## CHAPTER III

# CONCERNING THE RESOLUTION OF EQUATIONS IN WHICH EACH OF THE TWO DIFFERENTIAL FORMULAS IS GIVEN IN TERMS OF THE OTHER IN SOME MANNER

### PROBLEM 10

**73.** *If z should be a function of the two variables x and y of this kind, so that the differential formulas* $\left(\frac{dz}{dx}\right)$ *and* $\left(\frac{dz}{dy}\right)$ *are made equal to each other, to find the general nature of this function.*

### SOLUTIO

There may be put in place $\left(\frac{dz}{dx}\right)=p$ and $\left(\frac{dz}{dy}\right)=q$, so that there shall be $dz=pdx+qdy$ and this formula $pdx+qdy$ admits integration at once. Therefore as it is required that there shall be $q=p$, there will be $dz=p(dx+dy)$ and on putting $x+y=u$ there becomes $dz=pdu$; which formula since it must itself be integrable, it is necessary that $p$ shall be a function of the variable $u$ involving no additional variable ; and hence on integrating that quantity $z=\int pdu$ will be equal to a function of $u$ or there will arise $z=f:u$, which entirely arbitrary function is left by us, so that for $z$ some assumed function of $u$ either continuous or also discontinuous satisfies the problem. Whereby since there shall be $u=x+y$, for the solution of our problem there shall be $z=f:(x+y)$. Which form, so that it may become more apparent, just as for a prescribed condition it may satisfy, there shall be $d.f:u=du\,f':u$ and thus on account of $u=x+y$ there will be

$$dz=(dx+dy)\,f':(x+y)=dxf':(x+y)+dyf':(x+y)$$

and thus also

$$\left(\frac{dz}{dx}\right)=p=f':(x+y) \text{ and } \left(\frac{dz}{dy}\right)=q=f':(x+y)$$

and therefore $\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dy}\right)$ or $q=p$, entirely as the problem postulates.

### COROLLARY 1

**74.** Therefore whatever function of the quantity $x+y$ may be formed, that assumed for *z* will have the prescribed property, so that there shall be $\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dy}\right)$. But such a function we have indicated by the sign $f:(x+y)$, thus so that there shall be $z=f:(x+y)$.

### COROLLARY 2

**75.** Thus this solution may be referred to geometrically. With some curved line described on the axis either regular or irregular, if the abscissa is expressed by $x+y$, the applied line always will show a suitable value for the function *z*.

### COROLLARY 3

**76.** The universality of this solution elicited by integration is in agreement with this, since whatever kind of function $x+y$ we may have found for $z$, either continuous or discontinuous, certainly always satisfies the condition of that problem.

### SCHOLlUM 1

**77.** The basis of the solution depends on this principle, that the differential formula $pdu$ is unable to be integrated, unless the quantity *p* shall be a function of *u* or in turn *u* is a function of *p,* thus so that it may proceed with no other variables in the computation. Moreover, whatever kind the function *p* should be of *u,* the integration, unless actually shown, can still be considered always. For if *u* denotes the abscissa and *p* the applied line [*i. e.* the *y* co-ord.] of some curve either regular or irregular, by which account certainly any function of $u$ can be represented in the widest sense, the area $\int pdu$ of this curve is given by the value of the integral $\int pdu$, which again can be considered as a function of *u*; from which in turn any function of *u* draws out the nature of the formula of the integral $\int pdu$. But because any function of the quantity $x+y$ taken for $z$ satisfies the condition, so that in the differential $dz=pdx+qdy$ there becomes $p=q$ or $\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dy}\right)$, thus it is evident naturally, that it may not [even] be required to be illustrated by an example. For if there is put for example

$$z=aa+b(x+y)+(x+y)^2=aa+bx+by+xx+2xy+yy,$$

there becomes on differentiation

$$\left(\frac{dz}{dx}\right)=b+2x+2y \text{ and } \left(\frac{dz}{dy}\right)=b+2x+2y,$$

which values certainly are equal to each other.

**SCHOLIUM 2**

**78.** Since *z* shall be a function of the two variables *x* and *y* and there is put

$$dz = pdx + qdy ,$$

so that there shall be $\left(\frac{dz}{dx}\right) = p$ and $\left(\frac{dz}{dy}\right) = q$, in this chapter it has been proposed to solve questions of this kind, in which some equation between *p* and *q* will be prescribed, in which none of the remaining variables *x, y* and *z* is present. Therefore for some proposed equation between the two formulas *p* and *q* and constants it is required to investigate the nature of the function *z* of the two variables *x* and *y,* so that thence the condition agrees with that prescribed from the formulas there arises by differentiation $p = \left(\frac{dz}{dx}\right)$ and $q = \left(\frac{dz}{dy}\right)$. As indeed we have the treatment of the introductory preamble from the simplest example $p = q$, the solution of this also can only be made with the aid of the principle. But truly the same principle will suffice to be extended to the resolution of more general problems.

**PROBLEM 11**

**79.** *If z should be a function of the two variables x and y of the this kind, so that there becomes* $\alpha\left(\frac{dz}{dx}\right) + \beta\left(\frac{dz}{dy}\right) = \gamma$, *to define the nature of this function z in general.*

**SOLUTION**

On putting $dz = pdx + qdy$ it is required that there shall be $\alpha p + \beta q = \gamma$. Hence since there shall be $q = \frac{\gamma - \alpha p}{\beta}$, there will be

$$dz = pdx + \frac{(\gamma - \alpha p)}{\beta}dy \text{ or } dz = \frac{\gamma}{\beta}dy + \frac{p}{\beta}(\beta dx - \alpha dy),$$

as the integral formula is required to become. But since the part $\frac{\gamma}{\beta}dy$ naturally shall be integrable, the other part also by necessity shall be integrable, from which on putting $\beta x - \alpha y = u$, as the other part becomes $\frac{p}{\beta}du$, it is evident that *p* must be some function of *u* and thence also the integral to be produced will be a function of $u = \beta x - \alpha y$. Whereby we may put

$$\int p(\beta dx - \alpha dy) = f:(\beta x - \alpha y)$$

and there will be

$$z=\frac{\gamma}{\beta}y+\frac{1}{\beta}f:(\beta x-\alpha y)$$

or the nature of the equation *z* being determined shall be

$$\beta z=\gamma y+f:(\beta x-\alpha y)$$

with the sign $f:$ denoting some function either continuous or discontinuous of the formula of the attached $\beta x-\alpha y$. And on indicating the differential of the formula $f:u$ by $du\, f':u$ there will be

$$p=f':(\beta x-\alpha y) \text{ and } q=\frac{\gamma}{\beta}-\frac{\alpha}{\beta}f':(\beta x-\alpha y),$$

from which clearly it results that $\alpha p+\beta q=\gamma$.

**COROLLARY 1**

**80.** The very same solution is returned, if for *p* we may substitute the value of this $p=\frac{\gamma-\beta q}{\alpha}$ , from which there becomes

$$dz=\frac{\gamma}{\alpha}dx+\frac{q}{\alpha}(\alpha dy-\beta dx)$$

and hence in the same manner

$$z=\frac{\gamma x}{\alpha}+\frac{1}{\alpha}f:(\alpha y-\beta x)$$

For even if this form is seen to differ from the preceding, yet it is easily reduced to that on putting there

$$f:(\beta x-\alpha y)=\frac{\gamma(\beta x-\alpha y)}{\alpha}+\frac{\beta}{\alpha}\varphi:(\alpha y-\beta x),$$

which form certainly is a function of $\beta x-\alpha y$.

**COROLLARY 2**

**81.** Therefore if in the form $dz=pdx+qdy$ there must be $p+q=1$, on account of $\alpha=1$, $\beta=1$ and $\gamma=1$ the solution is reduced to this, so that there becomes $z=y+f:(x-y)$. Therefore with some curve constructed if for the abscissa $x-y$ there corresponds the applied line *v*, there will be $z=y+v$.

### SCHOLIUM

**82.** If another relation is proposed between $p$ and $q$, it is not allowed to obtain a solution by the same method, but it is appropriate for another principle to be used, the truth of which indeed is evident from the first elements of integral calculus. Evidently it is required to be observed that

$$\int pdx = px - \int xdp$$

and in a similar manner

$$\int qdy = qy - \int ydq,$$

thus so that, since there shall be

$$z = \int (pdx + qdy),$$

there shall become

$$z = px + qy - \int (xdp + ydq).$$

But how this principle shall be applied to the solution of questions of this kind, which are referred to in this chapter, will be shown in the following problems.

### PROBLEM 12

**83.** *If z should be a function of the two variables x and y of this kind, so that on putting* $dz = pdx + qdy$ *there becomes* $pq = 1$, *to define in general the nature of this function z.*

### SOLUTION

From the principle established before we may note to be

$$z = px + qy - \int (xdp + ydq).$$

Now since on account of $pq = 1$ there shall be $q = \frac{1}{p}$, then there shall be

$$z = px + \frac{y}{p} - \int \left( xdp - \frac{ydp}{pp} \right).$$

Therefore this form $\int \left( x - \frac{y}{pp} \right) dp$ must be integrable. But in general the formula $\int udp$ does not admit to integration, unless $u$ shall be a function of $p$; whereby in our case it is necessary that the quantity $x - \frac{y}{pp}$ shall be a function of $p$ only, from which also the integral $\int dp \left( x - \frac{y}{pp} \right)$ will be a function of $p$ only; which if it were indicated by $f:p$ and the differential of this by $dp\ f':p$, there will be

$$z = px + \frac{y}{p} - f:p \quad \text{and} \quad x - \frac{y}{pp} = f':p.$$

Whereby towards solving our problem a new variable $p$ must be introduced, from which since with each taken with $y$ the two remaining $x$ and $z$ may be determined. Evidently with the variable taken $p$ and some function of this $f:p$, and thence by differentiation with the derivative $f':p$ taken, in the first place

$$x = \frac{2y}{pp} + f':p$$

and thence there will be

$$z = \frac{2y}{p} + pf':p - f:p,$$

which is the solution of the general problem sought.

## COROLLARY 1

**84.** Therefore here the function sought $z$ cannot involve the variables $x$ and $y$ explicitly, because the quantity $p$ in general cannot be defined by $x$ and $y$ from the equation $x - \frac{2y}{pp} = f':p$.

## COROLLARY 2

**85.** Truly a suitable and complete a solution may be considered from nothing less, because on introducing the new variable $p$ with the two $y$ and $p$ themselves not in turn dependent, both the remaining $x$ and $z$ are defined.

## COROLLARY 3

**86**. If we assume $f':p = \alpha + \frac{\beta}{pp}$, there will be

$$f:p = \alpha p - \frac{\beta}{p} \text{ and } x - \alpha = \frac{\beta + y}{pp} ,$$

hence $p = \sqrt{\frac{\beta+y}{x-\alpha}}$, from which the function sought $z$ thus may be had

$$z = \frac{2y\sqrt{(x-\alpha)}}{\sqrt{(\beta+y)}} + \frac{\alpha y+\beta x}{\sqrt{(x-\alpha)(\beta+y)}} - \frac{\alpha y-\beta x+2\alpha\beta}{\sqrt{(x-\alpha)(\beta+y)}}$$

or $z = 2\sqrt{(x-\alpha)(y+\beta)}$, which is a particular solution, and the simplest shall be $z = 2\sqrt{xy}$.

## SCHOLIUM 1

**87.** Just as the solution of this problem has been deduced from the other principle, thus also the form of the solution disagrees with the preceding, because here the equation between *x*, *y* and *z* may not be allowed to be shown explicitly, but a new variable *p* is introduced. Therefore since before a single equation might put in place the solution between the three variables *x, y* and *z,* now with the new variable *p* added the solution demands a twin equation between these four variables, and thus in our case we find

$$z = px + \frac{y}{p} - f{:}p \text{ and } x - \frac{y}{pp} = f'{:}p$$

with $d.f{:}p = dp\, f'{:}p$ arising, where the indefinite sign of the function $f{:}$ , because it allows discontinuous functions also, makes good the generality of the solution. But if hence the letter *p* be allowed to be eliminated, an equation established between *x, y* and *z* is obtained ; but this elimination succeeds only as often as the function for $f{:}\,p$ is taken as an algebraic function of *p*, but in general in no way can it be expected to happen. Truly with nothing less than with the help of a curve assumed as you please the problem can be constructed; for on taking some curve either regular or irregular the abscissa is put $= p$ and there shall be the applied line $f'{:}p = r$ *;* then there will be $f{:}p = \int rdp$ the area of this curve; which if it is called $= s$, the two equations

$$x - \frac{y}{pp} = r \quad \text{and} \quad z = px + \frac{y}{p} - s$$

give the complete solution to the problem. Evidently on taking some value for *x* there will be $y = pp(x - r)$ and hence there becomes $z = 2px - pr - s$ , in which solution nothing can be desired regarding an example.

Hence it is perhaps apparent also possible, that two new variables should be introduced and then the solution may be contained in three equations ; nor also then will it lack practical use.

## SCHOLlUM 2

**88.** Since it is required for the formula $dz = pdx + qdy$ , that there shall be $pq = 1$, on introducing the indefinite angle $\varphi$ another more precise solution can be elicited. For on putting $p = \text{tang.}\varphi$ there will be $q = \cot.\varphi$ and on account of $dz = dx\,\text{tang.}\varphi + dy\cot.\varphi$ there becomes by the reduction indicated above

$$z = x\text{tang.}\varphi + y\cot.\varphi - \int d\varphi\left(\frac{x}{cos^2\varphi} - \frac{y}{sin^2\varphi}\right),$$

from which it is apparent that the formula $\frac{x}{cos^2\varphi}-\frac{y}{sin^2\varphi}$ must be a function of $\varphi$; which if there is put $f':\varphi$ and the formula of the integral $\int d\varphi f':\varphi = f:\varphi$, then the two equations containing the solution will be

$$\frac{x}{cos^2\varphi}-\frac{y}{sin^2\varphi}=f':\varphi \text{ et } z=x\text{tang.}\varphi+y\text{cot.}\varphi-f:\varphi,$$

from which now it is possible to eliminate *x* or *y* as it pleases. Therefore also we are able to eliminate each and both *x* and *y* thus are expressed by the two variables *z* and $\varphi$

$$x=\tfrac{1}{2}z\text{cot.}\varphi \;\; +\tfrac{1}{2}\text{cot.}\varphi\cdot f:\varphi+\tfrac{1}{2}\text{cos.}^2\varphi\cdot f':\varphi,$$
$$y=\tfrac{1}{2}z\text{tang.}\varphi+\tfrac{1}{2}\text{tang.}\varphi{:}f'.\varphi-\tfrac{1}{2}\text{sin.}^2\varphi\cdot f':\varphi.$$

But if hence therefore the differentials are taken and there is put $dy = 0$, from the latter the relation will be given between $dz$ and $d\varphi$, from which if the value of $d\varphi$ is substituted into the first, by necessity there is produced $dz=dx\,\text{tang.}\varphi$; moreover in a similar manner if there is put $dx=0$, from the other there arises $dz=dy\,\text{cot.}\varphi$.

**PROBLEM 13**

**89.** *If z should be a function of the two variables x and y of this kind, so that on putting* $dz=pdx+qdy$ *there becomes* $pp+qq=1$, *to investigate the nature of this function z in general.*

**SOLUTION**

Since by reduction there becomes

$$z=px+qy-\int\left(xdp+ydq\right),$$

in order that we may avoid irrationality, we may put

$$p=\tfrac{1-rr}{1+rr} \quad \text{and} \quad q=\tfrac{2r}{1+rr},$$

if indeed there shall be hence $pp+qq=1$. But there shall be

$$dp=\tfrac{-4rdr}{(1+rr)^2} \text{ and } dq=\tfrac{2dr(1-rr)}{(1+rr)^2}$$

and hence there becomes

$$z=\frac{(1-rr)x+2ry}{1+rr}+2\int\frac{2xrdr-ydr(1-rr)}{(1+rr)^2};$$

which form of the integral since it shall be a function of $r$, that is put in place $=f:r$ and the differential of this $=drf':r$, from which we will obtain

$$\frac{2xr-y(1-rr)}{(1+rr)^2}=f':r \text{ and } z=\frac{(1-rr)x+2ry}{1+rr}+2f:r.$$

From which id we elicit

$$x=\frac{(1-rr)y}{2r}+\frac{(1+rr)^2}{2r}f':r,$$

there will be

$$z=\frac{(1+rr)y}{2r}+\frac{1-r^4}{2r}f':r+2f:r$$

## COROLLARY 1

**90.** If we are not afraid of irrationality, on account of

$$q=\sqrt{(1-pp)} \text{ and } dq=\frac{-pdp}{\sqrt{(1-pp)}}$$

there will be

$$z=px+y\sqrt{(1-pp)}-\int dp\left(x-\frac{py}{\sqrt{(1-pp)}}\right).$$

Hence on putting $z=px+y\sqrt{(1-pp)}-f:p$ there will be $x-\frac{py}{\sqrt{(1-pp)}}=f':p$.

## COROLLARY 2

**91.** The simplest solution without doubt will be produced on assuming $f:p=0$; from which since there shall be $x=\frac{py}{\sqrt{(1-pp)}}$, will be

$$p=\frac{x}{\sqrt{(xx+yy)}} \quad \text{and} \quad \sqrt{(1-pp)}=\frac{y}{\sqrt{(xx+yy)}}$$

and hence

$$z=\frac{xx+yy}{\sqrt{(xx+yy)}}=\sqrt{(xx+yy)}.$$

From which value there becomes

$$\left(\frac{dz}{dx}\right)=p=\frac{x}{\sqrt{(xx+yy)}} \quad \text{and} \quad \left(\frac{dz}{dy}\right)=q=\frac{y}{\sqrt{(xx+yy)}}$$

and thus $pp+qq=1$.

## COROLLARY 3

**92.** If we put $p=\sin.\varphi$, there will be $q=\cos.\varphi$ and hence

$$z=x\sin.\varphi+y\cos.\varphi-\int d\varphi\left(x\cos.\varphi-y\sin.\varphi\right);$$

there will be this integral $=f:\varphi$ and the differential of this $d\varphi f':\varphi$. From which we will have

$$z=x\sin.\varphi+y\cos.\varphi-f:\varphi \text{ and } x\cos.\varphi-y\sin.\varphi=f':\varphi.$$

## PROBLEM 14

**93.** *If z should be a function of this kind of x and y, so that on putting* $dz=pdx+qdy$ *the quantity q is equal to a given function of* $p$, *to investigate in general the nature of this function z.*

## SOLUTION

Since $q$ shall be a given function of $p$, there is put $dq=rdp$; also $r$ will be a given function of $p$. Therefore our general equation providing the solution can adopt this form

$$z=px+qy-\int dp\left(x+ry\right),$$

from which it is apparent that the integral $\int dp\left(x+ry\right)$ becomes a function of $p$; which if generally set out by $f:p$ and the differential of this by $dpf':p$, we will have

$$z=px+qy-f:p \text{ and } x+ry=f':p,$$

which two equations are including the most general solution of the problem, if indeed $f:p$ can denote some function of $p$ either continuous or discontinuous.

## COROLLARY 1

**94.** Since $q$ shall be a given function of $p$ and thence $r=\frac{dq}{dp}$, if an indefinite function of $p$ is put in place $f:p=P$, on account of $f':p=\frac{dP}{dp}$ the solution will be contained by these equations

$$z=px+qy-P \text{ et } xdp+ydq=dP.$$

## COROLLARY 2

**95.** If we may use some curve in the construction, in which, if the abscissa is taken $= p$, the applied line shall be $= f':p$, the area of this curve will give the value of $f:p$. But if the applied line is indicated by $f:p$, then $f':p$ expresses the tangent of the angle that the tangent of the curve makes with the axis.

## SCHOLIUM

**96.** Therefore some curve can be described in two ways as it pleases, either it shall be continuous or contained analytically by some equation, or drawn by hand freely and delineated in some manner, able to be used in the construction of the problem. For either the abscissa can be indicated by $p$ with the applied line able to be taken according to $f:p$, or not said to be expressed easily according to $f':p$, either shall become more useful in practice. But where actual problems of this kind occur, the remaining circumstances usually determine the solution, from which for whatever case the construction best suited is easily deduced. But mechanics problems are always demanding this part of the integral calculus according to differential formulas deduced of the second or higher orders, the resolution of which cannot indeed be undertaken before it is seen, how the method for differential formulas of the first order should be revealed.

Indeed thus far it is clear how to resolve proposed problems ; but now when the prescribed condition defines a relation of the formulas $\left(\frac{dz}{dx}\right)$ and $\left(\frac{dz}{dy}\right)$ by the remaining variables $x$, $y$ and $z$, the work in general does not advance further, unless a prescribed relation connects only a single variable with the two differential formulas.

## CAPUT III

# DE RESOLUTIONE AEQUATIONUM QUIBUS BINARUM FORMULARUM DIFFERENTIALIUM ALTERA PER ALTERAM UTCUNQUE DATUR

### PROBLEMA 10

**73.** *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut formulae differentiales* $\left(\frac{dz}{dx}\right)$ *et* $\left(\frac{dz}{dy}\right)$ *inter se fiant aequales, indolem istius functionis in genere determinare.*

### SOLUTIO

Ponatur $\left(\frac{dz}{dx}\right)=p$ et $\left(\frac{dz}{dy}\right)=q$, ut sit $dz=pdx+qdy$ haecque formula $pdx+qdy$ integrationem sponte admittat. Quoniam igitur requiritur, ut sit $q=p$, erit $dz=p(dx+dy)$ et posito $x+y=u$ fiet $dz=pdu$; quae formula cum debeat esse per se integrabilis, necesse est, ut *p* sit functio quantitatis variabilis *u* nullam praeterea aliam variabilem involvens; hincque integrando ipsa quantitas $z=\int pdu$ aequabitur functioni ipsius *u* seu prodibit $z=f:u$, quae functio omnino arbitrio nostro relinquitur, ita ut pro *z* functio quaecunque ipsius *u* sive continua sive etiam discontinua assumta problemati satisfaciat. Quare cum sit $u=x+y$, erit pro solutione nostri problematis $z=f:(x+y)$. Quae forma, quo facilius appareat, quomodo conditioni praescriptae satisfaciat, sit $d.f:u=du\,f':u$ ideoque ob $u=x+y$ erit

$$dz=(dx+dy)\,f':(x+y)=dxf':(x+y)+dyf':(x+y)$$

ideoque et

$$\left(\frac{dz}{dx}\right)=p=f':(x+y)\ \text{ et }\ \left(\frac{dz}{dy}\right)=q=f':(x+y)$$

ac propterea $\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dy}\right)$ seu $q=p$, omnino uti problema postulat.

### COROLLARlUM 1

**74.** Quaecunque ergo functio quantitatis $x+y$ formetur, ea pro *z* assumta praescriptam habebit proprietatem, ut sit $\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dy}\right)$. Talem autem functionen indicamus signo $f:(x+y)$, ita ut sit $z=f:(x+y)$.

### COROLLARlUM 2

**75.** Geometrice haec solutio ita referri potest. Descripta super axe linea curva quacunque sive regulari sive irregulari si abscissa exprimatur per $x+y$, applicata semper idoneum valorem pro functione *z* exhibebit.

### COROLLARlUM 3

**76.** Universalitas huius solutionis per integrationem erutae in hoc consistit, quod quantitatis $x+y$ functionem qualemcunque sive continuam sive etiam discontinuam pro *z* invenerimus, quippe quae conditioni problematis semper satisfacit.

### SCHOLlON 1

**77.** Fundamentum solutionis hoc nititur principio, quod formula differentialis *pdu* integrabilis esse nequeat, nisi quantitas *p* sit functio ipsius *u* vel vicissim *u* functio ipsius *p*, ita ut nulla alia variabilis in computum ingrediatur. Quin etiam, qualiscunque fuerit *p* functio ipsius *u*, integrale, nisi actu exhiberi, semper tamen concipi potest. Si enim *u* denotet abscissam et *p* applicatam curvae cuiuscunque sive regularis sive irregularis, qua ratione utique functio quaecunque ipsius *u* in sensu latissimo repraesentari potest, eius curvae area $\int pdu$ praebet valorem formulae integralis $\int pdu$, quae iterum ut functio ipsius *u* spectari potest; ex quo vicissim functio quaecunque ipsius *u* naturam formulae integralis $\int pdu$ exhaurit. Quod autem functio quaecunque quantitatis $x+y$ pro *z* assumta satisfaciat conditioni, ut in differentiali $dz=pdx+qdy$ fiat $p=q$ seu $\left(\frac{dz}{dx}\right)=\left(\frac{dz}{dy}\right)$, ita per se est perspicuum, ut illustratione per exempla non egeat. Si enim verbi gratia ponatur

$$z=aa+b(x+y)+(x+y)^2=aa+bx+by+xx+2xy+yy,$$

erit differentiando

$$\left(\frac{dz}{dx}\right)=b+2x+2y \text{ et } \left(\frac{dz}{dy}\right)=b+2x+2y,$$

qui valores inter se utique sunt aequales.

### SCHOLION 2

**78.** Cum *z* sit functio binarum variabilium *x* et *y* ac ponatur

$$dz = pdx + qdy\,,$$

ut sit $\left(\frac{dz}{dx}\right) = p$ et $\left(\frac{dz}{dy}\right) = q$, in hoc capite eiusmodi quaestiones evolvere est propositum, in quibus aequatio quaecunque inter $p$ et $q$ praescribitur, in quam reliquarum variabilium $x$, $y$ et $z$ nulla ingrediatur. Proposita ergo aequatione quacunque inter binas formulas $p$ et $q$ et constantes quaeri oportet indolem functionis $z$ binarum variabilium $x$ et $y$, ut formulis inde per differentiationem natis $p = \left(\frac{dz}{dx}\right)$ et $q = \left(\frac{dz}{dy}\right)$ praescripta illa conditio conveniat. Quam tractationem quidem exorsi sumus ab exemplo simplicissimo $p = q$, cuius solutio etiam ope principii modo expositi confici potest. At vero idem principium sufficit problemati sequenti latius patenti resolvendo.

## PROBLEMA 11

**79.** *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut fiat* $\alpha\left(\frac{dz}{dx}\right) + \beta\left(\frac{dz}{dy}\right) = \gamma$, *indolem istius functionis z in genere definire.*

## SOLUTIO

Posito $dz = pdx + qdy$ requiritur, ut sit $\alpha p + \beta q = \gamma$. Hinc cum sit $q = \frac{\gamma - \alpha p}{\beta}$, erit

$$dz = pdx + \frac{(\gamma - \alpha p)}{\beta}dy \text{ seu } dz = \frac{\gamma}{\beta}dy + \frac{p}{\beta}\left(\beta dx - \alpha dy\right),$$

quam formulam integrabilem esse oportet. Cum autem pars $\frac{\gamma}{\beta}dy$ per se sit integrabilis, altera pars etiam integrabilis sit necesse est, unde posito $\beta x - \alpha y = u$, ut altera pars fiat $\frac{p}{\beta}du$, evidens est $p$ functionem esse debere ipsius $u$ indeque etiam integrale proditurum esse functionem ipsius $u = \beta x - \alpha y$.
Quare ponamus

$$\int p\left(\beta dx - \alpha dy\right) = f{:}\left(\beta x - \alpha y\right)$$

eritque

$$z = \frac{\gamma}{\beta}y + \frac{1}{\beta}f{:}\left(\beta x - \alpha y\right)$$

seu aequatio quaesita indolem functionis $z$ determinans erit

$$\beta z = \gamma y + f{:}\left(\beta x - \alpha y\right)$$

denotante signo $f{:}$ functionem quamcunque sive continuam sive discontinuam formulae suffixae $\beta x - \alpha y$. Atque indicando formulae $f{:}u$ differentiale per $du\, f'{:}u$ erit

$$p = f':(\beta x - \alpha y) \text{ et } q = \tfrac{\gamma}{\beta} - \tfrac{\alpha}{\beta} f':(\beta x - \alpha y),$$

unde manifesto resultat $\alpha p + \beta q = \gamma$.

**COROLLARIUM 1**

**80.** Eodem solutio redit, si pro *p* eius valorem $p = \frac{\gamma - \beta q}{\alpha}$ substituamus, unde fit

$$dz = \tfrac{\gamma}{\alpha} dx + \tfrac{q}{\alpha}(\alpha dy - \beta dx)$$

hincque eodem modo

$$z = \tfrac{\gamma x}{\alpha} + \tfrac{1}{\alpha} f:(\alpha y - \beta x)$$

Etsi enim haec forma a praecedente differre videtur, tamen facile eo reducitur ponendo ibi

$$f:(\beta x - \alpha y) = \frac{\gamma(\beta x - \alpha y)}{\alpha} + \frac{\beta}{\alpha}\varphi:(\alpha y - \beta x),$$

quae forma utique est functio ipsius $\beta x - \alpha y$.

**COROLLARIUM 2**

**81.** Si ergo in forma $dz = pdx + qdy$ debeat esse $p + q = 1$, ob $\alpha = 1$, $\beta = 1$ et $\gamma = 1$ solutio huc redit, ut fiat

$$z = y + f:(x - y).$$

Constructa ergo curva quacunque si abscissae $x - y$ respondeat applicata *v*, erit $z = y + v$.

**SCHOLION**

**82.** Si alia proponatur relatio inter *p* et *q*, eadem methodo solutionem obtinere non licet, sed alio principio uti convenit, cuius quidem veritas ex primis calculi integralis elementis est manifesta. Notari scilicet oportet esse

$$\int pdx = px - \int xdp$$

similique modo

$$\int qdy = qy - \int ydq,$$

ita ut, cum sit

$$z = \int(pdx + qdy),$$

futurum sit

$$z = px + qy - \int (xdp + ydq).$$

Quomodo autem hoc principium ad solutionem huiusmodi quaestionum, quae ad hoc caput sint referendae, applicandum sit, in sequentibus problematibus docebitur.

## PROBLEMA 12

**83.** *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut posito* $dz = pdx + qdy$ *fiat* $pq = 1$, *indolem istius functionis z in genere definire.*

## SOLUTIO

Ex principio ante stabilito notemus fore

$$z = px + qy - \int (xdp + ydq).$$

Cum iam ob $pq = 1$ sit $q = \frac{1}{p}$, erit

$$z = px + \frac{y}{p} - \int \left( xdp - \frac{ydp}{pp} \right).$$

Integrabilis ergo esse debet haec forma $\int \left( x - \frac{y}{pp} \right) dp$. At in genere formula $\int udp$ integrationem non admittit, nisi sit *u* functio ipsius *p;* quare in nostro casu necesse est sit quantitas $x - \frac{y}{pp}$ functio ipsius *p* tantum, unde etiam integrale $\int dp \left( x - \frac{y}{pp} \right)$ erit functio ipsius *p* tantum; quae si indicetur per $f:p$ eiusque differentiale per $dp\, f':p$, erit

$$z = px + \frac{y}{p} - f:p \quad \text{et} \quad x - \frac{y}{pp} = f':p.$$

Quare ad problema nostrum solvendum nova variabilis *p* introduci debet, ex qua cum altera *y* coniuncta binae reliquae *x* et *z* determinentur. Sumta scilicet variabili *p* eiusque functione quacunque $f:p$ indeque per differentiationem derivata $f':p$ capiatur primo

$$x = \frac{2y}{pp} + f':p$$

indeque erit

$$z = \frac{2y}{p} + pf':p - f:p,$$

quae est solutio problematis quaesita generalis.

## COROLLARIUM 1

**84.** Hic igitur functio quaesita $z$ per ipsas variabiles $x$ et $y$ explicite evolvi nequit, propterea quod quantitatem $p$ ex aequatione $x-\frac{2y}{pp}=f':p$ in genere per $x$ et $y$ definire non licet.

### COROLLARIUM 2

**85.** Nihilo vero minus solutio pro idonea et completa est habenda, quoniam introducendo novam variabilem $p$ ex binis $y$ et $p$ a se invicem non pendentibus ambae reliquae $x$ et $z$ definiuntur.

### COROLLARIUM 3

**86**. Si sumamus $f':p=\alpha+\frac{\beta}{pp}$, erit

$$f:p=\alpha p-\frac{\beta}{p} \text{ et } x-\alpha=\frac{\beta+y}{pp},$$

hinc $p=\sqrt{\frac{\beta+y}{x-\alpha}}$. unde functio quaesita $z$ ita se habebit

$$z=\frac{2y\sqrt{(x-\alpha)}}{\sqrt{(\beta+y)}}+\frac{\alpha y+\beta x}{\sqrt{(x-\alpha)(\beta+y)}}-\frac{\alpha y-\beta x+2\alpha\beta}{\sqrt{(x-\alpha)(\beta+y)}}$$

seu $z=2\sqrt{(x-\alpha)(y+\beta)}$, quae est solutio particularis, et simplicissima est $z=2\sqrt{xy}$.

### SCHOLION 1

**87.** Quemadmodum solutio huius problematis ex alio principio est deducta, ita etiam forma solutionis a praecedentibus discrepat, quod hic aequationem inter $x$, $y$ et $z$ explicitam exhibere non liceat, sed nova variabilis $p$ introducatur. Cum igitur ante una aequatio inter ternas variabiles $x$, $y$ et $z$ solutionem continuisset, nunc accedente nova variabili $p$ solutio geminam aequationem inter has quatuor variabiles postulat sicque pro nostro casu invenimus

$$z=px+\frac{y}{p}-f:p \text{ et } x-\frac{y}{pp}=f':p$$

existente $d.f:p=dp\, f':p$, ubi functionis signum indefinitum $f:$, quod etiam functiones discontinuas admittit, universalitatem solutionis praestat. Quodsi hinc litteram $p$ eliminare liceret, aequatio evoluta inter $x$, $y$ et $z$ obtineretur; haec autem eliminatio, succedit, quoties pro $f:p$ functio algebraica ipsius $p$ assumitur, in genere autem nullo modo sperari potest. Nihilo vero minus ope curvae pro lubitu assumtae problema construi potest; sumta enim curva quacunque sive regulari sive irregulari ponatur abscissa $=p$

sitque applicata $f':p = r$; erit $f:p = \int rdp$ area eius curvae; quae si dicatur = $s$, aequationes binae

$$x - \tfrac{y}{pp} = r \quad \text{et} \quad z = px + \tfrac{y}{p} - s$$

solutionem completam problematis praebebunt. Scilicet sumto pro $x$ valore quocunque erit $y = pp(x - r)$ hincque fit $z = 2px - pr - s$, in qua solutione nihil ad praxin spectans desiderari potest.

Hinc patet etiam fortasse fieri posse, ut duae novae variabiles sint introducendae ac tum solutio tribus aequationibus contineatur; neque etiam tum quicquam deerit ad usum practicum.

## SCHOLlON 2

**88.** Cum pro formula $dz = pdx + qdy$ requiratur, ut sit $pq = 1$, introducendo angulum indefinitum $\varphi$ alia solutio concinnior elici potest. Posito enim $p = \text{tang.}\varphi$ erit $q = \cot.\varphi$ et ob $dz = dx\,\text{tang.}\varphi + dy\cot.\varphi$ fiet per reductionem supra indicatam

$$z = x\text{tang.}\varphi + y\cot.\varphi - \int d\varphi\left(\tfrac{x}{cos^2\varphi} - \tfrac{y}{sin^2\varphi}\right),$$

unde patet formulam $\frac{x}{cos^2\varphi} - \frac{y}{sin^2\varphi}$ esse debere functionem ipsius $\varphi$; quae si ponatur $f':\varphi$ et formula integralis $\int d\varphi f':\varphi = f:\varphi$, binae aequationes solutionem continentes erunt

$$\tfrac{x}{cos^2\varphi} - \tfrac{y}{sin^2\varphi} = f':\varphi \ \text{ et } \ z = x\text{tang.}\varphi + y\cot.\varphi - f:\varphi,$$

unde iam pro lubitu $x$ vel $y$ eliminare licet. Quin etiam utramque eliminare possumus ac per binas variabiles $z$ et $\varphi$ binae reliquae $x$ et $y$ ita exprimentur

$$x = \tfrac{1}{2}z\cot.\varphi \ + \tfrac{1}{2}\cot.\varphi \cdot f:\varphi + \tfrac{1}{2}\cos.^2\varphi \cdot f':\varphi,$$
$$y = \tfrac{1}{2}z\text{tang.}\varphi + \tfrac{1}{2}\text{tang.}\varphi : f'.\varphi - \tfrac{1}{2}\sin.^2\varphi \cdot f':\varphi.$$

Quodsi igitur hinc differentialia capiantur ac ponatur $dy = 0$, ex posteriori dabitur relatio inter $dz$ et $d\varphi$, unde, si ipsius $d\varphi$ valor in priori substituatur, necesse est prodeat $dz = dx\,\text{tang.}\varphi$; simili autem modo si ponatur $dx = 0$, ex altera orietur $dz = dy\cot.\varphi$.

## PROBLEMA 13

**89.** *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut posito* $dz = pdx + qdy$ *fiat* $pp + qq = 1$, *indolem istius functionis z in genere investigare.*

## SOLUTIO

Cum per reductionem fiat

$$z = px + qy - \int (xdp + ydq),$$

ut irrationalia evitemus, ponamus

$$p = \tfrac{1-rr}{1+rr} \quad \text{et} \quad q = \tfrac{2r}{1+rr},$$

siquidem hinc fit $pp + qq = 1$. Erit autem

$$dp = \frac{-4rdr}{(1+rr)^2} \quad \text{et} \quad dq = \frac{2dr(1-rr)}{(1+rr)^2}$$

hincque fit

$$z = \frac{(1-rr)x + 2ry}{1+rr} + 2\int \frac{2xrdr - ydr(1-rr)}{(1+rr)^2};$$

quae forma integralis cum sit functio ipsius *r*, statuatur ea $= f:r$ eiusque differentiale $= drf':r$, ex quo obtinebimus

$$\frac{2xr - y(1-rr)}{(1+rr)^2} = f':r \text{ et } z = \frac{(1-rr)x + 2ry}{1+rr} + 2f:r.$$

Unde si eliciamus

$$x = \frac{(1-rr)y}{2r} + \frac{(1+rr)^2}{2r} f':r,$$

erit

$$z = \frac{(1+rr)y}{2r} + \frac{1-r^4}{2r} f':r + 2f:r$$

## COROLLARIUM 1

**90.** Si irrationalitatem non pertimescamus, ob

$$q = \sqrt{(1-pp)} \text{ et } dq = \frac{-pdp}{\sqrt{(1-pp)}}$$

erit

$$p = \frac{x}{\sqrt{(xx+yy)}} \quad \text{et} \quad \sqrt{(1-pp)} = \frac{y}{\sqrt{(xx+yy)}}$$

hincque

$$z = px + y\sqrt{(1-pp)} - \int dp\left(x - \frac{py}{\sqrt{(1-pp)}}\right).$$

Posito ergo $z = px + y\sqrt{(1-pp)} - f:p$ erit $x - \frac{py}{\sqrt{(1-pp)}} = f':p$.

### COROLLARIUM 2

**91.** Solutio simplicissima sine dubio prodit sumendo $f:p = 0$; unde cum sit $x = \frac{py}{\sqrt{(1-pp)}}$, erit

$$p = \frac{x}{\sqrt{(xx+yy)}} \quad \text{et} \quad \sqrt{(1-pp)} = \frac{y}{\sqrt{(xx+yy)}}$$

hincque

$$z = \frac{xx+yy}{\sqrt{(xx+yy)}} = \sqrt{(xx+yy)}.$$

Ex quo valore fit

$$\left(\frac{dz}{dx}\right) = p = \frac{x}{\sqrt{(xx+yy)}} \quad \text{et} \quad \left(\frac{dz}{dy}\right) = q = \frac{y}{\sqrt{(xx+yy)}}$$

ideoque $pp + qq = 1$.

### COROLLARIUM 3

**92.** Si ponamus $p = \sin.\varphi$, erit $q = \cos.\varphi$ hincque

$$z = x\sin.\varphi + y\cos.\varphi - \int d\varphi\left(x\cos.\varphi - y\sin.\varphi\right);$$

erit hoc integrale $= f:\varphi$ eiusque differentiale $d\varphi f':\varphi$. Ex quo habebimus

$$z = x\sin.\varphi + y\cos.\varphi - f:\varphi \quad \text{et} \quad x\cos.\varphi - y\sin.\varphi = f':\varphi.$$

### PROBLEMA 14

**93.** *Si z eiusmodi esse debeat functio binarum variabilium x et y, ut posito $dz = pdx + qdy$ quantitas q aequetur functioni datae ipsius p, indolem huius functionis z in genere investigare.*

### SOLUTIO

Cum $q$ sit functio data ipsius $p$, ponatur $dq = rdp$; erit etiam $r$ functio data ipsius $p$. Aequatio ergo nostra generalis solutionem suppeditans induet hanc formam

$$z = px + qy - \int dp(x + ry),$$

unde patet integrale $\int dp(x+ry)$ fore functionem ipsius *p;* quae si generatim per $f:p$ exponatur eiusque differentiale per $dpf':p$, habebimus

$$z = px + qy - f:p \text{ et } x + ry = f':p,$$

quae duae aequationes solutionem problematis universalissime complectuntur, siquidem $f:p$ functionem quamcunque ipsius *p* sive continuam sive discontinuam denotare potest.

### COROLLARIUM 1

**94.** Cum sit *q* functio data ipsius *p* indeque $r = \frac{dq}{dp}$, si functio indefinita ipsius *p* ponatur $f:p = P$, ob $f':p = \frac{dP}{dp}$ solutio his aequationibus continebitur

$$z = px + qy - P \text{ et } xdp + ydq = dP.$$

### COROLLARIUM 2

**95.** Si ad constructionem utamur curva quacunque, in qua, si abscissa capiatur $= p$, applicata sit $= f':p$, area eius curvae dabit valorem ipsius $f:p$. Sin autem applicata indicetur per $f:p$, tum $f':p$ exprimet tangentem anguli, quem tangens curvae faciet cum axe.

### SCHOLION

**96.** Duplici ergo modo curva quaecunque ad libitum descripta, sive sit continua seu aequatione quapiam analytica contenta sive libero manus ductu utcunque delineata, ad constructionem problematis adhiberi potest. Vel enim abscissa per *p* indicata applicata sumi potest ad $f:p$ vel ad $f':p$ exprimendum nec facile dici potest, utrum ad praxin commodius sit futurum. Ubi autem huiusmodi problemata realia occurrunt, reliquae circumstantiae solutionem determinare solent, unde pro quovis casu constructio maxime idonea facile colligetur. Problemata autem mechanica hanc calculi integralis partem postulantia semper ad formulas differentiales secundi altiorumque ordinum deducunt, quarum resolutio ne suscipi quidem posse ante videtur, quam methodus pro formulis differentialibus primi gradus fuerit patefacta.

Hactenus quidem problemata proposita absolute resolvere licuit; nunc autem quando conditio praescripta relationem formularum $\left(\frac{dz}{dx}\right)$ et $\left(\frac{dz}{dy}\right)$ per reliquas variabiles *x, y* et *z* definit, negotium in genere non amplius succedit, nisi relatio praescripta unicam tantum variabilem cum binis formulis differentialibus coniungat.