CHAPTER IV

CONCERNING THE RESOLUTION OF EQUATIONS IN WHICH A RELATION IS PROPOSED BETWEEN THE TWO DIFFERENTIAL FORMULAS AND A SINGLE THIRD VARIABLE QUANTITY

PROBLEM 15

97. *If z should be a function of this kind of the two variables x and y, so that on putting* $dz = pdx + qdy$ *there shall be* $q = \frac{px}{a}$, *to investigate the nature of this function in general.*

SOLUTION

Since there shall be

$$dz = pdx + \frac{pxdy}{a} = px\left(\frac{dx}{x} + \frac{dy}{a}\right)$$

and this formula must become integrable, it is necessary that *px* and hence also *z* shall be a function of the quantity $lx + \frac{y}{a}$. Whereby the solution to our problem thus shall be had in general, so that there shall be

$$z = f:\left(lx + \frac{y}{a}\right) \text{ and } px = f':\left(lx + \frac{y}{a}\right)$$

clearly always on taking $d.f:u = duf':u$. And moreover there will be thus $q = \frac{px}{a}$, entirely as required.

COROLLARY 1

98. Since there shall be

$$z = px - \int xdp + \int \frac{pxdy}{a} = px + \int px\left(\frac{dy}{a} - \frac{dp}{p}\right),$$

hence a different solution can be deduced. For if we put

$$\int px\left(\frac{dy}{a} - \frac{dp}{p}\right) = f:\left(\frac{y}{a} - lp\right),$$

there will be $px = f':\left(\frac{y}{a} - lp\right)$, and hence

$$z = f':\left(\tfrac{y}{a} - lp\right) + f:\left(\tfrac{y}{a} - lp\right).$$

COROLLARIUM 2

99. Therefore from this new solution the variable p is introduced, from which together with y there is defined initially

$$x = \tfrac{1}{p} f':\left(\tfrac{y}{a} - lp\right),$$

then truly the function itself is sought

$$z = px + f:\left(\tfrac{y}{a} - lp\right).$$

But without doubt the preceding solution surpasses this one, since that can express the quantity z immediately by x and y.

SCHOLIUM

100. So that we are able to compare these two solutions between themselves, since an arbitrary function occurs naturally in each separate solution, we may use each with a different character. Therefore since the first solution gives

$$z = f:\left(\tfrac{y}{a} + lx\right) \text{ and } px = f':\left(\tfrac{y}{a} + lx\right),$$

while the other gives

$$z = F:\left(\tfrac{y}{a} - lp\right) + F':\left(\tfrac{y}{a} - lp\right) \text{ and } px = F':\left(\tfrac{y}{a} - lp\right),$$

it is evident that

$$f':\left(\tfrac{y}{a} + lx\right) = F':\left(\tfrac{y}{a} - lp\right) \text{ and } f:\left(\tfrac{y}{a} + lx\right) = F:\left(\tfrac{y}{a} - lp\right) + F':\left(\tfrac{y}{a} - lp\right),$$

from which not only the relation between the nature of each of the functions f and F is defined , but also thence it must follow that

$$px = f':\left(\tfrac{y}{a} + lx\right),$$

which is seen perhaps to be a little obscure. Truly on this account that problem itself is therefore seen to be more noteworthy, because the other solution, in which the new variable p is introduced, agrees with the first, where z is defined at once by x and y, and nor yet is it possible to show clearly the agreement of these solutions. On this account when we come to solutions of this kind, as it usually came about in the latter problems in the preceding chapters, in which a new variable was introduced, that we must not immediately abandon all hope of the elimination of this variable, as with the latter case the solution clearly will be reducible to the former, even if the method of reduction should not be clear at once, as we will show yet below in § 119.

PROBLEM 16

101. *If z should be a function of the two variables x and y, so that on putting $dz = pdx + qdy$ there shall be $q = pX + T$ with some functions X and T of x present, to investigate the nature of this function z in general.*

SOLUTION

Therefore since there shall be $dz = pdx + pXdy + Tdy$, there is put in place $p = r - \frac{T}{X}$ so that there is produced

$$dz = rdx - \frac{Tdx}{X} + rXdy = \frac{-Tdx}{X} + rX\left(\frac{dx}{X} + dy\right),$$

with which reduction made it is evident that rX as well as $\int rX\left(\frac{dx}{X} + dy\right)$ shall be a function of $y + \int \frac{dx}{X}$. Whereby if we put

$$\int rX\left(\frac{dx}{X} + dy\right) = f:\left(y + \int \frac{dx}{X}\right),$$

there will be

$$rX = f':\left(y + \int \frac{dx}{X}\right)$$

and then the function sought will be

$$z = -\int \frac{Tdx}{X} + f:\left(y + \int \frac{dx}{X}\right),$$

which on account of the indefinite function $f:$ is a complete integral. Then indeed there shall be [on eliminating r]

$$p = \frac{-T}{X} + \frac{1}{X} f':\left(y + \int \frac{dx}{X}\right) \text{ and } q = [rx] = f':\left(y + \int \frac{dx}{X}\right),$$

from which it is apparent that everywhere $q = pX + T$. Now since X and T are given functions of x, the integral formulas $\int \frac{dx}{X}$ and $\int \frac{Tdx}{X}$ do not upset the solution.

COROLLARY 1

102. The solution is returned somewhat easier by assuming the prescribed condition $p=\frac{q}{X}-\frac{T}{X}$, from which there becomes

$$dz=-T\frac{dX}{X}+\frac{qdx}{X}+qdy \text{ and } z=-\int\frac{TdX}{X}+\int q\left(dy+\frac{dx}{X}\right)$$

Now evidently there shall be

$$\int q\left(dy+\frac{dx}{X}\right)=f:\left(y+\int\frac{dx}{X}\right)$$

and thus the preceding solution itself results.

COROLLARY 2

103. In the same manner the problem is resolved, if there is put in place the condition $q=pY+V$ for given functions Y and V of y arising. Then indeed there shall be

$$dz=pdx+pYdy+Vdy \text{ and } z=\int Vdy+\int p\left(dx+Ydy\right).$$

Therefore here there shall be

$$\int p\left(dx+Ydy\right)=f:\left(x+\int Ydy\right)$$

and the solution will be

$$z=\int Vdy+f:\left(x+\int Ydy\right),$$

from which there becomes

$$p=f':\left(x+\int Ydy\right) \text{ and } q=V+Yf':\left(x+\int Ydy\right).$$

SCHOLIUM

104. We are able to discern from the form of the solution found here, in what manner the problem must be prepared, so that the solution of this can be completed by this method, and a function z can be shown by the two variables x and y. For let there be some functions K and V of x and y, and thence on differentiation,

$$dK=Ldx+Mdy \text{ and } dV=Pdx+Qdy;$$

now we may start from the solution, and put

$$z=K+f:V$$

and there will be on different ion

$$dz = Ldx + Mdy + (Pdx + Qdy)\, f':V .$$

Now when this form

$$dz = pdx + qdy$$

shall be compared with the assumed

$$p = L + Pf':V \text{ and } q = M + Qf':V ,$$

and there will be

$$Qp - Pq = LQ - MP .$$

Whereby if this problem is proposed, so that on putting $dz = pdx + qdy$ there should become

$$q = \tfrac{Q}{P}p + M - \tfrac{LQ}{P} ,$$

the solution will be $z = K + f:V$, as long as M and L and likewise P and Q shall be prepared thus, so that there will be

$$Ldx + Mdy = dK \text{ and } Pdx + Qdy = dV ;$$

indeed these cases are to be referred to the following chapter [espec. §146].

PROBLEM 17

105. *If z should be a function of the two variables x and y of this kind, so that on putting $dz = pdx + qdy$ there shall be $q = Px + \Pi$ with the given functions P and Π of p, to investigate the general nature of this function z.*

SOLUTION

Therefore since there shall be $dz = pdx + Pxdy + \Pi dy$, there will be

$$z = px + \int (Pxdy + \Pi dy - xdp).$$

There is put in place $Px + \Pi = v$, so that there shall be $x = \tfrac{v-\Pi}{P}$, and there becomes

$$z = px + \int \left(vdy - \tfrac{vdp}{P} + \tfrac{\Pi dp}{P} \right).$$

Whereby since P and Π shall be functions of p and likewise with the formula $\int \tfrac{\Pi dp}{P}$ given, there will be found

$$z = px + \int \frac{\Pi dp}{P} + \int v\left(dy - \frac{dp}{P}\right),$$

from which it is apparent that v as well as $\int v\left(dy - \frac{dp}{P}\right)$ must be a function of the formula $y - \int \frac{dp}{P}$. Therefore we may put

$$\int v\left(dy - \frac{dp}{P}\right) = f:\left(y - \int \frac{dp}{P}\right)$$

and there will be

$$v = Px + \Pi = f':\left(y - \int \frac{dp}{P}\right)$$

and hence

$$x = \frac{-\Pi}{P} + \frac{1}{P} f':\left(y - \int \frac{dp}{P}\right),$$

then truly

$$z = \int \frac{\Pi dp}{P} - \frac{\Pi p}{P} + \frac{p}{P} f':\left(y - \int \frac{dp}{P}\right) + f:\left(y - \int \frac{dp}{P}\right).$$

COROLLARY 1

106. In the solution of this problem again a new variable p is introduced, from which in the first place jointly the variable x, as well as truly the function sought z may be determined in terms of y.

COROLLARY 2

107. Nor indeed is it possible to remove this new variable p from the calculation, as usually came about before, as because here P and Π denote functions of p, the nature of which is present now in the problem itself.

COROLLARY 3

108. In a similar manner the problem may be resolved , if with x and y interchanged the quantity p thus may be given by y and q, so that there shall be $p = Qy + \Xi$ with Q and Ξ denoting given functions of q.

SCHOLIUM

109. In this chapter we have established problems of this kind to be treated, in which a condition is expressed by some equation between the two differential formulas $\left(\frac{dz}{dx}\right) = p, \left(\frac{dz}{dy}\right) = q$ and one of the three variables x, y and z. But the two problems set out from this general case are complementary, the solution of which can be set out by a particular method and likewise leading to simpler formulas. Finally indeed thus we have assumed a certain relation between p, q and x, so that there shall be $q = Px + \Pi$ or so that in the value of q expressed by p and x the quantity x does not exceed a single dimension; now previously thus, as there may arise $q = pX + T$ or, so that in the value of q expressed by p and x, the quantity p prevails with a single dimension.

Moreover in general it is helpful to note that both the quantities p and x as well as q and y can be permuted between each other. For since there shall be $\int pdx = px - \int xdp$, in place of

$$z = \int (pdx + qdy)$$

there will be

$$z = px + \int (qdy - xdp).$$

In a similar manner there is

$$z = qy + \int (pdx - ydq),$$

then indeed also

$$z = px + qy - \int (xdp + ydq).$$

In which cases therefore with a single one of these [differential equations] returned integrable, from the same the three remaining can be integrated. Therefore since in the above chapter we have resolved the first formula, if p or q may be given in some way by x and y, thus in the same way the second formula may be resolved, if q may be given by p and y, moreover in the third case, if p may be given by x and q, and moreover in the fourth case, if either x may be given in some manner by p and q or y by p and q ; which questions since they may be set out generally, we will pursue in the following problem.

PROBLEM 18

110. *On putting $dz = pdx + qdy$, if a relation between p, q and x may be defined by some equation, to investigate in general the nature of the function z, in whatever way it may be determined from the two variables x and y.*

SOLUTION

From the proposed equation between p, q and x the value is sought of x, which will be equal to a certain function of p and q. Since now there shall be

$$z = px + qy - \int (xdp + ydq),$$

because x is a given function of p and q, the formula xdp may be integrated on taking the quantity q constant and there shall be

$$\int xdp = V + f:q$$

and V shall be a known function of p and q, from which differentiated there may emerge

$$dV = xdp + Sdq,$$

where S also will be a given function of p and q. Because now the $\int(xdp+ydq)$ must be allowed to be integrated, it will be equal to the form $V+f:q$, from which on differentiation there is concluded

$$xdp+ydq=xdp+Sdq+dq\,f':q$$

and thus

$$y=S+f':q \quad \text{and} \quad z=px+qy-V-f:q$$

or

$$z=px+Sq+qf':q-f:q-V.$$

Therefore the solution itself thus may be found:

In the first place from the prescribed condition x is given in terms of p and q; then on taking q constant there shall be $V=\int xdp$ and in turn $dV=xdp+Sdq$; but with V and S found in terms of p and q the remaining quantities y and z are expressed in the same manner, so that there shall be

$$y=S+f':q \quad \text{and} \quad z=px+Sq+q\,f':q-f:q-V$$

which solution, because denotes some function $f:q$ of q either continuous or discontinuous, is to be considered extended the widest for the complete solution.

OTHERWISE

111. Or from the given equation between p, q and x the value of p may be sought expressed by x and q, thus so that p is equal to a function of the two given variables x and q, by which also we may attempt to define the remaining quantities y and z. Accordingly we use the formula

$$z=qy+\int(pdx-ydq),$$

and because p is a function of x and q, a function V of these of the same kind will be given, so that there shall be

$$dV=pdx+Rdq.$$

Therefore there is taken

$$\int(pdx-ydq)=V+f:q$$

and there shall be

$$y=-R-f':q \quad \text{and} \quad z=qy+V+f:q.$$

COROLLARY 1

112. Each solution can be used equally conveniently, if from the proposed relation between p, q and x both the quantity x as well as p are allowed to be defined with equal convenience. But if either of these can be defined more easily, that with the solution which is more suitable for the case will be used.

COROLLARY 2

113. But if neither p nor x is able to be elicited conveniently, then here the resolution of this equation may be taken by conceding nothing less than transcending quantities of each order.
[One presumes Euler means that the integral itself defines a function not in the known repertoire of functions available at the time, which consisted only of the algebraic, trig. and log. functions and their inverses, and combinations of these.]
Likewise even if q may be defined easily by p and x, hence the calculation is helped to no purpose.

COROLLARY 3

114. From this problem extended as widely as possible also the two preceding problems can be resolved ; but the solution hence found from the preceding will be in disagreement, since that shall be deduced by a particular method. But it will be worth the effort to compare these two solutions with each other.

EXAMPLE 1

115. *If there should be* $q = pX + T$ *, with X and T being functions of x, to investigate the nature of the function z.*

Here with the solution being used from the former [Problem 10], for which there is $p = \frac{q-T}{X}$; now on putting q constant there emerges

$$V = \int pdx = q\int \frac{dx}{X} - \int \frac{Tdx}{X}$$

and hence

$$R = \left(\frac{dV}{dq}\right) = \int \frac{dx}{X},$$

from which the solution may be present in these formulas

$$q = pX + T,\quad y = -\int \frac{dx}{X} - f':q,\quad z = -\int \frac{Tdx}{X} - q\, f':q + f:q\,;$$

but the above solution thus will be found itself [§ 101]

$$q = pX + T,\ \ q = f':\left(y + \int \frac{dx}{X}\right) \quad \text{and} \quad z = -\int \frac{Tdx}{X} + f:\left(y + \int \frac{dx}{X}\right).$$

SCHOLIUM

116. The agreement of these two solutions thus must be shown, so that from the one we have found here, the previous may be formed from legitimate logical consequences.

Since indeed there shall be

$$f':q = -y - \int \frac{dx}{X},$$

for the sake of brevity there is put in place $y + \int \frac{dx}{X} = v$, so that there shall be $f':q = -v$; therefore q in turn will be equal to a certain function of v, which is put as $q = F':v$, from which there becomes $dq = dvF'':v$, hence

$$dq\ f':q = -vdvF'':v = -vd.F':v,$$

therefore on integration

$$f:q = -\int vd.F':v = -vF':v + \int dvF':v = -vF':v + F:v.$$

Whereby since there shall be

$$z = -\int \frac{Tdx}{X} - qf':q + f:q,$$

there will be

$$z = -\int \frac{Tdx}{X} + vF':v - vF':v + F:v \quad \text{or} \quad z = -\int \frac{Tdx}{X} + F:\left(y + \int \frac{dx}{X}\right),$$

which is the preceding solution itself.

EXAMPLE 2

117. *If there should be* $q = Px + \Pi$ *with the given functions of p present P and* Π *, to investigate the nature of the function z, so that there shall be* $dz = pdx + qdy$ *.*

Here with the earlier solution being used [Prob. 18], as there shall be $x = \frac{q-\Pi}{p}$. Therefore on taking q constant there is sought

$$V = \int xdp = q\int \frac{dp}{P} - \int \frac{\Pi dp}{P},$$

from which there shall be

$$S = \left(\frac{dV}{dq}\right) = \int \frac{dp}{P}.$$

Therefore the solution may be presented [from $y = S + f':q$ and $z = px + qy - V - f:q$]

$$y = \int \frac{dp}{P} + f':q$$

et

$$z = \frac{pq}{P} - \frac{p\Pi}{P} + q\int \frac{dp}{P} + qf':q - f:q - q\int \frac{dp}{P} + \int \frac{\Pi dp}{P}$$

or

$$z = \frac{p(q-\Pi)}{P} + \int \frac{\Pi dp}{P} + q\, f':q - f:q\,.$$

But the solution of the same case found above (§ 105) was

$$x = -\frac{\Pi}{P} + \frac{1}{P} f': \left(y - \int \frac{dp}{P}\right) \text{ and } q = Px + \Pi$$

and

$$z = \frac{-p\Pi}{P} + \int \frac{\Pi dp}{P} + \frac{p}{P} f':\left(y - \int \frac{dp}{P}\right) + f:\left(y - \int \frac{dp}{P}\right).$$

SCHOLIUM 1

118. We may consider, how the solution found here can be reduced to the above.
Since there we came upon

$$y - \int \frac{dp}{P} = f':q\,,$$

q in turn will be equal to a function of the quantity $y - \int \frac{dp}{P}$*;* therefore there is put

$$q = F':\left(y - \int \frac{dp}{P}\right)$$

and there will be at once

$$x = \left[\frac{q-\Pi}{p}\right] = \frac{-\Pi}{p} + \frac{1}{P} F':\left(y - \int \frac{dp}{P}\right).$$

For the sake of brevity let $y - \int \frac{dp}{P} = v$, so that there becomes $q = F': v$ and $v = f': q$; there will be

$$F:v = \int q dv = qv - \int v dq = qv - \int dq\, f':q\,.$$

Therefore $F:v = qv - f:q$, thus so that there shall be

$$f:q = q\left(y - \int \frac{dp}{P}\right) - F:\left(y - \int \frac{dp}{P}\right)$$

or

$$f:q=\left(y-\int\tfrac{dp}{P}\right)F':\left(y-\int\tfrac{dp}{P}\right)-F:\left(y-\int\tfrac{dp}{P}\right)$$

With which values substituted we will have

$$x=\tfrac{-\Pi}{p}+\tfrac{1}{P}F':\left(y-\int\tfrac{dp}{P}\right)$$

and

$$z=\tfrac{-p\Pi}{P}+\tfrac{p}{P}F':\left(y-\int\tfrac{dp}{P}\right)+\int\tfrac{\Pi dp}{P}+\left(y-\int\tfrac{dp}{P}\right)F':\left(y-\int\tfrac{dp}{P}\right)$$
$$-\left(y-\int\tfrac{dp}{P}\right)F':\left(y-\int\tfrac{dp}{P}\right)+F:\left(y-\int\tfrac{dp}{P}\right)$$

or

$$z=\tfrac{-p\Pi}{P}+\tfrac{p}{P}F':\left(y-\int\tfrac{dp}{P}\right)+\int\tfrac{\Pi dp}{P}+F:\left(y-\int\tfrac{dp}{P}\right)$$

which is that solution found before.

SCHOLIUM 2

119. With this agreement shown we are able also to demonstrate agreement on the above observation (§ 100), which we may see to be much more obscure.

But the first solution found there was

$$px=F':\left(\tfrac{y}{a}-lp\right)\quad\text{and}\quad z=px+F:\left(\tfrac{y}{a}-lp\right),$$

from the first formula of which in turn it is apparent that $\frac{y}{a}-lp$ is a function of px. Hence also $\frac{y}{a}-lp+lpx$ or $\frac{y}{a}+lx$ will be equal to a function of px; therefore again in turn px will be equal to a certain function of $\frac{y}{a}+lx$. Therefore there is put $px=f':\left(\frac{y}{a}+lx\right)$, and since there shall be

$$d.F:\left(\tfrac{y}{a}-lp\right)=\left(\tfrac{dy}{a}-\tfrac{dp}{p}\right)F':\left(\tfrac{y}{a}-lp\right),$$

then there will be

$$F:\left(\tfrac{y}{a}-lp\right)=\int px\left(\tfrac{dy}{a}-\tfrac{dp}{p}\right)=\int px\left(\tfrac{dy}{a}+\tfrac{dx}{x}\right)-\int px\left(\tfrac{dx}{x}+\tfrac{dp}{p}\right)=\int px\left(\tfrac{dy}{a}+\tfrac{dx}{x}\right)-px.$$

Now with the value $f':\left(\frac{y}{a}+lx\right)$ substituted for px there will be found

$$F:\left(\tfrac{y}{a}-lp\right)=-px+\int\left(\tfrac{dy}{a}+\tfrac{dx}{x}\right)f':\left(\tfrac{y}{a}+lx\right)=-px+f:\left(\tfrac{y}{a}+lx\right),$$

thus so that there hence becomes $z = f:\left(\frac{y}{a} + lx\right)$, which is itself the other solution.

Therefore by this reduction some light is shone on the other mysteries of this kind to be investigated. But the sum of this reasoning recounted here shall be that : if there should be $r = f':s$ also there shall be $r = F':(s+R)$ with R denoting a function of r, which indeed is evident by itself, because each r is determined by s. Therefore since there shall be

$$f':s = r = F':(s+R),$$

there will be

$$f:s = \int dsf':s = \int rds = \int r\left(ds + dR - dR\right) = \int \left(ds + dR\right)F':(s+R) - \int rdR$$

and therefore

$$f:s = F:(s+R) - \int rdR,$$

from which, in place of the quantity s, functions of the quantity $s + R$ can be introduced. Evidently if there shall be $r = f':s$, it is possible to take $r = F':(s+R)$ with some function R of r, then truly there will be $f:s = F:(s+R) - \int rdR$.

EXAMPLE 3

120. *On putting $dz = pdx + qdy$ if x is equal to a homogeneous function of n dimensions of p and q, to investigate the nature of the function z.*

Since x may be given by p and q, there will be using the first solution, and on account of x being equal to a homogeneous function of n dimensions of p and q, there is put $p = qr$, and there becomes $x = q^n R$, with a function R of r arising. Now q is assumed constant and there is sought:

$$V = \int xdp = \int q^{n+1}Rdr,$$

on account of $dp = qdr$, and there will be

$$V = q^{n+1}\int Rdr$$

which integral is given. Hence on differentiation there will be

$$dV = q^{n+1}Rdr + (n+1)q^n dq\int Rdr;$$

which as it may be compared with

$$dV = xdp + Sdq = q^n Rdp + Sdq,$$

because on account of $dp = qdr + rdq$ there shall be

$$dV = q^{n+1}Rdr + q^n Rrdq + Sdq,$$

there will be

$$S = -q^n Rr + (n+1)q^n \int Rdr,$$

from which there becomes

$$y = -q^n Rr + (n+1)q^n \int Rdr + f':q \quad \text{and} \quad x = q^n R$$

and

$$z = nq^{n+1}\int Rdr + q\, f':q - f:q$$

with $p = qr$ arising.

COROLLARY 1

121. Let there be $x = \frac{p^m}{q^m}$ and on putting $p = qr$ there will be $x = r^m$ and thus $n = 0$ and $R = r^m$ from which there becomes

$$y = -r^{m+1} + \tfrac{r^{m+1}}{m+1} + f':q = \tfrac{-m}{m+1} r^{m+1} + f':q \quad \text{and} \quad z = q\, f':q - f:q.$$

Whereby on account of $r = x^{\frac{1}{m}}$ there will be

$$y = \tfrac{-m}{m+1} x^{\frac{m+1}{m}} + f':q.$$

COROLLARY 2

122. Therefore in the same case, in which $x = \frac{p^m}{q^m}$, q will be equal to a function of the quantity $y + \frac{m}{m+1} x^{\frac{m+1}{m}}$; which quantity, if it is put $= v$ and $q = F':v$, so that there shall be $v = f:q$, there will be

$$f:q = \int dq\, f':q = \int vdvF'':v$$

on account of $dq = dvF'':v$, from which it is concluded that

$$f:q = vF':v - F:v \quad \text{and} \quad z = F:v = F:\left(y + \tfrac{m}{m+1} x^{\frac{m+1}{m}}\right).$$

EXAMPLE 4

123. *To investigate the function z of the two variables x and y of this kind, so that on putting* $dz = pdx + qdy$ *there becomes* $p^3 + x^3 = 3pqx$.

The form may be considered

$$z = qy + \int (pdx - ydq),$$

where now the integral formula is required to be returned $dz = pdx + qdy$. There is put $p = ux$ and the prescribed condition gives $x(1+u^3) = 3qu$; from which there becomes

$$x = \frac{3qu}{1+u^3} \quad \text{and} \quad p = \frac{3quu}{1+u^3},$$

then truly

$$dx = \frac{3qdu(1-2u^3)}{(1+u^3)^2} + \frac{3udq}{1+u^3}$$

and thus there will be had

$$z = qy + \int \left(\frac{9qquudu(1-2u^3)}{(1+u^3)^3} + \frac{9qu^3dq}{(1+u^3)^2} - ydq \right),$$

but

$$\int \frac{9qquudu(1-2u^3)}{(1+u^3)^3} = \frac{3qq(1+4u^3)dq}{2(1+u^3)^2} - \int \frac{3q(1+4u^3)dq}{(1+u^3)^2}$$

therefore

$$z = qy + \frac{3qq(1+4u^3)}{2(1+u^3)^2} - \int dq\left(y + \frac{3q}{1+u^3}\right).$$

Whereby it is necessary that $y + \frac{3q}{1+u^3}$ is a function of q only, which shall be $-f':q$, from which there becomes

$$y = -\frac{3q}{1+u^3} - f':q \quad \text{and} \quad z = qy + \frac{3qq(1+4u^3)}{2(1+u^3)^2} + f:q$$

or

$$z = \frac{3qq(2u^3-1)}{2(1+u^3)^2} - q\, f':q + f:q$$

with $x = \frac{3qu}{1+u^3}$ arising. From which three equations, if the two quantities q and u are eliminated, there arises the equation which is sought between z and x, y.

COROLLARIUM 1

124. From the equation found for y there is deduced

$$\frac{3}{1+u^3} = \frac{-y - f':q}{q}$$

but the equation found for z will change into this

$$z = \frac{3qq}{1+u^3} - \frac{9qq}{2\left(1+u^3\right)^2} - q\,f':q + f:q,$$

which with u removed is changed into this

$$z = -qy - 2qf':q - \tfrac{1}{2}\left(y + f':q\right)^2 + f:q\ ;$$

then indeed there shall be

$$x = -u\left(y + f':q\right),$$

from which there is found $u = \frac{-x}{y+f':q}$ and hence

$$x^3 = 3q\left(y + f':q\right)^2 + \left(y + f':q\right)^3.$$

COROLLARY 2

125. If we assume that $f':q = a$, then there will be $f:q = aq + b$ and the latter equation gives $q = \frac{x^3 - (y+a)^3}{3(y+a)^2}$. Then for this case there becomes

$$z = -qy - aq - \tfrac{1}{2}\left(y + a\right)^2 + b,$$

and there emerges, on substituting in place of q the value found,

$$z = \frac{6b(y+a) - (y+a)^3 - 2x^3}{6(y+a)}.$$

COROLLARY 3

126. Since in general there shall be

$$x^3 = \left(y + f':q\right)^2 \left(y + 3q + f':q\right),$$

we may put $f':q = a - 3q$ and thus $f:q = b + aq - \frac{3}{2}:qq$, so that there becomes

$$\left(y + a - 3q\right)^2 = \frac{x^3}{y+a},$$

and thus

$$y + a - 3q = \frac{x\sqrt{x}}{\sqrt{(y+a)}} \quad \text{and} \quad q = \frac{1}{3}y\left(y + a\right) - \frac{x\sqrt{x}}{3\sqrt{(y+a)}}.$$

Hence there appears

$$f':q = \frac{x\sqrt{x}}{\sqrt{(y+a)}} - y$$

and

$$f:q = b + \frac{a(y+a)}{3} - \frac{ax\sqrt{x}}{3\sqrt{(y+a)}} - \frac{1}{6}\left(y + a\right)^2 + \frac{1}{3}x\sqrt{x\left(y + a\right)} - \frac{x^3}{6(y+a)}$$
$$= b + \frac{aa-yy}{6} + \frac{xy\sqrt{x}}{3\sqrt{(y+a)}} - \frac{x^3}{6(y+a)}$$

and

$$z = -\frac{1}{3}y\left(y + a\right) + \frac{yx\sqrt{x}}{3\sqrt{(y+a)}} - 2aq + 6qq - \frac{x^3}{2(y+a)} + b + aq - \frac{3}{2}qq$$

or

$$z = b - \frac{1}{3}y\left(y + a\right) + \frac{yx\sqrt{x}}{3\sqrt{(y+a)}} - \frac{x^3}{2(y+a)} - aq + \frac{9}{2}qq$$

and with the reduction made,

$$z = b + \frac{1}{6}\left(y + a\right)^2 - \frac{2}{3}x\sqrt{x\left(y + a\right)}$$

COROLLARARY 4

127. But if here there is taken $a = 0$ and $b = 0$, it will be satisfied by the simple expression

$$z = \frac{1}{6}yy - \frac{2}{3}x\sqrt{xy},$$

which thus makes apparent how the prescribed condition is satisfied. By differentiation it is deduced, that

$$p = \left(\frac{dz}{dx}\right) = -\sqrt{xy} \quad \text{and} \quad q = \left(\frac{dz}{dy}\right) = \frac{1}{3}y - \frac{x\sqrt{x}}{3\sqrt{y}}$$

and hence

$$p^3+q^3=-xy\sqrt{xy}+x^3,$$

but $3pq=xx-y\sqrt{xy}$ and thus $3pqx=x^3-xy\sqrt{xy}$, hence

$$p^3+x^3=3pqx.$$

SCHOLIUM

128. Therefore the solution has followed, when some equation between *p, q* and *x* is proposed, even if in cases, from which thence neither *x* nor *p* can be elicited, a certain difficulty remains, but which chiefly affects the resolution of finite equations, as here deservedly we have asked to concede. Meanwhile from the final example it is observed, how the operation may be put in place, if with the aid of a suitable substitution the proposed equation is able to be adapted for resolution, but concerning which business I shall not tarry here longer. And nor also these cases, in which a certain relation is prescribed between *p, q* and *y*, will I establish separately here, since on account of the permutability of *x* and *y* themselves, as also *p* and *q* are permutated, these cases are recalled at once to the preceding ones. Therefore there remains the case, in which an equation is proposed between *p, q* and *z*, where indeed it is obvious at once in the equation $dz=pdx+qdy$ that the quantities *p* and *q* are not to be regarded as functions of *x* and *y* themselves, since they also depend on *z*, and nor therefore will it be possible thence to determine the nature of these, so that the integrable formula $pdx+qdy$ prevails. Truly without distinction that condition must be defined, in order that the differential equation $dz-pdx-qdy=0$ becomes possible ; according to which from the principles established above (in § 6), it is required that on putting

$$\left(\frac{dq}{dz}\right)=L,\quad -\left(\frac{dp}{dz}\right)=M\quad\text{and}\quad\left(\frac{dp}{dy}\right)-\left(\frac{dq}{dx}\right)=N$$

there shall be

$$Lp+Mq-N=0\quad\text{or}\quad p\left(\frac{dq}{dz}\right)-q\left(\frac{dp}{dz}\right)+\left(\frac{dq}{dx}\right)-\left(\frac{dp}{dy}\right)=0$$

Whereby whatever the proposed equation between *p, q* and *z* it is necessary in general to investigate these conditions, so that this requirement is satisfied.

PROBLEM 19

129. *If on putting* $dz = pdx + qdy$ *there must be* $p + q = \frac{z}{a}$, *to investigate the relation of the function z to the variables x et y in general.*

SOLUTION

Since there shall be $q = \frac{z}{a} - p$, our equation adopts this form

$$dz = pdx - pdy + \frac{zdy}{a}$$

or

$$p(dx - dy) = \frac{adz - zdy}{a} = z\left(\frac{dz}{z} - \frac{dy}{a}\right).$$

Therefore since both the formulas $dx - dy$ and $\frac{dz}{z} - \frac{dy}{a}$ by themselves are integrable, on account of

$$\frac{dz}{z} - \frac{dy}{a} = \frac{p}{z}(dx - dy)$$

it is necessary that $\frac{p}{z}$ shall be a function of the quantity $x - y$; therefore there is put $\frac{p}{z} = f':(x - y)$, so that there becomes

$$lz - \frac{y}{a} = f:(x - y).$$

Therefore it is able to define *z* by *x* and *y*, and since there shall be $e^{f:(x-y)}$ also a function of $x - y$, if that is put $= F:(x - y)$, then there will be

$$z = e^{\frac{y}{a}} F:(x - y),$$

from which there becomes

$$\left(\frac{dz}{dx}\right) = p = e^{\frac{y}{a}} F':(x - y) \quad \text{and} \quad \left(\frac{dz}{dy}\right) = q = -e^{\frac{y}{a}} F':(x - y) + \frac{1}{a} e^{\frac{y}{a}} F:(x - y)$$

and thus

$$p + q = \frac{1}{a} e^{\frac{y}{a}} F:(x - y) = \frac{z}{a},$$

as is required.

COROLLARY 1

130. From this example it is understood, how a certain function of p and q can be equal to z, even if p and q shall be functions of x and y themselves. Likewise clearly an account of the integral formulas $dz = pdx + qdy$ is introduced into the calculation.

COROLLARY 2

131. The form $e^{\frac{y}{a}}F:(x-y)$ found for the value of z can be multiplied by some function of $x-y$. Therefore if it may be multiplied by $e^{\frac{x-y}{a}}$, there becomes $z = e^{\frac{x}{a}}F:(x-y)$. But if it is multiplied by $e^{\frac{x-y}{2a}}$, there becomes $z = e^{\frac{x+y}{2a}}F:(x-y)$, which forms satisfy the problem equally well.

PROBLEM 20

132. *If on putting $dz = pdx + qdy$ the quantity z must become equal to a given function of p and q, to investigate in general the nature by which z is defined by x and y.*

SOLUTION

From the proposed formula we will have $dy = \frac{dz}{q} - \frac{pdx}{q}$; there is put $p = qr$, so that z shall be equal to a function of q and r, and from $dy = \frac{dz}{q} - rdx$ there is elicited

$$y = \frac{z}{q} - rx + \int\left(\frac{zdq}{qq} + xdr\right),$$

as it is required that an integrable formula be returned. Therefore since z shall be a given function of q and r themselves, on putting r constant there is sought the integral of the formula $\frac{zdq}{qq}$ and let there be

$$\int \frac{zdq}{qq} = V + f:r,$$

from which on differentiating there emerges

$$dV = \frac{zdq}{qq} + Rdr,$$

and now it is apparent that there must be $x = R + f':r$ and thence there is obtained

$$y = \frac{z}{q} - Rr - rf':r + V + f:r,$$

from which two equations the relation between the proposed quantities is determined.

Therefore initially on putting $p = qr$, z is given by q and r. Then on assuming r constant the formula $\frac{zdq}{qq}$ may be integrated and the resulting integral $V = \int \frac{zdq}{qq}$, which also is given by q and r, from which on assuming q constant $R = \left(\frac{dV}{dr}\right)$ is deduced. With which found there will be

$$x = R + f':r \quad \text{and} \quad y = \tfrac{z}{q} - rx + V + f:r$$

and thus all the quantities are determined by the two variables q and r.

COROLLARY 1

133. Because the letters p and q are permuted with x and y, in a similar manner we may be able to begin our investigation from the equation $dx = \frac{dz}{p} - \frac{qdy}{p}$ and a similar solution may be produced, which indeed with different forms, but to be in agreement with the problem.

COROLLARY 2

134. Now evidently on putting $q = ps$, so that there shall be $dx = \frac{dz}{p} - sdy$, there will be

$$x = \tfrac{z}{p} - sy + \int\left(\frac{zdp}{pp} + yds\right).$$

Now on assuming s constant there is put $\int \frac{zdp}{pp} = U$, which quantity is determined by p and s, from that indeed there arises $\left(\frac{dU}{ds}\right) = S$; there will be

$$y = S + f':s \quad \text{and} \quad x = \tfrac{z}{p} - sy + U + f:s.$$

EXEMPLUM 1

135. *If there should be $p + q = \frac{z}{a}$, to show the solution for this case.*

On putting $p = qr$ there will be $z = aq(1+r)$; now on assuming r constant there will be

$$V = \int \frac{zdq}{qq} = a(1+r)lq \quad \text{et} \quad R = \left(\frac{dV}{dr}\right) = alq.$$

Hence there is found

$$x = alq + f':r \quad \text{and} \quad y = \tfrac{z}{q} - arlq - rf':r + a(1+r)lq + f:r$$

or

$$y = a(1+r) + alq - rf':r + f:r.$$

If hence we wish to eliminate q, on account of $q = \frac{z}{a(1+r)}$ the solution is contained by these two equations

$$y = al\frac{z}{a(1+r)} f':r$$

and

$$y = al\frac{z}{a(1+r)} + a(1+r) - rf':r + f:r.$$

From which in the following manner the preceding solution can be elicited [§ 129, 131] . From the first form there is

$$\frac{x}{a} - l\frac{z}{a} = -l(1+r) + \frac{1}{a} f':r = \text{funct.}r,$$

and truly from both

$$y - x = a(1+r) - (1+r) f':r + f:r = \text{ funct.}r.$$

Therefore since both $\frac{x}{a} - l\frac{z}{a}$ or $ze^{-\frac{x}{a}}$ as well as $y-x$ shall be a function of *r*, with one form equal to a function of the other, from which there can be put in place

$$ze^{-\frac{x}{a}} = F:(y-x) \text{ or } z = e^{\frac{x}{a}}F:(y-x),$$

which is the solution found before.

EXEMPLE 2

136. *If on putting* $dz = pdx + qdy$ *there must be* $z = apq$, *to investigate the relation between x, y and z.*

On putting $p = qr$ there will be $z = aqqr$ and on assuming r constant there becomes $V = \int\frac{zdq}{qq} = aqr$ and hence $R = \left(\frac{dV}{dr}\right) = aq$. On account of which we will have

$$x = aq + f':r \quad \text{and} \quad y = aqr - rf':r + f:r$$

or on account of $r = \frac{z}{aqq}$ there will be

$$x = aq + f':\frac{z}{aqq} \quad \text{and} \quad y = \frac{z}{q} - \frac{z}{aqq} f':\frac{z}{aqq} + f:\frac{z}{aqq}$$

Here we may note in general, if there shall be $f':r = v$ and we put $r = F':v$, on account of $dr = dvF'':v$ there becomes

$$f:r = \int drf':r = \int vdvF'':v = vF':v - F:v$$

or $f:r = vF':v - F:v$ and hence $f:r - rf':r = -F:v$. Whereby since there shall be $f':r = x - aq$, if we put $r = F':(x-aq)$, there will be

$$f:r - rf':r = -F:(x - aq) \quad \text{and} \quad y = aqF':(x - aq) - F:(x - aq)$$

and

$$z = aqqF':(x - aq).$$

SCHOLIUM

137. These latter formulas thus are able to be elicited at once from the condition of the question. For on account of $p = \frac{z}{aq}$ there will be

$$dz = \frac{zdx}{aq} + qdy \quad \text{and} \quad dy = \frac{dz}{q} - \frac{zdx}{aqq}$$

and hence

$$y = \frac{z}{q} + \int\left(\frac{zdq}{qq} - \frac{zdx}{aqq}\right) = \frac{z}{q} + \int \frac{z}{qq}\left(dq - \frac{dx}{q}\right),$$

where it is evident that $\frac{z}{qq}$ is a function of the quantity $q - \frac{x}{a}$. Whereby on putting $\frac{z}{qq} = F'\left(q - \frac{x}{a}\right)$ there will be

$$y = \frac{z}{q} + F:\left(q - \frac{x}{a}\right).$$

Also from the same place the other solution can be deduced on putting

$$dx = \frac{aq}{z}(dz - qdy),$$

which on putting $z = qv$ will change into

$$dx = \frac{a}{v}(vdq + qdv - qdy),$$

from which

$$x = aq + \int \frac{aq}{v}(dv - dy).$$

Whereby on putting $\frac{aq}{v} = f':(v - y)$ there shall be $x = aq + f:(v - y)$.
Now on restoring the value $v = \frac{z}{q}$ there will be had

$$\frac{aqq}{z} = f':\left(\frac{z}{q} - y\right) \quad \text{and} \quad x - aq = f:\left(\frac{z}{q} - y\right).$$

But the first solution is the most suitable for eliminating q and r in the examples.
If indeed there is put $f':r = \frac{b}{\sqrt{r}} + c$, there will be $f:r = 2b\sqrt{r} + cr + d$ *;* hence

$$z = aqqr \quad \text{et} \quad x = aq + \tfrac{b}{\sqrt{r}} + c, \quad y = aqr + b\sqrt{r} + d\,.$$

Now on account of $r = \frac{z}{aqq}$ there becomes

$$x = aq + bq\sqrt{\tfrac{a}{z}} + c \quad \text{and} \quad y = \tfrac{z}{q} + \tfrac{b}{q}\sqrt{\tfrac{z}{a}} + d.$$

Hence

$$x - c = q\left(a + \tfrac{b\sqrt{a}}{\sqrt{z}}\right) \quad \text{and} \quad y - d = \tfrac{z}{aq}\left(a + \tfrac{b\sqrt{a}}{\sqrt{z}}\right)$$

and on multiplying q is eliminated and there becomes

$$a(x-c)(y-d) = \tfrac{z}{a}\left(a + \tfrac{b\sqrt{a}}{\sqrt{z}}\right)^2 = \left(b + \sqrt{az}\right)^2,$$

thus so that there shall be

$$b + \sqrt{az} = \sqrt{(x-c)(y-d)}$$

and hence

$$z = \frac{(x-c)(y-d) - 2b\sqrt{(x-c)(y-d)} + bb}{a},$$

which, if $b = \text{c} = d = 0$, gives the most simple case $z = \frac{xy}{a}$.

CAPUT IV

DE RESOLUTIONE AEQUATIONUM QUIBUS RELATIO INTER BINAS FORMULAS DIFFERENTIALES ET UNICAM TRIUM QUANTITATUM VARIABILIUM PROPONITUR

PROBLEMA 15

97. *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut posito* $dz = pdx + qdy$ *sit* $q = \frac{px}{a}$, *indolem huius functionis in genere investigare.*

SOLUTIO

Cum sit

$$dz = pdx + \frac{pxdy}{a} = px\left(\frac{dx}{x} + \frac{dy}{a}\right)$$

haecque formula esse debeat integrabilis, necesse est, ut px ac proinde etiam z sit functio quantitatis $lx + \frac{y}{a}$. Quare solutio nostri problematis in genere ita se habebit, ut sit

$$z = f:\left(lx + \frac{y}{a}\right) \text{ et } px = f':\left(lx + \frac{y}{a}\right)$$

sumendo scilicet perpetuo $d.f:u = duf':u$. Hinc autem erit sicque $q = \frac{px}{a}$, omnino uti requiritur.

COROLLARIUM 1

98. Cum sit

$$z = px - \int xdp + \int \frac{pxdy}{a} = px + \int px \left(\frac{dy}{a} - \frac{dp}{p}\right),$$

hinc alia solutio deduci potest. Si enim ponamus

$$\int px \left(\frac{dy}{a} - \frac{dp}{p}\right) = f:\left(\frac{y}{a} - lp\right),$$

erit $px = f':\left(\frac{y}{a} - lp\right)$ indeque

$$z = f':\left(\frac{y}{a} - lp\right) + f:\left(\frac{y}{a} - lp\right).$$

COROLLARIUM 2

99. Hac ergo solutione nova introducitur variabilis *p*, ex qua cum *y* coniuncta definitur primo

$$x=\frac{1}{p}f':\left(\frac{y}{a}-lp\right),$$

tum vero ipsa functio quaesita

$$z=px+f:\left(\frac{y}{a}-lp\right).$$

Huic autem solutioni praecedens sine dubio antecellit, cum illa quantitatem *z* immediate per *x* et *y* exprimat.

SCHOLION

100. Quo has duas solutiones inter se comparare queamus, quoniam functio arbitraria in utraque diversae est indolis, etiam charactere diverso utamur. Cum igitur prima praebeat

$$z=f:\left(\frac{y}{a}+lx\right) \text{ et } px=f':\left(\frac{y}{a}+lx\right),$$

altera vero

$$z=F:\left(\frac{y}{a}-lp\right)+F':\left(\frac{y}{a}-lp\right) \text{ et } px=F':\left(\frac{y}{a}-lp\right),$$

patet fore

$$f':\left(\frac{y}{a}+lx\right)=F':\left(\frac{y}{a}-lp\right) \text{ et } f:\left(\frac{y}{a}+lx\right)=F:\left(\frac{y}{a}-lp\right)+F':\left(\frac{y}{a}-lp\right),$$

unde non solum relatio inter utriusque functionis *f* et *F* indolem definitur, sed etiam inde sequi debet fore

$$px=f':\left(\frac{y}{a}+lx\right),$$

id quod non parum videtur absconditum. Verum ob hoc ipsum istud problema eo magis est notatu dignum, quod solutio altera, qua nova variabilis *p* introducitur, congruit cum priore, ubi *z* per *x* et *y* immediate definitur, neque tamen consensus harum solutionum perspicue monstrari potest. Quamobrem quando ad eiusmodi solutiones pervenimus, uti in problematibus posterioribus capitis praecedentis usu venit, in quibus nova variabilis introducitur, non omnem *statim* spem eius eliminandae abiicere debemus, cum isto casu altera solutio ad priorem certe sit reductibilis, etiamsi methodus reducendi non perspiciatur, quam tamen infra § 119 exhibebimus.

PROBLEMA 16

101. *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut posito dz = pdx + qdy sit q = pX + T existentibus X et T functionibus quibuscunque ipsius x, indolem istius functionis z in genere investigare.*

SOLUTIO

Cum ergo sit $dz = pdx + pXdy + Tdy$, statuatur $p = r - \frac{T}{X}$ ut prodeat

$$dz = rdx - \frac{Tdx}{X} + rXdy = \frac{-Tdx}{X} + rX\left(\frac{dx}{X} + dy\right),$$

qua reductione facta perspicuum est tam rX quam $\int rX\left(\frac{dx}{X} + dy\right)$ fore functionem quantitatis $y + \int \frac{dx}{X}$. Quare si ponamus

$$\int rX\left(\frac{dx}{X} + dy\right) = f:\left(y + \int \frac{dx}{X}\right),$$

erit

$$rX = f':\left(y + \int \frac{dx}{X}\right)$$

ac tum functio quaesita erit

$$z = -\int \frac{Tdx}{X} + f:\left(y + \int \frac{dx}{X}\right),$$

quae ob functionem indefinitam $f:$ est completa. Tum vero erit

$$p = \frac{-T}{X} + \frac{1}{X} f':\left(y + \int \frac{dx}{X}\right) \text{ et } q = f':\left(y + \int \frac{dx}{X}\right),$$

unde patet fore utique $q = pX + T$. Quoniam vero X et T sunt functiones datae ipsius x, formulae integrales $\int \frac{dx}{X}$ et $\int \frac{Tdx}{X}$ solutionem non turbant.

COROLLARIUM 1

102. Solutio aliquanto facilior redditur sumendo ex conditione praescripta $p = \frac{q}{X} - \frac{T}{X}$ unde fit

$$dz = -T\frac{dX}{X} + \frac{qdx}{X} + qdy \text{ et } z = -\int \frac{TdX}{X} + \int q\left(dy + \frac{dx}{X}\right)$$

Iam manifesto est

$$\int q\left(dy + \frac{dx}{X}\right) = f:\left(y + \int \frac{dx}{X}\right)$$

sicque ipsa solutio praecedens resultat.

COROLLARIUM 2

103. Eodem modo resolvitur problema, si proponatur conditio $q = pY + V$ existentibus *Y* et *V* functionibus datis ipsius *y*. Tum enim erit

$$dz = pdx + pYdy + Vdy \text{ et } z = \int Vdy + \int p(dx + Ydy).$$

Hic ergo fit

$$\int p(dx + Ydy) = f:\left(x + \int Ydy\right)$$

et solutio erit

$$z = \int Vdy + f:\left(x + \int Ydy\right),$$

unde fit

$$p = f':\left(x + \int Ydy\right) \text{ et } q = V + Yf':\left(x + \int Ydy\right).$$

SCHOLION

104. Ex forma solutionis hic inventae discere poterimus, quomodo problema comparatum esse debeat, ut eius solutio hac ratione perfici et functio *z* per binas variabiles *x* et *y* exhiberi queat. Sint enim *K* et *V* functiones quaecunque ipsarum *x* et *y* indeque differentiando

$$dK = Ldx + Mdy \text{ et } dV = Pdx + Qdy;$$

iam a solutione incipiamus ponamusque

$$z = K + f:V$$

eritque differentiando

$$dz = Ldx + Mdy + (Pdx + Qdy)\, f':V.$$

Cum iam hanc formam cum assumta

$$dz = pdx + qdy$$

comparando sit

$$p = L + Pf':V \text{ et } q = M + Qf':V,$$

erit

$$Qp - Pq = LQ - MP.$$

Quare si hoc problema proponatur, ut posito $dz = pdx + qdy$ fieri debeat

$$q = \tfrac{Q}{P}p + M - \tfrac{LQ}{P},$$

solutio erit $z = K + f:V$, dummodo *M* et *L* itemque *P* et *Q* ita sint comparatae, ut sit

$$Ldx + Mdy = dK \text{ et } Pdx + Qdy = dV;$$

verum hi casus ad sequens caput sunt referendi.

PROBLEMA 17

105. *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut posito $dz = pdx + qdy$ sit $q = Px + \Pi$ existentibus P et Π functionibus datis ipsius p, indolem istius functionis z in genere investigare.*

SOLUTIO

Cum igitur sit $dz = pdx + Pxdy + \Pi dy$, erit

$$z = px + \int (Pxdy + \Pi dy - xdp).$$

Statuatur $Px + \Pi = v$, ut sit $x = \frac{v - \Pi}{P}$, fietque

$$z = px + \int \left(vdy - \frac{vdp}{P} + \frac{\Pi dp}{P} \right).$$

Quare cum P et Π sint functiones ipsius p ideoque formula $\int \frac{\Pi dp}{P}$ data, habebitur

$$z = px + \int \frac{\Pi dp}{P} + \int v \left(dy - \frac{dp}{P} \right),$$

unde patet tam v quam $\int v \left(dy - \frac{dp}{P} \right)$ functionem esse debere formulae $y - \int \frac{dp}{P}$. Ponamus ergo

$$\int v \left(dy - \frac{dp}{P} \right) = f : \left(y - \int \frac{dp}{P} \right)$$

eritque

$$v = Px + \Pi = f' : \left(y - \int \frac{dp}{P} \right)$$

et hinc

$$x = \frac{-\Pi}{P} + \frac{1}{P} f' : \left(y - \int \frac{dp}{P} \right),$$

tum vero

$$z = \int \frac{\Pi dp}{P} - \frac{\Pi p}{P} + \frac{p}{P} f' : \left(y - \int \frac{dp}{P} \right) + f : \left(y - \int \frac{dp}{P} \right).$$

COROLLARIUM 1

106. In solutione huius problematis iterum nova variabilis p introducitur, ex qua cum y coniunctim primo variabilis x, tum vero ipsa functio quaesita z determinatur.

COROLLARIUM 2

107. Neque vero hinc istam novam variabilem p ex calculo elidere licet, uti ante usu venit, propterea quod hic P et Π functiones ipsius p denotant, quarum indoles iam in ipsum problema ingreditur.

COROLLARIUM 3

108. Simili modo problema resolvetur, si permutandis x et y quantitas p ita per y et q detur, ut sit $p = Qy + \Xi$ denotantibus Q et Ξ functiones datas ipsius q.

SCHOLION

109. In hoc capite constituimus eiusmodi problemata tractare, quorum conditio aequatione inter binas formulas differentiales $\left(\frac{dz}{dx}\right) = p, \left(\frac{dz}{dy}\right) = q$ et unam ex tribus variabilibus x, y et z utcunque exprimitur. Problemata autem bina evoluta ex hoc genere certos casus complectuntur, quorum solutio peculiari methodo expediri potest simulque ad formulas simpliciores perducitur. In posteriori quidem relationem inter p, q et x ita assumsimus, ut sit $q = Px + \Pi$ seu ut in valore ipsius q per p et x expresso quantitas x
unam dimensionem non excedat; in priori vero ita, ut sit $q = pX + T$ seu ut in valore ipsius q per p et x expresso quantitas p unicam obtineat dimensionem.

In genere autem notasse iuvabit tam quantitates p et x quam q et y inter se esse permutabiles. Cum enim sit $\int pdx = px - \int xdp$, loco

$$z = \int (pdx + qdy)$$

erit

$$z = px + \int (qdy - xdp).$$

Simili modo est

$$z = qy + \int (pdx - ydq),$$

tum vero etiam

$$z = px + qy - \int (xdp + ydq).$$

Quibus ergo casibus una harum quatuor formularum integralium redditur integrabilis, iisdem ternae reliquae etiam integrationem admittent. Cum igitur in superiori capite primam formulam resolverimus, si p vel q quomodocunque detur per x et y, ita eodem modo resolvetur formula secunda, si q per p et y, tertia autem, si p per x et q, at quarta, si vel x per p et q vel y per p

et q utcunque datur; quae quaestiones cum generaliter expediri queant, eas in sequenti problemate evolvamus.

PROBLEMA 18

110. *Posito* $dz = pdx + qdy$ *si relatio inter p, q et x aequatione quacunque definiatur, indolem functionis z, quemadmodum ex binis variabilibus x et y determinetur, in genere investigare.*

SOLUTIO

Ex aequatione inter p, q et x proposita quaeratur valor ipsius x, qui functioni cuipiam ipsarum p et q aequabitur. Cum iam sit

$$z = px + qy - \int (xdp + ydq),$$

quoniam x est functio data ipsarum p et q, formula xdp integretur sumta quantitate q constante sitque

$$\int xdp = V + f:q$$

et erit V functio cognita ipsarum p et q, qua differentiata prodeat

$$dV = xdp + Sdq,$$

ubi S quoque erit functio data ipsarum p et q. Quia iam forma $\int (xdp + ydq)$ integrationem admittere debet, aequabitur formae $V + f:q$, unde differentiando concluditur

$$xdp + ydq = xdp + Sdq + dqf':q$$

ideoque

$$y = S + f':q \quad \text{et} \quad z = px + qy - V - f:q$$

seu

$$z = px + Sq + qf':q - f:q - V.$$

Solutio ergo ita se habet:

Primo ex conditione praescripta datur x per p et q; tum sumta q constante sit $V = \int xdp$ et vicissim $dV = xdp + Sdq$; inventis autem V et S per p et q reliquae quantitates y et z ita per easdem exprimentur, ut sit

$$y = S + f':q \quad \text{et} \quad z = px + Sq + qf':q - f:q - V$$

quae solutio, quia $f:q$ functionem quamcunque ipsius q sive continuam siva discontinuam denotat, utique pro completa latissimeque patente est habenda.

ALITER

111. Vel ex aequatione inter *p*, *q* et *x* data quaeratur valor ipsius *p* per *x* et *q* expressus, ita ut *p* aequetur functioni cuipiam datae binarum variabilium *x* et *q*, per quas etiam reliquas quantitates *y* et *z* definire conemur. Ad hoc utamur formula

$$z = qy + \int \left(pdx - ydq \right),$$

et quia *p* est functio ipsarum *x* et *q*, dabitur earundem eiusmodi functio *V*, ut sit

$$dV = pdx + Rdq\,.$$

Statuatur ergo

$$\int \left(pdx - ydq \right) = V + f:q$$

eritque

$$y = -R - f':q \quad \text{et} \quad z = qy + V + f:q\,.$$

COROLLARIUM 1

112. Utraque solutio aeque commode adhiberi potest, si ex relatione inter *p*, *q* et *x* proposita tam quantitatem *x* quam *p* aeque commode definire liceat. Sin autem earum altera commodius definiri queat, ea solutione, quae ad istum casum est accommodata, erit utendum.

COROLLARIUM 2

113. Sin autem neque *p* neque *x* commode elici queat, tum nihilo minus hic resolutio aequationum cuiusque ordinis quin etiam transcendentium tanquam concessa assumitur. Ceterum etiamsi *q* facile per *p* et *x* definiatur, hinc calculus nihil iuvatur.

COROLLARIUM 3

114. Ex hoc problemate utpote latissime patente etiam bina praecedentia resolvi possunt; solutio autem hinc inventa a praecedente discrepabit, cum illa ex methodo particulari sit deducta. Operae autem pretium erit has duplices solutiones inter se comparare.

EXEMPLUM 1

115. *Si fuerit* $q = pX + T$ *existentibus X et T functionibus ipsius x, indolem functionis z investigare.*

Hic solutione utendum est posteriori, pro qua est $p = \frac{q-T}{X}$; nunc posita q constante prodit

$$V = \int pdx = q\int \frac{dx}{X} - \int \frac{Tdx}{X}$$

hincque

$$R = \left(\frac{dV}{dq}\right) = \int \frac{dx}{X},$$

unde solutio his formulis continetur

$$q = pX + T, \quad y = -\int \frac{dx}{X} - f':q, \quad z = -\int \frac{Tdx}{X} - q\, f':q + f:q;$$

solutio autem superior [§ 101] ita se habebat

$$q = pX + T, \; q = f':\left(y + \int \frac{dx}{X}\right) \quad \text{et} \; z = -\int \frac{Tdx}{X} + f:\left(y + \int \frac{dx}{X}\right).$$

SCHOLION

116. Consensus harum duarum solutionum ita ostendi potest, ut ex ea, quam hic invenimus, antecedens per legitimam consequentiam formetur.

Cum enim sit

$$f':q = -y - \int \frac{dx}{X},$$

statuatur brevitatis gratia $y + \int \frac{dx}{X} = v$, ut sit $f':q = -v$; erit ergo vicissim q aequalis functioni cuidam ipsius v, quae ponatur $q = F':v$, unde fit $dq = dvF'':v$, ergo

$$dq\, f':q = -vdvF'':v = -vd.F':v,$$

ergo integrando

$$f:q = -\int vd.F':v = -vF':v + \int dvF':v = -vF':v + F:v..$$

Quare cum sit

$$z = -\int \frac{Tdx}{X} - qf':q + f:q,$$

erit

$$z = -\int \frac{Tdx}{X} + vF':v - vF':v + F:v \quad \text{seu} \; z = -\int \frac{Tdx}{X} + F:\left(y + \int \frac{dx}{X}\right),$$

quae est ipsa solutio praecedens.

EXEMPLUM 2

117. *Si fuerit* $q = Px + \Pi$ *existentibus P et* Π *functionibus datis ipsius p, indolem functionis z, ut sit* $dz = pdx + qdy$, *investigare.*

Hic solutione priori utendum, cum sit $x = \frac{q-\Pi}{p}$. Sumto ergo q constante quaeratur

$$V = \int xdp = q\int \frac{dp}{P} - \int \frac{\Pi dp}{P},$$

unde fit

$$S = \left(\frac{dV}{dq}\right) = \int \frac{dp}{P}.$$

Solutio ergo praebet

$$y = \int \frac{dp}{P} + f':q$$

et

$$z = \frac{pq}{P} - \frac{p\Pi}{P} + q\int \frac{dp}{P} + qf':q - f:q - q\int \frac{dp}{P} + \int \frac{\Pi dp}{P}$$

sive

$$z = \frac{p(q-\Pi)}{P} + \int \frac{\Pi dp}{P} + q\, f':q - f:q.$$

Solutio autem eiusdem casus supra (§ 105) inventa erat

$$x = -\frac{\Pi}{P} + \frac{1}{P} f': \left(y - \int \frac{dp}{P}\right) \text{ et } q = Px + \Pi$$

atque

$$z = \frac{-p\Pi}{P} + \int \frac{\Pi dp}{P} + \frac{p}{P} f':\left(y - \int \frac{dp}{P}\right) + f:\left(y - \int \frac{dp}{P}\right).$$

SCHOLION 1

118. Videamus, quomodo solutio hic inventa ad superiorem reduci queat.
Cum ibi invenerimus

$$y - \int \frac{dp}{P} = f':q,$$

vicissim q aequabitur functioni quantitatis $y - \int \frac{dp}{P}$; ponatur ergo

$$q = F':\left(y - \int \frac{dp}{P}\right)$$

eritque statim

$$x = \frac{-\Pi}{p} + \frac{1}{P} F':\left(y - \int \frac{dp}{P}\right).$$

Sit brevitatis gratia $y - \int \frac{dp}{P} = v$, ut fiat $q = F'{:}\, v$ et $v = f'{:}\, q$; erit

$$F:v = \int qdv = qv - \int vdq = qv - \int dq\, f':q.$$

Ergo $F:v = qv - f:q$, ita ut sit

$$f:q = q\left(y - \int \frac{dp}{P}\right) - F:\left(y - \int \frac{dp}{P}\right)$$

seu

$$f:q = \left(y - \int \frac{dp}{P}\right)F':\left(y - \int \frac{dp}{P}\right) - F:\left(y - \int \frac{dp}{P}\right)$$

Quibus valoribus substitutis habebimus

$$x = \frac{-\Pi}{p} + \frac{1}{P}F':\left(y - \int \frac{dp}{P}\right)$$

et

$$z = \frac{-p\Pi}{P} + \frac{p}{P}F':\left(y - \int \frac{dp}{P}\right) + \int \frac{\Pi dp}{P} + \left(y - \int \frac{dp}{P}\right)F':\left(y - \int \frac{dp}{P}\right)$$
$$-\left(y - \int \frac{dp}{P}\right)F':\left(y - \int \frac{dp}{P}\right) + F:\left(y - \int \frac{dp}{P}\right)$$

seu

$$z = \frac{-p\Pi}{P} + \frac{p}{P}F':\left(y - \int \frac{dp}{P}\right) + \int \frac{\Pi dp}{P} + F:\left(y - \int \frac{dp}{P}\right)$$

quae est ipsa solutio ante inventa.

SCHOLION 2

119. Hoc consensu ostenso etiam consensum supra observatum (§ 100) demonstrare poterimus, qui multo magis absconditus videtur.

Altera autem solutio ibi inventa erat

$$px = F':\left(\frac{y}{a} - lp\right) \quad \text{et} \quad z = px + F:\left(\frac{y}{a} - lp\right),$$

ex quarum formula priori patet fore vicissim $\frac{y}{a} - lp$ functionem ipsius *px.*

Hinc etiam $\frac{y}{a} - lp + lpx$ seu $\frac{y}{a} + lx$ aequabitur functioni ipsius *px;* denuo ergo vicissim px aequabitur functioni cuipiam ipsius $\frac{y}{a} + lx$. Ponatur ergo $px = f':\left(\frac{y}{a} + lx\right)$, et cum sit

$$d.F:\left(\frac{y}{a} - lp\right) = \left(\frac{dy}{a} - \frac{dp}{p}\right)F':\left(\frac{y}{a} - lp\right),$$

erit

$$F:\left(\frac{y}{a} - lp\right) = \int px\left(\frac{dy}{a} - \frac{dp}{p}\right) = \int px\left(\frac{dy}{a} + \frac{dx}{x}\right) - \int px\left(\frac{dx}{x} + \frac{dp}{p}\right) = \int px\left(\frac{dy}{a} + \frac{dx}{x}\right) - px.$$

Iam pro px substituto valore $f':\left(\frac{y}{a}+lx\right)$ obtinebitur

$$F:\left(\frac{y}{a}-lp\right)=-px+\int\left(\frac{dy}{a}+\frac{dx}{x}\right)f':\left(\frac{y}{a}+lx\right)=-px+f:\left(\frac{y}{a}+lx\right),$$

ita ut hinc fiat $z=f:\left(\frac{y}{a}+lx\right)$, quae est ipsa solutio altera.

Hac igitur reductione haud parum luminis accenditur ad alia mysteria huius generis investiganda. Summa autem huius ratiocinii huc redit, ut si fuerit $r=f':s$ fore etiam $r=F':(s+R)$ denotante R functionem ipsius r, quod quidem per se est evidens, quia utrinque r per s determinatur. Cum ergo sit

$$f':s=r=F':(s+R),$$

erit

$$f:s=\int dsf':s=\int rds=\int r(ds+dR-dR)=\int(ds+dR)F':(s+R)-\int rdR$$

ideoque

$$f:s=F:(s+R)-\int rdR,$$

unde loco functionum quantitatis s functiones quantitatis $s+R$ introduci possunt. Scilicet si sit $r=f':s$, sumi potest $r=F':(s+R)$ existente R functione quacunque ipsius r, tum vero erit $f:s=F:(s+R)-\int rdR$.

EXEMPLUM 3

120. *Posito $dz=pdx+qdy$ si x aequetur functioni homogeneae n dimensionum ipsarum p et q, indolem functionis z investigare.*

Cum x detur per p et q, utendum erit solutione priori et ob x = functioni homogeneae n dimensionum ipsarum p et q ponatur $p=qr$ fietque $x=q^nR$ existente R functione ipsius r tantum. Sumatur nunc q constans et quaeratur

$$V=\int xdp=\int q^{n+1}Rdr$$

ob $dp=qdr$ eritque

$$V=q^{n+1}\int Rdr$$

quod integrale datur. Hinc differentiando erit

$$dV = q^{n+1}Rdr + (n+1)q^n dq\int Rdr;$$

quae ut cum

$$dV = xdp + Sdq = q^n Rdp + Sdq$$

comparari possit, quia ob $dp = qdr + rdq$ est

$$dV = q^{n+1}Rdr + q^n Rrdq + Sdq,$$

erit

$$S = -q^n Rr + (n+1)q^n \int Rdr,$$

unde fit

$$y = -q^n Rr + (n+1)q^n \int Rdr + f':q \quad \text{et} \quad x = q^n R$$

atque

$$z = nq^{n+1}\int Rdr + q\, f':q - f:q$$

existente $p = qr$.

COROLLARIUM 1

121. Sit $x = \frac{p^m}{q^m}$ et posito $p = qr$ erit $x = r^m$ ideoque $n = 0$ et $R = r^m$ unde fit

$$y = -r^{m+1} + \tfrac{r^{m+1}}{m+1} + f':q = \tfrac{-m}{m+1} r^{m+1} + f':q \quad \text{et} \quad z = q\, f':q - f:q.$$

Quare ob $r = x^{\frac{1}{m}}$ erit

$$y = \tfrac{-m}{m+1} x^{\frac{m+1}{m}} + f':q.$$

COROLLARIUM 2

122. Eodem casu ergo, quo $x = \frac{p^m}{q^m}$, aequabitur q functioni quantitatis $y + \frac{m}{m+1} x^{\frac{m+1}{m}}$; quae quantitas si ponatur $= v$ et $q = F':v$, ut sit $v = f:q$, erit

$$f:q = \int dq\, f':q = \int vdvF'':v$$

ob $dq = dvF'':v$, unde concluditur

$$f:q = vF':v - F:v \quad \text{et} \quad z = F:v = F:\left(y + \tfrac{m}{m+1}x^{\frac{m+1}{m}}\right).$$

EXEMPLUM 4

123. *Duarum variabilium x et y eiusmodi functionem z investigare, ut posito* $dz = pdx + qdy$ *fiat* $p^3 + x^3 = 3pqx$.

Consideretur forma

$$z = qy + \int (pdx - ydq),$$

ubi iam formulam $dz = pdx + qdy$ integrabilem reddi oportet. Statuatur $p = ux$ et conditio praescripta dat $x(1+u^3) = 3qu$ *;* unde fit

$$x = \frac{3qu}{1+u^3} \quad \text{et} \quad p = \frac{3quu}{1+u^3},$$

tum vero

$$dx = \frac{3qdu(1-2u^3)}{(1+u^3)^2} + \frac{3udq}{1+u^3}$$

sicque habebitur

$$z = qy + \int\left(\frac{9qquudu(1-2u^3)}{(1+u^3)^3} + \frac{9qu^3dq}{(1+u^3)^2} - ydq\right),$$

at

$$\int \frac{9qquudu(1-2u^3)}{(1+u^3)^3} = \frac{3qq(1+4u^3)dq}{2(1+u^3)^2} - \int \frac{3q(1+4u^3)dq}{(1+u^3)^2}$$

ergo

$$z = qy + \frac{3qq(1+4u^3)}{2(1+u^3)^2} - \int dq\left(y + \frac{3q}{1+u^3}\right).$$

Quare necesse est esse $y + \frac{3q}{1+u^3}$ functionem ipsius *q* tantum, quae sit $-f':q$, unde fit

$$y = -\frac{3q}{1+u^3} - f':q \quad \text{et} \quad z = qy + \frac{3qq(1+4u^3)}{2(1+u^3)^2} + f:q$$

seu

$$z = \frac{3qq(2u^3-1)}{2(1+u^3)^2} - q\, f':q + f:q$$

existente $x = \frac{3qu}{1+u^3}$. Ex quibus tribus aequationibus si eliminentur binae quantitates *q* et *u*, orietur aequatio inter *z* et *x, y,* quae quaeritur.

COROLLARIUM 1

124. Ex aequatione pro *y* inventa colligitur

$$\frac{3}{1+u^3} = \frac{-y-f':q}{q}$$

aequatio autem pro *z* inventa abit in hanc

$$z = \frac{3qq}{1+u^3} - \frac{9qq}{2(1+u^3)^2} - q\, f':q + f:q,$$

quae eliso *u* transmutatur in hanc

$$z = -qy - 2qf':q - \tfrac{1}{2}(y + f':q)^2 + f:q\,;$$

tum vero est

$$x = -u(y + f':q),$$

unde reperitur $u = \frac{-x}{y+f':q}$ hincque

$$x^3 = 3q(y + f':q)^2 + (y + f':q)^3.$$

COROLLARIUM 2

125. Si sumamus $f':q = a$, erit $f:q = aq + b$ et postrema aequatio praebet $q = \frac{x^3-(y+a)^3}{3(y+a)^2}$. Cum deinde pro hoc casu fiat

$$z = -qy - aq - \tfrac{1}{2}(y+a)^2 + b,$$

proveniet loco *q* valorem inventum substituendo

$$z = \frac{6b(y+a)-(y+a)^3-2x^3}{6(y+a)}.$$

COROLLARIUM 3

126. Cum in genere sit

$$x^3 = (y + f':q)^2 (y + 3q + f':q),$$

ponamus $f':q = a - 3q$ ideoque $f:q = b + aq - \frac{3}{2}:qq$, ut fiat

$$(y + a - 3q)^2 = \frac{x^3}{y+a},$$

eritque

$$y + a - 3q = \frac{x\sqrt{x}}{\sqrt{(y+a)}} \quad \text{et} \quad q = \frac{1}{3} y(y+a) - \frac{x\sqrt{x}}{3\sqrt{(y+a)}}.$$

Hinc ergo prodit

$$f':q = \frac{x\sqrt{x}}{\sqrt{(y+a)}} - y$$

et

$$f:q = b + \frac{a(y+a)}{3} - \frac{ax\sqrt{x}}{3\sqrt{(y+a)}} - \frac{1}{6}(y+a)^2 + \frac{1}{3} x\sqrt{x(y+a)} - \frac{x^3}{6(y+a)}$$

$$= b + \frac{aa-yy}{6} + \frac{xy\sqrt{x}}{3\sqrt{(y+a)}} - \frac{x^3}{6(y+a)}$$

atque

$$z = -\frac{1}{3} y(y+a) + \frac{yx\sqrt{x}}{3\sqrt{(y+a)}} - 2aq + 6qq - \frac{x^3}{2(y+a)} + b + aq - \frac{3}{2} qq$$

seu

$$z = b - \frac{1}{3} y(y+a) + \frac{yx\sqrt{x}}{3\sqrt{(y+a)}} - \frac{x^3}{2(y+a)} - aq + \frac{9}{2} qq$$

et facta reductione

$$z = b + \frac{1}{6}(y+a)^2 - \frac{2}{3} x\sqrt{x(y+a)}$$

COROLLARIUM 4

127. Quodsi hic sumatur $a = 0$ et $b = 0$, erit per expressionem satis simplicem

$$z = \frac{1}{6} yy - \frac{2}{3} x\sqrt{xy},$$

quae quomodo conditioni praescriptae satisfaciat, ita apparet. Per differentiationem colligitur

$$p = \left(\frac{dz}{dx}\right) = -\sqrt{xy} \quad \text{et} \quad q = \left(\frac{dz}{dy}\right) = \frac{1}{3} y - \frac{x\sqrt{x}}{3\sqrt{y}}$$

hincque

$$p^3 + q^3 = -xy\sqrt{xy} + x^3,$$

at $3pq = xx - y\sqrt{xy}$ ideoque $3pqx = x^3 - xy\sqrt{xy}$, ergo

$$p^3 + x^3 = 3pqx.$$

SCHOLION

128. Successit ergo solutio, quando aequatio quaecunque inter *p, q* et *x* proponitur, etiamsi casibus, quibus inde neque *x* neque *p* elici potest, difficultas quaedam restat, quae autem resolutionem aequationum finitarum potissimum afficit, quam hic merito concedi postulamus. Interim ex postremo exemplo perspicitur, quomodo operatio sit instituenda, si ope substitutionis idoneae aequatio proposita ad resolutionem accommodari queat, cui autem negotio hic amplius non immoror. Neque etiam eos casus, quibus inter *p,q* et *y* relatio quaedam praescribitur, hic seorsim evolvam, cum ob permutabilitatem ipsarum *x* et *y*, qua etiam *p* et *q* permutantur, hi casus ad praecedentes sponte revocentur. Superest igitur casus, quo aequatio inter *p, q* et *z* proponitur, ubi quidem statim manifestum est in aequatione $dz = pdx + qdy$ quantitates *p* et *q* non uti functiones ipsarum *x* et *y* spectari
posse, quoniam etiam a *z* pendent, neque ergo earum indoles inde determinari poterit, ut formula $pdx + qdy$ integrabilis evadat. Verum sine discrimine conditio ea est definienda, ut aequatio differentialis $dz - pdx - qdy = 0$ fiat possibilis; ad quod ex principiis supra stabilitis (§ 6) requiritur, ut posito

$$\left(\frac{dq}{dz}\right) = L, \quad -\left(\frac{dp}{dz}\right) = M \quad \text{et} \quad \left(\frac{dp}{dy}\right) - \left(\frac{dq}{dx}\right) = N$$

sit

$$Lp + Mq - N = 0 \quad \text{seu} \quad p\left(\frac{dq}{dz}\right) - q\left(\frac{dp}{dz}\right) + \left(\frac{dq}{dx}\right) - \left(\frac{dp}{dy}\right) = 0$$

Quare proposita aequatione quacunque inter *p, q* et *z* eas conditiones in genere investigari oportet, ut huic requisito satisfiat.

PROBLEMA 19

129. *Si posito $dz = pdx + qdy$ debeat esse $p + q = \frac{z}{a}$, relationem functionis z ad variabiles x et y in genere investigare.*

SOLUTIO

Cum sit $q = \frac{z}{a} - p$, aequatio nostra hanc induet formam

$$dz = pdx - pdy + \frac{zdy}{a}$$

seu

$$p(dx - dy) = \frac{adz - zdy}{a} = z\left(\frac{dz}{z} - \frac{dy}{a}\right).$$

Quoniam igitur ambae formulae $dx-dy$ et $\frac{dz}{z}-\frac{dy}{a}$ per se sunt integrabiles, ob

$$\frac{dz}{z}-\frac{dy}{a}=\frac{p}{z}(dx-dy)$$

necesse est, ut $\frac{p}{z}$ sit funetio quantitatis $x-y$; ponatur ergo $\frac{p}{z}=f':(x-y)$, ut fiat

$$lz-\frac{y}{a}=f:(x-y).$$

Definiri ergo potest *z* per *x* et *y*, et cum sit $e^{f:(x-y)}$. etiam functio ipsius $x-y$, si ea ponatur $=F:(x-y)$, erit

$$z=e^{\frac{y}{a}}F:(x-y),$$

unde fit

$$\left(\frac{dz}{dx}\right)=p=e^{\frac{y}{a}}F':(x-y) \quad \text{et} \quad \left(\frac{dz}{dy}\right)=q=-e^{\frac{y}{a}}F':(x-y)+\frac{1}{a}e^{\frac{y}{a}}F:(x-y)$$

ideoque

$$p+q=\frac{1}{a}e^{\frac{y}{a}}F:(x-y)=\frac{z}{a},$$

uti requiritur.

COROLLARIUM 1

130. Ex hoc exemplo intelligitur, quomodo certa functio ipsarum *p* et *q* quantitati *z* aequari possit, etiamsi *p* et *q* sint functiones ipsarum *x* et *y*. Simul scilicet ratio integralis formulae $dz=pdx+qdy$ introducitur in calculum.

COROLLARIUM 2

131. Forma $e^{\frac{y}{a}}F:(x-y)$ pro valore ipsius *z* inventa per functionem quamvis ipsius $x-y$ multiplicari potest. Si ergo multiplicetur per $e^{\frac{x-y}{a}}$, fit $z=e^{\frac{x}{a}}F:(x-y)$. Sin autem multiplicetur per $e^{\frac{x-y}{2a}}$, fit $z=e^{\frac{x+y}{2a}}F:(x-y)$,quae formae problemati aeque satisfaciunt.

PROBLEMA 20

132. *Si posito $dz = pdx + qdy$ quantitas z aequari debeat functioni datae ipsarum p et q, indolem, qua z per x et y definitur, in genere investigare.*

SOLUTIO

Ex formula proposita habemus $dy = \frac{dz}{q} - \frac{pdx}{q}$; statuatur $p = qr$, ut sit z aequalis functioni ipsarum q et r, et ex $dy = \frac{dz}{q} - rdx$ elicitur

$$y = \frac{z}{q} - rx + \int\left(\frac{zdq}{qq} + xdr\right),$$

quam formulam integrabilem reddi oportet. Cum igitur z sit functio data ipsarum q et r, posito r constante quaeratur integrale formulae $\frac{zdq}{qq}$ sitque

$$\int\frac{zdq}{qq} = V + f:r,$$

unde differentiando prodeat

$$dV = \frac{zdq}{qq} + Rdr,$$

ac iam patet esse debere $x = R + f':r$ indeque obtineri

$$y = \frac{z}{q} - Rr - rf':r + V + f:r,$$

quibus duabus aequationibus relatio inter quantitates propositas determinatur.

Primo igitur posito $p = qr$ datur z per q et r. Deinde sumto r constante integretur formula $\frac{zdq}{qq}$ sitque integrale resultans $V = \int\frac{zdq}{qq}$, quod etiam per q et r datur, unde sumto q constante colligitur $R = \left(\frac{dV}{dr}\right)$. Quibus inventis erit

$$x = R + f':r \quad \text{et} \quad y = \frac{z}{q} - rx + V + f:r$$

sicque omnes quantitates per binas variabiles q et r determinantur.

COROLLARlUM 1

133. Quia permutatis x et y litterae p et q permutantur, simili modo nostram investigationem incipere potuissemus ab aequatione $dx = \frac{dz}{p} - \frac{qdy}{p}$ similisque solutio prodiisset, quae quidem forma diversa, at re congruens esset.

COROLLARIUM 2

134. Iam scilicet posito $q = ps$, ut sit $dx = \frac{dz}{p} - sdy$, erit

$$x = \frac{z}{p} - sy + \int\left(\frac{zdp}{pp} + yds\right).$$

Iam sumto s constante ponatur $\int\frac{zdp}{pp} = U$, quae quantitas per p et s determinatur, ex ea vero prodeat $\left(\frac{dU}{ds}\right) = S$; erit

$$y = S + f':s \quad \text{et} \quad x = \frac{z}{p} - sy + U + f:s.$$

EXEMPLUM 1

135. *Si esse debeat $p + q = \frac{z}{a}$, solutionem pro hoc casu exhibere.*

Posito $p = qr$ erit $z = aq(1+r)$; nunc sumto r constante erit

$$V = \int\frac{zdq}{qq} = a(1+r)lq \quad \text{et} \quad R = \left(\frac{dV}{dr}\right) = alq.$$

Hinc reperitur

$$x = alq + f':r \quad \text{et} \quad y = \frac{z}{q} - arlq - rf':r + a(1+r)lq + f:r$$

seu

$$y = a(1+r) + alq - rf':r + f:r.$$

Si hinc q elidere velimus, ob $q = \frac{z}{a(1+r)}$ solutio his duabus aequationibus continetur

$$y = al\frac{z}{a(1+r)}\, f':r$$

et

$$y = al\frac{z}{a(1+r)} + a(1+r) - rf':r + f:r.$$

Unde sequenti modo praecedens solutio [§ 129, 131] elici potest. Ex forma priori est

$$\frac{x}{a} - l\frac{z}{a} = -l(1+r) + \frac{1}{a}f':r = \text{funct.}r,$$

ex ambabus vero

$$y - x = a(1+r) - (1+r)f':r + f:r = \text{funct.}r.$$

Cum ergo tam $\frac{x}{a} - l\frac{z}{a}$ seu $ze^{-\frac{x}{a}}$ quam $y - x$ sit functio ipsius *r*, altera forma aequabitur functioni alterius, unde statui potest

$$ze^{-\frac{x}{a}} = F:(y-x) \text{ seu } z = e^{\frac{x}{a}}F:(y-x),$$

quae est solutio ante inventa.

EXEMPLUM 2

136. *Si posito* $dz = pdx + qdy$ *debeat esse* $z = apq$ *, relationem inter x, y et z investigare.*

Posito $p = qr$ erit $z = aqqr$ et sumto *r* constante fit $V = \int \frac{zdq}{qq} = aqr$ hincque $R = \left(\frac{dV}{dr}\right) = aq$. Quocirca habebimus

$$x = aq + f':r \quad \text{et} \quad y = aqr - rf':r + f:r$$

seu ob $r = \frac{z}{aqq}$ erit

$$x = aq + f':\tfrac{z}{aqq} \quad \text{et} \quad y = \tfrac{z}{q} - \tfrac{z}{aqq}f':\tfrac{z}{aqq} + f:\tfrac{z}{aqq}$$

Hic in genere notemus, si sit $f':r = v$ ponamusque $r = F':v$, ob $dr = dvF'':v$ fore

$$f:r = \int drf':r = \int vdvF'':v = vF':v - F:v$$

seu $f:r = vF':v - F:v$ hincque $f:r - rf':r = -F:v$. Quare cum sit $f':r = x - aq$, si ponamus $r = F':(x-aq)$, erit

$$f:r - rf':r = -F:(x-aq) \quad \text{et} \quad y = aqF':(x-aq) - F:(x-aq)$$

atque

$$z = aqqF':(x-aq).$$

SCHOLION

137. Hae postremae formulae ita statim ex conditione quaestionis elici possunt. Nam ob $p = \frac{z}{aq}$ erit

$$dz = \frac{zdx}{aq} + qdy \quad \text{et} \quad dy = \frac{dz}{q} - \frac{zdx}{aqq}$$

hincque

$$y=\frac{z}{q}+\int\left(\frac{zdq}{qq}-\frac{zdx}{aqq}\right)=\frac{z}{q}+\int\frac{z}{qq}\left(dq-\frac{dx}{q}\right),$$

ubi manifestum est esse $\frac{z}{qq}$ functionem quantitatis $q-\frac{x}{a}$. Quare posito $\frac{z}{qq}=F'\left(q-\frac{x}{a}\right)$ erit

$$y=\frac{z}{q}+F{:}\left(q-\frac{x}{a}\right).$$

Quin etiam indidem alia solutio deduci potest ponendo

$$dx=\frac{aq}{z}\left(dz-qdy\right),$$

quae posito $z=qv$ abit in

$$dx=\frac{a}{v}\left(vdq+qdv-qdy\right),$$

unde

$$x=aq+\int\frac{aq}{v}\left(dv-dy\right).$$

Quare ponatur $\frac{aq}{v}=f'{:}\left(v-y\right)$ eritque $x=aq+f{:}\left(v-y\right)$.
Iam restituto valore $v=\frac{z}{q}$ habebitur

$$\frac{aqq}{z}=f'{:}\left(\frac{z}{q}-y\right)\ \text{ et }\ x-aq=f{:}\left(\frac{z}{q}-y\right).$$

Prima autem solutio ad eliminanda q et r est aptissima in exemplis.
Si enim ponatur $f'{:}r=\frac{b}{\sqrt{r}}+c$, erit $f{:}r=2b\sqrt{r}+cr+d$; hinc

$$z=aqqr\quad\text{ et }\ x=aq+\frac{b}{\sqrt{r}}+c,\ \ y=aqr+b\sqrt{r}+d\,.$$

Iam ob $r=\frac{z}{aqq}$ fit

$$x=aq+bq\sqrt{\frac{a}{z}}+c\ \text{ et }\ y=\frac{z}{q}+\frac{b}{q}\sqrt{\frac{z}{a}}+d.$$

Hinc

$$x-c=q\left(a+\frac{b\sqrt{a}}{\sqrt{z}}\right)\ \text{ et }\ y-d=\frac{z}{aq}\left(a+\frac{b\sqrt{a}}{\sqrt{z}}\right)$$

et multiplicando eliditur q fitque

$$a\left(x-c\right)\left(y-d\right)=\frac{z}{a}\left(a+\frac{b\sqrt{a}}{\sqrt{z}}\right)^2=\left(b+\sqrt{az}\right)^2,$$

ita ut sit

$$b+\sqrt{az}=\sqrt{\left(x-c\right)\left(y-d\right)}$$

et proinde

$$z=\frac{\left(x-c\right)\left(y-d\right)-2b\sqrt{\left(x-c\right)\left(y-d\right)}+bb}{a},$$

quae, si $b=c=d=0$, dat casum simplicissimum $z=\frac{xy}{a}$.