{a}=-\text{tang.}\omega-\frac{du}{ud\omega},$$

from which

$$\int\Theta d\omega=l\cos.\omega-lu \quad\text{and}\quad e^{2\int\Theta d\omega}=\frac{\cos.^2\omega}{uu}$$

which is the multiplier for that equation, and thus there becomes

$$\frac{t\cos.^2\omega}{uu}=C-\int\frac{d\omega\cos.^2\omega}{uu}$$

But there is

$$u=2m\cos.m\omega\cos.\omega+2\sin.m\omega\ \sin.\omega$$

and thus

$$\frac{t}{(m\cos.m\omega+\sin.m\omega\text{tang.}\omega)^2}=A-\int\frac{d\omega}{(m\cos.m\omega+\sin.m\omega\text{tang.}\omega)^2}$$

of which last part the integral is taken :

$$\frac{-m\text{tang.}m\omega+\text{tang.}\omega}{m(mm-1)(m+\text{tang.}m\omega\,\text{tang.}\omega)}=\frac{-m\sin.m\omega+\text{tang.}\omega\cos.m\omega}{m(mm-1)(m\cos.m\omega+\sin.m\omega\,\text{tang.}\omega)},$$

thus so that there shall be

$$\frac{t}{(m\cos.m\omega+\sin.m\omega\,\text{tang}.\omega)^2}=A+\frac{\cos.m\omega\,\text{tang}.\omega-m\sin.m\omega}{m(mm-1)(m\cos.m\omega+\sin.m\omega\,\text{tang}.\omega)}$$

or

$$\frac{1}{t}=\frac{m(mm-1)}{\big(C(m\cos.m\omega+\sin.m\omega\,\text{tang}.\omega)+\cos.m\omega\,\text{tang}.\omega-m\sin.m\omega\big)(m\cos.m\omega+\sin.m\omega\,\text{tang}.\omega)},$$

to which there is added

$$\Theta=-\text{tang}.\omega+\frac{m(mm-1)\sin.m\omega}{m\cos.m\omega+\sin.m\omega\,\text{tang}.\omega},$$

so that $\frac{s}{a}$ may be produced, and there will be

$$\frac{s}{a}=-\text{tang}.\omega+\frac{m(mm-1)(C\sin.m\omega+\cos.m\omega)}{C(m\cos.m\omega+\sin.m\omega\,\text{tang}.\omega)+\cos.m\omega\,\text{tang}.\omega-m\sin.m\omega}$$

or

$$\frac{s}{a}=\frac{(mm-1-\text{tang}^2.\omega)(C\sin.m\omega+\cos.m\omega)-m\,\text{tang}.\omega(C\cos.m\omega-\sin.m\omega)}{C(m\cos.m\omega+\sin.m\omega\,\text{tang}.\omega)+\cos.m\omega\,\text{tang}.\omega-m\sin.m\omega}.$$

COROLLARY 1

361. Here particularly it is to be noted that the integral of this equation

$$\frac{ddu}{d\omega^2}+\frac{2du}{d\omega}\text{tang}.\omega+(mm-1)u=0$$

becomes

$$u=m\cos.m\omega\cos.\omega+\sin.m\omega\sin.\omega\ ;$$

truly another particular integral is found in a like manner,

$$u=m\sin.m\omega\cos.\omega-\cos.m\omega\sin.\omega,$$

from which it is concluded that the complete integral is

$$u=A(m\cos.m\omega\cos.\omega+\sin.m\omega\sin.\omega)+B(m\sin.m\omega\cos.\omega-\cos.m\omega\sin.\omega).$$

COROLLARY 2

362. If here there is put

$$A = C\cos.\alpha \quad \text{and} \quad B = -C\sin.\alpha\,,$$

this complete integral is reduced to this form

$$u = C\big(m\cos.(m\omega+\alpha)\cos.\omega + \sin.(m\omega+\alpha)\sin.\omega\big),$$

which indeed it is possible to conclude at once from the first particular integral found, as there in place of the angle $m\omega$ it is possible to write $m\omega+\alpha$.

COROLLARY 3

363. Hence the value

$$\frac{s}{a} = -\text{tang}.\omega - \frac{du}{ud\omega}\,;$$

is found much easier since indeed there shall be

$$\frac{du}{d\omega} = -C(mm-1)\sin.(m\omega+\alpha)\cos.\omega\,,$$

there becomes

$$\frac{s}{a} = -\text{tang}.\omega + \frac{(mm-1)\sin.(m\omega+\alpha)\cos.\omega}{m\cos.(m\omega+\alpha)\cos.\omega+\sin.(m\omega+\alpha)\sin.\omega}$$

and hence

$$\frac{ds}{ad\omega} = \frac{ds}{aadx} = \frac{-1}{\cos.^2\omega} + \frac{(mm-1)\left(m^2\cos.^2\omega-\sin.^2(m\omega+\alpha)\right)}{\left(m\cos.(m\omega+\alpha)\cos.\omega+\sin.(m\omega+\alpha)\sin.\omega\right)^2}$$

and the equation, of which we have found the integration [§ 360], will be

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{2(mm-1)aa\left(m^2\cos.^2\omega-\sin.^2(m\omega+\alpha)\right)}{\left(m\cos.(m\omega+\alpha)\cos.\omega+\sin.(m\omega+\alpha)\sin.\omega\right)^2}z$$

and the integral of this is deduced

$$z = \frac{maa\left(m\sin.(m\omega+\alpha)\sin.\omega+\cos.(m\omega+\alpha)\cos.\omega\right)}{m\cos.(m\omega+\alpha)\cos.\omega+\sin.(m\omega+\alpha)\sin.\omega}\big(\pi:(x+y)+\varphi:(x-y)\big)$$
$$+\frac{(mm-1)a\sin.(m\omega+\alpha)\cos.\omega}{m\cos.(m\omega+\alpha)\cos.\omega+\sin.(m\omega+\alpha)\sin.\omega}\big(\pi':(x+y)+\varphi':(x-y)\big)+\big(\pi'':(x+y)+\varphi'':(x-y)\big)$$

with $\omega = ax + b$ present.

SCHOLIUM 1

364. Completely worthy of note is the integration of this equation

$$\frac{ddu}{d\omega^2} + \frac{2du}{d\omega}\text{tang.}\omega + (mm-1)u = 0,$$

from which I seize the opportunity to examine this more general equation

$$\frac{ddu}{d\omega^2} + \frac{2fdu}{d\omega}\text{tang.}\omega + gu = 0,$$

as in the first place I note on putting

$$\frac{du}{u} = -(2f+1)d\omega\text{tang.}\omega + \frac{dv}{v}$$

so that there becomes

$$u = \cos.^{2f+1}\omega v,$$

to change into this form

$$\frac{ddv}{d\omega^2} - \frac{2(f+1)dv}{d\omega}\text{tang.}\omega + (g-2f-1)v = 0,$$

thus so that, if the integral exists in the case $f = n$, it shall be integrable also in the case $f = -n-1$.

Now for that first equation there is put

$$u = A\sin.\lambda\omega + B\sin.(\lambda+2)\omega + C\sin.(\lambda+4)\omega + D\sin.(\lambda+6)\omega + \text{etc.}$$

and with the substitution made in the equation

$$\frac{2ddu}{d\omega^2}\cos.\omega + \frac{4fdu}{d\omega}\sin.\omega + 2gu\cos.\omega = 0$$

there is found

$$0=-\lambda\lambda A\sin.(\lambda-1)\omega-(\lambda+2)^2 B\sin.(\lambda+1)\omega-(\lambda+4)^2 C\sin.(\lambda+3)\omega-(\lambda+6)^2 D\sin.(\lambda+5)\omega \quad \text{etc.}$$

$$\begin{array}{llll}
 & -\lambda\lambda A & -(\lambda+2)^2 B & -(\lambda+4)^2 C \\
 & +2\lambda Af & +(\lambda+2)Bf & +2(\lambda+4)Cf \\
-2\lambda Af & -2(\lambda+2)Bf & -2(\lambda+4)Cf & -2(\lambda+6)Df \\
 & +Ag & +Bg & +Cg \\
+Ag & +Bg & +Cg & +Dg
\end{array}$$

Therefore it is required that there shall be $g=\lambda\lambda+2\lambda f$; then truly the coefficients assumed thus are determined :

$$B=\frac{\lambda f}{\lambda+f+1}A,\ C=\frac{(\lambda+1)(f-1)}{2(\lambda+f+2)}B,\ D=\frac{(\lambda+2)(f-2)}{2(\lambda+f+3)}C \quad \text{etc.}$$

Therefore we may put in place $g=mm-ff$, so that there becomes $\lambda=m-f$ and our equations shall be

$$\frac{ddu}{d\omega^2}+\frac{2fdu}{d\omega}\text{tang.}\omega+(mm-ff)u=0$$

and

$$\frac{ddv}{d\omega^2}-\frac{2(f+1)dv}{d\omega}\text{tang.}\omega+\left(mm-f(f+1)^2\right)v=0$$

with

$$u=v\cos.^{2f+1}\omega, \text{ or } v=\frac{u}{\cos.^{2f+1}\omega} \text{ arising}$$

Now because our series is terminated, as often as f is a whole number, we come upon simpler cases.

I. Let $f=0;$ there will be $\lambda=m$ and

$$B=0,\quad C=0 \text{ etc.}$$

and thus

$$u=A\sin.m\omega \quad\text{and}\quad v=\frac{A\sin.m\omega}{\cos.\omega}.$$

II. Let $f=1;$ there will be $\lambda=m-1$ and

$$B=\frac{m-1}{m+1}A,\ C=0 \quad \text{etc.,}$$

therefore

$$\tfrac{u}{a}=(m+1)\sin.(m-1)\omega+(m-1)\sin.(m+1)\omega \text{ and } v=\tfrac{u}{\cos.^2\omega} \text{ or}$$

$$\tfrac{u}{2a}=m\sin.m\omega\cos.\omega-\cos.m\omega\sin.\omega.$$

III. Let $f=2$; there will be $\lambda=m-2$ and

$$B=\tfrac{2(m-2)}{m+1}A,\ C=\tfrac{m-1}{2(m+2)}B=\tfrac{(m-1)(m-2)}{(m+1)(m+2)}A,\quad D=0 \quad \text{etc.},$$

hence

$$\tfrac{u}{a}=(m+1)(m+2)\sin.(m-2)\omega+2(m-2)(m+2)\sin.m\omega+(m-1)(m-2)\sin.(m+2)\omega$$

and from that $v=\tfrac{u}{\cos.^5\omega}$ or

$$\tfrac{u}{2a}=(mm+2)\sin.m\omega\cos.2m\omega+(mm-4)\sin.m\omega-3m\cos.m\omega\sin.2\omega.$$

IV. Let $f=3$; there will be $\lambda=m-3$ and

$$B=\tfrac{3(m-3)}{m+1}A,\ C=\tfrac{2(m-2)}{2(m+2)}B \quad \text{and} \quad D=\tfrac{m-1}{3(m+3)}C,\quad E=0 \quad \text{etc.},$$

therefore

$$\begin{aligned}\tfrac{u}{a}&=(m+1)(m+2)(m+3)\sin.(m-3)\omega+3(m+2)(mm-9)\sin.(m-1)\omega\\&+(m-1)(m-2)(m-3)\sin.(m+3)\omega+3(m-2)(mm-9)\sin.(m+1)\omega\end{aligned}$$

with $v=\tfrac{u}{\cos.^7\omega}$ arising.

V. Let $f=4$; there will be $\lambda=m-4$ and there is found

$$\begin{aligned}\tfrac{u}{a}&=(m+1)(m+2)(m+3)(m+4)\sin.(m-4)\omega+4(m+2)(m+3)(mm-16)\sin.(m-2)\omega\\&+(m-1)(m-2)(m-3)(m-4)\sin.(m+4)\omega+4(m-2)(m-3)(mm-16)\sin.(m+2)\omega\\&+6(mm-9)(mm-16)\sin.m\omega\end{aligned}$$

with $v=\tfrac{u}{\cos.^9\omega}$ arising, from which the ratio of the progression evidently is shown.

But it is agreed to be noted, if we should put

$$u = A\cos.\lambda\omega + B\cos.(\lambda+2)\omega + C\cos.(\lambda+4)\omega + \text{ etc.},$$

the same determinations of the coefficients be produced, from which these two values taken together will show the complete integral; which also is deduced from the form found, just as if in place of the angle $m\omega$ there will be written more generally $m\omega+\alpha$.

SCHOLIUM 2

365. But the same equation

$$\frac{ddu}{d\omega^2} + \frac{2fdu}{d\omega}\text{tang.}\omega + gu = 0$$

can be treated by several other methods and the integral of these can be expressed by series, from which other integrable cases will be obtained.

According to this in the first place it may be noted on putting $u=\sin.^{\lambda}\omega$, to become

$$\frac{du}{d\omega} = \lambda\sin.^{\lambda-1}\omega\cos.\omega \text{ and hence } \frac{du}{d\omega}\text{tang.}\omega = \lambda\sin.^{\lambda}\omega$$

and

$$\frac{ddu}{d\omega^2} = \lambda(\lambda-1)\sin.^{\lambda-2}\omega\cos.^2\omega - \lambda\sin.^{\lambda}\omega = \lambda(\lambda-1)\sin.^{\lambda-2}\omega - \lambda\lambda\sin.^{\lambda}\omega.$$

Hence, if we put

$$u = A\sin.^{\lambda}\omega + B\sin.^{\lambda+2}\omega + C\sin.^{\lambda+4}\omega + D\sin.^{\lambda+6}\omega + \text{etc.},$$

with the substitution made we come upon

$$\begin{aligned} 0 = \lambda(\lambda-1)A\sin.^{\lambda-2}\omega &+ (\lambda+2)(\lambda+1)B\sin.^{\lambda}\omega &&+ (\lambda+4)(\lambda+3)C\sin.^{\lambda+2}\omega + \text{etc.} \\ &- \lambda\lambda A &&- (\lambda+2)^2 B \\ &+ 2\lambda fA &&+ 2(\lambda+2)fB \\ &+ gA &&+ gB \end{aligned}$$

from which it is required to take either $\lambda=0$ or $\lambda=1$; then truly there will be

$$B = \frac{\lambda\lambda-2\lambda f-g}{(\lambda+1)(\lambda+2)}A,\quad C = \frac{(\lambda+2)^2-2(\lambda+2)f-g}{(\lambda+3)(\lambda+4)}B,\quad \text{etc.}$$

[See § 976 vol. II & E678 for more details.]

Hence it comes about that two cases are to be set out :

$\lambda = 0$	$\lambda = 1$
$B = \frac{-g}{1\cdot 2}A$	$B = \frac{1-2f-g}{2\cdot 3}A$
$C = \frac{4-4f-g}{3\cdot 4}B$	$C = \frac{9-6f-g}{4\cdot 5}B$
$D = \frac{16-8f-g}{5\cdot 6}C$	$D = \frac{25-10f-g}{6\cdot 7}C$
$E = \frac{36-12f-g}{7\cdot 8}D$	$E = \frac{49-14f-g}{8\cdot 9}D$
etc.	etc.

Therefore the integration will be successful, as often as there should be $g = ii - 2if$ with i denoting a whole positive number. Whereby since on putting $u = v\cos.^{2f+1}\omega$ the transformed equation shall be

$$\frac{ddv}{d\omega^2} - \frac{2(f+1)dv}{d\omega}\text{tang.}\omega + (g-2f-1)v = 0\,,$$

and therefore that equation also will be integrable, as often as there should be

$$g = (i+1)^2 + 2(i+1)f\,,$$

which two cases thus can be combined into one, so that the integration is allowed, provided that there shall be

$$g = ii \pm 2if$$

SCHOLIUM 3

366. Still keeping to the same equation, since of putting $u = \cos.^{\lambda}\omega$ there shall be

$$\frac{du}{d\omega} = -\lambda\cos.^{\lambda-1}\omega\, sin.\omega$$

and thus

$$\frac{du}{d\omega}\text{tang.}\omega = -\lambda\cos.^{\lambda-2}\omega\, sin.\omega + \lambda\cos.^{\lambda}\omega$$

and

$$\frac{ddu}{d\omega^2} = \lambda(\lambda-1)\cos.^{\lambda-2}\omega - \lambda\lambda\cos.^{\lambda}\omega,.$$

on placing

$$u = A\cos.^{\lambda}\omega + B\cos.^{\lambda+2}\omega + C\cos.^{\lambda+4}\omega + D\cos.^{\lambda+6}\omega + \text{ etc.}$$

and with the substitution made there arises

$$\begin{array}{llll} 0 = \lambda(\lambda-1)A\cos.^{\lambda-2}\omega & +(\lambda+2)(\lambda+1)B\cos.^{\lambda}\omega & +(\lambda+4)(\lambda+3)C\cos.^{\lambda+2}\omega & +\text{etc.} \\ & -\lambda\lambda A & -(\lambda+2)^2 B & \\ -2\lambda fA & -2(\lambda+2)fB & -2(\lambda+4)fC & \\ & +2\lambda fA & +2(\lambda+2)fB & \\ & +gA & +gB & \end{array}$$

Therefore it is required that either $\lambda = 0$ or $\lambda = 2f + 1$; then truly

$$B = \frac{\lambda\lambda - 2\lambda f - g}{(\lambda+2)(\lambda+1-2f)}A, \quad C = \frac{(\lambda+2)^2 - 2(\lambda+2)f - g}{(\lambda+4)(\lambda+3-2f)}B, \quad \text{etc.}$$

and both cases thus may be themselves considered:

$\lambda = 0$	$\lambda = 2f + 1$
$B = \frac{-g}{2(1-2f)}A$	$B = \frac{1+2f-g}{2(2f+3)}A$
$C = \frac{4-4f-g}{4(3-2f)}B$	$C = \frac{9+6f-g}{4(2f+5)}B$
$D = \frac{16-8f-g}{6(5-2f)}C$	$D = \frac{25+10f-g}{6(2f+7)}C$
etc.	etc.

From the former, the integration will be successful, if

$$g = 4ii - 4if\ ,$$

from the latter, if

$$g = (2i+1)^2 + 2(2i+1)f\ ,$$

which cases with these, which arise from the transformation, taken together return the same as found in the preceding paragraph.

Therefore at this stage all the cases of integrability found are recalled thus, so that on putting $g = mm - ff$ there shall be either $f = \pm i$ or $m = i \pm f$, that is either $f = \pm i$ or $f = \pm i \pm m$. Moreover these latter cases also follow from the first resolution (§ 364), where the series also is terminated, if

$$\lambda = -i \text{ and thus } g = mm - ff = ii - 2if \text{ , hence } i - f = \pm m$$

and the aid from the transformation called upon $f = \pm i \pm m$. Indeed the opposite case first found does not occur in the resolution of the latter.

PROBLEM 59

367. *With the integration of this equation permitted*

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv,$$

to find an equation of this form

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz$$

for which there shall be

$$z = \left(\frac{ddv}{dx^2}\right) + r\left(\frac{dv}{dx}\right) + sv,$$

where F, G, H, P, Q, R and r, s are functions of x only.

SOLUTION

Since there shall be

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{d^4v}{dx^2dy^2}\right) + r\left(\frac{d^3v}{dxdy^2}\right) + s\left(\frac{ddv}{dy^2}\right)$$

on account of

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv$$

there will be

$$\left(\frac{d^3v}{dxdy^2}\right) = F\left(\frac{d^3v}{dx^3}\right) + \frac{dF}{dx}\left(\frac{ddv}{dx^2}\right) + \frac{dG}{dx}\left(\frac{dv}{dx}\right) + \frac{dH}{dx}v$$
$$+ \quad G \qquad + \quad H$$

and

$$\left(\frac{d^4v}{dx^2dy^2}\right) = F\left(\frac{d^4v}{dx^4}\right) + \frac{2dF}{dx}\left(\frac{d^3v}{dx^3}\right) + \frac{ddF}{dx^2}\left(\frac{ddv}{dx^2}\right) + \frac{ddG}{dx^2}\left(\frac{dv}{dx}\right) + \frac{ddH}{dx^2}v$$
$$+ \quad G \qquad\qquad + \frac{dG}{dx} \qquad\qquad + \frac{2dH}{dx}$$
$$+ H$$

Then indeed on account of

$$z = \left(\frac{ddv}{dx^2}\right) + r\left(\frac{dv}{dx}\right) + sv$$

there becomes

$$\left(\frac{dz}{dx}\right) = \left(\frac{d^3v}{dx^3}\right) + r\left(\frac{ddv}{dx^2}\right) + \frac{dr}{dx}\left(\frac{dv}{dx}\right) + \frac{ds}{dx}v$$
$$+ s$$

and

$$\left(\frac{ddz}{dx^2}\right) = \left(\frac{d^4v}{dx^4}\right) + r\left(\frac{d^3v}{dx^3}\right) + \frac{2dr}{dx}\left(\frac{ddv}{dx^2}\right) + \frac{ddr}{dx^2}\left(\frac{dv}{dx}\right) + \frac{dds}{dx^2}v$$
$$+ \quad s \qquad\qquad + \frac{2ds}{dx}$$

Now with these substituted it is necessary that all the terms affected by $\left(\frac{d^4v}{dx^4}\right), \left(\frac{d^3v}{dx^3}\right), \left(\frac{ddv}{dx^2}\right), \left(\frac{dv}{dx}\right)$ and v separately vanish, from which the following equations result :

from	
$\left(\frac{d^4v}{dx^4}\right)$	I. $F = P$,
$\left(\frac{d^3v}{dx^3}\right)$	II. $G + \frac{2dF}{dx} + Fr = Pr + Q$,
$\left(\frac{ddv}{dx^2}\right)$	III. $H + \frac{2dG}{dx} + \frac{ddF}{dx^2} + Gr + \frac{rdF}{dx} + Fs = P\left(s + \frac{2dr}{dx}\right) + Qr + R$,
$\left(\frac{dv}{dx}\right)$	IV. $\frac{2dH}{dx} + \frac{ddG}{dx^2} + Hr + \frac{rdG}{dx} + Gs = P\left(\frac{2ds}{dx} + \frac{ddr}{dx^2}\right) + Q\left(s + \frac{dr}{dx}\right) + Rr$,
v	V. $\frac{ddH}{dx^2} + \frac{rdH}{dx} + Hs = P\frac{dds}{dx^2} + Q\frac{ds}{dx} + Rs$.

From the first there becomes $F = P$, from the second $Q = G + \frac{2dF}{dx}$, from the third

$$R = H + \frac{2dG}{dx} + \frac{ddF}{dx^2} - \frac{rdF + 2Fdr}{dx}$$

which values substituted into the final equations give

$$\frac{2dH}{dx} + \frac{ddG}{dx^2} - \frac{rdG + Gdr}{dx} - \frac{rddF}{dx^2} - \frac{2dFdr}{dx^2} - \frac{2sdF + 2Fds}{dx} + \frac{rrdF + 2Frdr}{dx} - \frac{Fddr}{dx^2} = 0$$

and

$$\frac{ddH}{dx^2}+\frac{rdH}{dx}-\frac{sddF+2dFds+Fdds}{dx^2}-\frac{2sdG+Gds}{dx}+\frac{s(rdF+2Fdr)}{dx}=0\,,$$

of which the former is integrable at once giving

$$2H+\frac{dG}{dx}-Gr-\frac{rdF+2Fdr}{dx}-2Fs+Frr=A,$$

then with these two equations thus represented

$$\frac{dd.Fr}{dx^2}-\frac{2d.Fs}{dx}+\frac{d.Frr}{dx}+\frac{ddG}{dx^2}-\frac{d.Gr}{dx}+\frac{2dH}{dx}=0\,,$$

$$-\frac{dd.Fs}{dx^2}+\frac{s}{r}\frac{d.Frr}{dx}-\frac{2sdG+Gds}{dx}+\frac{rdH}{dx}+\frac{ddH}{dx^2}=0$$

or even in this manner

$$\frac{dd.(G-Fr)}{dx}-d.r(G-Fr)+2d.(H-Fs)=0,$$

$$\frac{dd.(H-Fs)}{dx}+2Fsdr+rsdF-Gds-2sdG+rdH=0$$

truly the latter can be represented thus :

$$\frac{dd.(H-Fs)}{dx}-2sd.(G-Fr)-ds(G-Fr)+rd.(H-Fs)=0.$$

Because if now the former should be multiplied by $H-Fs$, truly the latter by $-(G-Fr)$, the sum becomes

$$\begin{aligned}&\frac{(H-Fs)dd.(G-Fr)-(G-Fr)dd.(H-Fs)}{dx}-(G-Fr)(H-Fs)dr\\&+2(H-Fs)d.(H-Fs)-r(H-Fs)d.(G-Fr)\\&+2s(G-Fr)d.(G-Fr)+(G-Fr)^2ds-r(G-Fr)d.(H-Fs)=0,\end{aligned}$$

the integral of which evidently is

$$\frac{(H-Fs)d(G-Fr)-(G-Fr)d(H-Fs)}{dx}+(H-Fs)^2+(G-Fr)^2s-(G-Fr)(H-Fs)r=B.$$

But the integral first found above is

$$\frac{d.(G-Fr)}{dx}-(G-Fr)r+2(H-Fs)=A,$$

which multiplied by $H-Fs$ and taken from that one leaves

$$-\frac{(G-Fr)d.(H-Fs)}{dx}-(H-Fs)^2+(G-Fr)^2s=B-A(H-Fs),$$

and thus there may be considered simply two differential equations, from which it is required to define the two quantities *r* and *s*, from which they become known also from the known functions *P, Q* and *R*.

COROLLARY 1

368. If there shall be $F=1,\quad G=0$ and $H=0$, the equations found will be

$$-\frac{dr}{dx}+rr-2s=a \quad \text{and} \quad \frac{sdr-rds}{dx}+ss=b,$$

from which on eliminating *dx* there becomes

$$\frac{rds-sdr}{dr}=\frac{b-ss}{a+2s-rr} \quad \text{or} \quad \frac{rds}{dr}=\frac{b+as+ss-rrs}{a+2s-rr},$$

the resolution of which in general scarcely may be considered to be undertaken. But with the constants taken $a=0$ and $b=0$ the equation $\frac{rds}{dr}=\frac{ss-rrs}{2s-rr}$ on putting $s=rrt$ is transferred into

$$\frac{rdt+2tdr}{dr}=\frac{tt-t}{2t-1} \quad \text{or} \quad \frac{rdt}{dr}=\frac{-3tt+t}{2t-1},$$

from which there becomes

$$\frac{dr}{r}=\frac{dt(1-2t)}{t(3t-1)}=\frac{-dt}{t}+\frac{dt}{3t-1} \quad \text{and} \quad r=\frac{\alpha\sqrt[3]{(3t-1)}}{t},$$

hence

$$s=\frac{\alpha\alpha\sqrt[3]{(3t-1)2}}{t}.$$

COROLLARY 2

369. As with the same singular case we may put $3t-1=u^3$, so that there becomes

$$r=\frac{3au}{1+u^3} \quad \text{and} \quad s=\frac{3\alpha\alpha uu}{1+u^3}.$$

Now on account of $a=0$ there is

$$dx=\frac{dr}{rr-2s}=\frac{dr}{rr(1-2t)}=\frac{3dr}{rr(1-2u^3)},$$

but

$$\frac{dr}{rr}=\frac{du}{3\alpha uu}-\frac{2udu}{3\alpha}=\frac{du(1-2u^3)}{3\alpha uu}$$

thus so that there shall become $dx=\frac{du}{\alpha uu}$ and hence

$$\frac{1}{u}=\beta-\alpha x \quad \text{and} \quad u=\frac{1}{\beta-\alpha x},$$

which indeed with the generality saved, there can be taken $\beta=0$ and $u=\frac{-1}{\alpha x}$, from which there comes about

$$r=\frac{-3xx}{x^3+c^3}$$

on making $\alpha=-\frac{1}{c}$ and

$$s=\frac{3x}{x^3+c^3}.$$

Therefore finally there is deduced

$$P=1,\ Q=0 \quad \text{and} \quad R=-\frac{2dr}{dx}=\frac{6x(2c^3-x^3)}{(c^3+x^3)^2}$$

COROLLARY 3

370. Therefore with the proposed equation $\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)$, the integral of which is

$$v=\Gamma:(x+y)+\Delta:(x-y),$$

the integral of this equation can be assigned

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{6x(2c^3-x^3)}{(c^3+x^3)^2}z;$$

indeed it is

$$z=\left(\frac{ddv}{dx^2}\right)-\frac{3xx}{c^3+x^3}\left(\frac{dv}{dx}\right)+\frac{3x}{c^3+x^3}v.$$

SCHOLION 1

371. These are able to be computed much more easily for the case $F=1$, $G=0$ and $H=0$ and more generally they are able to be computed for any value of the quantity *a*, provided there shall be *b* = 0; then indeed the other equation gives at once

$$dx=\frac{rds-sdr}{ss}$$

and hence

$$x=\frac{-r}{s} \text{ and } s=\frac{-r}{x},$$

from which the first equation adopts this form [§368]

$$\frac{dr}{dx}-rr-\frac{2r}{x}+a=0.$$

We may put $r=\frac{a}{t}$; there becomes

$$dt+\frac{2tdx}{x}-ttdx+adx=0,$$

for which the equation is satisfied in particular by

$$t=\sqrt{a}+\frac{1}{x}.$$

Therefore there is put in place

$$t=\sqrt{a}+\frac{1}{x}+\frac{1}{u}$$

and there is produced

$$du+dx+2udx\sqrt{a}=0,$$

which multiplied by $e^{2x\sqrt{a}}$ and integrated gives

$$e^{2x\sqrt{a}}u+\frac{1}{2\sqrt{a}}e^{2x\sqrt{a}}=\frac{n}{2\sqrt{a}}$$

and thus

$$\frac{1}{u}=\frac{2e^{2x\sqrt{a}}\sqrt{a}}{n-e^{2x\sqrt{a}}}=\frac{2\sqrt{a}}{ne^{-2x\sqrt{a}}-1}$$

both

$$t=\frac{1}{x}+\frac{ne^{-2x\sqrt{a}}+1}{ne^{-2x\sqrt{a}}-1}\sqrt{a}=\frac{1}{x}+\frac{n+e^{2x\sqrt{a}}}{n-e^{2x\sqrt{a}}}\sqrt{a}$$

and [since $r=\frac{a}{t}$,]

$$r=\frac{ax\left(n-e^{2x\sqrt{a}}\right)}{n\left(x\sqrt{a}+1\right)+e^{2x\sqrt{a}}\left(x\sqrt{a}-1\right)}$$

and therefore [since $s=\frac{-r}{x}$,]

$$s=\frac{-a\left(n-e^{2x\sqrt{a}}\right)}{n\left(x\sqrt{a}+1\right)+e^{2x\sqrt{a}}\left(x\sqrt{a}-1\right)},$$

then indeed at last

$$P=1,\quad Q=0\quad\text{and}\quad R=-\frac{2dr}{dx}=-2rr-\frac{4r}{x}+2a$$

or

$$R=\frac{-2a\left(nn-4naxxe^{2x\sqrt{a}}-2ne^{2x\sqrt{a}}+e^{4x\sqrt{a}}\right)}{\left(n\left(x\sqrt{a}+1\right)+e^{2x\sqrt{a}}\left(x\sqrt{a}-1\right)\right)^2}=\frac{-2a\left(n-e^{2x\sqrt{a}}\right)^2+8naaxxe^{2x\sqrt{a}}}{\left(n\left(x\sqrt{a}+1\right)+e^{2x\sqrt{a}}\left(x\sqrt{a}-1\right)\right)^2}.$$

If now a is taken vanishing and $n=1+\frac{2}{3}ac^3\sqrt{a}$, the formulas found before [§ 369] result. But if a shall be a negative quantity, on putting $a=-m^2$, and there is taken $n=\frac{\alpha\sqrt{-1}+\beta}{\alpha\sqrt{-1}-\beta}$, then there is found that

$$r=\frac{-mmx(\beta\cos.mx+\alpha\sin.mx)}{\beta\cos.mx+\alpha\sin.mx-mx(\alpha\cos.mx-\beta\sin.mx)}=\frac{-mmx\cos.(mx+\gamma)}{\cos.(mx+\gamma)+mx\sin.(mx+\gamma)}$$

and

$$s=\frac{mmx\cos.(mx+\gamma)}{\cos.(mx+\gamma)+mx\sin.(mx+\gamma)}$$

and hence

$$R=\frac{2mm\left(\cos.(mx+\gamma)-mmxx\right)}{\left(\cos.(mx+\gamma)+mx\sin.(mx+\gamma)\right)^2}$$

The quantity R is reduced to this

$$R=\frac{8naaxx-2a\left(ne^{-x\sqrt{a}}-e^{x\sqrt{a}}\right)^2}{\left(n\left(1+x\sqrt{a}\right)e^{-x\sqrt{a}}-\left(1-x\sqrt{a}\right)e^{x\sqrt{a}}\right)^2},$$

which form, with a taken very small, will change into

$$R=\frac{8naaxx-2a\left(n-1-(n+1)x\sqrt{a}+\frac{n-1}{2}axx-\frac{n+1}{6}ax^3\sqrt{a}+\text{etc.}\right)^2}{\left(n-1-\frac{1}{2}(n-1)axx+\frac{1}{3}(n+1)ax^3\sqrt{a}+\text{etc.}\right)^2}.$$

There is put in place $n=1+\beta a\sqrt{a}$, so that there shall be

$$n-1=\beta a\sqrt{a} \text{ and } n+1=2+\beta a\sqrt{a};$$

there becomes

$$R=\frac{8naaxx-2a\left(\beta a\sqrt{a}-2x\sqrt{a}-\beta axx+\frac{\beta aaxx\sqrt{a}}{2}-\frac{1}{3}ax^3\sqrt{a}\right)^2}{\left(\beta a\sqrt{a}-\frac{1}{2}\beta aaxx\sqrt{a}+\frac{2}{3}ax^3\sqrt{a}\right)^2},$$

where the numerator becomes

$$8aaxx+8\beta a^3xx\sqrt{a}-2a\left(\beta\beta a^3-4\beta aax-2\beta\beta a^3x\sqrt{a}+4axx+\frac{4}{3}aax^4\right)$$

where since the terms containing *aa* cancel each other, these remain only, which contain a^3; by observing likewise in the denominator

$$R=\frac{8\beta a^3x-\frac{8}{3}a^3x^4}{a^3\left(\beta+\frac{2}{3}x^3\right)^2}=\frac{8x\left(\beta-\frac{1}{3}x^3\right)}{\left(\beta+\frac{2}{3}x^3\right)^2},$$

which now easily is reduced to the form

$$R=\frac{6x\left(2c^3-x^3\right)}{\left(c^3+x^3\right)^2}$$

on taking $3\beta=2c^3$, so that there shall be $\beta=\frac{2}{3}c^3$. Whereby the case arises here on taking *a* vanishing, and

$$n=1+\frac{2}{3}c^3a\sqrt{a}.$$

SCHOLIUM 2

372. Since the development of the solution found shall be most difficult neither is it apparent in any way, how both the unknown quantities *r* and *s* extracted from the two equations are able to be defined, it will be most pleasing to observe the same problem from the science of increments by a repetition of the transformation in the first problem [§ 349] of this chapter, by which it can be solved, and nor therefore will it be without the use of these two solutions being compared with each other.

Therefore for the proposed equation

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv$$

we may put initially

$$u = \left(\frac{dv}{dx}\right) + pv$$

and p may be determined from this equation [§ 351]

$$Fdp + Gpdx - Fppdx + (\mathrm{C} - H)dx = 0$$

and then this equation will result

$$\left(\frac{ddu}{dy^2}\right) = F\left(\frac{ddu}{dx^2}\right) + \left(G + \frac{dF}{dx}\right)\left(\frac{du}{dx}\right) + \left(H + \frac{dG}{dx} - \frac{2Fdp + pdF}{dx}\right)u.$$

Now for this equation again we may put in place by transforming in a similar manner

$$z = \left(\frac{du}{dx}\right) + qu\ ,$$

thus so that there shall be also

$$z = \left(\frac{ddv}{dx^2}\right) + (p+q)\left(\frac{dv}{dx}\right) + \left(\frac{dP}{dx} + pq\right)v\ ,$$

and with the quantity q defined from that equation

$$Fdq + \left(G + \frac{dF}{dx}\right)qdx - Fqqdx + \left(D - H - \frac{dG}{dx} + \frac{2Fdp + PdF}{dx}\right)dx = 0$$

this equation may arise

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz\ ,$$

the quantities P, Q, R of which may be considered themselves

$$P = F\,,\quad Q = G + \frac{2dF}{dx}$$

and

$$R = H + \frac{2dG}{dx} - \frac{2Fdp + pdF}{dx} + \frac{ddF}{dx^2} - \frac{2Fdq + qdF}{dx}.$$

Therefore since that must agree with that solution, as the final problem [§ 367] required; in which since we could put in place at once

$$z = \left(\frac{ddv}{dx^2}\right) + r\left(\frac{dv}{dx}\right) + sv$$

certainly there will be

$$r = p + q \quad \text{and} \quad s = \frac{dp}{dx} + pq\,,$$

from which indeed at once the values for *P*, *Q* and *R* evidently appear the same.

Truly it is much less evident, if for *r* and *s* these values *p* and *q* are substituted, then these two equations

$$\frac{d.(G-Fr)}{dx} - (G - Fr)r + 2(H - Fs) = A$$

and

$$\frac{(G-Fr)d.(H-Fs)}{dx} + (H - FS)^2 - (G - Fr)^2 s - A(H - Fs) = B$$

are reduced to these, which we found before

$$\frac{Fdp}{dx} + Gp - Fpp - H + C = 0$$

and

$$\frac{Fdq}{dx} + \left(G + \frac{dF}{dx}\right)q - Fqq - H - \frac{dG}{dx} + \frac{2Fdp + pdF}{dx} + D = 0,$$

thus so that these constants *C* and *D* maintain a certain relation to these *A* and *B*. Meanwhile it is apparent that these latter equations are much simpler, as long as the first two includes only the variable *p* that must be determined and *x* is included and thence *p* in terms of *x*, of which *F*, *G* and *H* are given functions , with which found the quantity *q* is required to be elicited from the other equation in a similar manner. Truly in both the above equations the two variables *r* and *s* thus have been interchanged between themselves, so that no method can resolve these or to the extent that an equation between the two variables only can be considered to be arrived at. Therefore since it is certain that the first are the most difficult to produce a solution, and accordingly the latter much easier with the aid of the designated substitutions, without doubt a method bringing about this reduction with help not to be scorned, and soon to be considered in analysis.

SCHOLIUM 3

373. Since at this stage the agreement of these two solutions shall be greatly obscured, a special case will be set out to be considered most carefully.

Therefore let $F=1$, $G=0$ and $H=0$ and both the former equations adopt these forms between *r* and *s*:

$$\text{I.}\quad \frac{-dr}{dx}+rr-2s=A \text{ and II.}\quad \frac{rds}{dx}+ss-rrs+As=B,$$

truly the latter, these :

$$\text{III.}\quad \frac{dp}{dx}-pp+C=0 \quad\text{and}\quad \text{IV.}\quad \frac{dq}{dx}-qq+\frac{2dp}{dx}+D=0,$$

which since it is certain thus from these taken together, that there shall be

$$r=p+q \quad\text{and}\quad s=\frac{dp}{dx}+pq.$$

So that in any case we are ignorant of the agreement with the latter, let $C=-mm$ and the third gives

$$dx=\frac{dp}{mm+pp},$$

hence

$$x=\frac{1}{m}\text{ang.tang.}\frac{p}{m} \quad\text{and}\quad p=m\text{tang.}mx.$$

Hence since there shall be $\frac{dp}{dx}=mm+pp$, there will be

$$s=mm+pp+pq=mm+pr=m\left(m+r\text{ tang.}mx\right),$$

which value substituted in I gives

$$\frac{-dr}{dx}+rr-2mr\text{ tang.}mx-mm=A,$$

or

$$\frac{dr}{dx}=rr-2mr\text{ tang.}mx-mm-A.$$

the second truly on account of

$$\frac{ds}{dx}=\frac{mdr}{dx}\text{tang.}mx+\frac{mmr}{\cos.^2 mx}$$

will change into

$$\frac{mrdr}{dx}\text{tang.}mx = mr^3\text{tang.}mx - 2mmrr\,\text{tang.}^2mx - m(A+2mm)r\,\text{tang.}mx - m^4 - Amm + B,$$

from which on eliminating *dr* there becomes $B = Amm + m^4$. Truly for the fourth on account of

$$q = r - p = r - m\,\text{tang.}mx$$

there results

$$\frac{dr}{dx} = rr - 2mr\,\text{tang.}mx - mm - D,$$

thus so that there shall be $D = mm + A$. Therefore the agreement of our equations consists in this relation of the constants, so that on account of $mm = -C$ there shall be

$$D = A - C \quad \text{and} \quad B = -C(A - C) = -CD.$$

Truly in general also the same relations are in place; for if III and IV are gathered into one sum, on account of $C + D = A$ and $p + q = r$ there will be

$$\frac{Fdr}{dx} + Gr + \frac{rdF}{dx} - Fpp - Fqq - 2H - \frac{dG}{dx} + 2Fdp + A = 0;$$

since truly there shall be $\frac{dp}{dx} = s - pq$, there becomes

$$\frac{Fdr + rdF - dG}{dx} + Gr - Frr - 2H + 2Fs + A = 0$$

or

$$\frac{d.(G-Fr)}{dx} - (G - Fr)r + 2(H - Fs) = A,$$

which is that first equation itself.

Again the third equation on account of $\frac{dp}{dx} = s - pq$ gives

$$Fs - Fpr + Gp - H + C = 0 \text{ or } C = H - Fs - p(G - Fr);$$

truly the fourth is reduced to this form

$$\frac{Fdr}{dx} + Gq + \frac{qdF}{dx} - Fqq - H - \frac{dG}{dx} + Fs - Fpq + \frac{pdF}{dx} + D = 0$$

or

$$\frac{d.(Fr-G)}{dx} + q(G - Fr)r - H + Fs + D = 0$$

and hence

$$D = \frac{d.(G-Fr)}{dx} - q(G-Fr) + H - Fs,$$

from which it is concluded

$$CD = \frac{(H-Fs)d.(G-Fr)}{dx} - q(G-Fr)(H-Fs) + (H-Fs)^2$$
$$-\frac{p(G-Fr)d.(G-Fr)}{dx} + pq(G-Fr)^2 - p(G-Fr)(H-Fs).$$

Truly from the second we have

$$B = \frac{(G-Fr)d.(H-Fs)}{dx} - \frac{(H-Fs)d.(G-Fr)}{dx} - (H-FS)^2 + (G-Fr)(H-Fs)r - (G-Fr)^2 s,$$

with which expression joined together there becomes

$$\frac{CD+B}{(G-Fr)} = \frac{d.(H-Fs)}{dx} - \frac{pd.(G-Fr)}{dx} - \frac{dp(G-Fr)}{dx}$$
$$= \frac{d.(H-Fs)-d.p(G-Fr)}{dx} = 0,$$

accordingly there is $C = H - Fs - p(G-Fr)$, from which also in general there is

$$B = -CD \quad \text{and} \quad A = C + D.$$

Yet meanwhile it is not hence evident, how the two remaining equations III and IV are able to be derived from the equations I and II.

SCHOLIUM 4

374. With everything made evident from these careful considerations, the whole business is able to be put together satisfactorily with the aid of a simple substitution. Because that can be shown easier, for brevity we put $G - Fr = R$ and $H - Fs = S$, so that these two equations may be considered :

$$\text{I.} \quad A = \frac{dR}{dx} - \frac{GR}{F} + \frac{RR}{F} + 2S.$$

$$\text{II.} \quad B = \frac{RdS-SdR}{dx} - \frac{HRR}{F} + \frac{GRS}{F} - SS,$$

from which it is required to elicit the two quantities R and S, while F, G, H are any functions of x, but A and B are constant quantities. Accordingly this substitution may be used $S = C + Rp$ thus to be equipped, so that both these equations merge into one, in which besides x with a single new variable present p, then it is required to be investigated by known methods.

Hence on account of $dS = Rdp + pdR$ there will be considered

I. $A = \frac{dR}{dx} - \frac{GR}{F} + \frac{RR}{F} + 2C + 2Rp,$

II. $B = \frac{RRdp}{dx} - \frac{CdR}{dx} - \frac{HRR}{F} + \frac{CGR}{F} + \frac{GRRp}{F} - CC - 2CRp - RRpp,$

from which at first on eliminating dR there is concluded

$$B + AC = \frac{RRdp}{dx} + \frac{CRR}{F} + CC - \frac{HRR}{F} - RRpp + \frac{GRRp}{F};$$

provided therefore we assume C constant, so that there shall be $CC = B + AC$, also by division the quantity R is removed and this equation comes about

$$0 = \frac{dp}{dx} + \frac{C}{F} - \frac{H}{F} - pp + \frac{Gp}{F}.$$

the resolution of which relates to better recognised methods.

Therefore since that method shall be of the greatest concern, even if it shall be referring to the first part of the integral calculus, here it is considered worthwhile to be added.

PROBLEM 60

375. *With two differential equations of this kind proposed*

$$\text{I. } 0 = \frac{dy}{dx} + F + Gy + Hz + Iyy + Kyz + Lzz,$$
$$\text{II. } 0 = \frac{ydz - zdy}{dx} + P + Qy + Rz + Syy + Tyz + Vzz,$$

where F, G, H etc., P, Q, R etc. shall be functions of x, to set out the method by which these equations are to be resolved, if indeed that is possible to come about.

SOLUTION

The indicated method is formed from this, so that with the aid of the substitution $z = a + yv$ ex a single equation can be elicited from these equation involving only two variables x and v. Therefore since there is $ydz - zdy = yydv - ady$, this equation arises from Ia + II

$$0 = \frac{yydv}{dx} + P + Qy + Rz + Syy + Tyz + Vzz$$
$$+ aF + aGy + aHz + aIyy + aKyz + aLzz,$$

which with the value $a + yv$ substituted in place of z thus may be shown following the powers of y

$$0=\frac{yydv}{dx}+y^0\left(P+aF+a\left(R+aH\right)+aa\left(V+aL\right)\right)$$
$$+y^1\left(Q+aG+v\left(R+aH\right)+a\left(T+aK\right)+2av\left(V+aL\right)\right)$$
$$+y^2\left(S+aI\ +v\left(T+aK\right)+vv\left(V+aL\right)\right),$$

and now it is to be effected, that the whole equation is able to be divided by *yy* and thus the parts pertaining to y^0 and y^1 vanish. Therefore from the part corresponding to y^0 there is required to become

$$P+aF+a\left(R+aH\right)+aa\left(V+aL\right)=0\,,$$

moreover from the part y^1, because it is a new variable introduced into the calculation, these two conditions arise

$$Q+aG+a\left(T+aK\right)=0 \quad \text{and} \quad R+aH+2a\left(V+aL\right)=0\,,$$

from the first there will be given

$$P+\ aF-aa\left(V+aL\right)=0\,.$$

The conditions according to that reduction required are these three

$$\text{1.}\quad P+aF-aa\left(V+aL\right)=0,$$
$$\text{II.}\quad Q+aG+\ \ a\left(T+aK\right)=0,$$
$$\text{III.}\quad R+aH+2a\left(V+aL\right)=\ 0,$$

from which either *P*, *Q* and *R* or *F*, *G* and *H* are defined conveniently.
But with these conditions established the whole business is recalled to the solution of this equation

$$0=\tfrac{dv}{dx}+S+aI+v\left(T+aK\right)+vv\left(V+aL\right),$$

which contains only the two variables *x* and *v*, from which *v* is required to be determined through *x*, since then on putting $z=a+yv$, the first equation adopts this form

$$0=\tfrac{dy}{dx}+F+aH+aaL+y\left(G+Hv+aK+2aLv\right)+yy\left(I+Kv+Lvv\right),$$

truly the second this :

$$0=\frac{yydv}{dx}-\frac{ady}{dx}+P+aR+aaV+y\left(Q+Rv+aT+2aVv\right)+yy\left(S+Tv+Vvv\right),$$

or hence the above multiplied by *yy* on subtraction,

$$0=\frac{-ady}{dx}+P+aR+aaV+y\left(Q+Rv+aT+2aVv\right)-yy\left(Ia+aKv+aLvv\right),$$

which indeed agrees with that , as the nature of the argument demands.

COROLLARY 1

376. Therefore if the two equations proposed of this kind were

$$\begin{aligned}0&=\frac{dy}{dx}+F-Gy+Hz+Iyy+Kyz+Lzz,\\0&=\frac{ydz-zdy}{dx}-aF-aGy-aHz+Syy+Tyz+Vzz\\&\quad+a^3L-aaKy-2aaLz\\&\quad+aaV-aTy-2aVz,\end{aligned}$$

on making $z=a+yv$ in the first place this equation must be resolved

$$0=\frac{dv}{dx}+S+aI+v\left(T+aK\right)+vv\left(V+aL\right),$$

from which with *v* defined through *x*, this equation is required to be treated :

$$0=\frac{dy}{dx}+F+aH+aaL+y\left(G+aK\right)+yy\left(I+Kv+Lvv\right)+vy\left(H+2aL\right),$$

with which done there will also be had $z=a+vy$.

COROLLARY 2

377. If $F=A,\ K=0,\ L=0,\ H=-2b,\ V=b$ and $T=-G$, the above case treated of these equations § 374 results :

$$\begin{aligned}0&=\frac{dy}{dx}+A+Gy-2bz+Iyy,\\0&=\frac{ydz-zdy}{dx}-aA+aab+Syy-Gyz+bzz,\end{aligned}$$

where *G, I* and *S* are some functions of *x*, and the resolution thus may itself be considered, so that on putting $z=a+vy$ these equations must successively be arranged

$$0 = \frac{dv}{dx} + S + aI - Gv + bvv$$

and

$$0 = \frac{dy}{dx} + A - 2ab + y(G - 2bv) + Iyy.$$

COROLLARY 3

378. It is evident that the last equation works with no difficulty also in general, as long as there shall be

$$F + aH + aaL = 0;$$

but the solution of the first is brought out, if there shall be either $S + aI = 0$ or $V + aL = 0$.

CAPUT V

TRANSFORMATIO SINGULARIS EARUNDEM AEQUATIONUM

PROBLEMA 56

349. *Proposita hac aequatione*

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz\,,$$

in qua P, Q, R sint functiones ipsius x tantum, eam ope substitutionis

$$z = M\left(\frac{dv}{dx}\right) + Nv\,,$$

ubi quoque sint M et N functiones ipsius x tantum, in aliam eiusdem formae transmutare, ut prodeat

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv$$

existentibus F, G, H functionibus solius x.

SOLUTIO

Quia quantitates M et N ab y sunt immunes, erit

$$\left(\frac{ddz}{dy^2}\right) = M\left(\frac{d^3v}{dxdy^2}\right) + N\left(\frac{ddv}{dy^2}\right),$$

quae forma per aequationem, quam tandem resultare assumimus, abit in hanc

$$\begin{aligned}\left(\frac{ddz}{dy^2}\right) = MF\left(\frac{d^3v}{dx^3}\right) &+ \frac{MdF}{dx}\left(\frac{ddv}{dx^2}\right) + \frac{MdG}{dx}\left(\frac{dv}{dx}\right) + \frac{MdH}{dx}v\\ &+ MG \qquad\quad + MH \qquad\quad + NH\\ &+ NF \qquad\quad\; + NG\end{aligned}$$

Deinde vero pro altero aequationis propositae membro nostra substitutio praebet

$$\left(\frac{dz}{dx}\right) = M\left(\frac{ddv}{dx^2}\right) + \frac{dM}{dx}\left(\frac{dv}{dx}\right) + \frac{dN}{dx}v + N$$

hincque porro

$$\left(\frac{ddz}{dx^2}\right) = M\left(\frac{d^3v}{dx^3}\right) + \left(\frac{2dM}{dx} + N\right)\left(\frac{ddv}{dx^2}\right) + \left(\frac{ddM}{dx^2} + \frac{2dN}{dx}\right)\left(\frac{dv}{dx}\right) + \frac{ddN}{dx^2}v.$$

Cum nunc sit per hypothesin

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{d^2z}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz\,,$$

si hic valores modo inventi substituantur singulaque membra $\left(\frac{d^3v}{d^3x}\right)$, $\left(\frac{ddv}{dx^2}\right)$, $\left(\frac{dv}{dx}\right)$ et v seorsim ad nihilum redigantur, quatuor sequentes aequationes orientur, scilicet

ex	colligitur aequatio
$\left(\frac{d^3v}{dx^3}\right)$	$MF = MP\,,$
$\left(\frac{ddv}{dx^2}\right)$	$\frac{MdF}{dx} + MG + NF = \left(\frac{2dM}{dx} + N\right)P + MQ,$
$\left(\frac{dv}{dx}\right)$	$\frac{MdG}{dx} + MH + NG = \left(\frac{ddM}{dx^2} + \frac{2dN}{dx}\right)P + \left(\frac{dM}{dx} + N\right)Q + MR,$
v	$\frac{MdH}{dx} + NH = \frac{ddN}{dx^2}P + \frac{dN}{dx}Q + NR,$

ex quibus commodissime primo quaeruntur P, Q et R.
Verum prima dat statim $P = F$, unde secunda fit

$$\frac{MdF - 2FdM}{Mdx} + G = Q\,.$$

Ex binis ultimis autem eliminando R colligitur

$$\frac{M(NdG - MdH)}{dx} + NNG = \left(\frac{NddM - MddN}{dx^2} + \frac{2NdN}{dx}\right)F + \left(\frac{NdM - MdN}{dx} + \frac{NdN}{dx}\right)Q$$

et illum valorem pro Q substituendo

$$0 = \frac{MMdH}{dx} - \frac{MNdG}{dx} + \frac{NddM - MddN}{dx^2}F + \frac{2NFdN}{dx} + \frac{NdM - MdN}{dx}G$$
$$+ \frac{NdM - MdN}{dx^2}dF + \frac{NNdF}{dx} - \frac{2FdM(NdM - MdN)}{Mdx^2} - \frac{2NNFdM}{Mdx}\,,$$

quae aequatio per $\frac{dx}{MM}$ multiplicata commode integrabilis redditur, inveniturque integrale

$$C = H - \frac{N}{M}G + \frac{NdM - MdN}{MMdx}F + \frac{NNF}{MM}.$$

Quodsi ergo brevitatis gratia ponamus $N = Ms$, erit

$$C = H - Gs - \frac{Fds}{dx} + Fss$$

seu

$$ds + \frac{G}{F}sdx - ssdx + \frac{(C-H)dx}{F} = 0.$$

Sive iam hinc definiatur quantitas $s = \frac{N}{M}$ sive una functionum *F*, *G* et *H*, pro ipsa aequatione proposita litterae *P*, *Q* et R ita determinabuntur, ut sit

$$\text{I. } P = F,$$
$$\text{II. } Q = G + \frac{dF}{dx} - \frac{2FdM}{Mdx},$$

et ex ultima aequatione derivatur

$$R = H + \frac{MdH}{Ndx} - \frac{FddN}{Ndx^2} - \frac{dN}{Ndx}\left(G + \frac{dF}{dx} - \frac{2FdM}{Mdx}\right),$$

qui valor ob $N = Ms$ evadit

$$R = H + \frac{dH}{sdx} - \frac{Gds}{sdx} - \frac{GdM}{Mdx} - \frac{Fdds}{sdx^2} - \frac{FddM}{Mdx^2} + \frac{2FdM^2}{MMdx^2} - \frac{dFds}{sdx^2} - \frac{dFdM}{Mdx^2},$$

et cum aequatio inventa, si differentietur, det

$$0 = dH - Gds - sdG - \frac{Fdds}{dx} - \frac{dFds}{dx} + 2Fsds + ssdF,$$

obtinebimus

$$\text{III. } R = H - \frac{GdM}{Mdx} + \frac{dG}{dx} - \frac{FddM}{Mdx^2} - \frac{2Fds}{dx} + \frac{2FdM^2}{MMdx^2} - \frac{sdF}{dx} - \frac{dFdM}{Mdx^2};$$

unde, si aequatio

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv$$

resolutionem admittat, etiam resolutio succedet huius aequationis

$$\left(\frac{ddz}{dy^2}\right)=P\left(\frac{ddz}{dx^2}\right)+Q\left(\frac{dz}{dx}\right)+Rz,$$

cum sit

$$z=M\left(\frac{dv}{dx}\right)+Nv=M\left(sv+\left(\frac{dv}{dx}\right)\right).$$

COROLLARIUM 1

350. Si ponatur $M=1$, ut fiat $z=sv+\left(\frac{dv}{dx}\right)$, erit

$$P=F,\ Q=\ G+\frac{dF}{dx}\ \text{ et }\ R=H+\frac{dG}{dx}-\frac{2Fds+sdF}{dx}$$

neque hoc modo usus istius reductionis restringitur, quoniam, si deinceps loco z ponatur Mz, etiam aequationis hinc ortae resolutio est in promtu.

COROLLARIUM 2

351. Quoties ergo aequationis

$$\left(\frac{ddv}{dy^2}\right)=\ F\left(\frac{ddV}{dx^2}\right)+G\left(\frac{dv}{dx}\right)+Hv$$

resolutio est in potestate, toties etiam huius aequationis

$$\left(\frac{ddz}{dy^2}\right)=F\left(\frac{ddz}{dx^2}\right)+\left(G+\frac{dF}{dx}\right)\left(\frac{dz}{dx}\right)+\left(H+\frac{dG}{dx}-\frac{2Fds+sdF}{dx}\right)z$$

resolutio succedit, si modo capiatur s ex hac aequatione

$$Fds+Gsdx-Fssdx+\left(C-H\right)dx=0;$$

tum enim erit $z=sv+\left(\frac{dv}{dx}\right)$. Sunt autem litterae F, G, H functiones ipsius x tantum.

SCHOLION

352. Haec reductio methodum maxime naturalem suppeditare videtur eiusmodi integrationes perficiendi, quae simul functionum differentialia involvunt. Si enim aequationis pro v datae integrale sit $v=\varphi{:}t$ existente t functione ipsarum x et y, ob $dv=dt\varphi'{:}t$ erit $\left(\frac{dv}{dx}\right)=\left(\frac{dt}{dx}\right)\varphi'{:}t$ et aequationis inde derivatae pro z habebimus integrale

$$z=s\varphi{:}t+\left(\frac{dt}{dx}\right)\varphi'{:}t\,.$$

Deinde si fuerit generalius $v=u\varphi{:}t$, fiet

$$z = su\varphi{:}t + \left(\frac{du}{dx}\right)\varphi{:}t + u\left(\frac{dt}{dx}\right)\varphi'{:}t,$$

unde ratio perspicitur ad eiusmodi aequationes perveniendi, quarum integralia praeter functionem $\varphi{:}t$ etiam functiones ex eius differentiatione natas $\varphi'{:}t$ atque adeo etiam sequentes $\varphi''{:}t, \varphi'''{:}t$ etc. complectantur. Quamobrem operae pretium erit hanc reductionem accuratius evolvere.

PROBLEMA 57

353. *Concessa resolutione huius aequationis*

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) + \frac{m}{x}\left(\frac{dv}{dx}\right) + \frac{n}{xx}v$$

invenire aliam aequationem huius formae

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz$$

pro qua sit

$$z = sv + \left(\frac{dv}{dx}\right).$$

SOLUTIO

Facta comparatione cum praecedente problemate habemus

$$F = 1, G = \frac{m}{x} \quad \text{et} \quad H = \frac{n}{xx},$$

unde quantitatem *s* ex hac aequatione definiri oportet

$$ds + \frac{msdx}{x} - ssdx + \left(f - \frac{n}{xx}\right)dx = 0,$$

qua inventa ob $\frac{dG}{dx} = -\frac{m}{xx}$ aequatio quaesita erit

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) + \left(\frac{n-m}{xx} - \frac{2ds}{dx}\right)z$$

seu loco *ds* valore inde substituto

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) + \left(2f - \frac{n+m}{xx} + \frac{2ms}{x} - 2ss\right)z$$

pro qua est

$$z = sv + \left(\frac{dv}{dx}\right).$$

I. Ponamus primo quantitatem constantem $f = 0$, ut sit

$$ds + \frac{msdx}{x} - ssdx - \frac{ndx}{xx} = 0,$$

cuius integrale particulare est $s = \frac{\alpha}{x}$ existente

$$-\alpha + m\alpha - \alpha\alpha - n = 0 \text{ seu } \alpha\alpha - (m-1)\alpha + n = 0,$$

ex quo ob $\frac{ds}{dx} = \frac{-\alpha}{xx}$ oritur haec aequatio

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) + \frac{2\alpha - m + n}{xx} z$$

pro qua est

$$z = \frac{\alpha}{x} v + \left(\frac{dv}{dx}\right),$$

seu exclusa $n = \alpha(m - 1 - \alpha)$, si constet resolutio huius aequationis

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) + \frac{m}{x}\left(\frac{dv}{dx}\right) + \frac{\alpha(m-1-\alpha)}{xx} v$$

pro hac

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) + \frac{m}{x}\left(\frac{dz}{dx}\right) + \frac{(\alpha-1)(m-\alpha)}{xx} z$$

erit

$$z = \frac{\alpha}{x} v + \left(\frac{dv}{dx}\right).$$

II. Maneat $f = 0$ et quaeramus pro s valorem completum ponendo

$s = \frac{\alpha}{x} + \frac{1}{t}$ fietque ob $n = (m-1)\alpha - \alpha\alpha$

$$dt + \frac{(2a-m)tdx}{x} + dx = 0,$$

quae per $x^{2\alpha-m}$ multiplicata et integrata praebet

$$t=\frac{cx^{m-2\alpha}}{2\alpha-m+1}-\frac{x}{2\alpha-m+1}$$

hincque

$$s=\frac{\alpha cx^{m-2\alpha-1}+\alpha-m+1}{x\left(cx^{m-2\alpha-1}-1\right)}=\frac{\alpha}{x}+\frac{2\alpha-m+1}{x\left(cx^{m-2\alpha-1}-1\right)},$$

unde fit

$$\frac{ds}{dx}=\frac{-\alpha}{xx}+\frac{(m-2\alpha-1)(m-2\alpha)}{xx\left(cx^{m-2\alpha-1}-1\right)}+\frac{(m-2\alpha-1)^2}{xx\left(cx^{m-2\alpha-1}-1\right)^2}.$$

Hic praecipue notetur casus $c=0$, quo fit

$$s=\frac{m-\alpha-1}{x} \quad \text{et} \quad \frac{ds}{dx}=\frac{-m+\alpha+1}{xx}$$

ita ut data aequatione

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{m}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(m-1-\alpha)}{xx}v$$

pro hac aequatione

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\frac{(\alpha+1)(m-2-\alpha)}{xx}z$$

futurum sit

$$z=\frac{m-\alpha-1}{x}v+\left(\frac{dv}{dx}\right).$$

Pro generali autem valore sit $m-2\alpha-1=\beta$, ut habeatur

$$s=\frac{\alpha}{x}-\frac{\beta}{x\left(cx^{\beta}-1\right)} \quad \text{et} \quad \frac{ds}{dx}=\frac{-\alpha}{xx}+\frac{\beta(\beta+1)}{xx\left(cx^{\beta}-1\right)}+\frac{\beta\beta}{xx\left(cx^{\beta}-1\right)^2},$$

unde, si detur haec aequatio

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{2\alpha+\beta+1}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(\alpha+\beta)}{xx}v,$$

eius ope resolvetur haec

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{2\alpha+\beta+1}{x}\left(\frac{dz}{dx}\right)+\left((\alpha-1)(\alpha+\beta+1)-\frac{2\beta(\beta+1)}{cx^\beta-1}-\frac{2\beta\beta}{\left(cx^\beta-1\right)^2}\right)\frac{z}{xx},$$

cum sit

$$z=\left(\alpha-\frac{\beta}{cx^\beta-1}\right)\frac{v}{x}+\left(\frac{dv}{dx}\right)$$

III. Rationem quoque habeamus constantis f ponamusque $f=\frac{1}{aa}$, ut facto $n=\alpha(m-1-a)$ habeamus

$$ds+\frac{msdx}{x}-ssdx-\frac{\alpha(m-1-\alpha)dx}{xx}+\frac{dx}{aa}=0,$$

quae posito $s=\frac{\alpha}{x}+\frac{1}{t}$ abit in

$$dt-\frac{(m-2\alpha)tdx}{x}+dx=\frac{tt}{aa}dx.$$

Sit $m-2\alpha=\gamma$, ut aequatio data sit

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)+\frac{2\alpha+\gamma}{x}\left(\frac{dv}{dx}\right)+\frac{\alpha(\alpha+\gamma-1)}{xx}v$$

et inventa quantitate *s* prodeat haec aequatio

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{2\alpha+\gamma}{x}\left(\frac{dz}{dx}\right)+\left(\frac{\alpha\alpha-3\alpha+\alpha\gamma-\gamma}{xx}-\frac{2ds}{dx}\right)z$$

seu

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{2\alpha+\gamma}{x}\left(\frac{dz}{dx}\right)+\left(\frac{(\alpha-1)(\alpha+\gamma)}{xx}+\frac{2dt}{ttdx}\right)z,$$

pro qua est

$$z=\left(\frac{\alpha}{x}+\frac{1}{t}\right)v+\left(\frac{dv}{dx}\right),$$

ubi totum negotium ad inventionem quantitatis *t* redit ex aequatione

$$dt-\frac{\gamma tdx}{x}+dx=\frac{tt}{aa}dx.$$

Hunc in finem statuatur $t = a - \frac{aadu}{udx}$ ac reperitur

$$\frac{ddu}{dx^2} - \frac{\gamma du}{xdx} - \frac{2du}{adx} + \frac{\gamma u}{ax} = 0,$$

cuius duplex resolutio datur, altera ponendo

$$u = A + Bx + Cx^2 + Dx^3 + Ex^4 + Fx^5 + \text{ etc.},$$

existente

$$B = \frac{\gamma}{\gamma a}A, \quad C = \frac{\gamma-2}{2(\gamma-1)a}B, \quad D = \frac{\gamma-4}{3(\gamma-2)a}C, \quad E = \frac{\gamma-6}{4(\gamma-3)a}D \quad \text{etc.},$$

altera vero ponendo

$$u = Ax^{\gamma+1} + Bx^{\gamma+2} + Cx^{\gamma+3} + Dx^{\gamma+4} + Ex^4 + \text{etc.},$$

ubi

$$B = \frac{\gamma+2}{(\gamma+2)a}A, \quad C = \frac{\gamma+4}{2(\gamma+3)a}B, \quad D = \frac{\gamma+6}{3(\gamma+4)a}C, \quad E = \frac{\gamma+8}{4(\gamma+5)a}D \quad \text{etc.},$$

quarum illa abrumpitur, si sit γ numerus integer par positivus, haec vero, si negativus. Qui valores etsi sunt particulares, tamen supra iam ostendimus, quomodo inde valores completi sint eliciendi.

COROLLARIUM 1

354. Supra autem vidimus (§ 333) hanc aequationem

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) + \frac{2m}{x}\left(\frac{dv}{dx}\right) + \frac{(m+i)(m-i-1)}{xx}v$$

esse integrabilem, si sit *i* numerus integer quicunque, unde colligimus hanc aequationem

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) + \frac{m}{x}\left(\frac{dv}{dx}\right) + \frac{\alpha(m-i-\alpha)}{xx}v$$

integrationem admittere, quoties fuerit vel $\alpha = \frac{1}{2}m + i$ vel $\alpha = \frac{1}{2}m - i - 1$ seu $m - 2\alpha$ numerus integer par sive positivus sive negativus, qui casus ob $m - 2\alpha = \gamma$ cum casibus integrabilitatis pro valore generali ipsius *s* inveniendo congruunt.

COROLLARIUM 2

355. Quando autem ex hac aequatione functionem v definire licet, tum etiam hae duae sequentes aequationes [§ 353] illi similes resolvi poterunt

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\frac{(\alpha-1)(m-\alpha)}{xx}z$$

et

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{m}{x}\left(\frac{dz}{dx}\right)+\frac{(\alpha+1)(m-\alpha-2)}{xx}z\,,$$

cum pro illa sit

$$z=\frac{\alpha}{x}v+\left(\frac{dv}{dx}\right)$$

pro hac vero

$$z=\frac{m-\alpha-1}{x}v+\left(\frac{dv}{dx}\right).$$

COROLLARIUM 3

356. Praeterea vera etiam aequationes alius generis, ubi postremus terminus non est formae $\frac{n}{xx}z$, resolvi possunt, quae inveniuntur, si quantitatis s valor generalius investigatur atque adeo constantis f ratio habetur.

EXEMPLUM 1

357. *Proposita aequatione* $\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)$, *pro qua est*

$$v=\pi:(x+y)+\varphi:(x-y),$$

invenire aequationes magis complicatas, quae huius ope integrari queant.

Cum hic sit $F=1$, $G=0$ et $H=0$, resolvatur haec aequatio

$$ds-ssdx+Cdx=0$$

et huius aequationis

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{2ds}{dx}z$$

integrale erit

$$z=sv+\left(\frac{dv}{dx}\right)$$

Sumta autem primo constante $C=0$ fit $\frac{ds}{ss}=dx$ et $\frac{1}{s}=c-x$ seu $s=\frac{1}{c-x}$ atque $\frac{ds}{dx}=\frac{1}{(c-x)^2}$, ubi quidem sine ulla restrictione poni potest $c=0$, ut huius aequationis

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{2}{xx}z$$

integrale sit

$$z=-\tfrac{1}{x}\left(\pi:(x+y)+\varphi:(x-y)\right)+\pi':(x+y)+\varphi':(x-y).$$

Sit deinde $C=aa$ et ob $ds=dx(ss-aa)$ fiet $x=\frac{1}{2a}\frac{s-a}{s+a}-\frac{1}{2a}lA$ hincque

$$\tfrac{s-a}{s+a}=Ae^{2ax}\quad \text{et}\quad s=\frac{a\left(1+Ae^{2ax}\right)}{1-Ae^{2ax}}$$

unde

$$\frac{ds}{dx}=\frac{4Aaae^{2ax}}{\left(1-Ae^{2ax}\right)^2}$$

et aequationis

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{8Aaae^{2ax}}{\left(1-Ae^{2ax}\right)^2}z$$

integrale est

$$z=\frac{a\left(1+Ae^{2ax}\right)}{1-Ae^{2ax}}v+\left(\frac{dv}{dx}\right).$$

Sit tandem $C=-aa$ et ob $ds=dx(aa+ss)$ fit $ax+b=\text{Ang.tang.}\frac{s}{a}$ hincque

$$s=a\,\text{tang.}(ax+b)\quad \text{et}\quad \frac{ds}{dx}=\frac{aa}{\cos.^2(ax+b)},$$

quocirca huius aequationis

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{aa}{\cos.^2(ax+b)}z$$

integrale est

$$z=\frac{a\sin.(ax+b)}{\cos.(ax+b)}v+\left(\frac{dv}{dx}\right)$$

EXEMPLUM 2

358. *Proposita aequatione*

$$\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)-\frac{2}{xx}v,$$

cuius integrale constat [§ 357], *invenire alias eius ope integrabiles.*

Pro hoc casu habemus

$$ds - ssdx + \left(C + \tfrac{2}{xx}\right)dx = 0\,,$$

qua resoluta erit, huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - 2\left(\frac{1}{xx} + \frac{ds}{dx}\right)z$$

integrale

$$z = sv + \left(\frac{dv}{dx}\right).$$

1. Sit primo $C = 0$ et ex aequatione $ds - ssdx + \frac{2dx}{xx} = 0$ fit particulariter $s = \frac{1}{x}$ vel $s = -\frac{2}{x}$. Ponatur ergo $s = \frac{1}{x} + \frac{1}{t}$ eritque

$$dt + \frac{2tdx}{x} + dx = 0\,,$$

hinc $txx + \frac{1}{3}x^3 = \frac{1}{3}a^3$. Ergo

$t = \frac{a^3 - x^3}{3xx}$ et $s = \frac{a^3 + 2x^3}{x\left(a^3 - x^3\right)}$ ideoque $\frac{ds}{dx} + \frac{1}{xx} = \frac{3x\left(2a^3 + x^3\right)}{\left(a^3 - x^3\right)^2}$,

unde huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{6x\left(2a^3 + x^3\right)}{\left(a^3 - x^3\right)^2}z$$

integrale est

$$z = \frac{a^3 + 2x^3}{x\left(a^3 - x^3\right)}v + \left(\frac{dv}{dx}\right).$$

II. Sit $C = \frac{1}{cc}$ et posito $s = \frac{1}{x} + \frac{1}{t}$ fit

$$dt + \frac{2tdx}{x} + dx = \frac{ttdx}{cc}\,,$$

cui particulariter satisfacit $t = c + \frac{cc}{x}$, ut sit

$$s = \frac{cc + cx + xx}{cx(c + x)} \quad \text{et} \quad \frac{ds}{dx} + \frac{1}{xx} = \frac{1}{(c + x)^2}$$

atque huius aequationis

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{ddz}{dx^2}\right) - \frac{2}{(c + x)^2}z$$

integrale sit

$$z = \frac{cc+cx+xx}{cx(c+x)}v + \left(\frac{dv}{dx}\right).$$

Ad integrale autem pro t completum inveniendum statuatur

$$t = c + \frac{cc}{x} + \frac{1}{u}$$

fietque

$$du + \frac{2udx}{c} + \frac{dx}{cc} = 0 \quad \text{seu} \quad dx = \frac{-ccdu}{1+2cu},$$

hinc

$$x = b - 2l(1+2cu),$$

ergo

$$u = \frac{e^{\frac{2(b-x)}{c}} - 1}{2c},$$

unde

$$t = c + \frac{cc}{x} + \frac{2c}{e^{\frac{2(b-x)}{c}} - 1} \quad \text{et} \quad s = \frac{1}{x} + \frac{x\left(e^{\frac{2(b-x)}{c}} - 1\right)}{c\left((c+x)e^{\frac{2(b-x)}{c}} + x - c\right)}$$

atque

$$\frac{ds}{dx} + \frac{1}{xx} = \frac{-dt}{ttdx} = \frac{1}{tt}\left(1 + \frac{2t}{x} - \frac{tt}{cc}\right) = \frac{1}{tt}\left(\frac{tt}{cc} - \frac{4e^{\frac{2(b-x)}{c}}}{\left(e^{\frac{2(b-x)}{c}} - 1\right)^2}\right)$$

SCHOLION

359. Quoniam supra [§ 333] invenimus hanc aequationem

$$\left(\frac{ddv}{dy^2}\right) = \left(\frac{ddv}{dx^2}\right) - \frac{i(i+1)}{xx}v$$

integrationem admittere, quippe qui casus oritur ex generali forma (§ 354) sumto $m = 0$, erit problemate huc translato

$$ds - ssdx + \left(f + \frac{i(i+1)}{xx}\right)dx = 0$$

hincque inventa quantitate s huius aequationis

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\left(2f+\frac{i(i+1)}{xx}-2ss\right)z$$

integrale erit

$$z=sv+\left(\frac{dv}{dx}\right)$$

I. Quodsi iam capiamus $f=0$, erit particulariter $s=\frac{i}{x}$ vel $s=\frac{-i-1}{x}$ unde quidem aequationis integrabilis forma non mutatur. At facto $s=\frac{i}{x}+\frac{1}{t}$
oritur

$$dt+\frac{2itdx}{x}+dx=0,$$

cuius integrale est

$$x^{2i}t+\frac{1}{2i+1}x^{2i+1}=\frac{g}{2i+1}$$

ideoque

$$s=\frac{ig+(i+1)x^{2i+1}}{x\left(g-x^{2i+1}\right)}$$

et aequatio integrabilis fit

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{i(i-1)gg+6i(i+1)gx^{2i+1}+(i+1)(i+2)x^{4i+2}}{xx\left(g-x^{2i+1}\right)^2}z$$

II. At non reiecto f sit $s=\frac{i}{x}+u$ fietque

$$-du+\frac{2iudx}{x}+\frac{uudx}{x}=fdx;$$

quae ut in aequationem differentialem secundi gradus facile per seriem resolubilem convertatur, ponatur

$$u=\sqrt{f}-\frac{i}{x}-\frac{dr}{rdx}$$

et prodit

$$\frac{ddr}{dx^2}-\frac{2dr\sqrt{f}}{dx}-\frac{i(i+1)r}{xx}=0.$$

Sit $\sqrt{f}=a$ et statuatur

$$r=Ax^{i+1}+Bx^{i+2}+Cx^{i+3}+Dx^{i+4}+\text{etc.}$$

ac reperitur

$$B=\frac{2(i+1)a}{1(2i+2)}A,\quad C=\frac{2(i+2)a}{2(2i+3)}B,\quad D=\frac{2(i+3)a}{3(2i+4)}C,\quad E=\frac{2(i+4)a}{4(2i+5)}D,\quad \text{etc.},$$

quae abrumpitur, quoties i est numerus integer negativus.

Sin autem statuatur

$$r = Ax^{-i} + Bx^{1-i} + Cx^{2-i} + Dx^{3-i} + \text{etc.},$$

sequens relatio nascitur

$$B = \tfrac{2ia}{2i}A, \quad C = \tfrac{2(i-1)a}{2(2i-1)}B, \quad D = \tfrac{2(i-2)a}{3(2i-2)}C, \quad E = \tfrac{2(i-3)a}{4(2i-3)}D, \quad \text{etc.},$$

quae abrumpitur, quoties i est numerus integer positivus.

PROBLEMA 58

360. *Proposita aequatione*

$$\left(\tfrac{ddv}{dy^2}\right) = \left(\tfrac{ddv}{dx^2}\right) - \tfrac{2aa}{\cos.^2(ax+b)}v,$$

cuius integrale est [§ 357]

$v = a\text{tang.}(ax+b).(\pi:(x+y)+\varphi:(x-y)) + \pi':(x+y)+\varphi':(x-y)$, *per transformationem hic traditam alias invenire aequationes eius ope integrabiles.*

SOLUTIO

Ponamus brevitatis gratia angulum $ax+b=\omega$, ut sit $d\omega = adx$, et ex § 351, cum sit $F=1, \quad G=0, \quad H=\tfrac{-2aa}{\cos.^2\omega}$, quaeratur quantitas s ex hac aequatione

$$ds - ssdx + \left(C + \tfrac{2aa}{\cos.^2\omega}\right)dx = 0$$

eritque huius aequationis

$$\left(\tfrac{ddz}{dy^2}\right) = \left(\tfrac{ddz}{dx^2}\right) - \left(\tfrac{2aa}{\cos.^2\omega} + \tfrac{2ds}{dx}\right)z$$

integrale $z = sv + \left(\tfrac{dv}{dx}\right)$ seu

$$\begin{aligned} z &= as\text{tang.}\omega\cdot(\pi:(x+y)+\varphi:(x-y)) + s(\pi':(x+y)+\varphi':(x-y)) \\ &+ \tfrac{aa}{\cos.^2\omega}(\pi:(x+y)+\varphi:(x-y)) + a\text{tang.}\omega\cdot(\pi':(x+y)+\varphi':(x-y)) + (\pi'':(x+y)+\varphi'':(x-y)). \end{aligned}$$

Totum ergo negotium ad inventionem quantitatis s reducitur, quem in finem ponamus

$$s = \alpha\text{tang.}\omega - \tfrac{du}{udx},$$

fietque

$$\frac{ds}{dx} = \frac{\alpha a}{\cos.^2\omega} - \frac{ddu}{udx^2} + \frac{du^2}{uudx^2}$$

et facta substitutione prodit

$$\frac{\alpha a}{\cos.^2\omega} - \frac{\alpha\alpha \sin.^2\omega}{\cos.^2\omega} + C + \frac{2aa}{\cos.^2\omega} - \frac{ddu}{udx^2} + \frac{2\alpha du}{udx}\operatorname{tang}.\omega = 0.$$

Iam ob

$$-\frac{\alpha\alpha \sin.^2\omega}{\cos.^2\omega} = -\frac{\alpha\alpha}{\cos.^2\omega} + \alpha\alpha$$

sumatur α ita, ut fiat

$$-\alpha\alpha + a\alpha + 2aa = 0.$$

Capiatur ergo $\alpha = -a$, ut sit

$$s = -a\operatorname{tang}.\omega - \frac{du}{udx},$$

et pro quantitate u invenienda haec habetur aequatio

$$\frac{ddu}{udx^2} + \frac{2adu}{udx}\operatorname{tang}.\omega + naau = 0$$

posito $C = -aa - naa$ seu

$$\frac{ddu}{d\omega^2} + \frac{2du}{d\omega}\operatorname{tang}.\omega + nu = 0$$

ob $dx = \frac{d\omega}{a}$; cuius resolutio non parum ardua videtur, inter complures autem modos eam tractandi hic ad institutum maxime idoneus videtur.

Fingatur

$$u = A\cos.\lambda\omega + B\cos.(\lambda+2)\omega + C\cos.(\lambda+4)\omega + \text{etc.}$$

eritque

$$\frac{du}{d\omega} = -\lambda A\sin.\lambda\omega - (\lambda+2)B\sin.(\lambda+2)\omega - (\lambda+4)C\sin.(\lambda+4)\omega - \text{etc.}$$

$$\frac{ddu}{d\omega^2} = -\lambda\lambda A\cos.\lambda\omega - (\lambda+2)^2 B\cos.(\lambda+2)\omega - (\lambda+4)^2 C\cos.(\lambda+4)\omega - \text{etc.}$$

et aequatio hac forma repraesentata

$$\frac{ddu}{d\omega^2}\cos.\omega + \frac{4du}{d\omega}\sin.\omega + 2nu\cos.\omega = 0$$

dabit

$$0=-\lambda\lambda A\cos.(\lambda-1)\omega-(\lambda+2)^2 B\cos.(\lambda+1)\omega-(\lambda+4)^2 C\cos.(\lambda+3)\omega-\text{etc.}$$
$$\begin{array}{lll} & -\lambda\lambda A & -(\lambda+2)^2 B \\ -2\lambda A & -2(\lambda+2)B & -2(\lambda+4)C \\ & +2\lambda A & +2(\lambda+2)B \\ +nA & +nB & +nC \\ & +nA & +nB \end{array}$$

unde λ ita capi oportet, ut sit

$$\lambda\lambda+2\lambda=n \quad \text{seu} \quad \lambda=-1\pm\sqrt{(n+1)}$$

duplexque pro λ habeatur valor. Praeterea vero secundus terminus ob $n=\lambda\lambda+2\lambda$ praebet $B=\frac{\lambda}{\lambda+2}A$, tertius vero commode dat $C=0$, unde et sequentes omnes evanescunt.

Sumamus $n=mm-1$, ut sit

$$\lambda=-1\pm m \text{ et } B=\frac{-1\pm m}{1\pm m}A,$$

atque integrale completum concludi videtur

$$u=A\left(\cos.(m-1)\omega+\tfrac{m-1}{m+1}\cos.(m+1)\omega\right)+$$
$$\mathfrak{A}\left(\cos.(m+1)\omega+\tfrac{m+1}{m-1}\cos.(m-1)\omega\right).$$

Sit

$$A=(m+1)B \quad \text{et} \quad \mathfrak{A}=(m-1)\mathfrak{B};$$

fiet

$$u=(m+1)(B+\mathfrak{B})\cos.(m-1)\omega+(m-1)(B+\mathfrak{B})\cos.(m+1)\omega;$$

ubi cum binae constantes in unam coalescant, hoc integrale tantum est particulare, ex quo autem deinceps completum elici poterit [§ 361, §§ 362]. Cum ergo sit

$$\frac{du}{ud\omega}=\frac{-(mm-1)\sin.(m-1)\omega-(mm-1)\sin.(m+1)\omega}{(m+1)\cos.(m-1)\omega+(m-1)\cos.(m+1)\omega}$$

est

$$\frac{s}{a}=-\text{tang.}\omega+\frac{(mm-1)\left(\sin.(m-1)\omega+\sin.(m+1)\omega\right)}{(m+1)\cos.(m-1)\omega+(m-1)\cos.(m+1)\omega}$$

pro aequatione

$$\frac{ds}{ad\omega}-\frac{ss}{aa}-mm+\frac{2}{\cos.^2\omega}=0$$

ob $C=-(n+1)aa=-mmaa$.

Illud autem integrale inventum ad hanc formam reducitur

$$\frac{s}{a}=-\text{tang.}w+\frac{(mm-1)\text{tang.}m\omega}{m+\text{tang.}m\omega\ \text{tang.}\omega},$$

quae expressio substituta illi aequationi egregie satisfacere deprehenditur. Scribamus eius loco Θ ac ponamus $\frac{s}{a}=\Theta+\frac{1}{t}$ pro integrali completo eliciendo prodibitque

$$-\frac{dt}{ttd\omega}-\frac{2\Theta}{t}-\frac{1}{tt}=0 \ \text{ seu } \ dt+2\Theta td\omega+d\omega=0.$$

Erat autem modo ante

$$\Theta=\frac{s}{a}=-\text{tang.}\omega-\frac{du}{ud\omega},$$

unde

$$\int\Theta d\omega=l\cos.\omega-lu \quad \text{et} \quad e^{2\int\Theta d\omega}=\frac{\cos.^2\omega}{uu}$$

qui est multiplicator pro illa aequatione, sicque fit

$$\frac{t\cos.^2\omega}{uu}=C-\int\frac{d\omega\cos.^2\omega}{uu}$$

At est

$$u=2m\cos.m\omega\cos.\omega+2\ \sin.m\omega\ \sin.\omega$$

ideoque

$$\frac{t}{(m\cos.m\omega+\sin.m\omega\text{tang.}\omega)^2}=A-\int\frac{d\omega}{(m\cos.m\omega+\sin.m\omega\text{tang.}\omega)^2}$$

cuius postremi membri integrale deprehenditur

$$\frac{-\ m\text{tang.}m\omega+\text{tang.}\omega}{m(mm-1)(m+\text{tang.}m\omega\,\text{tang.}\omega)}=\frac{-m\sin.m\omega+\text{tang.}\omega\cos.m\omega}{m(mm-1)(m\cos.m\omega+\sin.m\omega\,\text{tang.}\omega)},$$

ita ut sit

$$\frac{t}{(m\cos.m\omega+\sin.m\omega\text{tang.}\omega)^2}=A+\frac{\cos.m\omega\,\text{tang.}\omega-m\sin.m\omega}{m(mm-1)(m\cos.m\omega+\sin.m\omega\,\text{tang.}\omega)}$$

seu

$$\frac{1}{t}=\frac{m(mm-1)}{\left(C(m\cos.m\omega+\sin.m\omega\,\text{tang}.\omega)+\cos.m\omega\,\text{tang}.\omega-m\sin.m\omega\right)(m\cos.m\omega+\sin.m\omega\,\text{tang}.\omega)},$$

cui addatur

$$\Theta=\frac{s}{a}=-\text{tang}.\omega+\frac{m(mm-1)\sin.m\omega}{m\cos.m\omega+\sin.m\omega\,\text{tang}.\omega},$$

ut prodeat $\frac{s}{a}$, eritque

$$\frac{s}{a}=-\text{tang}.\omega+\frac{m(mm-1)(C\sin.m\omega+\cos.m\omega)}{C(m\cos.m\omega+\sin.m\omega\,\text{tang}.\omega)+\cos.m\omega\,\text{tang}.\omega-m\sin.m\omega}$$

seu

$$\frac{s}{a}=\frac{(mm-1-\text{tang}^2.\omega)(C\sin.m\omega+\cos.m\omega)-m\,\text{tang}.\omega(C\cos.m\omega-\sin.m\omega)}{C(m\cos.m\omega+\sin.m\omega\,\text{tang}.\omega)+\cos.m\omega\,\text{tang}.\omega-m\sin.m\omega}.$$

COROLLARIUM 1

361. Hic praecipue notandum est huius aequationis

$$\frac{ddu}{d\omega^2}+\frac{2du}{d\omega}\text{tang}.\omega+(mm-1)u=0$$

integrale particulare esse

$$u=m\cos.m\omega\cos.\omega+\sin.m\omega\sin.\omega\ ;$$

aliud vero integrale particulare reperitur simili modo

$$u=m\sin.m\omega\cos.\omega-\cos.m\omega\sin.\omega,$$

unde concluditur completum

$$u=A(m\cos.m\omega\cos.\omega+\sin.m\omega\sin.\omega)+B(m\sin.m\omega\cos.\omega-\cos.m\omega\sin.\omega).$$

COROLLARIUM 2

362. Si hic ponatur

$$A=C\cos.\alpha \quad \text{et} \quad B=-C\sin.\alpha,$$

hoc integrale completum ad hanc formam redigitur

$$u=C\left(m\cos.(m\omega+\alpha)\cos.\omega+\sin.(m\omega+\alpha)\sin.\omega\right),$$

quod quidem ex integrali particulari primum invento statim concludi potuisset, cum ibi loco anguli $m\omega$ scribere liceat $m\omega+\alpha$.

COROLLARIUM 3

363. Hinc multo facilius reperitur valor

$$\frac{s}{a}=-\text{tang.}\omega-\frac{du}{ud\omega};$$

cum enim sit

$$\frac{du}{d\omega}=-C(mm-1)\sin.(m\omega+\alpha)\cos.\omega,$$

erit

$$\frac{s}{a}=-\text{tang.}\omega+\frac{(mm-1)\sin.(m\omega+\alpha)\cos.\omega}{m\cos.(m\omega+\alpha)\cos.\omega+\sin.(m\omega+\alpha)\sin.\omega}$$

hincque

$$\frac{ds}{ad\omega}=\frac{ds}{aadx}=\frac{-1}{\cos.^2\omega}+\frac{(mm-1)\left(m^2\cos.^2\omega-\sin.^2(m\omega+\alpha)\right)}{\left(m\cos.(m\omega+\alpha)\cos.\omega+\sin.(m\omega+\alpha)\sin.\omega\right)^2}$$

et aequatio, cuius integrationem invenimus, erit [§ 360]

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)-\frac{2(mm-1)aa\left(m^2\cos.^2\omega-\sin.^2(m\omega+\alpha)\right)}{\left(m\cos.(m\omega+\alpha)\cos.\omega+\sin.(m\omega+\alpha)\sin.\omega\right)^2}z$$

eiusque integrale colligitur

$$z=\frac{maa\left(m\sin.(m\omega+\alpha)\sin.\omega+\cos.(m\omega+\alpha)\cos.\omega\right)}{m\cos.(m\omega+\alpha)\cos.\omega+\sin.(m\omega+\alpha)\sin.\omega}\left(\pi:(x+y)+\varphi:(x-y)\right)$$
$$+\frac{(mm-1)a\sin.(m\omega+\alpha)\cos.\omega}{m\cos.(m\omega+\alpha)\cos.\omega+\sin.(m\omega+\alpha)\sin.\omega}\left(\pi':(x+y)+\varphi':(x-y)\right)+\left(\pi'':(x+y)+\varphi'':(x-y)\right)$$

existente $\omega=ax+b$.

SCHOLlON 1

364. Omnino memoratu digna est integratio huius aequationis

$$\frac{ddu}{d\omega^2}+\frac{2du}{d\omega}\text{tang.}\omega+(mm-1)u=0,$$

unde occasionem carpo hanc aequationem generaliorem tractandi

$$\frac{ddu}{d\omega^2}+\frac{2fdu}{d\omega}\operatorname{tang}.\omega+gu=0,$$

quam primum observo posito

$$\frac{du}{u}=-(2f+1)d\omega\operatorname{tang}.\omega+\frac{dv}{v}$$

ut sit

$$u=\cos.^{2f+1}\omega v,$$

abire in hanc formam

$$\frac{ddv}{d\omega^2}-\frac{2(f+1)dv}{d\omega}\operatorname{tang}.\omega+(g-2f-1)v=0,$$

ita ut, si illa integrabilis existat casu $f=n$, integrabilis quoque sit casu $f=-n-1$.

Iam pro illa aequatione ponatur

$$u=A\sin.\lambda\omega+B\sin.(\lambda+2)\omega+C\sin.(\lambda+4)\omega+D\sin.(\lambda+6)\omega+\text{etc.}$$

et facta substitutione in aequatione

$$\frac{2ddu}{d\omega^2}\cos.\omega+\frac{4fdu}{d\omega}\sin.\omega+2gu\cos.\omega=0$$

reperitur

$$\begin{array}{lllll}
0=-\lambda\lambda A\sin.(\lambda-1)\omega & -(\lambda+2)^2B\sin.(\lambda+1)\omega & -(\lambda+4)^2C\sin.(\lambda+3)\omega & -(\lambda+6)^2D\sin.(\lambda+5)\omega & \text{etc.}\\
 & -\lambda\lambda A & -(\lambda+2)^2B & -(\lambda+4)^2C & \\
 & +2\lambda Af & +(\lambda+2)Bf & +2(\lambda+4)Cf & \\
-2\lambda Af & -2(\lambda+2)Bf & -2(\lambda+4)Cf & -2(\lambda+6)Df & \\
 & +Ag & +Bg & +Cg & \\
+Ag & +Bg & +Cg & +Dg &
\end{array}$$

Oportet ergo sit $g=\lambda\lambda+2\lambda f$; tum vero coefficientes assumti ita determinantur

$$B=\frac{\lambda f}{\lambda+f+1}A,\ C=\frac{(\lambda+1)(f-1)}{2(\lambda+f+2)}B,\ D=\frac{(\lambda+2)(f-2)}{2(\lambda+f+3)}C\quad\text{etc.}$$

Statuamus ergo $g=mm-ff$, ut fiat $\lambda=m-f$ et aequationes nostrae sint

$$\frac{ddu}{d\omega^2}+\frac{2fdu}{d\omega}\operatorname{tang}.\omega+(mm-ff)u=0$$

et

$$\frac{ddv}{d\omega^2}-\frac{2(f+1)dv}{d\omega}\text{tang.}\omega+\left(mm-f(f+1)^2\right)v=0$$

existente

$$u=v\cos.^{2f+1}\omega \quad\text{seu}\quad v=\frac{u}{\cos.^{2f+1}\omega}$$

Quoniam nunc series nostra abrumpitur, quoties est *f* numerus integer, percurramus casus simpliciores.

I. Sit $f=0$; erit $\lambda=m$ et

$$B=0,\quad C=0\ \text{etc.}$$

ideoque

$$u=A\sin.m\omega \quad\text{et}\quad v=\frac{A\sin.m\omega}{\cos.\omega}.$$

II. Sit $f=1$; erit $\lambda=m-1$ et

$$B=\frac{m-1}{m+1}A,\ C=0\quad\text{etc.},$$

ergo

$$\frac{u}{a}=(m+1)\sin.(m-1)\omega+(m-1)\sin.(m+1)\omega\ \ \text{et}\ \ v=\frac{u}{\cos.^2\omega}\ \text{seu}$$

$$\frac{u}{2a}=m\sin.m\omega\cos.\omega-\cos.m\omega\sin.\omega.$$

III. Sit $f=2$; erit $\lambda=m-2$ et

$$B=\frac{2(m-2)}{m+1}A,\ C=\frac{m-1}{2(m+2)}B=\frac{(m-1)(m-2)}{(m+1)(m+2)}A,\quad D=0\quad\text{etc.},$$

hinc

$$\frac{u}{a}=(m+1)(m+2)\sin.(m-2)\omega+2(m-2)(m+2)\sin.m\omega+(m-1)(m-2)\sin.(m+2)\omega$$

indeque $v=\frac{u}{\cos.^5\omega}$ seu

$$\frac{u}{2a}=(mm+2)\sin.m\omega\cos.2m\omega+(mm-4)\sin.m\omega-3m\cos.m\omega\sin.2\omega.$$

IV. Sit $f=3$; erit $\lambda=m-3$ et

$$B=\frac{3(m-3)}{m+1}A,\ C=\frac{2(m-2)}{2(m+2)}B\quad\text{et}\quad D=\frac{m-1}{3(m+3)}C,\quad E=0\quad\text{etc.},$$

ergo

$$\tfrac{u}{a}=(m+1)(m+2)(m+3)\sin.(m-3)\omega+3(m+2)(mm-9)\sin.(m-1)\omega$$
$$+(m-1)(m-2)(m-3)\sin.(m+3)\omega+3(m-2)(mm-9)\sin.(m+1)\omega$$

existente $v=\frac{u}{\cos.^7\omega}$.

V. Sit $f=4$; erit $\lambda=m-4$ ac reperitur

$$\tfrac{u}{a}=(m+1)(m+2)(m+3)(m+4)\sin.(m-4)\omega+4(m+2)(m+3)(mm-16)\sin.(m-2)\omega$$
$$+(m-1)(m-2)(m-3)(m-4)\sin.(m+4)\omega+4(m-2)(m-3)(mm-16)\sin.(m+2)\omega$$
$$+6(mm-9)(mm-16)\sin.m\omega$$

existente $v=\frac{u}{\cos.^9\omega}$, unde ratio progressionis per se est manifesta.

Notari autem convenit, si posuissemus

$$u=A\cos.\lambda\omega+B\cos.(\lambda+2)\omega+C\cos.(\lambda+4)\omega+\text{ etc.},$$

easdem coefficientium determinationes prodituras fuisse, ex qua hi duo valores coniuncti integrale completum exhibebunt; quod etiam ex forma inventa colligitur, si modo loco anguli $m\omega$ generalius scribatur $m\omega+\alpha$.

SCHOLION 2

365. Pluribus autem aliis modis eadem aequatio

$$\tfrac{ddu}{d\omega^2}+\tfrac{2fdu}{d\omega}\text{tang.}\omega+gu=0$$

tractari et eius integrale per series exprimi potest, unde alii casus integrabilitatis obtinentur.

Ad hoc primum notetur posito $u=\sin.^{\lambda}\omega$, fore

$$\tfrac{du}{d\omega}=\lambda\sin.^{\lambda-1}\omega\cos.\omega \text{ hincque } \tfrac{du}{d\omega}\text{tang.}\omega=\lambda\sin.^{\lambda}\omega$$

et

$$\tfrac{ddu}{d\omega^2}=\lambda(\lambda-1)\sin.^{\lambda-2}\omega\cos.^2\omega-\lambda\sin.^{\lambda}\omega=\lambda(\lambda-1)\sin.^{\lambda-2}\omega-\lambda\lambda\sin.^{\lambda}\omega.$$

Hinc, si ponamus

$$u=A\sin.^{\lambda}\omega+B\sin.^{\lambda+2}\omega+C\sin.^{\lambda+4}\omega+D\sin.^{\lambda+6}\omega+\text{etc.},$$

facta substitutione adipiscimur

$$\begin{aligned}0 = \lambda(\lambda-1)A\sin.^{\lambda-2}\omega &+ (\lambda+2)(\lambda+1)B\sin.^{\lambda}\omega + (\lambda+4)(\lambda+3)C\sin.^{\lambda+2}\omega + \text{etc.}\\ &- \lambda\lambda A \qquad - (\lambda+2)^2 B\\ &+ 2\lambda fA \qquad + 2(\lambda+2)fB\\ &+ gA \qquad + gB\end{aligned}$$

unde sumi oportet vel $\lambda = 0$ vel $\lambda = 1$; tum vero erit

$$B = \frac{\lambda\lambda - 2\lambda f - g}{(\lambda+1)(\lambda+2)}A, \quad C = \frac{(\lambda+2)^2 - 2(\lambda+2)f - g}{(\lambda+3)(\lambda+4)}B, \quad \text{etc.}$$

Hinc duo casus evolvi convenit :

$\lambda = 0$	$\lambda = 1$
$B = \frac{-g}{1\cdot 2}A$	$B = \frac{1-2f-g}{2\cdot 3}A$
$C = \frac{4-4f-g}{3\cdot 4}B$	$C = \frac{9-6f-g}{4\cdot 5}B$
$D = \frac{16-8f-g}{5\cdot 6}C$	$D = \frac{25-10f-g}{6\cdot 7}C$
$E = \frac{36-12f-g}{7\cdot 8}D$	$E = \frac{49-14f-g}{8\cdot 9}D$
etc.	etc.

Integratio ergo succedit, quoties fuerit $g = ii - 2if$ denotante i numerum integrum positivum. Quare cum posito $u = v\cos.^{2f+1}\omega$ aequatio transformata sit

$$\frac{ddv}{d\omega^2} - \frac{2(f+1)dv}{d\omega}\text{tang.}\omega + (g - 2f - 1)v = 0,$$

haec ideoque et illa erit integrabilis, quoties fuerit

$$g = (i+1)^2 + 2(i+1)f,$$

quos binos casus ita uno complecti licet, ut integratio succedat, dum sit

$$g = ii \pm 2if$$

SCHOLION 3

366. Eidem aequationi adhuc inhaerens, cum posito $u=\cos.^{\lambda}\omega$ sit

$$\frac{du}{d\omega}=-\lambda\cos.^{\lambda-1}\omega\sin.\omega$$

ideoque

$$\frac{du}{d\omega}\text{tang.}\omega=-\lambda\cos.^{\lambda-2}\omega\sin.\omega+\lambda\cos.^{\lambda}\omega$$

et

$$\frac{ddu}{d\omega^2}=\lambda(\lambda-1)\cos.^{\lambda-2}\omega-\lambda\lambda\cos.^{\lambda}\omega,.$$

statuo

$$u=A\cos.^{\lambda}\omega+B\cos.^{\lambda+2}\omega+C\cos.^{\lambda+4}\omega+D\cos.^{\lambda+6}\omega+\text{ etc.}$$

et facta substitutione orietur

$$\begin{array}{llll}
0=\lambda(\lambda-1)A\cos.^{\lambda-2}\omega & +(\lambda+2)(\lambda+1)B\cos.^{\lambda}\omega & +(\lambda+4)(\lambda+3)C\cos.^{\lambda+2}\omega & +\text{etc.}\\
 & -\lambda\lambda A & -(\lambda+2)^2B & \\
-2\lambda fA & -2(\lambda+2)fB & -2(\lambda+4)fC & \\
 & +2\lambda fA & +2(\lambda+2)fB & \\
 & +gA & +gB &
\end{array}$$

Oportet ergo vel $\lambda=0$ vel $\lambda=2f+1$; tum vero

$$B=\frac{\lambda\lambda-2\lambda f-g}{(\lambda+2)(\lambda+1-2f)}A,\quad C=\frac{(\lambda+2)^2-2(\lambda+2)f-g}{(\lambda+4)(\lambda+3-2f)}B,\quad \text{etc.}$$

et ambo casus ita se habebunt:

$\lambda=0$	$\lambda=2f+1$
$B=\frac{-g}{2(1-2f)}A$	$B=\frac{1+2f-g}{2(2f+3)}A$
$C=\frac{4-4f-g}{4(3-2f)}B$	$C=\frac{9+6f-g}{4(2f+5)}B$
$D=\frac{16-8f-g}{6(5-2f)}C$	$D=\frac{25+10f-g}{6(2f+7)}C$
etc.	etc.

Ex priori integratio succedit, si

$$g = 4ii - 4if \ ,$$

ex posteriori, si

$$g = (2i+1)^2 + 2(2i+1)f \ ,$$

qui casus cum iis, qui ex transformata nascuntur, iuncti eodem redeunt ac in paragrapho praecedente inventi.

Omnes ergo hactenus inventi integrabilitatis casus huc revocantur, ut posito $g = mm - ff$ sit vel $f = \pm i$ vel $m = i \pm f$, hoc est vel $f = \pm i$ vel $f = \pm i \pm m$. Ceterum hi posteriores casus etiam ex prima resolutione (§ 364) sequuntur, ubi series quoque abrumpitur, si

$$\lambda = -i \text{ ideoque } g = mm - ff = ii - 2if \text{, ergo } i - f = \pm m$$

et transformatione in subsidium vocata $f = \pm i \pm m$. Contra vero casus primo inventi in resolutionibus posterioribus non occurrunt.

PROBLEMA 59

367. *Concessa huius aequationis integratione*

$$\left(\frac{ddv}{dy^2}\right) = F\left(\frac{ddv}{dx^2}\right) + G\left(\frac{dv}{dx}\right) + Hv$$

invenire aequationem huius formae

$$\left(\frac{ddz}{dy^2}\right) = P\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dx}\right) + Rz$$

pro qua sit

$$z = \left(\frac{ddv}{dx^2}\right) + r\left(\frac{dv}{dx}\right) + sv \ ,$$

ubi F, G, H, P, Q, R et r, s sunt functiones ipsius x tantum.

SOLUTIO

Cum sit

$$\left(\frac{ddz}{dy^2}\right) = \left(\frac{d^4v}{dx^2dy^2}\right) + r\left(\frac{d^3v}{dxdy^2}\right) + s\left(\frac{ddv}{dy^2}\right)$$

ob

$$\left(\frac{ddv}{dy^2}\right)=F\left(\frac{ddv}{dx^2}\right)+G\left(\frac{dv}{dx}\right)+Hv$$

erit

$$\left(\frac{d^3v}{dxdy^2}\right)=F\left(\frac{d^3v}{dx^3}\right)+\frac{dF}{dx}\left(\frac{ddv}{dx^2}\right)+\frac{dG}{dx}\left(\frac{dv}{dx}\right)+\frac{dH}{dx}v$$
$$+\;G \qquad\qquad +\;H$$

et

$$\left(\frac{d^4v}{dx^2dy^2}\right)=F\left(\frac{d^4v}{dx^4}\right)+\frac{2dF}{dx}\left(\frac{d^3v}{dx^3}\right)+\frac{ddF}{dx^2}\left(\frac{ddv}{dx^2}\right)+\frac{ddG}{dx^2}\left(\frac{dv}{dx}\right)+\frac{ddH}{dx^2}v$$
$$+\;G \qquad\qquad +\frac{dG}{dx} \qquad\qquad +\;\frac{2dH}{dx}$$
$$+\;H$$

Deinde vero ob

$$z=\left(\frac{ddv}{dx^2}\right)+r\left(\frac{dv}{dx}\right)+sv$$

fit

$$\left(\frac{dz}{dx}\right)=\left(\frac{d^3v}{dx^3}\right)+r\left(\frac{ddv}{dx^2}\right)+\frac{dr}{dx}\left(\frac{dv}{dx}\right)+\frac{ds}{dx}v$$
$$+\;s$$

et

$$\left(\frac{ddz}{dx^2}\right)=\left(\frac{d^4v}{dx^4}\right)+r\left(\frac{d^3v}{dx^3}\right)+\frac{2dr}{dx}\left(\frac{ddv}{dx^2}\right)+\frac{ddr}{dx^2}\left(\frac{dv}{dx}\right)+\frac{dds}{dx^2}v$$
$$+\;s \qquad\qquad +\frac{2ds}{dx}$$

His iam substitutis necesse est, ut omnes termini affecti per $\left(\frac{d^4v}{dx^4}\right),\left(\frac{d^3v}{dx^3}\right),\left(\frac{ddv}{dx^2}\right),\left(\frac{dv}{dx}\right)$ et v seorsim evanescant, unde sequentes resultant aequationes:

ex	
$\left(\frac{d^4v}{dx^4}\right)$	I. $F=P$,
$\left(\frac{d^3v}{dx^3}\right)$	II. $G+\frac{2dF}{dx}+Fr=Pr+Q$,
$\left(\frac{ddv}{dx^2}\right)$	III. $H+\frac{2dG}{dx}+\frac{ddF}{dx^2}+Gr+\frac{rdF}{dx}+Fs=P\left(s+\frac{2dr}{dx}\right)+Qr+R$,
$\left(\frac{dv}{dx}\right)$	IV. $\frac{2dH}{dx}+\frac{ddG}{dx^2}+Hr+\frac{rdG}{dx}+Gs=P\left(\frac{2ds}{dx}+\frac{ddr}{dx^2}\right)+Q\left(s+\frac{dr}{dx}\right)+Rr$,
v	V. $\frac{ddH}{dx^2}+\frac{rdH}{dx}+Hs=P\frac{dds}{dx^2}+Q\frac{ds}{dx}+Rs$.

Ex prima fit $F = P$, ex secunda $Q = G + \frac{2dF}{dx}$, ex tertia

$$R = H + \frac{2dG}{dx} + \frac{ddF}{dx^2} - \frac{rdF + 2Fdr}{dx}$$

qui valores in binis ultimis substituti praebent

$$\frac{2dH}{dx} + \frac{ddG}{dx^2} - \frac{rdG + Gdr}{dx} - \frac{rddF}{dx^2} - \frac{2dFdr}{dx^2} - \frac{2sdF + 2Fds}{dx} + \frac{rrdF + 2Frdr}{dx} - \frac{Fddr}{dx^2} = 0$$

et

$$\frac{ddH}{dx^2} + \frac{rdH}{dx} - \frac{sddF + 2dFds + Fdds}{dx^2} - \frac{2sdG + Gds}{dx} + \frac{s(rdF + 2Fdr)}{dx} = 0$$

quarum illa sponte est integrabilis praebens

$$2H + \frac{dG}{dx} - Gr - \frac{rdF + 2Fdr}{dx} - 2Fs + Frr = A,$$

deinde binis illis aequationibus ita repraesentatis

$$\frac{dd.Fr}{dx^2} - \frac{2d.Fs}{dx} + \frac{d.Frr}{dx} + \frac{ddG}{dx^2} - \frac{d.Gr}{dx} + \frac{2dH}{dx} = 0$$

$$-\frac{dd.Fs}{dx^2} + \frac{s}{r}\frac{d.Frr}{dx} - \frac{2sdG + Gds}{dx} + \frac{rdH}{dx} + \frac{ddH}{dx^2} = 0$$

vel adeo hoc modo

$$\frac{dd.(G - Fr)}{dx} - d.r(G - Fr) + 2d.(H - Fs) = 0,$$

$$\frac{dd.(H - Fs)}{dx} + 2Fsdr + rsdF - Gds - 2sdG + rdH = 0$$

ultima vero ita repraesentari potest

$$\frac{dd.(H - Fs)}{dx} - 2sd.(G - Fr) - ds(G - Fr) + rd.(H - Fs) = 0.$$

Quodsi iam prior per $H - Fs$, haec vero per $-(G - Fr)$ multiplicetur,
summa fit

$$\frac{(H - Fs)dd.(G - Fr) - (G - Fr)dd.(H - Fs)}{dx} - (G - Fr)(H - Fs)dr$$
$$+2(H - Fs)d.(H - Fs) - r(H - Fs)d.(G - Fr)$$
$$+2s(G - Fr)d.(G - Fr) + (G - Fr)^2 ds - r(G - Fr)d.(H - Fs) = 0,$$

cuius integrale manifesto est

$$\frac{(H-Fs)d(G-Fr)-(G-Fr)d(H-Fs)}{dx}+(H-Fs)^2+(G-Fr)^2 s-(G-Fr)(H-Fs)r=B.$$

Integrale autem prius inventum est

$$\frac{d.(G-Fr)}{dx}-(G-Fr)r+2(H-Fs)=A,$$

quae per $H-Fs$ multiplicata et ab illa subtracta relinquit

$$-\frac{(G-Fr)d.(H-Fs)}{dx}-(H-Fs)^2+(G-Fr)^2 s=B-A(H-Fs),$$

sicque habentur duae aequationes simpliciter differentiales, ex quibus binas quantitates r et s definiri oportet, quibus cognitis etiam functiones P, Q et R innotescunt.

COROLLARlUM 1

368. Si sit $F=1$, $G=0$ et $H=0$, aequationes inventae erunt

$$-\frac{dr}{dx}+rr-2s=a \quad \text{et} \quad \frac{sdr-rds}{dx}+ss=b,$$

unde dx eliminando fit

$$\frac{rds-sdr}{dr}=\frac{b-ss}{a+2s-rr} \quad \text{seu} \quad \frac{rds}{dr}=\frac{b+as+ss-rrs}{a+2s-rr},$$

cuius resolutio in genere vix suscipienda videtur. Sumtis autem constantibus $a=0$ et $b=0$ aequatio $\frac{rds}{dr}=\frac{ss-rrs}{2s-rr}$ positio $s=rrt$ transit in

$$\frac{rdt+2tdr}{dr}=\frac{tt-t}{2t-1} \quad \text{seu} \quad \frac{rdt}{dr}=\frac{-3tt+t}{2t-1},$$

unde fit

$$\frac{dr}{r}=\frac{dt(1-2t)}{t(3t-1)}=\frac{-dt}{t}+\frac{dt}{3t-1} \quad \text{et} \quad r=\frac{\alpha\sqrt[3]{(3t-1)}}{t},$$

hinc

$$s=\frac{\alpha\alpha\sqrt[3]{(3t-1)2}}{t}.$$

COROLLARlUM 2

369. Pro eodem casu singulari ponamus $3t-1=u^3$, ut fiat

$$r=\frac{3au}{1+u^3} \quad \text{et} \quad s=\frac{3\alpha\alpha uu}{1+u^3}.$$

Iam ob $a=0$ est

$$dx=\frac{dr}{rr-2s}=\frac{dr}{rr(1-2t)}=\frac{3dr}{rr(1-2u^3)},$$

at

$$\frac{dr}{rr}=\frac{du}{3\alpha uu}-\frac{2udu}{3\alpha}=\frac{du(1-2u^3)}{3\alpha uu}$$

ita ut sit $dx=\frac{du}{\alpha uu}$ hincque

$$\frac{1}{u}=\beta-\alpha x \quad \text{et} \quad u=\frac{1}{\beta-\alpha x},$$

qui quidem salva generalitate sumi potest $\beta=0$ et $u=\frac{-1}{\alpha x}$, unde fit

$$r=\frac{-3xx}{x^3+c^3}$$

facto $\alpha=-\frac{1}{c}$ et

$$s=\frac{3x}{x^3+c^3}.$$

Tandem ergo colligitur

$$P=1,\ Q=0 \quad \text{et} \quad R=-\frac{2dr}{dx}=\frac{6x(2c^3-x^3)}{(c^3+x^3)^2}$$

COROLLARIUM 3

370. Proposita ergo aequatione $\left(\frac{ddv}{dy^2}\right)=\left(\frac{ddv}{dx^2}\right)$,cuius integrale est

$$v=\Gamma:(x+y)+\Delta:(x-y),$$

huius aequationis integrale assignari poterit

$$\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right)+\frac{6x(2c^3-x^3)}{(c^3+x^3)^2}z;$$

est enim

$$z=\left(\frac{ddv}{dx^2}\right)-\frac{3xx}{c^3+x^3}\left(\frac{dv}{dx}\right)+\frac{3x}{c^3+x^3}v.$$

SCHOLION 1

371. Haec pro casu $F=1$, $G=0$ et $H=0$ multo facilius atque generalius computari possunt pro quocunque valore quantitatis *a*, dum sit *b* = 0; tum enim altera aequatio statim dat

$$dx=\frac{rds-sdr}{ss}$$

hincque

$$x=\frac{-r}{s} \quad \text{et} \quad s=\frac{-r}{x},$$

ex quo prima aequatio hane induit formam

$$\frac{dr}{dx}-rr-\frac{2r}{x}+a=0.$$

Ponamus $r=\frac{a}{t}$; fiet

$$dt+\frac{2tdx}{x}-ttdx+adx=0,$$

cui particulariter satisfacit

$$t=\sqrt{a}+\frac{1}{x}.$$

Statuatur ergo

$$t=\sqrt{a}+\frac{1}{x}+\frac{1}{u}$$

ac prodit

$$du+dx+2udx\sqrt{a}=0,$$

quae per $e^{2x\sqrt{a}}$ multiplicata et integrata praebet

$$e^{2x\sqrt{a}}u+\frac{1}{2\sqrt{a}}e^{2x\sqrt{a}}=\frac{n}{2\sqrt{a}}$$

ideoque

$$\frac{1}{u}=\frac{2e^{2x\sqrt{a}}\sqrt{a}}{n-e^{2x\sqrt{a}}}=\frac{2\sqrt{a}}{ne^{-2x\sqrt{a}}-1}$$

et

$$t=\frac{1}{x}+\frac{ne^{-2x\sqrt{a}}+1}{ne^{-2x\sqrt{a}}-1}\sqrt{a}=\frac{1}{x}+\frac{n+e^{2x\sqrt{a}}}{n-e^{2x\sqrt{a}}}\sqrt{a}$$

et

$$r=\frac{ax\left(n-e^{2x\sqrt{a}}\right)}{n\left(x\sqrt{a}+1\right)+e^{2x\sqrt{a}}\left(x\sqrt{a}-1\right)}$$

ae propterea

$$s=\frac{-a\left(n-e^{2x\sqrt{a}}\right)}{n\left(x\sqrt{a}+1\right)+e^{2x\sqrt{a}}\left(x\sqrt{a}-1\right)},$$

tum vero postremo

$$P=1,\quad Q=0\quad \text{et}\quad R=-\tfrac{2dr}{dx}=-2rr-\tfrac{4r}{x}+2a$$

seu

$$R=\frac{-2a\left(nn-4naxxe^{2x\sqrt{a}}-2ne^{2x\sqrt{a}}+e^{4x\sqrt{a}}\right)}{\left(n\left(x\sqrt{a}+1\right)+e^{2x\sqrt{a}}\left(x\sqrt{a}-1\right)\right)^2}=\frac{-2a\left(n-e^{2x\sqrt{a}}\right)^2+8naaxxe^{2x\sqrt{a}}}{\left(n\left(x\sqrt{a}+1\right)+e^{2x\sqrt{a}}\left(x\sqrt{a}-1\right)\right)^2}.$$

Si iam sumatur a evanescens et $n=1+\tfrac{2}{3}ac^3\sqrt{a}$, formulae ante [§ 369] inventae resultant. At si a sit quantitas negativa, puta $a=-m^2$, capiaturque

$$n=\frac{\alpha\sqrt{-1}+\beta}{\alpha\sqrt{-1}-\beta},\quad \text{reperitur}$$

$$r=\frac{-mmx(\beta\cos.mx+\alpha\sin.mx)}{\beta\cos.mx+\alpha\sin.mx-mx(\alpha\cos.mx-\beta\sin.mx)}=\frac{-mmx\cos.(mx+\gamma)}{\cos.(mx+\gamma)+mx\sin.(mx+\gamma)}$$

et

$$s=\frac{mmx\cos.(mx+\gamma)}{\cos.(mx+\gamma)+mx\sin.(mx+\gamma)}$$

indeque

$$R=\frac{2mm\left(\cos.(mx+\gamma)-mmxx\right)}{\left(\cos.(mx+\gamma)+mx\sin.(mx+\gamma)\right)^2}$$

Quantitas R reducitur ad hanc

$$R=\frac{8naaxx-2a\left(ne^{-x\sqrt{a}}-e^{x\sqrt{a}}\right)^2}{\left(n\left(1+x\sqrt{a}\right)e^{-x\sqrt{a}}-\left(1-x\sqrt{a}\right)e^{x\sqrt{a}}\right)^2},$$

quae forma sumto a valde parvo abit in

$$R=\frac{8naaxx-2a\left(n-1-(n+1)x\sqrt{a}+\frac{n-1}{2}axx-\frac{n+1}{6}ax^3\sqrt{a}+\text{etc.}\right)^2}{\left(n-1-\frac{1}{2}(n-1)axx+\frac{1}{3}(n+1)ax^3\sqrt{a}+\text{etc.}\right)^2}.$$

Statuatur $n=1+\beta a\sqrt{a}$, ut sit

$$n-1=\beta a\sqrt{a}\ \text{ et }\ n+1=2+\beta a\sqrt{a};$$

erit

$$R=\frac{8naaxx-2a\left(\beta a\sqrt{a}-2x\sqrt{a}-\beta axx+\frac{\beta aaxx\sqrt{a}}{2}-\frac{1}{3}ax^3\sqrt{a}\right)^2}{\left(\beta a\sqrt{a}-\frac{1}{2}\beta aaxx\sqrt{a}+\frac{2}{3}ax^3\sqrt{a}\right)^2},$$

ubi numerator fit

$$8aaxx+8\beta a^3xx\sqrt{a}-2a\left(\beta\beta a^3-4\beta aax-2\beta\beta a^3x\sqrt{a}+4axx+\frac{4}{3}aax^4\right)$$

ubi cum termini per *aa* affecti se destruant, retineantur ii soli, qui per a^3 sunt affecti; erit idem in denominatore observando

$$R=\frac{8\beta a^3x-\frac{8}{3}a^3x^4}{a^3\left(\beta+\frac{2}{3}x^3\right)^2}=\frac{8x\left(\beta-\frac{1}{3}x^3\right)}{\left(\beta+\frac{2}{3}x^3\right)^2},$$

quae iam facile ad formam

$$R=\frac{6x\left(2c^3-x^3\right)}{\left(c^3+x^3\right)^2}$$

reducitur sumendo $3\beta=2c^3$, ut sit $\beta=\frac{2}{3}c^3$. Quare hic casus oritur sumendo *a* evanescens et

$$n=1+\frac{2}{3}c^3a\sqrt{a}.$$

SCHOLION 2

372. Cum evolutio solutionis inventae sit difficillima neque ulla via pateat, quomodo ambae quantitates incognitae *r* et *s* ex binis aequationibus erutis definiri queant, in scientiae incrementum haud parum iuvabit observasse idem problema per repetitionem transformationis in primo problemate [§ 349] huius capitis quoque solvi posse neque proinde usu carebit has duas solutiones inter se comparasse.

Proposita ergo aequatione

$$\left(\frac{ddv}{dy^2}\right)=F\left(\frac{ddv}{dx^2}\right)+G\left(\frac{dv}{dx}\right)+Hv$$

ponamus primo

$$u=\left(\frac{dv}{dx}\right)+pv$$

ac p ex hac aequatione determinetur [§ 351]

$$Fdp+Gpdx-Fppdx+(\mathrm{C}-H)dx=0$$

ac tum ista resultabit aequatio

$$\left(\frac{ddu}{dy^2}\right)=F\left(\frac{ddu}{dx^2}\right)+\left(G+\frac{dF}{dx}\right)\left(\frac{du}{dx}\right)+\left(H+\frac{dG}{dx}-\frac{2Fdp+pdF}{dx}\right)u.$$

Nunc pro hac aequatione porro transformanda statuamus simili modo

$$z=\left(\frac{du}{dx}\right)+qu\ ,$$

ita ut sit quoque

$$z=\left(\frac{ddv}{dx^2}\right)+(p+q)\left(\frac{dv}{dx}\right)+\left(\frac{dP}{dx}+pq\right)v\ ,$$

et quantite q ex hac aequatione definita

$$Fdq+\left(G+\frac{dF}{dx}\right)qdx-Fqqdx+\left(D-H-\frac{dG}{dx}+\frac{2Fdp+PdF}{dx}\right)dx=0$$

orietur haec aequatione

$$\left(\frac{ddz}{dy^2}\right)=P\left(\frac{ddz}{dx^2}\right)+Q\left(\frac{dz}{dx}\right)+Rz\ ,$$

cuius quantitates P, Q, R ita se habent

$$P=F,\quad Q=G+\frac{2dF}{dx}$$

et

$$R=H+\frac{2dG}{dx}-\frac{2Fdp+pdF}{dx}+\frac{ddF}{dx^2}-\frac{2Fdq+qdF}{dx}.$$

Cum hac ergo solutione convenire debet ea, quam postremum problema [§ 367] suppeditavit; in quo cum statim posuerimus

$$z=\left(\frac{ddv}{dx^2}\right)+r\left(\frac{dv}{dx}\right)+sv$$

erit utique

$$r = p + q \quad \text{et} \quad s = \frac{dp}{dx} + pq\,,$$

unde quidem statim valores pro *P*, *Q* et *R* manifesto prodeunt iidem.

Verum multo minus apparet, si pro *r* et *s* isti valores *p* et *q* substituantur, tum istas binas aequationes

$$\frac{d.(G-Fr)}{dx} - (G-Fr)r + 2(H-Fs) = A$$

et

$$\frac{(G-Fr)d.(H-Fs)}{dx} + (H-FS)^2 - (G-Fr)^2 s - A(H-Fs) = B$$

ad eas, quas ante invenimus, reduci

$$\frac{Fdp}{dx} + Gp - Fpp - H + C = 0$$

et

$$\frac{Fdq}{dx} + \left(G + \frac{dF}{dx}\right)q - Fqq - H - \frac{dG}{dx} + \frac{2Fdp + pdF}{dx} + D = 0,$$

ita ut hae constantes *C* et *D* ad illas *A* et *B* certam teneant relationem. Interim patet has postremas aequationes multo esse simpliciores, dum prior duas tantum variabiles *p* et *x* complectitur indeque *p* per *x*, cuius *F*, *G* et *H* sunt functiones datae, determinari debet, qua inventa quantitatem *q* simili modo ex altera aequatione elici oportet. Verum in ambabus superioribus aequationibus binae variabiles *r* et *s* ita inter se sunt permixtae, ut nulla methodus eas resolvendi vel adeo ad aequationem inter duas tantum variabiles perveniendi habeatur. Cum igitur certum sit priores solutu difficillimas ad posteriores multo faciliores ope substitutionum assignatarum perduci posse, sine dubio methodus hanc reductionem efficiendi haud contemnenda subsidia in Analysin esse allatura videtur.

SCHOLION 3

373. Cum adeo consensus harum duarum solutionum maxime sit absconditus, casum specialem accuratius perpendi expediet.

Sit igitur $F = 1$, $G = 0$ et $H = 0$ ac binae priores aequationes inter *r* et s has induent formas

$$\text{I.} \quad \frac{-dr}{dx} + rr - 2s = A \quad \text{et} \quad \text{II.} \quad \frac{rds}{dx} + ss - rrs + As = B\,,$$

posteriores vero istas

$$\text{III.} \quad \frac{dp}{dx} - pp + C = 0 \quad \text{et} \quad \text{IV.} \quad \frac{dq}{dx} - qq + \frac{2dp}{dx} + D = 0,$$

quas cum illis certum est ita cohaerere, ut sit

$$r = p+q \quad \text{et} \quad s = \frac{dp}{dx} + pq\,.$$

Ut saltem consensum a posteriori agnoscamus, sit $C = -mm$ et tertia dat

$$dx = \frac{dp}{mm+pp}\,,$$

hinc

$$x = \frac{1}{m}\,\text{Ang.tang.}\frac{p}{m} \quad \text{et} \quad p = m\,\text{tang.}mx.$$

Hinc cum sit $\frac{dp}{dx} = mm+pp$, erit

$$s = mm+pp+pq = mm+pr = m\left(m+r\,\text{tang.}mx\right),$$

qui valor in I substitutus dat

$$\frac{-dr}{dx} + rr - 2mr\,\text{tang.}mx - mm = A,$$

seu

$$\frac{dr}{dx} = rr - 2mr\,\text{tang.}mx - mm - A\,.$$

secunda vero ob

$$\frac{ds}{dx} = \frac{mdr}{dx}\,\text{tang.}mx + \frac{mmr}{\cos.^2 mx}$$

abit in

$$\frac{mrdr}{dx}\,\text{tang.}mx = mr^3\text{tang.}mx - 2mmrr\,\text{tang.}^2 mx - m\left(A+2mm\right)r\,\text{tang.}mx - m^4 - Amm + B,$$

ex quibus *dr* eliminando fit $B = Amm + m^4$. Pro quarta vero ob

$$q = r - p = r - m\,\text{tang.}mx$$

resultat

$$\frac{dr}{dx} = rr - 2mr\,\text{tang.}mx - mm - D,$$

ita ut sit $D = mm + A$. Consensus ergo nostrarum aequationum in hac constantium relatione consistit, ut ob $mm = -C$ sit

$$D = A - C \quad \text{et} \quad B = -C\left(A-C\right) = -CD.$$

In genere vero etiam eaedem relationes locum habent; nam si III et IV in unam summam colligantur, ob $C+D=A$ et $p+q=r$ erit

$$\frac{Fdr}{dx}+Gr+\frac{rdF}{dx}-Fpp-Fqq-2H-\frac{dG}{dx}+2Fdp+A=0$$

cum vero sit $\frac{dp}{dx}=s-pq$, fit

$$\frac{Fdr+rdF-dG}{dx}+Gr-Frr-2H+2Fs+A=0$$

seu

$$\frac{d.(G-Fr)}{dx}-(G-Fr)r+2(H-Fs)=A,$$

quae est ipsa aequatio prima.

Porro aequatio tertia ob $\frac{dp}{dx}=s-pq$ dat

$$Fs-Fpr+Gp-H+C=0 \text{ seu } C=H-Fs-p(G-Fr);$$

quarta vero reducitur ad hanc formam

$$\frac{Fdr}{dx}+Gq+\frac{qdF}{dx}-Fqq-H-\frac{dG}{dx}+Fs-Fpq+\frac{pdF}{dx}+D=0$$

seu

$$\frac{d.(Fr-G)}{dx}+q(G-Fr)r-H+Fs+D=0$$

hincque

$$D=\frac{d.(G-Fr)}{dx}-q(G-Fr)+H-Fs\,,$$

ex quibus concluditur

$$CD=\frac{(H-Fs)d.(G-Fr)}{dx}-q(G-Fr)(H-Fs)+(H-Fs)^2$$
$$-\frac{p(G-Fr)d.(G-Fr)}{dx}+pq(G-Fr)^2-p(G-Fr)(H-Fs).$$

Ex secunda vero habemus

$$B=\frac{(G-Fr)d.(H-Fs)}{dx}-\frac{(H-Fs)d.(G-Fr)}{dx}-(H-FS)^2+(G-Fr)(H-Fs)r-(G-Fr)^2 s\,,$$

quibus expressionibus coniunctis fit

$$\frac{CD+B}{(G-Fr)} = \frac{d.(H-Fs)}{dx} - \frac{pd.(G-Fr)}{dx} - \frac{dp(G-Fr)}{dx}$$
$$= \frac{d.(H-Fs)-d.p(G-Fr)}{dx} = 0,$$

siquidem est $C = H - Fs - p(G - Fr)$, ex quo etiam in genere est

$$B = -CD \quad \text{et} \quad A = C + D.$$

Interim tamen hinc non perspicitur, quomodo ex aequationibus I et II binae reliquae III et IV derivari queant.

SCHOLION 4

374. Omnibus his diligenter pensitatis manifestum fiet totum negotium ope substitutionis satis simplicis confici posse. Quod quo facilius ostendatur, ponamus brevitatis causa $G - Fr = R$ et $H - Fs = S$, ut habeantur hae duae aequationes

I. $A = \frac{dR}{dx} - \frac{GR}{F} + \frac{RR}{F} + 2S.$

II. $B = \frac{RdS-SdR}{dx} - \frac{HRR}{F} + \frac{GRS}{F} - SS,$

ex quibus duas quantitates *R* et S erui oporteat, dum *F*, *G*, *H* sunt functiones quaecunque ipsius *x*, at *A* et *B* quantitates constantes. Ad hoc adhibeatur ista substitutio $S = C + Rp$ ita adornanda, ut binae illae aequationes coalescant in unam, in qua praeter *x* unica insit nova variabilis *p* deinceps per methodos cognitas investiganda.

Hinc ob $dS = Rdp + pdR$ habebitur

I. $A = \frac{dR}{dx} - \frac{GR}{F} + \frac{RR}{F} + 2C + 2Rp,$

ll. $B = \frac{RRdp}{dx} - \frac{CdR}{dx} - \frac{HRR}{F} + \frac{CGR}{F} + \frac{GRRp}{F} - CC - 2CRp - RRpp,$

unde primo eliminando *dR* concluditur

$$B + AC = \frac{RRdp}{dx} + \frac{CRR}{F} + CC - \frac{HRR}{F} - RRpp + \frac{GRRp}{F};$$

dummodo ergo constantem *C* ita assumamus, ut sit $CC = B + AC$, per divisionem etiam ipsa quantitas *R* tolletur resultabitque haec aequatio

$$0 = \frac{dp}{dx} + \frac{C}{F} - \frac{H}{F} - pp + \frac{Gp}{F}.$$

cuius resolutio ad methodos magis cognitas pertinet.

Cum igitur ista methodus maximi sit momenti, sequens problema, etiamsi ad primam partem calcun integralis sit referendum, hic adiicere operae pretium videtur.

PROBLEMA 60

375. *Propositis huiusmodi duabus aequationibus differentialibus*

1. $0 = \frac{dy}{dx} + F + Gy + Hz + Iyy + Kyz + Lzz,$

1I. $0 = \frac{ydz - zdy}{dx} + P + Qy + Rz + Syy + Tyz + Vzz,$

ubi F, G, H etc., P, Q, R etc. sint functiones ipsius x, methodum exponere has aequationes, siquidem fieri licet, resolvendi.

SOLUTIO

Methodus indicata in hoc consistit, ut ope substitutionis $z = a + yv$ ex illis aequationibus una elici queat duas tantum variabiles *x* et *v* implicans. Quoniam igitur est $ydz - zdy = yydv - ady$, ex I*a* + II nascitur haec aequatio

$$0 = \frac{yydv}{dx} + P + Qy + Rz + Syy + Tyz + Vzz$$
$$+ aF + aGy + aHz + aIyy + aKyz + aLzz,$$

quae loco *z* substituto valore $a + yv$ ita exhibeatur secundum potestates ipsius *y*

$$0 = \frac{yydv}{dx} + y^0\big(P + aF + a(R + aH) + aa(V + aL)\big)$$
$$+ y^1\big(Q + aG + v(R + aH) + a(T + aK) + 2av(V + aL)\big)$$
$$+ y^2\big(S + aI + v(T + aK) + vv(V + aL)\big),$$

nuncque efficiendum est, ut tota aequatio per *yy* dividi queat ideoque partes per y^0 et y^1 affectae evanescant. Ex parte ergo y^0 fieri oportet

$$P + aF + a(R + aH) + aa(V + aL) = 0,$$

ex parte autem y^1, quia vest nova variabilis in calculum inducta, hae duae conditiones nascuntur

$$Q + aG + a(T + aK) = 0 \quad \text{et} \quad R + aH + 2a(V + aL) = 0,$$

unde prima dabit

$$P+ aF-aa(V+aL)=0.$$

Conditiones ad istam reductionem requisitae sunt hae tres

$$\text{I.}\quad P+aF-aa(V+aL)=0,$$
$$\text{II.}\quad Q+aG+ a(T+aK)=0,$$
$$\text{III.}\quad R+aH+2a(V+aL)= 0,$$

unde vel *P*, *Q* et *R* vel *F*, *G* et *H* commode definiuntur.
His autem conditionibus stabilitis totum negotium ad resolutionem huius aequationis revocatur

$$0=\tfrac{dv}{dx}+S+aI+v(T+aK)+vv(V+aL),$$

quae duas tantum continet variabiles *x* et *v*, ex qua *v* per *x* determinari oportet, cum deinde posito $z=a+yv$ prima aequatio induat hanc formam

$$0=\tfrac{dy}{dx}+F+aH+aaL+y(G+Hv+aK+2aLv)+yy(I+Kv+Lvv),$$

secunda vero istam

$$0=\tfrac{ydv}{dx}-\tfrac{ady}{dx}+P+aR+aaV+y(Q+Rv+aT+2aVv)+yy(S+Tv+Vvv),$$

seu hinc superiorem per *yy* multiplicatam subtrahendo

$$0=\tfrac{-ady}{dx}+P+aR+aaV+y(Q+Rv+aT+2aVv)-yy(Ia+aKv+aLvv),$$

quae quidem cum illa congruit, ut natura rei postulate

COROLLARIUM 1

376. Si ergo huiusmodi binae aequationes fuerint propositae

$$0=\tfrac{dy}{dx}+F-Gy+Hz+Iyy+Kyz+Lzz,$$
$$\begin{aligned}0=\tfrac{ydz-zdy}{dx}&-aF-aGy-aHz+Syy+Tyz+Vzz\\&+a^3L-aaKy-2aaLz\\&+aaV-aTy-2aVz,\end{aligned}$$

facto $z=a+yv$ primo resolvi debet haec aequatio

$$0=\tfrac{dv}{dx}+S+aI+v\left(T+aK\right)+vv\left(V+aL\right),$$

unde definita *v* per *x* hanc aequationem tractari oportet

$$0=\tfrac{dy}{dx}+F+aH+aaL+y\left(G\ +aK\right)+yy\left(I+Kv+Lvv\right)+vy\left(H+2aL\right),$$

quo facto habebitur quoque $z=a+vy$.

COROLLARIUM 2

377. Si $F=A,\ K=0,\ L=0,\ H=-2b,\ V=b$ et $T=-G$, casus supra § 374 tractatus resultat harum aequationum

$$0=\tfrac{dy}{dx}+A+Gy-2bz+Iyy,$$
$$0=\tfrac{ydz-zdy}{dx}-aA+aab+Syy-Gyz+bzz,$$

ubi *G, I* et *S* sunt functiones quaecunque ipsius *x*, et resolutio ita se habet, ut posito $z=a+vy$ hae aequationes successive debeant expediri

$$0=\tfrac{dv}{dx}+S+aI-Gv+bvv$$

et

$$0=\tfrac{dy}{dx}+A-2ab+y\left(G-2bv\right)+Iyy.$$

COROLLARIUM 3

378. Evidens est postremam aequationem nulla laborare difficultate etiam in genere, dum sit

$$F+aH+aaL=0\,;$$

prioris autem solutio in promtu est, si sit vel $S+aI=0$ vel $V+aL=0$.