CHAPTER II

CONCERNING THE RESOLUTION OF EQUATIONS IN WHICH EITHER DIFFERENTIAL FORMULA IS GIVEN BY SOME FINITE QUANTITY

PROBLEM 4a [repeated number]

33. *To investigate the nature of a function z of the two variables x and y, so that the differential formula* $\left(\frac{dz}{dx}\right) = p$ *shall be a constant quantity* $= a$.

SOLUTION

Hence on putting $dz = pdx + qdy$ that nature of the function z is sought, so that there shall be $p = a$ or $dz = adx + qdy$; in order that this may be found y is assumed constant ; and there will be $dz = adx$ and on integrating $z = ax$ + Const., where it is required to be noted that this constant can involve the quantity y everywhere. Whereby so that we may show the general solution, there will be $z = ax + f{:}y$ with $f{:}y$ denoting some function of y, which by itself in no manner can be determined , but depends completely on our choice. Which differential also in turn is indicated ; for if we indicate the differential of this function $f{:}y$ by $dyf'{:}y$, there will be everywhere $dz = adx + dyf'{:}y$ and thus $\left(\frac{dz}{dx}\right) = a$, in short as the question demands ; from which it is apparent in this case that the other differential formula $q = \left(\frac{dz}{dy}\right)$ is equal to a function of y only, since there shall be $q = f'{:}y$.

[Recall from the previous chapter that a derivative in brackets is a partial derivative differentiated with respect to that variable, with the other variables considered as constants; note also Euler's notation for an arbitrary function of x at this time $f{:}x$. See Scholium 2 below relating the usage of this to the shape of a stretched string.]

COROLLARY 1

34. Therefore if a function z of this kind of the two variables x and y is sought, so that there shall be $\left(\frac{dz}{dx}\right) = a$, then it will be $z = ax + f{:}y$ and the other differential formula $\left(\frac{dz}{dy}\right)$ by necessity is equal to a function of y only.

COROLLARY 2

35. If such a function is required, so that there shall be $\left(\frac{dz}{dx}\right)=0$, by necessity that will be a function of *y* only or the quantity *x* plainly is not involved ; for since from the variation of *x* no change ought to be apparent, evidently this quantity *x* is not present in the determination of this also.

COROLLARY 3

36. Hence it is apparent also that the differential equation $dz = adx + qdy$ cannot be valid, unless *q* shall be a function of *y* only. Which characteristic has been stated above ; indeed with an equation reduced to this form $adx + qdy - dz = 0$ on account of $P = a$, $Q = q$ and $R = -1$, there will be

$$L = \left(\frac{dq}{dz}\right),\ M = 0 \text{ and } N = -\left(\frac{dq}{dx}\right)$$

and thus the validity demands that there shall be $a\left(\frac{dq}{dz}\right)+\left(\frac{dq}{dx}\right)=0$. But by hypothesis *q* does not depend on *z*, from which on account of $\left(\frac{dq}{dz}\right)=0$ there will be $\left(\frac{dq}{dx}\right)=0$, and thus also *q* does not depend on *x*.

SCHOLIUM 1

37. From what has emerged it is clear enough that the operation, by which we have determined the function *z*, is in truth integration, by which as in common integration some indeterminate is introduced. Evidently here the quantity advanced is some function of *y*, the nature of which cannot be determined from itself in any way; that also is permitted to be considered, so that some curve is described, if the abscissae of this may be indicated by *y*, the applied lines [read as 'coordinates'] show a function of some kind of *y*. Nor truly is there a need, that this curve shall be regular and described by some equation; with any curve drawn freely by hand performing the same effect, even if it should be greatly irregular and put together from many parts of different curves. Irregular functions of this kind are allowed to be called discontinuous or held together by discontinuities [lit. *by ties destitute of continuation*] ; initially from which this occurs which is noteworthy, since integrations of the first kind besides do not allow other continuous functions, here also discontinuous functions may enter into the calculation, as has been considered by many distinguished Geometers, to be thus opposed to the principles of the calculus. Now in this following book on the treatment of the principles of integration, in this a particularly strong argument is advanced that discontinuous functions shall be allowed also; by which , from this as if new calculus, the limits of analysis are agreed to be greatly advanced.

SCHOLIUM 2

38. Hence just as in common integrations, the arbitrary constant introduced always is determined from the nature of the problem, and the solution of which thus may be produced; thus here also from the nature of the problem, the solution of which is resolved by integration, the nature of the arbitrary functions introduced will always be determined by integration. Thus if some shape is introduced for a stretched string with that suddenly set free, so that it may perform oscillations, with the aid of the principles of mechanics, the shape that the string shall then be considered to have at some time can be defined, and this shall be done by an integration of this kind, where a certain arbitrary function is introduced; but thus as it is agreed to be determined hence, so that the initial motion itself may be produced from that shape of the string induced ; and since the solution must be general, so that any of the initial shapes may be satisfied, it is necessary, as it is also apparent for these cases, in which an initial irregular shape of the string is induced with no connection from the aforementioned continuity, since that cannot happen unless by integration, an arbitrary function of this kind may be introduced from our arbitrary function left, as it is permitted also to be adapted into irregular shapes. Arbitrary functions of this kind, as I have made here, I will indicate by being marked in this way $f:y$, from which there will be a caution, without the letter f for the quantity that may be considered, concerning which a *colon* is considered sufficient. In a similar manner in the following this text $f:(x+y)$ will denote an arbitrary function of the quantity $x+y$; and where more such functions may be present in the calculation, also besides the letter f I will make use of these letters φ, ψ, θ etc. with a similar significance.

PROBLEM 5

39. *To investigate the nature of functions z of the two variables x and y, so that the formula of the differential* $\left(\frac{dz}{dx}\right) = p$ *becomes equal to a given function of x, which shall be X, thus so that there shall be* $p = X$.

SOLUTION

On putting $dz = pdx + qdy$ on account of $p = X$ there will be $dz = Xdx + qdy$. Now because the part of the differential Xdx of this has been given, the constant y may be taken on finding the integral, and since there shall be $dz = Xdx$, on integrating there will be $z = \int Xdx + \text{Const.}$; which constant since it can imply some quantity y, for that it will be allowed to assume some arbitrary function of y and therefore there will be for the integral sought

$$z = \int Xdx + f:y,$$

which by differentiation gives $dz = Xdx + dyf':y$, thus so that there shall be $q = f':y$ and $\left(\frac{dz}{dx}\right) = X$, plainly as was required.

COROLLARY 1

40. Therefore the integral of the equation $\left(\frac{dz}{dx}\right) = X$, with the function *z* of the two variables *x* and *y* present, is $z = \int Xdx + f:y$, where on account of X given the formula of the integral $\int Xdx$ [*i. e.* the argument] denotes a given function of x itself, since the constant introduced from this integration can be taken as the arbitrary function $f:y$.

COROLLARY 2

41. Hence it follows that the differential equation $dz = Xdx + qdy$ cannot be valid [*i. e.* a total derivative], unless *q* shall be a function of *y;* because indeed since with this limitation it is to be understood, unless *q* also may involve the quantity *z;* but hence we may remove that case.

SCHOLIUM

42. If indeed q also is able to depend on *z*, the equation $dz = Xdx + qdy$ will be valid, if *q* were some function of the two quantities $z - \int Xdx$ and *y;* because that can be easily made apparent, if there is put $z - \int Xdx = u$, thus so that now *q* will become a function of the two variables *u* and *y*. Then indeed the differential equation, which shall become $du = qdy$, only contains the two variables *u* and *y* and thus certainly is valid; and just as the integral of this may be had itself, from that *u* is always equal to a certain function of *y*, from which there becomes $u = z - \int Xdx = f:y$ just as before. Therefore just as often as there must be $\left(\frac{dz}{dx}\right) = X$, also lest indeed with this case excepted, in which perhaps *q* itself involves the quantity *z*, the integral will be

$$z = \int Xdx + f:y$$

nor at any time can another solution be put in place.

Hence this will be the complete integral, therefore because it has involved an arbitrary function, because that is to be considered as the most certain criterion for a complete integral. This therefore is required to complete the integral, so that not only a certain arbitrary constant, but thus an arbitrary function of the variable is introduced ; thus, if for which for the case $\left(\frac{dz}{dx}\right) = axx$ this integral may be shown

$$z = \tfrac{1}{3}ax^3 + A + By + Cy^2 + \text{etc.},$$

yet that will still be a particular integral, even if several arbitrary constants *A*, *B*, *C* etc. and perhaps even an infinite number are included ; truly it is apparent for the complete integral

$$z = \frac{1}{3}ax^3 + f:y$$

of boundless width; which is required to be properly observed in understanding the following correctly.

But cases may occur everywhere, in which on account of a defect in the investigation of the complete integral, we must be content with particular integrals, which, even if thus an infinitude of arbitrary constants are understood, yet they may be considered as particular constants only. This observation is required to be remembered always in the following, lest at any time we should be concerned with particular integrals and we have failed to grasp complete integrals.

PROBLEM 6

43. *If z should be a function of the two variables x and y of this kind, so that the formula of the differential* $\left(\frac{dz}{dx}\right) = p$ *is equal to a function of some given x and y, to define in general the nature of the function sought z.*

SOLUTION

Let *V* be that given function of x and *y*, to which the formula of the differential $\left(\frac{dz}{dx}\right) = p$ must be equal, and on putting $dz = pdx + qdy$ it is required, that there shall be $p = V$. Now towards finding the form of the function z there may be considered as it were a constant quantity *y* and there will be $dz = Vdx$. Therefore the formula $\int Vdx$ may be integrated with regard of x as variable, because *y* is taken for constant, thus so that in this formula with the single variable *x* present and thus the integration of this shall be liable to no difficulty; only by holding it constant is it possible to involve the other quantity *y* in some manner, and thus for the function sought *z* will have this expression

$$z = \int Vdx + f:y$$

for the integral thus taken $\int Vdx$, as if the quantity *y* should be constant and with only x variable ; but $f:y$ denotes some arbitrary function of *y* without indeed excluding discontinuous forms, which are not able to be shown by analytic expressions, and on this account this arbitrary function itself is to be considered for the complete integral.

COROLLARY 1

44. Since V shall be a function given of x and y, the formula of the integral $\int Vdx$ will also be a known function and with the determination of the same quantities x and y; since indeed it enters the integration by choice, $f:y$ is understood to be in the other part.

COROLLARY 2

45. Hence also the other part qdy of the differential dz is defined arising from the variability of y. For by § 27 it is of the form $\int Vdx$ with the differential arising from each of the variables x and y

$$Vdx + dy\int dx\left(\tfrac{dV}{dy}\right),$$

and if the differential of the function $f:y$ may be indicated by $dyf':y$, there will be

$$dz = Vdx + dy\int dx\left(\tfrac{dV}{dy}\right) + dyf':y.$$

COROLLARY 3

46. Therefore since we have put $dz = pdx + qdy$ and there shall be $p = V$, there will be

$$q = \int dx\left(\tfrac{dV}{dy}\right) + dyf':y,$$

where on account of the given function V of x and y themselves, also $\left(\tfrac{dV}{dy}\right)$ will be a given function, and in the integration $\int dx\left(\tfrac{dV}{dy}\right)$ only x will be considered to be variable.

EXAMPLE 1

47. *There is sought a function z of x and y of this kind , such that there shall be*

$$\left(\tfrac{dz}{dx}\right) = \frac{x}{\sqrt{(xx+yy)}} .$$

On account of $V = \frac{x}{\sqrt{(xx+yy)}}$ there will be

$$\int Vdx = \sqrt{(xx+yy)}$$

and thus we may consider

$$z = \sqrt{(xx+yy)} + f:y$$

from which there shall be

$$\left(\frac{dz}{dy}\right) = q = \frac{y}{\sqrt{(xx+yy)}} + f':y,$$

that which also emerges from the rule given. For there will be

$$\left(\frac{dV}{dy}\right) = \frac{-xy}{(xx+yy)^{\frac{3}{2}}},$$

hence on taking *y* constant :

$$\int dx\left(\frac{dV}{dy}\right) = -y\int\frac{xdx}{(xx+yy)^{\frac{3}{2}}} = \frac{y}{\sqrt{(xx+yy)}}.$$

EXAMPLE 2

48. *There is sought a function z of x and y of this kind, so that there shall be*

$$\left(\frac{dz}{dx}\right) = \frac{y}{\sqrt{(yy-xx)}}.$$

Since there shall be $V = \frac{y}{\sqrt{(yy-xx)}}$, there will be

$$\int Vdx = y\text{Ang.sin.}\frac{x}{y};$$

and hence

$$z = y\text{Ang.sin.}\frac{x}{y} + f:y;$$

the differential of which arising from the variability of *y* if we so wish, on account of

$$\left(\frac{dV}{dy}\right) = \frac{-xx}{(yy-xx)^{\frac{3}{2}}}$$

will be

$$\int dx\left(\frac{dV}{dy}\right) = -\int\frac{xxdx}{(yy-xx)^{\frac{3}{2}}} = \int\frac{dx}{\sqrt{(yy-xx)}} - yy\int\frac{dx}{(yy-xx)^{\frac{3}{2}}}$$

and thus

$$\int dx\left(\frac{dV}{dy}\right) = \text{Ang.sin.}\frac{x}{y} - \frac{x}{\sqrt{(yy-xx)}}$$

and

$$q = \text{Ang.sin.}\frac{x}{y} - \frac{x}{\sqrt{(yy-xx)}} + f':y.$$

Likewise there is found from the differentiation of the expression for *z* :

$$dz = dy \text{ Ang.sin.} \tfrac{x}{y} + \tfrac{ydx - xdy}{\sqrt{(yy - xx)}} + dyf\,' : y,$$

from which the same value emerges for $q = \left(\frac{dz}{dy}\right)$.

EXAMPLE 3

49. *There is sought a function of z of this kind of x and y, so that there shall be*

$$\left(\tfrac{dz}{dx}\right) = \tfrac{a}{\sqrt{(aa - yy - xx)}}.$$

On account of $V = \frac{a}{\sqrt{(aa - yy - xx)}}$ there will be

$$\int Vdx = a\text{Ang.sin.}\tfrac{x}{\sqrt{(aa - yy)}},$$

from which the form of the function *z* sought is

$$z = a\text{Ang.sin.}\tfrac{x}{\sqrt{(aa - yy)}} + f:y$$

from which because

$$\left(\tfrac{dV}{dy}\right) = \tfrac{ay}{(aa - yy - xx)^{\frac{3}{2}}},$$

there will be

$$\int dx\left(\tfrac{dV}{dy}\right) = ay\int \tfrac{dx}{(aa - yy - xx)^{\frac{3}{2}}} = \tfrac{ay}{aa - yy} \cdot \tfrac{x}{\sqrt{(aa - yy - xx)}}$$

and thus

$$\left(\tfrac{dz}{dy}\right) = q = \tfrac{axy}{(aa - yy)\sqrt{(aa - yy - xx)}} + f\,':y,$$

which same expression is elicited also from the differentiation of *z* itself.

SCHOLIUM 1

50. Yet at this point in this calculation a certain uncertainty is left, by which the value of the quantity *q* is affected. For since the value of $z = \int Vdx + f:y$ shall be determined, whenever the integral $\int Vdx$ with respect to *x* itself thus should have been determined, so that from the value of *x* given also a given value [of *y*] should be obtained, and thus with the differential of this satisfied no uncertainty should be present, but it is necessary that the value of q produced equally gives rise to the value of *p*, yet meanwhile the formula of the integral $\int dx\left(\frac{dV}{dy}\right)$ is not determined, but it is seen from the beginning that an independent new arbitrary quantity is introduced. Therefore so that such a significant uncertainty may be avoided, it is necessary to consider a condition, by which the integral $\int Vdx$ may be determined, and the same condition in the integration of the formula $\int dx\left(\frac{dV}{dy}\right)$ must be used. For now we may put the integral $\int Vdx$ thus taken, so that it vanishes on putting $x=a$, and the value of this will be determined $\int Vdx = S$, and that power therefore may have perhaps a factor $a-x$ or $a^n - x^n$; which since it does not contain *y*, also $\left(\frac{dS}{dy}\right)$ will contain the same factor and thus $\left(\frac{dS}{dy}\right)$ vanishes on putting $x=a$. Now truly there is $\left(\frac{dS}{dy}\right) = \int dx\left(\frac{dV}{dy}\right)$, from which it is observed, if the integral $\int Vdx$ is taken thus, so that it vanishes on putting $x=a$, also the other integral $\int dx\left(\frac{dV}{dy}\right)$ must be taken thus, so that it may vanish on putting $x=a$.

In bringing up the last two examples each integration has been put in place thus, so that it vanishes on putting $x=a$, but in the first no rule of this kind has been observed ; but if we use the same rule, we will have

$$\int Vdx = \sqrt{(xx+yy)} - y \quad \text{and} \quad \int dx\left(\frac{dV}{dy}\right) = \frac{y}{\sqrt{(xx+yy)}} - 1,$$

from which indeed the same solution appears, because here –*y* is present in $f:y$ and thus –1 in $f':y$. But it is in the same way, by whatever rule provided the former integration may be determined, so also we use the same rule in the latter.

SCHOLIUM 2

51. The principle of this determination rests on a theorem equally elegant and noteworthy:
If S shall be a function of the two variables x and y of this kind, which vanishes on putting $x=a$, and there shall be

$$dS = Pdx + Qdy,$$

then also the quantity Q vanishes on putting $x=a$.

From which likewise it is deduced, if *S* should vanish on putting $y=b$, then also there becomes $P=Q$, if there is put $y=b$. But here it is to be observed properly, which are understood from the similar determination of the two integral formulas $\int Vdx$ and $\int dx\left(\frac{dV}{dy}\right)$, only prevails, if the value *a* being attributed to *x* should be constant ; nor also can the above theorem be considered, if for example the function *S* should vanish on putting $x = y$ *;* for thence by no means does it follow in the same case that the quantity Q to be vanishing. For even if the function S may have a factor $x - y$ or $x^n - y^n$, it does not follow at all that the formula $\left(\frac{dS}{dy}\right)$ or Q should have the same factor, just as it comes about in use, if the factor were $x - a$ or $x^n - a^n$.

But I have said that it is not necessary that such a factor actually should be present, as long as if a power should be present in the function S. Just as if there should be

$$S = a - x + y - \sqrt{(aa - xx + yy)}\ ,$$

which function on putting $x = a$ certainly vanishes, even if it should contain neither of the factors $x = a$ or $x^n - a^n$; likewise indeed also

$$\left(\frac{dS}{dy}\right) = 1 - \frac{y}{\sqrt{(aa - xx + yy)}}$$

vanishes on putting $x = a$.

Therefore in a calculation of this kind, that we have used in these problems, where the integral of the formula $\int V dx$ must be shown, we have noted that it is composed always of two parts, the one indeterminate, indicated by the function $f{:}y$, but the other determined which we have expressed properly by $\int V dx$, which clearly vanishes on putting $x = a$; and this always is in the same way, whatever constant is taken for a, provided the distinction with the other indeterminate part is always involved.

PROBLEMA 7

52. *If z must be determined thus by the two variables x and y, so that the formula of the differential* $\left(\frac{dz}{dx}\right) = p$ *is equal to some given function of y and z, which shall be = V, in general to define the nature of the function z by x and y.*

SOLUTION

Since on putting $dz = pdx + qdy$ there shall be $p = V$, if we take the quantity y as constant, there will be $dz = Vdx$ *;* where since V shall be a given function of y and z and y is considered as constant, the equation $\frac{dz}{V} = dx$ will be integrable, from the complete integration of which there arises

$$\int \frac{dz}{V} = x + f{:}y\ ,$$

from which equation the relation between the three variables x, y and z thus is expressed in general, so that from that z is defined by x and y and the nature of the function z can be assigned.

But if hence we wish to investigate also the other part of the differential qdy or the function $q=\left(\frac{dz}{dy}\right)$, we may put the integral $\int\frac{dz}{V}$, where y is considered as a constant, thus to be taken so that it vanishes on putting $z=c$, and again on differentiating the quantity $\int\frac{dz}{V}$, so that also y is assumed variable,

$$d.\int\frac{dz}{V}=\frac{dz}{V}+dy\int dz\left(\frac{d(1:V)}{dy}\right)$$

or

$$d.\int\frac{dz}{V}=\frac{dz}{V}-dy\int\frac{dz}{VV}\left(\frac{dV}{dy}\right),$$

where in the integral $\int\frac{dz}{VV}\left(\frac{dV}{dy}\right)$ the quantity y may be considered again as constant, and this integral can thus be taken, so that it may vanish on putting $z=c$. With which accomplished since the differential of the equation found shall be

$$\frac{dz}{V}-dy\int\frac{dz}{VV}\left(\frac{dV}{dy}\right)=dx+dyf':y$$

for the form proposed we will consider

$$dz=Vdx+dy\left(V\int\frac{dz}{VV}\left(\frac{dV}{dy}\right)+Vf':y\right),$$

from which the quantity q will become known.

COROLLARY 1

53. In this problem it is most easily defined what kind of function the quantity x will become of the two remaining variables y and z, since there shall be

$$x=\int\frac{dz}{V}-f:y,$$

if indeed V is given by y and z. But likewise there is either z determined by x and y, or x determined by y and z.

COROLLARY 2

54. Since the relation between the three variables x, y and z may be determined thus, so that there becomes $\left(\frac{dz}{dx}\right)=V$, for a given function [$x$] of y and z, on account of $dx=\frac{dz}{V}$ on taking y constant, x will be a function of this kind of y and z, so that there shall be $\left(\frac{dx}{dz}\right)=\frac{1}{V}$ and thus $\left(\frac{dx}{dz}\right)\cdot\left(\frac{dz}{dx}\right)=1$.

SCHOLIUM

55. But in general any relation being proposed between the three variables x, y and z, from which each one is to be determined by the two remaining variables, and as it can be considered as function of these, there will be always $\left(\frac{dx}{dz}\right)\cdot\left(\frac{dz}{dx}\right)=1$. For we may put that relation with the equation expressing the differentials to become

$$Pdx+Qdy+Rdz=0$$

and it is evident on taking y constant to become

$$Pdx+Rdz=0$$

and thus as $\left(\frac{dz}{dx}\right)=\frac{-P}{R}$ so also $\left(\frac{dx}{dz}\right)=\frac{-R}{P}$; in a similar manner there will be

$$\left(\frac{dx}{dy}\right)=\frac{-Q}{R},\quad \left(\frac{dy}{dx}\right)=\frac{-P}{Q},\quad \left(\frac{dz}{dy}\right)=\frac{-Q}{R},\quad \left(\frac{dy}{dz}\right)=\frac{-R}{Q},$$

from which the proposition is apparent, even if a relation between more than three variables should be present.

In addition this case differs from the preceding cases, because here the nature of the function z, in as much as it is formed from the two remaining x and y, is not shown explicitly, but it may be defined finally from the resolution of the equation found, concerning which is it pleasing to set out a number of examples.

EXAMPLE 1

56. *A function z of x and y of this kind is sought, so that there shall be* $\left(\frac{dz}{dx}\right)=\frac{y}{z}$.

Therefore since there shall be $dz=y\frac{dx}{z}+qdy$, with y taken as constant there will be $zdz=ydx$ and

$$\tfrac{1}{2}zz=xy+f:y.$$

For finding q, this equation is differentiated generally

$$zdz=ydx+xdy+dyf':y$$

and there shall be

$$q=\frac{x}{z}+\frac{1}{z}f':y,$$

as the same is found by the rule given. For on account of $V = \frac{y}{z}$ there will be $\int \frac{dz}{V} = \frac{zz}{2y}$ with the integral thus taken, so that it vanishes on putting $z = 0$, then truly on account of $\left(\frac{dV}{dy}\right) = \frac{1}{z}$ there will be

$$\int \frac{dz}{VV}\left(\frac{dV}{dy}\right) = \int \frac{zdz}{yy} = \frac{zz}{2yy}$$

with the same rule of integration observed. Hence there becomes

$$dz = \frac{ydx}{z} + \frac{ydy}{z}\left(\frac{zz}{2yy} + f':y\right) \text{ and } q = \frac{z}{2y} + \frac{y}{z} f':y,$$

which expression agrees with the preceding ; from the comparison there becomes indeed

$$x + f':y = \frac{zz}{2y} + yf':y,$$

from which *x* is equal as before to the quantity $\frac{zz}{2y}$ together with a function of *y*. Only this should be noted, so that according to the perfect agreement here we ought to have written $yf:y$ for $f:y$.

EXAMPLE 2

57. *A function z of the two variables x and y is sought of this kind, such that there shall be*

$$\left(\frac{dz}{dx}\right) = \frac{\sqrt{(yy-zz)}}{z}.$$

Therefore since there shall be $dz = \frac{dx\sqrt{(yy-zz)}}{z} + qdy$, on taking *y* constant there becomes

$$dx = \frac{zdz}{\sqrt{(yy-zz)}}$$

and on integrating

$$x = y - \sqrt{(yy-zz)} - f:y;$$

from which in turn on differentiation [generally] there arises

$$dx = dy - \frac{ydy-zdz}{\sqrt{(yy-zz)}} - dyf':y$$

or

$$dz = \frac{dx\sqrt{(yy-zz)}}{z} + dy\left(\frac{y}{z} - \frac{\sqrt{(yy-zz)}}{z}(1-f':y)\right)$$

But by the rule given, on account of $V = \frac{\sqrt{(yy-zz)}}{z}$ there is $\int \frac{dz}{V} = y - \sqrt{(yy-zz)}$ for the integral to be taken thus, so that it vanishes on putting $z = 0$. Now indeed there is

$$\left(\frac{dV}{dy}\right) = \frac{y}{z\sqrt{(yy-zz)}} \quad \text{and} \quad \frac{1}{VV}\left(\frac{dV}{dy}\right) = \frac{yz}{(yy-zz)^{\frac{3}{2}}},$$

hence

$$\int \frac{dz}{VV}\left(\frac{dV}{dy}\right) = \frac{y}{\sqrt{(yy-zz)}} - 1$$

for the integral assumed for the same rule. On account of which there is deduced

$$q = \frac{\sqrt{(yy-zz)}}{z}\left(\frac{y}{\sqrt{(yy-zz)}} - 1 + f':y\right) = \frac{y}{z} - \frac{\sqrt{(yy-zz)}}{z}(1-f':y)$$

in short as before.

PROBLEM 8

58. *If z must be determined thus by the two variables x and y, so that the formula of the differential* $\left(\frac{dz}{dx}\right) = p$ *is equal to a certain given function of x and z, which shall be* $= V$, *to define in general the nature of the function z in terms of x and y.*

SOLUTION

There is put $dz = pdx + qdy$, and since here there shall be $p = V$; the quantity *y* is assumed constant and there will be $dz - Vdx = 0$, which equation may be reduced to an integral containing two variables *x* and *z* only with the aid of a multiplier of this kind, which shall be $= M$, thus in order that $Mdz - MVdx$ shall be the differential truly of a certain function of *x* and *z*, which function shall be $= S$ not involving the quantity *y*. From which our equation of the integral will be $S = f:y$, from which the nature of the function *z*, as determined by *x* and *y*, becomes known. We may differentiate this equation by taking besides *x* and *z*, *y* also to be variable and there shall be

$$dS = Mdz - MVdx = dyf':y$$

or

$$dz = Vdx + \frac{dy}{M}f':y,$$

thus so that there shall be $q = \frac{1}{M}f':y$.

COROLLARY 1

59. Also the multiplier M rendering the formula $dz - Vdx$ integrable will not contain the quantity y, since in the given function V, y is not present. But immediately, with this multiplier found, the value of $q = \frac{1}{M} f':y$ is deduced.

COROLLARY 2

60. If the integral of the differential formula $Mdz - MVdx$ were S, a function of x and z, from the solution of the problem we will have $S = f:y$, from which it is apparent that the constant, as which perhaps it might be wished to add to S, now contains an arbitrary function $f:y$.

EXAMPLE 1

61. *A function z of x and y of this kind is sought, so that there shall be* $\left(\frac{dz}{dx}\right) = \frac{nz}{x}$.

On putting $dz = \frac{nzdx}{x} + qdy$ on assuming y constant there will be $dz - \frac{nzdx}{x} = 0$, which equation multiplied by $\frac{1}{z}$ becomes integrable, thus so that there shall be the multiplier $M = \frac{1}{z}$ and hence the integral $S = lz - lx^n$; hence our equation sought of the integral will be $l\frac{z}{x^n} = f:y$, from which also $\frac{z}{x^n}$ will be equal to some function of y, thus so that there shall be $z = x^n f:y$.

EXAMPLE 2

62. *A function z of the two variables x and y is sought, so that the formula of the differential* $\left(\frac{dz}{dx}\right) = nx - z$.

On putting $dz = (nx - z)dx + qdy$ on assuming y constant there will be $dz + zdx - nxdx = 0$, which with the aid of the multiplier $M = e^x$ gives

$$S = e^x z - n\int e^x xdx = e^x z - ne^x x + ne^x,$$

from which equation the relation sought expressing the relation between x, y and z is

$$e^x z - ne^x x + ne^x = f:y \text{ or } z = n(x-1) + e^{-x} f:y,$$

then truly there will be

$$q = \left(\frac{dz}{dy}\right) = e^{-x} f':y.$$

EXAMPLE 3

63. *A function of z is sought of the two variables x and y, so that the formula of the differential shall be* $\left(\frac{dz}{dx}\right)=\frac{xz}{xx+zz}$.

Therefore there is put $dz=\frac{xzdx}{xx+zz}+qdy$ and on putting *y* constant there is sought the integral of this differential equation :

$$dz-\frac{xzdx}{xx+zz}=0,$$

which is rendered integrable with the aid of a certain divisor, which on account of the homogeneity is found on writing *x* and *z* in place of the differentials *dx* and *dz,* thus so that here the divisor shall be

$$z-\frac{xxz}{xx+zz}=\frac{z^3}{xx+zz}$$

and hence the multiplier $M=\frac{xx+zz}{z^3}$. Whereby there shall be

$$dS=\frac{(xx+zz)dz}{z^3}-\frac{xdx}{zz}$$

and thus

$$S=\frac{-xx}{2zz}+lz,$$

from which our equation sought will be

$$lz-\frac{xx}{2zz}=f:y \quad \text{and} \quad q=\frac{z^3}{xx+zz}f':y,$$

from which, since on putting $lz-\frac{xx}{2zz}=u$ there shall be $u=f:y$, also in turn it is possible to be concluded that $y=f:u$.

PROBLEM 9

64. *If z must be determined thus by the two variables x and y, so that the differential formula* $\left(\frac{dz}{dx}\right)$ *is equal to a certain given function involving all three variables x, y and z, which shall be* $=V$, *to define in general the nature of the functions z in terms of x and y.*

SOLUTION

Since there shall be $dz=Vdx+qdy$, if we assume *y* constant, there will be $dz=Vdx$, which equation therefore contains only the variables *x* and *z*, but involving the letter *y* in the function V. Therefore there will be given a multiplier M rendering this equation integrable, thus so that there shall be

$$Mdz - MVdx = dS\ ,$$

from which the equation of the integral expressing the relation between *x*, *y* and *z* will be $S = f{:}y$, where *S* will be a certain function of *x*, *y* and *z*, that can be made, so that *M* also includes all these three letters. But it is appropriate for the function *S* found by integration to be given a determined value, because the indeterminate part is included in the arbitrary function $f:y$. Therefore we may put *S* taken thus, so that it vanishes if there is put in place $x = a$ and $z = c$. [Thus, at this stage, for whatever *y*, $S(a, y, c) = 0$.]

But if we wish to find hence the other part of the proposed differential equation *qdy*, we may differentiate the function *S* on assuming that *y* shall be variable also, and

$$dS = Mdz - MVdx + Qdy = dyf'{:}y;$$

where since there shall be

$$\left(\frac{dQ}{dz}\right) = \left(\frac{dM}{dy}\right) \quad \text{and} \quad \left(\frac{dQ}{dx}\right) = -\left(\frac{d.MV}{dy}\right),$$

there will be on taking *y* constant again

$$dQ = dz\left(\frac{dQ}{dz}\right) + dx\left(\frac{dQ}{dx}\right) = dz\left(\frac{dM}{dy}\right) - dx\left(\frac{d.MV}{dy}\right),$$

which formula certainly will be integral. But *Q* must be taken by same rule as by which we have selected *S*, thus so that it shall vanish on putting $x = a$ and $z = c$, and with this quantity *Q* found, since we may consider

$$dz = Vdx - \frac{Qdy}{M} + \frac{dy}{M}\ f'{:}y\ ,$$

there will be

$$q = \left(\frac{dz}{dy}\right) = \frac{-Q + f'{:}y}{M}.$$

This determination depends on that fundamental idea itself, that if *S* should be a function of some kind of *x*, *y* and *z*, which vanishes on putting $x = a$ and $z = c$, the formula of the differential $\left(\frac{dS}{dy}\right)$ also vanishes in the same case.

[Thus, we are given $S(x, y, z)$ such that $S(a, y, c) = 0$ for all *y*, and hence for vanishing $h > 0$ near *y*, which is arbitrary, we have $S(a, y+h, c) = 0 = S(a, y, c) + h\frac{\partial S}{\partial y} = 0 + h\frac{\partial S}{\partial y}$, which implies that $\frac{\partial S}{\partial y} = \left(\frac{dS}{dy}\right) = 0$, as required.]

COROLLARY 1

65. Therefore the resolution of this problem is reduced to the integration of this differential equation

$$dz - Vdx = 0,$$

in which the quantity *y* is considered as constant, even if *V* should contain all three letters *x*, *y* and *z*. Therefore there will certainly be given the multiplier *M*, which may render this equation integrable, so that there shall be

$$Mdz - MVdx = dS$$

with a certain function *S* of *x*, *y* and *z* present.

COROLLARY 2

66. But with this multiplier *M* found and thence the integral of the quantity *S* , the quantity *z* thus may be defined by the two variables *x* and *y* so that there shall be $S = f{:}y$, where $f{:}y$ denotes some function of *y* either continuous or discontinuous, on account of which the integration may be had for completion.

COROLLARY 3

67. Since the relation between *z*, *x*, *y* has been defined in this manner, there will be with that thus differentiated, so that all three letters *x*, *y* and *z* are taken,

$$dz = Vdx + \frac{f'{:}y - Q}{M} dy ,$$

where the quantity *Q* from its own differential

$$dQ = dz\left(\frac{dM}{dy}\right) - dx\left(\frac{d.MV}{dy}\right)$$

must be defined with *y* assumed constant, with the integration thus moderated, so that if *S* vanishes in the case $x = a$ and $z = c$, also Q vanishes in the same case.

SCHOLIUM

68. Therefore here we deduce this notable theorem :

If there should be a function S of this kind of x, y and z, which vanishes on putting $x = a$ and $z = c$, then also for the same position the formula $\left(\frac{dS}{dy}\right)$ shall be vanishing.

Just as if there should be

$$S = Axx + Bxyz + Czz - Aaa - Bacy - Ccc,$$

there will be

$$\left(\frac{dS}{dy}\right) = Bxz - Bac,$$

of which the expression vanishes in each case where $x = a$ and $z = c$. Moreover with many more examples shown of this kind the truth of the theorem thus is apparent, so that the usual demonstration may not be desired. Meanwhile a function of this kind always containing the quantity *y* alone thus may be set out separated from the rest, so that it may be changed into such a form

$$S = PY + QY' + R\ Y'' + \text{etc.},$$

where by hypothesis $P,\ Q, R$ etc. are functions of *x* and *z* only and such indeed, which on putting $x = a$ et $z = c$ vanish individually. Hence now it is evident that there becomes

$$\left(\frac{dS}{dy}\right) = P\frac{dY}{dy} + Q\frac{dY'}{dy} + R\frac{dY''}{dy} + \text{etc.},$$

which forms evidently vanish under the same conditions. But however if a function *S* with this aforementioned nature should involve both irrational as well as transcending functions, that always is allowed to be set out in a form of this kind ; which even if it progresses indefinitely, this demonstration still retains its strength.

EXAMPLE 1

69. *A function z is sought of the two variables x and y of this kind, so that there shall be the differential formula* $\left(\frac{dz}{dx}\right)=\frac{xz}{ay}$.

Therefore we may put $dz=\frac{xzdx}{ay}+qdy$ and on taking y constant this equation will be considered :

$dz-\frac{xzdx}{ay}=0$, so that there shall be $V=\frac{xz}{ay}$, and the multiplier will be $M=\frac{1}{z}$; from which there becomes

$$S=l\frac{z}{c}-\frac{xx-aa}{2ay}$$

and the equation of the complete integral determining the function z will be

$$l\frac{z}{c}+\frac{aa-xx}{2ay}=f:y$$

Again in order to find the quantity q on account of $M=\frac{1}{z}$ and $MV=\frac{x}{ay}$ there will be $dQ=\frac{xdx}{ayy}$ and $Q=\frac{xx-aa}{2ayy}$ from which there becomes $q=zf':y-\frac{z(xx-aa)}{2ayy}$.

But here the same value is elicited by differentiation of the equation found $\left[i.e.\quad l\frac{z}{c}+\frac{aa-xx}{2ay}=f:y\right]$, which gives

$$\frac{dz}{z}-\frac{xdx}{ay}-\frac{aa-xx}{2ayy}dy=dyf':y$$

and thus

$$dz=\frac{xzdx}{ay}+\frac{z(xx-aa)}{2ayy}dy+zdyf':y\,,$$

thus so that there shall be

$$q=\frac{z(xx-aa)}{2ayy}+zf':y\,.$$

EXAMPLE 2

70. *A function z is sought of the two variables x and y of this kind, so that there shall be*

$$\left(\frac{dz}{dx}\right)=\frac{y}{x+z}\,.$$

Since there shall be $V=\frac{y}{x+z}$ on taking y constant this equation may be considered :

$$dz - \frac{ydx}{x+z} = 0,$$

towards finding the multiplier of which in the first place the equation is multiplied by $x+z$, so that there emerges

$$xdz + zdz - ydx = 0 \text{ or } dx - \frac{xdz}{y} = \frac{zdz}{y},$$

which multiplied by $e^{-\frac{z}{y}}$ prevails integrable and there is produced

$$e^{-\frac{z}{y}}x = \int e^{-\frac{z}{y}}\frac{zdz}{y} = -e^{-\frac{z}{y}}z + \int e^{-\frac{z}{y}}dz$$

and hence

$$e^{-\frac{z}{y}}x = -e^{-\frac{z}{y}}z - ye^{-\frac{z}{y}} + C.$$

On account of which

$$M = (x+z)\cdot -\frac{1}{y}\cdot e^{-\frac{z}{y}} = -\frac{x+z}{y}e^{-\frac{z}{y}}$$

and

$$S = e^{-\frac{z}{y}}(x+z+y) - e^{-\frac{c}{y}}(a+c+y),$$

from which the complete integral will be

$$e^{-\frac{z}{y}}(x+z+y) - e^{-\frac{c}{y}}(a+c+y) = f:y.$$

Now again since there shall be $MV = -e^{-\frac{z}{y}}$, there will be

$$\left(\frac{dM}{dy}\right) = e^{-\frac{z}{y}}\left(\frac{x+z}{yy} - \frac{z(x+z)}{y^3}\right) = e^{-\frac{z}{y}}(x+z)\left(\frac{1}{yy} - \frac{z}{y^3}\right)$$

and

$$\left(\frac{d.MV}{dy}\right) = -e^{-\frac{z}{y}}\cdot\frac{z}{yy}$$

and hence

$$dQ = e^{-\frac{z}{y}}\left(dz(x+z)\left(\frac{1}{yy} - \frac{z}{y^3}\right) + \frac{zdx}{yy}\right)$$

on taking *y* constant, from which the integrand will be obtained

$$Q = e^{-\frac{z}{y}}\left(\frac{xz}{yy} + 1 + \frac{z}{y} + \frac{zz}{yy}\right) - e^{-\frac{c}{y}}\left(\frac{ac}{yy} + 1 + \frac{c}{y} + \frac{cc}{yy}\right),$$

hence

$$q=\tfrac{z}{y}+\tfrac{y+z}{x+z}-e^{\frac{z-c}{y}}\left(\tfrac{ac+cc+cy+yy}{y(x+z)}\right)-\tfrac{y}{x+z}e^{\frac{c}{y}}f':y,$$

thus so that there shall be

$$dz=\tfrac{ydx}{x+z}+qdy.$$

But the equation found if it should be differentiated, gives

$$-e^{-\frac{z}{y}}\tfrac{(x+z)dz}{y}+e^{-\frac{z}{y}}dx+e^{-\frac{z}{y}}dy\left(1+\tfrac{z}{y}+\tfrac{xz}{yy}+\tfrac{zz}{yy}\right)$$
$$-e^{-\frac{c}{y}}dy\left(1+\tfrac{c}{y}+c\tfrac{(a+c)}{yy}\right)=dyf':y,$$

from which the value for q is concluded directly.

EXAMPLE 3

71. *A function sought z is sought of the two variables x and y of this kind, so that there shall be*

$$\left(\tfrac{dz}{dx}\right)=\tfrac{yy+zz}{yy+xx}.$$

On putting $dz=\tfrac{yy+zz}{yy+xx}dx+qdy$ the quantity y is taken constant, and since there shall be $dz-\tfrac{(yy+zz)dx}{yy+xx}=0$, it is evident that the nature of the multiplier shall be $M=\tfrac{y}{yy+zz}$, from which, since there shall be $\tfrac{ydz}{yy+zz}-\tfrac{ydx}{yy+xx}=0$, and on integrating

$$S=\text{Atang.}\tfrac{z}{y}-\text{Atang.}\tfrac{x}{y}+C=\text{Atang.}\tfrac{yz-yx}{yy+xz}-\text{Atang.}\tfrac{(c-a)y}{ac+yy}$$

and the function sought z may be defined by this equation

$$\text{Atang.}\tfrac{y(z-x)}{yy+xz}-\text{Atang.}\tfrac{(c-a)y}{ac+yy}=f:y.$$

Since again there shall be $MV=\tfrac{y}{yy+xx}$, then there will be

$$\left(\tfrac{dM}{dy}\right)=\tfrac{zz-yy}{(yy+xx)^2}\quad\text{and}\quad\left(\tfrac{d.MV}{dy}\right)=\tfrac{xx-yy}{(yy+xx)^2}$$

and hence

$$dQ = \frac{(zz-yy)dz}{(yy+zz)^2} - \frac{(xx-yy)dx}{(yy+xx)^2}$$

on taking y constant. Therefore

$$Q = \frac{-z}{yy+zz} + \frac{x}{yy+xx} + \frac{c}{yy+cc} - \frac{a}{yy+aa} \text{ and } q = \frac{-Q+f':y}{M},$$

which produces the same value also from differentiation.

Moreover since the constants *a* and *c* can be taken as it pleases, with these taken with these taken equal to zero or at least $c = a$ the equation of the integral will be

$$\text{Atang.}\frac{y(z-x)}{yy+xz} = f:y,$$

from which also there will be $\frac{y(z-x)}{yy+xz} = \text{funct.}y$ and $\frac{yy+xz}{z-x} = \text{funct.}y$

which function if it is called *Y*, will give $z = \frac{yy+xY}{Y-x}$.

SCHOLIUM

72. Scarcely is there a need to be noted that is often possible, that a solution of this kind of question surpasses the powers of analysis, evidently when the differential equation cannot be resolved at this time by known artifices. Just as if the case should be proposed

$$\left(\frac{dz}{dx}\right) = \frac{yy}{xx+zz},$$

from which on taking *y* must become $yydx = xxdz + zzdz$, the integration of which is not yet possible to be brought about. Meanwhile because the integral can be shown by a series, only that shall be made completely, also a solution by a series shall be obtained. Evidently on putting $x = \frac{-yydu}{udz}$ and on taking the element *dz* constant, this second order differential equation arises

$$y^4ddu + uzzdz^2 = 0,$$

from which by a series on integrating there is found [See e.g. Vol. II, §§ 929]

$$u = A\left(1 - \frac{z^4}{3.4\,y^4} + \frac{z^8}{3.\,4.7.8y^8} - \text{etc.}\right) + Bz\left(1 - \frac{z^4}{4.5\,y^4} + \frac{z^8}{4.5.8.9y^8} - \text{etc.}\right),$$

where any functions of y can be taken for the functions A and B. Whereby if there is put $\frac{A}{B} = f:y$, there will be

$$x=\frac{yyf:y\left(\frac{z^3}{3y^4}-\frac{z^7}{3.\,4.7y^8}+\text{etc.}\right)-yy\left(1-\frac{z^4}{4y^4}+\frac{z^8}{4.5.8y^8}-\text{etc.}\right)}{f:y\left(1-\frac{z^4}{3.4\,y^4}+\frac{z^8}{3.\,4.7.8y^8}-\text{etc.}\right)+z\left(1-\frac{z^4}{4.5\,y^4}+\frac{z^8}{4.5.8.9y^8}-\text{etc.}\right)},$$

from which equation the function sought z is expressed generally by the two variables x and y.

Therefore since we have uncovered methods, any differential equations can be integrated by approximations and that completely, the help from calling on these methods enables all relevant problems to be resolved at any rate by approximations. Moreover in this part we are able to concede assuming more advanced analysis in the resolution of differential equations pertaining to the first part as entirely useful, so that the further we progress into analysis, that always, which pertains to the preceding parts, even if they do not depend on the argument, we are accustomed consider as done.

CAPUT II

DE RESOLUTIONE AEQUATIONUM QUIBUS ALTERA FORMULA DIFFERENTIALIS PER QUANTITATES FINITAS UTCUNQUE DATUR

PROBLEMA 4[a]

33. *Investigare indolem functionis z binarum variabilium x et y, ut formula differentialis* $\left(\frac{dz}{dx}\right) = p$ *sit quantitas constans* $= a$.

SOLUTIO

Posito ergo $dz = pdx + qdy$ ea functionis *z* indoles quaeritur, ut sit $p = a$ seu $dz = adx + qdy$; ad quam inveniendam sumatur *y* pro constante; erit $dz = adx$ et integrando $z = ax$ + Const., ubi notari oportet hanc constantem utcunque involvere posse quantitatem *y*. Quare ut solutionem generalem exhibeamus, erit $z = ax + f:y$ denotante $f:y$ functionem quamcunque ipsius *y*, quae per se nullo modo determinatur, sed penitus ab arbitrio nostro pendet. Quod etiam differentiatio vicissim declarat; si enim huius functionis $f:y$ differentiale per $dyf':y$ indicemus, erit utique $dz = adx + dyf':y$ ideoque $\left(\frac{dz}{dx}\right) = a$, prorsus uti quaestio postulat; unde patet hoc casu alteram formulam differentialem $q = \left(\frac{dz}{dy}\right)$ functioni solius *y* aequari, cum sit $q = f':y$.

COROLLARIUM 1

34. Si ergo eiusmodi quaeratur functio *z* binarum variabilium *x* et *y*,ut sit $\left(\frac{dz}{dx}\right) = a$, erit $z = ax + f:y$ et altera formula differentialis $\left(\frac{dz}{dy}\right)$ necessario aequatur functioni ipsius *y* tantum.

COROLLARIUM 2

35. Si talis requiratur functio, ut sit $\left(\frac{dz}{dx}\right) = 0$, ea necessario erit functio ipsius *y* tantum seu quantitatem *x* plane non involvet; cum enim a variatione ipsius *x* nullam mutationem pati debeat, haec quantitas *x* quoque in eius determinationem plane non ingredietur.

COROLLARIUM 3

36. Hinc etiam patet aequationem differentialem $dz = adx + qdy$ realem esse non posse, nisi *q* sit functio ipsius *y* tantum. Quod etiam character supra expositus declarat;

aequatione enim ad hanc formam $adx + qdy - dz = 0$ reducta ob $P = a$, $Q = q$ et $R = -1$ erit

$$L = \left(\frac{dq}{dz}\right),\ M = 0 \text{ et } N = -\left(\frac{dq}{dx}\right)$$

ideoque realitas postulat, ut sit $a\left(\frac{dq}{dz}\right) + \left(\frac{dq}{dx}\right) = 0$. At per hypothesin q non pendet a z, unde ob $\left(\frac{dq}{dz}\right) = 0$ erit $\left(\frac{dq}{dx}\right) = 0$ ideoque etiam q ab x non pendet.

SCHOLION 1

37. Ex allatis satis patet hanc operationem, qua functionem z determinavimus, veram esse integrationem, qua uti in vulgaribus integrationibus aliquid indeterminati introducitur. Hic scilicet ingressa est functio quaecunque ipsius y, cuius indoles per se nullo modo determinatur; eam quoque ita concipere licet, ut descripta curva quacunque, si eius abscissae per y indicentur, applicatae exhibeant eiusmodi functionem ipsius y. Neque vero opus est, ut haec curva sit regularis et aequatione quapiam contenta; sed curva quaecunque libero manus ductu descripta eundem praestat effectum, etiamsi sit maxime irregularis et ex pluribus partibus diversarum curvarum conflata. Huiusmodi functiones irregulares appellare licet discontinuas seu nexu continuitatis destitutas; unde hoc imprimis notatu dignum occurrit, quod, cum prioris generis integrationes alias functiones praeter continuas non admittant, hic etiam functiones discontinuae calculo subiiciantur, quod pluribus insignibus Geometris adeo calculi principiis adversari est visum. Verum integrationum in hoc secundo libro tradendarum vis praecipua in eo consistit, quod etiam functionum discontinuarum sint capaces; ex quo per hunc quasi novum calculum fines Analyseos maxime proferri sunt censendi.

SCHOLION 2

38. Quemadmodum deinde in vulgaribus integrationibus constans arbitraria ingressa semper ex indole problematis, cuius solutio eo perduxerat, determinatur, ita etiam hic natura problematis, cuius solutio huiusmodi integratione absolvitur, semper indolem functionis arbitrariae per integrationem ingressae determinabit. Ita si cordae tensae figura quaecunque inducatur eaque subito dimittatur, ut oscillationes peragat, ope principiorum mechanicorum ad quodvis tempus figura, quam corda tum sit habitura, definiri potest hocque fit eiusmodi integratione, qua functio quaedam arbitraria introducitur; quam autem deinceps ita determinari convenit, ut pro ipso motus initio ipsa illa figura cordae inducta prodeat; et cum solutio debeat esse generalis, ut satisfaciat figurae cuicunque initiali, necesse est, ut etiam ad eos casus pateat, quibus cordae initio figura irregularis nullo continuitatis nexu praedita inducatur, quod fieri non posset, nisi per integrationem eiusmodi functio arbitrio nostro relicta ingrederetur, quam etiam ad figuras irregulares adaptare liceret. Huiusmodi functiones arbitrarias, prouti hie feci, eiusmodi signandi modo $f:y$ indicabo, unde cavendum erit, ne littera f pro quantitate habeatur, quocirca ipsi *colon* suffigere visum est. Simili modo in sequentibus haec scriptio $f:(x+y)$ denotabit functionem arbitrariam quantitatis $x+y$; ac ubi plures tales functiones in calculum ingredientur, praeter litteram *f* etiam his characteribus φ, ψ, θ etc. cum simili significatione utar.

PROBLEMA 5

39. *Investigare indolem functionis z binarum variabilium x et y, ut formula differentialis* $\left(\frac{dz}{dx}\right)=p$ *aequalis fiat functioni datae ipsius x, quae sit X, ita ut sit* $p=X$.

SOLUTIO

Posito $dz=pdx+qdy$ ob $p=X$ erit $dz=Xdx+qdy$. Quia iam huius differentialis pars *Xdx* est data, ad integrale inveniendum accipiatur *y* constans, et cum sit $dz=Xdx$, erit integrando $z=\int Xdx+\text{Const.}$; quae constans cum etiam quantitatem *y* utcunque implicare possit, pro ea assumere licebit functionem quamcunque arbitrariam ipsius *y* eritque ergo integrale quaesitum

$$z=\int Xdx+f:y,$$

quae per differentiationem praebet $dz=Xdx+dyf':y$, ita ut sit $q=f':y$ atque $\left(\frac{dz}{dx}\right)=X$, plane ut requirebatur.

COROLLARIUM 1

40. Aequationis ergo $\left(\frac{dz}{dx}\right)=X$ existente *z* functione duarum variabilium *x* et *y* integrale est $z=\int Xdx+f:y,$ ubi ob *X* datum formula integralis $\int Xdx$ datam functionem ipsius *x* denotat, quandoquidem constans hac integratione ingressa in functione arbitraria $f:y$ comprehendi potest.

COROLLARIUM 2

41. Hinc sequitur aequationem differentialem $dz=Xdx+qdy$ realem esse non posse, nisi *q* sit functio ipsius *y;* quod quidem cum hac limitatione est intelligendum, nisi *q* etiam involvat quantitatem *z;* quem casum autem hinc removemus.

SCHOLION

42. Si enim *q* etiam a *z* pendere queat, aequatio $dz=Xdx+qdy$ realis erit, si *q* fuerit functio quaecunque binarum quantitatum $z-\int Xdx$ et *y;* id quod hinc facillime patet, si ponatur $z-\int Xdx=u$, ita ut iam q futura sit functio binarum quantitatum u et y. Tum enim aequatio differentialis, quae fit $du=qdy$, duas tantum continet variabiles u et y ideoque certo est realis; et quomodocunque eius integrale se habeat, inde semper u aequabitur certae functioni ipsius y, unde fit $u=z-\int Xdx=f:y$ prorsus ut ante. Quoties ergo esse debet $\left(\frac{dz}{dx}\right)=X$, etiam ne hoc quidem casu excepto, quo forte q ipsam quantitatem z implicat, integrale erit

$$z=\int Xdx+f:y$$

neque unquam alia solutio locum habere potest.

Erit ergo hoc integrale completum, propterea quod functionem arbitrariam involvit, id quod pro certissimo criterio integralis completi est habendum. Hic igitur ad integrale completum requiritur, ut non tam constans quaedam arbitraria, sed functio adeo variabilis arbitraria ingrediatur; ita, si quis pro casu $\left(\frac{dz}{dx}\right)=axx$ exhibeat hoc integrale

$$z=\tfrac{1}{3}ax^3+A+By+Cy^2+\text{etc.},$$

id tantum erit particulare, etiamsi plures constantes arbitrarias A, B, C etc. ac fortasse infinitas complectatur; verum enim integrale completum

$$z=\tfrac{1}{3}ax^3+f:y$$

infinite latius patet; id quod ad sequentia recte intelligenda probe notari oportet.

Occurrent autem utique casus, quibus ob defectum methodi integrale completum investigandi integralibus particularibus contenti esse debemus, quae, etiamsi adeo infinitas constantes arbitrarias comprehendant, tamen pro solutionibus particularibus tantum sunt habenda. Hanc observationem in sequentibus perpetuo meminisse oportet, ne circa integralia particularia et completa unquam decipiamur.

PROBLEMA 6

43. *Si z debeat esse eiusmodi functio binarum variabilium x et y, ut formula differentialis* $\left(\frac{dz}{dx}\right)=p$ *aequetur functioni cuipiam datae ipsarum x et y, definire in genere indolem functionis quaesitae z.*

SOLUTIO

Sit V functio ista data ipsarum x et y, cui formula differentialis $\left(\frac{dz}{dx}\right)=p$ aequalis esse debet, ac posito $dz=pdx+qdy$ requiritur, ut sit $p=V$. Iam ad formam functionis z inveniendam consideretur quantitas y tanquam constans eritque $dz=Vdx$. Integretur igitur formula $\int Vdx$ spectata sola x ut variabili, quia y pro constante sumitur, ita ut in hac formula unica insit variabilis x ideoque eius integratio nulli obnoxia sit difficultati; id tantum est tenendum constantem integratione ingressam utcunque involvere posse alteram quantitatem y sicque pro functione quaesita z haec habebitur expressio

$$z = \int V dx + f:y$$

integrali $\int V dx$ ita sumto, quasi quantitas y esset constans solaque x variabilis; at $f:y$ denotat functionem quamcunque arbitrariam ipsius y ne exclusis quidem formis discontinuis, quae nullis expressionibus analyticis exhiberi queant, atque ob hanc ipsam functionem arbitrariam integratio pro completa est habenda.

COROLLARIUM 1

44. Cum V sit functio data ipsarum x et y, formula integralis $\int V dx$ erit etiam functio cognita et determinata earundem quantitatum x et y; quod enim per integrationem arbitrarii ingreditur, in altera parte $f:y$ comprehenditur.

COROLLARIUM 2

45. Hinc etiam differentialis dz altera pars qdy ex variabilitate ipsius y oriunda definitur. Nam per § 27 est formae $\int V dx$ differentiale ex utraque variabili x et y ortum

$$V dx + dy \int dx \left(\frac{dV}{dy}\right),$$

ac si functionis $f:y$ differentiale indicetur per $dyf':y$, erit

$$dz = V dx + dy \int dx \left(\frac{dV}{dy}\right) + dyf':y.$$

COROLLARIUM 3

46. Cum ergo posuerimus $dz = pdx + qdy$ sitque $p = V$, erit

$$q = \int dx \left(\frac{dV}{dy}\right) + dyf':y,$$

ubi ob V functionem datam ipsarum x et y etiam $\left(\frac{dV}{dy}\right)$ erit functio data et in integratione $\int dx \left(\frac{dV}{dy}\right)$ sola x pro variabili habetur.

EXEMPLUM 1

47. *Quaeratur eiusmodi functio z ipsarum x et y, ut sit* $\left(\frac{dz}{dx}\right)=\frac{x}{\sqrt{(xx+yy)}}$.

Ob $V=\frac{x}{\sqrt{(xx+yy)}}$ erit

$$\int Vdx=\sqrt{(xx+yy)}$$

ideoque habemus

$$z=\sqrt{(xx+yy)}+f:y$$

unde fit

$$\left(\frac{dz}{dy}\right)=q=\frac{y}{\sqrt{(xx+yy)}}+f':y,$$

id quod etiam per regulam datam prodit. Erit enim

$$\left(\frac{dV}{dy}\right)=\frac{-xy}{(xx+yy)^{\frac{3}{2}}},$$

hinc sumta *y* constante

$$\int dx\left(\frac{dV}{dy}\right)=-y\int\frac{xdx}{(xx+yy)^{\frac{3}{2}}}=\frac{y}{\sqrt{(xx+yy)}}.$$

EXEMPLUM 2

48. *Quaeratur eiusmodi functio z ipsarum x et y, ut sit* $\left(\frac{dz}{dx}\right)=\frac{y}{\sqrt{(yy-xx)}}$.

Cum sit $V=\frac{y}{\sqrt{(yy-xx)}}$ erit

$$\int Vdx=y\text{Ang.sin.}\tfrac{x}{y};$$

hincque

$$z=y\text{Ang.sin.}\tfrac{x}{y}+f:y;$$

cuius differentiale ex ipsius *y* variabilitate oriundum si desideremus, ob

$$\left(\frac{dV}{dy}\right)=\frac{-xx}{(yy-xx)^{\frac{3}{2}}}$$

erit

$$\int dx\left(\frac{dV}{dy}\right)=-\int\frac{xxdx}{(yy-xx)^{\frac{3}{2}}}=\int\frac{dx}{\sqrt{(yy-xx)}}-yy\int\frac{dx}{(yy-xx)^{\frac{3}{2}}}$$

ideoque

$$\int dx\left(\frac{dV}{dy}\right) = \text{Ang.sin.}\frac{x}{y} - \frac{x}{\sqrt{(yy-xx)}}$$

et

$$q = \text{Ang.sin.}\frac{x}{y} - \frac{x}{\sqrt{(yy-xx)}} + f':y.$$

Idem reperitur ex differentiatione expressionis pro z inventae

$$dz = dy\ \text{Ang.sin.}\frac{x}{y} + \frac{ydx-xdy}{\sqrt{(yy-xx)}} + dyf':y,$$

unde pro $q = \left(\frac{dz}{dy}\right)$ idem valor prodit.

EXEMPLUM 3

49. *Quaeratur eiusmodi functio z ipsarum x et y, ut sit* $\left(\frac{dz}{dx}\right) = \frac{a}{\sqrt{(aa-yy-xx)}}$.

Ob $V = \frac{a}{\sqrt{(aa-yy-xx)}}$ erit

$$\int Vdx = a\text{Ang.sin.}\frac{x}{\sqrt{(aa-yy)}},$$

unde functionis z forma quaesita est

$$z = a\text{Ang.sin.}\frac{x}{\sqrt{(aa-yy)}} + f:y$$

deinde quia

$$\left(\frac{dV}{dy}\right) = \frac{ay}{(aa-yy-xx)^{\frac{3}{2}}},$$

erit

$$\int dx\left(\frac{dV}{dy}\right) = ay\int\frac{dx}{(aa-yy-xx)^{\frac{3}{2}}} = \frac{ay}{aa-yy}\cdot\frac{x}{\sqrt{(aa-yy-xx)}}$$

ideoque

$$\left(\frac{dz}{dy}\right) = q = \frac{axy}{(aa-yy)\sqrt{(aa-yy-xx)}} + f':y,$$

quae eadem expressio etiam ex ipsa differentiatione ipsius z eruitur.

SCHOLION 1

50. In hoc calculo tamen adhuc quaedam incertitudo relinquitur, qua valor quantitatis q afficitur. Cum enim valor ipsius $z = \int Vdx + f:y$ sit determinatus, quandoquidem integrale $\int Vdx$ respectu ipsius *x* ita fuerit determinatum, ut pro dato ipsius *x* valore etiam datum valorem obtineat, adeoque in eius differentiali pleno nulla incertitudo inesse potest, sed necesse est, ut valor ipsius *q* aeque prodeat determinatus atque ipsius *p*, interim tamen formula integralis $\int dx\left(\frac{dV}{dy}\right)$ non determinatur, sed novam arbitrariam a priori non pendentem introducere videtur. Ut igitur talis significatus vagus evitetur, spectari oportet conditionem, qua integrale $\int Vdx$ determinatur, eademque conditio in formulae $\int dx\left(\frac{dV}{dy}\right)$ integratione adhiberi debet. Nam ponamus integrale $\int Vdx$ ita capi, ut evanescat posito $x=a$, sitque eius valor determinatus $\int Vdx = S$, isque igitur potentia saltem habebit factorem $a-x$ seu $a^n - x^n$; qui cum non contineat *y*, etiam $\left(\frac{dS}{dy}\right)$ eundem factorem continebit ideoque $\left(\frac{dS}{dy}\right)$ evanescet posito $x=a$. Est vero $\left(\frac{dS}{dy}\right) = \int dx\left(\frac{dV}{dy}\right)$, ex quo perspicitur, si integrale $\int Vdx$ ita capiatur, ut evaneseat posito $x=a$, etiam alterum integrale $\int dx\left(\frac{dV}{dy}\right)$ ita capi debere, ut evaneseat posito $x=a$.

In allatis binis postremis exemplis utraque integratio ita est instituta, ut evanescat posito $x=a$, in primo autem nulla huiusmodi regula est observata; sin autem eandem legem adhibeamus, habebimus

$$\int Vdx = \sqrt{(xx+yy)} - y \quad \text{et} \quad \int dx\left(\frac{dV}{dy}\right) = \frac{y}{\sqrt{(xx+yy)}} - 1,$$

unde quidem eadem solutio emergit, quia ibi –*y* continetur in $f:y$ et hic –1 in $f':y$. Perinde autem est, quacunque lege prior integratio determinetur, dummodo eadem lege et in posteriori utamur.

SCHOLION 2

51. Principium huius determinationis isto innititur theoremate aeque elegante ae notatu digno:

Si S sit eiusmodi functio binarum variabilium x et y, quae evanescat posito $x=a$, *fueritque*

$$dS = Pdx + Qdy,$$

tum etiam quantitas Q evanescet posito $x=a$.

Unde simul colligitur, si *S* evanescat posito $y=b$, tum etiam fieri $P=Q$, si ponatur $y=b$. Hic autem probe observandum est, quae de simili determinatione binarum formularum integralium $\int Vdx$ et $\int dx\left(\frac{dV}{dy}\right)$ sunt praecepta, tantum valere, si valor *a* ipsi

x tribuendus fuerit constans; neque etiam superius theorema locum habet, si verbi gratia functio S evanescat posito $x = y$; inde enim neutiquam sequitur eodem casu quantitatem Q esse evanituram. Etiamsi enim functio S factorem habeat $x - y$ vel $x^n - y^n$, minime sequitur formulam $\left(\frac{dS}{dy}\right)$ seu Q eundem factorem esse habituram, quemadmodum usu venit, si factor fuerit $x - a$ seu $x^n - a^n$.

Dixi autem non opus esse, ut talis factor revera adsit, dummodo quasi potentia in functione S contineatur. Veluti si fuerit

$$S = a - x + y - \sqrt{(aa - xx + yy)},$$

quae functio posito $x = a$ utique evanescit, etiamsi neque factorem $x = a$ neque $x^n - a^n$ contineat; simul vero etiam

$$\left(\frac{dS}{dy}\right) = 1 - \frac{y}{\sqrt{(aa - xx + yy)}}$$

posito $x = a$ evanescit.

In huiusmodi ergo calculo, quo in his problematibus utimur, ubi integrale formulae $\int Vdx$ exhiberi debet, id semper ex duabus partibus compositum spectamus, altera indeterminata, per functionem $f:y$ indicata, altera autem, quam proprie per $\int Vdx$ exprimimus, determinata, quae scilicet posito $x = a$ evanescat; hicque semper perinde est, qualis constans pro a assumatur, dum discrimen perpetuo alteri parti indeterminatae involvitur.

PROBLEMA 7

52. *Si z debeat ita determinari per binas variabiles x et y, ut formula differentialis* $\left(\frac{dz}{dx}\right) = p$ *aequetur datae cuipiam functioni ipsarum y et z, quae sit = V, definire in genere indolem functionis z per x et y.*

SOLUTIO

Cum posito $dz = pdx + qdy$ sit $p = V$, si quantitatem y pro constante capiamus, erit $dz = Vdx$; ubi cum V sit functio data ipsarum y et z et y pro constante habeatur, aequatio $\frac{dz}{V} = dx$ erit integrabilis, ex cuius integratione completa oritur

$$\int \frac{dz}{V} = x + f:y,$$

qua aequatione relatio inter ternas variabiles x, y et z ita in genere exprimitur, ut ex ea z per x et y definiri indolesque functionis z assignari possit.

Quodsi hinc alteram quoque differentialis partem qdy seu functionem $q = \left(\frac{dz}{dy}\right)$ indagare velimus, ponamus integrale $\int \frac{dz}{V}$, ubi y ut constans spectatur, ita capi, ut evanescat posito $z = c$, eritque quantitatem $\int \frac{dz}{V}$ denuo differentiando, ut etiam y variabilis assumatur,

$$d.\int \frac{dz}{V} = \frac{dz}{V} + dy\int dz\left(\frac{d(1:V)}{dy}\right)$$

seu

$$d.\int \frac{dz}{V} = \frac{dz}{V} - dy\int \frac{dz}{VV}\left(\frac{dV}{dy}\right),$$

ubi in integrali $\int \frac{dz}{VV}\left(\frac{dV}{dy}\right)$ quantitas y iterum pro constante habetur hocque integrale ita capi debet, ut posito $z = c$ evanescat. Quo facto cum aequationis inventae differentiale sit

$$\frac{dz}{V} - dy\int \frac{dz}{VV}\left(\frac{dV}{dy}\right) = dx + dyf':y$$

pro forma proposita habebimus

$$dz = Vdx + dy\left(V\int \frac{dz}{VV}\left(\frac{dV}{dy}\right) + V f':y\right),$$

unde quantitas q innotescit.

COROLLARIUM 1

53. In hoc problemate facillime definitur, qualis functio quantitas x futura sit binarum reliquarum y et z, cum sit

$$x = \int \frac{dz}{V} - f:y,$$

siquidem V per y et z detnr. Perinde autem est, sive z per x et y sive x per y et z determinetur.

COROLLARIUM 2

54. Cum relatio inter ternas variabiles x, y et z ita sit determinata, ut fiat $\left(\frac{dz}{dx}\right) = V$ functioni datae ipsarum y et z, ob $dx = \frac{dz}{V}$ sumto y constante erit x eiusmodi functio ipsarum y et z, ut sit $\left(\frac{dx}{dz}\right) = \frac{1}{V}$ ideoque $\left(\frac{dx}{dz}\right)\cdot\left(\frac{dz}{dx}\right) = 1$.

SCHOLION

55. In genere autem quaecunque relatio inter ternas variabiles x, y et z proponatur, unde unaquaeque per binas reliquas determinari et tanquam earundem functio spectari possit, semper erit $\left(\frac{dx}{dz}\right)\cdot\left(\frac{dz}{dx}\right)=1$. Ponamus enim aequatione illam relationem exprimente differentiata prodire

$$Pdx+Qdy+Rdz=0$$

ac manifestum est sumta y pro constante fore

$$Pdx+Rdz=0$$

ideoque tam $\left(\frac{dz}{dx}\right)=\frac{-P}{R}$ quam $\left(\frac{dx}{dz}\right)=\frac{-R}{P}$; simili autem modo erit

$$\left(\frac{dx}{dy}\right)=\frac{-Q}{R},\quad \left(\frac{dy}{dx}\right)=\frac{-P}{Q},\quad \left(\frac{dz}{dy}\right)=\frac{-Q}{R},\quad \left(\frac{dy}{dz}\right)=\frac{-R}{Q},$$

unde propositum patet, etiamsi relatio inter plures tribus variabiles locum habeat.

Ceterum hic casus a praecedentibus differt, quod hic natura functionis z, quatenus ex binis reliquis x et y formatur, non explicite exhibeatur, sed per resolutionem demum aequationis inventae definiri debet, cuius rei aliquot exempla evolvisse iuvabit.

EXEMPLUM 1

56. *Quaeratur eiusmodi functio z ipsarum x et y, ut sit* $\left(\frac{dz}{dx}\right)=\frac{y}{z}$.

Cum ergo sit $dz=y\frac{dx}{z}+qdy$, erit y pro constante sumendo $zdz=ydx$ et

$$\tfrac{1}{2}zz=xy+f:y.$$

Pro q inveniendo differentietur haec aequatio generaliter

$$zdz=ydx+xdy+dyf':y$$

eritque

$$q=\frac{x}{z}+\frac{1}{z}f':y,$$

quod idem per regulam datam reperitur. Nam ob $V=\frac{y}{z}$ erit $\int\frac{dz}{V}=\frac{zz}{2y}$ integrali ita sumto, ut evanescat posito $z=0$, tum vero ob $\left(\frac{dV}{dy}\right)=\frac{1}{z}$ erit

$$\int\frac{dz}{VV}\left(\frac{dV}{dy}\right)=\int\frac{zdz}{yy}=\frac{zz}{2yy}$$

eadem integrationis lege observata. Hinc fit $dz=\frac{ydx}{z}+\frac{ydy}{z}\left(\frac{zz}{2yy}+f':y\right)$ et

$$q = \frac{z}{2y} + \frac{y}{z} f':y,$$

quae expressio cum praecedente convenit; ex comparatione enim fit

$$x + f':y = \frac{zz}{2y} + yf':y,$$

unde *x* aequatur ut ante qllantitati $\frac{zz}{2y}$ una cum functione ipsius *y*. Hoc tantum notetur, quod ad consensum perfectum hic pro $f:y$ scribere debuissemus $yf:y$.

EXEMPLUM 2

57. *Quaeratur eiusmodi functio z binarum variabilium x et y, ut sit*

$$\left(\frac{dz}{dx}\right) = \frac{\sqrt{(yy-zz)}}{z}.$$

Cum ergo sit $dz = \frac{dx\sqrt{(yy-zz)}}{z} + qdy$, sumta *y* constante fit

$$dx = \frac{zdz}{\sqrt{(yy-zz)}}$$

et integrando

$$x = y - \sqrt{(yy-zz)} - f:y;$$

unde vicissim differentiando oritur

$$dx = dy - \frac{ydy - zdz}{\sqrt{(yy-zz)}} - dyf':y$$

seu

$$dz = \frac{dx\sqrt{(yy-zz)}}{z} + dy\left(\frac{y}{z} - \frac{\sqrt{(yy-zz)}}{z}(1 - f':y)\right)$$

Per regulam autem datam ob $V = \frac{\sqrt{(yy-zz)}}{z}$ est $\int \frac{dz}{V} = y - \sqrt{(yy-zz)}$ integrali ita sumto, ut evanescat posito $z = 0$. Iam vero est

$$\left(\frac{dV}{dy}\right) = \frac{y}{z\sqrt{(yy-zz)}} \quad \text{et} \quad \frac{1}{VV}\left(\frac{dV}{dy}\right) = \frac{yz}{(yy-zz)^{\frac{3}{2}}},$$

hinc

$$\int \frac{dz}{VV}\left(\frac{dV}{dy}\right) = \frac{y}{\sqrt{(yy-zz)}} - 1$$

integrali eadem lege sumto. Quocirca colligitur

$$q = \frac{\sqrt{(yy-zz)}}{z}\left(\frac{y}{\sqrt{(yy-zz)}} - 1 + f':y\right) = \frac{y}{z} - \frac{\sqrt{(yy-zz)}}{z}\left(1 - f':y\right)$$

prorsus ut ante.

PROBLEMA 8

58. *Si z ita debeat determinari per binas variabiles x et y, ut formula differentialis* $\left(\frac{dz}{dx}\right) = p$ *aequetur functioni cuipiam datae ipsarum x et z, quae sit* $=V$ *definire in genere indolem functionis z per x et y.*

SOLUTIO

Ponatur $dz = pdx + qdy$, et cum sit $p = V$; sumatur quantitas y constans eritque $dz - Vdx = 0$, quae aequatio duas tantum quantitates variabiles x et z continens integrabilis reddetur ope cuiusdam multiplicatoris, qui sit $= M$, ita ut $Mdz - MVdx$ sit differentiale verum cuiuspiam functionis ipsarum x et z, quae functio sit $= S$ quantitatem y non involvens. Ex quo aequatio nostra integralis erit $S = f:y$, unde indoles functionis z, quemadmodum per x et y determinatur, innotescit. Differentiemus hanc aequationem sumto praeter x et z etiam y variabili eritque

$$dS = Mdz - MVdx = dyf':y$$

seu

$$dz = Vdx + \frac{dy}{M} f':y,$$

ita ut sit $q = \frac{1}{M} f':y$.

COROLLARIUM 1

59. Multiplicator etiam M formulam $dz - Vdx$ integrabilem reddens quantitatem y non continebit, quia in functione data V non inest y. Statim autem hoc multiplicatore invento valor ipsius $q = \frac{1}{M} f':y$ colligitur.

COROLLARIUM 2

60. Si formulae differentialis $Mdz - MVdx$ integrale fuerit S, functio ipsarum x et z, pro solutione problematis habebimus $S = f:y$, unde patet constantem, quam quis forte ad S adiicere voluerit, iam in functione arbitraria $f:y$ contineri.

EXEMPLUM 1

61. *Quaeratur eiusmodi functio z ipsarum x et y, ut sit* $\left(\frac{dz}{dx}\right)=\frac{nz}{x}$.

Positio $dz=\frac{nzdx}{x}+qdy$ sumto *y* constante erit $dz-\frac{nzdx}{x}=0$, quae aequatio per $\frac{1}{z}$ multiplicata fit integrabilis, ita ut sit multiplicator $M=\frac{1}{z}$ hincque integrale $S=lz-lx^n$; ergo aequatio nostra integralis quaesita erit $l\frac{z}{x^n}=f:y$, unde etiam $\frac{z}{x^n}$ aequabitur functioni cuicunque ipsius *y*, ita ut sit $z=x^n f:y$.

EXEMPLUM 2

62. *Quaeratur eiusmodi functio z binarum variabilium x et y, ut sit formula differentialis* $\left(\frac{dz}{dx}\right)=nx-z$.

Posito $dz=(nx-z)dx+qdy$ sumto *y* constante erit $dz+zdx-nxdx=0$, quae ope multiplicatoris $M=e^x$ dat

$$S=e^x z-n\int e^x xdx=e^x z-ne^x x+ne^x,$$

unde aequatio quaesitam relationem inter *x*, *y* et *z* exprimens est

$$e^x z-ne^x x+ne^x=f:y \text{ sive } z=n(x-1)+e^{-x}f:y,$$

tum vero erit

$$q=\left(\frac{dz}{dy}\right)=e^{-x}f':y.$$

EXEMPLUM 3

63. *Quaeratur eiusmodi functio z binarum variabilium x et y, ut sit formula differentialis* $\left(\frac{dz}{dx}\right)=\frac{xz}{xx+zz}$.

Ponatur ergo $dz=\frac{xzdx}{xx+zz}+qdy$ et posito *y* constante quaeratur integrale huius aequationis differentialis

$$dz-\frac{xzdx}{xx+zz}=0,$$

quae integrabilis redditur ope cuiusdam divisoris, qui ob homogeneitatem reperitur scribendo *x* et *z* loco differentialium *dx* et *dz*, ita ut hic divisor sit

$$z - \frac{xxz}{xx+zz} = \frac{z^3}{xx+zz}$$

hincque multiplicator $M = \frac{xx+zz}{z^3}$. Quare erit

$$dS = \frac{(xx+zz)dz}{z^3} - \frac{xdx}{zz}$$

ideoque

$$S = \frac{-xx}{2zz} + lz,$$

unde aequatio nostra quaesita erit

$$lz - \frac{xx}{2zz} = f:y \quad \text{et} \quad q = \frac{z^3}{xx+zz} f':y,$$

ex qua, cum posito $lz - \frac{xx}{2zz} = u$ sit $u = f:y$, etiam vicissim concludi potest fore $y = f:u$.

PROBLEMA 9

64. *Si z ita debeat determinari per binas variabiles x et y, ut formula differentialis* $\left(\frac{dz}{dx}\right)$ *aequetur functioni cuipiam datae omnes tres variabiles x, y et z, implicanti, quae sit* $=V$, *definire in genere indolem functionis z per x et y.*

SOLUTIO

Cum sit $dz = Vdx + qdy$, si sumamus y constans, erit $dz = Vdx$, quae ergo aequatio duas tantum continet variabiles *x* et *z*, litteram autem *y* in functione *V* involvens. Dabitur ergo multiplicator *M* hanc aequationem integrabilem reddens, ita ut sit

$$Mdz - MVdx = dS,$$

unde aequatio integralis relationem inter *x, y* et *z* exprimens erit $S = f:y$, ubi *S* erit functio certa ipsarum *x, y* et *z*, fierique potest, ut etiam *M* omnes has tres litteras comprehendat. Convenit autem functioni *S* per integrationem inventae valorem determinatum tribui, quoniam pars indeterminata in functione arbitraria $f:y$ includitur. Ponamus ergo *S* ita capi, ut evanescat, si ponatur $x = a$ et $z = c$.

Quodsi hinc aequationis differentialis propositae alteram partem *qdy* invenire velimus, differentiemus functionem *S* sumto etiam *y* variabili sitque

$$dS = Mdz - MVdx + Qdy = dyf':y;$$

ubi cum sit

$$\left(\frac{dQ}{dz}\right) = \left(\frac{dM}{dy}\right) \quad \text{et} \quad \left(\frac{dQ}{dx}\right) = -\left(\frac{d.MV}{dy}\right),$$

erit sumto iterum *y* constante

$$dQ = dz\left(\frac{dQ}{dz}\right) + dx\left(\frac{dQ}{dx}\right) = dz\left(\frac{dM}{dy}\right) - dx\left(\frac{d.MV}{dy}\right),$$

quae formula eerto erit integrabilis. Capi autem Q eadem lege debet, qua S sumsimus, ita ut evaneseat posito $x = a$ et $z = c$, atque inventa hac quantitate Q, cum habeamus

$$dz = Vdx - \frac{Qdy}{M} + \frac{dy}{M}\, f':y\,,$$

erit

$$q = \left(\frac{dz}{dy}\right) = \frac{-Q + f':y}{M}.$$

Haec determinatio isto nititur fundamento, quod, si S fuerit eiusmodi functio ipsarum x, y et z, quae posito $x = a$ et $z = c$ evanescat, etiam formula differentialis $\left(\frac{dS}{dy}\right)$ eodem casu evanescat.

COROLLARY 1

65. Reducitur ergo resolutio huius problematis ad integrationem aequationis differentialis

$$dz - Vdx = 0\,,$$

In qua quantitas y ut constans spectatur, etiamsi V contineat omnes tres litteras x, y et z. Dabitur ergo utique multiplicator M, qui hanc aequationem integrabilem reddat, ut sit

$$Mdz - MVdx = dS$$

existente S certa quadam functione ipsarum x, y et z.

COROLLARY 2

66. Invento autem hoc multiplicatore M indeque integrali S quantitas z ita per binas variabiles x et y definietur, ut sit $S = f:y$, ubi $f:y$ denotat functionem quamcunque ipsius y sive continuam sive etiam discontinuam, ob cuius naturam integratio pro completa est habenda.

COROLLARIUM 3

67. Cum hoc modo relatio inter *z, x, y* fuerit definita, erit ea ita differentiata, ut omnes tres litterae *x, y* et *z* variabiles sumantur,

$$dz = Vdx + \frac{f':y-Q}{M}dy\,,$$

ubi quantitas *Q* ex suo differentiali

$$dQ = dz\left(\frac{dM}{dy}\right) - dx\left(\frac{d.MV}{dy}\right)$$

definiri debet sumta *y* constante, integrationem ita temperando, ut, si *S* evanescat casu $x = a$ et $z = c$, etiam *Q* eodem casu evanescat.

SCHOLION

68. Hic ergo ad insigne hoc theorema deducimur:

Quodsi fuerit S eiusmodi functio ipsarum x, y et z, quae evanescat ponendo $x = a$ *et* $z = c$, *tum etiam pro eadem positione formulam* $\left(\frac{dS}{dy}\right)$ *esse evanituram.*

Veluti si fuerit

$$S = Axx + Bxyz + Czz - Aaa - Bacy - Ccc\,,$$

erit

$$\left(\frac{dS}{dy}\right) = Bxz - Bac\,,$$

quarum utraque expressio casu $x = a$ et $z = c$ evanescit. Pluribus autem huiusmodi exemplis evolutis veritas theorematis ita patet, ut demonstratio solennis non desideretur. Interim huiusmodi functio semper quantitates solam *y* continentes a reliquis separando ita evolvi potest, ut in talem formam transmutetur

$$S = PY + QY' + R\ Y'' + \text{etc.},$$

ubi per hypothesin $P,\ Q, R$ etc. sunt functiones ipsarum *x* et *z* tantum et tales quidem, quae ponendo $x = a$ et $z = c$ singulae evanescant. Hinc iam perspicuum est fore

$$\left(\frac{dS}{dy}\right) = P\frac{dY}{dy} + Q\frac{dY'}{dy} + R\frac{dY''}{dy} + \text{etc.},$$

quae forma manifesto sub iisdem conditionibus evanescit. Quomodocunque autem functio *S* hac indole praedita fuerit complicata tam formulis irrationalibus quam transcendentibus, eam semper in eiusmodi formam evolvere licet; quae etsi in infinitum progrediatur, haec demonstratio tamen vim suam retinet.

EXEMPLUM 1

69. *Quaeratur eiusmodi functio z duarum variabilium x et y, ut sit formula differentialis* $\left(\frac{dz}{dx}\right)=\frac{xz}{ay}$.

Ponamus ergo $dz=\frac{xzdx}{ay}+qdy$ et sumta y constante habebitur aequatio $dz-\frac{xzdx}{ay}=0$, ut sit $V=\frac{xz}{ay}$, et multiplicator erit $M=\frac{1}{z}$; unde fit

$$S=l\frac{z}{c}-\frac{xx-aa}{2ay}$$

et aequatio integralis completa functionem z determinans erit

$$l\frac{z}{c}+\frac{aa-xx}{2ay}=f:y$$

Porro ad quantitatem q inveniendam ob $M=\frac{1}{z}$ et $MV=\frac{x}{ay}$ erit $dQ=\frac{xdx}{ayy}$ et $Q=\frac{xx-aa}{2ayy}$ unde fit $q=zf':y-\frac{z(xx-aa)}{2ayy}$.

Hic idem autem valor ex differentiatione aequationis inventae eruitur, quae praebet

$$\frac{dz}{z}-\frac{xdx}{ay}-\frac{aa-xx}{2ayy}dy=dyf':y$$

ideoque

$$dz=\frac{xzdx}{ay}+\frac{z(xx-aa)}{2ayy}dy+zdyf':y\,,$$

ita ut sit

$$q=\frac{z(xx-aa)}{2ayy}+zf':y\,.$$

EXEMPLUM 2

70. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{dz}{dx}\right)=\frac{y}{x+z}\,.$$

Cum sit $V=\frac{y}{x+z}$ habebitur sumto y constante haec aequatio

$$dz-\frac{ydx}{x+z}=0\,,$$

ad cuius multiplicatorem inveniendum multiplicetur primo per $x+z$, ut prodeat

$$xdz+zdz-ydx=0 \text{ seu } dx-\frac{xdz}{y}=\frac{zdz}{y}\,,$$

quae multiplicata per $e^{-\frac{z}{y}}$ integrabilis evadit proditque

$$e^{-\frac{z}{y}}x=\int e^{-\frac{z}{y}}\frac{zdz}{y}=-e^{-\frac{z}{y}}z+\int e^{-\frac{z}{y}}dz$$

hincque

$$e^{-\frac{z}{y}}x=-e^{-\frac{z}{y}}z-ye^{-\frac{z}{y}}+C.$$

Quocirca erit multiplicator

$$M=(x+z)\cdot-\frac{1}{y}\cdot e^{-\frac{z}{y}}=-\frac{x+z}{y}e^{-\frac{z}{y}}$$

et

$$S=e^{-\frac{z}{y}}(x+z+y)-e^{-\frac{c}{y}}(a+c+y),$$

ex quo aequatio integralis completa erit

$$e^{-\frac{z}{y}}(x+z+y)-e^{-\frac{c}{y}}(a+c+y)=f:y.$$

Nunc porro cum sit $MV=-e^{-\frac{z}{y}}$, erit

$$\left(\frac{dM}{dy}\right)=e^{-\frac{z}{y}}\left(\frac{x+z}{yy}-\frac{z(x+z)}{y^3}\right)=e^{-\frac{z}{y}}(x+z)\left(\frac{1}{yy}-\frac{z}{y^3}\right)$$

et

$$\left(\frac{d.MV}{dy}\right)=-e^{-\frac{z}{y}}\cdot\frac{z}{yy}$$

hincque

$$dQ=e^{-\frac{z}{y}}\left(dz(x+z)\left(\frac{1}{yy}-\frac{z}{y^3}\right)+\frac{zdx}{yy}\right)$$

sumto y constante, unde integrando obtinebitur

$$Q=e^{-\frac{z}{y}}\left(\frac{xz}{yy}+1+\frac{z}{y}+\frac{zz}{yy}\right)-e^{-\frac{c}{y}}\left(\frac{ac}{yy}+1+\frac{c}{y}+\frac{cc}{yy}\right),$$

hinc

$$q=\frac{z}{y}+\frac{y+z}{x+z}-e^{\frac{z-c}{y}}\left(\frac{ac+cc+cy+yy}{y(x+z)}\right)-\frac{y}{x+z}e^{\frac{c}{y}}f':y,$$

ita ut sit

$$dz=\frac{ydx}{x+z}+qdy.$$

Aequatio autem inventa si differentietur, dat

$$-e^{-\frac{z}{y}}\frac{(x+z)dz}{y}+e^{-\frac{z}{y}}dx+e^{-\frac{z}{y}}dy\left(1+\frac{z}{y}+\frac{xz}{yy}+\frac{zz}{yy}\right)$$
$$-e^{-\frac{c}{y}}dy\left(1+\frac{c}{y}+c\frac{(a+c)}{yy}\right)=dyf':y,$$

unde idem prorsus valor pro q concluditur.

EXEMPLUM 3

71. *Quaeratur binarum variabilium x et y eiusmodi functio z, ut sit*

$$\left(\frac{dz}{dx}\right)=\frac{yy+zz}{yy+xx}.$$

Posito $dz=\frac{yy+zz}{yy+xx}dx+qdy$ sumatur quantitas y constans, et cum sit $dz-\frac{(yy+zz)dx}{yy+xx}=0$, evidens est multiplicatorem idoneum esse $M=\frac{y}{yy+zz}$,
unde, cum sit $\frac{ydz}{yy+zz}-\frac{ydx}{yy+xx}=0$, erit per integrationem

$$S=\text{Atang.}\frac{z}{y}-\text{Atang.}\frac{x}{y}+C=\text{Atang.}\frac{yz-yx}{yy+xz}-\text{Atang.}\frac{(c-a)y}{ac+yy}$$

et functio quaesita z hac aequatione definietur

$$\text{Atang.}\frac{y(z-x)}{yy+xz}-\text{Atang.}\frac{(c-a)y}{ac+yy}=f:y.$$

Cum porro sit $MV=\frac{y}{yy+xx}$, erit

$$\left(\frac{dM}{dy}\right)=\frac{zz-yy}{(yy+xx)^2}\quad\text{et}\quad\left(\frac{d.MV}{dy}\right)=\frac{xx-yy}{(yy+xx)^2}$$

hincque

$$dQ=\frac{(zz-yy)dz}{(yy+zz)^2}-\frac{(xx-yy)dx}{(yy+xx)^2}$$

sumto y constante. Ergo

$$Q=\frac{-z}{yy+zz}+\frac{x}{yy+xx}+\frac{c}{yy+cc}-\frac{a}{yy+aa}\quad\text{et}\quad q=\frac{-Q+f':y}{M},$$

qui idem valor etiam ex differentiatione prodit.

Ceterum cum constantes a et c pro lubitu accipi queant, sumtis iis nihilo aequalibus seu saltem $c=a$ erit aequatio integralis

$$\text{Atang.}\frac{y(z-x)}{yy+xz}=f:y,$$

unde erit etiam

$$\frac{y(z-x)}{yy+xz} = \text{funct.}y \quad \text{et} \quad \frac{yy+xz}{z-x} = \text{funct.}y$$

quae functio si dicatur Y, habebitur $z = \frac{yy+xY}{Y-x}$.

SCHOLION

72. Vix opus est notari saepe fieri posse, ut solutio huiusmodi quaestionum superet vires Analyseos, quando scilicet aequatio differentialis resolvenda artificiis adhuc cognitis integrari nequit. Veluti si proponatur casus

$$\left(\frac{dz}{dx}\right) = \frac{yy}{xx+zz},$$

unde sumpto y constante fieri debet $yydx = xxdz + zzdz$, cuius integrationem nondum expedire licet. Interim quia integrale per seriem exhiberi potest, modo id fiat complete, etiam solutio per seriem obtinebitur. Posito scilicet $x = \frac{-yydu}{udz}$ et sumto elemento dz constante oritur haec aequatio differentio- differentialis

$$y^4 ddu + uzzdz^2 = 0,$$

unde per series integrando reperitur

$$u = A\left(1 - \frac{z^4}{3.4\,y^4} + \frac{z^8}{3.\,4.7.8y^8} - \text{etc.}\right) + Bz\left(1 - \frac{z^4}{4.5\,y^4} + \frac{z^8}{4.5.8.9y^8} - \text{etc.}\right),$$

ubi pro A et B functiones quaecunque ipsius y accipi possunt. Quare si ponatur $\frac{A}{B} = f:y$, erit

$$x = \frac{yyf:y\left(\frac{z^3}{3y^4} - \frac{z^7}{3.\,4.7y^8} + \text{etc.}\right) - yy\left(1 - \frac{z^4}{4y^4} + \frac{z^8}{4.5.8y^8} - \text{etc.}\right)}{f:y\left(1 - \frac{z^4}{3.4\,y^4} + \frac{z^8}{3.\,4.7.8y^8} - \text{etc.}\right) + z\left(1 - \frac{z^4}{4.5\,y^4} + \frac{z^8}{4.5.8.9y^8} - \text{etc.}\right)},$$

qua aequatione functio quaesita z per binas variabiles x et y generalissime exprimitur.

Quoniam ergo methodos aperuimus aequationes differentiales quascunque per approximationes integrandi idque complete, his methodis in subsidium vocandis omnia problemata huc pertinentia saltem per approximationem resolvi poterunt. Ceterum in hac parte Analyseos sublimiori resolutionem aequationum differentialium ad priorem partem Analysis pertinentium pro concessa assumere possumus, omnino uti, quo longius in Analyst progredimur, ea semper, quae ad partes praecedentes pertinent, etiamsi non penitus sunt evoluta, tanquam confecta spectare solemus.