CHAPTER III

IF TWO OR ALL OF THE SECOND ORDER FORMULAS ARE DETERMINED BY THE REMAINING QUANTITIES

PROBLEM 48

296. *If z should be a function of x and y of this kind, so that* $\left(\frac{ddz}{dy^2}\right)=aa\left(\frac{ddz}{dx^2}\right)$, *to determine the nature of the function z.*

SOLUTION

Two new variables t and u are introduced, so that there shall be

$$t=\alpha x+\beta y \text{ and } u=\gamma x+\delta y,$$

and from §233 all the differential formulas will undergo changes

$$\left(\frac{dz}{dx}\right)=\alpha\left(\frac{dz}{dt}\right)+\gamma\left(\frac{dz}{du}\right),\ \left(\frac{dz}{dy}\right)=\beta\left(\frac{dz}{dt}\right)+\delta\left(\frac{dz}{du}\right),$$
$$\left(\frac{ddz}{dx^2}\right)=\alpha\alpha\left(\frac{ddz}{dt^2}\right)+2\alpha\gamma\left(\frac{ddz}{dtdu}\right)+\gamma\gamma\left(\frac{ddz}{du^2}\right),$$
$$\left(\frac{ddz}{dxdy}\right)=\alpha\beta\left(\frac{ddz}{dt^2}\right)+\left(\alpha\delta+\beta\gamma\right)\left(\frac{ddz}{dtdu}\right)+\gamma\delta\left(\frac{ddz}{du^2}\right),$$
$$\left(\frac{ddz}{dy^2}\right)=\beta\beta\left(\frac{ddz}{dt^2}\right)+2\beta\delta\left(\frac{ddz}{dtdu}\right)+\delta\delta\left(\frac{ddz}{du^2}\right),$$

from which our equation will be changed into this

$$\left(\beta\beta-\alpha\alpha aa\right)\left(\frac{ddz}{dt^2}\right)+2\left(\beta\delta-\alpha\gamma aa\right)\left(\frac{ddz}{dtdu}\right)+\left(\delta\delta-\gamma\gamma aa\right)\left(\frac{ddz}{du^2}\right)=0.$$

Therefore there is put in place

$$\beta\beta=\alpha\alpha aa \quad \text{and} \quad \delta\delta=\gamma\gamma aa$$

or

$$\alpha=1,\quad \gamma=1,\quad \beta=a \quad \text{and} \quad \delta=-a,$$

so that the two end formulas vanish, as there becomes

$$-2(aa+aa)\left(\frac{ddz}{dtdu}\right)=0, \quad \text{or} \quad \left(\frac{ddz}{dtdu}\right)=0,$$

on putting $t=x+ay$ and $u=x-ay$,

from which by § 270 the complete integral is deduced

$$z=f:t+F:u$$

and with the values restored for t and u

$$z=f:(x+ay)+F:(x-ay),$$

by which form it is clearly satisfied, since there shall be

$$\left(\frac{dz}{dx}\right)=f':(x+ay)+F':(x-ay), \quad \left(\frac{dz}{dy}\right)=af':(x+ay)-aF':(x-ay),$$
$$\left(\frac{ddz}{dx^2}\right)=f'':(x+ay)+F'':(x-ay), \quad \left(\frac{ddz}{dy^2}\right)=aaf'':(x+ay)+aaF'':(x-ay).$$

COROLLARY 1

297. Therefore the value *z* of this is equal to the sum of two arbitrary functions, the one is of $x+ay$, and other of $x-ay$, and both these functions thus can be assumed at will, so that also discontinuous functions are able to be taken in place of these.

COROLLARIUM 2

298. Therefore any two curves described freely by hand as it pleases are able to be taken according to this usage. Evidently if in one the abscissa is taken as $=x+ay$, and in the other truly the abscissa = $x-ay$, then the sum of the applied lines [*i.e.* the *y*-coordinates] will always put in place a suitable value for the function *z*.

SCHOLIUM 1

299. This nearly is the first occurance of a problem, because in this a new kind of calculation occurs which has to be solved; moreover the solution of the general problem concerning vibrating strings leads to this equation, that we have treated here. [The reader no doubt has noted that on putting the variable *y* as the time, and *a* as the speed, then one has a one dimensional wave equation such as the small vibrations of a tight string, and the two solutions represent travelling waves in opposite directions]The celebrated d'Alembert, who was the first to attack this problem successfully, integrated the equation by an unusual method; evidently since there is required to be

$\left(\frac{ddz}{dy^2}\right) = a^2\left(\frac{ddz}{dx^2}\right)$, on putting $dz = pdx + qdy$ and thence both $dp = rdx + sdy$ and $dq = sdx + tdy$ postulate that equation, so that there shall be $t = aar$. Again with these equations considered

$$dp = rdx + sdy, \quad dq = sdx + aardy$$

on combining there is elicited

$$adp + dq = ar\left(dx + ady\right) + s\left(ady + dx\right)$$

or

$$adp + dq = \left(ar + s\right)\left(dx + ady\right),$$

from which it is apparent $ar + s$ must be equal to a function of $x + ay$, from which also $ap + q$ is equal to such a function. And because a can be taken equally negative or positive, two solutions of this kind may be considered

$$ap + q = 2af':\left(x + ay\right) \text{ and } q - ap = 2aF':\left(x - ay\right),$$

from which it is deduced

$$q = af':\left(x + ay\right) + aF':\left(x - ay\right) \text{ and } p = f':\left(x + ay\right) - F':\left(x - ay\right),$$

and hence the equation $dz = pdx + gdy$ is integrated at once and there becomes

$$\mathrm{z} = f:\left(x + ay\right) - F:\left(x - ay\right).$$

In this manner the most sagacious of men has attained the complete solution, but he has not noticed that in place of the function of these introduced not only continuous functions of every kind are allowed, but also all those without the continuity law can to be taken into account.

[d'Alembert gave a special case in his solution corresponding to $a = 1$, while Euler gave a more general result involving discontinuous functions; this led to some hostility between the two, who at the time were both employed at the Berlin Academy of Science; Daniel Bernoulli also gave a solution involving multiples of sines and cosines. See E119 & E140. The problem was finally led to rest with the establishment of Fourier Analysis. This note is an abbreviated version of that in the *O.O*. edition at this point where more early references are given, if you are interested.

We may add that Frederick the Great and Euler never really got on well together, and Euler was never made director of the Academy, which was perhaps part of the reason why he returned to St. Petersburg in 1766, after working in Berlin for 25 years. See e.g. Ch.4 of *Euler and Modern Scienc* for the details]

SCHOLIUM 2

300. Since a great deal may be present generally in this new kind of calculation so that many methods are to be pursued, thus from these a solution of our problem is to be attempted, so that there is put $\left(\frac{dz}{dy}\right)=k\left(\frac{dz}{dx}\right)$, from which initially there becomes

$$\left(\frac{ddz}{dxdy}\right)=k\left(\frac{ddz}{dx^2}\right),$$

then truly

$$\left(\frac{ddz}{dy^2}\right)=k\left(\frac{ddz}{dxdy}\right),$$

from which there is deduced $\left(\frac{ddz}{dy^2}\right)=kk\left(\frac{ddz}{dx^2}\right)$.

Therefore it is evident for our case there must be taken $kk=aa$ or $k=\pm a$. Therefore there may be $k=a$ and on account of $\left(\frac{dz}{dy}\right)=a\left(\frac{dz}{dx}\right)$ there becomes

$$dz=dx\left(\frac{dz}{dx}\right)+dy\left(\frac{dz}{dy}\right)=\left(\frac{dz}{dx}\right)(dx+ady)$$

and hence evidently there shall be $z=f:(x+ay)$ and on account of the choice of *a*, because two values separately are satisfying, also taken together they satisfy the differential equation, and it is concluded that the solution has been found.

Also the problem can be tackled in this way. There is put in place

$$\left(\frac{ddz}{dy^2}\right)=aa\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddv}{dxdy}\right)$$

and there will be

$$\left(\frac{dz}{dy}\right)=\left(\frac{dv}{dx}\right) \quad \text{and} \quad aa\left(\frac{dz}{dx}\right)=\left(\frac{dv}{dy}\right).$$

Now with the formulas of the first order $\left(\frac{dv}{dx}\right)$ and $\left(\frac{dv}{dy}\right)$ found and

$$dv=dx\left(\frac{dv}{dx}\right)+dy\left(\frac{dv}{dy}\right)$$

we will have these equations

$$dz = dx\left(\tfrac{dz}{dx}\right) + dy\left(\tfrac{dz}{dy}\right) \text{ and } dv = dx\left(\tfrac{dz}{dy}\right) + aady\left(\tfrac{dz}{dx}\right),$$

and from the combination of which we deduce

$$dv + adz = (dx + ady)\left(\left(\tfrac{dz}{dy}\right) + a\left(\tfrac{dz}{dx}\right)\right)$$

and hence

$$v + az = f:(x + ay) \text{ and } v - az = F:(x - ay),$$

and thus the same form for *z* comes into being.

Truly the method, as in the solution I have followed, is seen to be better adapted according to the nature of the problem, since also in other more complicated problems it may bring a conspicuous usefulness.

SCHOLIUM 3

301. But our solution has this disadvantage, because it leads to an imaginary expression for this equation

$$\left(\tfrac{ddz}{dy^2}\right) + aa\left(\tfrac{ddz}{dx^2}\right) = 0,$$

evidently

$$z = f:\left(x + ay\sqrt{-1}\right) + F:\left(x - ay\sqrt{-1}\right);$$

but wherever the functions *f* and *F* are continuous, of whatever nature they may be finally, the values of these can be reduced to the form $P \pm Q\sqrt{-1}$ always, from which the following form, deduced from that easily, always will show a real value

$$z = \tfrac{1}{2}\, f:\left(x + ay\sqrt{-1}\right) + \tfrac{1}{2}\, f:\left(x - ay\sqrt{-1}\right) + \tfrac{1}{2\sqrt{-1}}\, F:\left(x + ay\sqrt{-1}\right) - \tfrac{1}{2\sqrt{-1}} F:\left(x - ay\sqrt{-1}\right);$$

for the reduction of this to real numbers , it will be helpful to note on putting

$$x = s\cos.\varphi \text{ and } ay = s\sin.\varphi$$

that there becomes

$$\left(x \pm ay\sqrt{-1}\right)^n = s^n\left(\cos.n\varphi \pm \sqrt{-1}\sin.n\varphi\right).$$

Whereby whenever proposed functions are put together by analytical operations, that is by continuous functions, the values of these really can be shown by the values of the cosine and sine of multiples of *p*. But when functions of these are discontinuous, such a reduction by no means has a place, even if evidently it may be seen that then also the reported form can be arrived at. But for which some curve described, drawn freely by hand, to imagine at any rate in the mind the corresponding abscissas

$$x+ay\sqrt{-1} \text{ and } x-ay\sqrt{-1},$$

and that it may emerge that a real value be assigned to the sum of these, or to the difference divided by $\sqrt{-1}$, that will be real also ? This therefore is discerned to be a great defect in the calculation, that cannot be remedied in any manner at present; and on account of this defect a great many general solutions of this kind lose their power.

PROBLEM 49

302. *With the proposed equation* $\left(\frac{ddz}{dy^2}\right)=PP\left(\frac{ddz}{dx^2}\right)$, *to examine the kind of functions of x and y it is allowed to assume for P, so that the integration is successful with the aid of a reduction.*

SOLUTION

I assume this reduction to come about thus, so that two other variables *t* and *u* are introduced in place of *x* and *y*, with which substitution done following § 232 generally this equation will be produced

$$\left.\begin{aligned}&\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right)+\left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right)+2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dy}\right)^2\left(\frac{ddz}{du^2}\right)\\&-P^2\left(\frac{ddt}{dx^2}\right)-P^2\left(\frac{ddu}{dx^2}\right)-P^2\left(\frac{dt}{dx}\right)^2-2P^2\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)-P^2\left(\frac{du}{dx}\right)^2\end{aligned}\right\}=0.$$

Now a relation of this kind is put in place between the two variables *t, u* and the preceding *x, y*, so that the two formulas $\left(\frac{ddz}{dt^2}\right)$ and $\left(\frac{ddz}{du^2}\right)$ are removed from the calculation, that which comes about on putting

$$\left(\frac{dt}{dy}\right)+P\left(\frac{dt}{dx}\right)=0 \quad\text{and}\quad \left(\frac{du}{dy}\right)-P\left(\frac{du}{dx}\right)=0.$$

Moreover then there shall be

$$\left(\frac{ddt}{dy^2}\right)=-P\left(\frac{ddt}{dxdy}\right)-\left(\frac{dP}{dy}\right)\left(\frac{dt}{dx}\right),$$

but since there will be in the same matter

$$\left(\frac{ddt}{dxdy}\right)=-P\left(\frac{ddt}{dx^2}\right)-\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right),$$

there will be

$$\left(\frac{ddt}{dy^2}\right)=PP\left(\frac{ddt}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right)-\left(\frac{dP}{dy}\right)\left(\frac{dt}{dx}\right)$$

and in a like manner on assuming *P* negative,

$$\left(\frac{ddu}{dy^2}\right)=PP\left(\frac{ddu}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{du}{dx}\right)+\left(\frac{dP}{dy}\right)\left(\frac{du}{dx}\right)$$

With which substitutions made our equation adopts this form

$$\left(P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)\right)\left(\frac{dt}{dx}\right)\left(\frac{dz}{dt}\right)+\left(P\left(\frac{dP}{dx}\right)+\left(\frac{dP}{dy}\right)\right)\left(\frac{du}{dx}\right)\left(\frac{dz}{du}\right)-4PP\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)\left(\frac{ddz}{dtdu}\right)=0;$$

which is allowed to be integrated, since it contains the single formula of the second order $\left(\frac{ddz}{dtdu}\right)$, if either $\left(\frac{dz}{dt}\right)$ or $\left(\frac{dz}{du}\right)$ departs from the calculation. Therefore above we put in place

$$P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)=0,$$

from which equation the nature of the function sought *P* is defined; with which accomplished the equation to be integrated divided by $2P\left(\frac{du}{dx}\right)$ will become

$$\left(\frac{dP}{dx}\right)\left(\frac{dz}{du}\right)-2P\left(\frac{dt}{dx}\right)\left(\frac{ddz}{dtdu}\right)=0,$$

the integral of which on putting $\left(\frac{dz}{du}\right)=v$ becomes

$$2lv=\int\frac{dt\left(\frac{dP}{dx}\right)}{P\left(\frac{dt}{dx}\right)}=2l\left(\frac{dz}{du}\right).$$

Truly it is required to define the earlier function *P* in terms of x and y.
Since there shall be $\left(\frac{dP}{dy}\right)=P\left(\frac{dP}{dx}\right)$, then there will be

$$dP=dx\left(\frac{dP}{dx}\right)+Pdy\left(\frac{dP}{dx}\right)$$

and therefore on putting for the sake of brevity $\left(\frac{dP}{dx}\right)=p$ there becomes $dx=\frac{dP}{p}-Pdy$ and

$$x=-Py+\int dP\left(y+\tfrac{1}{p}\right).$$

Therefore there is put in place $y+\frac{1}{p}=f':P$, and there is found

$$x+Py=f:P$$

and

$$p=\left(\tfrac{dP}{dx}\right)=\tfrac{1}{f':P-y} \quad \text{and} \quad \left(\tfrac{dP}{dy}\right)=\tfrac{P}{f':P-y},$$

from which an account of the determination of the quantity P in terms of x and y is defined.

But for the new variables t and u, on account of $\left(\frac{dt}{dy}\right)=-P\left(\frac{dt}{dx}\right)$, there will be

$$dt=\left(\tfrac{dt}{dx}\right)\left(dx-Pdy\right)$$

and on account of $x=-Py+f:P$ there becomes

$$dt=\left(\tfrac{dt}{dx}\right)\left(dPf':P-2Pdy-ydP\right)=P^{\frac{1}{2}}\left(\tfrac{dt}{dx}\right)\left(\tfrac{dP}{\sqrt{P}}f':P-2dy\sqrt{P}-\tfrac{ydP}{\sqrt{P}}\right);$$

since the integral of which last part of the formula shall be $\int\frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}$, there becomes

$$t=F:\left(\int\tfrac{dP}{\sqrt{P}}f':P-2y\sqrt{P}\right).$$

Thereupon on account of $\left(\frac{du}{dy}\right)=P\left(\frac{du}{dx}\right)$ there is had

$$du=\left(\tfrac{du}{dx}\right)\left(dx+Pdy\right)=\left(\tfrac{du}{dx}\right)\left(dPf':P-ydP\right)$$

and therefore

$$du=\left(\tfrac{du}{dx}\right)\left(f':P-y\right)dP$$

whereby u is equal to a function of P. But in this calculation any functions are allowed to be taken, as from the following integration, the general solution may be obtained at last. Whereby we may put

$$t=\int\tfrac{dP}{\sqrt{P}}f':P-2y\sqrt{P} \quad \text{and} \quad u=P$$

with $x+Py=f:P$ present.

Finally towards finding the integral itself, because there is

$$2l\left(\frac{dz}{du}\right)=\int\frac{dt\left(\frac{dP}{dx}\right)}{P\left(\frac{dt}{dx}\right)},$$

in which integration u or P is taken constant, by what has gone above there will be

$$\frac{dt}{\left(\frac{dt}{dx}\right)}=dPf':P-2Pdy-ydP=-2Pdy$$

on account of constant P and $\left(\frac{dP}{dx}\right)=\frac{1}{f':P-y}$, from which there becomes

$$2l\left(\frac{dz}{dP}\right)=\int\frac{-2dy}{f':P-y}=2l\left(f':P-y\right)+2lF:P$$

or

$$\left(\frac{dz}{dP}\right)=\left(f':P-y\right)F:P$$

and hence again

$$z=\int dP\left(f':P-y\right)F:P$$

here on taking t constant. Therefore since there will be

$$y=\frac{1}{2\sqrt{P}}\int\frac{dP}{\sqrt{P}}f':P-\frac{t}{2\sqrt{P}},$$

and thus

$$f':P-y=f':P-\frac{1}{2\sqrt{P}}\int\frac{dP}{\sqrt{P}}f':P+\frac{t}{2\sqrt{P}},$$

from which there is completed

$$z=\int dP\left(f':P-\frac{1}{2\sqrt{P}}\int\frac{dP}{\sqrt{P}}f':P\right)F:P+\left(\frac{1}{2}\int\frac{dP}{\sqrt{P}}f':P-y\sqrt{P}\right)\int\frac{dP}{\sqrt{P}}F:P+\Phi:\left(\int\frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}\right),$$

which expression contains two arbitrary functions F and Φ.

COROLLARY 1

303. The first member of this form can be transformed thus

$$\int \frac{dP}{\sqrt{P}}\left(\sqrt{P}\cdot f':P-\tfrac{1}{2}\int \frac{dP}{\sqrt{P}}f':P\right)F:P,$$

but

$$\sqrt{P}\cdot f':P-\tfrac{1}{2}\int \frac{dP}{\sqrt{P}}f':P=\int dP\sqrt{P}\cdot f'':P,$$

from which the first member will become

$$\int \frac{dP}{\sqrt{P}}F:P\int dP\sqrt{P}\cdot f'':P .$$

COROLLARY 2

304. But since this first member shall be an indefinite function of P, if that is indicated by $\varPi:P$, there will be

$$\frac{dP}{\sqrt{P}}\ F:P\ =\ \frac{dP\varPi':P}{\int dP\sqrt{P}\cdot f'':P}$$

from which the form of the integral becomes

$$z=\varPi:P+\varPhi:\left(\int \frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}\right)+\left(\int \frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}\right)\int \frac{dP\varPi':P}{2\int dP\sqrt{P}\cdot f'':P}.$$

COROLLARY 3

305. A more special solution arises on taking $\varPi:P=0$ and hence z will be equal to some function of the quantity $\int \frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}$, which on account of $x+Py=f:P$, can be considered to be shown in terms of x and y.

SCHOLIUM

306. Though I have used the same method here and in the preceding problem [§ 296], yet, as it is amazing to see, the case of the preceding problem, from which there was $P=a$, is not contained in this case. An account of this paradox lies in the resolution of the equation $\left(\frac{dP}{dy}\right)=P\left(\frac{dP}{dx}\right)$, for which the value $P=a$ evidently is satisfactory, even if it is not contained in the form thence derived $x+Py=f:P$. Clearly this comes from a certain use, because now we have observed above [Volume I, §§ 546], that often a certain value of a differential equation is able to be satisfactory, which is not contained in the integral, just as we may see the value $x=a$ to satisfy the equation $dy\sqrt{(a-x)}=dx$, which still excludes the form of the integral $y=\mathrm{C}-2\sqrt{(a-x)}$.

Whereby also in our case the value $P = a$ demands a special derivation in the first problem completed.

Concerning the rest, where for $f\!: P$ a certain sure function of P is assumed, we explain by certain examples.

EXAMPLE 1

307. *On taking* $f:P = 0$, *so that there shall be* $P = -\frac{x}{y}$, *to investigate the complete integral of this equation*

$$\left(\frac{ddz}{dy^2}\right) = \frac{xx}{yy}\left(\frac{ddz}{dx^2}\right).$$

[Recall from above for these examples, that $x + Py = f:P$.]

Since there shall be $f':P = 0$, the solution found on account of $\int \frac{dP}{\sqrt{P}} f':P = C$ gives [presumably on can justify this procedure by differentiation, which of course gives zero]

$$z = \frac{-C}{2}\int \frac{dP}{\sqrt{P}} F:P + \left(\tfrac{1}{2}C - y\sqrt{P}\right)\int \frac{dP}{\sqrt{P}} F:P + \Phi:\left(C - 2y\sqrt{P}\right).$$

There is put in place $\int \frac{dP}{\sqrt{P}} F:P = \Pi:P$ and there will be produced

$$z = -y\sqrt{P} \cdot \Pi:P + \Phi:y\sqrt{P}.$$

The value $-\frac{x}{y}$ for P is restored, and on account of $y\sqrt{P} = \sqrt{-xy}$ by involving functions of the imaginary $\sqrt{-1}$, there will be

$$z = \sqrt{xy} \cdot \Pi:\frac{x}{y} + \Phi:\sqrt{xy},$$

which form easily is transferred into this

$$z = \frac{x}{y}\Gamma:\frac{x}{y} + \Theta:xy,$$

where $x\Gamma:\frac{x}{y}$ denotes some homogeneous function of one dimension of x and y.

But the resolution may be put in place on introducing these new variables t and u in place of x and y, so that there shall be $t = C - 2\sqrt{-xy}$ and $u = -\frac{x}{y}$ or also more simply $t = 2\sqrt{xy}$ and $u = \frac{x}{y}$, from which there becomes

$$\left(\frac{dt}{dx}\right) = \frac{\sqrt{y}}{\sqrt{x}}, \quad \left(\frac{dt}{dy}\right) = \frac{\sqrt{x}}{\sqrt{y}}, \quad \left(\frac{ddt}{dx^2}\right) = \frac{-\sqrt{y}}{2x\sqrt{x}}, \quad \left(\frac{ddt}{dy^2}\right) = \frac{-\sqrt{x}}{2y\sqrt{y}},$$

$$\left(\frac{du}{dx}\right) = \frac{1}{y}, \quad \left(\frac{du}{dy}\right) = \frac{-x}{yy}, \quad \left(\frac{ddu}{dx^2}\right) = 0, \quad \left(\frac{ddu}{dy^2}\right) = \frac{2x}{y^3},$$

and on account of $PP = \frac{xx}{yy}$ the proposed equation adopts this form

$$0.\left(\frac{dz}{dt}\right) + \frac{2x}{y^3}\left(\frac{dz}{du}\right) - \frac{4x\sqrt{x}}{yy\sqrt{y}}\left(\frac{ddz}{dtdu}\right) = 0.$$

Now since there shall be $ttu = 4xx$ and $x = \frac{1}{2}t\sqrt{u}$ and also $y = \frac{t}{2\sqrt{u}}$, we will have

$$\frac{8uu}{tt}\left(\frac{dz}{du}\right) - \frac{8uu}{t}\left(\frac{ddz}{dtdu}\right) = 0 \quad \text{or} \quad \left(\frac{dz}{du}\right) = t\left(\frac{ddz}{dtdu}\right).$$

There is made $\left(\frac{dz}{du}\right) = v$, so that there shall be $v = t\left(\frac{dv}{dt}\right)$ and on taking u constant, $\frac{dt}{t} = \frac{dv}{v}$, therefore $v = \left(\frac{dz}{du}\right) = tf':u$. Now let t be constant, and there becomes

$$z = tf:u + F:t = 2\sqrt{xy} \cdot f:\frac{x}{y} + F:\sqrt{xy}$$

as before.

COROLLARY

308. But just as the expression found $z = x\Gamma:\frac{x}{y} + \Theta:xy$ is satisfactory, with the differentials duly taken there may be seen

$$\left(\frac{dz}{dx}\right) = \Gamma:\frac{x}{y} + \frac{x}{y}\Gamma':\frac{x}{y} + y\Theta':xy, \quad \left(\frac{dz}{dy}\right) = \frac{-xx}{yy}\Gamma':\frac{x}{y} + x\Theta':xy,$$

from which again there becomes

$$\left(\frac{ddz}{dx^2}\right) = \frac{2}{y}\Gamma':\frac{x}{y} + \frac{x}{yy}\Gamma'':\frac{x}{y} + yy\Theta'':xy, \quad \text{and} \quad \left(\frac{ddz}{dy^2}\right) = \frac{2xx}{y^3}\Gamma':\frac{x}{y} + \frac{x^3}{y^4}\Gamma'':\frac{x}{y} + xx\Theta'':xy,$$

EXAMPLE 2

309. *Put* $f:P = \frac{PP}{2a}$*, so that there shall be*

$$PP = 2aPy + 2ax \quad \textit{and} \quad P = ay + \sqrt{(aayy + 2ax)},$$

to investigate the complete integral of this equation

$$\left(\frac{ddz}{dy^2}\right) = (2ayy + 2ax)\sqrt{(aayy + 2ax)}\left(\frac{ddz}{dx^2}\right).$$

Since there shall be $f:P=\frac{PP}{2a}$, then there will be $f':P=\frac{P}{a}$ and

$$\int\frac{dP}{\sqrt{P}}f':P=\int\frac{1}{a}dP\sqrt{P}=\frac{2}{3a}P\sqrt{P}$$

from which the general form found above [§ 302] will change into

$$z=\int dP\frac{2P}{3a}F:P+\left(\frac{P\sqrt{P}}{3a}-y\sqrt{P}\right)\int\frac{dP}{\sqrt{P}}F:P+\Phi:\left(\frac{2}{3a}P\sqrt{P}-2y\sqrt{P}\right),$$

there is put

$$\int\frac{dP}{\sqrt{P}}F:P=\Pi:P\,;$$

then there will be $dPF:P=dP\sqrt{P}\cdot\Pi':P$ and

$$z=\frac{2}{3a}\int P^{\frac{3}{2}}dP\Pi':P+\left(\frac{P\sqrt{P}}{3a}-y\sqrt{P}\right)\Pi:P+\Phi:\left(\frac{P\sqrt{P}}{3a}-y\sqrt{P}\right).$$

But there shall be

$$\frac{P}{3a}-y=\frac{-2}{3}y+\frac{1}{3}\sqrt{\left(yy+\frac{2x}{a}\right)}\,;$$

the working out of which formulas leads to exceedingly intricate expressions. But the substitutions leading to the goal are

$$t=\frac{2}{3a}P\sqrt{P}-2y\sqrt{P}\quad\text{and}\quad u=P.$$

COROLLARY

310. If for a more restricted solution there is put $\Pi:P=P^{n-\frac{1}{2}}$, there will be

$$\Pi':P=\left(n-\tfrac{1}{2}\right)P^{n-\frac{3}{2}}$$

and hence there is deduced

$$z=\frac{n}{(n+1)a}P^{n+1}-P^{n}y+\Phi:\left(\frac{P\sqrt{P}}{3a}-y\sqrt{P}\right).$$

Let there be n = 1 and the function Φ may vanish; then there will be

$$z=\frac{1}{2a}PP-Py=x;$$

but the case n = 2 gives

$$z=\frac{2}{3a}P^{3}-P^{2}y=\frac{2}{3}axy+\frac{2}{3}P\left(2x+ayy\right)$$

or

$$z=\frac{2}{3}aay^{3}+2axy+\frac{2}{3}\left(ayy+2x\right)\sqrt{\left(aayy+2ax\right)}\,.$$

SCHOLIUM

311. The form of the integral found [§ 302] can be made simpler in the following way. There is put $\int \frac{dP}{\sqrt{P}} F{:}P = \Pi{:}P$; then there will be

$$F{:}P = \sqrt{P}\cdot \Pi'{:}P$$

and on omitting the latter member

$$\Phi{:}\left(\int \frac{dP}{\sqrt{P}} f'{:}P - 2y\sqrt{P}\right),$$

because it needs to be reduced to zero,

$$z = \int dP\left(\sqrt{P}\cdot f'{:}P - \tfrac{1}{2}\int \frac{dP}{\sqrt{P}} f'{:}P\right)\Pi'{:}P + \tfrac{1}{2}\Pi{:}P\int \frac{dP}{\sqrt{P}} f'{:}P - y\sqrt{P}\cdot \Pi{:}P\,;$$

but

$$\tfrac{1}{2}\Pi{:}P\int \frac{dP}{\sqrt{P}} f'{:}P = \int\left(\tfrac{1}{2}dP\,\Pi'{:}P\int \frac{dP}{\sqrt{P}} f'{:}P + \tfrac{1}{2}\frac{dP}{\sqrt{P}}\Pi{:}Pf'{:}P\right),$$

from which there becomes

$$z = \int \Pi'{:}PdP\sqrt{P}\cdot f'{:}P + \tfrac{1}{2}\int \Pi{:}P\frac{dP}{\sqrt{P}} f'{:}P - y\sqrt{P}\cdot \Pi{:}P.$$

again there is

$$\int dP\,\Pi'{:}P\sqrt{P}\cdot f'{:}P = \Pi{:}P\sqrt{P}f'{:}P - \int \Pi{:}P\left(\frac{dP}{2\sqrt{P}} f'{:}P + dP\sqrt{P}\cdot f''{:}P\right)$$

and thus

$$z = \Pi{:}P\sqrt{P}\cdot f'{:}P - \int dP\,\Pi{:}P\sqrt{P}\cdot f''{:}P - y\sqrt{P}\cdot \Pi{:}P\,;$$

again there is established

$$\int dP\,\Pi{:}P\sqrt{P}\cdot f''{:}P = \Theta{:}P\,;$$

there will be

$$\Pi{:}P = \frac{\Theta'{:}P}{\sqrt{P}\cdot f''{:}P} \text{ and } z = \frac{\Theta'{:}P}{\sqrt{P}\cdot f''{:}P}\left(f'{:}P - y\right) - \Theta{:}P + \Phi{:}\left(\int \frac{dP}{\sqrt{P}} f'{:}P - 2y\sqrt{P}\right),$$

which form without doubt is much simpler than the first found.

PROBLEM 50

312. *With the proposed equation*

$$\left(\frac{ddz}{dy^2}\right) - PP\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dy}\right) + R\left(\frac{dz}{dx}\right) = 0$$

to find the cases of the quantities P, Q, R, for which the integration follows with the aid of the reduction used before.

SOLUTION

With the two new variables t and u introduced we will have [§ 232]

$$\begin{aligned}0 = &\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right) + \left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right) + \left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right) + 2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right) + \left(\frac{du}{dy}\right)^2\left(\frac{ddz}{du^2}\right)\\ &-P^2\left(\frac{ddt}{dx^2}\right) - P^2\left(\frac{ddu}{dx^2}\right) - P^2\left(\frac{dt}{dx}\right)^2 - 2P^2\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right) - P^2\left(\frac{du}{dx}\right)^2\\ &+Q\left(\frac{dt}{dy}\right) + Q\left(\frac{du}{dy}\right)\\ &+R\left(\frac{dt}{dx}\right) + R\left(\frac{du}{dx}\right)\end{aligned}$$

Therefore as before we may establish

$$\left(\frac{dt}{dy}\right) = P\left(\frac{dt}{dx}\right) \quad \text{and} \quad \left(\frac{du}{dy}\right) = -P\left(\frac{du}{dx}\right),$$

from which there becomes

$$\left(\frac{ddt}{dxdy}\right) = P\left(\frac{ddt}{dx^2}\right) + \left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right)$$

both

$$\left(\frac{ddt}{dy^2}\right) = PP\left(\frac{ddt}{dx^2}\right) + P\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right) + \left(\frac{dP}{dy}\right)\left(\frac{dt}{dx}\right)$$

and

$$\left(\frac{ddu}{dy^2}\right) = PP\left(\frac{ddu}{dx^2}\right) + P\left(\frac{dP}{dx}\right)\left(\frac{du}{dx}\right) - \left(\frac{dP}{dy}\right)\left(\frac{du}{dx}\right),$$

and the equation to be resolved will be

$$0 = \left(P\left(\frac{dP}{dx}\right) + \left(\frac{dP}{dy}\right) + PQ + R\right)\left(\frac{dt}{dx}\right)\left(\frac{dz}{dt}\right) - 4PP\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)\left(\frac{ddz}{dtdu}\right) + \left(P\left(\frac{dP}{dx}\right) - \left(\frac{dP}{dy}\right) - PQ + R\right)\left(\frac{du}{dx}\right)\left(\frac{dz}{du}\right).$$

Now it is evident that the integration can be put in place, if either the formula $\left(\frac{dz}{dt}\right)$ or $\left(\frac{dz}{du}\right)$ is not present in the calculation. Therefore we may put in place to be

$$P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)-PQ+R=0 \quad \text{or} \quad R=PQ+\left(\frac{dP}{dy}\right)-P\left(\frac{dP}{dx}\right)$$

and the resulting equation divided by $\left(\frac{dt}{dx}\right)$ becomes

$$0=2\left(PQ+\left(\frac{dP}{dy}\right)\right)\left(\frac{dz}{dt}\right)-4PP\left(\frac{du}{dx}\right)\left(\frac{ddz}{dtdu}\right).$$

There becomes $\left(\frac{dz}{dt}\right)=v$ *;* there will be

$$\left(PQ+\left(\frac{dP}{dy}\right)\right)v-2PP\left(\frac{du}{dx}\right)\left(\frac{dv}{du}\right)=0\,;$$

there is assumed t constant, so that there comes about

$$\frac{dv}{v}=\frac{\left(PQ+\left(\frac{dP}{dy}\right)\right)du}{2PP\left(\frac{du}{dx}\right)}\,,$$

where it is necessary that the quantities P, Q, $\left(\frac{dP}{dy}\right)$ and $\left(\frac{du}{dx}\right)$ may be expressed by the new variables t and u. Therefore in the first place it is convenient to define these. Since there shall be

$$\left(\frac{dt}{dy}\right)=P\left(\frac{dt}{dx}\right) \quad \text{and} \quad \left(\frac{du}{dy}\right)=-P\left(\frac{du}{dx}\right),$$

there will be

$$dt=\left(\frac{dt}{dx}\right)(dx+Pdy) \ \text{ and } \ du=\left(\frac{du}{dx}\right)(dx-Pdy)\,;$$

therefore $\left(\frac{dt}{dx}\right)$ and $\left(\frac{du}{dx}\right)$ are the integrable factors returning the formulas $dx+Pdy$ and $dx-Pdy$; indeed there is no need, that hence the values t and u are defined more generally. Let p and q be such multipliers given in terms of x and y and let there be

$$t=\int p(dx+Pdy) \ \text{ and } \ u=\int q(dx-Pdy)\,,$$

from which the above integration becomes

$$\frac{dv}{v}=\frac{\left(PQ+\left(\frac{dP}{dy}\right)\right)du}{2PPq},$$

in which integration the quantity $t=\int p(dx+Pdy)$ is required to be observed as constant. Or on account of $du=q(dx-Pdy)$ there will be

$$\frac{dv}{v}=\frac{\left(PQ+\left(\frac{dP}{dy}\right)\right)(dx-Pdy)}{2PP}$$

Truly on account of $dt=0$ there is $dx=-Pdy$, thus so that there is produced

$$\frac{dv}{v}=-\frac{dy}{P}\left(PQ+\left(\frac{dP}{dy}\right)\right),$$

where on account of constant t and the value given by x and y an expression of x can be substituted in terms of y and t, so that only the variable y is present, and on finding the integral

$$-\int\frac{dy}{P}\left(PQ+\left(\frac{dP}{dy}\right)\right)=lV$$

there will be $v=Vf:t=\left(\frac{dz}{dt}\right)$. Now on putting u constant; there will be

$$z=\int Vdtf:t+F:u.$$

But the condition, under which this integration has a place, demands that there shall be

$$R=PQ+\left(\frac{dP}{dy}\right)-P\left(\frac{dP}{dx}\right).$$

COROLLARIUM 1

313. In the same manner the proposed equation may be allowed to be resolved, if there should be

$$R=-PQ-\left(\frac{dP}{dy}\right)-P\left(\frac{dP}{dx}\right)$$

and there remains as before

$$t=\int p(dx+Pdy)\ \text{ and }\ u=\int q(dx-Pdy).$$

Then indeed there shall be

$$0=-\left(PQ+\left(\frac{dP}{dy}\right)\right)\left(\frac{dz}{du}\right)-2PP\left(\frac{dt}{dx}\right)\left(\frac{ddt}{dtdu}\right)$$

which on putting $\left(\frac{dz}{du}\right)=v$ and on taking u constant gives

$$\frac{dv}{v}=\frac{-\left(PQ+\left(\frac{dP}{dy}\right)\right)dt}{2PP\left(\frac{dt}{dx}\right)}=\frac{-\left(PQ+\left(\frac{dP}{dy}\right)\right)(dx+Pdy)}{2PP}.$$

COROLLARY 2

314. If again for the reason given, because $u=\int q(dx-Pdy)$ shall be constant and $dx=Pdy$, there is put

$$\int-\frac{dy\left(PQ+\left(\frac{dP}{dy}\right)\right)}{P}=lV,$$

there will be

$$v=Vf:u=\left(\frac{dz}{du}\right),$$

from which now at last on taking $t=\int p(dx+Pdy)$ constant there is deduced

$$z=\int Vduf:u+F:t.$$

EXAMPLE 1

315. *If there is taken $P=a$ and $R=aQ$, whatever the function Q should be of x and y, to integrate the equation*

$$\left(\frac{ddz}{dy^2}\right)-aa\left(\frac{ddz}{dx^2}\right)+Q\left(\frac{dz}{dy}\right)+R\left(\frac{dz}{dx}\right)=0$$

Since here there shall be $P=a$, there will be $p=1$, $q=1$ and $t=x+ay$ and also $u=x-ay$, from which on putting $\left(\frac{dz}{dt}\right)=v$ there becomes

$$\frac{dv}{v}=\frac{aQdu}{2aa}=\frac{Qdu}{2a}.$$

Therefore since there shall be

$$x=\frac{t+u}{2} \text{ and } y=\frac{t-u}{2a},$$

with these values substituted Q is made a function of t and u and with t considered as constant there will be

$$lv=\frac{1}{2a}\int Qdu+lf:t \quad \text{or} \quad \left(\frac{dz}{dt}\right)=e^{\frac{1}{2a}\int Qdu}f:t$$

and now on taking u constant

$$z=\int e^{\frac{1}{2a}\int Qdu}dtf:t+F:u.$$

COROLLARY 1

316. If Q shall be a constant $=2ab$, the integral of this equation

$$\left(\frac{ddz}{dy^2}\right)-aa\left(\frac{ddz}{dx^2}\right)+2ab\left(\frac{dz}{dy}\right)+2aab\left(\frac{dz}{dx}\right)=0$$

shall be

$$z=e^{bu}f:t+F:u=e^{b(x-ay)}f:(x+ay)+F:(x-ay)$$

or

$$z=e^{b(x-ay)}\left(f:(x+ay)+F:(x-ay)\right).$$

COROLLARY 2

317. If $Q=\frac{a}{x}$ then the integral of this equation

$$\left(\frac{ddz}{dy^2}\right)-aa\left(\frac{ddz}{dx^2}\right)+\frac{a}{x}\left(\frac{dz}{dy}\right)+\frac{aa}{x}\left(\frac{dz}{dx}\right)=0$$

on account of

$$\int Qdu=\int\frac{adu}{x}=\int\frac{2adu}{t+u}=2al(t+u)$$

will be

$$z=\int(t+u)dtf:t+F:u=\int tdtf:t+u\int dtf:t+F:u.$$

Or if there shall be $f:t=\Pi'':t;$ then there will be $\int dtf:t=\Pi':t$ and

$$\int tdtf:t=\int td.\Pi':t=t\Pi':t-\int dt\Pi':t=t\Pi':t-\Pi:t,$$

therefore

$$z=(t+u)\Pi':t-\Pi:t+F:u$$

or

$$z=2x\Pi':(x+ay)-\Pi:(x+ay)+F:(x-ay).$$

EXAMPLE 2

318. *Let there be* $P=\frac{x}{y}$ *and* $R=\frac{-x}{y}Q+\frac{x}{yy}-\frac{x}{yy}=\frac{-x}{y}Q$ *and there is taken* $Q=\frac{1}{x}$, *so that there shall be* $R=\frac{-1}{y}$ *and this equation to be integrated must become*

$$\left(\frac{ddz}{dy^2}\right)-\frac{xx}{yy}\left(\frac{ddz}{dx^2}\right)+\frac{1}{x}\left(\frac{dz}{dy}\right)-\frac{1}{y}\left(\frac{dz}{dx}\right)=0.$$

Therefore since there shall be

$$t=\int p\left(dx+\frac{xdy}{y}\right) \text{ and } u=\int q\left(dx-\frac{xdy}{y}\right)$$

there is assumed $p=y$ and $q=\frac{1}{y}$, so that there becomes $t=xy$ and $u=\frac{x}{y}$. Now on putting $\left(\frac{dz}{du}\right)=v$ and on taking u constant, from Corollary 1 there becomes

$$\frac{dv}{v}=\frac{-\left(\frac{1}{y}-\frac{x}{yy}\right)dt}{\frac{2xx}{yy}y}=\frac{-(y-x)dt}{2xxy}.$$

Truly there is $tu=xx$ and hence $x=\sqrt{tu}$ and $y=\sqrt{\frac{t}{u}}$ and also $2xxy=2t\sqrt{tu}$, from which there becomes

$$\frac{dv}{v}=\frac{\left(\sqrt{tu}-\sqrt{\frac{t}{u}}\right)dt}{2t\sqrt{tu}}=\frac{dt}{2t}-\frac{dt}{2tu}$$

and on account of constant u $lv=\frac{1}{2}lt-\frac{1}{2u}lt$, hence

$$\left(\frac{dz}{du}\right)=t^{\frac{1}{2}-\frac{1}{2u}}f:u.$$

Whereby now on taking t constant there will be

$$z=t^{\frac{1}{2}}\int t^{-\frac{1}{2u}}duf:u+F:t.$$

Or on putting $-\frac{1}{2u}=s$, so that there shall be $s=-\frac{y}{2x}$, and there will be

$$z=t^{\frac{1}{2}}\int t^{s}dsf:s+F:t.$$

In this integration $\int t^s dsf:s$, s alone is variable and at last for the assumed integral to be restored, there must be $t = xy$ and $s = \frac{-y}{2x}$.

Otherwise it is apparent [that the integral] be satisfied particularly by any function of *xy*.

PROBLEM 51

319. *For the proposed general equation*

$$\left(\frac{ddz}{dy^2}\right) - 2P\left(\frac{ddz}{dxdy}\right) + (PP - QQ)\left(\frac{ddz}{dx^2}\right) + R\left(\frac{dz}{dy}\right) + S\left(\frac{dz}{dx}\right) + Tz + V = 0$$

to find the conditions of the quantities P, Q, R, S, T, so that the integration follows with the help of the given reduction.

SOLUTION

From the same substitution made [§ 232] with two new variables *t* and *u* being introduced, our equation adopts the following form

$$0 = V + Tz$$

$$\begin{aligned}
&+\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right) + \left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right) + \left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right) + 2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right) + \left(\frac{du}{dy}\right)^2\left(\frac{ddz}{du^2}\right)\\
&-2P\left(\frac{ddt}{dxdy}\right) - 2P\left(\frac{ddu}{dxdy}\right) - 2P\left(\frac{dt}{dx}\right)\left(\frac{dt}{dy}\right) - 2P\left(\frac{dt}{dx}\right)\left(\frac{du}{dy}\right) - 2P\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)\\
&\qquad - 2P\left(\frac{du}{dx}\right)\left(\frac{dt}{dy}\right)\\
&+\left(P^2 - Q^2\right)\left(\frac{ddt}{dx^2}\right) + \left(P^2 - Q^2\right)\left(\frac{ddu}{dx^2}\right) + \left(P^2 - Q^2\right)\left(\frac{dt}{dx}\right)^2 + 2\left(P^2 - Q^2\right)\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right) + \left(P^2 - Q^2\right)\left(\frac{du}{dx}\right)^2\\
&\qquad R\left(\frac{dt}{dy}\right) + R\left(\frac{du}{dy}\right)\\
&\qquad S\left(\frac{dt}{dx}\right) + S\left(\frac{du}{dx}\right)
\end{aligned}$$

Now these two new variables *t* and *u* thus may be determined in terms of *x* and *y*, so that the formulas
$\left(\frac{ddz}{dt^2}\right)$ and $\left(\frac{ddz}{du^2}\right)$ vanish, and there must become

$$\left(\frac{dt}{dy}\right) = (P + Q)\left(\frac{dt}{dx}\right) \text{ and } \left(\frac{du}{dy}\right) = (P - Q)\left(\frac{du}{dx}\right)$$

from which it is apparent these variables must be determined in the following way

$$t=\int p\left(dx+(P+Q)dy\right)\text{ and } u=\int q\left(dx+(P-Q)dy\right)$$

on taking p and q thus, so that these formulas may admit integration [as above]. Now since there shall be

$$\left(\frac{ddt}{dxdy}\right)=(P+Q)\left(\frac{ddt}{dx^2}\right)\quad+\left(\left(\frac{dP}{dx}\right)+\left(\frac{dQ}{dx}\right)\right)\left(\frac{dt}{dx}\right),$$
$$\left(\frac{ddt}{dy^2}\right)=(P+Q)^2\left(\frac{ddt}{dx^2}\right)\quad+(P+Q)\left(\left(\frac{dP}{dx}\right)+\left(\frac{dQ}{dx}\right)\right)\left(\frac{dt}{dx}\right)+\left(\left(\frac{dP}{dy}\right)+\left(\frac{dQ}{dy}\right)\right)\left(\frac{dt}{dx}\right),$$
$$\left(\frac{ddu}{dxdy}\right)=(P-Q)\left(\frac{ddu}{dx^2}\right)\quad+\left(\left(\frac{dP}{dx}\right)-\left(\frac{dQ}{dx}\right)\right)\left(\frac{du}{dx}\right),$$
$$\left(\frac{ddu}{dy^2}\right)=(P-Q)^2\left(\frac{ddu}{dx^2}\right)\quad+(P-Q)\left(\left(\frac{dP}{dx}\right)-\left(\frac{dQ}{dx}\right)\right)\left(\frac{du}{dx}\right)+\left(\left(\frac{dP}{dy}\right)-\left(\frac{dQ}{dy}\right)\right)\left(\frac{du}{dx}\right),$$

hence the coefficient of the formula $2\left(\frac{ddz}{dtdu}\right)$ is found

$$-2QQ\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right),$$

the coefficient of the term $\left(\frac{dz}{dt}\right)$

$$\left(-(P-Q)\left(\frac{dP+dQ}{dx}\right)+\left(\frac{dP+dQ}{dy}\right)+R(P+Q)+S\right)\left(\frac{dt}{dx}\right),$$

truly the coefficient of the term $\left(\frac{dz}{du}\right)$

$$\left(-(P+Q)\left(\frac{dP-dQ}{dx}\right)+\left(\frac{dP-dQ}{dy}\right)+R(P-Q)+S\right)\left(\frac{du}{dx}\right).$$

Truly there is

$$\left(\frac{dt}{dx}\right)=p\quad\text{and}\quad\left(\frac{du}{dx}\right)=q$$

from which, if for the sake of brevity there may be called

$$S+R(P+Q)+\left(\frac{dP+dQ}{dy}\right)-(P-Q)\left(\frac{dP+dQ}{dx}\right)=M$$

and

$$S+R(P-Q)+\left(\frac{dP-dQ}{dy}\right)-(P+Q)\left(\frac{dP-dQ}{dx}\right)=N,$$

our equation to be resolved will be

$$0 = V + Tz + Mp\left(\frac{dz}{dt}\right) + Nq\left(\frac{dz}{du}\right) - 4QQpq\left(\frac{ddz}{dtdu}\right)$$

or, so that it can be compared with the forms shown above § 294 and § 295,

$$\left(\frac{ddz}{dtdu}\right) - \frac{M}{4QQq}\left(\frac{dz}{dt}\right) - \frac{N}{4QQp}\left(\frac{dz}{du}\right) - \frac{T}{4QQpq}z - \frac{V}{4QQpq} = 0,$$

which, if for the sake of brevity there is put

$$\frac{M}{4QQq} = K \quad \text{and} \quad \frac{N}{4QQp} = L,$$

which admits integration in the twofold case: the one, if there should be

$$-\frac{T}{4QQpq} = KL - \left(\frac{dL}{du}\right) \quad \text{or} \quad T = 4QQpq\left(\frac{dL}{du}\right) - \frac{MN}{4QQ},$$

truly the other, if there should be

$$-\frac{T}{4QQpq} = KL - \left(\frac{dK}{dt}\right) \quad \text{or} \quad T = 4QQpq\left(\frac{dK}{dt}\right) - \frac{MN}{4QQ}.$$

Truly since K and L are given by x and y, these formulas $\left(\frac{dK}{dt}\right)$ and $\left(\frac{dL}{du}\right)$ can be reduced thus, so that there shall be

$$\left(\frac{dK}{dt}\right) = \frac{Q-P}{2Qp}\left(\frac{dK}{dx}\right) + \frac{1}{2Qq}\left(\frac{dK}{dy}\right) \quad \text{and} \quad \left(\frac{dL}{du}\right) = \frac{P+Q}{2Qq}\left(\frac{dL}{dx}\right) - \frac{1}{2Qq}\left(\frac{dL}{dy}\right).$$

Moreover just as the integrals must be found for these cases, that indeed has been declared above [§ 294, 295], from which it would be superfluous to repeat these tedious calculations here; indeed for any case offered the solution thence can be desired.

SCHOLIUM 1

320. As it pertains to that reduction of the formulas, that is put in place in the following manner. Since in general there will be

$$dz = dx\left(\frac{dz}{dx}\right) + dy\left(\frac{dz}{dy}\right),$$

from the formulas

$$dt = pdx + p(P+Q)dy \quad \text{and} \quad du = qdx + q(P-Q)dy$$

there will be

$$qdt - pdu = 2pqQdy \quad \text{or} \quad dy = \frac{qdt - pdu}{2Qpq}$$

and

$$q(P-Q)dt-p(P+Q)du=-2Qpqdx \quad \text{or} \quad dx=\frac{p(P+Q)du-q(P-Q)dt}{2Qpq}.$$

With which values substituted there will be obtained

$$dz=\left(\frac{(P+Q)du}{2Qq}-\frac{(P-Q)dt}{2Qp}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dt}{2Qp}-\frac{du}{2Qq}\right)\left(\frac{dz}{dy}\right),$$

thus so that dz may be expressed by the differentials dt and du. Therefore on putting u constant and $du=0$ there will be

$$\left(\frac{dz}{dt}\right)=\frac{Q-P}{2Qp}\left(\frac{dz}{dx}\right)+\frac{1}{2Qp}\left(\frac{dz}{dy}\right),$$

but on putting t constant and $dt = 0$ there will be

$$\left(\frac{dz}{du}\right)=\frac{P+Q}{2Qq}\left(\frac{dz}{dx}\right)-\frac{1}{2Qq}\left(\frac{dz}{dy}\right).$$

SCHOLlUM 2

321. Therefore the method treated in this chapter consists of this, that equations of this kind with the help of two new variables t and u introduced are reduced to this form

$$\left(\frac{ddz}{dtdu}\right)+P\left(\frac{dz}{dt}\right)+Q\left(\frac{dz}{du}\right)+Rz+S=0,$$

from which we have seen in the preceding chapter [§ 275, 278, 287, 294, 295], which cases there that can be integrated. Therefore also all the equations for the same cases, which are allowed to be reduced to such a form, admit integration.

Truly certain cases of the same form of great singularity, the integral of which can be absolved, from which again an infinite multitude of other equations arise, indeed are able to be reduced to that, and which equally admit to integration. Which on that account we shall explain more carefully in the following chapter.

CAPUT III

SI DUAE VEL OMNES FORMULAE SECUNDI GRADUS PER RELIQUAS QUANTITATES DETERMINANTUR

PROBLEMA 48

296. *Si z eiusmodi debeat esse functio ipsarum x et y, ut fiat* $\left(\frac{ddz}{dy^2}\right)=aa\left(\frac{ddz}{dx^2}\right)$, *indolem functionis z determinare.*

SOLUTIO

Introducantur binae novae variabiles t et u, ut sit

$$t=\alpha x+\beta y \text{ et } u=\gamma x+\delta y,$$

atque ex §233 omnes formulae differentiales sequentes mutationes subibunt

$$\left(\frac{dz}{dx}\right)=\alpha\left(\frac{dz}{dt}\right)+\gamma\left(\frac{dz}{du}\right),\ \left(\frac{dz}{dy}\right)=\beta\left(\frac{dz}{dt}\right)+\delta\left(\frac{dz}{du}\right),$$

$$\left(\frac{ddz}{dx^2}\right)=\alpha\alpha\left(\frac{ddz}{dt^2}\right)+2\alpha\gamma\left(\frac{ddz}{dtdu}\right)+\gamma\gamma\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dxdy}\right)=\alpha\beta\left(\frac{ddz}{dt^2}\right)+(\alpha\delta+\beta\gamma)\left(\frac{ddz}{dtdu}\right)+\gamma\delta\left(\frac{ddz}{du^2}\right),$$

$$\left(\frac{ddz}{dy^2}\right)=\beta\beta\left(\frac{ddz}{dt^2}\right)+2\beta\delta\left(\frac{ddz}{dtdu}\right)+\delta\delta\left(\frac{ddz}{du^2}\right),$$

unde nostra aequatio transibit in hanc

$$(\beta\beta-\alpha\alpha aa)\left(\frac{ddz}{dt^2}\right)+2(\beta\delta-\alpha\gamma aa)\left(\frac{ddz}{dtdu}\right)+(\delta\delta-\gamma\gamma aa)\left(\frac{ddz}{du^2}\right)=0.$$

Ponatur ergo

$$\beta\beta=\alpha\alpha aa \quad \text{et} \quad \delta\delta=\gamma\gamma aa$$

seu

$$\alpha=1,\quad \gamma=1,\quad \beta=a \quad \text{et} \quad \delta=-a,$$

ut binae formulae extremae evanescant, quod fit ponendo

$$t=x+ay \quad \text{et} \quad u=x-ay,$$

$$-2(aa+aa)\left(\frac{ddz}{dtdu}\right)=0, \quad \text{seu} \quad \left(\frac{ddz}{dtdu}\right)=0$$

unde per § 270 colligitur integrale completum

$$z=f:t+F:u$$

ac pro *t* et *u* restitutis valoribus

$$z=f:(x+ay)+F:(x-ay),$$

quae forma manifesto satisfacit, cum sit

$$\left(\frac{dz}{dx}\right)=f':(x+ay)+F':(x-ay), \quad \left(\frac{dz}{dy}\right)=af':(x+ay)-aF':(x-ay),$$

$$\left(\frac{ddz}{dx^2}\right)=f'':(x+ay)+F'':(x-ay), \quad \left(\frac{ddz}{dy^2}\right)=aaf'':(x+ay)+aaF'':(x-ay).$$

COROLLARIUM 1

297. Valor igitur ipsius *z* aequatur aggregato duarum functionum arbitrariarum, alterius ipsius $x+ay$, alterius ipsius $x-ay$, atque ambae hae functiones ita ad arbitrium assumi possunt, ut etiam functiones discontinuas earum loco capere liceat.

COROLLARIUM 2

298. Pro lubitu ergo binae curvae quaecunque etiam libero manus tractu descriptae ad hunc usum adhiberi possunt. Scilicet si in una abscissa capiatur = $x+ay$, in altera vero abscissa = $x-ay$, summa applicatarum semper valorem idoneum pro functione *z* suppeditabit.

SCHOLION 1

299. Hoc fere primum est problema, quod in hoc novo calculi genere solvendum occurrit; perduxerat autem solutio generalis problematis de cordis vibrantibus ad hanc ipsam aequationem, quam hic tractavimus. Celeb. Alembertus, qui hoc problema primus felici successu est aggressus, methodo singulari aequationem integravit; scilicet cum esse oporteat $\left(\frac{ddz}{dy^2}\right)=a^2\left(\frac{ddz}{dx^2}\right)$, posito $dz=pdx+qdy$ indeque $dp=rdx+sdy$ et $dq=sdx+tdy$ illa aequatio postulat, ut sit $t=aar$. Consideratis porro istis aequationibus

$$dp=rdx+sdy, \quad dq=sdx+aardy$$

elicitur combinando

$$adp+dq=ar(dx+ady)+s(ady+dx)$$

seu

$$adp+dq=(ar+s)(dx+ady),$$

unde patet $ar+s$ functioni ipsius $x+ay$ aequari debere, ex quo etiam $ap+q$ tali functioni aequatur. Atque quia a aeque negative ac positive accipi potest, habentur duae huiusmodi aequationes

$$ap+q=2af':(x+ay) \text{ et } q-ap=2aF':(x-ay),$$

unde colligitur

$$q=af':(x+ay)+aF':(x-ay) \quad \text{et} \quad p=f':(x+ay)-F':(x-ay),$$

hincque aequatio $dz=pdx+gdy$ sponte integratur fitque

$$z=f:(x+ay)-F:(x-ay).$$

Hoc modo sagacissimus Vir integrale completum est adeptus, sed non animadvertit loco functionum harum introductarum non solum omnis generis functiones continuas, sed etiam omni continuitatis lege destitutas accipi licere.

SCHOLION 2

300. Cum plurimum intersit in hoc novo calculi genere quam plurimas methodos persequi, ab aliis solutio nostrae aequationis ita est tentata, ut ponerent $\left(\frac{dz}{dy}\right)=k\left(\frac{dz}{dx}\right)$, unde fit primo

$$\left(\frac{ddz}{dxdy}\right)=k\left(\frac{ddz}{dx^2}\right),$$

tum vero

$$\left(\frac{ddz}{dy^2}\right)=k\left(\frac{ddz}{dxdy}\right),$$

ex quo colligitur $\left(\frac{ddz}{dy^2}\right)=kk\left(\frac{ddz}{dx^2}\right)$.

Evidens ergo est pro nostro casu capi debere $kk=aa$ seu $k=\pm a$. Sit ergo $k=a$ et ob $\left(\frac{dz}{dy}\right)=a\left(\frac{dz}{dx}\right)$ fiet

$$dz=dx\left(\frac{dz}{dx}\right)+dy\left(\frac{dz}{dy}\right)=\left(\frac{dz}{dx}\right)(dx+ady)$$

hincque manifestum est fore $z=f:(x+ay)$ et ob a ambiguum, quoniam bini valores seorsim satisfacientes etiam iuncti satisfacient, concluditur ipsa solutio inventa.

Hoc etiam modo negotium confici potest. Statuatur

$$\left(\frac{ddz}{dy^2}\right)=aa\left(\frac{ddz}{dx^2}\right)=\left(\frac{ddv}{dxdy}\right)$$

eritque

$$\left(\frac{dz}{dy}\right)=\left(\frac{dv}{dx}\right) \quad \text{et} \quad aa\left(\frac{dz}{dx}\right)=\left(\frac{dv}{dy}\right).$$

Inventis nunc formulis primi gradus $\left(\frac{dv}{dx}\right)$ et $\left(\frac{dv}{dy}\right)$ ob

$$dv=dx\left(\frac{dv}{dx}\right)+dy\left(\frac{dv}{dy}\right)$$

habebimus has aequationes

$$dz=dx\left(\frac{dz}{dx}\right)+dy\left(\frac{dz}{dy}\right) \;\text{ et }\; dv=dx\left(\frac{dz}{dy}\right)+aady\left(\frac{dz}{dx}\right),$$

ex quarum combinatione colligimus

$$dv+adz=\left(dx+ady\right)\left(\left(\frac{dz}{dy}\right)+a\left(\frac{dz}{dx}\right)\right)$$

hincque

$$v+az=f:\left(x+ay\right) \;\text{ et }\; v-az=F:\left(x-ay\right),$$

sicque pro z eadem forma exsurgit.

Methodus vero, quam in solutione sum secutus, ad naturam rei magis videtur accommodata, cum etiam in aliis problematibus magis complicatis insignem utilitatem afferat.

SCHOLION 3

301. Nostra autem solutio hoc habet incommodi, quod pro hac aequatione

$$\left(\frac{ddz}{dy^2}\right)+aa\left(\frac{ddz}{dx^2}\right)=0$$

ad expressionem imaginariam deducit, scilicet

$$z=f:\left(x+ay\sqrt{-1}\right)+F:\left(x-ay\sqrt{-1}\right);$$

quoties autem functiones f et F sunt continuae, cuiuscunque demum fuerint indolis, semper earum valores ad hanc formam $P\pm Q\sqrt{-1}$ reduci possunt, unde sequens forma ex illa facile deducenda semper valorem realem exhibebit

$$z=\tfrac{1}{2}f:\left(x+ay\sqrt{-1}\right)+\tfrac{1}{2}f:\left(x-ay\sqrt{-1}\right)+\tfrac{1}{2\sqrt{-1}}F:\left(x+ay\sqrt{-1}\right)-\tfrac{1}{2\sqrt{-1}}F:\left(x-ay\sqrt{-1}\right),$$

pro cuius ad realitatem reductione notasse iuvabit posito

$$x = s\cos.\varphi \quad \text{et} \quad ay = s\sin.\varphi$$

fore

$$\left(x \pm ay\sqrt{-1}\right)^n = s^n\left(\cos.n\varphi \pm \sqrt{-1}\sin.n\varphi\right).$$

Quare quoties functiones propositae per operationes analyticas sunt conflatae, hoc est continuae, earum valores realiter per cosinus et sinus angulorum multiplorum ipsius p exhiberi possunt. Quando autem functiones illae sunt discontinuae, talis reductio neutiquam locum habet, etiamsi certum videatur etiam tunc formam allatam valorem realem esse adepturam. Quis autem in curva quacunque libero manus ductu descripta applicatas abscissis

$$x + ay\sqrt{-1} \quad \text{et} \quad x - ay\sqrt{-1}$$

respondentes animo saltem imaginari ac summam earum realem assignare valuerit aut differentiam, quae per $\sqrt{-1}$ divisa etiam erit realis? Hic ergo haud exiguus defectus calculi cernitur, quem nullo adhuc modo supplere licet; atque ob hunc ipsum defectum huiusmodi solutiones universales plurimum de sua vi perdunt.

PROBLEMA 49

302. *Proposita aequatione* $\left(\frac{ddz}{dy^2}\right) = PP\left(\frac{ddz}{dx^2}\right)$ *inquirere, quales functiones ipsarum x et y pro P assumere liceat, ut integratio ope reductionis succedat.*

SOLUTIO

Reductionem hanc ita fieri assumo, ut loco x et y binae aliae variabiles t et u introducantur, qua substitutione secundum § 232 in genere facta prodit haec aequatio

$$\left.\begin{aligned}&\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right)+\left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right)+\left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right)+2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right)+\left(\frac{du}{dy}\right)^2\left(\frac{ddz}{du^2}\right)\\&-P^2\left(\frac{ddt}{dx^2}\right)-P^2\left(\frac{ddu}{dx^2}\right)-P^2\left(\frac{dt}{dx}\right)^2-2P^2\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)-P^2\left(\frac{du}{dx}\right)^2\end{aligned}\right\}=0.$$

Iam relatio inter binas variabiles t, u et praecedentes x, y eiusmodi statuatur, ut binae formulae $\left(\frac{ddz}{dt^2}\right)$ et $\left(\frac{ddz}{du^2}\right)$ ex calculo egrediantur, id quod fiet ponendo

$$\left(\frac{dt}{dy}\right)+P\left(\frac{dt}{dx}\right)=0 \quad \text{et} \quad \left(\frac{du}{dy}\right)-P\left(\frac{du}{dx}\right)=0.$$

Tum autem erit

$$\left(\frac{ddt}{dy^2}\right)=-P\left(\frac{ddt}{dxdy}\right)-\left(\frac{dP}{dy}\right)\left(\frac{dt}{dx}\right),$$

at cum sit indidem

$$\left(\frac{ddt}{dxdy}\right)=-P\left(\frac{ddt}{dx^2}\right)-\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right)$$

erit

$$\left(\frac{ddt}{dy^2}\right)=PP\left(\frac{ddt}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right)-\left(\frac{dP}{dy}\right)\left(\frac{dt}{dx}\right)$$

similique modo sumendo P negative

$$\left(\frac{ddu}{dy^2}\right)=PP\left(\frac{ddu}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{du}{dx}\right)+\left(\frac{dP}{dy}\right)\left(\frac{du}{dx}\right)$$

Quibus substitutis nostra aequatio hanc induet formam

$$\left(P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)\right)\left(\frac{dt}{dx}\right)\left(\frac{dz}{dt}\right)+\left(P\left(\frac{dP}{dx}\right)+\left(\frac{dP}{dy}\right)\right)\left(\frac{du}{dx}\right)\left(\frac{dz}{du}\right)-4PP\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)\left(\frac{ddz}{dtdu}\right)=0;$$

quae cum unicam formulam secundi gradus $\left(\frac{ddz}{dtdu}\right)$ contineat, integrationem admittit, si vel $\left(\frac{dz}{dt}\right)$ vel $\left(\frac{dz}{du}\right)$ e calculo excesserit. Ponamus ergo insuper

$$P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)=0\,,$$

qua aequatione indoles quaesitae functionis P definitur; quo facto aequatio integranda per $2P\left(\frac{du}{dx}\right)$ divisa erit

$$\left(\frac{dP}{dx}\right)\left(\frac{dz}{du}\right)-2P\left(\frac{dt}{dx}\right)\left(\frac{ddz}{dtdu}\right)=0\,,$$

cuius integrale posito $\left(\frac{dz}{du}\right)=v$ fit

$$2lv=\int\frac{dt\left(\frac{dP}{dx}\right)}{P\left(\frac{dt}{dx}\right)}=2l\left(\frac{dz}{du}\right).$$

Verum prius ipsam functionem P per x et y definiri oportet.
Cum sit $\left(\frac{dP}{dy}\right)=P\left(\frac{dP}{dx}\right)$, erit

$$dP=dx\left(\frac{dP}{dx}\right)+Pdy\left(\frac{dP}{dx}\right)$$

hincque ponendo brevitatis ergo $\left(\frac{dP}{dx}\right)=p$ fit $dx=\frac{dP}{p}-Pdy$ atque

$$x=-Py+\int dP\left(y+\frac{1}{p}\right).$$

Statuatur ergo $y+\frac{1}{p}=f':P$ ac reperitur

$$x+Py=f:P$$

et

$$p=\left(\frac{dP}{dx}\right)=\frac{1}{f':P-y} \quad \text{ac} \quad \left(\frac{dP}{dy}\right)=\frac{P}{f':P-y},$$

unde ratio determinationis quantitatis P per x et Y definitur.

Pro novis autem variabilibus t et u ob $\left(\frac{dt}{dy}\right)=-P\left(\frac{dt}{dx}\right)$ erit

$$dt=\left(\frac{dt}{dx}\right)\left(dx-Pdy\right)$$

et ob $x=-Py+f:P$ fit

$$dt=\left(\frac{dt}{dx}\right)\left(dPf':P-2Pdy-ydP\right)=P^{\frac{1}{2}}\left(\frac{dt}{dx}\right)\left(\frac{dP}{\sqrt{P}}f':P-2dy\sqrt{P}-\frac{ydP}{\sqrt{P}}\right);$$

cuius postremae formulae cum integrale sit $\int\frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}$, erit

$$t=F:\left(\int\frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}\right).$$

Deinde ob $\left(\frac{du}{dy}\right)=P\left(\frac{du}{dx}\right)$ habetur

$$du=\left(\frac{du}{dx}\right)\left(dx+Pdy\right)=\left(\frac{du}{dx}\right)\left(dPf':P-ydP\right)$$

ideoque

$$du=\left(\frac{du}{dx}\right)\left(f':P-y\right)dP$$

quare u aequabitur functioni ipsius P. In hoc autem negotio functiones quascunque accipere licet, quia sequente demum integratione universalitas solutionis obtinetur. Quare ponamus

$$t=\int\frac{dP}{\sqrt{P}}f':P-2y\sqrt{P} \quad \text{et} \quad u=P$$

existente $x+Py=f:P$.

Denique ad ipsum integrale inveniendum, quia est

$$2l\left(\frac{dz}{du}\right)=\int\frac{dt\left(\frac{dP}{dx}\right)}{P\left(\frac{dt}{dx}\right)},$$

in qua integratione u seu P sumitur constans, per superiora est

$$\frac{dt}{\left(\frac{dt}{dx}\right)}=dPf':P-2Pdy-ydP=-2Pdy$$

ob P constans et $\left(\frac{dP}{dx}\right)=\frac{1}{f':P-y}$, unde fit

$$2l\left(\frac{dz}{dP}\right)=\int\frac{-2dy}{f':P-y}=2l\left(f':P-y\right)+2lF:P$$

seu

$$\frac{dz}{dP}=\left(f':P-y\right)F:P$$

hincque porro

$$z=\int dP\left(f':P-y\right)F:P$$

sumendo hic t constans. Cum igitur sit

$$y=\frac{1}{2\sqrt{P}}\int\frac{dP}{\sqrt{P}}f':P-\frac{t}{2\sqrt{P}},$$

ideoque

$$f':P-y=f':P-\frac{1}{2\sqrt{P}}\int\frac{dP}{\sqrt{P}}f':P+\frac{t}{2\sqrt{P}},$$

unde conficitur

$$z=\int dP\left(f':P-\frac{1}{2\sqrt{P}}\int\frac{dP}{\sqrt{P}}f':P\right)F:P+\left(\frac{1}{2}\int\frac{dP}{\sqrt{P}}f':P-y\sqrt{P}\right)\int\frac{dP}{\sqrt{P}}F:P+\Phi:\left(\int\frac{dP}{\sqrt{P}}f':P-2y\sqrt{P}\right),$$

quae expressio duas continet functiones arbitrarias F et Φ.

COROLLARIUM 1

303. Primum huius formae membrum ita transformari potest

$$\int\frac{dP}{\sqrt{P}}\left(\sqrt{P}\cdot f':P-\frac{1}{2}\int\frac{dP}{\sqrt{P}}f':P\right)F:P,$$

at

$$\sqrt{P}\cdot f':P-\frac{1}{2}\int\frac{dP}{\sqrt{P}}f':P=\int dP\sqrt{P}\cdot f'':P,$$

unde primum membrum erit

$$\int \frac{dP}{\sqrt{P}} F{:}P \int dP\sqrt{P} \cdot f''{:}P\,.$$

COROLLARIUM 2

304. Cum autem hoc primum membrum sit functio indefinita ipsius *P*, si ea indicetur per $\Pi{:}P$, erit

$$\frac{dP}{\sqrt{P}}\; F{:}P \;=\; \frac{dP\,\Pi'{:}P}{\int dP\sqrt{P}\cdot f''{:}P}$$

unde forma integralis fit

$$z=\Pi{:}P+\Phi{:}\left(\int \frac{dP}{\sqrt{P}}\, f'{:}P-2y\sqrt{P}\right)+\left(\int \frac{dP}{\sqrt{P}}\, f'{:}P-2y\sqrt{P}\right)\int \frac{dP\,\Pi'{:}P}{2\int dP\sqrt{P}\cdot f''{:}P}.$$

COROLLARIUM 3

305. Solutio magis particularis nascitur sumendo $\Pi{:}P=0$ hincque *z* aequabitur functioni cuicunque quantitatis $\int \frac{dP}{\sqrt{P}}\, f'{:}P-2y\sqrt{P}$, quae ob $x+Py=f{:}P$ per *x* et *y* exhiberi censenda est.

SCHOLION

306. Quanquam hic eadem methodo sum usus atque in problemate praecedente [§ 296], tamen, quod mirum videatur, casus praecedentis problematis, quo erat $P=a$, in hac solutione non continetur. Ratio huius paradoxi in resolutione aequationis $\left(\frac{dP}{dy}\right)=P\left(\frac{dP}{dx}\right)$ est sita, cui manifesto satisfacit valor $P=a$, etiamsi in forma inde derivata $x+Py=f{:}P$ non contineatur. Hic scilicet simile quiddam usu venit, quod iam supra observavimus, saepe aequationi differentiali valorem quendam satisfacere posse, qui in integrali non contineatur, veluti aequationi $dy\sqrt{(a-x)}=dx$ satisfacere videmus valorem $x=a$, quem tamen [forma] integralis $y=\mathrm{C}-2\sqrt{(a-x)}$ excludit. Quare etiam nostro casu valor $P=a$ peculiarem evolutionem postulat in priore problemate peractam.

De reliquis, ubi pro *f: P* certa quaedam functio ipsius *P* assumitur, exempla quaedam evolvamus.

EXEMPLUM 1

307. *Sumto* $f{:}P=0$, *ut sit* $P=-\frac{x}{y}$, *integrale completum huius aequationis y*

$$\left(\frac{ddz}{dy^2}\right)=\frac{xx}{yy}\left(\frac{ddz}{dx^2}\right)$$

investigare.

Cum sit $f'{:}P=0$, solutio inventa ob $\int \frac{dP}{\sqrt{P}}\, f'{:}P=C$ praebet

$$z = \frac{-C}{2}\int \frac{dP}{\sqrt{P}} F{:}P + \left(\tfrac{1}{2}C - y\sqrt{P}\right)\int \frac{dP}{\sqrt{P}} F{:}P + \Phi : \left(C - 2y\sqrt{P}\right).$$

Statuatur $\int \frac{dP}{\sqrt{P}} F{:}P = \Pi{:}P$ prodibitque

$$z = -y\sqrt{P}\cdot \Pi{:}P + \Phi{:}y\sqrt{P}.$$

Restituatur pro P valor $-\frac{x}{y}$ et ob $y\sqrt{P} = \sqrt{-xy}$ imaginarium $\sqrt{-1}$ in functiones involvendo erit

$$z = \sqrt{xy}\cdot \Pi{:}\tfrac{x}{y} + \Phi{:}\sqrt{xy},$$

quae forma facile in hanc transfunditur

$$z = \tfrac{x}{y}\Gamma{:}\tfrac{x}{y} + \Theta{:}xy,$$

ubi $x\Gamma{:}\frac{x}{y}$ denotat functionem quamcunque homogeneam unius dimensionis ipsarum x et y.

Resolutio autem instituetur loco x et y has novas variabiles t et u introducendo, ut sit $t = C - 2\sqrt{-xy}$ et $u = -\frac{x}{y}$ vel etiam simplicius $t = 2\sqrt{xy}$ et $u = \frac{x}{y}$, unde fit

$$\left(\frac{dt}{dx}\right) = \frac{\sqrt{y}}{\sqrt{x}},\quad \left(\frac{dt}{dy}\right) = \frac{\sqrt{x}}{\sqrt{y}},\quad \left(\frac{ddt}{dx^2}\right) = \frac{-\sqrt{y}}{2x\sqrt{x}},\quad \left(\frac{ddt}{dy^2}\right) = \frac{-\sqrt{x}}{2y\sqrt{y}},$$

$$\left(\frac{du}{dx}\right) = \frac{1}{y},\quad \left(\frac{du}{dy}\right) = \frac{-x}{yy},\quad \left(\frac{ddu}{dx^2}\right) = 0,\qquad \left(\frac{ddu}{dy^2}\right) = \frac{2x}{y^3},$$

et ob $PP = \frac{xx}{yy}$ aequatio proposita hanc induit formam

$$0.\left(\frac{dz}{dt}\right) + \frac{2x}{y^3}\left(\frac{dz}{du}\right) - \frac{4x\sqrt{x}}{yy\sqrt{y}}\left(\frac{ddz}{dtdu}\right) = 0,$$

Nunc cum sit $ttu = 4xx$ et $x = \frac{1}{2}t\sqrt{u}$ atque $y = \frac{t}{2\sqrt{u}}$, habebimus

$$\frac{8uu}{tt}\left(\frac{dz}{du}\right) - \frac{8uu}{t}\left(\frac{ddz}{dtdu}\right) = 0 \quad \text{seu} \quad \left(\frac{dz}{du}\right) = t\left(\frac{ddz}{dtdu}\right).$$

Fiat $\left(\frac{dz}{du}\right) = v$, ut sit $v = t\left(\frac{dv}{dt}\right)$ et sumto u constante $\frac{dt}{t} = \frac{dv}{v}$, ergo $v = \left(\frac{dz}{du}\right) = tf'{:}u$. Sit iam t constans fietque

$$z = tf{:}u + F{:}t = 2\sqrt{xy}\cdot f{:}\tfrac{x}{y} + F{:}\sqrt{xy}$$

ut ante.

COROLLARIUM

308. Quemadmodum autem expressio inventa $z = x\Gamma{:}\frac{x}{y} + \Theta{:}xy$ satisfaciat, differentialibus rite sumtis perspicietur

$$\left(\frac{dz}{dx}\right)=\Gamma:\frac{x}{y}+\frac{x}{y}\Gamma':\frac{x}{y}+y\Theta':xy,\quad \left(\frac{dz}{dy}\right)=\frac{-xx}{yy}\Gamma':\frac{x}{y}+x\Theta':xy,$$

unde porro fit

$$\left(\frac{ddz}{dx^2}\right)=\frac{2}{y}\Gamma':\frac{x}{y}+\frac{x}{yy}\Gamma'':\frac{x}{y}+yy\Theta'':xy,\quad \text{et}\quad \left(\frac{ddz}{dy^2}\right)=\frac{2xx}{y^3}\Gamma':\frac{x}{y}+\frac{x^3}{y^4}\Gamma'':\frac{x}{y}+xx\Theta'':xy,$$

EXEMPLUM 2

309. *Sumto* $f:P=\frac{PP}{2a}$, *ut sit*

$$PP=2aPy+2ax\ \text{ et }\ P=\ ay+\sqrt{(aayy+2ax)},$$

huius aequationis

$$\left(\frac{ddz}{dy^2}\right)=(2ayy+2ax)\sqrt{(aayy+2ax)}\left(\frac{ddz}{dx^2}\right)$$

integrale completum investigare.

Cum sit $f:P=\frac{PP}{2a}$, erit $f':P=\frac{P}{a}$ et

$$\int\frac{dP}{\sqrt{P}}\ f':P=\int\frac{1}{a}dP\sqrt{P}=\frac{2}{3a}P\sqrt{P}$$

unde forma generalis supra [§ 302] inventa abit in

$$z=\int dP\,\frac{2P}{3a}F:P+\left(\frac{P\sqrt{P}}{3a}-y\sqrt{P}\right)\int\frac{dP}{\sqrt{P}}F:P+\Phi:\left(\frac{2}{3a}P\sqrt{P}-2y\sqrt{P}\right),$$

statuatur

$$\int\frac{dP}{\sqrt{P}}F:P=\Pi:P\ ;$$

erit $dPF:P=dP\sqrt{P}\cdot\Pi':P$ atque

$$z=\frac{2}{3a}\int P^{\frac{3}{2}}dP\Pi':P+\left(\frac{P\sqrt{P}}{3a}-y\sqrt{P}\right)\Pi:P+\Phi:\left(\frac{P\sqrt{P}}{3a}-y\sqrt{P}\right).$$

Est autem

$$\frac{P}{3a}-y=\frac{-2}{3}y+\frac{1}{3}\sqrt{\left(yy+\frac{2x}{a}\right)}\ ;$$

quarum formularum evolutio deducit ad expressiones nimis perplexas. At substitutiones ad scopum perducentes sunt

$$t=\tfrac{2}{3a}P\sqrt{P}-2y\sqrt{P} \quad \text{et} \quad u=P.$$

COROLLARIUM

310. Si pro solutione magis restricta ponatur $\Pi{:}P=P^{n-\frac{1}{2}}$, erit

$$\Pi'{:}P=\left(n-\tfrac{1}{2}\right)P^{n-\frac{3}{2}}$$

hincque colligitur

$$z=\tfrac{n}{(n+1)a}P^{n+1}-P^{n}y+\Phi{:}\left(\tfrac{P\sqrt{P}}{3a}-y\sqrt{P}\right).$$

Sit n = 1 et functio Φ evanescat; erit

$$z=\tfrac{1}{2a}PP-Py=x;$$

at casus n = 2 dat

$$z=\tfrac{2}{3a}P^{3}-P^{2}y=\tfrac{2}{3}axy+\tfrac{2}{3}P\left(2x+ayy\right)$$

seu

$$z=\tfrac{2}{3}aay^{3}+2axy+\tfrac{2}{3}\left(ayy+2x\right)\sqrt{\left(aayy+2ax\right)}\,.$$

SCHOLION

311. Forma integralis inventa [§ 302] sequenti modo simplicior effici potest. Ponatur $\int\frac{dP}{\sqrt{P}}F{:}P=\Pi{:}P$; erit

$$F{:}P=\sqrt{P}\cdot\Pi'{:}P$$

eritque omittendo postremum membrum

$$\Phi{:}\left(\int\tfrac{dP}{\sqrt{P}}\,f'{:}P-2y\sqrt{P}\right),$$

quod nulla reductione indiget,

$$z=\int dP\left(\sqrt{P}\cdot f'{:}P-\tfrac{1}{2}\int\tfrac{dP}{\sqrt{P}}\,f'{:}P\right)\Pi'{:}P+\tfrac{1}{2}\Pi{:}P\int\tfrac{dP}{\sqrt{P}}\,f'{:}P-y\sqrt{P}\cdot\Pi{:}P\,;$$

at

$$\tfrac{1}{2}\Pi{:}P\int\tfrac{dP}{\sqrt{P}}\,f'{:}P=\int\left(\tfrac{1}{2}dP\Pi'{:}P\int\tfrac{dP}{\sqrt{P}}\,f'{:}P+\tfrac{1}{2}\tfrac{dP}{\sqrt{P}}\Pi{:}Pf'{:}P\right),$$

unde fit

$$z=\int\Pi'{:}PdP\sqrt{P}\cdot f'{:}P+\tfrac{1}{2}\int\Pi{:}P\tfrac{dP}{\sqrt{P}}\,f'{:}P-y\sqrt{P}\cdot\Pi{:}P.$$

porro est

$$\int dP\Pi':P\sqrt{P}\cdot f':P = \Pi:P\sqrt{P}\,f':P - \int \Pi:P\left(\tfrac{dP}{2\sqrt{P}}\,f':P + dP\sqrt{P}\cdot f'':P\right)$$

ideoque

$$z = \Pi:P\sqrt{P}\cdot f':P - \int dP\Pi:P\sqrt{P}\cdot f'':P - y\sqrt{P}\cdot\Pi:P\,;$$

statuatur porro

$$\int dP\Pi:P\sqrt{P}\cdot f'':P = \Theta:P\,;$$

erit

$$\Pi:P = \tfrac{\Theta':P}{\sqrt{P}\cdot f'':P} \text{ et } z = \tfrac{\Theta':P}{\sqrt{P}\cdot f'':P}\left(f':P - y\right) - \Theta:P + \Phi:\left(\int \tfrac{dP}{\sqrt{P}}\,f':P - 2y\sqrt{P}\right),$$

quae forma sine dubio multo est simplicior quam primo inventa.

PROBLEMA 50

312. *Proposita aequatione*

$$\left(\tfrac{ddz}{dy^2}\right) - PP\left(\tfrac{ddz}{dx^2}\right) + Q\left(\tfrac{dz}{dy}\right) + R\left(\tfrac{dz}{dx}\right) = 0$$

invenire casus quantitatum P, Q, R, quibus integratio ope reductionis ante adhibitae succedit.

SOLUTIO

Introductis binis novis variabilibus *t* et *u* habebimus [§ 232]

$$\begin{aligned}0 = &\left(\tfrac{ddt}{dy^2}\right)\left(\tfrac{dz}{dt}\right) + \left(\tfrac{ddu}{dy^2}\right)\left(\tfrac{dz}{du}\right) + \left(\tfrac{dt}{dy}\right)^2\left(\tfrac{ddz}{dt^2}\right) + 2\left(\tfrac{dt}{dy}\right)\left(\tfrac{du}{dy}\right)\left(\tfrac{ddz}{dtdu}\right) + \left(\tfrac{du}{dy}\right)^2\left(\tfrac{ddz}{du^2}\right)\\ &-P^2\left(\tfrac{ddt}{dx^2}\right) - P^2\left(\tfrac{ddu}{dx^2}\right) - P^2\left(\tfrac{dt}{dx}\right)^2 - 2P^2\left(\tfrac{dt}{dx}\right)\left(\tfrac{du}{dx}\right) - P^2\left(\tfrac{du}{dx}\right)^2\\ &+Q\left(\tfrac{dt}{dy}\right) + Q\left(\tfrac{du}{dy}\right)\\ &+R\left(\tfrac{dt}{dx}\right) + R\left(\tfrac{du}{dx}\right)\end{aligned}$$

Statuamus ergo ut ante

$$\left(\tfrac{dt}{dy}\right) = P\left(\tfrac{dt}{dx}\right) \quad \text{et} \quad \left(\tfrac{du}{dy}\right) = -P\left(\tfrac{du}{dx}\right),$$

unde fit

$$\left(\frac{ddt}{dxdy}\right)=P\left(\frac{ddt}{dx^2}\right)+\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right)$$

et

$$\left(\frac{ddt}{dy^2}\right)=PP\left(\frac{ddt}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{dt}{dx}\right)+\left(\frac{dP}{dy}\right)\left(\frac{dt}{dx}\right)$$

atque

$$\left(\frac{ddu}{dy^2}\right)=PP\left(\frac{ddu}{dx^2}\right)+P\left(\frac{dP}{dx}\right)\left(\frac{du}{dx}\right)-\left(\frac{dP}{dy}\right)\left(\frac{du}{dx}\right),$$

et aequatio resolvenda erit

$$0=\left(P\left(\frac{dP}{dx}\right)+\left(\frac{dP}{dy}\right)+PQ+R\right)\left(\frac{dt}{dx}\right)\left(\frac{dz}{dt}\right)-4PP\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right)\left(\frac{ddz}{dtdu}\right)+\left(P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)-PQ+R\right)\left(\frac{du}{dx}\right)\left(\frac{dz}{du}\right).$$

Iam evidens est integrationem institui posse, si alterutra formula $\left(\frac{dz}{dt}\right)$ vel $\left(\frac{dz}{du}\right)$ ex calculo abeat. Ponamus ergo esse

$$P\left(\frac{dP}{dx}\right)-\left(\frac{dP}{dy}\right)-PQ+R=0 \quad \text{seu} \quad R=PQ+\left(\frac{dP}{dy}\right)-P\left(\frac{dP}{dx}\right)$$

et aequatio resultans per $\left(\frac{dt}{dx}\right)$ divisa fit

$$0=2\left(PQ+\left(\frac{dP}{dy}\right)\right)\left(\frac{dz}{dt}\right)-4PP\left(\frac{du}{dx}\right)\left(\frac{ddz}{dtdu}\right).$$

Fiat $\left(\frac{dz}{dt}\right)=v$ *;* erit

$$\left(PQ+\left(\frac{dP}{dy}\right)\right)v-2PP\left(\frac{du}{dx}\right)\left(\frac{dv}{du}\right)=0\,;$$

sumatur *t* constans, ut fiat

$$\frac{dv}{v}=\frac{\left(PQ+\left(\frac{dP}{dy}\right)\right)du}{2PP\left(\frac{du}{dx}\right)},$$

ubi necesse est, ut quantitates *P*, *Q*, $\left(\frac{dP}{dy}\right)$ et $\left(\frac{du}{dx}\right)$ per novas variabiles *t* et *u*

exprimantur. Has ergo primum definiri convenit. Cum sit

$$\left(\frac{dt}{dy}\right)=P\left(\frac{dt}{dx}\right) \quad \text{et} \quad \left(\frac{du}{dy}\right)=-P\left(\frac{du}{dx}\right),$$

erit

$$dt=\left(\frac{dt}{dx}\right)(dx+Pdy) \text{ et } du=\left(\frac{du}{dx}\right)(dx-Pdy);$$

sunt ergo $\left(\frac{dt}{dx}\right)$ et $\left(\frac{du}{dx}\right)$ factores integrabiles reddentes formulas $dx+Pdy$ et $dx-Pdy$; non enim opus est, ut hinc valores *t* et *u* generalissime definiantur. Sint *p* et *q* tales multiplicatores per *x* et *y* dati eritque

$$t=\int p(dx+Pdy) \text{ et } u=\int q(dx-Pdy),$$

unde superior integratio fit

$$\frac{dv}{v}=\frac{\left(PQ+\left(\frac{dP}{dy}\right)\right)du}{2PPq},$$

in qua integratione quantitas $t=\int p(dx+Pdy)$ constans est spectanda. Seu ob $du=q(dx-Pdy)$ erit

$$\frac{dv}{v}=\frac{\left(PQ+\left(\frac{dP}{dy}\right)\right)(dx-Pdy)}{2PP}$$

Verum ob $dt=0$ est $dx=-Pdy$, ita ut prodeat

$$\frac{dv}{v}=-\frac{dy}{P}\left(PQ+\left(\frac{dP}{dy}\right)\right),$$

ubi ob *t* constans et datum per *x* et *y* valor ipsius *x* per *y* et *t* expressus substitui potest, ut sola *y* variabilis insit, et invento integrali

$$-\int\frac{dy}{P}\left(PQ+\left(\frac{dP}{dy}\right)\right)=lV$$

erit $v=Vf:t=\left(\frac{dz}{dt}\right)$. Nunc ponatur *u* constans; erit

$$z=\int Vdtf:t+F:u.$$

Conditio autem, sub qua haec integratio locum habet, postulat, ut sit

$$R=PQ+\left(\frac{dP}{dy}\right)-P\left(\frac{dP}{dx}\right).$$

COROLLARIUM 1

313. Eodem modo aequatio proposita resolutionem admittet, si fuerit

$$R=-PQ-\left(\frac{dP}{dy}\right)-P\left(\frac{dP}{dx}\right)$$

manetque ut ante

$$t=\int p\left(dx+Pdy\right) \text{ et } u=\int q\left(dx-Pdy\right).$$

Tum vero fit

$$0=-\left(PQ+\left(\frac{dP}{dy}\right)\right)\left(\frac{dz}{du}\right)-2PP\left(\frac{dt}{dx}\right)\left(\frac{ddt}{dtdu}\right)$$

quae posito $\left(\frac{dz}{du}\right)=v$ sumtoque u constante dat

$$\frac{dv}{v}=\frac{-\left(PQ+\left(\frac{dP}{dy}\right)\right)dt}{2PP\left(\frac{dt}{dx}\right)}=\frac{-\left(PQ+\left(\frac{dP}{dy}\right)\right)\left(dx+Pdy\right)}{2PP}.$$

COROLLARIUM 2

314. Si porro habita ratione, quod $u=\int q\left(dx-Pdy\right)$ sit constans et $dx=Pdy$, ponatur

$$\int-\frac{dy\left(PQ+\left(\frac{dP}{dy}\right)\right)}{P}=lV,$$

erit

$$v=Vf:u=\left(\frac{dz}{du}\right),$$

unde tandem sumendo iam $t=\int p\left(dx+Pdy\right)$ [constans] colligitur

$$z=\int Vduf:u+F:t.$$

EXEMPLUM 1

315. *Si sumatur $P = a$ et $R = aQ$, quaecunque fuerit Q functio ipsarum x et y, integrare aequationem*

$$\left(\frac{ddz}{dy^2}\right) - aa\left(\frac{ddz}{dx^2}\right) + Q\left(\frac{dz}{dy}\right) + R\left(\frac{dz}{dx}\right) = 0$$

Cum hic sit $P = a$, erit $p = 1$, $q = 1$ et $t = x + ay$ atque $u = x - ay$, unde posito $\left(\frac{dz}{dt}\right) = v$ fit

$$\frac{dv}{v} = \frac{aQdu}{2aa} = \frac{Qdu}{2a}.$$

Quoniam igitur est

$$x = \frac{t+u}{2} \text{ et } y = \frac{t-u}{2a},$$

his valoribus substitutis fit Q functio ipsarum t et u ac spectata t ut constante erit

$$lv = \frac{1}{2a}\int Qdu + lf:t \quad \text{seu} \quad \left(\frac{dz}{dt}\right) = e^{\frac{1}{2a}\int Qdu} f:t,$$

et sumta iam u constante

$$z = \int e^{\frac{1}{2a}\int Qdu} dtf:t + F:u.$$

COROLLARIUM 1

316. Si Q sit constans $= 2ab$, aequationis huius

$$\left(\frac{ddz}{dy^2}\right) - aa\left(\frac{ddz}{dx^2}\right) + 2ab\left(\frac{dz}{dy}\right) + 2aab\left(\frac{dz}{dx}\right) = 0$$

integrale erit

$$z = e^{bu} f:t + F:u = e^{b(x-ay)} f:(x+ay) + F:(x-ay)$$

sive

$$z = e^{b(x-ay)}\left(f:(x+ay) + F:(x-ay)\right).$$

COROLLARIUM 2

317. Si $Q=\frac{a}{x}$ huius aequationis

$$\left(\frac{ddz}{dy^2}\right)-aa\left(\frac{ddz}{dx^2}\right)+\frac{a}{x}\left(\frac{dz}{dy}\right)+\frac{aa}{x}\left(\frac{dz}{dx}\right)=0$$

integrale ob

$$\int Qdu=\int\frac{adu}{x}=\int\frac{2adu}{t+u}=2al(t+u)$$

erit

$$z=\int(t+u)dtf:t+F:u=\int tdtf:t+u\int dtf:t+F:u.$$

Vel sit $f:t=\Pi'':t;$ erit $\int dtf:t=\Pi':t$ et

$$\int tdtf:t=\int td.\Pi':t=t\Pi':t-\int dt\Pi':t=t\Pi':t-\Pi:t,$$

ergo

$$z=(t+u)\Pi':t-\Pi:t+F:u$$

seu

$$z=2x\Pi':(x+ay)-\Pi:(x+ay)+F:(x-ay).$$

EXEMPLUM 2

318. *Sit* $P=\frac{x}{y}$ et $R=\frac{-x}{y}Q+\frac{x}{yy}-\frac{x}{yy}=\frac{-x}{y}Q$ *sumaturque* $Q=\frac{1}{x}$, *ut sit* $R=\frac{-1}{y}$ *et haec aequatio integrari debeat y*

$$\left(\frac{ddz}{dy^2}\right)-\frac{xx}{yy}\left(\frac{ddz}{dx^2}\right)+\frac{1}{x}\left(\frac{dz}{dy}\right)-\frac{1}{y}\left(\frac{dz}{dx}\right)=0$$

Cum ergo sit

$$t=\int p\left(dx+\frac{xdy}{y}\right)\text{ et }u=\int q\left(dx-\frac{xdy}{y}\right)$$

sumatur $p=y$ et $q=\frac{1}{y}$, ut fiat $t=xy$ et $u=\frac{x}{y}$;. Posito nunc $\left(\frac{dz}{du}\right)=v$ sumtoque u constante ex Corollario 1 fit

$$\frac{dv}{v}=\frac{-\left(\frac{1}{y}-\frac{x}{yy}\right)dt}{\frac{2xx}{yy}y}=\frac{-(y-x)dt}{2xxy}.$$

Est vero $tu = xx$ hincque $x = \sqrt{tu}$ et $y = \sqrt{\frac{t}{u}}$ atque $2xxy = 2t\sqrt{tu}$, unde fit

$$\frac{dv}{v} = \frac{\left(\sqrt{tu} - \sqrt{\frac{t}{u}}\right)dt}{2t\sqrt{tu}} = \frac{dt}{2t} - \frac{dt}{2tu}$$

et ob u constans $lv = \frac{1}{2}lt - \frac{1}{2u}lt$, ergo

$$\left(\frac{dz}{du}\right) = t^{\frac{1}{2} - \frac{1}{2u}} f:u$$

Quare sumto iam t constante erit

$$z = t^{\frac{1}{2}} \int t^{-\frac{1}{2u}} du f:u + F:t.$$

Vel ponatur $-\frac{1}{2u} = s$, ut sit $s = -\frac{y}{2x}$, eritque

$$z = t^{\frac{1}{2}} \int t^{s} ds f:s + F:t.$$

In hac integratione $\int t^{s} ds f:s$ sola s est variabilis ac demum integrali sumto restitui debet $t = xy$ et $s = \frac{-y}{2x}$.

Ceterum patet functionem quamcunque ipsius xy particulariter satisfacere.

PROBLEMA 51

319. *Proposita aequatione generali*

$$\left(\frac{ddz}{dy^2}\right)-2P\left(\frac{ddz}{dxdy}\right)+(PP-QQ)\left(\frac{ddz}{dx^2}\right)+R\left(\frac{dz}{dy}\right)+S\left(\frac{dz}{dx}\right)+Tz+V=0$$

invenire conditiones quantitatum P, Q, R, S, T, ut integratio ope reductionis adhibitae succedat.

SOLUTIO

Facta eadem substitutione [§ 232] introducendis binis novis variabilibus t et u aequatio nostra sequentem induet formam

$$\begin{aligned}0=V+Tz\\
&+\left(\frac{ddt}{dy^2}\right)\left(\frac{dz}{dt}\right) &&+\left(\frac{ddu}{dy^2}\right)\left(\frac{dz}{du}\right) &&+\left(\frac{dt}{dy}\right)^2\left(\frac{ddz}{dt^2}\right) &&+2\left(\frac{dt}{dy}\right)\left(\frac{du}{dy}\right)\left(\frac{ddz}{dtdu}\right) &&+\left(\frac{du}{dy}\right)^2\left(\frac{ddz}{du^2}\right)\\
&-2P\left(\frac{ddt}{dxdy}\right) &&-2P\left(\frac{ddu}{dxy}\right) &&-2P\left(\frac{dt}{dx}\right)\left(\frac{dt}{dy}\right) &&-2P\left(\frac{dt}{dx}\right)\left(\frac{du}{dy}\right) &&-2P\left(\frac{du}{dx}\right)\left(\frac{du}{dy}\right)\\
& && && &&-2P\left(\frac{du}{dx}\right)\left(\frac{dt}{dy}\right)\\
&+\left(P^2-Q^2\right)\left(\frac{ddt}{dx^2}\right) &&+\left(P^2-Q^2\right)\left(\frac{ddu}{dx^2}\right) &&+\left(P^2-Q^2\right)\left(\frac{dt}{dx}\right)^2 &&+2\left(P^2-Q^2\right)\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right) &&+\left(P^2-Q^2\right)\left(\frac{du}{dx}\right)^2\\
&R\left(\frac{dt}{dy}\right) &&+R\left(\frac{du}{dy}\right)\\
&S\left(\frac{dt}{dx}\right) &&+S\left(\frac{du}{dx}\right)
\end{aligned}$$

Determinentur iam hae duae novae variabiles t et u ita per x et y, ut formulae $\left(\frac{ddz}{dt^2}\right)$ et $\left(\frac{ddz}{du^2}\right)$ evanescant, debebitque esse

$$\left(\frac{dt}{dy}\right)=(P+Q)\left(\frac{dt}{dx}\right) \text{ et } \left(\frac{du}{dy}\right)=(P-Q)\left(\frac{du}{dx}\right)$$

unde patet has variabiles sequenti modo determinari

$$t=\int p\left(dx+(P+Q)dy\right) \text{ et } u=\int q\left(dx+(P-Q)dy\right)$$

sumendo p et q ita, ut hae formulae integrationem admittant. Cum nunc sit

$$\left(\frac{ddt}{dxdy}\right)=(P+Q)\left(\frac{ddt}{dx^2}\right)\quad+\left(\left(\frac{dP}{dx}\right)+\left(\frac{dQ}{dx}\right)\right)\left(\frac{dt}{dx}\right),$$

$$\left(\frac{ddt}{dy^2}\right)=(P+Q)^2\left(\frac{ddt}{dx^2}\right)\quad+(P+Q)\left(\left(\frac{dP}{dx}\right)+\left(\frac{dQ}{dx}\right)\right)\left(\frac{dt}{dx}\right)+\left(\left(\frac{dP}{dy}\right)+\left(\frac{dQ}{dy}\right)\right)\left(\frac{dt}{dx}\right),$$

$$\left(\frac{ddu}{dxdy}\right)=(P-Q)\left(\frac{ddu}{dx^2}\right)\quad+\left(\left(\frac{dP}{dx}\right)-\left(\frac{dQ}{dx}\right)\right)\left(\frac{du}{dx}\right),$$

$$\left(\frac{ddu}{dy^2}\right)=(P-Q)^2\left(\frac{ddu}{dx^2}\right)\quad+(P-Q)\left(\left(\frac{dP}{dx}\right)-\left(\frac{dQ}{dx}\right)\right)\left(\frac{du}{dx}\right)+\left(\left(\frac{dP}{dy}\right)-\left(\frac{dQ}{dy}\right)\right)\left(\frac{du}{dx}\right),$$

hinc reperitur formulae $2\left(\frac{ddz}{dtdu}\right)$ coefficiens

$$-2QQ\left(\frac{dt}{dx}\right)\left(\frac{du}{dx}\right),$$

termini $\left(\frac{dz}{dt}\right)$ coefficiens

$$\left(-(P-Q)\left(\frac{dP+dQ}{dx}\right)+\left(\frac{dP+dQ}{dy}\right)+R(P+Q)+S\right)\left(\frac{dt}{dx}\right),$$

termini vero $\left(\frac{dz}{du}\right)$ coefficiens

$$\left(-(P+Q)\left(\frac{dP-dQ}{dx}\right)+\left(\frac{dP-dQ}{dy}\right)+R(P-Q)+S\right)\left(\frac{du}{dx}\right).$$

Est vero

$$\left(\frac{dt}{dx}\right)=p\quad\text{et}\quad\left(\frac{du}{dx}\right)=q$$

unde, si brevitatis gratia vocetur

$$S+R(P+Q)+\left(\frac{dP+dQ}{dy}\right)-(P-Q)\left(\frac{dP+dQ}{dx}\right)=M$$

et

$$S+R(P-Q)+\left(\frac{dP-dQ}{dy}\right)-(P+Q)\left(\frac{dP-dQ}{dx}\right)=N,$$

aequatio nostra resolvenda erit

$$0=V+Tz+Mp\left(\frac{dz}{dt}\right)+Nq\left(\frac{dz}{du}\right)-4QQpq\left(\frac{ddz}{dtdu}\right)$$

seu, ut cum formis supra § 294 et § 295 exhibitis comparari queat,

$$\left(\frac{ddz}{dtdu}\right)-\frac{M}{4QQq}\left(\frac{dz}{dt}\right)-\frac{N}{4QQp}\left(\frac{dz}{du}\right)-\frac{T}{4QQpq}z-\frac{V}{4QQpq}=0,$$

quae, si porro brevitatis gratia ponatur

$$\frac{M}{4QQq}=K \quad \text{et} \quad \frac{N}{4QQp}=L,$$

duplici casu integrationem admittit: altero, si fuerit

$$-\frac{T}{4QQpq}=KL-\left(\frac{dL}{du}\right) \quad \text{seu} \quad T=4QQpq\left(\frac{dL}{du}\right)-\frac{MN}{4QQ},$$

altero vero, si fuerit

$$-\frac{T}{4QQpq}=KL-\left(\frac{dK}{dt}\right) \quad \text{seu} \quad T=4QQpq\left(\frac{dK}{dt}\right)-\frac{MN}{4QQ}.$$

Quoniam vero K et L per x et y dantur, formulae illae $\left(\frac{dK}{dt}\right)$ et $\left(\frac{dL}{du}\right)$ ita reduci possunt, ut sit

$$\left(\frac{dK}{dt}\right)=\frac{Q-P}{2Qp}\left(\frac{dK}{dx}\right)+\frac{1}{2Qq}\left(\frac{dK}{dy}\right) \quad \text{et} \quad \left(\frac{dL}{du}\right)=\frac{P+Q}{2Qq}\left(\frac{dL}{dx}\right)-\frac{1}{2Qq}\left(\frac{dL}{dy}\right).$$

Quemadmodum autem ipsa integralia his casibus inveniri debeant, id quidem supra [§ 294, 295] est declaratum, unde superfluum foret calculos illos taediosos hic repetere; quovis enim casu oblato solutio inde peti poterit.

SCHOLION 1

320. Quod ad hanc reductionem formularum attinet, ea sequenti modo instituitur. Cum sit in genere

$$dz=dx\left(\frac{dz}{dx}\right)+dy\left(\frac{dz}{dy}\right),$$

ex formulis

$$dt=pdx+p(P+Q)dy \quad \text{et} \quad du=qdx+q(P-Q)dy$$

erit

$$qdt-pdu=2pqQdy \quad \text{seu} \quad dy=\frac{qdt-pdu}{2Qpq}$$

et

$$q(P-Q)dt-p(P+Q)du=-2Qpqdx \quad \text{seu} \quad dx=\frac{p(P+Q)du-q(P-Q)dt}{2Qpq}.$$

Quibus valoribus substitutis obtinebitur

$$dz=\left(\frac{(P+Q)du}{2Qq}-\frac{(P-Q)dt}{2Qp}\right)\left(\frac{dz}{dx}\right)+\left(\frac{dt}{2Qp}-\frac{du}{2Qq}\right)\left(\frac{dz}{dy}\right),$$

ita ut dz per differentialia dt et du exprimatur. Posito ergo u constante et $du=0$ erit

$$\left(\frac{dz}{dt}\right)=\frac{Q-P}{2Qp}\left(\frac{dz}{dx}\right)+\frac{1}{2Qp}\left(\frac{dz}{dy}\right),$$

at posito t constante et $dt = 0$ erit

$$\left(\frac{dz}{du}\right) = \frac{P+Q}{2Qq}\left(\frac{dz}{dx}\right) - \frac{1}{2Qq}\left(\frac{dz}{dy}\right).$$

SCHOLION 2

321. Methodus igitur hoc capite tradita in hoc consistit, ut huiusmodi aequationes ope introductionis binarum novarum variabilium t et u ad hanc formam reducantur

$$\left(\frac{ddz}{dtdu}\right) + P\left(\frac{dz}{dt}\right) + Q\left(\frac{dz}{du}\right) + Rz + S = 0,$$

de qua in praecedente capite [§ 275, 278, 287, 294, 295] vidimus, quibusnam casibus ea integrari queat. Iisdem igitur quoque casibus omnes aequationes, quae ad talem formam se reduci patiuntur, integrationem admittent.

Est vero eiusdem formae casus quidam maxime singularis, cuius integratio absolvi potest, unde denuo infinita multitudo aliarum aequationum, quae quidem eo reduci queant, oritur integrationem pariter admittentium. Quem propterea casum sequenti capite diligentius evolvamus.