CHAPTER IV

A SPECIAL METHOD OF INTEGRATING EQUATIONS OF THIS KIND IN ANOTHER WAY.

PROBLEM 52

322. *If the proposed equation to be integrated should have this form*

$$(x+y)^2\left(\frac{ddz}{dxdy}\right)+m(x+y)\left(\frac{dz}{dx}\right)+m(x+y)\left(\frac{dz}{dy}\right)+nz=0\,,$$

to investigate the complete integral of this form.

SOLUTION

Since here the two variables x and y are involved equally, in the first place there is put

$$z=A(x+y)^{\lambda}\ f{:}x+\ B(x+y)^{\lambda+1}\ f'{:}x+C(x+y)^{\lambda+2}\ f''{:}x+D(x+y)^{\lambda+3}\ f'''{:}x+\text{etc.},$$

in which to be made easier by putting in place the substitution $v=(x+y)^{\mu}\,F{:}x$, there may be noted to become

$$\left(\frac{dv}{dx}\right)=\mu(x+y)^{\mu-1}F{:}x+(x+y)^{\mu-1}F'{:}x\,,$$

$$\left(\frac{dv}{dy}\right)=\mu(x+y)^{\mu-1}F{:}x$$

and

$$\left(\frac{ddv}{dxdy}\right)=\mu(\mu-1)(x+y)^{\mu-2}F{:}x+\mu(x+y)^{\mu-1}F'{:}x\,.$$

Therefore with the substitution made we will obtain this equation [note the suppression of $f(x)(x+y)^{\lambda}$, $f'(x)(x+y)^{\lambda+1}$, etc. in the second and subsequent rows in this table]

$0=$	$nA(x+y)^{\lambda}\ f{:}x+$		$nB(x+y)^{\lambda+1}\ f'{:}x+$		$nC(x+y)^{\lambda+2}\ f''{:}x+\text{etc.}$
		$+$	mA	$+$	mB
$+$	$2m\lambda A$	$+$	$2m(\lambda+1)B$	$+$	$2m(\lambda+2)C$
		$+$	λA	$+$	$(\lambda+1)B$
	$+\lambda(\lambda-1)A$		$+(\lambda+1)\lambda B$		$+(\lambda+2)(\lambda+1)C$

in which the whole exercise is reduced to the determination of the coefficients *A, B, C, D* etc. ; moreover it was easy to foresee from the form assumed above that equal powers of $(x+y)$ are to be produced in the individual members. Therefore it is necessary that

$$n+2m\lambda+\lambda\lambda-\lambda=0,$$
$$(n+2m\lambda+2m+\lambda\lambda+\lambda)B+(m+\lambda)A=0,$$
$$(n+2m\lambda+4m+\lambda\lambda+3\lambda+2)C+(m+\lambda+1)B=0,$$
$$(n+2m\lambda+6m+\lambda\lambda+5\lambda+6)D+(m+\lambda+2)C=0$$
$$\text{etc.,}$$

which determinations thus can be more conveniently expressed with the aid of the first $n+2m\lambda+\lambda\lambda-\lambda=0$:

$$B=-\tfrac{m+\lambda}{2m+2\lambda}A,\quad F=-\tfrac{m+\lambda+4}{5(2m+2\lambda+4)}E,$$
$$C=-\tfrac{m+\lambda+1}{2(2m+2\lambda+1)}B,\quad G=-\tfrac{m+\lambda+5}{6(2m+2\lambda+5)}F,$$
$$D=-\tfrac{m+\lambda+2}{3(2m+2\lambda+2)}C,\quad H=-\tfrac{m+\lambda+6}{7(2m+2\lambda+6)}G,$$
$$E=-\tfrac{m+\lambda+3}{4(2m+2\lambda+3)}D,\quad \text{etc.,}$$

from which the law of the progression is evident.

But for the exponent λ we can extract the twofold value

$$\lambda=\tfrac{1}{2}-m\pm\sqrt{\left(\tfrac{1}{4}-m-n+mm\right)},$$

each of which is allowed equally to be accepted for λ. But here there are special cases to be noted in particular, in which the assumed series breaks off, which occurs whenever $m+\lambda+i=0$ with i denoting some whole positive number not excluding zero.
Therefore this comes about, as often as there should be

$$\tfrac{1}{2}+i\pm\sqrt{\left(\tfrac{1}{4}-m-n+mm\right)}=0,$$

as that cannot happen, unless $\tfrac{1}{4}-m-n+mm$ should be a square number. Moreover another series of this kind either finite or running off to infinity can be found for the functions of y, from which the expressed value of z thus may be found

$$z = A(x+y)^{\lambda}(f:x+F:y)+B(x+y)^{\lambda+1}(f':x+F':y)$$
$$+C(x+y)^{\lambda+2}(f'':x+F'':y)+D(x+y)^{\lambda+3}(f''':x+F''':y)$$
$$+E(x+y)^{\lambda+4}(f^{IV}:x+F^{IV}:y)+F(x+y)^{\lambda+5}(f^{V}:x+F^{V}:y)$$
etc.;

in which since the two arbitrary functions are present, that certainly is the sign that the form is to be the complete integral of the proposed equation.

COROLLARY 1

323. If there should be $\lambda=-m$, that is $n-mm+m=0$ or $n=mm-m$, the integral will correspond to a single member on account of $B=0$, and the integral will be

$$z = A(x+y)^{-m}(f:x+F:y).$$

COROLLARY 2

324. Moreover the integral will correspond to two members, if

$$\lambda=-m-1 \text{ or } n=mm-m-2=(m+1)(m-2);$$

then there will be $B=-\frac{1}{2}A$ and the integral will be

$$z=(x+y)^{-m-1}(f:x+F:y)-\tfrac{1}{2}(x+y)^{-m}(f':x+F':y).$$

COROLLARY 3

325. The integral will correspond to three terms, if

$$\lambda=-m-2 \text{ or } n=(m+2)(m-3);$$

then there will be

$$B=-\tfrac{1}{2}A \text{ and } C=-\tfrac{1}{6}B=+\tfrac{1}{12}A,$$

truly the integral will be :

$$z=(x+y)^{-m-2}(f:x+F:y)-\tfrac{1}{2}(x+y)^{-m-1}(f':x+F':y)+\tfrac{1}{12}(x+y)^{-m}(f'':x+F'':y).$$

COROLLARY 4

326. Moreover the integral will correspond to four terms, if there should be

$$\lambda=-m-3 \text{ or } n=(m+3)(m-4);$$

but then there will be

$$B=-\tfrac{1}{2}A,\ \ C=-\tfrac{1}{5}B=+\tfrac{1}{10}A,\ \ D=-\tfrac{1}{12}C=-\tfrac{1}{120}A$$

and the integral will be

$$z=(x+y)^{-m-3}(f:x+F:y)-\tfrac{1}{2}(x+y)^{-m-2}(f':x+F':y)$$
$$+\tfrac{1}{10}(x+y)^{-m-1}(f'':x+F'':y)-\tfrac{1}{120}(x+y)^{-m}(f''':x+F''':y).$$

SCHOLIUM

327. Because if in general we put $\lambda+m=-i$, there will be $n=(m+i)(m-i-1)$, then indeed

$$B=-\tfrac{1}{2}A,\ \ C=-\tfrac{i-1}{2(2i-1)}B,\ \ D=-\tfrac{i-2}{3(2i-2)}C,\ \ E=-\tfrac{i-3}{4(2i-3)}D,$$

from which everything becomes on reduction to the first

$$B=-\tfrac{1}{2}A,\ \ C=\tfrac{i-1}{2\cdot 2(2i-1)}A,\ \ D=\tfrac{-(i-2)}{2\cdot 2\cdot 2\cdot 3(2i-1)}A,$$
$$E=\tfrac{+(i-2)(i-3)}{2\cdot 2\cdot 2\cdot 3\cdot 4(2i-1)(2i-3)}A,\quad F=\tfrac{-(i-3)(i-4)}{2^4\cdot 3\cdot 4\cdot 5(2i-1)(2i-3)}A\ \text{ etc.},$$

which thus themselves become :

	A	B	C	D	E	F	G
$i=1$	1	$-\frac{1}{2}$	0	0	0	0	0
$i=2$	1	$-\frac{1}{2}$	$\frac{1}{12}$	0	0	0	0
$i=3$	1	$-\frac{1}{2}$	$\frac{2}{20}$	$-\frac{1}{120}$	0	0	0
$i=4$	1	$-\frac{1}{2}$	$\frac{3}{28}$	$-\frac{2}{7\cdot 24}$	$\frac{2\cdot 1}{96\cdot 9\cdot 5}$	0	0
$i=5$	1	$-\frac{1}{2}$	$\frac{4}{36}$	$-\frac{3}{9\cdot 24}$	$\frac{3\cdot 2}{96\cdot 9\cdot 7}$	$-\frac{2\cdot 1}{960\cdot 9\cdot 7}$	0
$i=6$	1	$-\frac{1}{2}$	$\frac{5}{44}$	$-\frac{4}{11\cdot 24}$	$\frac{4\cdot 3}{96\cdot 11\cdot 9}$	$-\frac{3\cdot 2}{960\cdot 11\cdot 9}$	$\frac{3\cdot 2\cdot 1}{5760\cdot 11\cdot 9\cdot 7}$

Thus the complete integral of this equation

$$\left(\frac{ddz}{dxdy}\right)+\frac{m}{x+y}\left(\frac{dz}{dx}\right)+\frac{m}{x+y}\left(\frac{dz}{dy}\right)+\frac{(m+i)(m-i-1)}{(x+y)^2}z=0$$

will be

$$z=\left(x+y\right)^{-m-i}\left(f{:}x+F{:}y\right)-\tfrac{i}{2i}\left(x+y\right)^{-m-i+1}\left(f'{:}\,x+F'{:}\,y\right)$$
$$+\tfrac{i(i-1)}{2i\cdot 2(2i-1)}\left(x+y\right)^{-m-i+2}\left(f''{:}x+F''{:}y\right)-\tfrac{i(i-1)(i-2)}{2i\cdot 2(2i-1)\cdot 3(2i-2)}\left(x+y\right)^{-m-i+3}\left(f'''{:}x+F'''{:}y\right)$$
$$+\tfrac{i(i-1)(i-2)(i-3)}{2i\cdot 2(2i-1)\cdot 3(2i-2)\cdot 4(2i-3)}\left(x+y\right)^{-m-i+4}\left(f^{IV}{:}x+F^{IV}{:}y\right)$$
$$-\tfrac{i(i-1)(i-2)(i-3)(i-4)}{2i\cdot 2(2i-1)\cdot 3(2i-2)\cdot 4(2i-3)\cdot 5(2i-4)}\left(x+y\right)^{-m-i+5}\left(f^{V}{:}x+F^{V}{:}y\right)$$
etc.,

which form, as often as *i* should be a positive whole number, depends on a finite number of terms; but otherwise extends to infinity.

But that integration besides has this especial singularity, because not only does it involve these arbitrary functions *f*: *x* and F*: y*, but also the differential formulas of these.

EXAMPLE

328. *If this equation*

$$\left(\tfrac{ddz}{dxdy}\right)+\tfrac{m}{x+y}\left(\tfrac{dz}{dx}\right)+\tfrac{m}{x+y}\left(\tfrac{dz}{dy}\right)=0$$

should occur, to define the cases, in which the integral of this can be shown by a finite form.

Since here there shall be $n=(m+i)(m-i-1)=0$, on taking positive whole numbers for *i* two orders of cases will be had, from which the integration succeeds, the one, in which there is $m=-i$, the other, in which $m=i+1$, thus so that in general there is a place for a finite integration, as often as *m* should be a positive or negative integer.

Therefore in the first place if there shall be $m=-i$, then there will be

$$z=1\left(f{:}x+F{:}y\right)-\tfrac{i}{2i}\left(x+y\right)\left(f'{:}\,x+F'{:}\,y\right)$$
$$+\tfrac{1}{2}\cdot\tfrac{i(i-1)}{2i(2i-1)}\left(x+y\right)^{2}\left(f''{:}x+F''{:}y\right)-\tfrac{1}{6}\cdot\tfrac{i(i-1)(i-2)}{2i(2i-1)(2i-2)}\left(x+y\right)^{3}\left(f'''{:}x+F'''{:}y\right)$$
$$+\tfrac{1}{24}\cdot\tfrac{i(i-1)(i-2)(i-3)}{2i(2i-1)(2i-2)(2i-3)}\left(x+y\right)^{4}\left(f^{IV}{:}x+F^{IV}{:}y\right)$$
etc.,

Then if there shall be $m=i+1$, then

$$(x+y)^{2i+1}z = 1(f:x+F:y) - \tfrac{i}{2i}(x+y)(f':x+F':y)$$
$$+\tfrac{1}{2}\cdot\tfrac{i(i-1)}{2i(2i-1)}(x+y)^2(f'':x+F'':y) - \tfrac{1}{6}\cdot\tfrac{i(i-1)(i-2)}{2i(2i-1)(2i-2)}(x+y)^3(f''':x+F''':y)$$
$$+\tfrac{1}{24}\cdot\tfrac{i(i-1)(i-2)(i-3)}{2i(2i-1)(2i-2)(2i-3)}(x+y)^4(f^{IV}:x+F^{IV}:y)$$
etc.,

Clearly each has the same expression, to which for the first case that is equal to the quantity z, for the following the quantity $(x+y)^{2i+1}z$.

Towards setting out these cases more distinctly we put

$$A = (f:x+F:y),$$
$$B = (f:x+F:y) - \tfrac{1}{2}(x+y)(f':x+F':y),$$
$$C = (f:x+F:y) - \tfrac{2}{4}(x+y)(f':x+F':y) + \tfrac{1}{4\cdot 3}(x+y)^2(f'':x+F'':y),$$
$$D = (f:x+F:y) - \tfrac{3}{6}(x+y)(f':x+F':y) + \tfrac{3}{6\cdot 5}(x+y)^2(f'':x+F'':y) - \tfrac{1}{6\cdot 5\cdot 4}(x+y)^3(f''':x+F''':y)$$
etc.

or on putting for the sake of brevity

$$\mathfrak{A} = f:x+F:y,$$
$$\mathfrak{B} = (x+y)(f':x+F':y),$$
$$\mathfrak{C} = (x+y)^2(f'':x+F'':y),$$
$$\mathfrak{D} = (x+y)^3(f''':x+F''':y),$$
$$\mathfrak{E} = (x+y)^4(f^{IV}:x+F^{IV}:y)$$
etc.

there shall be

$$A = \mathfrak{A},$$
$$B = \mathfrak{A} - \tfrac{1}{2}\mathfrak{B},$$
$$C = \mathfrak{A} - \tfrac{2}{4}\mathfrak{B} + \tfrac{1}{4\cdot 3}\mathfrak{C},$$
$$D = \mathfrak{A} - \tfrac{3}{6}\mathfrak{B} + \tfrac{3}{6\cdot 5}\mathfrak{C} - \tfrac{1}{6\cdot 5\cdot 4}\mathfrak{D},$$
$$E = \mathfrak{A} - \tfrac{4}{8}\mathfrak{B} + \tfrac{6}{8\cdot 7}\mathfrak{C} - \tfrac{4}{8\cdot 7\cdot 6}\mathfrak{D} + \tfrac{1}{8\cdot 7\cdot 6\cdot 5}\mathfrak{E},$$
$$F = \mathfrak{A} - \tfrac{5}{10}\mathfrak{B} + \tfrac{10}{10\cdot 9}\mathfrak{C} - \tfrac{10}{10\cdot 9\cdot 8}\mathfrak{D} + \tfrac{5}{10\cdot 9\cdot 8\cdot 7}\mathfrak{E} - \tfrac{1}{10\cdot 9\cdot 8\cdot 7\cdot 6}\mathfrak{F},$$
$$G = \mathfrak{A} - \tfrac{6}{12}\mathfrak{B} + \tfrac{15}{12\cdot 11}\mathfrak{C} - \tfrac{20}{12\cdot 11\cdot 10}\mathfrak{D} + \tfrac{15}{12\cdot 11\cdot 10\cdot 9}\mathfrak{E} - \tfrac{6}{12\cdot 11\cdot 10\cdot 9\cdot 8}\mathfrak{F} + \tfrac{1}{12\cdot 11\cdot 10\cdot 9\cdot 8\cdot 7}\mathfrak{G}$$
etc.

With which values found, there will be for the twofold order:

For,	there shall be	For,	there shall be
$m = 0,$	$z = A$	$m = 1,$	$(x+y)z = A$
$m = -1,$	$z = B$	$m = 2,$	$(x+y)^3 z = B$
$m = -2,$	$z = C$	$m = 3,$	$(x+y)^5 z = C$
$m = -3,$	$z = D$	$m = 4,$	$(x+y)^7 z = D$
$m = -4,$	$z = E$	$m = 5,$	$(x+y)^9 z = E$
$m = -5,$	$z = F$	$m = 6,$	$(x+y)^{11} z = F$
$m = -6,$	$z = G$	$m = 7,$	$(x+y)^{13} z = G$
etc.		etc.	

SCHOLIUM

329. If a negative number is taken for *i*, the expression extends to infinity. For let $i = -k$ and from the first formula there will be $m = k$ and thus

$$z = \mathfrak{A} - \tfrac{k}{2k}\mathfrak{B} + \tfrac{1}{2}\cdot\tfrac{k(k+1)}{2k(2k+1)}\mathfrak{C} - \tfrac{1}{6}\cdot\tfrac{k(k+1)(k+2)}{2k(2k+1)(2k+2)}\mathfrak{D} + \text{ etc. to infinity.}$$

But for the same case $m = k$ the other form on account of $i = k-1$ gives

$$(x+y)^{2k-1} z$$
$$= \mathfrak{A} - \tfrac{k-1}{2k-2}\mathfrak{B} + \tfrac{1}{2}\cdot\tfrac{(k-1)(k-2)}{(2k-2)(2k-3)}\mathfrak{C} - \tfrac{1}{6}\cdot\tfrac{(k-1)(k-2)(k-3)}{(2k-2)(2k-3)(2k-4)}\mathfrak{D} + \text{ etc.},$$

which moreover are to be agreed not to have absolutely equal forms, but in the other functions *f: x* and *F: y* other forms will be considered, so that nevertheless both forms are satisfied equally. Indeed for the case $k = \frac{1}{2}$ both agree perfectly. Moreover we may put $k = 0$, so that the former may give

$$z = \mathfrak{A} = f{:}x + F{:}y\,;$$

but the latter gives

$$\tfrac{z}{x+y} = \mathfrak{A} - \tfrac{1}{2}\mathfrak{B} + \tfrac{1}{6}\mathfrak{C} - \tfrac{1}{24}\mathfrak{D} + \tfrac{1}{120}\mathfrak{E} - \text{etc.}$$

So that the agreement of which it may become apparent, let there be here in the latter

$$f:x = ax^3 \text{ and } F:y = by^2;$$

there will be

$$\mathfrak{A} = ax^3 + by^2, \quad \mathfrak{B} = (x+y)(3axx + 2by), \quad \mathfrak{C} = (x+y)^2(6ax+2b), \quad \mathfrak{D} = (x+y)^3 6a,$$

but the remaining parts vanish. Therefore we will obtain from the latter

$$z = (x+y)\left(ax^3 + by^2\right) - \tfrac{1}{2}(x+y)^2(3axx+2by) + \tfrac{1}{3}(x+y)^3(3ax+b) - \tfrac{1}{4}(x+y)^4 a,$$

which evaluated gives

$$\tfrac{1}{4}ax^4 - \tfrac{1}{4}a^4 y + \tfrac{1}{3}bx^3 + \tfrac{1}{3}b^3 y = z,$$

which form certainly may be contained in the former $z = f:x + F:y$. Therefore the agreement of the two more general forms is more worthy of note there.

PROBLEM 53

330. *To find the cases, in which this general equation*

$$\left(\frac{ddz}{dy^2}\right) - QQ\left(\frac{ddz}{dx^2}\right) + R\left(\frac{dz}{dy}\right) + S\left(\frac{dz}{dx}\right) + Tz = 0$$

can be reduced to the preceding form and thus is able to be integrated in the same cases.

SOLUTION

On introducing the two new variables *t* and *u*, so that there shall be, just as the reduction § 319 used, where there is declared $P = 0$ and $V = 0$,

$$t = \int p(dx + Qdy) \text{ and } u = \int q(dx - Qdy),$$

if we put as an abbreviation

$$M = S + QR + \left(\frac{dQ}{dy}\right) + Q\left(\frac{dQ}{dx}\right),$$

$$N = S - QR - \left(\frac{dQ}{dy}\right) + Q\left(\frac{dQ}{dx}\right),$$

this equation will be produced

$$\left(\frac{ddz}{dtdu}\right) - \frac{M}{4QQq}\left(\frac{dz}{dt}\right) - \frac{N}{4QQp}\left(\frac{dz}{du}\right) - \frac{T}{4QQpq}z = 0,$$

that hence it is required to recall to that form

$$\left(\frac{ddz}{dtdu}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)+\frac{n}{(t+u)^2}z=0,$$

we have designated before the case of the integrability of which [§ 327], clearly as often as there should be

$$n=(m+i)(m-i-1)$$

with i denoting some positive whole number with zero not excluded.
Therefore according to this it is necessary that there becomes

$$M=\frac{-4mQQq}{t+u},\quad N=\frac{-4mQQp}{t+u}\quad\text{and}\quad T=\frac{-4nQQpq}{(t+u)^2}$$

But because here an ratio must be had of the integrability formulas t and u , we may take

$$Q=\frac{\varphi':y}{\pi':x}$$

and there shall be

$$p=a\pi':x\quad\text{and}\quad q=b\pi':x$$

and there shall be

$$t=a\pi:x+a\varphi:y\quad\text{and}\quad u=b\pi:x-b\varphi:y.$$

Hence there becomes

$$M+N=2S+2Q\left(\frac{dQ}{dx}\right)=\frac{-4m(a+b)QQ\pi':x}{t+u}$$

and

$$M-N=2QR+2\left(\frac{dQ}{dy}\right)=\frac{4m(a-b)QQ\pi':x}{t+u}$$

and thus

$$R=\frac{2m(a-b)Q\pi':x}{t+u}-\frac{1}{Q}\left(\frac{dQ}{dy}\right),\quad S=\frac{-2m(a+b)QQ\pi':x}{t+u}-Q\left(\frac{dQ}{dx}\right)$$

and

$$T=\frac{-4nabQQ\pi':x\cdot\pi':x}{(t+u)^2}=\frac{-4nab\varphi':y\cdot\varphi':y}{(t+u)^2}$$

on account of $Q=\frac{\varphi':y}{\pi':x}$, from which there is

$$\left(\frac{dQ}{dy}\right)=\frac{\varphi'':y}{\pi':x}\quad\text{and}\quad\left(\frac{dQ}{dx}\right)=\frac{-\pi'':y\cdot\varphi':y}{\pi':x\cdot\pi':x}$$

and

$$t+u=(a+b)\pi:x+(a-b)\varphi:y.$$

Thus we will have

$$R=\frac{2m(a-b)\varphi':y}{t+u}-\frac{\varphi'':y}{\varphi':y}\quad\text{and}\quad\frac{S}{QQ}=\frac{-2m(a+b)\pi':x}{t+u}+\frac{\pi'':x}{\pi':x}.$$

So that the equation becomes simpler, two cases especially are to be considered, the one, where $b = a$, the other where $b = -a$. In the first there is $t + u = 2a\pi{:}x$ and our equation will be

$$\left(\frac{ddz}{dy^2}\right)-\left(\frac{\varphi'{:}y}{\pi'{:}x}\right)^2\left(\frac{ddz}{dx^2}\right)-\frac{\varphi''{:}y}{\varphi'{:}y}\left(\frac{dz}{dy}\right)+\left(\frac{\varphi'{:}y}{\pi'{:}x}\right)^2\left(\frac{\pi''{:}x}{\pi'{:}x}-\frac{2m\pi'{:}x}{\pi{:}x}\right)\left(\frac{dz}{dx}\right)-n\left(\frac{\varphi'{:}y}{\pi{:}x}\right)^2 z=0\,;$$

truly in the other case $b = -a$ there shall be $t + u = 2a\varphi{:}y$ and

$$\left(\frac{ddz}{dy^2}\right)-\left(\frac{\varphi'{:}y}{\pi'{:}x}\right)^2\left(\frac{ddz}{dx^2}\right)+\left(\frac{2m\varphi'{:}y}{\varphi{:}y}-\frac{\varphi''{:}y}{\varphi'{:}y}\right)\left(\frac{dz}{dy}\right)+\left(\frac{\varphi'{:}y}{\pi'{:}x}\right)^2\cdot\frac{\pi''{:}x}{\pi'{:}x}\left(\frac{dz}{dx}\right)+n\left(\frac{\varphi'{:}y}{\varphi{:}y}\right)^2 z=0\,,$$

which both equations allow integration in the cases

$$n=(m+i)(m-i-1).$$

COROLLARY 1

331. The final equations found do not differ from each other, except that the two variables x and y are interchanged in turn, from which it suffices that either alone be considered. But the former [on taking $a = 1$] is transformed on putting

$$t=\pi{:}x+\varphi{:}y \quad \text{and} \quad u=\pi{:}x-\varphi{:}y\,,$$

and truly the latter on putting

$$t=\pi{:}x+\varphi{:}y \quad \text{and} \quad u=\varphi{:}y-\pi{:}x\,.$$

COROLLARIUM 2

332. These equations also can be represented in the more transparent form, indeed the former

$$\frac{1}{(\varphi'{:}y)^2}\left(\frac{ddz}{dy^2}\right)-\left(\frac{1}{\pi'{:}x}\right)^2\left(\frac{ddz}{dx^2}\right)-\frac{\varphi''{:}y}{(\varphi'{:}y)^3}\left(\frac{dz}{dy}\right)+\left(\frac{\pi''{:}x}{(\pi'{:}x)^3}-\frac{2m}{\pi{:}x\cdot\pi'{:}x}\right)\left(\frac{dz}{dx}\right)-\frac{n}{(\pi{:}x)^2}z=0$$

and the latter

$$\frac{1}{(\varphi'{:}y)^2}\left(\frac{ddz}{dy^2}\right)-\left(\frac{1}{\pi'{:}x}\right)^2\left(\frac{ddz}{dx^2}\right)+\left(\frac{2m}{\varphi{:}y\cdot\varphi'{:}y}-\frac{\varphi''{:}y}{(\varphi'{:}y)^3}\right)\left(\frac{dz}{dy}\right)+\frac{\pi''{:}x}{(\pi'{:}x)^3}\left(\frac{dz}{dx}\right)+\frac{n}{(\varphi{:}y)^2}z=0.$$

CASE 1

333. We may put $\pi':x=a$ and $\varphi':y=b$; there will be

$$\pi:x=ax \quad \text{and} \quad \varphi:y=by,$$

then truly $\pi'':x=0$ and $\varphi'':y=0$, from which the first form will produce

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aa}\left(\frac{ddz}{dx^2}\right)-\frac{2m}{aax}\left(\frac{dz}{dx}\right)-\frac{n}{aaxx}z=0,$$

which is reduced to the above resolved form on putting

$$t=ax+by \quad \text{and} \quad u=ax-by.$$

Now the latter form is

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aa}\left(\frac{ddz}{dx^2}\right)+\frac{2m}{bby}\left(\frac{dz}{dy}\right)+\frac{n}{bbyy}z=0,$$

which is reduced to the above resolved form on putting

$$t=ax+by \quad \text{and} \quad u=by-ax;$$

moreover each is integrable in the case

$$n=(m+i)(m-i-1).$$

Indeed from the reduction made to the variables *t* and *u* this equation arises

$$\left(\frac{ddz}{dtdu}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)+\frac{n}{(t+u)^2}z=0.$$

COROLLARY 1

334. If there is put $n=0$, both these equations

$$\frac{aa}{bb}\left(\frac{ddz}{dy^2}\right)-\left(\frac{ddz}{dx^2}\right)-\frac{2m}{x}\left(\frac{dz}{dx}\right)=0 \quad \text{and} \quad \left(\frac{ddz}{dy^2}\right)-\frac{bb}{aa}\left(\frac{ddz}{dx^2}\right)+\frac{2m}{y}\left(\frac{dz}{dy}\right)=0$$

are integrable, as long as *m* should be a whole number, and thus *2m* an even number.

COROLLARY 2

335. Behold therefore equations, noteworthy on account of simplicity with three term constant terms only, which admit integration for an infinite number of cases. Moreover the integral can be shown for any case from § 328, but only if there in place of x and y there may be written t and u.

CASE 2

336. Let $\pi':x = ax^{\mu}$ and $\varphi':y = b$;then there will be

$$\pi:x = \tfrac{1}{\mu+1}ax^{\mu+1} \text{ and } \varphi:y = by,$$

then truly $\pi'':x = \mu ax^{\mu-1}$ and $\varphi'':y = 0$. From which the first form arises

$$\tfrac{1}{bb}\left(\tfrac{ddz}{dy^2}\right) - \tfrac{1}{aax^{2\mu}}\left(\tfrac{ddz}{dx^2}\right) + \tfrac{\mu-2m\mu-2m}{aax^{2\mu+1}}\left(\tfrac{dz}{dx}\right) - \tfrac{n(\mu+1)^2}{aax^{2\mu+2}}z = 0,$$

which is reduced to the above resolved form on putting

$$t = \tfrac{1}{\mu+1}ax^{\mu+1} + by \quad \text{and} \quad u = \tfrac{1}{\mu+1}ax^{\mu+1} - by$$

Now the latter form will become

$$\tfrac{1}{bb}\left(\tfrac{ddz}{dy^2}\right) - \tfrac{1}{aax^{2\mu}}\left(\tfrac{ddz}{dx^2}\right) + \tfrac{2m}{bby}\left(\tfrac{dz}{dy}\right) + \tfrac{\mu}{aax^{2\mu+1}}\left(\tfrac{dz}{dx}\right) + \tfrac{n}{bbyy}z = 0,$$

the reduction of which is completed on putting

$$t = \tfrac{1}{\mu+1}ax^{\mu+1} + by \quad \text{and} \quad u = by - \tfrac{1}{\mu+1}ax^{\mu+1}.$$

And both these equations admit integration, as often as there should be

$$n = (m+i)(m-i-1).$$

COROLLARY 1

337. From the first form a most noteworthy case arises, if there is taken $m=\frac{\mu}{2\mu+2}$ and $n=0$; then indeed there will be

$$\frac{aa}{bb}x^{2\mu}\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right),$$

which is integrable, as often as $\frac{\mu}{2\mu+2}$ should be a whole number *m*, either positive or negative.

COROLLARY 2

338. Or since there may be $m=\frac{-\mu}{2\mu-1}$, this equation

$$\frac{aa}{bb}x^{\frac{-4m}{2m-1}}\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right) \quad\text{or}\quad \left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right)$$

will be integrable, as often as *m* should be a whole number either positive or negative ; moreover the reduction will be made on putting

$$t=-(2m-1)ax^{\frac{-1}{2m-1}}+by \quad\text{and}\quad u=-(2m-1)ax^{\frac{-1}{2m-1}}-by$$

CASE 3

339. Let there be $\pi':x=ax^{\mu}$ and $\varphi':y=by^{\nu}$; there will be

$$\pi:x=\frac{1}{\mu+1}ax^{\mu+1} \quad\text{and}\quad \varphi:y=\frac{1}{\nu+1}by^{\nu+1},$$

then now there will be $\pi'':x=\mu ax^{\mu-1}$ and $\varphi'':y=\nu by^{\nu-1}$. Hence the first form comes about

$$\frac{1}{bby^{2\nu}}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right)-\frac{\nu}{bby^{2\nu+1}}\left(\frac{dz}{dy}\right)+\frac{\mu-2m\mu-2m}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right)-\frac{n(\mu+1)^2}{aax^{2\mu+2}}z=0,$$

which is reduced on putting

$$t=\frac{1}{\mu+1}ax^{\mu+1}+\frac{1}{\nu+1}by^{\nu+1} \quad\text{and}\quad u=\frac{1}{\mu+1}ax^{\mu+1}-\frac{1}{\nu+1}by^{\nu+1}$$

Now the latter form arises

$$\frac{1}{bby^{2\nu}}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right)+\frac{2m\nu+2m-\nu}{bby^{2\nu+1}}\left(\frac{dz}{dy}\right)+\frac{\mu}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right)+\frac{n(\nu+1)}{bby^{2\nu+2}}z=0.$$

the reduction of which shall be made from this substitution

$$t=\tfrac{1}{\mu+1}ax^{\mu+1}+\tfrac{1}{\nu+1}by^{\nu+1}\quad\text{and}\quad u=\tfrac{-1}{\mu+1}ax^{\mu+1}+\tfrac{1}{\nu+1}by^{\nu+1}$$

Or since here only the ration between a and b is present in the computation, there can be put for the former

$$t=\tfrac{1}{2}x^{\mu+1}+\frac{(\mu+1)b}{2(\nu+1)a}y^{\nu+1}\quad\text{and}\quad u=\tfrac{1}{2}ax^{\mu+1}-\frac{(\mu+1)b}{2(\nu+1)a}y^{\nu+1},$$

so that there becomes $t+u=x^{\mu+1}$, from which the expression of the integral becomes simpler.

COROLLARY 1

340. If $\mu=\frac{-2m}{2m-1}$ is put into the first form, that is diminished by a single term and there becomes

$$\frac{1}{bby^{2\nu}}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aa}x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right)-\frac{\nu}{bby^{2\nu+1}}\left(\frac{dz}{dy}\right)-\frac{n}{(2m-1)^2aa}x^{\frac{2m}{2m-1}}z=0.$$

There is put in place $a=b$ and there is taken also $\nu=\frac{-2m}{2m-1}$, so that there arises

$$y^{\frac{4m}{2m-1}}\left(\frac{ddz}{dy^2}\right)-x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right)+\frac{2m}{2m-1}y^{\frac{2m+1}{2m-1}}\left(\frac{dz}{dy}\right)-\frac{n}{(2m-1)^2}x^{\frac{2}{2m-1}}z=0.$$

COROLLARY 2

341. Again there is taken in the first form $\nu=\mu$ and there becomes

$$\mu-2m\mu-2m=-\mu\quad\text{or }\nu=\frac{\mu}{\mu+1},$$

so that there appears

$$\frac{1}{bby^{2\mu}}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right)-\frac{\mu}{bby^{2\mu+1}}\left(\frac{dz}{dy}\right)-\frac{\mu}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right)-\frac{n(\mu+1)^2}{aax^{2\mu+2}}z=0,$$

which integral arises, as often as there should be

$$n=-\frac{(\mu+(\mu+1)i)((\mu+1)i+1)}{(\mu+1)^2}\ \text{or}\ n=-\left(i+\frac{\mu}{\mu+1}\right)\left(i+\frac{1}{\mu+1}\right).$$

SCHOLIUM

342. Hence therefore the most plentiful supply of well ordered equations supplies our needs, which can be integrated with the help of the method treated here. And here in the first place two cases are observed, of which one

$$\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right)$$

has been found for the determining the motion of strings provided with unequal thickness [see E287 and E442, to be found in Vol. 8 & 9 of Series II, *Opera Omnia*], moreover the other satisfied by this equation

$$\frac{aa}{bb}\left(\frac{ddz}{dy^2}\right)-\left(\frac{ddz}{dx^2}\right)-\frac{2m}{x}\left(\frac{dz}{dx}\right)=0$$

[§ 334] thus is remarkable, since such a formula is come upon in the analysis put in place for the propagation of sound. [see E306, E307, and E319, to be found in Vol. 1 of Series III, & Vol. 23 of Series I of the *Opera Omnia*] Therefore these two equations deserve merit before the rest, so that we may present the integrals for these integrable cases.

PROBLEM 54

343. *With the proposed cases for this differential equation*

$$\frac{aa}{bb}\left(\frac{ddz}{dy^2}\right)-\left(\frac{ddz}{dx^2}\right)-\frac{2m}{x}\left(\frac{dz}{dx}\right)=0,$$

for which m is either a positive or negative integer, the show the complete integral of this.

SOLUTION

With the substitution made $t=\frac{1}{2}x+\frac{b}{2a}y$ and $u=\frac{1}{2}x-\frac{b}{2a}y$, our equation adopts this form

$$\left(\frac{ddz}{dtdu}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)=0.$$

Therefore since there shall be $t+u=x$, if we put

$$\mathfrak{A} = f:\tfrac{ax+by}{2a} + F:\tfrac{ax-by}{2a},$$
$$\mathfrak{B} = x\left(f':\tfrac{ax+by}{2a} + F':\tfrac{ax-by}{2a}\right),$$
$$\mathfrak{C} = x^2\left(f'':\tfrac{ax+by}{2a} + F'':\tfrac{ax-by}{2a}\right),$$
$$\mathfrak{D} = x^3\left(f''':\tfrac{ax+by}{2a} + F''':\tfrac{ax-by}{2a}\right),$$
$$\mathfrak{E} = x^4\left(f^{IV}:\tfrac{ax+by}{2a} + F^{IV}:\tfrac{ax-by}{2a}\right),$$
$$\mathfrak{E} = x^5\left(f^{V}:\tfrac{ax+by}{2a} + F^{V}:\tfrac{ax-by}{2a}\right),$$
etc.,

the integrable cases thus will be had themselves [§ 328], in the first place the negative:

if $m=0$, $z = \mathfrak{A}$,
if $m=-1$, $z = \mathfrak{A} - \frac{1}{2}\mathfrak{B}$
if $m=-2$, $z = \mathfrak{A} - \frac{2}{4}\mathfrak{B} + \frac{1}{4\cdot 3}\mathfrak{C}$,
if $m=-3$, $z = \mathfrak{A} - \frac{3}{6}\mathfrak{B} + \frac{3}{6\cdot 5}\mathfrak{C} - \frac{1}{6\cdot 5\cdot 4}\mathfrak{D}$,
if $m=-4$, $z = \mathfrak{A} - \frac{4}{8}\mathfrak{B} + \frac{6}{8\cdot 7}\mathfrak{C} - \frac{4}{8\cdot 7\cdot 6}\mathfrak{D} + \frac{1}{8\cdot 7\cdot 6\cdot 5}\mathfrak{E}$,
if $m=-5$, $z = \mathfrak{A} - \frac{5}{10}\mathfrak{B} + \frac{10}{10\cdot 9}\mathfrak{C} - \frac{10}{10\cdot 9\cdot 8}\mathfrak{D} + \frac{5}{10\cdot 9\cdot 8\cdot 7}\mathfrak{E} - \frac{1}{10\cdot 9\cdot 8\cdot 7\cdot 6}\mathfrak{F}$,
etc.,

then indeed for the positive values of *m*:

si $m=1$, $xz = \mathfrak{A}$,
si $m=2$, $x^3 z = \mathfrak{A} - \frac{1}{2}\mathfrak{B}$
si $m=3$, $x^5 z = \mathfrak{A} - \frac{2}{4}\mathfrak{B} + \frac{1}{4\cdot 3}\mathfrak{C}$,
si $m=4$, $x^7 z = \mathfrak{A} - \frac{3}{6}\mathfrak{B} + \frac{3}{6\cdot 5}\mathfrak{C} - \frac{1}{6\cdot 5\cdot 4}\mathfrak{D}$,
si $m=5$, $x^9 z = \mathfrak{A} - \frac{4}{8}\mathfrak{B} + \frac{6}{8\cdot 7}\mathfrak{C} - \frac{4}{8\cdot 7\cdot 6}\mathfrak{D} + \frac{1}{8\cdot 7\cdot 6\cdot 5}\mathfrak{E}$,
si $m=6$, $x^{11} z = \mathfrak{A} - \frac{5}{10}\mathfrak{B} + \frac{10}{10\cdot 9}\mathfrak{C} - \frac{10}{10\cdot 9\cdot 8}\mathfrak{D} + \frac{5}{10\cdot 9\cdot 8\cdot 7}\mathfrak{E} - \frac{1}{10\cdot 9\cdot 8\cdot 7\cdot 6}\mathfrak{F}$,
etc.,

Therefore the value *z* is equal to the expression in the case $m=-i$ in the case, likewise in the case $m=i+1$.the value is equal to $x^{2i+1}z$

SCHOLIUM

344. The values of *t* and *u* thus are to be assumed here, so that there becomes $t+u=x$, and it is required to use the same values also in the functions. For even if $f:\frac{ax+by}{2a}$ is a function also of $ax+by$, yet the functions thence derived by differentiation disagree. In as much as if we put

$$f:\frac{ax+by}{2a}=\varphi:(ax+by),$$

there will be on differentiating

$$\frac{adx+bdy}{2a}f':\frac{ax+by}{2a}=(adx+bdy)\varphi':(ax+by),$$

from which there will be

$$f':\frac{ax+by}{2a}=2a\varphi':(ax+by)$$

therefore nor are these differential functions equal, even if the main parts assumed shall be equal. In a like manner there will be

$$f'':\frac{ax+by}{2a}=4aa\varphi'':(ax+by)$$

and

$$f''':\frac{ax+by}{2a}=8a^3\varphi''':(ax+by)$$

and thus henceforth.

PROBLEM 55

345. *With the proposed differential equation cases*

$$\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right),$$

in which m is either a positive or negative integer, to show the complete integral m.

SOLUTION

With the new variables *t* and *u* introduced, thus so that there shall be [§ 338]

$$t=\frac{1}{2}x^{\frac{-1}{2m-1}}-\frac{b}{2(2m-1)a}y \quad \text{and} \quad u=\frac{1}{2}x^{\frac{-1}{2m-1}}+\frac{b}{2(2m-1)a}y,$$

our equation adopts this form

$$\left(\frac{ddz}{dtdu}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)=0,$$

where there is

$$t+u=x^{\frac{-1}{2m-1}}$$

Therefore on putting

$$\mathfrak{A}=f:t+F:u,\quad \mathfrak{B}=x^{\frac{-1}{2m-1}}\left(f':t+F':u\right),$$
$$\mathfrak{C}=x^{\frac{-2}{2m-1}}\left(f'':t+F'':u\right),\ \mathfrak{D}=x^{\frac{-3}{2m-1}}\left(f''':t+F''':u\right),$$
$$\mathfrak{E}=x^{\frac{-4}{2m-1}}\left(f^{IV}:t+F^{IV}:u\right),\quad \mathfrak{E}=x^{\frac{-5}{2m-1}}\left(f^{V}:t+F^{V}:u\right),$$
etc.,

we may run through the first case, in which *m* decreases from nothing through the negative numbers.

I. If $m=0$, the integral of the equation

$$\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}\left(\frac{ddz}{dx^2}\right),$$

will be

$$z=f:\left(\tfrac{1}{2}x+\tfrac{b}{2a}y\right)+F:\left(\tfrac{1}{2}x-\tfrac{b}{2a}y\right).$$

II. If $m=-1$, on account of

$$t=\tfrac{1}{2}x^{\frac{1}{3}}-\tfrac{b}{6a}y \quad\text{and}\quad u=\tfrac{1}{2}x^{\frac{1}{3}}-\tfrac{b}{6a}y$$

the integral of the equation

$$\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{4}{3}}\left(\frac{ddz}{dx^2}\right)$$

will be

$$z=f:t+F:u-\tfrac{1}{2}x^{\frac{1}{3}}\left(f':t+F':u\right).$$

III. If $m=-2$, on account of

$$t=\tfrac{1}{2}x^{\frac{1}{5}}+\tfrac{b}{10a}y \quad\text{and}\quad u=\tfrac{1}{2}x^{\frac{1}{5}}-\tfrac{b}{10a}y$$

the integral of the equation

$$\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{8}{5}}\left(\frac{ddz}{dx^2}\right)$$

will be

$$z=f:t+F:u-\tfrac{2}{4}x^{\frac{1}{5}}\left(f':t+F':u\right)+\tfrac{1}{4\cdot 3}x^{\frac{2}{5}}\left(f'':t+F'':u\right)..$$

IV. If $m=-3$, on account of

$$t=\tfrac{1}{2}x^{\frac{1}{7}}+\tfrac{b}{14a}y \quad \text{and} \quad u=\tfrac{1}{2}x^{\frac{1}{7}}-\tfrac{b}{14a}y$$

the integral of the equation

$$\left(\tfrac{ddz}{dy^2}\right)=\tfrac{bb}{aa}x^{\frac{12}{7}}\left(\tfrac{ddz}{dx^2}\right)$$

will be

$$z=f:t+F:u-\tfrac{3}{6}x^{\frac{1}{7}}\left(f':t+F':u\right)+\tfrac{3}{6\cdot5}x^{\frac{2}{7}}\left(f'':t+F'':u\right)-\tfrac{1}{6\cdot5\cdot4}x^{\frac{3}{7}}\left(f''':t+F''':u\right).$$

V. If $m=-4$, on account of

$$t=\tfrac{1}{2}x^{\frac{1}{9}}+\tfrac{b}{18a}y \quad \text{and} \quad u=\tfrac{1}{2}x^{\frac{1}{9}}-\tfrac{b}{18a}y$$

the integral of the equation

$$\left(\tfrac{ddz}{dy^2}\right)=\tfrac{bb}{aa}x^{\frac{16}{9}}\left(\tfrac{ddz}{dx^2}\right)$$

will be

$$z=f:t+F:u-\tfrac{4}{8}x^{\frac{1}{9}}\left(f':t+F':u\right)+\tfrac{6}{8\cdot7}x^{\frac{2}{9}}\left(f'':t+F'':u\right)$$
$$-\tfrac{4}{8\cdot7\cdot6}x^{\frac{3}{9}}\left(f''':t+F''':u\right)+\tfrac{1}{8\cdot7\cdot6\cdot5}x^{\frac{4}{9}}\left(f^{IV}:t+F^{IV}:u\right)$$

and thus henceforth.

Now for the other case, where *m* has positive values, the integrals can be expressed in the following manner :

I. If there shall be $m=1$ or

$$\left(\tfrac{ddz}{dy^2}\right)=\tfrac{bb}{aa}x^{4}\left(\tfrac{ddz}{dx^2}\right)$$

on account of

$$t=\tfrac{1}{2}x^{-1}-\tfrac{b}{2a}y \quad \text{and} \quad u=\tfrac{1}{2}x^{-1}+\tfrac{b}{2a}y$$

the integral will be

$$x^{-1}z=f:t+F:u \quad \text{or} \quad z=x\left(f:t+F:u\right).$$

II. If there shall be $m=2$ or

$$\left(\tfrac{ddz}{dy^2}\right)=\tfrac{bb}{aa}x^{\frac{8}{3}}\left(\tfrac{ddz}{dx^2}\right),$$

on account of

$$t=\tfrac{1}{2}x^{-\frac{1}{3}}-\tfrac{b}{6a}y \quad \text{and} \quad u=\tfrac{1}{2}x^{-\frac{1}{3}}+\tfrac{b}{6a}y$$

the integral will be

$$z = x\left(f:t + F:u \right) - \tfrac{1}{2} x^{\frac{2}{3}} \left(f':t + F':u \right).$$

III. If there shall be $m = 3$ or

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{12}{5}} \left(\frac{ddz}{dx^2}\right)$$

on account of

$$t = \tfrac{1}{2} x^{-\frac{1}{5}} - \tfrac{b}{10a} y \quad \text{and} \quad u = \tfrac{1}{2} x^{-\frac{1}{5}} + \tfrac{b}{10a} y$$

the integral will be

$$z = x\left(f:t + F:u \right) - \tfrac{2}{4} x^{\frac{4}{5}} \left(f':t + F':u \right) + \tfrac{1}{4\cdot 3} x^{\frac{3}{5}} \left(f'':t + F'':u \right).$$

IV. If there shall be $m = 4$ or

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{16}{7}} \left(\frac{ddz}{dx^2}\right)$$

on account of

$$t = \tfrac{1}{2} x^{-\frac{1}{7}} - \tfrac{b}{14a} y \quad \text{and} \quad u = \tfrac{1}{2} x^{-\frac{1}{7}} + \tfrac{b}{14a} y$$

the integral will by

$$z = x\left(f:t + F:u \right) - \tfrac{3}{6} x^{\frac{6}{7}} \left(f':t + F':u \right) + \tfrac{3}{6\cdot 5} x^{\frac{5}{7}} \left(f'':t + F'':u \right) - \tfrac{1}{6\cdot 5\cdot 4} x^{\frac{4}{7}} \left(f''':t + F''':u \right).$$

V. If there shall be $m = 5$ or

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{20}{9}} \left(\frac{ddz}{dx^2}\right)$$

on account of

$$t = \tfrac{1}{2} x^{-\frac{1}{9}} - \tfrac{b}{18a} y \quad \text{and} \quad u = \tfrac{1}{2} x^{-\frac{1}{9}} + \tfrac{b}{18a} y$$

the integral will be

$$\begin{aligned} z = {} & x\left(f:t + F:u \right) - \tfrac{4}{8} x^{\frac{8}{9}} \left(f':t + F':u \right) + \tfrac{6}{8\cdot 7} x^{\frac{7}{9}} \left(f'':t + F'':u \right) \\ & - \tfrac{4}{8\cdot 7\cdot 6} x^{\frac{6}{9}} \left(f''':t + F''':u \right) + \tfrac{1}{8\cdot 7\cdot 6\cdot 5} x^{\frac{5}{9}} \left(f^{IV}:t + F^{IV}:u \right) \quad \text{etc.,} \end{aligned}$$

from which the law, by which it is allowed to continue these expressions further, is obvious through these.

SCHOLIUM 1

346. These cases of integrability agree with these which are observed in the equation said to be Riccatian; clearly we know that this equation

$$dy + yydx = ax^{\frac{-4m}{2m-1}}dx$$

to be integrable, as often as *m* is a positive or negative integer [see §§ 436 in vol. I and § 943 in vol. II.]. But this equation is not connected our equation in a trivial manner, which can be shown thus.

With the proposed general form

$$\left(\frac{ddz}{dy^2}\right) = X\left(\frac{ddz}{dx^2}\right)$$

for finding particular integrals there is put in place $z = e^{\alpha y}v$, so that *v* shall be function of *x* only; there will be

$$\left(\frac{dz}{dx}\right) = e^{\alpha y}\cdot\frac{dv}{dx} \quad \text{and} \quad \left(\frac{ddz}{dx^2}\right) = e^{\alpha y}\cdot\frac{ddv}{dx^2};$$

then truly there shall be $\left(\frac{ddz}{dy^2}\right) = \alpha\alpha e^{\alpha y}v$, from which this equation will emerge

$$\alpha\alpha v = \frac{Xddv}{dx^2};$$

in which if again there is put $v = e^{\int pdx}$, there arises

$$\frac{\alpha\alpha dx}{X} = dp + ppdx,$$

and if $X = Ax^{\frac{4m}{2m-1}}$ so that in our case, this equation arises

$$dp + ppdx = \alpha x^{\frac{-4m}{2m-1}}dx.$$

Therefore it is not easily considered how each equation admits to the same cases to be allowed to be integrated.

Yet meanwhile it comes to mind to be worthy of note, that the case $m = \infty$, which in the Riccatian form becomes easier, likewise in our equation by no means is permitted to be integrated. Certainly this equation may be considered

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}xx\left(\frac{ddz}{dx^2}\right),$$

the reduction of which does not succeed by the method used above in § 330. For on account of

$$Q=\frac{bx}{a},\ R=0,\ S=0 \text{ and } T=0,$$

there is put for new variables

$$t=\int p\left(dx+\frac{bxdy}{a}\right) \text{ and } u=\int q\left(dx-\frac{bxdy}{a}\right),$$

from which on account of $M=\frac{bbx}{aa}=N$, this equation arises

$$\left(\frac{ddz}{dtdu}\right)-\frac{1}{4qx}\left(\frac{dz}{dt}\right)-\frac{1}{4px}\left(\frac{dz}{du}\right)=0,$$

from which on taking

$$p=\frac{1}{x} \text{ and } q=\frac{1}{x},$$

so that there shall be

$$t=lx+\frac{by}{a} \text{ and } u=lx-\frac{by}{a},$$

turns into

$$\left(\frac{ddz}{dtdu}\right)-\frac{1}{4}\left(\frac{dz}{dt}\right)-\frac{1}{4}\left(\frac{dz}{du}\right)=0,$$

the integral of which is not transparent.

SCHOLIUM 2

347. But an infinite number of particular integrals of the equation $\left(\frac{ddz}{dy^2}\right)=xx\left(\frac{ddz}{dx^2}\right)$ can be shown satisfied by this form $z=Ax^{\lambda}e^{\mu y}$. For since hence there shall be

$$\left(\frac{dz}{dy}\right)=\mu Ax^{\lambda}e^{\mu y} \quad\text{and}\quad \left(\frac{dz}{dx}\right)=\lambda Ax^{\lambda-1}e^{\mu y},$$

there will be

$$\mu\mu Ax^{\lambda}e^{\mu y}=\lambda(\lambda-1)Ax^{\lambda}e^{\mu y}$$

and thus $\mu=\sqrt{\lambda(\lambda-1)}$, from which with some number assumed for λ two values arise for μ, thus so that there may be considered

$$z=Ax^{\lambda}e^{y\sqrt{\lambda(\lambda-1)}}+Bx^{\lambda}e^{-y\sqrt{\lambda(\lambda-1)}},$$

and the number of members of this kind by changing λ is able to be multiplied indefinitely.

Yet meanwhile besides, these individual members can be returned more generally. For on putting $z = Ax^{\lambda}e^{\mu y}v$ we may see, or with v by necessity must be considered constant; hence moreover there arises

$$\left(\frac{dz}{dy}\right) = \mu x^{\lambda}e^{\mu y}v + x^{\lambda}e^{\mu y}\left(\frac{dv}{dy}\right) \quad \text{and} \quad \left(\frac{dz}{dx}\right) = \lambda x^{\lambda-1}e^{\mu y}v + x^{\lambda}e^{\mu y}\left(\frac{dv}{dx}\right)$$

and thus our equation on division by $x^{\lambda}e^{\mu y}$ gives

$$\mu\mu v + 2\mu\left(\frac{dv}{dy}\right) + \left(\frac{ddv}{dy^2}\right) = \lambda(\lambda-1)v + 2x\lambda\left(\frac{dv}{dx}\right) + xx\left(\frac{ddv}{dx^2}\right).$$

Putting in place as before $\mu\mu = \lambda(\lambda-1)$ and there shall be $v = \alpha lx + \beta y$; there will be

$$2\beta\mu = 2\alpha\lambda - \alpha \quad \text{or} \quad \frac{\alpha}{\beta} = \frac{2\mu}{2\lambda-1} = \frac{2\sqrt{\lambda(\lambda-1)}}{2\lambda-1},$$

from which the form of each member arising from the number λ will be

$$z = x^{\lambda}\left\{\begin{aligned} &e^{y\sqrt{\lambda(\lambda-1)}}\left(A + \frac{2\sqrt{\lambda(\lambda-1)}}{\mathfrak{A}}lx + \frac{2\lambda-1}{\mathfrak{A}}y\right) \\ &+e^{-y\sqrt{\lambda(\lambda-1)}}\left(B - \frac{2\sqrt{\lambda(\lambda-1)}}{\mathfrak{B}}lx + \frac{2\lambda-1}{\mathfrak{B}}y\right)\end{aligned}\right\}.$$

Therefore however not only the exponent λ, but also the quantities $A, \mathfrak{A}, B, \mathfrak{B}$ may be varied, an boundless members of this kind can be formed, which all taken together are agreed to give the complete value of the function z.

But also imaginary numbers can be assumed for λ; for on putting $\lambda = a + b\sqrt{-1}$ there becomes $\mu = p + q\sqrt{-1}$ with

$$pp - qq = aa - a - bb \quad \text{and} \quad pp + qq = \sqrt{(aa+bb)(aa-2a+1+bb)},$$

being present, then there is now

$$x^{\lambda} = x^{a}\left(\cos.blx + \sqrt{-1}\cdot\sin.blx\right) \text{ and } x^{\mu y} = x^{py}\left(\cos.qy + \sqrt{-1}\cdot\sin.qy\right),$$

from which the real form is deduced

$$z = x^a e^{py} \left\{ \begin{array}{l} A\cos.(blx+qy) + B\big(2plx+(2a-1)y\big)\cos.(blx+qy) \\ \qquad\qquad -\ B(2qlx+2by)\sin.(blx+qy) \\ +\mathfrak{A}\sin.(blx+qy) + \mathfrak{B}\big(2plx+(2a-1)y\big)\sin.(blx+qy) \\ \qquad\qquad +\mathfrak{B}(2qlx+2by)\cos.(blx+qy) \end{array} \right\}$$

where the quantities *a* and *b* can be taken as you please, from which *p* and *q* are defined at the same time.

But if we regard here the letters *b* and *q* as given, the two remaining *a* and *p* are determined from these, so that there shall be

$$2a-1 = q\sqrt{\left(\tfrac{1}{qq-bb} - 4\right)} \quad \text{and} \quad p = \tfrac{b}{2}\sqrt{\left(\tfrac{1}{qq-bb} - 4\right)};$$

therefore here by necessity there shall be $qq > bb$ and $qq < bb + \frac{1}{4}$, or *qq* must be contained between these strict limits *bb* and $bb + \frac{1}{4}$. There is put in place

$$q = c \quad \text{and} \quad \sqrt{\left(\tfrac{1}{qq-bb} - 4\right)} = 2f,$$

so that there shall be

$$\tfrac{1}{qq-bb} = 4(1+ff) \quad \text{or} \quad cc - bb = \tfrac{1}{4(1+ff)}$$

and

$$2a-1 = 2cf \quad \text{and} \quad p = bf,$$

from which forms of the particular integrals there shall be

$$z = x^{cf+\frac{1}{2}} e^{bfy} \left\{ \begin{array}{l} A\cos.(blx+cy) + 2Bf(blx+cy)\cos.(blx+cy) \\ \qquad\qquad -\ 2B(clx+by)\sin.(blx+cy) \\ +\mathfrak{A}\sin.(blx+cy) + 2\mathfrak{B}f(blx+cy)\sin.(blx+cy) \\ \qquad\qquad +2\mathfrak{B}(clx+by)\cos.(blx+cy) \end{array} \right\}$$

which on putting for the sake of brevity the angle $blx + cy = \varphi$ is transformed into this

$$z = x^{cf+\frac{1}{2}} e^{bfy} \big(A\cos.(\varphi+\alpha) + Bf(blx+cy)\sin.(\varphi+\beta) + B(clx+by)\cos.(\varphi+\beta) \big),$$

where the quantities $b, c, A, B, \alpha, \beta$ depend on our choice.

SCHOLIUM 3

348. Therefore the resolution of the equation

$$\left(\frac{ddz}{dy^2}\right) = xx\left(\frac{ddz}{dx^2}\right)$$

can be put in place thus, so that there is produced

$$z = x^{\lambda}e^{\mu y}\left(mlx + ny\right),$$

from which there becomes

$$\left(\frac{dz}{dx}\right) = \lambda x^{\lambda-1}e^{\mu y}\left(mlx + ny\right) + mx^{\lambda-1}e^{\mu y}$$

and

$$\left(\frac{dz}{dy}\right) = \mu x^{\lambda}e^{\mu y}\left(mlx + ny\right) + nx^{\lambda}e^{\mu y}$$

and hence on further differentiation

$$\left(\frac{ddz}{dx^2}\right) = x^{\lambda-2}e^{\mu y}\left(m\left(2\lambda-1\right) + \lambda\left(\lambda-1\right)mlx + \lambda\left(\lambda-1\right)ny\right)$$

and

$$\left(\frac{ddz}{dy^2}\right) = x^{\lambda}e^{\mu y}\left(2\mu n + \mu\mu mlx + \mu\mu ny\right).$$

From which it is deduced in the first place that $\mu = \sqrt{\lambda\left(\lambda-1\right)}$, then $2n\sqrt{\lambda\left(\lambda-1\right)} = m\left(2\lambda-1\right)$, so that there shall be $\frac{m}{n} = \frac{2\sqrt{\lambda(\lambda-1)}}{(2\lambda-1)}$, and thus the same integration emerges, just as we have given before.

CAPUT IV

ALIA METHODUS PECULIARIS HUIUSMODI AEQUATIONES INTEGRANDI

PROBLEMA 52

322. *Si aequatio proposita hanc habuerit formam*

$$(x+y)^2\left(\tfrac{ddz}{dxdy}\right)+m(x+y)\left(\tfrac{dz}{dx}\right)+m(x+y)\left(\tfrac{dz}{dy}\right)+nz=0,$$

eius integrale completum investigare.

SOLUTIO

Cum hic binae variabiles x et y aequaliter insint, ponatur primo

$$z=A(x+y)^{\lambda}\, f{:}x+B(x+y)^{\lambda+1}\, f'{:}x+C(x+y)^{\lambda+2}\, f''{:}x+D(x+y)^{\lambda+3}\, f'''{:}x+\text{etc.},$$

ubi pro faciliori substitutione notetur posito $v=(x+y)^{\mu}F{:}x$ fore

$$\left(\tfrac{dv}{dx}\right)=\mu(x+y)^{\mu-1}F{:}x+(x+y)^{\mu-1}F'{:}x,$$

$$\left(\tfrac{dv}{dy}\right)=\mu(x+y)^{\mu-1}F{:}x$$

et

$$\left(\tfrac{ddv}{dxdy}\right)=\mu(\mu-1)(x+y)^{\mu-2}F{:}x+\mu(x+y)^{\mu-1}F'{:}x.$$

Facta ergo substitutione obtinebimus hanc aequationem

$$\begin{array}{llll}
0= nA(x+y)^{\lambda}\, f{:}x+ & nB(x+y)^{\lambda+1}\, f'{:}x+ & nC(x+y)^{\lambda+2}\, f''{:}x+\text{etc.} \\
 & + \quad mA & + \quad mB \\
+\, 2m\lambda A & +2m(\lambda+1)B & +2m(\lambda+2)C \\
 & +\, \lambda A & +(\lambda+1)B \\
+\lambda(\lambda-1)A & +\, (\lambda+1)\lambda B & +(\lambda+2)(\lambda+1)C
\end{array}$$

ubi totum negotium ad coefficientium *A, B, C, D* etc. determinationem revocatur; facile autem erat praevidere forma superiori assumta potestates ipsius $(x+y)$ in singulis membris pares esse prodituras. Fieri igitur necesse est

$$\begin{aligned}
&n+2m\lambda+\lambda\lambda-\lambda=0,\\
&(n+2m\lambda+2m+\lambda\lambda+\lambda)B+(m+\lambda)A=0,\\
&(n+2m\lambda+4m+\lambda\lambda+3\lambda+2)C+(m+\lambda+1)B=0,\\
&(n+2m\lambda+6m+\lambda\lambda+5\lambda+6)D+(m+\lambda+2)C=0\\
&\qquad\text{etc.},
\end{aligned}$$

quae determinationes ope primae $n+2m\lambda+\lambda\lambda-\lambda=0$ ita commodius exprimuntur

$$\begin{aligned}
B&=-\tfrac{m+\lambda}{2m+2\lambda}A, & F&=-\tfrac{m+\lambda+4}{5(2m+2\lambda+4)}E,\\
C&=-\tfrac{m+\lambda+1}{2(2m+2\lambda+1)}B, & G&=-\tfrac{m+\lambda+5}{6(2m+2\lambda+5)}F,\\
D&=-\tfrac{m+\lambda+2}{3(2m+2\lambda+2)}C, & H&=-\tfrac{m+\lambda+6}{7(2m+2\lambda+6)}G,\\
E&=-\tfrac{m+\lambda+3}{4(2m+2\lambda+3)}D, & &\text{etc.},
\end{aligned}$$

unde lex progressionis est manifesta.

At pro exponente λ duplicem eruimus valorem

$$\lambda=\tfrac{1}{2}-m\pm\sqrt{\left(\tfrac{1}{4}-m-n+mm\right)},$$

quorum utrumque aeque pro λ accipere licet. Hic autem praecipue notandi sunt casus, quibus series assumta abrumpitur, quod fit, quoties $m+\lambda+i=0$ denotante *i* numerum quemcunque integrum positivum cyphra non exclusa.
Hoc ergo evenit, quoties fuerit

$$\tfrac{1}{2}+i\pm\sqrt{\left(\tfrac{1}{4}-m-n+mm\right)}=0,$$

id quod fieri nequit, nisi $\frac{1}{4}-m-n+mm$ fuerit quadratum. Inventa autem huiusmodi serie sive finita sive in infinitum excurrente alia similis pro functionibus ipsius *y* reperitur, unde valor ipsius *z* ita reperietur expressus

$$\begin{aligned}
z=&A(x+y)^{\lambda}(f:x+F:y)+B(x+y)^{\lambda+1}(f':x+F':y)\\
&+C(x+y)^{\lambda+2}(f'':x+F'':y)+D(x+y)^{\lambda+3}(f''':x+F''':y)\\
&+E(x+y)^{\lambda+4}(f^{IV}:x+F^{IV}:y)+F(x+y)^{\lambda+5}(f^{V}:x+F^{V}:y)\\
&\qquad\text{etc.};
\end{aligned}$$

ubi cum binae functiones arbitrariae adsint, id certum est signum hanc formam esse integrale completum aequationis propositae.

COROLLARIUM 1

323. Si fuerit $\lambda = -m$, hoc est $n - mm + m = 0$ seu $n = mm - m$, integrale ex unico membro constabit ob $B = 0$ eritque integrale

$$z = A(x+y)^{-m}(f:x + F:y).$$

COROLLARIUM 2

324. Integrale autem duo membra continebit, si

$$\lambda = -m-1 \text{ vel } n = mm - m - 2 = (m+1)(m-2);$$

tum erit $B = -\frac{1}{2}A$ et integrale erit

$$z = (x+y)^{-m-1}(f:x + F:y) - \tfrac{1}{2}(x+y)^{-m}(f':x + F':y).$$

COROLLARIUM 3

325. Integrale tribus terminis constabit, si

$$\lambda = -m-2 \text{ vel } n = (m+2)(m-3);$$

tum erit

$$B = -\tfrac{1}{2}A \text{ et } C = -\tfrac{1}{6}B = +\tfrac{1}{12}A,$$

integrale vero

$$z = (x+y)^{-m-2}(f:x + F:y) - \tfrac{1}{2}(x+y)^{-m-1}(f':x + F':y) + \tfrac{1}{12}(x+y)^{-m}(f'':x + F'':y).$$

COROLLARIUM 4

326. Ex quatuor autem membris integrale constabit, si fuerit

$$\lambda = -m-3 \text{ seu } n = (m+3)(m-4);$$

tum autem erit

$$B = -\tfrac{1}{2}A,\ C = -\tfrac{1}{5}B = +\tfrac{1}{10}A,\ D = -\tfrac{1}{12}C = -\tfrac{1}{120}A$$

et integrale

$$z = (x+y)^{-m-3}(f:x+F:y) - \tfrac{1}{2}(x+y)^{-m-2}(f':x+F':y)$$
$$+\tfrac{1}{10}(x+y)^{-m-1}(f'':x+F'':y) - \tfrac{1}{120}(x+y)^{-m}(f''':x+F''':y).$$

SCHOLION

327. Quodsi in genere ponamus $\lambda + m = -i$, erit $n = (m+i)(m-i-1)$, tum vero

$$B = -\tfrac{1}{2}A,\ \ C = -\tfrac{i-1}{2(2i-1)}B,\ \ D = -\tfrac{i-2}{3(2i-2)}C,\ \ E = -\tfrac{i-3}{4(2i-3)}D,$$

unde fit omnes ad primum reducendo

$$B = -\tfrac{1}{2}A,\ \ C = \tfrac{i-1}{2\cdot 2(2i-1)}A,\ \ D = \tfrac{-(i-2)}{2\cdot 2\cdot 2\cdot 3(2i-1)}A,$$
$$E = \tfrac{+(i-2)(i-3)}{2\cdot 2\cdot 2\cdot 3\cdot 4(2i-1)(2i-3)}A,\quad F = \tfrac{-(i-3)(i-4)}{2^4\cdot 3\cdot 4\cdot 5(2i-1)(2i-3)}A \ \text{ etc.},$$

qui ita se habent:

	A	B	C	D	E	F	G
$i=1$	1	$-\frac{1}{2}$	0	0	0	0	0
$i=2$	1	$-\frac{1}{2}$	$\frac{1}{12}$	0	0	0	0
$i=3$	1	$-\frac{1}{2}$	$\frac{2}{20}$	$-\frac{1}{120}$	0	0	0
$i=4$	1	$-\frac{1}{2}$	$\frac{3}{28}$	$-\frac{2}{7\cdot 24}$	$\frac{2\cdot 1}{96\cdot 9\cdot 5}$	0	0
$i=5$	1	$-\frac{1}{2}$	$\frac{4}{36}$	$-\frac{3}{9\cdot 24}$	$\frac{3\cdot 2}{96\cdot 9\cdot 7}$	$-\frac{2\cdot 1}{960\cdot 9\cdot 7}$	0
$i=6$	1	$-\frac{1}{2}$	$\frac{5}{44}$	$-\frac{4}{11\cdot 24}$	$\frac{4\cdot 3}{96\cdot 11\cdot 9}$	$-\frac{3\cdot 2}{960\cdot 11\cdot 9}$	$\frac{3\cdot 2\cdot 1}{5760\cdot 11\cdot 9\cdot 7}$

Ita huius aequationis

$$\left(\frac{ddz}{dxdy}\right) + \frac{m}{x+y}\left(\frac{dz}{dx}\right) + \frac{m}{x+y}\left(\frac{dz}{dy}\right) + \frac{(m+i)(m-i-1)}{(x+y)^2}z = 0$$

integrale completum erit

$$z=(x+y)^{-m-i}(f:x+F:y)-\frac{i}{2i}(x+y)^{-m-i+1}(f':x+F':y)$$
$$+\frac{i(i-1)}{2i\cdot 2(2i-1)}(x+y)^{-m-i+2}(f'':x+F'':y)-\frac{i(i-1)(i-2)}{2i\cdot 2(2i-1)\cdot 3(2i-2)}(x+y)^{-m-i+3}(f''':x+F''':y)$$
$$+\frac{i(i-1)(i-2)(i-3)}{2i\cdot 2(2i-1)\cdot 3(2i-2)\cdot 4(2i-3)}(x+y)^{-m-i+4}(f^{IV}:x+F^{IV}:y)$$
$$-\frac{i(i-1)(i-2)(i-3)(i-4)}{2i\cdot 2(2i-1)\cdot 3(2i-2)\cdot 4(2i-3)\cdot 5(2i-4)}(x+y)^{-m-i+5}(f^{V}:x+F^{V}:y)$$
$$\text{etc.,}$$

quae forma, quoties *i* fuerit numerus integer positivus, finito constat terminorum numero; secus autem in infinitum excurrit.

Imprimis autem ista integratio hoc habet singulare, quod non solum ipsas functiones arbitrarias *f*: *x* et F*: y* complectatur, sed etiam earum formulas differentiales.

EXEMPLUM

328. *Si occurrat ista aequatio*

$$\left(\frac{ddz}{dxdy}\right)+\frac{m}{x+y}\left(\frac{dz}{dx}\right)+\frac{m}{x+y}\left(\frac{dz}{dy}\right)=0,$$

definire casus, quibus eius integrale per formam finitam exhiberi potest.

Cum hic sit $n=(m+i)(m-i-1)=0$, sumendo pro *i* numeros integros positivos duo ordines habebuntur casuum, quibus integratio succedit, alter, quo est $m=-i$, alter, quo $m=i+1$, ita ut in genere integratio finita locum habeat, quoties *m* fuerit numerus integer sive positivus sive negativus.

Primo ergo si sit $m=-i$, erit

$$z=1(f:x+F:y)-\frac{i}{2i}(x+y)(f':x+F':y)$$
$$+\frac{1}{2}\cdot\frac{i(i-1)}{2i(2i-1)}(x+y)^{2}(f'':x+F'':y)-\frac{1}{6}\cdot\frac{i(i-1)(i-2)}{2i(2i-1)(2i-2)}(x+y)^{3}(f''':x+F''':y)$$
$$+\frac{1}{24}\cdot\frac{i(i-1)(i-2)(i-3)}{2i(2i-1)(2i-2)(2i-3)}(x+y)^{4}(f^{IV}:x+F^{IV}:y)$$
$$\text{etc.,}$$

Deinde si sit $m=i+1$, erit

$$\begin{aligned}&(x+y)^{2i+1}z=1(f:x+F:y)-\tfrac{i}{2i}(x+y)(f':x+F':y)\\&+\tfrac{1}{2}\cdot\tfrac{i(i-1)}{2i(2i-1)}(x+y)^2(f'':x+F'':y)-\tfrac{1}{6}\cdot\tfrac{i(i-1)(i-2)}{2i(2i-1)(2i-2)}(x+y)^3(f''':x+F''':y)\\&+\tfrac{1}{24}\cdot\tfrac{i(i-1)(i-2)(i-3)}{2i(2i-1)(2i-2)(2i-3)}(x+y)^4(f^{IV}:x+F^{IV}:y)\\&\text{etc.},\end{aligned}$$

Utrinque scilicet eadem habetur expressio, cui casu priori ipsa quantitas z, posteriori quantitas $(x+y)^{2i+1}z$ aequatur.
Ad singulos hos casus distinctius evolvendos ponamus

$$\begin{aligned}&A=(f:x+F:y),\\&B=(f:x+F:y)-\tfrac{1}{2}(x+y)(f':x+F':y),\\&C=(f:x+F:y)-\tfrac{2}{4}(x+y)(f':x+F':y)+\tfrac{1}{4\cdot3}(x+y)^2(f'':x+F'':y),\\&D=(f:x+F:y)-\tfrac{3}{6}(x+y)(f':x+F':y)+\tfrac{3}{6\cdot5}(x+y)^2(f'':x+F'':y)-\tfrac{1}{6\cdot5\cdot4}(x+y)^3(f''':x+F''':y)\\&\text{etc.}\end{aligned}$$

vel posito brevitatis gratia

$$\begin{aligned}&\mathfrak{A}=f:x+F:y,\\&\mathfrak{B}=(x+y)(f':x+F':y),\\&\mathfrak{C}=(x+y)^2(f'':x+F'':y),\\&\mathfrak{D}=(x+y)^3(f''':x+F''':y),\\&\mathfrak{E}=(x+y)^4(f^{IV}:x+F^{IV}:y)\\&\text{etc.}\end{aligned}$$

sit

$$\begin{aligned}&A=\mathfrak{A},\\&B=\mathfrak{A}-\tfrac{1}{2}\mathfrak{B},\\&C=\mathfrak{A}-\tfrac{2}{4}\mathfrak{B}+\tfrac{1}{4\cdot3}\mathfrak{C},\\&D=\mathfrak{A}-\tfrac{3}{6}\mathfrak{B}+\tfrac{3}{6\cdot5}\mathfrak{C}-\tfrac{1}{6\cdot5\cdot4}\mathfrak{D},\\&E=\mathfrak{A}-\tfrac{4}{8}\mathfrak{B}+\tfrac{6}{8\cdot7}\mathfrak{C}-\tfrac{4}{8\cdot7\cdot6}\mathfrak{D}+\tfrac{1}{8\cdot7\cdot6\cdot5}\mathfrak{E},\\&F=\mathfrak{A}-\tfrac{5}{10}\mathfrak{B}+\tfrac{10}{10\cdot9}\mathfrak{C}-\tfrac{10}{10\cdot9\cdot8}\mathfrak{D}+\tfrac{5}{10\cdot9\cdot8\cdot7}\mathfrak{E}-\tfrac{1}{10\cdot9\cdot8\cdot7\cdot6}\mathfrak{F},\\&G=\mathfrak{A}-\tfrac{6}{12}\mathfrak{B}+\tfrac{15}{12\cdot11}\mathfrak{C}-\tfrac{20}{12\cdot11\cdot10}\mathfrak{D}+\tfrac{15}{12\cdot11\cdot10\cdot9}\mathfrak{E}-\tfrac{6}{12\cdot11\cdot10\cdot9\cdot8}\mathfrak{F}+\tfrac{1}{12\cdot11\cdot10\cdot9\cdot8\cdot7}\mathfrak{G}\end{aligned}$$

etc.

Quibus valoribus inventis erit pro duplici ordine:

Si,	erit	Si,	erit
$m=0,$	$z=A$	$m=1,$	$(x+y)z=A$
$m=-1,$	$z=B$	$m=2,$	$(x+y)^3 z=B$
$m=-2,$	$z=C$	$m=3,$	$(x+y)^5 z=C$
$m=-3,$	$z=D$	$m=4,$	$(x+y)^7 z=D$
$m=-4,$	$z=E$	$m=5,$	$(x+y)^9 z=E$
$m=-5,$	$z=F$	$m=6,$	$(x+y)^{11} z=F$
$m=-6,$	$z=G$	$m=7,$	$(x+y)^{13} z=G$
etc.		etc.	

SCHOLION

329. Si pro i sumatur numerus negativus, expressio in infinitum excurrit. Sit enim $i=-k$ et ex formula prima erit $m=k$ ideoque

$$z=\mathfrak{A}-\tfrac{k}{2k}\mathfrak{B}+\tfrac{1}{2}\cdot\tfrac{k(k+1)}{2k(2k+1)}\mathfrak{C}-\tfrac{1}{6}\cdot\tfrac{k(k+1)(k+2)}{2k(2k+1)(2k+2)}\mathfrak{D}+\text{ etc. in infinitum.}$$

Pro eodem autem casu $m=k$ altera forma ob $i=k-1$ dat

$$(x+y)^{2k-1}z$$
$$=\mathfrak{A}-\tfrac{k-1}{2k-2}\mathfrak{B}+\tfrac{1}{2}\cdot\tfrac{(k-1)(k-2)}{(2k-2)(2k-3)}\mathfrak{C}-\tfrac{1}{6}\cdot\tfrac{(k-1)(k-2)(k-3)}{(2k-2)(2k-3)(2k-4)}\mathfrak{D}+\text{ etc.,}$$

quae autem formae non absolute aequales sunt censendae, sed in altera functiones *f: x* et *F: y* alias formas habebunt, ut nihilominus ambae aeque satisfaciant. Casu quidem $k=\frac{1}{2}$ ambae conveniunt perfecte. Ponamus autem $k=0$, ut prior det

$$z=\mathfrak{A}=f{:}x+F{:}y\,;$$

at posterior praebet

$$\tfrac{z}{x+y}=\mathfrak{A}-\tfrac{1}{2}\mathfrak{B}+\tfrac{1}{6}\mathfrak{C}-\tfrac{1}{24}\mathfrak{D}+\tfrac{1}{120}\mathfrak{E}-\text{etc.}$$

Quarum consensus ut appareat, sit in hac posteriori

$$f:x = ax^3 \text{ et } F:y = by^2;$$

erit

$$\mathfrak{A} = ax^3 + by^2, \quad \mathfrak{B} = (x+y)(3axx + 2by), \quad \mathfrak{C} = (x+y)^2(6ax + 2b), \quad \mathfrak{D} = (x+y)^3 6a,$$

at reliquae partes evanescunt. Obtinebimus ergo ex posteriori

$$z = (x+y)\left(ax^3 + by^2\right) - \tfrac{1}{2}(x+y)^2(3axx + 2by) + \tfrac{1}{3}(x+y)^3(3ax + b) - \tfrac{1}{4}(x+y)^4 a,$$

quae evoluta praebet

$$\tfrac{1}{4}ax^4 - \tfrac{1}{4}a^4 y + \tfrac{1}{3}bx^3 + \tfrac{1}{3}b^3 y = z,$$

quae forma utique in priori $z = f:x + F:y$ continetur. Consensus ergo binarum illarum formarum generalium eo magis est notatu dignus.

PROBLEMA 53

330. *Invenire casus, quibus haec aequatio generalis*

$$\left(\frac{ddz}{dy^2}\right) - QQ\left(\frac{ddz}{dx^2}\right) + R\left(\frac{dz}{dy}\right) + S\left(\frac{dz}{dx}\right) + Tz = 0$$

ad formam praecedentem reduci ideoque iisdem casibus integrari potest.

SOLUTIO

Introducendo binas novas variabiles t et u, ut sit, quemadmodum reductio § 319 adhibita, ubi $P = 0$ et $V = 0$, declarat,

$$t = \int p(dx + Qdy) \text{ et } u = \int q(dx - Qdy),$$

si ponamus ad abbreviandum

$$M = S + QR + \left(\frac{dQ}{dy}\right) + Q\left(\frac{dQ}{dx}\right),$$

$$N = S - QR - \left(\frac{dQ}{dy}\right) + Q\left(\frac{dQ}{dx}\right),$$

prodibit haec aequatio

$$\left(\frac{ddz}{dtdu}\right) - \frac{M}{4QQq}\left(\frac{dz}{dt}\right) - \frac{N}{4QQp}\left(\frac{dz}{du}\right) - \frac{T}{4QQpq}z = 0,$$

quam ergo ad hanc formam revocari oportet

$$\left(\frac{ddz}{dtdu}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)+\frac{n}{(t+u)^2}z=0,$$

cuius casus integrabilitatis ante [§ 327] designavimus, scilicet quoties fuerit

$$n=(m+i)(m-i-1)$$

denotante i numerum integrum quemcunque positivum cyphra non exclusa.
Ad hoc ergo necesse est, ut fiat

$$M=\frac{-4mQQq}{t+u},\quad N=\frac{-4mQQp}{t+u}\quad \text{et}\quad T=\frac{-4nQQpq}{(t+u)^2}$$

Quia autem hic integrabilitatis formularum t et u ratio haberi debet, sumamus

$$Q=\frac{\varphi':y}{\pi':x}$$

sitque

$$p=a\pi':x\quad \text{et}\quad q=b\pi':x$$

eritque

$$t=a\pi:x+a\varphi:y\quad \text{et}\quad u=b\pi:x-b\varphi:y\,.$$

Hinc fit

$$M+N=2S+2Q\left(\frac{dQ}{dx}\right)=\frac{-4m(a+b)QQ\pi':x}{t+u}$$

et

$$M-N=2QR+2\left(\frac{dQ}{dy}\right)=\frac{4m(a-b)QQ\pi':x}{t+u}$$

ideoque

$$R=\frac{2m(a-b)Q\pi':x}{t+u}-\frac{1}{Q}\left(\frac{dQ}{dy}\right),\quad S=\frac{-2m(a+b)QQ\pi':x}{t+u}-Q\left(\frac{dQ}{dx}\right)$$

et

$$T=\frac{-4nabQQ\pi':x\cdot\pi':x}{(t+u)^2}=\frac{-4nab\varphi':y\cdot\varphi':y}{(t+u)^2}$$

ob $Q=\frac{\varphi':y}{\pi':x}$ unde est

$$\left(\frac{dQ}{dy}\right)=\frac{\varphi'':y}{\pi':x}\quad \text{et}\quad \left(\frac{dQ}{dx}\right)=\frac{-\pi'':y\cdot\varphi':y}{\pi':x\cdot\pi':x}$$

et

$$t+u=(a+b)\pi:x+(a-b)\varphi:y.$$

Ideoque habebimus

$$R=\frac{2m(a-b)\varphi':y}{t+u}-\frac{\varphi'':y}{\varphi':y}\quad \text{et}\quad \frac{S}{QQ}=\frac{-2m(a+b)\pi':x}{t+u}+\frac{\pi'':x}{\pi':x}.$$

Quo aequatio fiat simplicior, duo casus praecipue sunt considerandi, alter, ubi $b = a$, alter $b = -a$. Priori est $t + u = 2a\pi{:}x$ et aequatio nostra erit

$$\left(\frac{ddz}{dy^2}\right) - \left(\frac{\varphi'{:}y}{\pi'{:}x}\right)^2 \left(\frac{ddz}{dx^2}\right) - \frac{\varphi''{:}y}{\varphi'{:}y}\left(\frac{dz}{dy}\right) + \left(\frac{\varphi'{:}y}{\pi'{:}x}\right)^2 \left(\frac{\pi''{:}x}{\pi'{:}x} - \frac{2m\pi'{:}x}{\pi{:}x}\right)\left(\frac{dz}{dx}\right) - n\left(\frac{\varphi'{:}y}{\pi{:}x}\right)^2 z = 0$$

altero vero casu $b = -a$ fit $t + u = 2a\varphi{:}y$ et

$$\left(\frac{ddz}{dy^2}\right) - \left(\frac{\varphi'{:}y}{\pi'{:}x}\right)^2 \left(\frac{ddz}{dx^2}\right) + \left(\frac{2m\varphi'{:}y}{\varphi{:}y} - \frac{\varphi''{:}y}{\varphi'{:}y}\right)\left(\frac{dz}{dy}\right) + \left(\frac{\varphi'{:}y}{\pi'{:}x}\right)^2 \cdot \frac{\pi''{:}x}{\pi'{:}x}\left(\frac{dz}{dx}\right) + n\left(\frac{\varphi'{:}y}{\varphi{:}y}\right)^2 z = 0$$

quae ambae aequationes integrationem admittunt casibus

$$n = (m+i)(m-i-1).$$

COROLLARIUM 1

331. Aequationes postremo inventae a se invicem non differunt, nisi quod binae variabiles *x* et *y* iuvicem permutantur, unde sufficit alterutram solam considerasse. Prior autem [sumto $a = 1$] transformatur ponendo

$$t = \pi{:}x + \varphi{:}y \quad \text{et} \quad u = \pi{:}x - \varphi{:}y\,,$$

posterior vero ponendo

$$t = \pi{:}x + \varphi{:}y \quad \text{et} \quad u = \varphi{:}y - \pi{:}x\,.$$

COROLLARIUM 2

332. Hae aequationes etiam sequenti forma magis perspicua repraesentari possunt, prior quidem

$$\frac{1}{(\varphi'{:}y)^2}\left(\frac{ddz}{dy^2}\right) - \left(\frac{1}{\pi'{:}x}\right)^2 \left(\frac{ddz}{dx^2}\right) - \frac{\varphi''{:}y}{(\varphi'{:}y)^3}\left(\frac{dz}{dy}\right) + \left(\frac{\pi''{:}x}{(\pi'{:}x)^3} - \frac{2m}{\pi{:}x\cdot\pi'{:}x}\right)\left(\frac{dz}{dx}\right) - \frac{n}{(\pi{:}x)^2} z = 0$$

posterior

$$\frac{1}{(\varphi'{:}y)^2}\left(\frac{ddz}{dy^2}\right) - \left(\frac{1}{\pi'{:}x}\right)^2 \left(\frac{ddz}{dx^2}\right) + \left(\frac{2m}{\varphi{:}y\cdot\varphi'{:}y} - \frac{\varphi''{:}y}{(\varphi'{:}y)^3}\right)\left(\frac{dz}{dy}\right) + \frac{\pi''{:}x}{(\pi'{:}x)^3}\left(\frac{dz}{dx}\right) + \frac{n}{(\varphi{:}y)^2} z = 0.$$

CASUS 1

333. Ponamus $\pi'{:}x = a$ et $\varphi'{:}y = b$; erit

$$\pi{:}x = ax \quad \text{et} \quad \varphi{:}y = by\,,$$

tum vero $\pi''{:}\,x = 0$ et $\varphi''{:}\,y = 0$, unde forma prior prodibit

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aa}\left(\frac{ddz}{dx^2}\right)-\frac{2m}{aax}\left(\frac{dz}{dx}\right)-\frac{n}{aaxx}z=0\,,$$

quae reducitur ad formam supra resolutam ponendo

$$t=ax+by \quad \text{et} \quad u=ax-by\,.$$

Posterior vero forma est

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aa}\left(\frac{ddz}{dx^2}\right)+\frac{2m}{bby}\left(\frac{dz}{dy}\right)+\frac{n}{bbyy}z=0.$$

quae reducitur ad formam supra resolutam ponendo

$$t=ax+by \quad \text{et} \quad u=by-ax\,;$$

utraque autem est integrabilis casu

$$n=(m+i)(m-i-1)\,.$$

Reductione enim ad variabiles *t* et *u* facta oritur haec aequatio

$$\left(\frac{ddz}{dtdu}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)+\frac{n}{(t+u)^2}z=0\,.$$

COROLLARIUM 1

334. Si sumatur $n=0$, hae ambae aequationes

$$\frac{aa}{bb}\left(\frac{ddz}{dy^2}\right)-\left(\frac{ddz}{dx^2}\right)-\frac{2m}{x}\left(\frac{dz}{dx}\right)=0 \quad \text{et} \quad \left(\frac{ddz}{dy^2}\right)-\frac{bb}{aa}\left(\frac{ddz}{dx^2}\right)+\frac{2m}{y}\left(\frac{dz}{dy}\right)=0$$

sunt integrabiles, quoties *m* fuerit numerus integer ideoque *2m* numerus par.

COROLLARIUM 2

335. En ergo aequationes ob simplicitatem notatu dignas ex tribus tantum terminis constantes, quae infinitis casibus integrationem admittunt. Integrale autem quovis casu facile exhibetur ex § 328, si modo ibi loco *x* et *y* scribatur *t* et *u*.

CASUS 2

336. Sit $\pi':x=ax^{\mu}$ et $\varphi':y=b$; erit

$$\pi:x=\frac{1}{\mu+1}ax^{\mu+1} \quad \text{et} \quad \varphi:y=by\,,$$

tum vero $\pi'':x=\mu a x^{\mu-1}$ et $\varphi'':y=0$. Unde forma prior provenit

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right)+\frac{\mu-2m\mu-2m}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right)-\frac{n(\mu+1)^2}{aax^{2\mu+2}}z=0,$$

quae reducitur ad formam supra resolutam ponendo

$$t=\frac{1}{\mu+1}ax^{\mu+1}+by \quad \text{et} \quad u=\frac{1}{\mu+1}ax^{\mu+1}-by$$

Posterior vero forma fit

$$\frac{1}{bb}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right)+\frac{2m}{bby}\left(\frac{dz}{dy}\right)+\frac{\mu}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right)+\frac{n}{bbyy}z=0,$$

cuius reductio absolvitur ponendo

$$t=\frac{1}{\mu+1}ax^{\mu+1}+by \quad \text{et} \quad u=by-\frac{1}{\mu+1}ax^{\mu+1}.$$

Haeque ambae aequationes integrationem admittunt, quoties fuerit

$$n=(m+i)(m-i-1).$$

COROLLARIUM 1

337. Ex priori forma casus maxime notabilis existit, si capiatur $m=\frac{\mu}{2\mu+2}$ et $n=0$; tum enim erit

$$\frac{aa}{bb}x^{2\mu}\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right),$$

quae est integrabilis, quoties $\frac{\mu}{2\mu+2}$ fuerit numerus integer *m* sive positivus sive negativus.

COROLLARIUM 2

338. Vel cum sit $m=\frac{-\mu}{2\mu-1}$, haec aequatio

$$\frac{aa}{bb}x^{\frac{-4m}{2m-1}}\left(\frac{ddz}{dy^2}\right)=\left(\frac{ddz}{dx^2}\right) \quad \text{seu} \quad \left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right)$$

erit integrabilis, quoties *m* fuerit numerus integer sive positivus sive negativus;
reductio autem fit ponendo

$$t=-(2m-1)ax^{\frac{-1}{2m-1}}+by \quad \text{et} \quad u=-(2m-1)ax^{\frac{-1}{2m-1}}-by$$

CASUS 3

339. Sit $\pi':x=ax^{\mu}$ et $\varphi':y=by^{\nu}$; erit

$$\pi:x=\frac{1}{\mu+1}ax^{\mu+1} \text{ et } \varphi:y=\frac{1}{\nu+1}by^{\nu+1},$$

tum vero $\pi'':x=\mu ax^{\mu-1}$ et $\varphi'':y=\nu by^{\nu-1}$. Hinc prior forma resultat

$$\frac{1}{bby^{2\nu}}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right)-\frac{\nu}{bby^{2\nu+1}}\left(\frac{dz}{dy}\right)+\frac{\mu-2m\mu-2m}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right)-\frac{n(\mu+1)^2}{aax^{2\mu+2}}z=0,$$

quae reducitur ponendo

$$t=\frac{1}{\mu+1}ax^{\mu+1}+\frac{1}{\nu+1}by^{\nu+1} \quad \text{et} \quad u=\frac{1}{\mu+1}ax^{\mu+1}-\frac{1}{\nu+1}by^{\nu+1}$$

Posterior vero forma evadit

$$\frac{1}{bby^{2\nu}}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right)+\frac{2m\nu+2m-\nu}{bby^{2\nu+1}}\left(\frac{dz}{dy}\right)+\frac{\mu}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right)+\frac{n(\nu+1)}{bby^{2\nu+2}}z=0.$$

cuius reductio fit hac substitutione

$$t=\frac{1}{\mu+1}ax^{\mu+1}+\frac{1}{\nu+1}by^{\nu+1} \quad \text{et} \quad u=\frac{-1}{\mu+1}ax^{\mu+1}+\frac{1}{\nu+1}by^{\nu+1}$$

Vel cum hic tantum ratio inter *a* et *b* in computum ingrediatur, pro priori poni poterit

$$t=\frac{1}{2}x^{\mu+1}+\frac{(\mu+1)b}{2(\nu+1)a}y^{\nu+1} \quad \text{et} \quad u=\frac{1}{2}ax^{\mu+1}-\frac{(\mu+1)b}{2(\nu+1)a}y^{\nu+1},$$

ut fiat $t+u=x^{\mu+1}$ quo expressio integralis fiat simplicior.

COROLLARIUM 1

340. Si ponatur in forma priori $\mu=\frac{-2m}{2m-1}$, minuetur ea uno termino fietque

$$\frac{1}{bby^{2\nu}}\left(\frac{ddz}{dy^2}\right)-\frac{1}{aa}x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right)-\frac{\nu}{bby^{2\nu+1}}\left(\frac{dz}{dy}\right)-\frac{n}{(2m-1)^2aa}x^{\frac{2m}{2m-1}}z=0.$$

Statuatur $a = b$ et capiatur quoque $v = \frac{-2m}{2m-1}$, ut prodeat

$$y^{\frac{4m}{2m-1}}\left(\frac{ddz}{dy^2}\right) - x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right) + \frac{2m}{2m-1}y^{\frac{2m+1}{2m-1}}\left(\frac{dz}{dy}\right) - \frac{n}{(2m-1)^2}x^{\frac{2}{2m-1}}z = 0.$$

COROLLARIUM 2

341. Sumatur porro in priori forma $v = \mu$ et fiat

$$\mu - 2m\mu - 2m = -\mu \quad \text{seu } v = \frac{\mu}{\mu+1},$$

ut prodeat

$$\frac{1}{bby^{2\mu}}\left(\frac{ddz}{dy^2}\right) - \frac{1}{aax^{2\mu}}\left(\frac{ddz}{dx^2}\right) - \frac{\mu}{bby^{2\mu+1}}\left(\frac{dz}{dy}\right) - \frac{\mu}{aax^{2\mu+1}}\left(\frac{dz}{dx}\right) - \frac{n(\mu+1)^2}{aax^{2\mu+2}}z = 0,$$

quae integrabilis existit, quoties fuerit

$$n = \frac{-(\mu+(\mu+1)i)((\mu+1)i+1)}{(\mu+1)^2} \text{ seu } n = -\left(i + \frac{\mu}{\mu+1}\right)\left(i + \frac{1}{\mu+1}\right).$$

SCHOLION

342. Largissima ergo hinc nobis suppeditatur copia aequationum satis concinnarum, quas ope methodi hic traditae integrare licet. Atque hic imprimis duo casus conspiciuntur, quorum alter

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right)$$

pro motu cordarum inaequali crassitie praeditarum determinando est inventus, alter autem hac aequatione

$$\frac{aa}{bb}\left(\frac{ddz}{dy^2}\right) - \left(\frac{ddz}{dx^2}\right) - \frac{2m}{x}\left(\frac{dz}{dx}\right) = 0$$

contentus [§ 334] ideo est memorabilis, quod in analysi pro soni propagatione instituta ad talem formam pervenitur. Hae igitur binae aequationes prae ceteris merentur, ut pro casibus integrabilitatis integralia exhibeamus.

PROBLEMA 54

343. *Proposita aequatione differentiali*

$$\frac{aa}{bb}\left(\frac{ddz}{dy^2}\right)-\left(\frac{ddz}{dx^2}\right)-\frac{2m}{x}\left(\frac{dz}{dx}\right)=0$$

casibus, quibus m est numerus integer sive positivus sive negativus, eius integrale completum exhibere.

SOLUTIO

Facta substitutione $t=\frac{1}{2}x+\frac{b}{2a}y$ et $u=\frac{1}{2}x-\frac{b}{2a}y$ aequatio nostra hanc induit formam

$$\left(\frac{ddz}{dtdu}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)=0.$$

Cum igitur sit $t+u=x$, si ponamus

$$\mathfrak{A}=f:\frac{ax+by}{2a}+F:\frac{ax-by}{2a},$$
$$\mathfrak{B}=x\left(f':\frac{ax+by}{2a}+F':\frac{ax-by}{2a}\right),$$
$$\mathfrak{C}=x^2\left(f'':\frac{ax+by}{2a}+F'':\frac{ax-by}{2a}\right),$$
$$\mathfrak{D}=x^3\left(f''':\frac{ax+by}{2a}+F''':\frac{ax-by}{2a}\right),$$
$$\mathfrak{E}=x^4\left(f^{IV}:\frac{ax+by}{2a}+F^{IV}:\frac{ax-by}{2a}\right),$$
$$\mathfrak{E}=x^5\left(f^{V}:\frac{ax+by}{2a}+F^{V}:\frac{ax-by}{2a}\right),$$
etc.,

casus integrabiles [§ 328] ita se habebunt, primo negativi:

si $m=0$, $\quad z=\mathfrak{A}$,
si $m=-1$, $\quad z=\mathfrak{A}-\frac{1}{2}\mathfrak{B}$
si $m=-2$, $\quad z=\mathfrak{A}-\frac{2}{4}\mathfrak{B}+\frac{1}{4\cdot 3}\mathfrak{C}$,
si $m=-3$, $\quad z=\mathfrak{A}-\frac{3}{6}\mathfrak{B}+\frac{3}{6\cdot 5}\mathfrak{C}-\frac{1}{6\cdot 5\cdot 4}\mathfrak{D}$,
si $m=-4$, $\quad z=\mathfrak{A}-\frac{4}{8}\mathfrak{B}+\frac{6}{8\cdot 7}\mathfrak{C}-\frac{4}{8\cdot 7\cdot 6}\mathfrak{D}+\frac{1}{8\cdot 7\cdot 6\cdot 5}\mathfrak{E}$,
si $m=-5$, $\quad z=\mathfrak{A}-\frac{5}{10}\mathfrak{B}+\frac{10}{10\cdot 9}\mathfrak{C}-\frac{10}{10\cdot 9\cdot 8}\mathfrak{D}+\frac{5}{10\cdot 9\cdot 8\cdot 7}\mathfrak{E}-\frac{1}{10\cdot 9\cdot 8\cdot 7\cdot 6}\mathfrak{F}$,
etc.,

tum vero pro valoribus positivis ipsius m:

si $m=1$, $xz=\mathfrak{A}$,

si $m=2$, $x^3z=\mathfrak{A}-\frac{1}{2}\mathfrak{B}$

si $m=3$, $x^5z=\mathfrak{A}-\frac{2}{4}\mathfrak{B}+\frac{1}{4\cdot 3}\mathfrak{C}$,

si $m=4$, $x^7z=\mathfrak{A}-\frac{3}{6}\mathfrak{B}+\frac{3}{6\cdot 5}\mathfrak{C}-\frac{1}{6\cdot 5\cdot 4}\mathfrak{D}$,

si $m=5$, $x^9z=\mathfrak{A}-\frac{4}{8}\mathfrak{B}+\frac{6}{8\cdot 7}\mathfrak{C}-\frac{4}{8\cdot 7\cdot 6}\mathfrak{D}+\frac{1}{8\cdot 7\cdot 6\cdot 5}\mathfrak{E}$,

si $m=6$, $x^{11}z=\mathfrak{A}-\frac{5}{10}\mathfrak{B}+\frac{10}{10\cdot 9}\mathfrak{C}-\frac{10}{10\cdot 9\cdot 8}\mathfrak{D}+\frac{5}{10\cdot 9\cdot 8\cdot 7}\mathfrak{E}-\frac{1}{10\cdot 9\cdot 8\cdot 7\cdot 6}\mathfrak{F}$,

etc.,

Cui ergo expressioni casu $m=-i$ aequatur valor z, eidem aequatur casu $m=i+1$ valor ipsius $x^{2i+1}z$.

SCHOLION

344. Valores ipsarum t et u ita hic assumsi, ut fieret $t+u=x$, atque eosdem valores quoque in functionibus adhiberi oportet. Etsi enim $f:\frac{ax+by}{2a}$ etiam est functio ipsius $ax+by$, tamen functiones per differentiationem inde derivatae discrepant. Namque si ponamus

$$f:\tfrac{ax+by}{2a}=\varphi:(ax+by),$$

erit differentiando

$$\tfrac{adx+bdy}{2a}\,f':\tfrac{ax+by}{2a}=(adx+bdy)\varphi':(ax+by),$$

unde erit

$$f':\tfrac{ax+by}{2a}=2a\varphi':(ax+by)$$

neque ergo hae functiones differentiales sunt aequales, etiamsi principales assumtae sint aequales. Simili modo erit

$$f'':\tfrac{ax+by}{2a}=4aa\varphi'':(ax+by)$$

et

$$f''':\tfrac{ax+by}{2a}=8a^3\varphi''':(ax+by)$$

et ita porro.

PROBLEMA 55

345. *Proposita aequatione differentiali*

$$\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{4m}{2m-1}}\left(\frac{ddz}{dx^2}\right)$$

casibus, quibus m est numerus integer sive positivus sive negativus, integrale completum exhibere.

SOLUTIO

Introductis novis variabilibus *t* et *u*, ita ut sit [§ 338]

$$t=\tfrac{1}{2}x^{\frac{-1}{2m-1}}-\frac{b}{2(2m-1)a}y \quad \text{et} \quad u=\tfrac{1}{2}x^{\frac{-1}{2m-1}}+\frac{b}{2(2m-1)a}y,$$

aequatio nostra hanc induit formam

$$\left(\frac{ddz}{dtdu}\right)+\frac{m}{t+u}\left(\frac{dz}{dt}\right)+\frac{m}{t+u}\left(\frac{dz}{du}\right)=0,$$

ubi est

$$t+u=x^{\frac{-1}{2m-1}}$$

Posito igitur

$$\mathfrak{A}=f:t+F:u,\quad \mathfrak{B}=x^{\frac{-1}{2m-1}}\left(f':t+F':u\right),$$
$$\mathfrak{C}=x^{\frac{-2}{2m-1}}\left(f'':t+F'':u\right),\ \mathfrak{D}=x^{\frac{-3}{2m-1}}\left(f''':t+F''':u\right),$$
$$\mathfrak{E}=x^{\frac{-4}{2m-1}}\left(f^{IV}:t+F^{IV}:u\right),\quad \mathfrak{E}=x^{\frac{-5}{2m-1}}\left(f^{V}:t+F^{V}:u\right),$$

etc.,

percurramus primo casus, quibus *m* a cyphra per numeros negativos decrescit.

I. Si $m=0$, erit aequationis

$$\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}\left(\frac{ddz}{dx^2}\right)$$

integrale

$$z=f:\left(\tfrac{1}{2}x+\frac{b}{2a}y\right)+F:\left(\tfrac{1}{2}x-\frac{b}{2a}y\right).$$

II. Si $m=-1$, ob

$$t=\tfrac{1}{2}x^{\frac{1}{3}}-\frac{b}{6a}y \quad \text{et} \quad u=\tfrac{1}{2}x^{\frac{1}{3}}-\frac{b}{6a}y$$

erit aequationis

$$\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{4}{3}}\left(\frac{ddz}{dx^2}\right)$$

integrale

$$z=f:t+F:u-\tfrac{1}{2}x^{\frac{1}{3}}\left(f':t+F':u\right).$$

III. Si $m=-2$, ob

$$t=\tfrac{1}{2}x^{\frac{1}{5}}+\tfrac{b}{10a}y \quad \text{et} \quad u=\tfrac{1}{2}x^{\frac{1}{5}}-\tfrac{b}{10a}y$$

erit aequationis

$$\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{8}{5}}\left(\frac{ddz}{dx^2}\right)$$

integrale

$$z=f:t+F:u-\tfrac{2}{4}x^{\frac{1}{5}}\left(f':t+F':u\right)+\tfrac{1}{4\cdot 3}x^{\frac{2}{5}}\left(f'':t+F'':u\right)..$$

IV. Si $m=-3$, ob

$$t=\tfrac{1}{2}x^{\frac{1}{7}}+\tfrac{b}{14a}y \quad \text{et} \quad u=\tfrac{1}{2}x^{\frac{1}{7}}-\tfrac{b}{14a}y$$

erit aequationis

$$\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{12}{7}}\left(\frac{ddz}{dx^2}\right)$$

integrale

$$z=f:t+F:u-\tfrac{3}{6}x^{\frac{1}{7}}\left(f':t+F':u\right)+\tfrac{3}{6\cdot 5}x^{\frac{2}{7}}\left(f'':t+F'':u\right)-\tfrac{1}{6\cdot 5\cdot 4}x^{\frac{3}{7}}\left(f''':t+F''':u\right).$$

V. Si $m=-4$, ob

$$t=\tfrac{1}{2}x^{\frac{1}{9}}+\tfrac{b}{18a}y \quad \text{et} \quad u=\tfrac{1}{2}x^{\frac{1}{9}}-\tfrac{b}{18a}y$$

erit aequationis

$$\left(\frac{ddz}{dy^2}\right)=\frac{bb}{aa}x^{\frac{16}{9}}\left(\frac{ddz}{dx^2}\right)$$

integrale

$$\begin{aligned}z=&f:t+F:u-\tfrac{4}{8}x^{\frac{1}{9}}\left(f':t+F':u\right)+\tfrac{6}{8\cdot 7}x^{\frac{2}{9}}\left(f'':t+F'':u\right)\\&-\tfrac{4}{8\cdot 7\cdot 6}x^{\frac{3}{9}}\left(f''':t+F''':u\right)+\tfrac{1}{8\cdot 7\cdot 6\cdot 5}x^{\frac{4}{9}}\left(f^{IV}:t+F^{IV}:u\right)\end{aligned}$$

et ita porro.

Pro altero vero casu, ubi *m* habet valores positivos, integralia sequenti modo exprimentur:

I. Si sit $m = 1$ seu

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}x^4\left(\frac{ddz}{dx^2}\right)$$

ob

$$t = \tfrac{1}{2}x^{-1} - \tfrac{b}{2a}y \quad \text{et} \quad u = \tfrac{1}{2}x^{-1} + \tfrac{b}{2a}y$$

erit integrale

$$x^{-1}z = f:t + F:u \quad \text{seu} \quad z = x\left(f:t + F:u\right).$$

II. Si sit $m = 2$ seu

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}x^{\frac{8}{3}}\left(\frac{ddz}{dx^2}\right),$$

ob

$$t = \tfrac{1}{2}x^{-\frac{1}{3}} - \tfrac{b}{6a}y \quad \text{et} \quad u = \tfrac{1}{2}x^{-\frac{1}{3}} + \tfrac{b}{6a}y$$

erit integrale

$$z = x\left(f:t + F:u\right) - \tfrac{1}{2}x^{\frac{2}{3}}\left(f':t + F':u\right).$$

III. Si sit $m = 3$ seu

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}x^{\frac{12}{5}}\left(\frac{ddz}{dx^2}\right)$$

ob

$$t = \tfrac{1}{2}x^{-\frac{1}{5}} - \tfrac{b}{10a}y \quad \text{et} \quad u = \tfrac{1}{2}x^{-\frac{1}{5}} + \tfrac{b}{10a}y$$

erit integrale

$$z = x\left(f:t + F:u\right) - \tfrac{2}{4}x^{\frac{4}{5}}\left(f':t + F':u\right) + \tfrac{1}{4\cdot 3}x^{\frac{3}{5}}\left(f'':t + F'':u\right).$$

IV. Si sit $m = 4$ seu

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa}x^{\frac{16}{7}}\left(\frac{ddz}{dx^2}\right)$$

ob

$$t = \tfrac{1}{2}x^{-\frac{1}{7}} - \tfrac{b}{14a}y \quad \text{et} \quad u = \tfrac{1}{2}x^{-\frac{1}{7}} + \tfrac{b}{14a}y$$

erit integrale

$$z = x\left(f:t + F:u\right) - \tfrac{3}{6}x^{\frac{6}{7}}\left(f':t + F':u\right) + \tfrac{3}{6\cdot 5}x^{\frac{5}{7}}\left(f'':t + F'':u\right) - \tfrac{1}{6\cdot 5\cdot 4}x^{\frac{4}{7}}\left(f''':t + F''':u\right).$$

V. Si sit $m = 5$ seu

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} x^{\frac{20}{9}} \left(\frac{ddz}{dx^2}\right)$$

ob

$$t = \frac{1}{2} x^{-\frac{1}{9}} - \frac{b}{18a} y \quad \text{et} \quad u = \frac{1}{2} x^{-\frac{1}{9}} + \frac{b}{18a} y$$

erit integrale

$$\begin{aligned} z = x\left(f:t + F:u\right) - \tfrac{4}{8} x^{\frac{8}{9}} \left(f':t + F':u\right) + \tfrac{6}{8\cdot 7} x^{\frac{7}{9}} \left(f'':t + F'':u\right) \\ - \tfrac{4}{8\cdot 7\cdot 6} x^{\frac{6}{9}} \left(f''':t + F''':u\right) + \tfrac{1}{8\cdot 7\cdot 6\cdot 5} x^{\frac{5}{9}} \left(f^{IV}:t + F^{IV}:u\right) \quad \text{etc.,} \end{aligned}$$

unde lex, qua has expressiones ulterius continuare licet, per se est manifesta.

SCHOLION 1

346. Casus isti integrabilitatis congruunt cum iis, qui in aequatione RICCATIANA dicta deprehenduntur; novimus scilicet aequationem hanc

$$dy + yydx = ax^{\frac{-4m}{2m-1}} dx$$

integrari posse, quoties *m* est numerus integer sive positivus sive negativus. Haec autem aequatio haud levi vinculo cum nostra forma est connexa, quod ita ostendi potest.

Proposita forma generali

$$\left(\frac{ddz}{dy^2}\right) = X \left(\frac{ddz}{dx^2}\right)$$

pro integralibus particularibus inveniendis statuatur $z = e^{\alpha y} v$, ut v sit functio ipsius x tantum; erit

$$\left(\frac{dz}{dx}\right) = e^{\alpha y} \cdot \frac{dv}{dx} \quad \text{et} \quad \left(\frac{ddz}{dx^2}\right) = e^{\alpha y} \cdot \frac{ddv}{dx^2};$$

tum vero $\left(\frac{ddz}{dy^2}\right) = \alpha\alpha e^{\alpha y} v$, unde prodit haec aequatio

$$\alpha\alpha v = \frac{Xddv}{dx^2};$$

in qua si porro statuatur $v = e^{\int pdx}$, oritur

$$\frac{\alpha\alpha dx}{X} = dp + ppdx,$$

ac si $X = Ax^{\frac{4m}{2m-1}}$ ut in nostro casu, haec aequatio fit

$$dp + ppdx = \alpha x^{\frac{-4m}{2m-1}} dx.$$

Haud temere igitur evenire putandum est, quod utraque aequatio iisdem casibus integrationem admittat.

Interim tamen notatu dignum occurrit, quod casus $m = \infty$, qui in forma RICCATIANA fit facillimus, idem in nostra aequatione neutiquam integrationem admittat. Habetur quippe haec aequatio

$$\left(\frac{ddz}{dy^2}\right) = \frac{bb}{aa} xx \left(\frac{ddz}{dx^2}\right),$$

cuius reductio modo supra § 330 adhibito non succedit. Nam ob $Q = \frac{bx}{a}$, $R = 0$, $S = 0$ et $T = 0$, pro novis variabilibus ponitur

$$t = \int p\left(dx + \frac{bxdy}{a}\right) \quad \text{et} \quad u = \int q\left(dx - \frac{bxdy}{a}\right),$$

unde ob $M = \frac{bbx}{aa} = N$ oritur haec aequatio

$$\left(\frac{ddz}{dtdu}\right) - \frac{1}{4qx}\left(\frac{dz}{dt}\right) - \frac{1}{4px}\left(\frac{dz}{du}\right) = 0,$$

quae sumendo

$$p = \frac{1}{x} \quad \text{et} \quad q = \frac{1}{x},$$

ut sit

$$t = lx + \frac{by}{a} \quad \text{et} \quad u = lx - \frac{by}{a},$$

transit in

$$\left(\frac{ddz}{dtdu}\right) - \frac{1}{4}\left(\frac{dz}{dt}\right) - \frac{1}{4}\left(\frac{dz}{du}\right) = 0,$$

cuius integratio haud perspicitur.

SCHOLION 2

347. Aequationis autem $\left(\frac{ddz}{dy^2}\right) = xx\left(\frac{ddz}{dx^2}\right)$ integralia particularia infinita exhibere licet in hac forma $z = Ax^{\lambda}e^{\mu y}$ contenta. Cum enim hinc sit

$$\left(\frac{dz}{dy}\right) = \mu A x^{\lambda} e^{\mu y} \quad \text{et} \quad \left(\frac{dz}{dx}\right) = \lambda A x^{\lambda-1} e^{\mu y},$$

erit

$$\mu\mu Ax^{\lambda}e^{\mu y}=\lambda(\lambda-1)Ax^{\lambda}e^{\mu y}$$

ideoque $\mu=\sqrt{\lambda(\lambda-1)}$, unde ex quovis numero pro λ assumto bini valores pro μ oriuntur, ita ut habeatur

$$z=Ax^{\lambda}e^{y\sqrt{\lambda(\lambda-1)}}+Bx^{\lambda}e^{-y\sqrt{\lambda(\lambda-1)}},$$

et huiusmodi membrorum numerus variando λ in infinitum multiplicari potest.

Interim tamen singula haec membra adhuc generaliora reddi possunt. Posito enim $z=Ax^{\lambda}e^{\mu y}v$ videamus, an v necessario constans esse debeat; hinc autem fit

$$\left(\frac{dz}{dy}\right)=\mu x^{\lambda}e^{\mu y}v+x^{\lambda}e^{\mu y}\left(\frac{dv}{dy}\right)\quad\text{et}\quad\left(\frac{dz}{dx}\right)=\lambda x^{\lambda-1}e^{\mu y}v+x^{\lambda}e^{\mu y}\left(\frac{dv}{dx}\right)$$

ideoque nostra aequatio praebet per $x^{\lambda}e^{\mu y}$ divisa

$$\mu\mu v+2\mu\left(\frac{dv}{dy}\right)+\left(\frac{ddv}{dy^2}\right)=\lambda(\lambda-1)v+2x\lambda\left(\frac{dv}{dx}\right)+xx\left(\frac{ddv}{dx^2}\right).$$

Statuatur ut ante $\mu\mu=\lambda(\lambda-1)$ sitque $v=\alpha lx+\beta y$; erit

$$2\beta\mu=2\alpha\lambda-\alpha\quad\text{seu}\quad\frac{\alpha}{\beta}=\frac{2\mu}{2\lambda-1}=\frac{2\sqrt{\lambda(\lambda-1)}}{2\lambda-1},$$

unde cuiusque membri ex numero λ nati forma erit

$$z=x^{\lambda}\left\{\begin{array}{l}e^{y\sqrt{\lambda(\lambda-1)}}\left(A+\frac{2\sqrt{\lambda(\lambda-1)}}{\mathfrak{A}}lx+\frac{2\lambda-1}{\mathfrak{A}}y\right)\\+e^{-y\sqrt{\lambda(\lambda-1)}}\left(B-\frac{2\sqrt{\lambda(\lambda-1)}}{\mathfrak{B}}lx+\frac{2\lambda-1}{\mathfrak{B}}y\right)\end{array}\right\}.$$

Quomodocunque igitur non solum exponens λ, sed etiam quantitates $A,\mathfrak{A},B,\mathfrak{B}$ varientur, infinita huiusmodi membra formari possunt, quae omnia iunctim sumta valorem completum functionis z praebere sunt censenda.

Quin etiam pro λ imaginaria assumi possunt; posito enim $\lambda=a+b\sqrt{-1}$ fit $\mu=p+q\sqrt{-1}$ existente

$$pp-qq=aa-a-bb \quad \text{et} \quad pp+qq=\sqrt{(aa+bb)(aa-2a+1+bb)}\,,$$

tum vero est

$$x^{\lambda}=x^{a}\left(\cos.blx+\sqrt{-1}\cdot\sin.blx\right) \text{ et } x^{\mu y}=x^{py}\left(\cos.qy+\sqrt{-1}\cdot\sin.qy\right),$$

unde colligitur forma realis

$$z=x^{a}e^{py}\left\{\begin{array}{l} A\cos.(blx+qy)+B\left(2plx+(2a-1)y\right)\cos.(blx+qy) \\ \qquad\qquad -\ B(2qlx+2by)\sin.(blx+qy) \\ +\mathfrak{A}\sin.(blx+qy)+\mathfrak{B}\left(2plx+(2a-1)y\right)\sin.(blx+qy) \\ \qquad\qquad +\mathfrak{B}(2qlx+2by)\cos.(blx+qy) \end{array}\right\}$$

ubi quantitates a et b pro lubitu assumere licet, unde simul p et q definiuntur.

Quodsi hic litteras b et q ut datas spectemus, binae reliquae a et p ex iis ita determinantur, ut sit

$$2a-1=q\sqrt{\left(\tfrac{1}{qq-bb}-4\right)} \quad \text{et} \quad p=\tfrac{b}{2}\sqrt{\left(\tfrac{1}{qq-bb}-4\right)};$$

hic ergo necesse est sit $qq>bb$ et $qq<bb+\frac{1}{4}$, seu qq inter hos arctos limites bb et $bb+\frac{1}{4}$ contineri debet. Statuatur

$$q=c \quad \text{et} \quad \sqrt{\left(\tfrac{1}{qq-bb}-4\right)}=2f\,,$$

ut sit

$$\tfrac{1}{qq-bb}=4(1+ff) \quad \text{seu} \quad cc-bb=\tfrac{1}{4(1+ff)}$$

atque

$$2a-1=2cf \quad \text{et} \quad p=bf\,,$$

ex quo forma integralium particularium erit

$$z = x^{cf+\frac{1}{2}} e^{bfy} \left\{ \begin{array}{r} A\cos.(blx+cy) + 2Bf(blx+cy)\cos.(blx+cy) \\ -\ 2B(clx+by)\sin.(blx+cy) \\ +\mathfrak{A}\sin.(blx+cy) + 2\mathfrak{B}f(blx+cy)\sin.(blx+cy) \\ +2\mathfrak{B}(clx+by)\cos.(blx+cy) \end{array} \right\}$$

quae posito brevitatis gratia angulo $blx+cy=\varphi$ transformatur in hanc

$$z = x^{cf+\frac{1}{2}} e^{bfy} \left(A\cos.(\varphi+\alpha) + Bf(blx+cy)\sin.(\varphi+\beta) + B(clx+by)\cos.(\varphi+\beta) \right),$$

ubi quantitates $b,\ c,\ A,\ B,\ \alpha, \beta$ ab arbitrio nostro pendent.

SCHOLION 3

348. Resolutio ergo aequationis

$$\left(\frac{ddz}{dy^2}\right) = xx\left(\frac{ddz}{dx^2}\right)$$

ita institui potest, ut fingatur

$$z = x^{\lambda} e^{\mu y}(mlx+ny),$$

unde fit

$$\left(\frac{dz}{dx}\right) = \lambda x^{\lambda-1} e^{\mu y}(mlx+ny) + mx^{\lambda-1} e^{\mu y}$$

et

$$\left(\frac{dz}{dy}\right) = \mu x^{\lambda} e^{\mu y}(mlx+ny) + nx^{\lambda} e^{\mu y}$$

hincque ulterius differentiando

$$\left(\frac{ddz}{dx^2}\right) = x^{\lambda-2} e^{\mu y}\left(m(2\lambda-1) + \lambda(\lambda-1)mlx + \lambda(\lambda-1)ny\right)$$

et

$$\left(\frac{ddz}{dy^2}\right) = x^{\lambda} e^{\mu y}(2\mu n + \mu\mu mlx + \mu\mu ny).$$

Ex quo colligitur primo $\mu = \sqrt{\lambda(\lambda-1)}$, deinde $2n\sqrt{\lambda(\lambda-1)} = m(2\lambda-1)$, ut sit $\frac{m}{n} = \frac{2\sqrt{\lambda(\lambda-1)}}{(2\lambda-1)}$, sicque eadem prodit integratio, quam modo ante dedimus.