CHAPTER V

CONCERNING THE RESOLUTION OF EQUATIONS FOR WHICH A RELATION IS GIVEN BETWEEN THE QUANTITIES $\left(\frac{dz}{dx}\right), \left(\frac{dz}{dy}\right)$ AND ANY TWO OF THE THREE VARIABLES x, y, z

PROBLEM 21

138. *If on putting $dz = pdx + qdy$ there must become $px + qy = 0$, to investigate in general the nature of the function z of x and y.*

SOLUTION

Since there shall be $q = -\frac{px}{y}$, then there becomes

$$dz = pdx - \frac{pxdy}{y} = px\left(\frac{dx}{x} - \frac{dy}{y}\right)$$

or

$$dz = py\left(\frac{dx}{y} - \frac{xdy}{yy}\right) = pyd.\frac{x}{y}.$$

From which it is apparent that *py* must be a function of $\frac{x}{y}$, and if there is put $py = f':\frac{x}{y}$, there becomes $z = f:\frac{x}{y}$. Clearly we will always make use of the rule in describing functions, so that there shall be $d.f:v = dvf':v$ and thus again $d.f':v = dvf'':v$ and $d.f'':v = dvf''':v$, etc. But $f:\frac{x}{y}$ denotes some homogeneous function of *x* and *y* of zero dimensions, and if *z* were such a function then on differentiation it should give rise to $dz = pdx + qdy$, and there will always be $px + qy = 0$.

COROLLARY 1

139. But if *z* were a homogeneous function of zero dimensions of *x* and *y* themselves, hence on account of $p = \left(\frac{dz}{dx}\right)$ and $q = \left(\frac{dz}{dy}\right)$ there will be

$$x\left(\frac{dz}{dx}\right) + y\left(\frac{dz}{dy}\right) = 0,$$

which truth indeed we have elicited above. [See §481 of Book I.]

COROLLARY 2

140. Then indeed since there shall be

$$p=\frac{1}{y}f':\frac{x}{y} \text{ and } q=\frac{-x}{yy}f':\frac{x}{y},$$

p will be a homogeneous function of x and y with dimensions $=-1$, and if there shall be $q=\frac{-px}{y}$, the function z itself is found from integration $z=\int pyd.\frac{x}{y}$.

SCHOLIUM

141. The problem is solved in the same manner, if on putting $dz = pdx+qdy$ there must become $mpx+nqy=a$. Then indeed on account of $q=\frac{a}{ny}-\frac{mpx}{ny}$ there will be

$$dz=\frac{ady}{ny}+pdx-\frac{mpxdy}{ny}$$

or

$$dz=\frac{ady}{ny}+\frac{pdx}{n}\left(\frac{ndx}{x}-\frac{mdy}{y}\right)=\frac{ady}{ny}+\frac{py^m}{nx^{n-1}}d.\frac{x^n}{y^m},$$

from which the solution provides

$$\frac{py^m}{nx^{n-1}}=f'\frac{x^n}{y^m} \quad\text{and}\quad z=\frac{a}{n}ly+f:\frac{x^n}{y^m}.$$

Also this more general solution can be resolved, in which there must be

$$pX+qY=A$$

with X being a function of x and Y a function of y. Since indeed thence there becomes $q=\frac{A}{Y}-\frac{pX}{Y}$, there will be

$$dz=\frac{Ady}{Y}+pdx-\frac{pXdy}{Y}=\frac{Ady}{Y}+pX\left(\frac{dx}{X}-\frac{dy}{Y}\right).$$

Hence there must be put in place

$$pX=f':\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)$$

and thence there becomes

$$z=A\int\frac{dy}{Y}+f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right).$$

PROBLEM 22

142. *If on putting $dz = pdx + qdy$, $\frac{q}{p}$ has to be equal to some given function of x and y, to investigate the nature of the function z in general.*

SOLUTION

Let V be that given function of x and y, so that there shall be $q = pV$ and there will be had $dz = p(dx + Vdy)$. Now a multiplier M will be given, likewise a function of x and y, so that $M(dx + Vdy)$ is made integrable. Therefore there is put $M(dx + Vdy) = dS$, and also S will be given a function of the same x and y. Hence since there shall be $dz = \frac{pdS}{M}$, it is evident that the quantity $\frac{p}{M}$ must be equal to a function of S; whereby if we put $\frac{p}{M} = f':S$, there becomes $z = f:S$ and thereupon there will be

$$p = Mf':S \quad \text{and} \quad q = MVf':S.$$

COROLLARY 1

143. Therefore in this case the function sought z is found at once expressed in terms of x and y, because S is given by x and y. But it can come about, that S gives rise to a transcending function, so that moreover by the methods known at present the multiplier M indeed cannot be found.

COROLLARY 2

144. If V shall be a function of x and y of zero dimensions, then there will be $M = \frac{1}{x+Vy}$. Or on putting $x = vy$, V becomes a function of v and

$$dS = M(ydv + vdy + Vdy).$$

There may be taken $M = \frac{1}{y(v+V)}$ and there shall be $dS = \frac{dy}{y} + \frac{dv}{v+V}$ from which [since $z = f:S$] there is found

$$z = f:\left(ly + \int \frac{dv}{v+V}\right).$$

SCHOLIUM

145. On account of the permutability of p and x, likewise q and y, the following problems can resolved.

I. If there should be $q = xV$ with V being some function of p and y present, the form may be considered

$$z = px + \int (qdy - xdp) = px + \int x(Vdy - dp).$$

A multiplier M is sought, so that there shall be

$$M(Vdy - dp) = dS;$$

S will be a function of p and y and

$$z = px + \int \tfrac{xdS}{M},$$

from which this solution is deduced

$$\tfrac{x}{M} = f':S \quad \text{and} \quad z = pMf':S + f:S.$$

II. If there should be $y = pV$ with V being some function of x and q, the form may be considered

$$z = qy + \int (pdx - ydq) = qy + \int p(dx - Vdq).$$

A multiplier M is sought, so that there shall be

$$M(dx - Vdq) = dS;$$

there will be a function S of x and q, and

$$z = qy + \int \tfrac{pdS}{M}.$$

Whereby there becomes

$$\tfrac{p}{M} = f':S \quad \text{and} \quad z = qy + f:S$$

or, on account of $p = \tfrac{y}{V}$, there will be

$$y = MVf':S \quad \text{and} \quad z = qMVf':S + f:S.$$

III. If there should be $y = xV$ with some function V present of the variables p and q, this form may be considered

$$z = px + qy - \int (xdp + xVdq).$$

A multiplier M is sought, so that there becomes

$$M(dp + Vdq) = dS;$$

there will be a function S of p and q and

$$z = px + qy - \int \tfrac{xdS}{M}$$

from which this solution arises

$$\frac{x}{M} = f':S \quad \text{and} \quad z = px + qy - f:S .$$

Thus all these cases can be reduced, so that of the four quantities *p, x, q, y* either $\frac{q}{p}$, or $\frac{q}{x}$, or $\frac{y}{p}$ or $\frac{y}{x}$ is equal to some function of the remaining two.

PROBLEM 23

146. *If on putting* $dz = pdx + qdy$ *, it is required that there shall be* $q = pV + U$ *with any functions V as well as U of the two variables x and y, to investigate the nature of the function z in general.*

SOLUTION

Since on account of $q = pV + U$ there shall be $dz = p(dx + Vdy) + Udy$, initially there is sought a multiplier *M* returning the formula $dx + Vdy$ integrable, and there shall be

$$M(dx + Vdy) = dS ;$$

M and *S* will be functions of *x* and *y* and there shall be

$$dz = \frac{pdS}{M} + Udy .$$

Now since *S* shall be a function of *x* and *y*, thereupon *x* can be defined in terms of *y* and *S*, with which value introduced, *U* and *M* become functions of *y* and *S*. Now on taking *S* constant the formula *Udy* may be integrated and there becomes

$$\int Udy = T + f:S \text{ , [where } f:S \text{ is the constant of integration,]}$$

and on putting [since *T* is a function of *y* and *S*,]

$$dT = Udy + WdS ,$$

there becomes

$$\frac{p}{M} = [\left(\frac{dz}{dS}\right)] = W + f':S \quad \text{and} \quad z = T + f:S ,$$

[as $\frac{p}{M}$ is equal to the partial derivative w.r.t. *S* of the function found on integrating *U* w.r.t. *y* .]

and thus everything can be expressed by the two variables *y* and *S*.

COROLLARY 1

147. Therefore with the functions *V* and *U* of the two given variables *x* and *y*, so that there shall be $q = pV + U$, the solution of the problem first postulates, that a multiplier *M* be investigated rendering the formula $dx + Vdy$ integrable, with which found a function *S* of the variables *x* and *y* will be considered, so that there shall be

$$S = \int M\left(dx + Vdy\right).$$

COROLLARY 2

148. To this end it is convenient to consider the differential equation $dx + Vdy = 0$; for if this can be integrated, likewise thereupon it will be possible to deduce the multiplier *M*, so that the formula $M\left(dx + Vdy\right)$ truly becomes the differential of a certain function *S*, which therefore hence may be found.

COROLLARY 3

149. Again with this function *S* found the quantity *x* must be expressed by *y* and *S*, thus so that *x* may be equal to a function of *y* and *S;* with which value substituted into the quantity *U* the integral $\int Udy = T$ is sought on regarding *S* as a constant, and thus a function *T* of *y* and *S* will be found.

COROLLARY 4

150. And then with this function *T* found there shall be $W = \left(\frac{dT}{dS}\right)$, from which finally the solution of the problem is deduced contained by these two formulas

$$\frac{p}{M} = W + f':S \text{ and } z = T + f:S\,;$$

where since *S* shall be a function of *x* and *y*, at once there is found a function *z* of *x* and *y*.

COROLLARIUM 5

151. If *U* shall be a function of *y* only, there is no need for that expression of *x* in terms of *y* and *S*, but $T = \int Udy$ will be also a function of *y* only, hence $W = \left(\frac{dT}{dS}\right) = 0$. But this case evidently is reduced to the preceding one [§ 142] on putting *z* in place of $z - \int Udy$.

EXAMPLE 1

152. *If on putting* $dz = pdx + qdy$ *there must be* $q = \frac{px}{y} + \frac{y}{x}$, *to investigate the nature of the function z.*

Therefore here there shall be

$$V = \frac{x}{y} \text{ and } U = \frac{y}{x}$$

from which on account of

$$dx + Vdy = dx + \frac{xdy}{y}$$

the multiplier will be $M = y$ and $dS = ydx + xdy$, hence $S = xy$ and thus there will be considered

$$x = \frac{S}{y} \text{ and } U = \frac{yy}{S}.$$

Now there will be

$$T = \int Udy = \int \frac{yydy}{S} = \frac{y^3}{3S} \text{ and } W = \frac{-y^3}{3SS}.$$

Whereby, for the solution of this example, we shall have

$$\frac{p}{y} = \frac{-y^3}{3SS} + f':S \quad \text{and} \quad z = \frac{y^3}{3S} + f:S$$

or on account of $S = xy$ there will be $z = \frac{yy}{3x} + f:xy$.

EXAMPLE 2

153. *If on putting $dz = pdx + qdy$ there must become $px + qy = n\sqrt{(xx + yy)}$, to investigate the nature of the function z.*

Since here there shall be $q = \frac{-px}{y} + \frac{n}{y}\sqrt{(xx + yy)}$, there will be

$$V = \frac{-x}{y} \text{ and } U = \frac{n}{y}\sqrt{(xx + yy)},$$

therefore $dS = M\left(dx - \frac{xdy}{y}\right)$, whereby there is taken $M = \frac{1}{y}$, so that there becomes $dS = \frac{dx}{y} - \frac{xdy}{yy}$ and $S = \frac{x}{y}$. Hence there arises

$$x = Sy \quad \text{and } U = n\left(\sqrt{1 + SS}\right)$$

and thus on putting S constant there will be

$$T = \int Udy = ny\sqrt{(1 + SS)} \text{ and } W = \left(\frac{dT}{dS}\right) = \frac{nyS}{\sqrt{(1+SS)}},$$

thus so that the solution to our question shall be

$$py = \frac{nyS}{\sqrt{(1+SS)}} + f':S \quad \text{and } z = ny\sqrt{(1 + SS)} + f'S..$$

Therefore since there shall be $S = \frac{x}{y}$, there will be

$$z = n\sqrt{(xx + yy)} + f:y$$

where $f:\frac{x}{y}$ denotes some function of zero dimensions of x and y.

EXEMPLUM 3

154. *If on putting* $dz = pdx + qdy$ *there must become* $pxx + qyy = nxy$ *, to investigate the nature of the function z.*

Since there shall be $q = \frac{-pxx}{yy} + \frac{nx}{y}$ there will be

$$V = \frac{-xx}{yy} \text{ and } U = \frac{nx}{y}.$$

Whereby on account of $dS = M\left(dx - \frac{xxdy}{yy}\right)$ there is taken $M = \frac{1}{xx}$ so that there is made

$$S = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}.$$

Hence there will be

$$\frac{1}{x} = \frac{1}{y} - S \text{ and } x = \frac{y}{1-Sy}$$

and thus $U = \frac{n}{1-Sy}$. Therefore on taking S constant we will have

$$T = \int \frac{ndy}{1-Sy} = -\frac{n}{S} l(1 - Sy) \text{ and } W = +\frac{n}{SS} l(1 - Sy) + \frac{ny}{S(1-Sy)}.$$

Consequently on account of $S = \frac{x-y}{xy}$ and $1 - Sy = \frac{y}{x}$ the solution presented

$$z = \frac{-nxy}{x-y} l \frac{y}{x} + f:\frac{x-y}{xy}.$$

SCHOLIUM

155. From the solution of this problem also this wider question appearing can be resolved. Let *P*, *Q*, likewise *V*, *U* be some given functions of *x* and *y* and it is required to find the function *z*, so that there shall be

$$dz = Pdx + Qdy + L(Vdx + Udy),$$

or, as it returns the same, the function *L* must be investigated, so that this differential formula is itself integrable.

According to making this good initially there is sought a multiplier M making the formula $Vdx + Udy$ integrable and there is put $dS = M(Vdx + Udy)$, from which the function *S* may be found expressed by *x* and *y*. From that the value expressed is sought of *x* in terms of *y* and *S*, and since there shall be

$$dz = Pdx + Qdy + \tfrac{LdS}{M}$$

here that value is substituted everywhere in place of *x*; but thereupon there shall be $dx = Edy + FdS$, from which also E and F may become known, and there will be

$$dz = EPdy + Qdy + FPdS + \tfrac{LdS}{M}.$$

The quantity *S* is assumed constant and there shall be

$$T = \int (EP + Q)dy\,;$$

and there will be

$$z = T + f{:}S\,,$$

which indeed suffices for the solution. But finding L this expression may be differentiated,

$$dz = (EP + Q)dy + dS\left(\tfrac{dT}{dS}\right) + dSf'{:}S$$

and by necessity there becomes

$$FP + \tfrac{L}{M} = \left(\tfrac{dT}{dS}\right) + f'{:}S$$

and thus

$$L = -FMP + M\left(\tfrac{dT}{dS}\right) + Mf'{:}S\,.$$

Otherwise on account of interchanging *p*, *x* and *q*, *y* also hence the following problems may be resolved, which therefore I will run through briefly.

PROBLEM 24

156. *If on putting* $dz = pdx + qdy$ *it is requires that there shall be* $q = Vx + U$ *with V as well as U being some given function of p and y, to investigate the nature of the function sought z.*

SOLUTION

The formula is used

$$z = px + \int (qdy - xdp),$$

and since in place of *q* with the value substituted there shall be

$$\int (qdy - xdp) = \int (Vxdy - xdp + Udy),$$

as it is required that an integrable formula is returned, that for brevity shall be called $\mathfrak{h}$, and since there shall be

$$d\mathfrak{h} = x(Vdy - dp) + Udy,$$

in the first place there is sought a multiplier *M* rendering the formula $Vdy - dp$ integrable and there is put

$$M(Vdy - dp) = dS$$

and thus *S* will be given by *y* and *p;* from which *p* may be elicited expressed by *y* and *S*, with which value substituted there, there will be

$$d\mathfrak{h} = \tfrac{xdS}{M} + Udy.$$

Now on assuming *S* constant the integral is may be obtained $\int Udy = T + f:S$ and there will be

$$\tfrac{x}{M} = \left(\tfrac{dT}{dS}\right) + f':S \quad \text{and} \quad \mathfrak{h} = T + f:S \quad .$$

Therefore the solution itself thus will be had in terms of the two variables *y* and *S*

$$x = M\left(\tfrac{dT}{dS}\right) + Mf':S \quad \text{and} \quad z = px + T + f:S,$$

where now indeed *S* is given by *p* and *y*.

PROBLEM 25

157. *If on putting* $dz = pdx + qdy$ *it is required that there shall be* $p = Vy + U$ *with the given functions V and U of x and q present, to investigate the nature of the function z.*

SOLUTION

Now we may use this form

$$z = qy + \int (pdx - ydq)$$

and the formula required to be integrated is put

$$\int (pdx - ydq) = \mathfrak{h}.$$

Hence on substituting the assumed value for *p* there will be

$$d\mathfrak{h} = Vydx + Udx - ydq = y\left(Vdx - dq\right) + Udx\,.$$

We may seek a multiplier *M*, so that there becomes

$$M\left(Vdx - dq\right) = dS\,,$$

and *M* as well as *S* will be functions of *x* and *q*, from the latter of which the value of *q* expressed in terms of *x* and *S* is elicited in the following operation, on substituting for *q*. Clearly since now there shall be

$$d\mathfrak{h} = \tfrac{ydS}{M} + Udx\,,$$

On taking *S* constant there is sought $T = \int Udx$ and there shall be

$$\mathfrak{h} = T + f{:}S\,,$$

from which it is deduced

$$\tfrac{y}{M} = \left(\tfrac{dT}{dS}\right) + f'{:}S \quad \text{and} \quad z = qy + T + f{:}S\,,$$

and now indeed the value for *S* in terms of *x* and *q* can be restored.

PROBLEMA 26

158. *If on putting $dz = pdx + qdy$ it is required that there shall be $y = Vx + U$ for any functions V and U of p and q present, to investigate the nature of the function z in general.*

SOLUTION

Here this formula is required to be used

$$z = px + qy - \int\left(xdp + ydq\right);$$

there is put in place $\int\left(xdp + ydq\right) = \mathfrak{h}$ and there will be on substituting the prescribed value for *y*

$$d\mathfrak{h} = xdp + Vxdq + Udq\,.$$

Now there is sought a multiplier *M* rendering the formula *dp* + *Vdq* integrable and there shall be

$$M\left(dp + Vdq\right) = dS\,,$$

where *M* and *S* will be given by *p* and *q*, and from the final equation the value expressed *p* in terms of *q* and *S* may be elicited, as it is requires to be use . Evidently there will be then

$$d\mathfrak{h} = \frac{xdS}{M} + Udq\,,$$

on assuming *S* constant the formula *Udq* may be integrated and there shall be $T = \int Udq$; there will be $\mathfrak{h} = T + f{:}S$ and hence

$$\frac{x}{M} = \left(\frac{dT}{dS}\right) + f'{:}S \quad \text{and} \quad z = px + qy - T - f{:}S\,.$$

Therefore everything in terms of *p* and *q*, from which *M*, *S* and *T* with $\left(\frac{dT}{dS}\right)$ are given, thus will be determined , so that there shall be

$$x = M\left(\frac{dT}{dS}\right) + Mf'{:}S, \quad y = Vx + U \quad \text{and} \quad z = px + qy - T - f{:}S\,.$$

EXAMPLE

159. *If on putting* $dz = pdx + qdy$ *there must become* $px + qy = apq$, *to investigate the nature of the function z.*

Therefore since there shall be $y = -\frac{px}{q} + ap$, then there shall be

$$V = \frac{-p}{q} \quad \text{and } U = ap.$$

Now because there must be $M\left(dp - \frac{pdq}{q}\right) = dS$, there may be taken $M = \frac{1}{q}$ and there becomes $S = \frac{p}{q}$ and $p = Sq$. Hence $U = aSq$ and on taking *S* constant

$$T = \int Udq = \frac{1}{2}aSqq$$

and therefore $\left(\frac{dT}{dS}\right) = \frac{1}{2}aqq$. On which account we will have for the solution

$$x = \frac{1}{2}aq + \frac{1}{q}f'{:}\frac{p}{q}\,, \quad y = \frac{1}{2}ap - \frac{p}{qq}f'{:}\frac{p}{q}$$

et

$$z = px + qy - \frac{1}{2}apq - f{:}\frac{p}{q} = \frac{1}{2}apq - f{:}\frac{p}{q}\,.$$

But from the reduction examined above [§116] we will have

$$y = (aq - x)F'{:}\left(qx - \frac{1}{2}aqq\right) \text{ and } z = qy + F{:}\left(qx - \frac{1}{2}aqq\right)$$

SCHOLIUM

160. These four problems jointly considered certainly extend widely and for the formula $dz = pdx + qdy$ all the relations between *p, q, x* and *y* are included, in which either *x* and *y*, or *p* and *y*, or *x* and *q*, or *p* and *q* nowhere are greater than a single dimension. From which often it can come about, that the same question will be resolved by two or more of these four problems, just as happened in this last example ; in which since not only *x* and *y*, but also *x* and *q*, and likewise *p* and *y* nowhere take more than one dimension, that the three previous problems refer to [§156–158] and this condition is only opposed in the first place in [§ 146]. But if moreover this relation is prescribed between *p, q, x, y*, so that there must be

$$\alpha px + \beta qy + ap + bq + mx + ny + c = 0,$$

the resolution can be put in place equally by all four methods. Truly also thereupon the resolutions arise, even if the forms differ, yet by the reduction set out previously are able to be recalled in agreement.

But the following case also shows the extent of the resolution admitted, that therefore it is convenient to establish.

PROBLEM 27

161. *If on putting $dz = pdx + qdy$ a relation of this kind between p, q and x, y may be given, so that a certain function of p and x should be equal to a certain function of q and y, to investigate the nature of the function z in general.*

SOLUTION

Let *P* be that function of *p* and *x* and *Q* that function of *q* and *y* themselves, which must be equal to each other. Therefore since there shall be *P* = *Q*, with each put = *v*, so that there shall be *P* = *v* and *Q*= *v*. From the former therefore *p* can be defined by *x* and *v*, from the latter *q* indeed by *y* and *v;* with which accomplished in formula $dz = pdx + qdy$, since *p* shall be a function of *x* and *v*, the part *pdx* can be integrated on assuming *v* constant and there shall be $\int pdx = R$; in a similar manner since *q* shall be a function of *y* and *v*, the other part too *qdy* can be integrated on assuming *v* constant and there shall be $\int qdy = S$; hence there shall be *R* equal to a function of *x* and *v* and *S* equal to a function of *y* and *v*. but on taking *v* to be variable also, there shall be

$$dR = pdx + Vdv \quad \text{and} \quad dS = qdy + Udv,$$

from which it is deduced

$$dz = dR + dS - dv(V + U);$$

which form since it must be integrable, it is required that there shall be $V + U = f':v$. Whereby the solution will be contained in these two equations

$$V+U=f':v \quad \text{and} \quad z=R+S-f:v.$$

Evidently since *p, R* and *V* may be given by *x* and *v* and also *q*, *S* and *U* by *y* and *v*, from the first equation *v* is defined from *x* and *y*, which value substituted into the other equation will determine the function sought *z* by *x* and *y*.

COROLLARY 1

162. Therefore whenever *q* should be equal to a function of this kind of *p, x, y*, so that thereupon the equation can be formed, from the one part of this only the two letters *x* and *p* may be found, and from the other part the two remaining letters *y* and *q* may be found, then the problem can be resolved.

COROLLARY 2

163. If a function of the two letters *p* and *x*, which I have put equal to *P*, thus shall be prepared, so that on putting that equal to *v* then it becomes easier for *x* to be defined by *p* and *v*, while it is convenient to use the formula

$$z=px+\int(qdy-xdp)$$

and likewise the setting out of the problem will be had as before.

COROLLARY 3

164. In a similar manner if from the other function $Q=v$ the quantity *y* is easier defined by *q* and *v*, the resolution will be desired from the form

$$z=qy+\int(pdx-ydq).$$

But if each comes about, so that *x* can be defined in terms of *p* and *v* as well as *y* by *q* and *v*, then the formula required to be used will be

$$z=px+qy-\int(xdp+ydq).$$

SCHOLIUM

165. This problem includes innumerable cases not dealt with in the preceding, and also the solution of this rests on a fundamental difference. Yet meanwhile at this stage we stand at a great distance from a general solution to the problem, which has been the aim of this chapter and in which in general a solution is desired, if some equation is proposed between the four-fold quantities *p, q, x, y*, but which on account of deficiencies in analysis it is considered indeed not to be hoped for. Therefore it is required that we be content, if we can instruct how most cases are to be resolved. But so that the strength of this problem be observed more, we may add several examples.

EXAMPLE 1

166. *If on putting* $dz = pdx + qdy$ *there must be* $q = \frac{xxyy}{a^4 p}$, *to investigate the general nature of the function z.*

Because here it is permitted to separate p, x and q, y, since there shall be $\frac{aaq}{yy} = \frac{xx}{aap}$, there may be put $\frac{xx}{aap} = v = \frac{aaq}{yy}$, from which p thus is defined by x and v and q by y and v, so that there shall be

$$p = \frac{xx}{aav} \quad \text{and} \quad q = \frac{vyy}{aa}$$

and thus

$$dz = \frac{xxdx}{aav} + \frac{vyydy}{aa}.$$

Hence we may deduce

$$z = \frac{x^3}{3aav} + \frac{vy^3}{3aa} + \frac{1}{3aa}\int\left(\frac{x^3dv}{vv} - y^3dv\right)$$

and thus $\frac{x^3}{vv} - y^3$ must be a function of v. And on putting

$$\frac{x^3}{vv} - y^3 = f':v \quad \text{or} \quad y^3 = \frac{x^3}{vv} - f':v$$

there will be

$$z = \frac{1}{3aa}\left(\frac{x^3}{v} + vy^3 + f:v\right).$$

COROLLARY

167. Hence v is easily eliminated, if there is put $f':v = \frac{b^3}{vv} - c^3$ and hence $f:v = \frac{-b^3}{v} - c^3 v$. Now the first equation gives $y^3 - c^3 = \frac{x^3 - b^3}{vv}$, from which $vv = \frac{x^3 - b^3}{y^3 - c^3}$, and on account of

$$3aaz = \frac{x^3 + vvy^3 - b^3 - c^3vv}{v} = 2v\left(y^3 - c^3\right)$$

there will be

$$z = \frac{2}{3aa}\sqrt{\left(x^3 - b^3\right)\left(y^3 - c^3\right)}.$$

EXAMPLE 2

168. *If on putting* $dz = pdx + qdy$ *there must be* $q = \frac{1}{b}\sqrt{(xx + yy - aapp)}$*, to investigate the nature of the function z.*

The prescribed condition reverts to

$$bbqq - yy = xx - aapp = v,$$

from which we deduce

$$q = \tfrac{1}{b}\sqrt{(yy + v)} \quad \text{and} \quad p = \tfrac{1}{a}\sqrt{(xx - v)}.$$

Now truly there shall be

$$\int pdx = \tfrac{1}{a}\int dx\sqrt{(xx - v)} = \tfrac{1}{2a}x\sqrt{(xx - v)} - \tfrac{v}{2a}\int \frac{dx}{\sqrt{(xx-v)}}$$
$$= \tfrac{x}{2a}\sqrt{(xx - v)} - \tfrac{v}{2a}l\left(x + \sqrt{(xx - v)}\right) = R;$$

in a similar manner there shall be

$$\int qdy = \tfrac{y}{2b}\sqrt{(yy + v)} + \tfrac{v}{2b}l\left(y + \sqrt{(yy + v)}\right) = S.$$

Whereby since there shall be

$$V = \left(\tfrac{dR}{dv}\right) = \frac{-x}{4a\sqrt{(xx-v)}} - \tfrac{1}{2a}l\left(x + \sqrt{(xx - v)}\right) + \frac{v}{4a\left(x+\sqrt{(xx-v)}\right)\sqrt{(xx-v)}},$$

which is reduced to

$$V = -\tfrac{1}{4a} - \tfrac{1}{2a}l\left(x + r\sqrt{(xx - v)}\right)$$

and in a like manner

$$U = \left(\tfrac{dS}{dv}\right) = +\tfrac{1}{4a} + \tfrac{1}{2b}l\left(y + r\sqrt{(yy + v)}\right),$$

whereby since $V + U = f':v$, there will be

$$\frac{a-b}{4ab} + l\frac{\left(y+\sqrt{(yy+v)}\right)^{\frac{1}{2b}}}{\left(x+\sqrt{(xx-v)}\right)^{\frac{1}{2a}}} = f':v$$

from which the value of v is determined by x and y. From which finally there is deduced

$$z = \tfrac{x}{2a}\sqrt{(xx - v)} + \tfrac{y}{2a}\sqrt{(yy + v)} + vl\frac{\left(y+\sqrt{(yy+v)}\right)^{\frac{1}{2b}}}{\left(x+\sqrt{(xx-v)}\right)^{\frac{1}{2a}}} - f:v$$

or

$$z=\tfrac{x}{2a}\sqrt{(xx-v)}+\tfrac{y}{2a}\sqrt{(yy+v)}-\tfrac{(a-b)v}{4ab}+vf':v-f:v.$$

SCHOLIUM

169. This solution can be freed from logarithms as follows :

Putting

$$f':v=lt+\tfrac{a-b}{4ab},$$

so that there shall be

$$t^{2ab}=\frac{\left(y+\sqrt{(yy+v)}\right)^{a}}{\left(x+\sqrt{(xx-v)}\right)^{b}},$$

from which v is given by t. Then truly let $v=tF':t$ and on account of $dvf'':v=\frac{dt}{t}$ there will be

$$\int vdvf'':v=vf':v-f:v=\int\tfrac{vdt}{t}=F:t$$

and thus there will be

$$z=\tfrac{x}{2a}\sqrt{(xx-v)}+\tfrac{y}{2b}\sqrt{(yy+v)}-\tfrac{(a-b)v}{4ab}+F:t,$$

where there is

$$v=tF':t \text{ and } t^{2ab}=\frac{\left(y+\sqrt{(yy+v)}\right)^{a}}{\left(x+\sqrt{(xx-v)}\right)^{b}},$$

from which t and v can be defined by x and y. Hence it appears at once, if $F':t=0$ were taken, to become $v=0$, $F:t=0$ and $z=\frac{xx}{2a}+\frac{yy}{2b}$ and hence $p=\frac{x}{a}$ and $q=\frac{y}{b}$, with which agreed upon the prescribed condition is satisfied everywhere.

Moreover this method of removing logarithms is especially worthy of note and it will have the greatest use in other cases.

EXAMPLE 3

170. *If on putting $dz=pdx+qdy$ there must become $x^{m}y^{n}=Ap^{\mu}q^{\nu}$, to investigate the nature of the function z.*

Therefore there is put in place

$$\frac{x^{m}}{p^{\mu}}=\frac{Aq^{\nu}}{y^{n}}=\upsilon^{\mu\nu}$$

and hence there is deduced

$$p = \frac{x^{\frac{m}{\mu}}}{\upsilon^{\nu}} \quad \text{and} \quad q = \tfrac{1}{a} y^{\frac{n}{\nu}} \upsilon^{\mu}$$

on putting $A = a^{\nu}$. From which we will have

$$\int p dx = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)\upsilon^{\nu}} + \frac{\mu\nu}{(m+\mu)} \int \frac{x^{\frac{m+\mu}{\mu}} d\upsilon}{\upsilon^{\nu+1}}$$

and

$$\int q dy = \frac{\nu y^{\frac{n+\nu}{\nu}}}{(n+\nu)a} \upsilon^{\mu} - \frac{\mu\nu}{(n+\nu)a} \int y^{\frac{n+\nu}{\nu}} \upsilon^{\nu-1} d\upsilon .$$

On which account there will be

$$z = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)\upsilon^{\nu}} + \frac{\nu y^{\frac{n+\nu}{\nu}} \upsilon^{\mu}}{(n+\nu)a} + \frac{\mu\nu}{(m+\mu)(n+\nu)a} \int d\upsilon \left(\frac{(n+\nu)a x^{\frac{m+\mu}{\mu}}}{\upsilon^{\nu+1}} - (m+\mu) y^{\frac{n+\nu}{\nu}} \upsilon^{\nu-1} \right),$$

thus so that, if we put in place

$$\frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)\upsilon^{\nu+1}} - \frac{y^{\frac{n+\nu}{\nu}} \upsilon^{\mu-1}}{(n+\nu)a} = f':\upsilon,$$

there will become

$$z = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)\upsilon^{\nu}} + \frac{\nu y^{\frac{n+\nu}{\nu}} \upsilon^{\mu}}{(n+\nu)a} + \mu\nu f:\upsilon.$$

For the simplest case we may put $f':\upsilon = 0$ et $f:\upsilon = 0$ and there will be

$$y^{\frac{n+\nu}{\nu}} \upsilon^{\mu} = \frac{(n+\nu)a}{m+\mu} x^{\frac{m+\mu}{\mu}} \quad \text{and} \quad \upsilon = \left(\frac{(n+\nu)a x^{\frac{m+\mu}{\mu}}}{(m+\mu) y^{\frac{n+\nu}{\nu}}} \right)^{\frac{1}{\mu+\nu}},$$

then indeed

$$z = \frac{1}{\upsilon^{\nu}} \left(\frac{\mu}{(m+\mu)} x^{\frac{m+\mu}{\mu}} + \frac{\nu}{(n+\nu)a} y^{\frac{n+\nu}{\nu}} \upsilon^{\mu+\nu} \right)$$

or

$$z = \frac{\mu+\nu}{(m+\mu)\upsilon^{\nu}} x^{\frac{m+\mu}{\mu}} = (\mu+\nu) \left(\frac{x^{\frac{m+\mu}{\mu}} y^{\frac{n+\nu}{\nu}}}{(m+\mu)^{\mu} (n+\nu)^{\nu} A} \right)^{\frac{1}{\mu+\nu}}.$$

PROBLEM 28

171. *If on putting* $dz = pdx + qdy$ *a relation is given between* p, q *and* x, y *of this kind, so that* p *and* q *may be equal to certain functions of x, y and of a new variable v, to examine the cases, in which the nature of the function z can be investigated..*

SOLUTION

Since p shall be a function of x, y and v, with y and v regarded as constants the integral is sought $\int pdx = P$, and it becomes with everything taken as variable

$$dP = pdx + Rdy + Mdv\,,$$

from which, since the value may be substituted for pdx, there will be

$$dz = dP + (q - R)dy - Mdv\,.$$

But if now it comes about, that $q - R$ shall be a function only of y and v with x excluded, on taking v constant there is sought $\int (q - R)dy = T$ and then there shall be

$$dT = (q - R)dy + Vdv\,.$$

Hence the value of $(q - R)dy$ substituted there will give

$$dz = dP + dT - (M + V)dv\,;$$

which form, since it must be integrable, $M + V = f':v$ is put in place, and there will be

$$z = P + T - f':v\,.$$

But from the operations undertaken there are given P, R, M in terms of x, y and v, but T and V are given by y and v only; and the resolution is successful, but only if in a form no longer containing x.

By similar reasoning the solution succeeds, if M is given in terms of y and v only; since then from y constant there is sought $\int Mdv = L$ and there shall be $dL = Mdv + Ndy$; then there shall be

$$dz = dP + (q - R + N)dy - dL$$

and it is convenient to put $q - R + N = f':y$, so that there becomes

$$z = P - L + f:y\,.$$

In a similar manner from the other part $\int qdy$ the calculation can begin and can be described in detail.

But on introducing an indefinite function *K* of x, y and v, a more general solution can be put in place. For let there be

$$dK = Fdx + Gdy + Hdv$$

and this form is considered

$$dz + dK = \left(p + F\right)dx + \left(q + G\right)dy + Hdv .$$

Now on taking *y* and *v* constant there is sought

$$\int \left(p + F\right)dx = P$$

and there shall be

$$dP = \left(p + F\right)dx + Rdy + Mdv,$$

from which there will be found

$$dz + dK = dP + \left(q + G - R\right)dy + \left(H - M\right)dv .$$

But now if it comes about, that either $q + G - R$ or $H - M$ shall contain only the two variables *y* and *v* with *x* excluded, the resolution can be completed, as has been shown before.

PROBLEM 29

172. *If on putting $dz = pdx + qdy$ a relation may be given between the two differential formulas p, q and the two variables x and z or y and z, the solution of the problem is to be perfected, as far as possible.*

SOLUTION

We may put a relation to be given between p, q and x, z and this case can be restored to the previous case. Indeed this formula may be considered :

$$dy = \tfrac{dz - pdx}{q}$$

and there may be called from the principle derived

$$\tfrac{1}{q} = m \text{ and } \tfrac{p}{q} = -n ,$$

so that there may be considered

$$dy = mdz + ndx ,$$

and on account of $\frac{1}{q}=m$ and $p=-\frac{n}{m}$ the proposed relation will turn into one between the four quantities *m*, *n*, *z* and *x* and thus everything sought is similar to that which we have treated previously, yet with this distinction, because here the quantity *y* may be defined, since before it was *z* being found. But because that determination is absolved from the equations, likewise it is whether finally we wish to elicit from those either *z* or *y* . But if therefore with that reduction made the question falls on the case treated before, it can be resolved by the method explained previously also.

EXAMPLE

173. *If on putting $dz = pdx + qdy$ there must become $qxz = aap$, to investigate the nature of the function z.*

The formula may be considered $dy=\frac{dz}{q}-\frac{pdz}{q}$. Now because $\frac{p}{q}=\frac{xz}{aa}$, there will be

$$dy=\frac{dz}{q}-\frac{xzdx}{aa} \text{ and } y=\int\left(\frac{dz}{q}-\frac{xzdx}{aa}\right),$$

but there is

$$\int\frac{xzdx}{aa}=\frac{xxz}{2aa}-\int\frac{xxdz}{2aa},$$

therefore

$$y=\int dz\left(\frac{1}{q}+\frac{xx}{2aa}\right)-\frac{xxz}{2aa}.$$

Therefore there is put

$$\frac{1}{q}+\frac{xx}{2aa}=f':z;$$

and there will be

$$y=\frac{-xxz}{2aa}+f:z,$$

from which equation certainly *z* is defined by *x* and *y*.

If for the simplest case we assume $f:z=b+\alpha z$, then there will be

$$y-b=\left(\alpha-\frac{xx}{2aa}\right)z \quad \text{and} \quad z=\frac{2aa(y-b)}{2\alpha aa-xx}.$$

and on taking $\alpha=0$ and $b=0$ for the simplest case there will be $z=\frac{-2aa}{xx}$. Moreover hence there becomes

$$p=\frac{+4aay}{x^3} \text{ and } q=\frac{-2aa}{xx},$$

and therefore

$$\frac{p}{q}=-\frac{2y}{x} \quad \text{and} \quad \frac{xz}{aa}=\frac{-2y}{x}.$$

CAPUT V

DE RESOLUTIONE AEQUATIONUM QUIBUS RELATIO INTER QUANTITATES $\left(\frac{dz}{dx}\right), \left(\frac{dz}{dy}\right)$ ET BINAS TRIUM VARIABILIUM x, y, z QUAECUNQUE DATUR

PROBLEMA 21

138. *Si posito* $dz = pdx + qdy$ *debeat esse* $px + qy = 0$, *functionis z indolem per x et y in genere investigare.*

SOLUTIO

Cum sit $q = -\frac{px}{y}$, erit

$$dz = pdx - \frac{pxdy}{y} = px\left(\frac{dx}{x} - \frac{dy}{y}\right)$$

seu

$$dz = py\left(\frac{dx}{y} - \frac{xdy}{yy}\right) = pyd.\frac{x}{y}.$$

Unde patet *py* esse debere functionem ipsius $\frac{x}{y}$, ac si ponatur $py = f':\frac{x}{y}$, fore $z = f:\frac{x}{y}$. Perpetuo scilicet in designandis functionibus hac lege utemur, ut sit $d.f:v = dvf':v$ sicque porro $d.f':v = dvf'':v$ et $d.f'':v = dvf''':v$ etc. At $f:\frac{x}{y}$ denotat functionem quamcunque homogeneam ipsarum *x* et *y* nullius *y* dimensionis, ac si *z* fuerit talis functio quaecunque et differentiando prodeat $dz = pdx + qdy$, semper erit $px + qy = 0$.

COROLLARIUM 1

139. Quodsi ergo *z* fuerit functio homogenea nullius dimensionis ipsarum *x* et *y*, ob $p = \left(\frac{dz}{dx}\right)$ et $q = \left(\frac{dz}{dy}\right)$ erit

$$x\left(\frac{dz}{dx}\right) + y\left(\frac{dz}{dy}\right) = 0,$$

quam veritatem quidem iam supra elicuimus.

COROLLARIUM 2

140. Tum vero cum sit

$$p=\frac{1}{y}f':\frac{x}{y} \quad \text{et} \quad q=\frac{-x}{yy}f':\frac{x}{y},$$

erit p functio homogenea ipsarum x et y numeri dimensionum $=-1$, et si sit $q=\frac{-px}{y}$, ipsa functio z reperitur ex integratione $z=\int pyd.\frac{x}{y}$.

SCHOLION

141. Simili modo solvitur problema, si posito $dz=pdx+qdy$ fieri debeat $mpx+nqy=a$. Tum enim ob $q=\frac{a}{ny}-\frac{mpx}{ny}$ erit

$$dz=\frac{ady}{ny}+pdx-\frac{mpxdy}{ny}$$

seu

$$dz=\frac{ady}{ny}+\frac{pdx}{n}\left(\frac{ndx}{x}-\frac{mdy}{y}\right)=\frac{ady}{ny}+\frac{py^m}{nx^{n-1}}d.\frac{x^n}{y^m},$$

unde solutio praebet

$$\frac{py^m}{nx^{n-1}}=f'\frac{x^n}{y^m} \quad \text{et} \quad z=\frac{a}{n}ly+f:\frac{x^n}{y^m}.$$

Quin etiam hoc generalius problema resolvi potest, quo esse debet

$$pX+qY=A$$

existente X functione ipsius x et Y ipsius y. Cum enim inde fiat $q=\frac{A}{Y}-\frac{pX}{Y}$, erit

$$dz=\frac{Ady}{Y}+pdx-\frac{pXdy}{Y}=\frac{Ady}{Y}+pX\left(\frac{dx}{X}-\frac{dy}{Y}\right).$$

Statui ergo debet

$$pX=f':\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right)$$

indeque fit

$$z=A\int\frac{dy}{Y}+f:\left(\int\frac{dx}{X}-\int\frac{dy}{Y}\right).$$

PROBLEMA 22

142. *Si posito* $dz = pdx + qdy$ *debeat esse* $\frac{q}{p}$ *aequale functioni datae cuicunque ipsarum x et y, indolem functionis z in genere investigare.*

SOLUTIO

Sit *V* ista functio data ipsarum *x* et *y*, ut sit $q = pV$ et habebitur $dz = p(dx + Vdy)$. Dabitur iam multiplicator *M*, itidem functio ipsarum *x* et *y*, ut $M(dx + Vdy)$ fiat integrabile. Ponatur ergo $M(dx + Vdy) = dS$ ac dabitur etiam *S*, functio ipsarum *x* et *y*. Cum ergo sit $dz = \frac{pdS}{M}$, perspicuum est quantitatem $\frac{p}{M}$ aequari debere functioni ipsius *S;* quare si ponamus $\frac{p}{M} = f':S$, fiet $z = f:S$ indeque erit

$$p = Mf':S \quad \text{et} \quad q = MVf':S.$$

COROLLARIUM 1

143. Hoc ergo casu functio quaesita *z* statim invenitur per *x* et *y* expressa, quoniam *S* per *x* et *y* datur. Fieri autem potest, ut *S* prodeat quantitas transcendens, quin etiam ut per methodos adhuc cognitas multiplicator *M* ne inveniri quidem possit.

COROLLARIUM 2

144. Si *V* sit functio nullius dimensionis ipsarum *x* et *y*, erit $M = \frac{1}{x+Vy}$. Seu posito $x = vy$ fiet *V* functio ipsius *v* et

$$dS = M(ydv + vdy + Vdy).$$

Capiatur $M = \frac{1}{y(v+V)}$ eritque $dS = \frac{dy}{y} + \frac{dv}{v+V}$ unde reperitur

$$z = f:\left(ly + \int \frac{dv}{v+V}\right).$$

SCHOLION

145. Ob permutabilitatem ipsarum *p* et *x*, item *q* et *y* simili modo sequentia problemata resolvi possunt.

I. Si debeat esse $q = xV$ existente *V* functione quacunque ipsarum *p* et *y*, consideretur forma

$$z = px + \int (qdy - xdp) = px + \int x(Vdy - dp).$$

Quaeratur multiplicator *M*, ut sit

$$M(Vdy - dp) = dS;$$

erit *S* functio ipsarum *p* et *y* atque

$$z = px + \int \frac{xdS}{M},$$

ex quo colligitur haec solutio

$$\frac{x}{M} = f':S \quad \text{et} \quad z = pMf':S + f:S.$$

II. Si debeat esse $y = pV$ existente V functione quacunque ipsarum x et q, consideretur forma

$$z = qy + \int (pdx - ydq) = qy + \int p(dx - Vdq).$$

Quaeratur multiplicator M, ut sit

$$M(dx - Vdq) = dS;$$

erit S functio ipsarum x et q et

$$z = qy + \int \frac{pdS}{M}.$$

Quare fit

$$\frac{p}{M} = f':S \quad \text{et} \quad z = qy + f:S$$

seu ob $p = \frac{y}{V}$ erit

$$y = MVf':S \quad \text{et} \quad z = qMVf':S + f:S.$$

III. Si debeat esse $y = xV$ existente V functione quacunque ipsarum p et q, consideretur haec forma

$$z = px + qy - \int (xdp + xVdq).$$

Quaeratur multiplicator M, ut fiat

$$M(dp + Vdq) = dS;$$

erit S functio ipsarum p et q et

$$z = px + qy - \int \frac{xdS}{M},$$

unde haec solutio nascitur

$$\frac{x}{M} = f':S \quad \text{et} \quad z = px + qy - f:S.$$

Omnes hi casus huc redeunt, ut quaternarum quantitatum p, x, q, y vel $\frac{q}{p}$ vel $\frac{q}{x}$ vel $\frac{y}{p}$ vel $\frac{y}{x}$ aequetur functioni cuicunque binarum reliquarum.

PROBLEMA 23

146. *Si posito $dz = pdx + qdy$ requiratur, ut sit $q = pV + U$ existente tam V quam U functione quacunque binarum variabilium x et y, indolem functionis z in genere investigare.*

SOLUTIO

Cum ob $q = pV + U$ sit $dz = p\left(dx + Vdy\right) + Udy$, quaeratur primo multiplicator M formulam $dx + Vdy$ reddens integrabilem sitque

$$M\left(dx + Vdy\right) = dS\ ;$$

erunt M et S functiones ipsarum x et y fietque

$$dz = \frac{pdS}{M} + Udy\ .$$

Cum iam sit S functio ipsarum x et y, inde x per y et S definiri potest, quo valore introducto fient U et M functiones ipsarum y et S. Nunc sumto S constante integretur formula Udy sitque

$$\int Udy = T + f:S$$

ac posito

$$dT = Udy + WdS$$

fiet

$$\frac{p}{M} = W + f':S \quad \text{et} \quad z = T + f:S$$

sicque omnia per binas variabiles y et S exprimentur.

COROLLARIUM 1

147. Datis ergo binarum variabilium x et y functionibus V et U, ut sit $q = pV + U$, solutio problematis primo postulat, ut multiplicator M investigetur formulam $dx + Vdy$ integrabilem reddens, quo invento habebitur functio S earundem variabilium x et y, ut sit

$$S = \int M\left(dx + Vdy\right).$$

COROLLARIUM 2

148. In hunc finem considerari conveniet aequationem differentialem $dx+Vdy=0$; haec enim si integrari poterit, simul inde colligi potest multiplicator *M*, ut formula $M\left(dx+Vdy\right)$ fiat verum differentiale cuiusdam functionis *S*, quae propterea hinc invenietur.

COROLLARIUM 3

149. Inventa porro hac functione *S* quantitas *x* per *y* et *S* exprimi debet, ita ut *x* aequetur functioni ipsarum *y* et *S;* quo valore in quantitate *U* substituto quaeratur integrale $\int Udy=T$ spectata *S* ut constante sicque obtinebitur ,*T* functio ipsarum *y* et *S.*

COROLLARIUM 4

150. Denique inventa hac functione *T* sit $W=\left(\frac{dT}{dS}\right)$, unde tandem colligitur solutio problematis his duabus formulis contenta

$$\frac{p}{M}=W+f':S \text{ et } z=T+f:S\,;$$

ubi cum *S* sit functio ipsarum *x* et *y*, pro *z* statim reperitur functio ipsarum *x* et *y*.

COROLLARIUM 5

151. Si *U* sit functio ipsius *y* tantum, non opus est illa expressione ipsius *x* per *y* et *S*, sed $T=\int Udy$ erit quoque functio ipsius *y* tantum, hinc $W=\left(\frac{dT}{dS}\right)=0$. Hic autem casus manifesto reducitur ad praecedentem [§ 142] ponendo *z* loco $z-\int Udy$.

EXEMPLUM 1

152. *Si posito $dz=pdx+qdy$ debeat esse $q=\frac{py}{x}+\frac{y}{x}$, indolem functionis z investigare.*

Hic ergo est

$$V=\frac{x}{y} \text{ et } U=\frac{y}{x}$$

unde ob

$$dx+Vdy=dx+\frac{xdy}{y}$$

erit multiplicator $M=y$ et $dS=ydx+xdy$, hinc $S=xy$ sicque habebitur

$$x=\frac{S}{y} \text{ et } U=\frac{yy}{S}.$$

Iam erit

$$T = \int U dy = \int \frac{yydy}{S} = \frac{y^3}{3S} \quad \text{et} \quad W = \frac{-y^3}{3SS}.$$

Quare pro solutione huius exempli habebimus

$$\frac{p}{y} = \frac{-y^3}{3SS} + f':S \quad \text{et} \quad z = \frac{y^3}{3S} + f:S$$

seu ob $S = xy$ erit $z = \frac{yy}{3x} + f:xy$.

EXEMPLUM 2

153. *Si posito* $dz = pdx + qdy$ *debeat esse* $px + qy = n\sqrt{(xx+yy)}$, *indolem functionis z investigare.*

Cum hic sit $q = \frac{-px}{y} + \frac{n}{y}\sqrt{(xx+yy)}$, erit

$$V = \frac{-x}{y} \quad \text{et} \quad U = \frac{n}{y}\sqrt{(xx+yy)},$$

ergo $dS = M\left(dx - \frac{xdy}{y}\right)$, quare capiatur $M = \frac{1}{y}$, ut fiat $dS = \frac{dx}{y} - \frac{xdy}{yy}$ et $S = \frac{x}{y}$. Hinc oritur

$$x = Sy \quad \text{et} \quad U = n\left(\sqrt{1+SS}\right)$$

ideoque posito *S* constante erit

$$T = \int U dy = ny\sqrt{(1+SS)} \ \text{et} \ W = \left(\frac{dT}{dS}\right) = \frac{nyS}{\sqrt{(1+SS)}},$$

ita ut solutio nostrae quaestionis sit

$$py = \frac{nyS}{\sqrt{(1+SS)}} + f':S \quad \text{et} \quad z = ny\sqrt{(1+SS)} + f'S..$$

Cum igitur sit $S = \frac{x}{y}$, erit

$$z = n\sqrt{(xx+yy)} + f:y$$

ubi $f:\frac{x}{y}$ denotat functionem quamcunque nullius dimensionis ipsarum *x* et *y*.

EXEMPLUM 3

154. *Si posito* $dz = pdx + qdy$ *debeat esse* $pxx + qyy = nxy$, *functionis z indolem investigare.*

Cum sit $q = \frac{-pxx}{yy} + \frac{nx}{y}$ erit

$$V = \frac{-xx}{yy} \quad \text{et} \quad U = \frac{nx}{y}.$$

Quare ob $dS = M\left(dx - \frac{xxdy}{yy}\right)$ capiatur $M = \frac{1}{xx}$ ut fiat $S = \frac{1}{y} - \frac{1}{x} = \frac{x-y}{xy}$.
Hinc erit

$$\frac{1}{x} = \frac{1}{y} - S \text{ et } x = \frac{y}{1-Sy}$$

ideoque $U = \frac{n}{1-Sy}$. Sumto igitur S constante habebimus

$$T = \int \frac{ndy}{1-Sy} = -\frac{n}{S} l(1-Sy) \text{ et } W = +\frac{n}{SS} l(1-Sy) + \frac{ny}{S(1-Sy)}.$$

Consequenter ob $S = \frac{x-y}{xy}$ et $1 - Sy = \frac{y}{x}$ solutio praebet

$$z = \frac{-nxy}{x-y} l \frac{y}{x} + f : \frac{x-y}{xy}.$$

SCHOLION

155. Ex solutione huius problematis etiam haec quaestio latius patens resolvi potest. Sint *P*, *Q*, item *V*, *U* functiones quaecunque datae ipsarum *x* et *y* et quaeri oporteat functionem *z*, ut sit

$$dz = Pdx + Qdy + L(Vdx + Udy),$$

seu, quod eodem redit, functio *L* investigari debet, ut ista formula differentialis integrationem admittat.

Ad hoc praestandum quaeratur primo multiplicator *M* formulam $Vdx + Udy$ integrabilem efficiens ponaturque $dS = M(Vdx + Udy)$, unde functio *S* reperietur per *x* et *y* expressa. Ex ea quaeratur valor ipsius *x* per *y* et *S* expressus, et cum sit

$$dz = Pdx + Qdy + \frac{LdS}{M}$$

hic ubique loco *x* valor ille substituatur; sit autem inde $dx = Edy + FdS$, unde etiam *E* et *F* innotescent, eritque

$$dz = EPdy + Qdy + FPdS + \frac{LdS}{M}.$$

Sumatur quantitas *S* pro constante sitque

$$T = \int (EP + Q)dy;$$

erit

$$z = T + f:S,$$

quod quidem ad solutionem sufficit. Sed ad *L* inveniendum differentietur haec expressio

$$dz = (EP+Q)dy + dS\left(\frac{dT}{dS}\right) + dSf':S$$

ac necesse est fiat

$$FP + \frac{L}{M} = \left(\frac{dT}{dS}\right) + f':S$$

ideoque

$$L = -FMP + M\left(\frac{dT}{dS}\right) + Mf':S .$$

Ceterum ob permutabilitatem ipsarum *p, x* et *q, y* etiam hinc sequentia problemata resolvi possunt, quae propterea strictim percurram.

PROBLEMA 24

156. *Si posito $dz = pdx + qdy$ requiratur, ut sit $q = Vx + U$ existente tam V quam U functione quacunque data ipsarum p et y, investigare indolem functionis quaesitae z.*

SOLUTIO

Utamur formula

$$z = px + \int (qdy - xdp),$$

et cum loco *q* valore substituto sit

$$\int (qdy - xdp) = \int (Vxdy - xdp + Udy),$$

quam formulam integrabilem reddi oportet, sit ea brevitatis gratia $\mathfrak{h}$, et cum sit

$$d\mathfrak{h} = x(Vdy - dp) + Udy,$$

quaeratur primo multiplicator *M* formulam $Vdy - dp$ integrabilem reddens ponaturque

$$M(Vdy - dp) = dS$$

sicque *S* dabitur per *y* et *p;* unde *p* eliciatur per *y* et *S* expressum, quo valore ibi substituto erit

$$d\mathfrak{h} = \frac{xdS}{M} + Udy .$$

Iam sumto *S* constante sumatur integrale $\int Udy = T + f:S$ eritque

$$\frac{x}{M} = \left(\frac{dT}{dS}\right) + f':S \quad \text{et} \quad \mathfrak{h} = T + f:S \quad .$$

Solutio igitur per binas variabiles y et S ita se habebit

$$x = M\left(\tfrac{dT}{dS}\right) + Mf':S \quad \text{et} \quad z = px + T + f:S\,,$$

ubi nunc quidem S per p et y datur.

PROBLEMA 25

157. *Si posito $dz = pdx + qdy$ requiratur, ut sit $p = Vy + U$ existentibus V et U functionibus datis ipsarum x et q, indolem functionis z investigare.*

SOLUTIO

Utamur iam forma

$$z = qy + \int (pdx - ydq)$$

ponaturque formula ad integrationem perducenda

$$\int (pdx - ydq) = \mathfrak{h}\,.$$

Hinc pro p valorem assumtum substituendo erit

$$d\mathfrak{h} = Vydx + Udx - ydq = y(Vdx - dq) + Udx\,.$$

Quaeramus multiplicatorem M, ut fiat

$$M(Vdx - dq) = dS\,,$$

ac tam M quam S erunt functiones ipsarum x et q, ex quarum posteriori valor ipsius q per x et S expressus eliciatur in sequenti operatione pro q substituendus. Scilicet cum nunc sit

$$d\mathfrak{h} = \tfrac{ydS}{M} + Udx\,,$$

Sumto S constante quaeratur $T = \int Udx$ sitque

$$\mathfrak{h} = T + f:S\,,$$

unde colligitur

$$\tfrac{y}{M} = \left(\tfrac{dT}{dS}\right) + f':S \quad \text{et} \quad z = qy + T + f:S\,,$$

ac nunc quidem pro S valorem in x et q restituere licet.

PROBLEMA 26

158. *Si posito* $dz = pdx + qdy$ *requiratur, ut sit* $y = Vx + U$ *existentibus V et U functionibus quibuscunque datis ipsarum p et q, indolem functionis z in genere investigare.*

SOLUTIO

Hic utendum est formula

$$z = px + qy - \int (xdp + ydq);$$

statuatur $\int (xdp + ydq) = \mathfrak{h}$ eritque pro *y* valorem praescriptum substituendo

$$d\mathfrak{h} = xdp + Vxdq + Udq.$$

Quaeratur iam multiplicator *M* formulam *dp* + *Vdq* integrabilem reddens sitque

$$M(dp + Vdq) = dS,$$

ubi *M* et *S* per *p* et *q* dabuntur, et ex posteriori eliciatur valor ipsius *p* per *q* et *S* expressus, quo deinceps uti oportet. Scilicet cum sit

$$d\mathfrak{h} = \tfrac{xdS}{M} + Udq,$$

sumto *S* constante integretur formula *Udq* sitque $T = \int Udq$; erit $\mathfrak{h} = T + f:S$ hincque

$$\tfrac{x}{M} = \left(\tfrac{dT}{dS}\right) + f':S \quad \text{et} \quad z = px + qy - T - f:S.$$

Omnia ergo per *p* et *q*, unde *M*, *S* et *T* cum $\left(\tfrac{dT}{dS}\right)$ dantur, ita determinabuntur, ut sit

$$x = M\left(\tfrac{dT}{dS}\right) + Mf':S, \quad y = Vx + U \quad \text{et} \quad z = px + qy - T - f:S.$$

EXEMPLUM

159. *Si posito* $dz = pdx + qdy$ *debeat esse* $px + qy = apq$, *indolem functionis z investigare.*

Cum ergo sit $y = -\tfrac{px}{q} + ap$, erit

$$V = \tfrac{-p}{q} \quad \text{et} \quad U = ap.$$

Quia nunc esse debet $M\left(dp - \tfrac{pdq}{q}\right) = dS$, capiatur $M = \tfrac{1}{q}$ fitque $S = \tfrac{p}{q}$ et $p = Sq$. Hinc $U = aSq$ et sumto *S* constante

$$T = \int Udq = \tfrac{1}{2}aSqq$$

ideoque $\left(\frac{dT}{dS}\right) = \frac{1}{2}aqq$. Quocirca pro solutione habebimus

$$x = \tfrac{1}{2}aq + \tfrac{1}{q}f':\tfrac{p}{q}\ , \quad y = \tfrac{1}{2}ap - \tfrac{p}{qq}f':\tfrac{p}{q}$$

et

$$z = px + qy - \tfrac{1}{2}apq - f:\tfrac{p}{q} = \tfrac{1}{2}apq - f:\tfrac{p}{q}.$$

Per reductionem autem supra [§116] traditam habebimus

$$y = (aq - x)F':\left(qx - \tfrac{1}{2}aqq\right) \text{ et } z = qy + F:\left(qx - \tfrac{1}{2}aqq\right)$$

SCHOLION

160. Quatuor problemata haec coniunctim considerata admodum late patent atque pro formula $dz = pdx + qdy$ omnes relationes inter *p, q, x* et *y* complectuntur, in quibus vel *x* et *y*, vel *p* et *y*, vel *x* et *q*, vel *p* et *q* nusquam unam dimensionem superant. Ex quo saepe fieri potest, ut eadem quaestio per duo plurave horum quatuor problematum resolvi possit, veluti evenit in exemplo hoc postremo; in quo cum non solum *x* et *y*, sed etiam *x* et *q*, itemque *p* et *y* nusquam plus una dimensione occupent, id ad tria praecedentia problemata [§156–158] referri queat haecque conditio primo tantum problemati [§ 146] adversatur. Quodsi autem inter *p, q, x, y* haec relatio praescribatur, ut esse debeat

$$\alpha px + \beta qy + ap + bq + mx + ny + c = 0,$$

resolutio per omnia quatuor problemata aeque institui potest. Verum etiam resolutiones inde ortae, etiamsi forma discrepent, tamen per reductionem ante expositam ad consensum revocari possunt.

At sequens casus latissime patens resolutionem quoque admittit, quem propterea evolvi conveniet.

PROBLEMA 27

161. *Si posito $dz = pdx + qdy$ inter p, q et x, y eiusmodi relatio detur, ut functio quaedam ipsarum p et x aequetur functioni cuipiam ipsarum q et y, functionis z indolem in genere investigare.*

SOLUTIO

Sit *P* functio illa ipsarum *p* et *x* et *Q* functio illa ipsarum *q* et *y*, quae inter se aequales esse debent. Cum igitur sit *P* = *Q*, ponatur utraque = *v*, ut sit *P* = *v* et *Q*= *v*. Ex priori ergo *p* definire licebit per *x* et *v*, ex posteriori vero *q* per *y* et *v;* quo facto in formula $dz = pdx + qdy$ cum *p* sit functio ipsarum *x* et *v*, integretur pars *pdx* sumto *v* constante sitque $\int pdx = R$; simili modo cum *q* sit functio ipsarum *y* et *v*, integretur quoque altera pars *qdy* sumto *v* constante sitque $\int qdy = S$; erit ergo *R*= functioni ipsarum *x* et *v* et *S* = functioni ipsarum *y* et *v*. At sumto etiam *v* variabili sit

$$dR = pdx + Vdv \quad \text{et} \quad dS = qdy + Udv,$$

unde colligitur

$$dz = dR + dS - dv(V + U);$$

quae forma quia integrabilis esse debet, oportet sit $V + U = f':v$. Quare solutio problematis his duabus aequationibus continebitur

$$V + U = f':v \quad \text{et} \quad z = R + S - f:v.$$

Scilicet cum *p, R* et *V* dentur per *x* et *v* atque *q, S* et *U* per *y* et *v*, per aequationem priorem definitur *v* ex *x* et *y*, qui valor in altera substitutus determinabit functionem quaesitam *z* per *x* et *y*.

COROLLARIUM 1

162. Quoties ergo *q* eiusmodi functioni ipsarum *p, x, y* aequari debet, ut inde aequatio formari possit, ex cuius altera parte tantum binae litterae *x* et *p*, ex altera tantum binae reliquae *y* et *q* reperiantur, problema resolvi poterit.

COROLLARIUM 2

163. Si functio illa binarum litterarum *p* et *x*, quam posui *P*, ita sit comparata, ut posita ea = *v* inde facilius *x* per *p* et *v* definiri possit, tum uti conveniet formula

$$z = px + \int (qdy - xdp)$$

et evolutio perinde se habebit atque ante.

COROLLARIUM 3

164. Simili modo si ex functione altera $Q = v$ quantitas *y* facilius per *q* et *v* definiatur, resolutio ex forma

$$z = qy + \int (pdx - ydq)$$

erit petenda. Sin autem utrumque eveniat, ut tam *x* per *p* et *v* quam *y* per *q* et *v* definiatur, utendum erit formula

$$z = px + qy - \int (xdp + ydq).$$

SCHOLION

165. Problema hoc innumerabiles complectitur casus in praecedentibus non comprehensos atque etiam eius solutio diverso nititur fundamento. Interim tamen longissime adhuc distamus a solutione problematis generalis, cui hoc caput est destinatum et quo in genere solutio desideratur, si inter quaternas quantitates *p, q, x, y* aequatio quaecunque proponatur, quae autem ob defectum Analyseos ne sperari quidem posse videtur. Contentos ergo nos esse oportet, si quam plurimos casus resolvere docuerimus. Quo autem vis huius problematis magis perspiciatur, aliquot exempla adiungamus.

EXEMPLUM 1

166. *Si posito* $dz = pdx + qdy$ *esse debeat* $q = \frac{xxyy}{a^4 p}$ *, indolem functionis z investigare.*

Quia hic *p*, *x* et *q*, *y* separare licet, cum sit, ponatur $\frac{xx}{aap} = v = \frac{aaq}{yy}$, unde *p* per *x* et *v* et *q* per *y* et *v* ita definitur, ut sit

$$p = \frac{xx}{aav} \quad \text{et} \quad q = \frac{vyy}{aa}$$

ideoque

$$dz = \frac{xxdx}{aav} + \frac{vyydy}{aa}.$$

Hinc colligimus

$$z = \frac{x^3}{3aav} + \frac{vy^3}{3aa} + \frac{1}{3aa}\int\left(\frac{x^3 dv}{vv} - y^3 dv\right)$$

sicque $\frac{x^3}{vv} - y^3$ debet esse functio ipsius *v*. Ac posito

$$\frac{x^3}{vv} - y^3 = f':v \quad \text{seu} \quad y^3 = \frac{x^3}{vv} - f':v$$

erit

$$z = \frac{1}{3aa}\left(\frac{x^3}{v} + vy^3 + f:v\right).$$

COROLLARlUM

167. Hinc facillime *v* eliminatur, si ponatur $f':v = \frac{b^3}{vv} - c^3$ hincque $f:v = \frac{-b^3}{v} - c^3 v$. Iam prior aequatio dat $y^3 - c^3 = \frac{x^3 - b^3}{vv}$,unde $vv = \frac{x^3 - b^3}{y^3 - c^3}$,et ob

$$3aaz = \frac{x^3 + vvy^3 - b^3 - c^3 vv}{v} = 2v\left(y^3 - c^3\right)$$

erit

$$z = \frac{2}{3aa}\sqrt{\left(x^3 - b^3\right)\left(y^3 - c^3\right)}\ .$$

EXEMPLUM 2

168. *Si posito* $dz = pdx + qdy$ *debeat esse* $q = \frac{1}{b}\sqrt{(xx+yy-aapp)}$ *, investigare indolem functionis z.*

Conditio praescripta redit ad

$$bbqq - yy = xx - aapp = v,$$

unde elicimus

$$q = \frac{1}{b}\sqrt{(yy+v)} \quad \text{et} \quad p = \frac{1}{a}\sqrt{(xx-v)}.$$

Nunc vero est

$$\int pdx = \frac{1}{a}\int dx\sqrt{(xx-v)} = \frac{1}{2a}x\sqrt{(xx-v)} - \frac{v}{2a}\int\frac{dx}{\sqrt{(xx-v)}}$$

$$= \frac{x}{2a}\sqrt{(xx-v)} - \frac{v}{2a}l\left(x+\sqrt{(xx-v)}\right) = R;$$

simili modo est

$$\int qdy = \frac{y}{2b}\sqrt{(yy+v)} + \frac{v}{2b}l\left(y+\sqrt{(yy+v)}\right) = S.$$

Quare cum sit

$$V = \left(\frac{dR}{dv}\right) = \frac{-x}{4a\sqrt{(xx-v)}} - \frac{1}{2a}l\left(x+\sqrt{(xx-v)}\right) + \frac{v}{4a\left(x+\sqrt{(xx-v)}\right)\sqrt{(xx-v)}},$$

quae reducitur ad

$$V = -\frac{1}{4a} - \frac{1}{2a}l\left(x+r\sqrt{(xx-v)}\right)$$

similique modo

$$U = \left(\frac{dS}{dv}\right) = +\frac{1}{4a} + \frac{1}{2b}l\left(y+r\sqrt{(yy+v)}\right),$$

ubi cum $V + U = f':v$, erit

$$\frac{a-b}{4ab} + l\frac{\left(y+\sqrt{(yy+v)}\right)^{\frac{1}{2b}}}{\left(x+\sqrt{(xx-v)}\right)^{\frac{1}{2a}}} = f':v$$

unde valor ipsius *v* per *x* et *y* determinatur. Ex quo tandem colligitur
seu

$$z = \frac{x}{2a}\sqrt{(xx-v)} + \frac{y}{2a}\sqrt{(yy+v)} + vl\frac{\left(y+\sqrt{(yy+v)}\right)^{\frac{1}{2b}}}{\left(x+\sqrt{(xx-v)}\right)^{\frac{1}{2a}}} - f:v$$

seu

$$z = \frac{x}{2a}\sqrt{(xx-v)} + \frac{y}{2a}\sqrt{(yy+v)} - \frac{(a-b)v}{4ab} + vf':v - f:v.$$

SCHOLION

169. Haec solutio a formulis logarithmicis liberari potest hoc modo.
Ponatur

$$f':v = lt + \frac{a-b}{4ab},$$

ut sit

$$t^{2ab} = \frac{\left(y+\sqrt{(yy+v)}\right)^a}{\left(x+\sqrt{(xx-v)}\right)^b},$$

unde v datur per t. Tum vero sit $v = tF':t$ et ob $dvf'':v = \frac{dt}{t}$ erit

$$\int vdvf'':v = vf':v - f:v = \int \frac{vdt}{t} = F:t$$

sicque erit

$$z = \frac{x}{2a}\sqrt{(xx-v)} + \frac{y}{2b}\sqrt{(yy+v)} - \frac{(a-b)v}{4ab} + F:t$$

ubi est

$$v = tF':t \text{ et } t^{2ab} = \frac{\left(y+\sqrt{(yy+v)}\right)^a}{\left(x+\sqrt{(xx-v)}\right)^b},$$

unde t et v per x et y definiri potest. Hinc statim patet, si capiatur $F':t = 0$,
fore $v = 0$, $F:t = 0$ et $z = \frac{xx}{2a} + \frac{yy}{2b}$ hincque $p = \frac{x}{a}$ et $q = \frac{y}{b}$, quo pacto utique conditioni praescriptae satisfit.

Ceterum haec ratio quantitates logarithmicas elidendi maxime est notatu digna et in aliis casibus usum amplissimum habere potest.

EXEMPLUM 3

170. *Si posito $dz = pdx + qdy$ debeat esse $x^m y^n = Ap^\mu q^\nu$, indolem functionis z investigare.*

Statuatur ergo

$$\frac{x^m}{p^\mu} = \frac{Aq^\nu}{y^n} = \upsilon^{\mu\nu}$$

et hinc deducitur

$$p = \frac{x^{\frac{m}{\mu}}}{\upsilon^\nu} \quad \text{et} \quad q = \frac{1}{a} y^{\frac{n}{\nu}} \upsilon^\mu$$

posito $A = a^\nu$. Unde habebimus

$$\int pdx = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)\upsilon^\nu} + \frac{\mu\nu}{(m+\mu)} \int \frac{x^{\frac{m+\mu}{\mu}} d\upsilon}{\upsilon^{\nu+1}}$$

et

$$\int qdy = \frac{vy^{\frac{n+v}{v}}}{(n+v)a}\upsilon^{\mu} - \frac{\mu v}{(n+v)a}\int y^{\frac{n+v}{v}}\upsilon^{v-1}d\upsilon .$$

Quocirca erit

$$z = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)\upsilon^{v}} + \frac{vy^{\frac{n+v}{v}}\upsilon^{\mu}}{(n+v)a} + \frac{\mu v}{(m+\mu)(n+v)a}\int d\upsilon\left(\frac{(n+v)ax^{\frac{m+\mu}{\mu}}}{\upsilon^{v+1}} - (m+\mu)y^{\frac{n+v}{v}}\upsilon^{v-1}\right),$$

ita ut, si statuamus

$$\frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)\upsilon^{v+1}} - \frac{y^{\frac{n+v}{v}}\upsilon^{\mu-1}}{(n+v)a} = f':\upsilon,$$

futurum sit

$$z = \frac{\mu x^{\frac{m+\mu}{\mu}}}{(m+\mu)\upsilon^{v}} + \frac{vy^{\frac{n+v}{v}}\upsilon^{\mu}}{(n+v)a} + \mu v f:\upsilon.$$

Pro casu simplicissimo ponamus $f':\upsilon = 0$ et $f:\upsilon = 0$ eritque

$$y^{\frac{n+v}{v}}\upsilon^{\mu} = \frac{(n+v)a}{m+\mu}x^{\frac{m+\mu}{\mu}} \text{ et } \upsilon = \left(\frac{(n+v)ax^{\frac{m+\mu}{\mu}}}{(m+\mu)y^{\frac{n+v}{v}}}\right)^{\frac{1}{\mu+v}},$$

tum vero

$$z = \frac{1}{\upsilon^{v}}\left(\frac{\mu}{(m+\mu)}x^{\frac{m+\mu}{\mu}} + \frac{v}{(n+v)a}y^{\frac{n+v}{v}}\upsilon^{\mu+v}\right)$$

seu

$$z = \frac{\mu+v}{(m+\mu)\upsilon^{v}}x^{\frac{m+\mu}{\mu}} = (\mu+v)\left(\frac{x^{\frac{m+\mu}{\mu}}y^{\frac{n+v}{v}}}{(m+\mu)^{\mu}(n+v)^{v}A}\right)^{\frac{1}{\mu+v}}.$$

PROBLEMA 28

171. *Si posito* $dz = pdx + qdy$ *inter p, q et x, y eiusmodi detur relatio, ut p et q aequentur functionibus quibusdam ipsarum x, y et novae variabilis v, explorare casus, quibus indolem functionis z investigare licet.*

SOLUTIO

Cum sit p functio ipsarum x, y et v, spectatis y et v ut constantibus quaeratur integrale $\int pdx = P$ sitque sumtis omnibus variabilibus

$$dP = pdx + Rdy + Mdv,$$

unde, si pro pdx valor substituatur, erit

$$dz = dP + (q-R)dy - Mdv.$$

Quodsi iam eveniat, ut $q-R$ sit tantum functio ipsarum *y* et *v* exclusa *x*, sumta *v* constante quaeratur $\int(q-R)dy=T$ sitque deinceps

$$dT=(q-R)dy+Vdv.$$

Hinc valor ipsius $(q-R)dy$ ibi substitutus dabit

$$dz=dP+dT-(M+V)dv;$$

quae forma quia integrabilis esse debet, statuatur $M+V=f':v$ eritque $z=P+T-f':v$.
Ex operationibus autem susceptis dantur *P, R, M* per *x, y* et *v*, at *T* et *V* per *y* et *v* tantum; ac resolutio succedit, si modo in forma non amplius *x* continetur.

Pari ratione solutio succedet, si *M* tantum per *y* et *v* detur; tum enim ex *y* constante quaeratur $\int Mdv=L$ sitque $dL=Mdv+Ndy$; erit

$$dz=dP+(q-R+N)dy-dL$$

ponique conveniet $q-R+N=f':y$, ut fiat

$$z=P-L+f:y.$$

Simili modo ab altera parte $\int qdy$ calculum incipere et prosequi licet.

Introducendo autem functionem ipsarum *x, y* et *v* indefinitam *K* negotium generalius confici poterit. Sit enim

$$dK=Fdx+Gdy+Hdv$$

ac consideretur haec forma

$$dz+dK=(p+F)dx+(q+G)dy+Hdv.$$

Nunc sumtis *y* et *v* constantibus quaeratur

$$\int(p+F)dx=P$$

sitque

$$dP=(p+F)dx+Rdy+Mdv,$$

unde habetur

$$dz+dK=dP+(q+G-R)dy+(H-M)dv.$$

Quodsi iam eveniat, ut vel $q+G-R$ vel $H-M$ tantum binas variabiles *y* et *v* exclusa *x* contineat, resolutio, ut ante est ostensum, absolvi poterit.

PROBLEMA 29

172. *Si posito $dz = pdx + qdy$ relatio detur inter binas formulas differentiales p, q et binas variabiles x et z vel y et z, solutionem problematis, quatenus fieri potest, perficere.*

SOLUTIO

Ponamus relationem dari inter *p, q* et *x, z* atque hunc casum facile ad praecedentem revocare licet. Consideretur enim haec formula

$$dy = \frac{dz - pdx}{q}$$

ex principali derivata voceturque

$$\frac{1}{q} = m \text{ et } \frac{p}{q} = -n,$$

ut habeatur

$$dy = mdz + ndx,$$

et ob $\frac{1}{q} = m$ et $p = -\frac{n}{m}$ relatio proposita versabitur inter quaternas quantitates *m, n, z* et *x* ideoque quaestio omnino similis est earum, quas antea tractavimus, hoc tantum discrimine, quod hic quantitas *y* definiatur, cum ante esset *z* investigata. Quoniam autem ista determinatio per aequationes absolvitur, perinde est, utrum tandem inde *z* an *y* elicere velimus. Quodsi ergo hac reductione facta quaestio in casus ante pertractatos incidat, methodis quoque expositis resolvi poterit.

EXEMPLUM

173. *Si posito $dz = pdx + qdy$ debeat esse $qxz = aap$, indolem functionis z investigare.*

Consideretur formula $dy = \frac{dz}{q} - \frac{pdz}{q}$. Iam quia $\frac{p}{q} = \frac{xz}{aa}$, erit

$$dy = \frac{dz}{q} - \frac{xzdx}{aa} \text{ et } y = \int\left(\frac{dz}{q} - \frac{xzdx}{aa}\right),$$

at est

$$\int \frac{xzdx}{aa} = \frac{xxz}{2aa} - \int \frac{xxdz}{2aa},$$

ergo

$$y = \int dz\left(\frac{1}{q} + \frac{xx}{2aa}\right) - \frac{xxz}{2aa}.$$

Ponatur ergo

$$\frac{1}{q} + \frac{xx}{2aa} = f':z;$$

erit

$$y = \frac{-xxz}{2aa} + f:z,$$

ex qua aequatione utique z per x et y definitur.

Si pro casu simpliciori sumamus $f:z = b + \alpha z$, erit

$$y - b = \left(\alpha - \frac{xx}{2aa}\right)z \quad \text{et} \quad z = \frac{2aa(y-b)}{2\alpha aa - xx}.$$

et sumtis $\alpha = 0$ et $b = 0$ pro casu simplicissimo erit $z = \frac{-2aa}{xx}$. Hinc autem fit

$$p = \frac{+4aay}{x^3} \quad \text{et} \quad q = \frac{-2aa}{xx},$$

ergo

$$\frac{p}{q} = -\frac{2y}{x} \quad \text{et} \quad \frac{xz}{aa} = \frac{-2y}{x}.$$