CHAPTER V

CONCERNING THE INTEGRATION OF DIFFERENTIAL EQUATIONS OF THE FORM

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \frac{Ex^4d^4y}{dx^4} + \text{etc.}$$

PROBLEM 164

1226. *With a proposed differential equation of this form :*

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \cdots + \frac{Nx^nd^ny}{dx^n},$$

to define a function of x multiplied by which that is made integrable.

SOLUTION

It soon becomes evident on examination that a simple power of x is best. Therefore this equation shall be integrable

$$Xx^\lambda dx = Ax^\lambda ydx + Bx^{\lambda+1}dy + \frac{Cx^{\lambda+2}ddy}{dx} + \cdots + \frac{Nx^{\lambda+n}d^ny}{dx^{n-1}},$$

the integral of which shall be

$$\int Xx^\lambda dx = A'x^{\lambda+1}y + \frac{B'x^{\lambda+2}dy}{dx} + \frac{C'x^{\lambda+3}ddy}{dx^2} + \cdots + \frac{M'x^{\lambda+n}d^{n-1}y}{dx^{n-1}}.$$

Therefore since the differential of this must be equal to that, we may obtain the following determinations [on equating equal powers of x and differentials of y] :

$$
\begin{aligned}
&A=(\lambda+1)A', &&\text{hence } (\lambda+1)A'=A,\\
&B=(\lambda+2)B'+A', &&\text{"} \quad (\lambda+1)(\lambda+2)B'=(\lambda+1)B-A,\\
&C=(\lambda+3)C'+B', &&\text{"} \quad (\lambda+1)(\lambda+2)(\lambda+3)C'=(\lambda+1)(\lambda+2)C\\
& && \qquad -(\lambda+1)B+A,\\
&D=(\lambda+4)D'+C', &&\text{"} \quad (\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)D'\\
& && \qquad =(\lambda+1)(\lambda+2)(\lambda+3)D\\
& && \qquad -(\lambda+1)(\lambda+2)C+(\lambda+1)B-A\\
&\vdots\\
&\vdots\\
&N=M' && \text{etc.;}
\end{aligned}
$$

indeed the following terms of the integral, which involve the order of the differential $d^n y$ and higher, must vanish, because otherwise the integration would not be successful. Therefore since in the integration the letter N' vanishes, we arrive at this equation

$$
\begin{aligned}
0= A-(\lambda+1)B+(\lambda+1)(\lambda+2)C-(\lambda+1)(\lambda+2)(\lambda+3)D+\ldots\\
\pm(\lambda+1)(\lambda+2)\ \ldots\ (\lambda+n)N,
\end{aligned}
$$

from which equation the exponent λ of power sought x^{λ} must be defined.

Therefore such an algebraic expression may be formed:

$$
\begin{aligned}
P=A+B(z-1)+C(z-1)(z-2)+D(z-1)(z-2)(z-3)+\ldots\\
+N(z-1)(z-2)\ \ldots\ (z-n)
\end{aligned}
$$

and all the simple factors of this may be sought, so that

$$P=(\alpha+z)(\beta+z)(\gamma+z)(\delta+z) \quad \text{etc.}$$

of which the number of factors present $=n$. Now from any factor $z+\alpha$ reduced to zero the value $z=-\alpha$ will give the power x^{α}, on multiplying by which the proposed equation becomes integrable, thus so that its integral shall become

$$x^{-\alpha-1}\int x^{\alpha}Xdx=A'y+\frac{B'xdy}{dx}+\frac{C'x^2ddy}{dx^2}+\frac{D'x^3d^3y}{dx^3}+\cdots+\frac{Nx^{n-1}d^{n-1}y}{dx^{n-1}},$$

where the order of the differentials is less by one. But now this equation is determined by the proposed integration, so that there shall be

$$A = (\alpha+1)A' \qquad B = (\alpha+2)B' + A'$$
$$C = (\alpha+3)C' + B' \qquad D = (\alpha+4)D' + C'$$
etc.

and hence the final coefficient N is arrived at, which is the same in both parts.

COROLLARY 1

1227. Because the integrated equation is similar to that proposed, that multiplied by a certain power of x becomes integrable again. Indeed towards finding that power it is required to consider this algebraic form :

$$Q = A' + B'(z-1) + C'(z-1)(z-2) + D'(z-1)(z-2)(z-3) + \ldots$$
$$+ N(z-1)(z-2)\ \ldots\ (z-n+1),$$

if there should be some simple factor of this $z+\mu$, x^{μ} will be that power of x rendering the equation integrable.

COROLLARY 2

1228. But if that proposed equation multiplied by the power x^{α} should be returned integrable, here it is agreed properly to be observed that the quantity Q formed from the integration thus depends on the former P from that proposed form, so that there shall be $Q = \frac{P}{\alpha+z}$ since by hypothesis $\alpha+z$ is a factor of P.

SCHOLIUM 1

1229. Towards demonstrating this notable property, which evidently shall be $P = (\alpha+z)Q$, it is necessary only that the quantity Q be multiplied by $\alpha+z$; now so that this conclusion becomes more apparent, for the individual terms of Q the multiplier is to be represented in two parts, and indeed for the first term in place of the term $\alpha+z$ there may be written $(\alpha+1)+(z-1)$, for the second $(\alpha+2)+(z-2)$, for the third $(\alpha+3)+(z-3)$, for the fourth $(\alpha+4)+(z-4)$ etc., thus so that there may be shown the product of each term by both parts, as I may put in place the whole operation here :

$Q = A'$	$+B'(z-1)$	$+C'(z-1)(z-2)$	$+D'(z-1)(z-2)(z-3)$ + etc.
Multiplier. $\alpha+1$ $z-1$	$\alpha+2$ $z-2$	$\alpha+3$ $z-3$	$\alpha+4$ $z-4$
Product $(\alpha+1)A'+$	$A'(z-1)+$	$B'(z-1)(z-2)+$	$C'(z-1)(z-2)(z-3)$

$$+(\alpha+2)B'(z-1)+(\alpha+3)C'(z-1)(z-2)$$
$$+(\alpha+4)D'(z-1)(z-2)(z-3)+\text{etc.}$$

But from the solution we may consider that there is

$$(\alpha+1)A' = A,\ \ (\alpha+2)B'+A' = B,\ \ (\alpha+3)C'+B' = C \quad \text{etc.},$$

wherefore this expression produced can be expressed in this form

$$A+B(z-1)+C(z-1)(z-2)+D(z-1)(z-2)(z-3)+\text{etc.},$$

to which the value of P is equal, and thus the notable property mentioned has been demonstrated, since there shall be $Q=\frac{P}{\alpha+z}$.

COROLLARIUM 3

1230. But if hence the value of P is thus represented resolved into simple factors

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z) \quad \text{etc.}$$

and from the factor $\alpha+z$ the proposed equation multiplied by x^{α} may be integrated, then from the integral in a like manner the value of Q may be formed, and it will be

$$Q = N(\beta+z)(\gamma+z)(\delta+z) \quad \text{etc.}$$

COROLLARY 4

1231. Therefore the integrated equation after it has been led to the prescribed form, so that on putting

$$x^{-\alpha-1}\int x^{\alpha}Xdx = X'$$

there may be had

$$X' = A'y+\frac{Bxdy}{dx}+\frac{C'x^2ddy}{dx^2}+\frac{D'x^3d^3y}{dx^3}+\text{etc},$$

this is rendered integrable again, if it is multiplied by some of these powers $x^{\beta}, x^{\gamma}, x^{\delta}$ etc., which also may render the proposed equation integrable.

SCHOLIUM 2

1232. Before I pursue further the continuation of these integrations, it may be appropriate to set out that case itself of the proposed general form separately, where the first member X of the equation goes to zero. Indeed in this case this comes ready for use, as the complete integral can be shown without a repeated integration and in a similar manner, by which I have used it above in Chapter II [§ 1125]. Indeed for this case, because now it can be treated much more easily, I have been unwilling to assign to a chapter of its own, lest the teaching precepts should appear to be multiplied exceedingly

PROBLEM 165

1233. *With this proposed differential equation of any order*

$$0 = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \text{etc.},$$

where the variable y with its differentials is nowhere greater by a single dimension than with the one dimension of the other variable x thus taken as zero, to find its complete integral.

SOLUTION

It is evident for this equation to be satisfied particularly, if y is equal to a certain power of x ; therefore we may put $y = x^{\mu}$ and with the substitution made, since we will have divided everywhere by x^{μ}, we reach this equation

$$0 = A + \mu B + \mu(\mu-1)C + \mu(\mu-1)(\mu-2)D + \text{etc.},$$

from which it is required to determine the exponent μ. Or if following the precepts of the previous problem , from the proposed equation we may form this algebraic form

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$$

and in that we may resolve all the factors, so that there shall be

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z) \quad \text{etc.},$$

and it is clear on putting $\mu = z - 1$ for the first equation to be satisfied on putting

$$\mu = -\alpha - 1 \quad \text{or} \quad \mu = -\beta - 1 \quad \text{or} \quad \mu = -\gamma - 1 \quad \text{etc.,}$$

thus so that each factor corresponds to a particular integral. Therefore since the number of factors is equal to the order of the differential I sum, hence it may be gathered that the complete integral of the proposed equation

$$y = \mathfrak{A}x^{-\alpha-1} + \mathfrak{B}x^{-\beta-1} + \mathfrak{C}x^{-\gamma-1} + \mathfrak{D}x^{-\delta-1} + \text{etc.,}$$

where it is so appropriate to observe, if two or more of the simple factors should be equal to each other, the form of the integral must be changed in a like manner, as I have used above in Chapter II. Clearly since the equations treated here are returned to the present form, in everywhere in place of *x* there is written *lx*, from which we may draw up these rules :

1. If the form *P* should have a factor $(\alpha + z)^2$, from this the part of the integral arises
$$= x^{-\alpha-1}\left(\mathfrak{A} + \mathfrak{B}lx\right).$$
2. If the form *P* should have a factor $(\alpha + z)^3$, from this the part of the integral arises
$$= x^{-\alpha-1}\left(\mathfrak{A} + \mathfrak{B}lx + \mathfrak{C}(lx)^2\right).$$
3. If the form *P* should have a factor $(\alpha + z)^4$, from this the part of the integral arises
$$= x^{-\alpha-1}\left(\mathfrak{A} + \mathfrak{B}lx + \mathfrak{C}(lx)^2 + \mathfrak{D}(lx)^3\right).$$

If imaginary factors arise, the parts arising are returned to form real parts by the usual reduction of imaginary parts, as I will show in the corollaries.

COROLLARY 1

1234. If the form *P* should have two simple imaginary factors present in the formula $ff + 2fz\cos.\theta + zz$, since compared with this product $(\alpha + z)(\beta + z)$, there will be

$$\alpha = f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right) \text{ et } \beta = f\left(\cos.\theta - \sqrt{-1}\cdot\sin.\theta\right),$$

from which there becomes

$$x^{-\alpha} = x^{-f\cos.\theta}x^{-f\sqrt{-1}\cdot\sin.\theta} = x^{-f\cos.\theta}e^{-\sqrt{-1}\cdot\sin.\theta\cdot lx}.$$

Now there is

$$e^{-u\sqrt{-1}} = \cos.u - \sqrt{-1}\cdot\sin.u$$

and thus there will be had

$$x^{-\alpha-1} = x^{-f\cos.\theta}\,\frac{\cos.(f\sin.\theta\cdot lx) - \sqrt{-1}\cdot\sin.(f\sin.\theta\cdot lx)}{x}$$

Whereby since $x^{-\beta-1}$ may be expressed in a like manner on changing the sign of $\sqrt{-1}$, from the twofold factor $ff+2fz\cos.\theta+zz$ this part of the integral arises

$$x^{-f\cos.\theta-1}\left(\mathfrak{A}\cos.\left(f\sin.\theta\cdot lx\right)+\mathfrak{B}\sin.\left(f\sin.\theta\cdot lx\right)\right),$$

which also can be represented thus

$$\mathfrak{A}x^{-f\cos.\theta-1}\cos.\left(\mathfrak{a}+f\sin.\theta\cdot lx\right),$$

with $\mathfrak{a}$ denoting some arbitrary constant angle.

COROLLARY 2

1235. In a similar manner if the form *P* should involve two equal factors of this kind, so that there shall be

$$(\alpha+z)^2(\beta+z)^2=(ff+2fz\cos.\theta+zz)^2,$$

the letters α and β will choose the same imaginary values as before, from the reduction of which it is deduced that this part of the integral hence is required to arise,

$$x^{-f\cos.\theta-1}\left(\mathfrak{A}\cos.\left(\mathfrak{a}+f\sin.\theta\cdot lx\right)+\mathfrak{B}lx\cos.\left(\mathfrak{b}+f\sin.\theta\cdot lx\right)\right)$$

containing four arbitrary constants $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{a}$ and $\mathfrak{b}$.

COROLLARY 3

1236. Hence it is therefore evident, how from the factors of the form *P*, whither they shall be simple or twofold, either unequal or equal, the individual parts of the integral are to be assigned, and from these the whole complete integral to be formed can be agreed upon.

SCHOLION

1237. Therefore the whole task reduces to this, that an algebraic quantity formed from the differential equation

$$P=A+B(z-1)+C(z-1)(z-2)+D(z-1)(z-2)(z-3)+\text{etc.}$$

can be resolved into its factors, either real, simple or twofold, in which generally great difficulty is involved, because forms of this kind are not usually treated. Truly since this resolution of the equation in general, as I have undertaken to set out in this chapter, is in common use, whatever has been clear to present here, is convenient to be shown rather for the general equation, to the resolution of which therefore I return. I begin by noting that only by necessity here, if for the general equation

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cxxddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \text{etc.}$$

in whatever manner a particular integral will have become known, for example $y = V$ with V being a function of x, then on putting $y = V + v$ to arrive at this equation

$$0 = Av + \frac{Bxdv}{dx} + \frac{Cxxddv}{dx^2} + \frac{Dx^3d^3v}{dx^3} + \text{etc.};$$

the complete integral of which, found by the precepts of this problem, if written in place of v, then the complete integral of this problem will be had, with which agreed upon certainly a significant gain to the calculation is obtained.

PROBLEM 166

1238. *With the proposed differential equation of any order n of this form*

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \cdots + \frac{Nx^nd^ny}{dx^n}$$

to find the integral of this by repeated integration n times in turn.

SOLUTION

This algebraic quantity may be formed from this equation

$$P = A + B(z-1) + C(z-1)(z-2) + \ldots + N(z-1)(z-2)\ \ldots\ (z-n),$$

of which all the simple factors are sought, having no regard as to whether they are real or imaginary, so that this may be expressed in the form

$$P = N(\alpha + z)(\beta + z)(\gamma + z)\cdots(\mu + z)(v + z)$$

with the number of factors present $= n$. With which done we have seen from the beginning of this chapter that any factor, for example $\alpha + z$, provides a power x^{α}, by which our equation is made integrable, and indeed we have shown the integral arising from this, if for shortness we put

$$x^{-\alpha-1}\int x^{\alpha}Xdx = X',$$

to become

$$X' = A'y + \frac{B'xdy}{dx} + \frac{C'x^2ddy}{dx^2} + \frac{D'x^3d^3y}{dx^3} + \cdots + \frac{Nx^{n-1}d^{n-1}y}{dx^{n-1}},$$

thus so that there shall be

$$A' = \frac{\alpha}{A+1}$$

and the other coefficients thus themselves may be had, as we have shown in that place ; but here it suffices mainly for the first to be considered.

Now with the first integration completed if by the same rule from the equation integrated we may form from the integral the quantity

$$P' = A' + B'(z-1) + C'(z-1)(z-2) + \cdots + N(z-1)(z-2)\ \ldots\ (z-n+1),$$

the resolution of which into factors now agrees from the first form *P*, to be shown after §1229 to be

$$P' = N(\beta+z)(\gamma+z)(\delta+z)\cdots(\mu+z)(\nu+z),$$

thus so that there shall be $P' = \frac{P}{\alpha+z}$, hence therefore in a similar manner the factor $\beta + z$ gives rise to the multiplier x^{β}, from which this integrable equation is returned, and on putting

$$x^{-\beta-1}\int x^{\beta} X' dx = X'',$$

so that there shall be

$$X'' = x^{-\beta-1}\int x^{\beta-\alpha-1}dx\int x^{\alpha} X dx,$$

the integral will be

$$X'' = A''y + \frac{B''xdy}{dx} + \frac{C''x^2ddy}{dx^2} + \cdots + \frac{Nx^{n-2}d^{n-2}y}{dx^{n-2}}$$

with here arising

$$A'' = \frac{A'}{\beta+1} = \frac{A}{(\alpha+1)(\beta+1)}.$$

If in this manner as many integrations are completed successfully as there are units present in the index *n*, and thus all the simple factors of the form *P* may be called into use, finally the equation is arrived at

$$X^{(n)} = A^{(n)}y,$$

which is the desired integral itself. But since this shall become

$$A^{(n)} = \frac{A}{(\alpha+1)(\beta+1)(\gamma+1)\cdots(\nu+1)},$$

it is evident that the denominator has come from the form $\frac{P}{N}$, if there is written unity in place of *z*; but then on assuming $z=1$ clearly the first form gives $P=A$, thus so that this denominator is made $=\frac{A}{N}$ and thus $A^{(n)}=N$, because it is apparent also from that, how of all the equations the final terms have the same coefficient *N*, from which therefore in the last integral the first term *y* must be affected. Then indeed there shall be

$$X^{(n)}=x^{-\nu-1}\int x^{\nu-\mu-1}dx\int x^{\mu-\lambda-1}dx\cdots\int x^{\beta-\alpha-1}dx\int x^{\alpha}Xdx,$$

where since the numbers α,β,γ etc. can be interchanged amongst themselves in any way, the integral sought can also be represented in this way

$$Ny=x^{-\alpha-1}\int x^{\alpha-\beta-1}dx\int x^{\beta-\gamma-1}dx\cdots\int x^{\mu-\nu-1}dx\int x^{\nu}Xdx.$$

COROLLARY 1

1239. Hence the whole business resorts to this, so that the algebraic form

$$P=A+B(z-1)+C(z-1)(z-2)+D(z-1)(z-2)(z-3)+\text{etc.}$$

may be resolved into its simple factors, with which found so that there shall be

$$P=N(\alpha+z)(\beta+z)(\gamma+z)\cdots(\nu+z),$$

hence the integral sought can be readily shown and indeed in several ways by the varied permutation of the factors, but which all express the same value, as will appear clearer from what follows.

COROLLARY 2

1240. Since this form of the integral found involves as many integrations, as the grade of the proposed differential equation should be, also just as many arbitrary constants are brought in, as the nature of the complete integral postulates.

SCHOLIUM

1241. Because the integral found has been complicated by several integrations, it may be convenient to resolve this form into parts for easier use, which may contain only a single integral sign. But this resolution is allowed to be put in place in a similar manner, to what we have used above, and here indeed the whole calculation can be recalled according to a formula of this kind :

$$\int x^{m-n-1}dx\int x^{n}Xdx,$$

which clearly may be reduced thus, to that there shall be

$$\frac{1}{m-n}x^{m-n}\int x^n Xdx - \frac{1}{m-n}\int x^m Xdx,$$

where yet is to be noted, if there should be $m=n$, there is need for a special reduction, and in this case it becomes

$$\int\frac{dx}{x}\int x^n Xdx = lx\int x^n Xdx - \int x^n Xdxlx.$$

Hence we may use this rule in the resolution of the following problem, in which successively we traverse all the orders of the differentials, indeed with the first order omitted, because the integral of the equation

$$X = Ay + \frac{Nxdy}{dx}$$

on account of $P = A + N(z-1) = N(\alpha + z)$ shall be

$$Ny = x^{-\alpha-1}\int x^{\alpha} Xdx,$$

which needs no reduction.

PROBLEM 167

1242. *With this proposed differential equation of the second order*

$$X = Ay + \frac{Bxdy}{dx} + \frac{Nx^2ddy}{dx^2}$$

to set out the integral of this by simple integral formulas.

SOLUTION

Since there shall be

$$P = A + B(z-1) + N(z-1)(z-2),$$

there is put in place

$$P = N(\alpha + z)(\beta + z)$$

and the integral found by the preceding method with $Ny = X''$ arising will be

$$x^{\beta+1}X'' = \int x^{\beta-\alpha-1}dx\int x^{\alpha} Xdx,$$

which form is changed into this

$$\tfrac{1}{\beta-\alpha}x^{\beta-\alpha}\int x^{\alpha}Xdx-\tfrac{1}{\beta-\alpha}\int x^{\beta}Xdx\,.$$

and thus there will be

$$Ny=\tfrac{x^{-\alpha-1}}{\beta-\alpha}\int x^{\alpha}Xdx+\tfrac{x^{-\beta-1}}{\alpha-\beta}\int x^{\beta}Xdx.$$

But hence it is required that the case in which $\beta=\alpha$ be excepted; then indeed there becomes

$$x^{\alpha+1}X''=\int\tfrac{dx}{x}\int x^{\alpha}Xdx=lx\int x^{\alpha}Xdx-\int x^{\alpha}Xdxlx,$$

therefore for this case we will have

$$Ny=x^{-\alpha-1}\int\tfrac{dx}{x}\int x^{\alpha}Xdx \quad\text{or}\quad x^{-\alpha-1}\left(lx\int x^{\alpha}Xdx-\int x^{\alpha}Xdxlx\right),$$

where indeed the first form is seen to be preferred.

COROLLARY 1

1243. If both simple forms shall be imaginary, there is put

$$(\alpha+z)(\beta+z)=ff+2fz\cos.\theta+zz$$

and there will be

$$\alpha=f\left(\cos.\theta+\sqrt{-1}\cdot\sin.\theta\right)\text{ and }\beta=f\left(\cos.\theta-\sqrt{-1}\cdot\sin.\theta\right)$$

and from this

$$\beta-\alpha=-2f\sqrt{-1}\cdot sin.\theta\,.$$

Then truly

$$x^{\alpha}=x^{f\cos.\theta}\left(\cos.(f\sin.\theta\cdot lx)+\sqrt{-1}\cdot\sin.(f\sin.\theta\cdot lx)\right)$$

and

$$x^{-\alpha}=x^{-f\cos.\theta}\left(\cos.(f\sin.\theta\cdot lx)-\sqrt{-1}\cdot\sin.(f\sin.\theta\cdot lx)\right),$$

which formulas by the change of sign of $\sqrt{-1}$ are transformed into x^{β} and $x^{-\beta}$.

COROLLARY 2

1244. For the sake of brevity the angle is put in place $f\sin.\theta\cdot lx=\varphi$ and with the substitution made we will have

$$Nxy=\frac{x^{-f\cos.\theta}\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)}{-2f\sqrt{-1}\cdot sin.\theta}\int x^{f\cos.\theta}Xdx\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)$$
$$+\frac{x^{-f\cos.\theta}\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)}{2f\sqrt{-1}\cdot sin.\theta}\int x^{f\cos.\theta}Xdx\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right),$$

where the imaginary parts cancel each other, and there becomes

$$Nxy=\frac{x^{-f\cos.\theta}}{fsin.\theta}\left(\sin.\varphi\int x^{f\cos.\theta}Xdx\cos.\varphi-\cos.\varphi\int x^{f\cos.\theta}Xdx\sin.\varphi\right).$$

COROLLARY 3

1245. But this form of real quantities is found,

$$Nxy=\frac{x^{-\alpha}}{\beta-\alpha}\int x^{\alpha}Xdx+\frac{x^{-\beta}}{\alpha-\beta}\int x^{\beta}Xdx,$$

equivalent to that involving imaginaries, if there should be

$$(\alpha+z)(\beta+z)=ff+2fz\cos.\theta+zz$$

and there is put $\varphi=f\sin.\theta\cdot lx$, which reduction once made will be better also to be used in what follows.

PROBLEM 168

1246. *With the proposed differential equation of the third order*

$$X=Ay+\frac{Bxdy}{dx}+\frac{Cxxddy}{dx^2}+\frac{Nx^3d^3y}{dx^3},$$

to set out the integral of this by simple integral formulas.

SOLUTION

Since here there shall be

$$P=A+B(z-1)+C(z-1)(z-2)+N(z-1)(z-2)(z-3),$$

there is put

$$P = N(\alpha + z)(\beta + z)(\gamma + z),$$

and since by general integration there is produced $Ny = X'''$, it is to be noted that $X''' = x^{-\gamma-1}\int x^{\gamma} X'' dx$, if indeed we can now find the value of X'' from the two factors $\alpha + z$ and $\beta + z$; hence indeed by the preceding problem there is found

$$X'' = \tfrac{x^{-\alpha-1}}{\beta-\alpha}\int x^{\alpha} Xdx + \tfrac{x^{-\beta-1}}{\alpha-\beta}\int x^{\beta} Xdx,$$

from which there is deduced

$$\int x^{\gamma} X'' dx = \tfrac{x^{\gamma-\alpha}}{(\beta-\alpha)(\gamma-\alpha)}\int x^{\alpha} Xdx - \tfrac{1}{(\beta-\alpha)(\gamma-\alpha)}\int x^{\gamma} Xdx$$
$$+ \tfrac{x^{\gamma-\beta}}{(\alpha-\beta)(\gamma-\beta)}\int x^{\beta} Xdx - \tfrac{1}{(\alpha-\beta)(\gamma-\beta)}\int x^{\gamma} Xdx.$$

Now there is the result

$$\tfrac{1}{(\beta-\alpha)(\gamma-\alpha)} + \tfrac{1}{(\alpha-\beta)(\gamma-\beta)} = \tfrac{-1}{(\alpha-\gamma)(\beta-\gamma)} \quad ;$$

because just as now it is evident by itself, then truly it is seen from the demonstration of Theorem § 1169. On account of which for the integral sought thus this expression may be obtained

$$Nxy = \tfrac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)}\int x^{\alpha} Xdx + \tfrac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)}\int x^{\beta} Xdx + \tfrac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)}\int x^{\gamma} Xdx.$$

COROLLARY 1

1247. If the form *P* should have two equal factors, so that there shall be $\beta = \alpha$ since there is then

$$X'' = x^{-\alpha-1}\int \tfrac{dx}{x}\int x^{\alpha} Xdx,$$

there will be

$$\int x^{\gamma} X'' dx = \tfrac{x^{\gamma-\alpha}}{\gamma-\alpha}\int \tfrac{dx}{x}\int x^{\alpha} Xdx - \tfrac{1}{\gamma-\alpha}\int x^{\gamma-\alpha-1}dx\int x^{\alpha} Xdx;$$

but

$$\int x^{\gamma-\alpha-1}dx\int x^{\alpha} Xdx = \tfrac{x^{\gamma-\alpha}}{\gamma-\alpha}\int x^{\alpha} Xdx - \tfrac{1}{\gamma-\alpha}\int x^{\gamma} Xdx,$$

from which it is deduced, that

$$Nxy = \frac{x^{-\alpha}}{\gamma-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx - \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int x^{\alpha}Xdx + \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int x^{\gamma}Xdx.$$

COROLLARY 2

1248. A similar form arises, if there is assumed $\gamma=\beta$ for then there becomes

$$\int x^{\gamma}X''dx = \frac{x^{\beta-\alpha}}{(\beta-\alpha)^2}\int x^{\alpha}Xdx - \frac{1}{(\beta-\alpha)^2}\int x^{\beta}dx + \frac{1}{\alpha-\beta}\int\frac{dx}{x}\int x^{\beta}Xdx$$

and thus

$$Nxy = \frac{x^{-\beta}}{\alpha-\beta}\int\frac{dx}{x}\int x^{\beta}Xdx - \frac{x^{-\beta}}{(\beta-\alpha)^2}\int x^{\beta}Xdx + \frac{x^{-\alpha}}{(\beta-\alpha)^2}\int x^{\alpha}Xdx.$$

COROLLARY 3

1249. But if now all three factors should be equal to each other, $\alpha=\beta=\gamma$, there will be

$$\int x^{\gamma}X''dx = \int\frac{dx}{x}\int\frac{dx}{x}\int x^{\alpha}Xdx$$

and thus in this case the integral can be expressed succinctly

$$Nxy = x^{-\alpha}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\alpha}Xdx.$$

COROLLARY 4

1250. If two factors shall be imaginary, as it were

$$(\alpha+z)(\beta+z) = ff + 2fz\cos.\theta + zz$$

on account of

$$\alpha = f\left(\cos.\theta+\sqrt{-1}\cdot\sin.\theta\right) \text{ and } \beta = f\left(\cos.\theta-\sqrt{-1}\cdot\sin.\theta\right)$$

indeed the last member of our integral remains real on account of

$$(\alpha-\gamma)(\beta-\gamma) = \gamma\gamma - 2f\gamma\cos.\theta + ff\ ,$$

but the two first become on putting $\varphi = f\sin.\theta\cdot lx$,

$$\frac{x^{-f\cos.\theta}\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)}{-2f\sqrt{-1}\cdot sin.\theta\left(\gamma-f\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)\right)}\int x^{f\cos.\theta}Xdx\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)$$
$$+\frac{x^{-f\cos.\theta}\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)}{2f\sqrt{-1}\cdot sin.\theta\left(\gamma-f\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)\right)}\int x^{f\cos.\theta}Xdx\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right),$$

which are reduced to this real form

$$\frac{x^{-f\cos.\theta}\left(\gamma\sin.\varphi-f\sin.(\theta+\varphi)\right)}{f sin.\theta\left(\gamma\gamma-2\gamma f\cos.\varphi+ff\right)}\int x^{f\cos.\theta}Xdx\cos.\varphi$$
$$-\frac{x^{-f\cos.\theta}\left(\gamma\cos.\varphi-f\cos.(\theta+\varphi)\right)}{f sin.\theta\left(\gamma\gamma-2\gamma f\cos.\varphi+ff\right)}\int x^{f\cos.\theta}Xdx\sin.\varphi.$$

SCHOLION

1251. Because the proposition extends to imaginary factors, the reduction of the integrals arising from these in general is easily put in place, accordingly I will not linger any further over that determination from the orders of the differentials. But it has been observed that here equal factors provide a need for individual orders to be pursued more carefully, because above [§1163, §1179], by hurrying exceedingly quickly to establish the general case, there has been a tendency to slide into a conspicuous error, that I might at once have avoided happily, if I should have proceeded there with the same method [that I will adopt here]. But the fault concerning imaginary factors is not a cause of concern here, since in this calculation nothing is encountered beyond neglecting [equal] infinitely small kinds of quantities [in the denominators]. Moreover these errors have arisen from that source which I have set out above ; which subtle error may be set out more clearly here, together with the necessary correction that I will establish. Clearly the question is returned here for the present case, so that the value of the two formulas

$$\frac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)}\int x^{\alpha}Xdx+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)}\int x^{\beta}Xdx$$

may be defined in the case where $\beta=\alpha$ and both members increase indefinitely ; hence in the end, on putting $\beta=\alpha+\omega$, with ω present as a small vanishing quantity, and since there shall be

$$x^{\beta}=x^{\alpha}x^{\omega}=x^{\alpha}e^{\omega lx}=x^{\alpha}\left(1+\omega lx\right)$$

and hence

$$x^{-\beta}=x^{-\alpha}x^{\omega}=x^{-\alpha}\left(1-\omega lx\right),$$

we will have

$$\frac{x^{-\alpha}}{\omega(\gamma-\alpha)}\int x^{\alpha}Xdx-\frac{x^{-\alpha}(1-\omega lx)}{\omega(\gamma-\beta)}\int x^{\alpha}Xdx\left(1+\omega lx\right).$$

Because now there is

$$\frac{1}{(\gamma-\alpha)}=\frac{1}{\gamma-\beta+\omega}=\frac{1}{\gamma-\beta}-\frac{\omega}{(\gamma-\beta)^2},$$

the first member adopts this form

$$\frac{x^{-\alpha}}{\omega(\gamma-\beta)}\int x^{\alpha}Xdx-\frac{x^{-\alpha}}{(\gamma-\beta)^2}\int x^{\alpha}Xdx,$$

now the latter set out so

$$-\frac{x^{-\alpha}}{\omega(\gamma-\beta)}\int x^{\alpha}Xdx+\frac{x^{-\alpha}}{\gamma-\beta}\left(lx\int x^{\alpha}Xdx-\int x^{\alpha}Xdxlx\right)$$

and thus the value sought in the case $\beta=\alpha$ is concluded [*i. e.* the parts tending to infinity cancel out]

$$\frac{x^{-\alpha}}{\gamma-\alpha}\left(lx\int x^{\alpha}Xdx-\int x^{\alpha}Xdxlx\right)-\frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int x^{\alpha}Xdx$$

or

$$\frac{x^{-\alpha}}{\gamma-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx-\frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int x^{\alpha}Xdx,$$

of which the latter member of the formula, which in error in that formula was omitted, thus results, because we have considered the difference here between these expressions $\gamma-\alpha$ and $\gamma-\beta$, which necessary caution we have ignored above.

PROBLEM 169

1252. *With this differential equation of the fourth order proposed*

$$X=Ay+\frac{Bxdy}{dx}+\frac{Cxxddy}{dx^2}+\frac{Dx^3d^3y}{dx^3}+\frac{Nx^4d^4y}{dx^4}$$

to establish the integral of this in terms of simple integral formulas.

SOLUTION

Hence with the algebraic expression formed

$$P=A+B(z-1)+C(z-1)(z-2)+D(z-1)(z-2)(z-3)$$
$$+N(z-1)(z-2)(z-3)(z-4)$$

there is put in place

$$P=N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)$$

and by the general precept there is

$$Ny=X^{IV}\text{, with present }X^{IV}=x^{-\delta-1}\int X'''x^{\delta}dx,$$

if indeed X''', may be determined from the three first factors, just as has been done in the preceding problem [§ 1246]. Clearly the value found there for Nxy is required to be multiplied here by $x^{\delta-1}dx$, from which there arises

$$\begin{aligned}\int x^{\delta}X'''dx=&+\tfrac{x^{\delta-\alpha}}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}\int x^{\alpha}Xdx-\tfrac{1}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}\int x^{\delta}Xdx\\&+\tfrac{x^{\delta-\beta}}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}\int x^{\beta}Xdx-\tfrac{1}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}\int x^{\delta}Xdx\\&+\tfrac{x^{\delta-\gamma}}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)}\int x^{\gamma}Xdx-\tfrac{1}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)}\int x^{\delta}Xdx,\end{aligned}$$

where on account of the reasons shown above [§ 1169] the three latter terms coalesce into

$$+\tfrac{1}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\int x^{\delta}Xdx,$$

thus so that the integral sought shall be

$$\begin{aligned}Nxy=&\tfrac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}\int x^{\alpha}Xdx+\tfrac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}\int x^{\beta}Xdx\\&+\tfrac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)}\int x^{\gamma}Xdx+\tfrac{x^{-\delta}}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\int x^{\delta}Xdx,\end{aligned}$$

if indeed all the factors shall be unequal to each other. But the case, in which two or more factors are equal, we will explore in the corollaries.

COROLLARY 1

1253. If there should be two equal factors, namely $\delta=\gamma$, or if there shall be

$$P=N(\alpha+z)(\beta+z)(\gamma+z)^{2}$$

the integral from the same form found before for X''' [§ 1246] arises :

$$Nxy = \frac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)^2}\int x^{\alpha} Xdx - \frac{x^{-\gamma}}{(\beta-\alpha)(\gamma-\alpha)^2}\int x^{\gamma} Xdx$$
$$+\frac{x^{-\beta}}{(\alpha-\gamma)(\gamma-\beta)^2}\int x^{\beta} Xdx - \frac{x^{-\gamma}}{(\alpha-\beta)(\gamma-\beta)^2}\int x^{\gamma} Xdx + \frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)}\int \frac{dx}{x}\int x^{\gamma} Xdx,$$

where the negative members can be represented thus :

$$\frac{x^{-\gamma}}{\alpha-\beta}\left(\frac{1}{(\gamma-\alpha)^2}-\frac{1}{(\gamma-\beta)^2}\right)\int x^{\gamma} Xdx$$

COROLLARY 2

1254. If there should be three equal factors, so that there shall be

$$P = N(\alpha+z)(\beta+z)^3$$

and thus $\delta=\gamma=\beta$, the integral is deduced from the formula § 1249 :

$$Nxy = \frac{x^{-\alpha}}{(\beta-\alpha)^3}\int x^{\alpha} Xdx - \frac{x^{-\beta}}{(\beta-\alpha)^3}\int x^{\beta} Xdx$$
$$-\frac{x^{-\beta}}{(\beta-\alpha)^2}\int \frac{dx}{x}\int x^{\beta} Xdx + \frac{x^{-\beta}}{(\alpha-\beta)}\int \frac{dx}{x}\int \frac{dx}{x}\int x^{\beta} Xdx.$$

COROLLARY 3

1255. If all four factors should be equal, so that there shall be

$$P = N(\alpha+z)^4$$

with $\delta=\gamma=\beta=\alpha$ present, from the form for the three equations found § 1249 the integral becomes

$$Nxy = x^{-\alpha}\int \frac{dx}{x}\int \frac{dx}{x}\int \frac{dx}{x}\int x^{\alpha} Xdx.$$

COROLLARIUM 4

1256. If there should be had $\beta=\alpha$ and $\delta=\gamma$, so that two factors of each shall be equal, clearly

$$P = N(\alpha+z)^2(\gamma+z)^2,$$

from § 1247, where the factors were $(\alpha+z)^2(\gamma+z)$ *(a +z)*, the integral is deduced

$$Nxy = \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\alpha}Xdx - \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int x^{\gamma-\alpha-1}dx\int x^{\alpha}Xdx - \frac{x^{-\alpha}}{(\gamma-\alpha)^3}\int x^{\alpha}Xdx$$
$$+\frac{x^{-\gamma}}{(\gamma-\alpha)^3}\int x^{\gamma}Xdx + \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx,$$

which on account of

$$\int x^{\gamma-\alpha-1}dx\int x^{\alpha}Xdx = \frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int x^{\alpha}Xdx - \frac{1}{\gamma-\alpha}\int x^{\gamma}Xdx$$

is collected together into this form

$$Nxy = \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\alpha}Xdx + \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx$$
$$-\frac{2x^{-\alpha}}{(\gamma-\alpha)^3}\int x^{\alpha}Xdx - \frac{2x^{-\gamma}}{(\alpha-\gamma)^3}\int x^{\gamma}Xdx.$$

PROBLEM 170

1257. *With this proposed differential equation of the fifth order,*

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \frac{Ex^4d^4y}{dx^4} + \frac{Nx^5d^5y}{dx^5},$$

to establish the integral of this in terms of simple integral formulas.

SOLUTION

Since here there shall be the algebraic quantity formed

$$P = A + B(z-1) + C(z-1)(z-2) + \cdots + N(z-1)(z-2)(z-3)(z-4)(z-5),$$

there is put in place

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)(\varepsilon+z),$$

and if these factors are all unequal to each other, from the preceding integral a new integration established will give rise to the integral sought :

$$\begin{aligned}Nxy=&\frac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha)}\int x^{\alpha}Xdx+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta)}\int x^{\beta}Xdx\\&+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)(\varepsilon-\gamma)}\int x^{\gamma}Xdx+\frac{x^{-\delta}}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)(\varepsilon-\delta)}\int x^{\delta}Xdx\\&+\frac{x^{-\varepsilon}}{(\alpha-\varepsilon)(\beta-\varepsilon)(\gamma-\varepsilon)(\delta-\varepsilon)}\int x^{\varepsilon}Xdx;\end{aligned}$$

the cases, in which two or more factors are equal, we will set out in the corollaries.

COROLLARIUM 1

1258. If there should be two equal factors, so that there shall be

$$P=N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)^2$$

and thus $\varepsilon=\delta$, from the preceding problem the integral is deduced :

$$\begin{aligned}Nxy=&\frac{x^{-\alpha}\int x^{\alpha}Xdx-x^{-\delta}\int x^{\delta}Xdx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)^2}+\frac{x^{-\beta}\int x^{\beta}Xdx-x^{-\delta}\int x^{\delta}Xdx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)^2}\\&+\frac{x^{-\gamma}\int x^{\gamma}Xdx-x^{-\delta}\int x^{\delta}Xdx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)^2}+\frac{x^{-\delta}}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\int\frac{dx}{x}\int x^{\delta}Xdx.\end{aligned}$$

COROLLARY 2

1259. If there should be three equal factors, so that there shall be

$$P=N(\alpha+z)(\beta+z)(\gamma+z)^3$$

and thus $\varepsilon=\delta=\gamma$, from Corollary 1 of the preceding problem it is deduced that

$$\begin{aligned}Nxy=&\frac{x^{-\alpha}\int x^{\alpha}Xdx-x^{-\gamma}\int x^{\gamma}Xdx}{(\beta-\alpha)(\gamma-\alpha)^3}-\frac{x^{-\gamma}}{(\beta-\alpha)(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx\\&+\frac{x^{-\beta}\int x^{\beta}Xdx-x^{-\gamma}\int x^{\gamma}Xdx}{(\alpha-\beta)(\gamma-\beta)^3}-\frac{x^{-\gamma}}{(\alpha-\beta)(\gamma-\beta)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx\\&+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}Xdx.\end{aligned}$$

COROLLARY 3

1260. If four factors shall be equal, so that there shall be

$$P=N(\alpha+z)(\beta+z)^4$$

and thus $\varepsilon=\delta=\gamma=\beta$, there will be by §1254

$$Nxy=\frac{x^{-\alpha}}{(\beta-\alpha)^4}\int x^{\alpha}Xdx-\frac{x^{-\beta}}{(\beta-\alpha)^4}\int x^{\beta}Xdx-\frac{x^{-\beta}}{(\beta-\alpha)^3}\int\frac{dx}{x}\int x^{\beta}Xdx$$
$$-\frac{x^{-\beta}}{(\beta-\alpha)^2}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx-\frac{x^{-\beta}}{\beta-\alpha}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx;$$

and if all five factors shall be equal to each other, or

$$P=N(\alpha+z)^5$$

then the integral will be

$$Nxy=x^{-\alpha}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\alpha}Xdx.$$

COROLLARY 4

1261. If *P* should have two square factors, so that there shall be

$$P=N(\alpha+z)(\beta+z)^2(\gamma+z)^2$$

and thus $\delta=\gamma$ and $\varepsilon=\beta$, there will be from § 1253 the integral with the necessary reduction made

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}Xdx-x^{-\beta}\int x^{\beta}Xdx}{(\beta-\alpha)^2(\gamma-\alpha)^2}-\frac{x^{-\gamma}\int x^{\gamma}Xdx-x^{-\beta}\int x^{\beta}Xdx}{(\beta-\alpha)(\alpha-\gamma)^2(\beta-\gamma)}$$
$$+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)^2}\int\frac{dx}{x}\int x^{\beta}Xdx-\frac{x^{-\gamma}\int x^{\gamma}Xdx-x^{-\beta}\int x^{\beta}Xdx}{(\alpha-\beta)(\beta-\gamma)^3}$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx-\frac{x^{-\gamma}\int x^{\gamma}Xdx-x^{-\beta}\int x^{\beta}Xdx}{(\alpha-\gamma)(\beta-\gamma)^3},$$

which again is reduced to this form

$$Nxy=\frac{x^{-\alpha}}{(\beta-\alpha)^2(\gamma-\alpha)^2}\int x^{\alpha}Xdx$$
$$+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)^2}\int\frac{dx}{x}\int x^{\beta}Xdx-\frac{x^{-\beta}}{(\alpha-\beta)^2(\gamma-\beta)^2}\int x^{\beta}Xdx-\frac{2x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)^3}\int x^{\beta}Xdx$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx-\frac{x^{-\gamma}}{(\alpha-\gamma)^2(\beta-\gamma)^2}\int x^{\gamma}Xdx-\frac{2x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)^3}\int x^{\gamma}Xdx$$

COROLLARY 5

1262. If P should have both a square and a cubic factor, so that there shall be

$$P = N(\alpha + z)^2 (\gamma + z)^3$$

and thus $\beta = \alpha$ and $\varepsilon = \delta = \gamma$, from § 1254 the integral is deduced :

$$\begin{aligned} Nxy = & \frac{x^{-\alpha}}{(\gamma-\alpha)^3}\int\frac{dx}{x}\int x^{\alpha}Xdx - \frac{3x^{-\alpha}}{(\gamma-\alpha)^4}\int x^{\alpha}Xdx \\ & + \frac{x^{-\gamma}}{(\alpha-\gamma)^2}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}Xdx - \frac{2x^{-\gamma}}{(\alpha-\gamma)^3}\int\frac{dx}{x}\int x^{\gamma}Xdx + \frac{3x^{-\gamma}}{(\alpha-\gamma)^4}\int x^{\gamma}Xdx. \end{aligned}$$

SCHOLION

1263. From these formulas of the pairs it is agreed, just as it is required to continue these further for a number with a greater number of factors, if indeed some of the factors should be equal to each other. Indeed the parts of the integrals, which correspond to unequal factors, show the rule they obey; but those which answer to equal parts, are able to be expressed by a more convenient reduction brought to bear. Just as for the case of Corollary 1 if for the sake of brevity there is put $\alpha - \delta = p$, $\beta - \delta = q$ and $\gamma - \delta = r$, the form $x^{-\delta}\int x^{\delta}Xdx$ is multiplied by

$$\frac{1}{(p-q)(r-p)pp} + \frac{1}{(p-q)(q-r)qq} + \frac{1}{(r-p)(q-r)rr}$$

or

$$\frac{(q-r)qqrr + (r-p)pprr + (p-q)ppqq}{(p-q)(q-r)(r-p)ppqqrr}$$

of which fraction the numerator is $-(p-q)(q-r)(r-p)(pq + pr + qr)$, thus so that this fraction may be reduced to that

$$\frac{-pq-pr-qr}{ppqqrr} = -\frac{1}{pqr}\left(\frac{1}{p} + \frac{1}{q} + \frac{1}{r}\right).$$

Therefore when there is

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z)^2,$$

the integral itself thus may be had

$$Nxy=\frac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)^2}\int x^{\alpha}Xdx+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)^2}\int x^{\beta}Xdx$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)^2}\int x^{\gamma}Xdx+\frac{x^{-\delta}}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\int\frac{dx}{x}\int x^{\delta}Xdx$$
$$-\frac{x^{-\delta}}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\left(\frac{1}{\alpha-\delta}+\frac{1}{\beta-\delta}+\frac{1}{\gamma-\delta}\right)\int x^{\delta}Xdx.$$

But for the case $P=N(\alpha+z)(\beta+z)(\gamma+z)^3$ there will be had [§ 1259]

$$Nxy=\frac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)^3}\int x^{\alpha}Xdx+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)^3}\int x^{\beta}Xdx$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}Xdx-\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{\alpha-\gamma}+\frac{1}{\beta-\gamma}\right)\int\frac{dx}{x}\int x^{\gamma}Xdx$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{(\alpha-\gamma)^2}+\frac{1}{(\alpha-\gamma)(\beta-\gamma)}+\frac{1}{(\beta-\gamma)^2}\right)\int x^{\gamma}Xdx.$$

Then truly for the case $P=N(\alpha+z)(\beta+z)^4$ there becomes [§ 1260]

$$Nxy=\frac{x^{-\alpha}}{(\beta-\alpha)^4}\int x^{\alpha}Xdx+\frac{x^{-\beta}}{(\alpha-\beta)}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx$$
$$-\frac{x^{-\beta}}{\alpha-\beta}\cdot\frac{1}{\alpha-\beta}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx+\frac{x^{-\beta}}{\alpha-\beta}\cdot\frac{1}{(\alpha-\beta)^2}\int\frac{dx}{x}\int x^{\beta}Xdx$$
$$-\frac{x^{-\beta}}{\alpha-\beta}\cdot\frac{1}{(\alpha-\beta)^3}\int x^{\beta}Xdx.$$

But for the case $P=N(\alpha+z)(\beta+z)^2(\gamma+z)^2$ the integral [§ 1261] thus may be had

$$Nxy=\frac{x^{-\alpha}}{(\beta-\alpha)^2(\gamma-\alpha)^2}\int x^{\alpha}Xdx$$
$$+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)^2}\int\frac{dx}{x}\int x^{\beta}Xdx-\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)^2}\left(\frac{1}{\alpha-\beta}+\frac{2}{\gamma-\beta}\right)\int x^{\beta}Xdx$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx-\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)^2}\left(\frac{1}{\alpha-\gamma}+\frac{2}{\beta-\gamma}\right)\int x^{\gamma}Xdx.$$

But for the case $P=N(\alpha+z)^2(\gamma+z)^3$ the integral [§ 1262] thus is had

$$Nxy = \frac{x^{-\alpha}}{(\gamma-\alpha)^3}\int\frac{dx}{x}\int x^{\alpha}Xdx - \frac{x^{-\alpha}}{(\gamma-\alpha)^3}\cdot\frac{3}{\gamma-\alpha}\int x^{\alpha}Xdx$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)^2}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}Xdx - \frac{x^{-\gamma}}{(\alpha-\gamma)^2}\cdot\frac{2}{\alpha-\gamma}\int\frac{dx}{x}\int x^{\gamma}Xdx$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)^2}\cdot\frac{3}{(\alpha-\gamma)^2}\int x^{\gamma}Xdx,$$

from which the nature of these formulas now shall be more evident and likewise the part of the integral is apparent from some factors arising not depending on the equality of the others. On account of which it will be permitted to approach the general problem.

PROBLEM 171

1264. *With the proposed differential equation of any order of this form*

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \frac{Ex^4d^4y}{dx^4} + \cdots + \frac{Nx^nd^ny}{dx^n},$$

from which the algebraic form arises by this rule

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \cdots$$
$$+ N(z-1)(z-2)\cdots(z-n)$$

all the factors are unequal to each other, to show the complete value of y by simple integral formulas.

SOLUTION

In the first place all the simple real factors of the form P shall be

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\cdots(\nu+z)$$

with the number of factors present $= n$ and from the preceding it is apparent that a part of the integral arises from any part. The following values may be elicited for the parts found:

1) on putting $z=-\alpha$ there shall be $\mathfrak{A} = \frac{P}{\alpha+z}$ or $\mathfrak{A} = \frac{dP}{dz}$,
2) on putting $z=-\beta$ there shall be $\mathfrak{B} = \frac{P}{\beta+z}$ or $\mathfrak{B} = \frac{dP}{dz}$,
3) on putting $z=-\gamma$ there shall be $\mathfrak{C} = \frac{P}{\gamma+z}$ or $\mathfrak{C} = \frac{dP}{dz}$

etc.

Therefore since there shall be

$$(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\cdots(v-\alpha)=\tfrac{\mathfrak{A}}{N}$$

the letter *N* from the above forms is removed by division and the integral sought is produced :

$$xy=\tfrac{1}{\mathfrak{A}}x^{-\alpha}\int x^{\alpha}Xdx+\tfrac{1}{\mathfrak{B}}x^{-\beta}\int x^{\beta}Xdx+\tfrac{1}{\mathfrak{C}}x^{-\gamma}\int x^{\gamma}Xdx+\text{etc.},$$

until the individual factors should be exhausted.

But if now the form *P* should have imaginary factors, then in the following manner the reduction of the imaginary parts arising to real parts may be put in place. Because two simple imaginary factors give a double real factor, we may put

$$(\alpha+z)(\beta+z)=ff+2fz\cos.\theta+zz,$$

thus so that there shall be

$$\alpha=f\left(\cos.\theta+\sqrt{-1}\cdot\sin.\theta\right) \text{ and } \beta=f\left(\cos.\theta-\sqrt{-1}\cdot\sin.\theta\right),$$

from which at first the values of the letters $\mathfrak{A}$ and $\mathfrak{B}$ may be defined ; since each of these may be derived from the form $\frac{dP}{dz}$, with that put as $z=-\alpha$, and truly on putting this as $z=-\beta$, into the form $\frac{dP}{dz}$ in place of z there may be written everywhere $-f\left(\cos.\theta\pm\sqrt{-1}\cdot\sin.\theta\right)$ and there is produced $\mathfrak{P}\pm\mathfrak{Q}\sqrt{-1}$ and it is evident that there becomes

$$\mathfrak{A}=\mathfrak{P}+\mathfrak{Q}\sqrt{-1} \quad \text{et} \quad \mathfrak{B}=\mathfrak{P}-\mathfrak{Q}\sqrt{-1},$$

where it is to be noted that the quantities $\mathfrak{A}$ and $\mathfrak{B}$ are real. Then since there shall be

$$x^{m+n\sqrt{-1}}=x^{m}e^{n\sqrt{-1}\cdot lx}=x^{m}\left(\cos.(nlx)+\sqrt{-1}\cdot\sin.(nlx)\right),$$

hence if for brevity we put the angle $f\sin.\theta\cdot lx=\varphi$, there will be

$$x^{\alpha}=x^{f\cos.\theta}\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right),\quad x^{-\alpha}=x^{-f\cos.\theta}\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right),$$
$$x^{\beta}=x^{f\cos.\theta}\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right),\quad x^{-\beta}=x^{-f\cos.\theta}\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right),$$

Whereby for the two parts

$$\tfrac{1}{\mathfrak{A}}x^{-\alpha}\int x^{\alpha}Xdx+\tfrac{1}{\mathfrak{A}}x^{-\beta}\int x^{\beta}Xdx$$

an account of $\mathfrak{A}\mathfrak{B}=\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}$ we will have

$$\tfrac{x^{-f\cos.\theta}}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}\left\{\begin{array}{l}\left(\mathfrak{P}-\mathfrak{Q}\sqrt{-1}\right)\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)\int x^{f\cos.\theta}\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)Xdx\\+\left(\mathfrak{P}+\mathfrak{Q}\sqrt{-1}\right)\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)\int x^{f\cos.\theta}\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)Xdx\end{array}\right\},$$

for which the form on account of the imaginary parts removing each other is reduced to this :

$$\frac{2x^{-f\cos.\theta}(\mathfrak{P}\cos.\varphi-\mathfrak{Q}\sin.\varphi)\int x^{f\cos.\theta}Xdx\cos.\varphi+2x^{-f\cos.\theta}(\mathfrak{P}\cos.\varphi+\mathfrak{Q}\sin.\varphi)\int x^{f\cos.\theta}Xdx\sin.\varphi}{\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}}$$

And such a form may be agreed upon for the integral, whenever the form P has a double factor $ff+2f z\cos.\theta+zz$.

COROLLARY 1

1265. Moreover, if certain of the simple factors of P are imaginary, the calculation of these which are real is not upset, as they depend minimally and may be deduced from the individual parts of the integral from the nature of the remaining factors.

COROLLARIY 2

1266. The part of the integral arising from two imaginary factors or from one duplicate factor in some manner can be written more succinctly, if there is put

$$\mathfrak{P}=\mathfrak{O}\cos.\zeta \quad\text{and}\quad \mathfrak{Q}=\mathfrak{O}\sin.\zeta\,;$$

since indeed that becomes

$$\tfrac{2}{\mathfrak{O}}x^{-f\cos.\theta}\left(\cos.(\zeta+\varphi)\int x^{f\cos.\theta}Xdx\cos.\varphi+\sin.(\zeta+\varphi)\int x^{f\cos.\theta}Xdx\sin.\varphi\right),$$

where ζ and θ are constant angle, φ truly is variable on account of $\varphi=f\sin.\theta\cdot lx$.

PROBLEM 172

1267. *If for the differential equation proposed in the preceding problem the algebraic quantity P thence formed should have two simple equal factors, to find the part of the integral thence arising.*

SOLUTION

Hence in the form found before

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z) \text{ etc.}$$

we may put $\beta = \alpha$. Because then truly each part of the integral becomes infinite, the one with the + sign, and the other affected by the – sign, thus so that taken together they constitute a finite part, towards eliciting which we may put $\beta = \alpha - \omega$ with ω denoting a vanishing quantity and there will be

$$\mathfrak{A} = -N\omega(\gamma - \alpha)(\delta - \alpha)(\varepsilon - \alpha) \text{ etc.}$$

and

$$\mathfrak{B} = +N\omega(\gamma - \beta)(\delta - \beta)(\varepsilon - \beta) \text{ etc.}$$

Now there is put

$$\frac{P}{(\alpha+z)(\beta+z)} = \frac{P}{(\alpha+z)^2} = Q,$$

so that there shall be

$$Q = N(\gamma + z)(\delta + z)(\varepsilon + z) \text{ etc.}$$

and it is evident there becomes $\mathfrak{A} = -\omega Q$ on putting $z = -\alpha$ and $\mathfrak{B} = \omega Q$ on putting $z = -\beta = -\alpha + \omega$, from which it is understood that the latter value of Q exceeds the first by its difference dQ, if there is made $z = -\alpha$ and $dz = \omega$, thus so that there shall be $\mathfrak{B} = \omega\left(\mathrm{Q} + \omega\frac{dQ}{dz}\right)$ on putting $z = -\alpha$, and hence

$$\frac{1}{\mathfrak{B}} = \frac{1}{\omega Q} - \frac{dQ}{QQdz} = \frac{1}{\omega Q} + \frac{1}{dz}d.\frac{1}{Q}$$

with $\frac{1}{\mathfrak{A}} = \frac{1}{\omega Q}$ present. Then indeed, since there shall be , $x^{\beta} = x^{\alpha}x^{-\omega} = x^{\alpha}(1 - \omega lx)$ and $x^{-\beta} = x^{-\alpha}(1 + \omega lx)$, the two parts of the integral sought will be

$$-\frac{1}{\omega Q}x^{-\alpha}\int x^{\alpha}Xdx + \left(\frac{1}{\omega Q} + \frac{1}{dz}d.\frac{1}{Q}\right)x^{-\alpha}(1 + \omega lx)\int x^{\alpha}Xdx(1 - \omega lx),$$

where since the members divided by ω cancel each other, there arises

$$\tfrac{1}{Q}x^{-\alpha}\left(lx\int x^{\alpha}Xdx-\int x^{\alpha}Xdxlx\right)+\tfrac{1}{dz}d.\tfrac{1}{Q}\cdot x^{-\alpha}\int x^{\alpha}Xdx$$

or

$$\tfrac{1}{Q}x^{-\alpha}\int\tfrac{dx}{x}\int x^{\alpha}Xdx+\tfrac{1}{dz}d.\tfrac{1}{Q}\cdot x^{-\alpha}\int x^{\alpha}Xdx,$$

if indeed both in the value for $\frac{1}{Q}$ as well as for $\frac{1}{dz}d.\frac{1}{Q}dz$ there is written everywhere $-\alpha$ in place of z. Now since there shall be $Q=\frac{P}{(\alpha+z)^2}$, these values hence are found easily.

COROLLARY 1

1268. But if hence the algebraic quantity P formed from the differential equation should have the quadratic factor $(\alpha+z)^2$, hence this part is to be transferred into the integral

$$\tfrac{(\alpha+z)^2}{P}x^{-\alpha}\int\tfrac{dx}{x}\int x^{\alpha}Xdx+\tfrac{1}{dz}d.\tfrac{(\alpha+z)^2}{P}\cdot x^{-\alpha}\int x^{\alpha}Xdx$$

on putting $z=-\alpha$; while if this factor $\alpha+z$ while if this factor should be on its own, the part of the integral arising becomes $\frac{\alpha+z}{P}x^{-\alpha}\int x^{\alpha}Xdx$ on putting $z=-\alpha$.

COROLLARY 2

1269. Since there shall be $Q=\frac{P}{(\alpha+z)^2}$, in the case $z=-\alpha$ there arises $Q=\frac{ddP}{2dz^2}$ [L'Hôpital's Rule again]; now since here the value of z now determined is to be attributed, hence $\frac{dQ}{dz}$ is not permitted to be deduced, but in the first place the form is to be used, for which there becomes $\frac{dQ}{dz}=\frac{(\alpha+z)dP-2Pdz}{(\alpha+z)^3dz}$; of which fraction since the numerator and the denominator vanish in the case $z=-\alpha$, for the same case there will be

$$\frac{dQ}{dz}=\frac{(\alpha+z)ddP-2dzdP}{3(\alpha+z)^2dz^2}=\frac{(\alpha+z)d^3P}{6(\alpha+z)dz^3}=\frac{d^3P}{6dz^3}$$

COROLLARY 3

1270. With this value found, because in the same case $z=-\alpha$ the quantity $Q=\frac{ddP}{2dz^2}$, there will be

$$\frac{1}{dz}d.\frac{1}{Q}=-\frac{dQ}{QQdz}=-\frac{2dzd^3P}{3ddP^2} \text{ or } \frac{1}{dz}d.\frac{1}{Q}=\frac{2dz}{3}d.\frac{1}{ddP},$$

from which formulas, if the factors of P shall not be set out, the parts of the integral can be found easily.

PROBLEM 173

1271. *If for the preceding differential equation the algebraic quantity P thus formed should have the cubic factor* $(\alpha+z)^3$, *to find the part of the integral arising from this.*

SOLUTION

Therefore we may put $P=(\alpha+z)^2(\gamma+z)R$ with $\gamma=\alpha-\omega$ present, where ω is assumed for some vanishing quantity. Since before there was Q, it hence becomes $Q=(\gamma+z)R$ and on making $z=-\alpha$ there will be $Q=-\omega R$, if also there is put into R $z=-\alpha$. Then since there shall be

$$\frac{dQ}{dz}=R+\frac{(\gamma+z)dR}{dz}=R-\frac{\omega dR}{dz}$$

in the same case there will be

$$\frac{1}{dz}d.\frac{1}{Q}=-\frac{1}{\omega\omega R}+\frac{dR}{\omega RRdz}=-\frac{1}{\omega\omega}\cdot\frac{1}{R}-\frac{1}{\omega dz}d.\frac{1}{R}.$$

On account of which from the square factor $(\alpha+z)^2$ by the preceding problem this part of the integral is obtained

$$-\frac{1}{\omega R}x^{-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx-\left(\frac{1}{\omega\omega R}+\frac{1}{\omega dz}d.\frac{1}{R}\right)x^{-\alpha}\int x^{\alpha}Xdx,$$

both members of which increase indefinitely on account of $\omega=0$. But we may add the part arising from the third factor $\gamma+z=\alpha-\omega+z$, which on account of $\frac{P}{\gamma+z}=(\alpha+z)^2R$ is

$$\frac{1}{(\alpha+z)^2R}x^{-\gamma}\int x^{\gamma}Xdx$$

on putting $z=-\gamma=-\alpha+\omega$. But if now as before R is that value, which arises on putting $z=-\alpha$, hence on increasing the value for this particular ω in place of $\frac{1}{R}$ there must be written

$$\frac{1}{R}+\frac{\omega}{dz}d.\frac{1}{R}+\frac{\omega^2}{1\cdot 2dz^2}dd.\frac{1}{R}+\text{etc.},$$

if indeed the value $z=-\alpha$ and we may retain here; from this the part of the integral on account of $\alpha+z=\omega$ will be

$$\left(\frac{1}{\omega\omega R}+\frac{1}{\omega dz}d.\frac{1}{R}+\frac{1}{2dz^2}dd.\frac{1}{R}\right)x^{-\alpha+\omega}\int x^{\alpha-\omega}Xdx$$

and thus it is evident that value of $\frac{1}{R}$ must be continued as far as the second power of ω and by the same law here it is appropriate to express the other part involving x. Regarding which I note, that if a formula of the same kind $x^{-\omega}\int x^{\omega}Vdx$ is considered, the following powers of ω are to be set out, that are made most conveniently by this method. On putting

$$v=x^{\omega}\int x^{-\omega}Vdx,$$

so that there shall be

$$x^{-\omega}v=\int x^{-\omega}Vdx,$$

on differentiating there shall be

$$dv-\frac{\omega v dx}{x}=Vdx,$$

whereby on putting

$$v=T+\omega T'+\omega^2T''+\omega^3T'''+\text{etc.}$$

we will have the terms following the powers of ω on setting out

$$\left.\begin{array}{l} dT+\omega dT'+\omega\omega T''+\omega^3 dT'''+\text{etc.}\\ -Vdx-\omega dT\frac{dx}{x}-\omega\omega T'\frac{dx}{x}-\omega^3 dT''\frac{dx}{x}-\text{etc.}\end{array}\right\}=0$$

and thus

$$T=\int Vdx,\ T'=\int\frac{dx}{x}\int Vdx,\ T''=\int\frac{dx}{x}\int\frac{dx}{x}\int Vdx\quad\text{etc.}$$

Consequently, since in the application there shall be $V = x^{\alpha}X$, the part of the integral arising from the factor $\gamma + z = \alpha - \omega + z$ will be

$$\left(\tfrac{1}{\omega\omega R} + \tfrac{1}{\omega dz}d.\tfrac{1}{R} + \tfrac{1}{2dz^2}dd.\tfrac{1}{R}\right)x^{-\alpha}\left(\int x^{\alpha}Xdx + \omega\int\tfrac{dx}{x}\int x^{\alpha}Xdx + \omega^2\int\tfrac{dx}{x}\int\tfrac{dx}{x}\int x^{\alpha}Xdx\right),$$

from which since the part arising from $(\alpha + z)^2$ taken together with all the infinite members mutually cancel each other, and for the quantity $P = (\alpha + z)^3 R$ with the cubic factor $(\alpha + z)^3$, this part is introduced into the integral

$$\tfrac{1}{R}x^{-\alpha}\int\tfrac{dx}{x}\int\tfrac{dx}{x}\int x^{\alpha}Xdx + \tfrac{1}{dz}d.\tfrac{1}{R}\cdot x^{-\alpha}\int\tfrac{dx}{x}\int x^{\alpha}Xdx + \tfrac{1}{2dz^2}dd.\tfrac{1}{R}\cdot x^{-\alpha}\int x^{\alpha}Xdx,$$

but only if everywhere in the quantity $R = \frac{P}{(\alpha+z)^3}$ there may be written $z = -\alpha$.

COROLLARY 1

1272. The method in the solution of this problem can be extended to any number of equal factors and applied easily. For if there should be the factor $(\alpha + z)^m$ of the quantity P and in this fraction $\frac{(\alpha+z)^m}{P}$ with continued differentiation of itself, after that has been worked out, there is put $z = -\alpha$, the parts of the integral thence arising thus shall be had themselves :

Factor of P	$\alpha + z$	$(\alpha + z)^2$	$(\alpha + z)^3$
Part of the integral	$\frac{\alpha+z}{P}x^{-\alpha}\int x^{\alpha}Xdx$	$\frac{(\alpha+z)^2}{P}x^{-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx$ $\frac{1}{dz}d.\frac{(\alpha+z)^2}{P}\cdot x^{-\alpha}\int x^{\alpha}Xdx$	$\frac{(\alpha+z)^3}{P}x^{-\alpha}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\alpha}Xdx$ $\frac{1}{dz}d.\frac{(\alpha+z)^3}{P}\cdot x^{-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx$ $\frac{1}{2dz^2}dd.\frac{(\alpha+z)^3}{P}\cdot x^{-\alpha}\int x^{\alpha}Xdx$

COROLLARY 2

1273. If there should be two or more double factors equal to each other, on taking

$$\alpha = f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right) \text{ and } \beta = f\left(\cos.\theta - \sqrt{-1}\cdot\sin.\theta\right)$$

the parts for $(\alpha + z)^2$ and $(\beta + z)^2$ set out separately by the above method can be applied without difficulty and gathered together and reduced to real parts.

SCHOLIUM

1274. By a similar method, to which this chapter has been investigated, in the workings of this section [§ 1163, 1178] of Chapter III as it was required then, nor was there then any danger of being scared of falling into errors. But now it appears superfluous that errors committed there are to be corrected here, since clearly not only is the method to be the same , but also the equation treated here easily was able there to be changed in form and in turn treated. For if indeed in the equation of Chapter III

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \frac{Ex^4d^4y}{dx^4} + \text{etc.}$$

there should be put $x = lv$, so that there becomes $dx = \frac{dv}{v}$, moreover the function *X* becomes a function of *v*, which shall be *V*, an equation of this form comes about which we have treated here. But then there the element *dx* has been taken for a constant, and we may lay aside that condition

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \text{ etc.},$$

so that this equation emerges

$$X = V = Ay + Bp + Cq + Dr + Es + \text{etc.}$$

But now on account of $dx = \frac{dv}{v}$ with the element *dv* assumed constant, we arrive at

$$p = \frac{dy}{dx} = \frac{vdy}{dv},$$
$$q = \frac{dp}{dx} = \frac{vvddy}{dv^2} + \frac{vdy}{dv},$$
$$r = \frac{dq}{dx} = \frac{v^3d^3y}{dv^3} + \frac{3vvddy}{dv^2} + \frac{vdy}{dv},$$
$$s = \frac{dr}{dx} = \frac{v^4d^4y}{dv^4} + \frac{6v^3d^3y}{dv^3} + \frac{7vvddy}{dv^2} + \frac{vdy}{dv},$$
$$t = \frac{ds}{dx} = \frac{v^5d^5y}{dv^5} + \frac{10v^4d^4y}{dv^4} + \frac{25v^3d^3y}{dv^3} + \frac{15vvddy}{dv^2} + \frac{vdy}{dv}$$
$$\text{etc.}$$

Whereby the equation between v and y will be this:

$$\begin{array}{llllll} V = Ay + & \frac{Bvdy}{dv} + & \frac{Cvvddy}{dv^2} + & \frac{Dv^3d^3y}{dv^3} + & \frac{Ev^4d^4y}{dv^4} + & \text{etc.} \\ & +C \vdots & +3D \vdots & +6E \vdots & +10F \vdots & \\ & +D \vdots & +7E \vdots & +25F \vdots & & \\ & +E \vdots & +15F \vdots & & & \\ & +F \vdots & & & & \end{array}$$

the integration of which we have set out here. But in the first place it is to be noted that the algebraic quantity hence P required to be formed

$$P = A + (B+C+D+E+F)(z-1) + (C+3D+7E+15F)(z-1)(z-2)$$
$$+ (D+6E+25F)(z-1)(z-2)(z-3) + \text{etc.}$$

to be reduced to this form

$$P = A + B(z-1) + C(z-1)^2 + D(z-1)^3 + E(z-1)^4 + F(z-1)^5 + \text{etc.},$$

which algebraic quantity from that, which in Chapter III we have used in the integration, only differs from that as there we have expressed with the letter z that which here we have expressed with the formula $z-1$; from which also the integration of both can be reduced easily from one to the other.

CONCLUSION OF THE FIRST BOOK

1275. And these are almost [all the propositions] required to be investigated, which have been considered to belong to the first book on the Integral Calculus, where a method is sought for treating functions of one variable from a relation between some given differentials of each order, because I consider there is thus indeed a need for me to have these investigated, so that scarcely any of these [propositions] concerning this argument which have been found so far by other people, and reported in journals, shall be passed over.

CAPUT V

DE INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM HUIUS FORMAE

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \frac{Ex^4d^4y}{dx^4} + \text{etc.}$$

PROBLEMA 164

1226. *Proposita aequatione differentiali huius formae*

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \cdots + \frac{Nx^nd^ny}{dx^n}$$

definire functionem ipsius x, per quam ea multiplicata fiat integrabilis.

SOLUTIO

Attendenti mox patebit, simplicem potestatem ipsius x hoc praestare. Sit igitur integrabilis haec aequatio

$$\int Xx^{\lambda}dx = A'x^{\lambda+1}y + \frac{B'x^{\lambda+2}dy}{dx} + \frac{C'x^{\lambda+3}ddy}{dx^2} + \cdots + \frac{M'x^{\lambda+n}d^{n-1}y}{dx^{n-1}}.$$

cuius integrale sit

$$\int Xx^{\lambda}dx = A'x^{\lambda+1}y + \frac{Bx^{\lambda+2}dy}{dx} + \frac{C'x^{\lambda+3}ddy}{dx^2} + \cdots + \frac{M'x^{\lambda+n}d^{n-1}y}{dx^{n-1}}.$$

Cum igitur huius differentiale illi debeat esse aequale, sequentes nanciscemur determinationes

$$\begin{aligned}
&A=(\lambda+1)A', &&\text{hinc } (\lambda+1)A'=A,\\
&B=(\lambda+2)B'+A', &&\;\;'' \;\;(\lambda+1)(\lambda+2)B'=(\lambda+1)B-A,\\
&C=(\lambda+3)C'+B', &&\;\;'' \;\;(\lambda+1)(\lambda+2)(\lambda+3)C'=(\lambda+1)(\lambda+2)C\\
& && \qquad\qquad -(\lambda+1)B+A,\\
&D=(\lambda+4)D'+C', &&\;\;'' \;\;(\lambda+1)(\lambda+2)(\lambda+3)(\lambda+4)D'\\
& && \qquad =(\lambda+1)(\lambda+2)(\lambda+3)D\\
& && \qquad -(\lambda+1)(\lambda+2)C+(\lambda+1)B-A \qquad \text{etc.};\\
&\quad\vdots\\
&\quad\vdots\\
&N=M' && \text{etc.};
\end{aligned}$$

integralis enim termini sequentes, qui involverent differentialis gradum $d^n y$ altioresque, evanescere debent, quia alioquin integratio non successisset. Cum igitur in integrali littera N' evanescat, pervenimus ad hanc aequationem

$$\begin{aligned}0= A-(\lambda+1)B+(\lambda+1)(\lambda+2)C-(\lambda+1)(\lambda+2)(\lambda+3)D+...\\ \pm(\lambda+1)(\lambda+2)\ ...\ (\lambda+n)N,\end{aligned}$$

ex qua aequatione exponens λ potestatis quaesitae x^λ definiri debet.

Formetur ergo talis expressio algebraica

$$\begin{aligned}P = A+B(z-1)+C(z-1)(z-2)+D(z-1)(z-2)(z-3)+...\\ +N(z-1)(z-2)\ ...\ (z-n)\end{aligned}$$

huiusque quaerantur omnes factores simplices, ut sit

$$P=(\alpha+z)(\beta+z)(\gamma+z)(\delta+z) \quad \text{etc.}$$

factorum horum numero existente = n. Iam ex quolibet factore $z+\alpha$ ad nihilum reducto valor $z=-\alpha$ dabit potestatem x^α, per quam proposita aequatio multiplicata integrabilis evadit, ita ut eius integrale sit futurum

$$x^{-\alpha-1}\int x^\alpha X dx = A'y+\frac{B'xdy}{dx}+\frac{C'x^2ddy}{dx^2}+\frac{D'x^3d^3y}{dx^3}+\cdots+\frac{Nx^{n-1}d^{n-1}y}{dx^{n-1}},$$

ubi differentialium gradus unitate est inferior. Ita autem haec aequatio integrata per propositam determinatur, ut sit

$$A=(\alpha+1)A' \qquad B=(\alpha+2)B'+A'$$
$$C=(\alpha+3)C'+B' \qquad D=(\alpha+4)D'+C'$$
etc.

donec perveniatur ad ultimum coefficientem N, qui utrobique est idem.

COROLLARIUM 1

1227. Quia aequatio integrata similis est ipsi propositae, ea per certam potestatem ipsius x multiplicata denuo fiet integrabilis. Ad hanc enim potestatem inveniendam considerare oportet hanc formam algebraicam

$$Q=A'+B'(z-1)+C'(z-1)(z-2)+D'(z-1)(z-2)(z-3)+\dots$$
$$+N(z-1)(z-2)\ \dots\ (z-n+1),$$

euius si fuerit factor simplex quicunque $z+\mu$, erit x^{μ} illa potestas ipsius x aequationem integrabilem reddens.

COROLLARIUM 2

1228. Quodsi ipsa aequatio proposita per potestatem x^{α} multiplicata reddita fuerit integrabilis, hic probe notari convenit quantitatem Q ex integrata formatam ita pendere a priori P ex ipsa proposita formata, ut sit $Q=\frac{P}{\alpha+z}$ quandoquidem per hypothesin $\alpha+z$ est factor ipsius P.

SCHOLION 1

1229. Ad hanc insignem proprietatem demonstrandam, quod scilicet sit $P=(\alpha+z)Q$, tantum opus est, ut quantitas Q per $\alpha+z$ multiplicetur; verum quo conclusio clarius in oculos occurrat, pro singulis terminis ipsius Q multiplicator bipartito est repraesentandus ac pro primo quidem termino loco $\alpha+z$ scribatur $(\alpha+1)+(z-1)$, pro secundo $(\alpha+2)+(z-2)$, pro tertio $(\alpha+3)+(z-3)$, pro quarto $(\alpha+4)+(z-4)$ etc., ita ut euiusque termini productum binis partibus exhibeatur, quam operationem hic totam apponam:

$$Q=A' \qquad +B'(z-1) \qquad +C'(z-1)(z-2)+D'(z-1)(z-2)(z-3) + \text{etc.}$$

Multipl. $\alpha+1$	$\alpha+2$	$\alpha+3$	$\alpha+4$
$z-1$	$z-2$	$z-3$	$z-4$

$$\text{Prod.}(\alpha+1)A' + \quad A'(z-1)+ \quad B'(z-1)(z-2)+ \quad C'(z-1)(z-2)(z-3)$$
$$+(\alpha+2)B'(z-1)+(\alpha+3)C'(z-1)(z-2)$$
$$+(\alpha+4)D'(z-1)(z-2)(z-3)+\text{etc.}$$

In solutione autem vidimus esse

$$(\alpha+1)A'=A,\quad (\alpha+2)B'+A'=B,\quad (\alpha+3)C'+B'=C \quad \text{etc.},$$

quocirca hoc productum hac forma exprimetur

$$A+B(z-1)+C(z-1)(z-2)+D(z-1)(z-2)(z-3)+\text{etc.},$$

cui valor ipsius P est aequalis, sicque demonstrata est insignis illa proprietas memorata, quod sit $Q=\frac{P}{\alpha+z}$

COROLLARIUM 3

1230. Quodsi ergo valor ipsius P in factores simplices resolutus ita repraesentetur

$$P=N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z) \quad \text{etc.}$$

et ex factore $\alpha+z$ aequatio proposita per x^{α} multiplicata integretur, tum vero ex integrata simili modo valor Q formetur, erit

$$Q=N(\beta+z)(\gamma+z)(\delta+z) \quad \text{etc.}$$

COROLLARIUM 4

1231. Aequatio ergo integrata postquam ad formam propositae fuerit perducta, ut posito

$$x^{-\alpha-1}\int x^{\alpha}Xdx = X'$$

habeatur

$$X' = A'y + \frac{Bxdy}{dx} + \frac{C'x^2ddy}{dx^2} + \frac{D'x^3d^3y}{dx^3} + \text{etc},$$

haec denuo integrabilis reddetur, si multiplicetur per quampiam harum potestatum $x^{\beta}, x^{\gamma}, x^{\delta}$ etc., quae etiam ipsam propositam integrabilem reddidissent.

SCHOLION 2

1232. Antequam continuationem harum integrationum ulterius prosequar, conveniet eum casum formae propositae generalis seorsim evolvi, quo prius aequationis membrum *X* in nihilum abit. Hoc enim casu hoc commodi usu venit, ut statim sine integrationibus repetitis integrale completum exhiberi queat idque simili modo, quo supra in Capite II [§ 1125] sum usus. Huic quidem casui, quia iam multo facilius tractari potest, proprium caput assignare nolui, ne praecepta nimis multiplicari videantur.

PROBLEMA 165

1233. *Proposita hac aequatione differentiali cuiuscunque ordinis*

$$0 = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \text{etc.},$$

ubi variabilis y cum suis differentialibus nusquam plus una dimensione, altera vero x adeo nullam obtineat, eius integrale completum invenire.

SOLUTIO

Particulariter huic aequationi satisfieri perspicuum est, si *y* certae ipsius *x* potestati aequetur; ponamus ergo esse $y = x^{\mu}$ et facta substitutione, cum ubique per x^{μ} diviserimus, perveniemus ad hanc aequationem

$$0 = A + \mu B + \mu(\mu-1)C + \mu(\mu-1)(\mu-2)D + \text{etc.},$$

unde exponentem μ determinari oportet. Vel si secundum praeceptum praecedentis problematis ex aequatione proposita hanc formemus formulam algebraicam

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \text{etc.}$$

eamque in factores simplices resolvamus, ut sit

$$P = N(\alpha + z)(\beta + z)(\gamma + z)(\delta + z) \quad \text{etc.},$$

evidens est posito $\mu = z - 1$ primae aequationi satisfieri sumendo

$$\mu = -\alpha - 1 \quad \text{vel} \quad \mu = -\beta - 1 \quad \text{vel} \quad \mu = -\gamma - 1 \quad \text{etc.},$$

ita ut quisque factor suppeditet integrale particulare. Cum igitur factorum numerus aequetur gradui differentialium summo, hinc colligetur integrale completum aequationis propositae

$$y = \mathfrak{A}x^{-\alpha-1} + \mathfrak{B}x^{-\beta-1} + \mathfrak{C}x^{-\gamma-1} + \mathfrak{D}x^{-\delta-1} + \text{etc.},$$

ubi tantum observari convenit, si factorum illorum simplicium duo pluresve fuerint inter se aequales, integralis formam simili modo immutari debere, quo supra Capite II sum usus. Scilicet cum aequationes ibi tractatae ad praesentem formam revocentur, si ibi loco *x* scribatur *lx*, inde has regulas haurimus:

1. Si forma *P* factorem habeat $(\alpha + z)^2$, inde nascitur pars integralis
$$= x^{-\alpha-1}(\mathfrak{A} + \mathfrak{B}lx).$$
2. Si forma *P* factorem habeat $(\alpha + z)^3$, pars integralis inde orta est
$$= x^{-\alpha-1}\left(\mathfrak{A} + \mathfrak{B}lx + \mathfrak{C}(lx)^2\right).$$
3. Si forma *P* factorem habeat $(\alpha + z)^4$ pars integralis inde orta erit
$$= x^{-\alpha-1}\left(\mathfrak{A} + \mathfrak{B}lx + \mathfrak{C}(lx)^2 + \mathfrak{D}(lx)^3\right).$$

Si factores occurrant imaginarii, partes inde oriundae per solitam imaginariorum reductionem facile ad formam realem revocabuntur, uti in corollariis docebo.

COROLLARIUM 1

1234. Si forma *P* duos habeat factores simplices imaginarios in formula $ff + 2fz\cos.\theta + zz$ contentos, hac cum producto $(\alpha + z)(\beta + z)$ comparata erit

$$\alpha = f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right) \text{ et } \beta = f\left(\cos.\theta - \sqrt{-1}\cdot\sin.\theta\right),$$

unde fit

$$x^{-\alpha} = x^{-f\cos.\theta}x^{-f\sqrt{-1}\cdot\sin.\theta} = x^{-f\cos.\theta}e^{-\sqrt{-1}\cdot\sin.\theta\cdot lx}.$$

Est vero

$$e^{-u\sqrt{-1}} = \cos.u - \sqrt{-1}\cdot\sin.u$$

ideoque habetur

$$x^{-\alpha-1} = x^{-f\cos.\theta}\ \frac{\cos.(f\sin.\theta\cdot lx)-\sqrt{-1}\cdot\sin.(f\sin.\theta\cdot lx)}{x}$$

Quare cum $x^{-\beta-1}$ simili modo exprimatur mutato signo ipsius $\sqrt{-1}$, ex factore duplici $ff+2fz\cos.\theta+zz$ haec nascitur pars integralis

$$x^{-f\cos.\theta-1}\left(\mathfrak{A}\cos.(f\sin.\theta\cdot lx)+\mathfrak{B}\sin.(f\sin.\theta\cdot lx)\right),$$

quae etiam ita potest repraesentari

$$\mathfrak{A}x^{-f\cos.\theta-1}\cos.\left(\mathfrak{a}+f\sin.\theta\cdot lx\right)$$

denotante $\mathfrak{a}$ angulum constantem arbitrarium.

COROLLARIUM 2

1235. Simili modo si forma P duos huiusmodi factores aequales involvat, ut sit

$$(\alpha+z)^2(\beta+z)^2 = (ff+2fz\cos.\theta+zz)^2,$$

litterae α et β eosdem quos ante sortientur valores imaginarios, ex quorum reductione colligitur haec pars integralis inde oriunda

$$x^{-f\cos.\theta-1}\left(\mathfrak{A}\cos.\left(\mathfrak{a}+f\sin.\theta\cdot lx\right)+\mathfrak{B}lx\cos.\left(\mathfrak{b}+f\sin.\theta\cdot lx\right)\right)$$

quatuor constantes arbitrarias $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{a}$ et $\mathfrak{b}$ continens.

COROLLARIUM 3

1236. Hinc ergo evidens est, quomodo ex factoribus formae P, sive sint simplices sive duplices, sive inaequales sive aequales, singulas integralis partes assignari indeque totum integrale completum formari conveniat.

SCHOLION

1237. Totum ergo negotium huc redit, ut quantitas algebraica ex aequatione differentiali formata

$$P = A+B(z-1)+C(z-1)(z-2)+D(z-1)(z-2)(z-3)+\text{etc.}$$

in suos factores reales vel simplices vel duplices resolvetur, in quo plerumque maxima difficultas versatur, quoniam huiusmodi formae minus tractari sunt solitae. Cum vero haec resolutio isti aequationi cum generali, quam hoc capite evolvere suscepi, est communis, quicquid hic praestare licuerit, potius in aequatione generali ostendi conveniet, ad quam resolvendam propterea revertor. Id tantum hic observasse necesse duco, quodsi pro aequatione generali

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cxxddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \text{etc.}$$

undecunque innotuerit integrale particulare, puta $y = V$ existente V certa functione ipsius x, tum posito $y = V + v$ perveniri ad hanc aequationem

$$0 = Av + \frac{Bxdv}{dx} + \frac{Cxxddv}{dx^2} + \frac{Dx^3d^3v}{dx^3} + \text{etc.}$$

cuius integrale completum per praecepta huius problematis inventum si loco v scribatur, habebitur integrale completum illius aequationis, quo pacto certe insigne calculi compendium obtinetur.

PROBLEMA 166

1238. *Proposita aequatione differentiali gradus cuiuscunque n kuius formae*

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \cdots + \frac{Nx^nd^ny}{dx^n}$$

eius integrale per integrationem n vicibus repetitam invenire.

SOLUTIO

Ex hac aequatione formetur haec quantitas algebraica

$$P = A + B(z-1) + C(z-1)(z-2) + \ldots + N(z-1)(z-2) \ \ldots \ (z-n),$$

cuius quaerantur omnes factores simplices nullo habito respectu, sive sint reales sive imaginarii, ut ea hoc modo exprimatur

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\cdots(\mu+z)(\nu+z)$$

factorum numero existente $= n$. Quo facto initio huius capitis vidimus quemlibet factorem, puta $\alpha + z$, praebere potestatem x^{α}, per quam nostra aequatio fiat integrabilis, atque adeo ostendimus integrale inde ortum, si compendii gratia ponamus

$$x^{-\alpha-1}\int x^{\alpha}Xdx = X',$$

fore

$$X' = A'y + \frac{B'xdy}{dx} + \frac{C'x^2ddy}{dx^2} + \frac{D'x^3d^3y}{dx^3} + \cdots + \frac{Nx^{n-1}d^{n-1}y}{dx^{n-1}},$$

ita ut sit

$$A' = \frac{\alpha}{A+1}$$

ceterique coefficientes ita se habeant, uti ibidem docuimus; hic autem sufficiet ad primum potissimum respexisse.

Absoluta iam prima integratione si eadem lege ex aequatione semel integrata formemus quantitatem

$$P' = A' + B'(z-1) + C'(z-1)(z-2) + \ldots + N(z-1)(z-2)\ \ldots\ (z-n+1),$$

cuius resolutio in factores iam ex prima forma P constat, postquam § 1229 demonstravi esse

$$P' = N(\beta+z)(\gamma+z)(\delta+z)\cdots(\mu+z)(\nu+z),$$

ita ut sit $P' = \frac{P}{\alpha+z}$, hinc ergo simili modo factor $\beta+z$ suppeditabit multiplicatorem x^{β}, quo haec aequatio integrabilis redditur, ac posito

$$x^{-\beta-1}\int x^{\beta}X'dx = X'',$$

ut sit

$$X'' = x^{-\beta-1}\int x^{\beta-\alpha-1}dx\int x^{\alpha}Xdx,$$

integrale erit

$$X'' = A''y + \frac{B''xdy}{dx} + \frac{C''x^2ddy}{dx^2} + \cdots + \frac{Nx^{n-2}d^{n-2}y}{dx^{n-2}}$$

existente hic

$$A'' = \frac{A'}{\beta+1} = \frac{A}{(\alpha+1)(\beta+1)}.$$

Quodsi hoc modo tot integrationes successive absolvantur, quot unitates continentur in indice n, sicque omnes factores simplices formae P in usum vocentur, tandem ad hanc pervenietur aequationem

$$X^{(n)} = A^{(n)}y,$$

quae est ipsa integralis desiderata. Cum autem hic futurum sit

$$A^{(n)} = \frac{A}{(\alpha+1)(\beta+1)(\gamma+1)\cdots(\nu+1)},$$

evidens est denominatorem nasci ex forma $\frac{P}{N}$, si loco z scribatur unitas; tum autem sumto $z=1$ prima forma manifesto dat $P=A$, ita ut denominator iste fiat $=\frac{A}{N}$ ideoque $A^{(n)}=N$, quod etiam inde patet, quod omnium aequationum ultimi termini habeant eundem coefficientem N, quo ergo in postrema integrali ipse primus terminus y debet esse affectus. Deinde vero est

$$X^{(n)}=x^{-\nu-1}\int x^{\nu-\mu-1}dx\int x^{\mu-\lambda-1}dx\cdots\int x^{\beta-\alpha-1}dx\int x^{\alpha}Xdx,$$

ubi cum numeri α,β,γ etc. utcunque inter se permutari possunt, integrale quaesitum etiam hoc modo repraesentari potest

$$Ny=x^{-\alpha-1}\int x^{\alpha-\beta-1}dx\int x^{\beta-\gamma-1}dx\cdots\int x^{\mu-\nu-1}dx\int x^{\nu}Xdx.$$

COROLLARIUM 1

1239. Totum ergo negotium huc redit, ut forma algebraica

$$P=A+B(z-1)+C(z-1)(z-2)+D(z-1)(z-2)(z-3)+\text{etc.}$$

in suos factores simplices resolvatur, quibus inventis ut sit

$$P=N(\alpha+z)(\beta+z)(\gamma+z)\cdots(\nu+z),$$

hinc integrale quaesitum facile exhibetur et quidem pro factorum varia permutatione pluribus modis, qui autem omnes eundem valorem exprimunt, uti ex sequentibus clarius patebit.

COROLLARIUM 2

1240. Cum haec forma integralis inventa tot involvat integrationes, quoti gradus fuerit aequatio differentialis proposita, totidem quoque constantes arbitrariae ingerentur, quemadmodum indoles integrationis completae postulat.

SCHOLION

1241. Quoniam integrale inventum pluribus integrationibus est involutum, ad usum faciliorem conveniet hanc formam in partes resolvi, quae singulae unicum tantum signum integrale contineant. Hanc autem resolutionem simili modo instituere licet, quo supra sumus usi, atque hic quidem totum negotium ad huiusmodi formulam revocatur

$$\int x^{m-n-1}dx\int x^{n}Xdx,$$

quae manifesto ita reducitur, ut sit

$$\frac{1}{m-n}x^{m-n}\int x^n Xdx-\frac{1}{m-n}\int x^m Xdx\,,$$

ubi tamen observandum est, si fuerit $m=n$, peculiari reductione opus esse hocque casu fore

$$\int\frac{dx}{x}\int x^n Xdx=lx\int x^n Xdx-\int x^n Xdxlx.$$

Hac ergo regula utemur in resolutione sequentium problematum, quibus successive omnes gradus differentialium percurramus, praetermisso quidem gradu prima, cum aequationis

$$X=Ay+\frac{Nxdy}{dx}$$

ob $P=A+N(z-1)=N(\alpha+z)$ integrale sit

$$Ny=x^{-\alpha-1}\int x^{\alpha}Xdx,$$

quod nulla reductione indiget.

PROBLEMA 167

1242. *Proposita hac aequatione differentiali secundi gradus*

$$X=Ay+\frac{Bxdy}{dx}+\frac{Nx^2ddy}{dx^2}$$

eius integrale per formulas integrales simplices evolvere.

SOLUTIO

Cum sit

$$P=A+B(z-1)+N(z-1)(z-2),$$

statuatur

$$P=N(\alpha+z)(\beta+z)$$

eritque integrale per methodum praecedentem inventum $Ny=X''$ existente

$$x^{\beta+1}X''=\int x^{\beta-\alpha-1}dx\int x^{\alpha}Xdx,$$

quae forma evolvitur in hanc

$$\frac{1}{\beta-\alpha}x^{\beta-\alpha}\int x^{\alpha}Xdx-\frac{1}{\beta-\alpha}\int x^{\beta}Xdx\,.$$

sicque erit

$$Ny=\frac{x^{-\alpha-1}}{\beta-\alpha}\int x^{\alpha}Xdx+\frac{x^{-\beta-1}}{\alpha-\beta}\int x^{\beta}Xdx.$$

Hinc autem casum excipi oportet, quo $\beta=\alpha$; tum enim fit

$$x^{\alpha+1}X''=\int\tfrac{dx}{x}\int x^{\alpha}Xdx=lx\int x^{\alpha}Xdx-\int x^{\alpha}Xdxlx,$$

pro hoc igitur casu habebimus

$$Ny=x^{-\alpha-1}\int\tfrac{dx}{x}\int x^{\alpha}Xdx \text{ seu } x^{-\alpha-1}\left(lx\int x^{\alpha}Xdx-\int x^{\alpha}Xdxlx\right),$$

ubi quidem prior forma praeferenda videtur.

COROLLARIUM 1

1243. Si ambo factores simplices sint imaginarii, ponatur

$$(\alpha+z)(\beta+z)=ff+2fz\cos.\theta+zz$$

eritque

$$\alpha=f\left(\cos.\theta+\sqrt{-1}\cdot\sin.\theta\right) \text{ et } \beta=f\left(\cos.\theta-\sqrt{-1}\cdot\sin.\theta\right)$$

indeque

$$\beta-\alpha=-2f\sqrt{-1}\cdot sin.\theta .$$

Tum vero

$$x^{\alpha}=x^{f\cos.\theta}\left(\cos.(f\sin.\theta\cdot lx)+\sqrt{-1}\cdot\sin.(f\sin.\theta\cdot lx)\right)$$

et

$$x^{-\alpha}=x^{-f\cos.\theta}\left(\cos.(f\sin.\theta\cdot lx)-\sqrt{-1}\cdot\sin.(f\sin.\theta\cdot lx)\right),$$

quae formulae mutato signo ipsius $\sqrt{-1}$ ad x^{β} et $x^{-\beta}$ transferuntur.

COROLLARIUM 2

1244. Ponatur brevitatis gratia angulus $f\sin.\theta\cdot lx=\varphi$ et facta substitutione habebimus

$$Nxy=\frac{x^{-f\cos.\theta}\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)}{-2f\sqrt{-1}\cdot sin.\theta}\int x^{f\cos.\theta}Xdx\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)$$
$$+\frac{x^{-f\cos.\theta}\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)}{2f\sqrt{-1}\cdot sin.\theta}\int x^{f\cos.\theta}Xdx\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right),$$

ubi partes imaginariae se sponte destruunt, fietque

$$Nxy = \frac{x^{-f\cos.\theta}}{f\sin.\theta}\left(\sin.\varphi\int x^{f\cos.\theta} Xdx\cos.\varphi - \cos.\varphi\int x^{f\cos.\theta} Xdx\sin.\varphi\right).$$

COROLLARIUM 3

1245. Forma ergo haec realis modo inventa aequivalet illi imaginaria implicanti

$$Nxy = \frac{x^{-\alpha}}{\beta-\alpha}\int x^{\alpha} Xdx + \frac{x^{-\beta}}{\alpha-\beta}\int x^{\beta} Xdx,$$

si fuerit

$$(\alpha+z)(\beta+z) = ff + 2fz\cos.\theta + zz$$

ponaturque $l\varphi = f\sin.\theta \cdot lx$, quae reductio semel facta etiam in sequentibus usum praestabit.

PROBLEMA 168

1246. *Proposita hac aequatione differentiali tertii gradus*

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cxxddy}{dx^2} + \frac{Nx^3d^3y}{dx^3}$$

eius integrale per formulas integrales simplices evolvere.

SOLUTIO

Cum hic sit

$$P = A + B(z-1) + C(z-1)(z-2) + N(z-1)(z-2)(z-3),$$

ponatur

$$P = N(\alpha+z)(\beta+z)(\gamma+z),$$

et cum per integrationem generalem prodeat $Ny = X'''$, notetur esse $X''' = x^{-\gamma-1}\int x^{\gamma} X''dx$, siquidem valorem ipsius X'' ex binis factoribus $\alpha+z$ et $\beta+z$ iam invenimus; hinc enim per problema praecedens habetur

$$X'' = \frac{x^{-\alpha-1}}{\beta-\alpha}\int x^{\alpha} Xdx + \frac{x^{-\beta-1}}{\alpha-\beta}\int x^{\beta} Xdx,$$

unde colligitur

$$\int x^{\gamma} X''dx = \frac{x^{\gamma-\alpha}}{(\beta-\alpha)(\gamma-\alpha)}\int x^{\alpha} Xdx - \frac{1}{(\beta-\alpha)(\gamma-\alpha)}\int x^{\gamma} Xdx$$
$$+ \frac{x^{\gamma-\beta}}{(\alpha-\beta)(\gamma-\beta)}\int x^{\beta} Xdx - \frac{1}{(\alpha-\beta)(\gamma-\beta)}\int x^{\gamma} Xdx.$$

Est vero

$$\frac{1}{(\beta-\alpha)(\gamma-\alpha)} + \frac{1}{(\alpha-\beta)(\gamma-\beta)} = \frac{-1}{(\alpha-\gamma)(\beta-\gamma)} \quad ;$$

quod quemadmodum iam per se liquet, tum vero ex Theoremate § 1169 demonstrato perspicitur. Quocirca integrale quaesitum ita obtinetur expressum

$$Nxy=\frac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)}\int x^{\alpha}Xdx+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)}\int x^{\beta}Xdx+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)}\int x^{\gamma}Xdx.$$

COROLLARIUM 1

1247. Si forma P duos habeat factores aequales, ut sit $\beta=\alpha$ quia tum est

$$X''=x^{-\alpha-1}\int\frac{dx}{x}\int x^{\alpha}Xdx,$$

erit

$$\int x^{\gamma}X''dx=\frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx-\frac{1}{\gamma-\alpha}\int x^{\gamma-\alpha-1}dx\int x^{\alpha}Xdx;$$

at

$$\int x^{\gamma-\alpha-1}dx\int x^{\alpha}Xdx=\frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int x^{\alpha}Xdx-\frac{1}{\gamma-\alpha}\int x^{\gamma}Xdx,$$

unde colligitur

$$Nxy=\frac{x^{-\alpha}}{\gamma-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx-\frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int x^{\alpha}Xdx+\frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int x^{\gamma}Xdx.$$

COROLLARIUM 2

1248. Similis forma oritur, si sumatur $\gamma=\beta$ tum enim fit

$$\int x^{\gamma}X''dx=\frac{x^{\beta-\alpha}}{(\beta-\alpha)^2}\int x^{\alpha}Xdx-\frac{1}{(\beta-\alpha)^2}\int x^{\beta}dx+\frac{1}{\alpha-\beta}\int\frac{dx}{x}\int x^{\beta}Xdx$$

ideoque

$$Nxy=\frac{x^{-\beta}}{\alpha-\beta}\int\frac{dx}{x}\int x^{\beta}Xdx-\frac{x^{-\beta}}{(\beta-\alpha)^2}\int x^{\beta}Xdx+\frac{x^{-\alpha}}{(\beta-\alpha)^2}\int x^{\alpha}Xdx.$$

COROLLARIUM 3

1249. Quodsi autem omnes tres factores inter se fuerint aequales, $\alpha=\beta=\gamma$, erit

$$\int x^{\gamma}X''dx=\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\alpha}Xdx$$

ideoque hoc casu integrale ita succincte exprimitur

$$Nxy = x^{-\alpha}\int \tfrac{dx}{x}\int \tfrac{dx}{x}\int x^{\alpha} Xdx.$$

COROLLARIUM 4

1250. Si duo factores sint imaginarii, scilicet

$$(\alpha + z)(\beta + z) = ff + 2fz\cos.\theta + zz$$

ob

$$\alpha = f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right) \text{ et } \beta = f\left(\cos.\theta - \sqrt{-1}\cdot\sin.\theta\right)$$

postremum quidem nostri integralis membrum manet reale ob

$$(\alpha - \gamma)(\beta - \gamma) = \gamma\gamma - 2f\gamma\cos.\theta + ff\ ,$$

at bina priora fient posito $\varphi = f\sin.\theta\cdot lx$,

$$\frac{x^{-f\cos.\theta}\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right)}{-2f\sqrt{-1}\cdot sin.\theta\left(\gamma - f\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right)\right)}\int x^{f\cos.\theta} Xdx\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right)$$
$$+\frac{x^{-f\cos.\theta}\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right)}{2f\sqrt{-1}\cdot sin.\theta\left(\gamma - f\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right)\right)}\int x^{f\cos.\theta} Xdx\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right),$$

quae reducuntur ad hanc formam realem

$$\frac{x^{-f\cos.\theta}\left(\gamma\sin.\varphi - f\sin.(\theta+\varphi)\right)}{f sin.\theta\left(\gamma\gamma - 2\gamma f\cos.\varphi + ff\right)}\int x^{f\cos.\theta} Xdx\cos.\varphi$$
$$-\frac{x^{-f\cos.\theta}\left(\gamma\cos.\varphi - f\cos.(\theta+\varphi)\right)}{f sin.\theta\left(\gamma\gamma - 2\gamma f\cos.\varphi + ff\right)}\int x^{f\cos.\theta} Xdx\sin.\varphi.$$

SCHOLION

1251. Quod ad factores imaginarios attinet, integralium inde natorum reductio facilius in genere instituetur, unde in his differentialium gradibus determinatis ei non amplius immorabor. Factores autem aequales hic data opera pro singulis gradibus accuratius persequi est visum, quia supra [§1163, §1179] nimis cito ad evolutionem generalem properanti in insignem errorem illabi contigit, quem statim feliciter evitassem, si eadem methodo ibi essem usus. Huiusmodi autem vitium circa factores imaginarios hic non est pertimescendum, cum in hoc negotio nihil sub specie infinite parvi negligendum occurrat. Ex hoc autem fonte errores illi, quos supra commisi, sunt nati; quod vitium subtile quo clarius ob oculos ponatur, una cum necessaria emendatione hic evolvam. Quaestio scilicet pro casu praesenti huc redit, ut valor harum duarum formularum

$$\frac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)}\int x^{\alpha}Xdx+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)}\int x^{\beta}Xdx$$

definiatur casu, quo $\beta=\alpha$ et ambo membra in infinitum excrescent; hunc in finem pono $\beta=\alpha+\omega$ existente ω particula evanescente, et cum sit

$$x^{\beta}=x^{\alpha}x^{\omega}=x^{\alpha}e^{\omega lx}=x^{\alpha}\left(1+\omega lx\right)$$

hincque

$$x^{-\beta}=x^{-\alpha}x^{\omega}=x^{-\alpha}\left(1-\omega lx\right),$$

habebimus

$$\frac{x^{-\alpha}}{\omega(\gamma-\alpha)}\int x^{\alpha}Xdx-\frac{x^{-\alpha}(1-\omega lx)}{\omega(\gamma-\beta)}\int x^{\alpha}Xdx\left(1+\omega lx\right).$$

Quia nunc est

$$\frac{1}{(\gamma-\alpha)}=\frac{1}{(\gamma-\beta+\omega)}=\frac{1}{\gamma-\beta}-\frac{\omega}{(\gamma-\beta)^{2}},$$

prius membrum induit hanc formam

$$\frac{x^{-\alpha}}{\omega(\gamma-\beta)}\int x^{\alpha}Xdx-\frac{x^{-\alpha}}{(\gamma-\beta)^{2}}\int x^{\alpha}Xdx,$$

posterius vero evolutum istam

$$-\frac{x^{-\alpha}}{\omega(\gamma-\beta)}\int x^{\alpha}Xdx+\frac{x^{-\alpha}}{\gamma-\beta}\left(lx\int x^{\alpha}Xdx-\int x^{\alpha}Xdxlx\right)$$

sicque valor quaesitus casu $\beta=\alpha$ concluditur

$$\frac{x^{-\alpha}}{\gamma-\alpha}\left(lx\int x^{\alpha}Xdx-\int x^{\alpha}Xdxlx\right)-\frac{x^{-\alpha}}{(\gamma-\alpha)^{2}}\int x^{\alpha}Xdx$$

seu

$$\frac{x^{-\alpha}}{\gamma-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx-\frac{x^{-\alpha}}{(\gamma-\alpha)^{2}}\int x^{\alpha}Xdx,$$

cuius formulae posterius membrum, quod in vitiosa ilIa methodo erat omissum, inde resultat, quod ad discrimen hic inter expressiones $\gamma-\alpha$ et $\gamma-\beta$ respeximus, quam necessariam cautionem supra negleximus.

PROBLEMA 169

1252. *Proposita hac aequatione differentiali quarti gradus*

$$X=Ay+\frac{Bxdy}{dx}+\frac{Cxxddy}{dx^{2}}+\frac{Dx^{3}d^{3}y}{dx^{3}}+\frac{Nx^{4}d^{4}y}{dx^{4}}$$

eius integrale per formulas integrales simplices evolvere.

SOLUTIO

Formata hinc expressione algebraica

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + N(z-1)(z-2)(z-3)(z-4)$$

statuatur

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)$$

et per praecepta generalia est

$$Ny = X^{IV} \text{ existente } X^{IV} = x^{-\delta-1}\int X'''x^{\delta}dx,$$

siquidem X''', ex tribus prioribus factoribus determinetur, quemadmodum in problemate praecedente [§ 1246] est factum. Valorem scilicet ibi pro Nxy inventum hic per $x^{\delta-1}dx$ multiplicari oportet, unde oritur

$$\begin{aligned}\int x^{\delta}X'''dx = &+\tfrac{x^{\delta-\alpha}}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}\int x^{\alpha}Xdx - \tfrac{1}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}\int x^{\delta}Xdx\\ &+\tfrac{x^{\delta-\beta}}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}\int x^{\beta}Xdx - \tfrac{1}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}\int x^{\delta}Xdx\\ &+\tfrac{x^{\delta-\gamma}}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)}\int x^{\gamma}Xdx - \tfrac{1}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)}\int x^{\delta}Xdx,\end{aligned}$$

ubi ob rationes supra [§ 1169] demonstratas tres postremi termini coalescunt in

$$+\tfrac{1}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\int x^{\delta}Xdx,$$

ita ut sit integrale quaesitum

$$\begin{aligned}Nxy = &\tfrac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}\int x^{\alpha}Xdx + \tfrac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}\int x^{\beta}Xdx\\ &+\tfrac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)}\int x^{\gamma}Xdx + \tfrac{x^{-\delta}}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\int x^{\delta}Xdx,\end{aligned}$$

siquidem omnes factores sint inter se inaequales. Casus autem, quibus duo pluresve sunt aequales, in corollariis explorabimus.

COROLLARIUM 1

1253. Si fuerint duo factores aequales, nempe $\delta=\gamma$, seu si sit

$$P=N(\alpha+z)(\beta+z)(\gamma+z)^2$$

ex eadem forma pro X''' ante [§ 1246] inventa oritur integrale

$$Nxy=\frac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)^2}\int x^{\alpha}Xdx-\frac{x^{-\gamma}}{(\beta-\alpha)(\gamma-\alpha)^2}\int x^{\gamma}Xdx$$
$$+\frac{x^{-\beta}}{(\alpha-\gamma)(\gamma-\beta)^2}\int x^{\beta}Xdx-\frac{x^{-\gamma}}{(\alpha-\beta)(\gamma-\beta)^2}\int x^{\gamma}Xdx+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)}\int\frac{dx}{x}\int x^{\gamma}Xdx,$$

ubi membra negativa ita repraesentari possunt

$$\frac{x^{-\gamma}}{\alpha-\beta}\left(\frac{1}{(\gamma-\alpha)^2}-\frac{1}{(\gamma-\beta)^2}\right)\int x^{\gamma}Xdx$$

COROLLARIUM 2

1254. Si fuerint tres factores aequales, ut sit

$$P=N(\alpha+z)(\beta+z)^3$$

ideoque $\delta=\gamma=\beta$, ex formula § 1249 inventa colligitur integrale

$$Nxy=\frac{x^{-\alpha}}{(\beta-\alpha)^3}\int x^{\alpha}Xdx-\frac{x^{-\beta}}{(\beta-\alpha)^3}\int x^{\beta}Xdx$$
$$-\frac{x^{-\beta}}{(\beta-\alpha)^2}\int\frac{dx}{x}\int x^{\beta}Xdx+\frac{x^{-\beta}}{(\alpha-\beta)}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx.$$

COROLLARIUM 3

1255. Si omnes quatuor factores fuerint aequales, ut sit

$$P=N(\alpha+z)^4$$

existente $\delta=\gamma=\beta=\alpha$, ex forma pro tribus aequalibus § 1249 inventa fit integrale

$$Nxy = x^{-\alpha}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\alpha}Xdx.$$

COROLLARIUM 4

1256. Si habeatur $\beta=\alpha$ et $\delta=\gamma$, ut bini factores sint aequales, scilicet

$$P = N(\alpha+z)^2(\gamma+z)^2,$$

ex § 1247, ubi factores erant $(\alpha+z)^2(\gamma+z)$ *(a +z)*, colligitur integrale

$$Nxy = \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\alpha}Xdx - \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int x^{\gamma-\alpha-1}dx\int x^{\alpha}Xdx - \frac{x^{-\alpha}}{(\gamma-\alpha)^3}\int x^{\alpha}Xdx$$
$$+\frac{x^{-\gamma}}{(\gamma-\alpha)^3}\int x^{\gamma}Xdx + \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx,$$

quae ob

$$\int x^{\gamma-\alpha-1}dx\int x^{\alpha}Xdx = \frac{x^{\gamma-\alpha}}{\gamma-\alpha}\int x^{\alpha}Xdx - \frac{1}{\gamma-\alpha}\int x^{\gamma}Xdx$$

contrahitur in hanc formam

$$Nxy = \frac{x^{-\alpha}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\alpha}Xdx + \frac{x^{-\gamma}}{(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx$$
$$-\frac{2x^{-\alpha}}{(\gamma-\alpha)^3}\int x^{\alpha}Xdx - \frac{2x^{-\gamma}}{(\alpha-\gamma)^3}\int x^{\gamma}Xdx.$$

PROBLEMA 170

1257. *Proposita hac aequatione differentiali quinti gradus*

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \frac{Ex^4d^4y}{dx^4} + \frac{Nx^5d^5y}{dx^5}$$

eius integrale per formulas integrales simplices evolvere.

SOLUTIO

Cum hic sit quantitas algebraica formanda

$$P = A + B(z-1) + C(z-1)(z-2) + \cdots + N(z-1)(z-2)(z-3)(z-4)(z-5),$$

statuatur

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)(\varepsilon+z),$$

ac si hi factores omnes sint inter se inaequales, ex integrali praecedente novam instituendo integrationem prodibit integrale quaesitum

$$Nxy = \frac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)}\int x^{\alpha}Xdx + \frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}\int x^{\beta}Xdx$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)}\int x^{\gamma}Xdx + \frac{x^{-\delta}}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\int x^{\delta}Xdx,$$

casus, quo duo pluresve factores sunt aequales, in corollariis evolvemus.

COROLLARIUM 1

1258. Si fuerint duo factores aequales, ut sit

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)^2$$

ideoque $\varepsilon=\delta$, ex praecedente problemate colligitur integrale

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}Xdx - x^{-\delta}\int x^{\delta}Xdx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)^2} + \frac{x^{-\beta}\int x^{\beta}Xdx - x^{-\delta}\int x^{\delta}Xdx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)^2}$$
$$+\frac{x^{-\gamma}\int x^{\gamma}Xdx - x^{-\delta}\int x^{\delta}Xdx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)^2} + \frac{x^{-\delta}}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\int\frac{dx}{x}\int x^{\delta}Xdx.$$

COROLLARIUM 2

1259. Si fuerint tres factores aequales, ut sit

$$P = N(\alpha+z)(\beta+z)(\gamma+z)^3$$

ideoque $\varepsilon=\delta=\gamma$, ex Corollario 1 problematis praecedentis colligitur

$$Nxy = \frac{x^{-\alpha}\int x^{\alpha}Xdx - x^{-\gamma}\int x^{\gamma}Xdx}{(\beta-\alpha)(\gamma-\alpha)^3} - \frac{x^{-\gamma}}{(\beta-\alpha)(\gamma-\alpha)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx$$
$$+\frac{x^{-\beta}\int x^{\beta}Xdx - x^{-\gamma}\int x^{\gamma}Xdx}{(\alpha-\beta)(\gamma-\beta)^3} - \frac{x^{-\gamma}}{(\alpha-\beta)(\gamma-\beta)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}Xdx.$$

COROLLARIUM 3

1260. Si quatuor factores sint aequales, ut sit

$$P = N(\alpha+z)(\beta+z)^4$$

ideoque $\varepsilon=\delta=\gamma=\beta$, erit per §1254

$$Nxy=\frac{x^{-\alpha}}{(\beta-\alpha)^4}\int x^{\alpha}Xdx-\frac{x^{-\beta}}{(\beta-\alpha)^4}\int x^{\beta}Xdx-\frac{x^{-\beta}}{(\beta-\alpha)^3}\int\frac{dx}{x}\int x^{\beta}Xdx$$
$$-\frac{x^{-\beta}}{(\beta-\alpha)^2}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx-\frac{x^{-\beta}}{(\beta-\alpha)}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx;$$

ac si omnes quinque sint inter se aequales seu

$$P=N(\alpha+z)^5$$

erit integrale

$$Nxy=x^{-\alpha}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\alpha}Xdx.$$

COROLLARIUM 4

1261. Si *P* habeat duos factores quadratos, ut sit

$$P=N(\alpha+z)(\beta+z)^2(\gamma+z)^2$$

ideoque $\delta=\gamma$ et $\varepsilon=\beta$, erit ex § 1253 integrale reductione necessaria facta

$$Nxy=\frac{x^{-\alpha}\int x^{\alpha}Xdx-x^{-\beta}\int x^{\beta}Xdx}{(\beta-\alpha)^2(\gamma-\alpha)^2}-\frac{x^{-\gamma}\int x^{\gamma}Xdx-x^{-\beta}\int x^{\beta}Xdx}{(\beta-\alpha)(\alpha-\gamma)^2(\beta-\gamma)}$$
$$+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)^2}\int\frac{dx}{x}\int x^{\beta}Xdx-\frac{x^{-\gamma}\int x^{\gamma}Xdx-x^{-\beta}\int x^{\beta}Xdx}{(\alpha-\beta)(\beta-\gamma)^3}$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx-\frac{x^{-\gamma}\int x^{\gamma}Xdx-x^{-\beta}\int x^{\beta}Xdx}{(\alpha-\gamma)(\beta-\gamma)^3},$$

quae porro redigitur ad hanc formam

$$Nxy=\frac{x^{-\alpha}}{(\beta-\alpha)^2(\gamma-\alpha)^2}\int x^{\alpha}Xdx$$
$$+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)^2}\int\frac{dx}{x}\int x^{\beta}Xdx-\frac{x^{-\beta}}{(\alpha-\beta)^2(\gamma-\beta)^2}\int x^{\beta}Xdx-\frac{2x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)^3}\int x^{\beta}Xdx$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx-\frac{x^{-\gamma}}{(\alpha-\gamma)^2(\beta-\gamma)^2}\int x^{\gamma}Xdx-\frac{2x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)^3}\int x^{\gamma}Xdx$$

COROLLARIUM 5

1262. Si P habeat et factorem quadratum et cubicum, ut sit

$$P = N(\alpha + z)^2 (\gamma + z)^3$$

ideoque $\beta = \alpha$ et $\varepsilon = \delta = \gamma$, ex § 1254 colligitur integrale

$$Nxy = \frac{x^{-\alpha}}{(\gamma-\alpha)^3}\int\frac{dx}{x}\int x^{\alpha}Xdx - \frac{3x^{-\alpha}}{(\gamma-\alpha)^4}\int x^{\alpha}Xdx$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)^2}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}Xdx - \frac{2x^{-\gamma}}{(\alpha-\gamma)^3}\int\frac{dx}{x}\int x^{\gamma}Xdx + \frac{3x^{-\gamma}}{(\alpha-\gamma)^4}\int x^{\gamma}Xdx.$$

SCHOLION

1263. Ex his formulis parum constat, quemadmodum eas ulterius pro maiori factorum numero continuari oporteat, siquidem factorum aliquot inter se fuerint aequales. Integralium enim partes, quae factoribus inaequalibus respondent, legem servant manifestam; quae autem partibus aequalibus respondent, adhibita certa reductione commodius exprimi possunt. Veluti pro casu Corollarii 1 si brevitatis gratia ponatur $\alpha - \delta = p$, $\beta - \delta = q$ et $\gamma - \delta = r$, forma $x^{-\delta}\int x^{\delta}Xdx$ ducta est in

$$\frac{1}{(p-q)(r-p)pp} + \frac{1}{(p-q)(q-r)qq} + \frac{1}{(r-p)(q-r)rr}$$

seu

$$\frac{(q-r)qqrr+(r-p)pprr+(p-q)ppqq}{(p-q)(q-r)(r-p)ppqqrr}$$

cuius fractionis numerator est $-(p-q)(q-r)(r-p)(pq+pr+qr)$, ita ut haec fractio reducatur ad istam

$$\frac{-pq-pr-qr}{ppqqrr} = -\frac{1}{pqr}\left(\frac{1}{p}+\frac{1}{q}+\frac{1}{r}\right).$$

Quando ergo est

$$P = N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)^2,$$

integrale ita se habet

$$\begin{aligned}Nxy &= \frac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)^2}\int x^{\alpha}Xdx+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)^2}\int x^{\beta}Xdx\\&+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)^2}\int x^{\gamma}Xdx+\frac{x^{-\delta}}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\int\frac{dx}{x}\int x^{\delta}Xdx\\&-\frac{x^{-\delta}}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}\left(\frac{1}{\alpha-\delta}+\frac{1}{\beta-\delta}+\frac{1}{\gamma-\delta}\right)\int x^{\delta}Xdx.\end{aligned}$$

Pro casu autem $P=N(\alpha+z)(\beta+z)(\gamma+z)^3$ habebitur [§ 1259]

$$\begin{aligned}Nxy &= \frac{x^{-\alpha}}{(\beta-\alpha)(\gamma-\alpha)^3}\int x^{\alpha}Xdx+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)^3}\int x^{\beta}Xdx\\&+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}Xdx-\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{\alpha-\gamma}+\frac{1}{\beta-\gamma}\right)\int\frac{dx}{x}\int x^{\gamma}Xdx\\&+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)}\left(\frac{1}{(\alpha-\gamma)^2}+\frac{1}{(\alpha-\gamma)(\beta-\gamma)}+\frac{1}{(\beta-\gamma)^2}\right)\int x^{\gamma}Xdx.\end{aligned}$$

Tum vero pro casu $P=N(\alpha+z)(\beta+z)^4$ fit [§ 1260]

$$\begin{aligned}Nxy &= \frac{x^{-\alpha}}{(\beta-\alpha)^4}\int x^{\alpha}Xdx+\frac{x^{-\beta}}{(\alpha-\beta)}\int\frac{dx}{x}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx\\&-\frac{x^{-\beta}}{\alpha-\beta}\cdot\frac{1}{\alpha-\beta}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\beta}Xdx+\frac{x^{-\beta}}{\alpha-\beta}\cdot\frac{1}{(\alpha-\beta)^2}\int\frac{dx}{x}\int x^{\beta}Xdx\\&-\frac{x^{-\beta}}{\alpha-\beta}\cdot\frac{1}{(\alpha-\beta)^3}\int x^{\beta}Xdx.\end{aligned}$$

At pro casu $P=N(\alpha+z)(\beta+z)^2(\gamma+z)^2$ integrale [§ 1261] ita se habet

$$\begin{aligned}Nxy &= \frac{x^{-\alpha}}{(\beta-\alpha)^2(\gamma-\alpha)^2}\int x^{\alpha}Xdx\\&+\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)^2}\int\frac{dx}{x}\int x^{\beta}Xdx-\frac{x^{-\beta}}{(\alpha-\beta)(\gamma-\beta)^2}\left(\frac{1}{\alpha-\beta}+\frac{2}{\gamma-\beta}\right)\int x^{\beta}Xdx\\&+\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)^2}\int\frac{dx}{x}\int x^{\gamma}Xdx-\frac{x^{-\gamma}}{(\alpha-\gamma)(\beta-\gamma)^2}\left(\frac{1}{\alpha-\gamma}+\frac{2}{\beta-\gamma}\right)\int x^{\gamma}Xdx.\end{aligned}$$

At pro casu $P=N(\alpha+z)^2(\gamma+z)^3$ integrale [§ 1262] ita se habet

$$Nxy = \frac{x^{-\alpha}}{(\gamma-\alpha)^3}\int\frac{dx}{x}\int x^{\alpha}Xdx - \frac{x^{-\alpha}}{(\gamma-\alpha)^3}\cdot\frac{3}{\gamma-\alpha}\int x^{\alpha}Xdx$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)^2}\int\frac{dx}{x}\int\frac{dx}{x}\int x^{\gamma}Xdx - \frac{x^{-\gamma}}{(\alpha-\gamma)^2}\cdot\frac{2}{\alpha-\gamma}\int\frac{dx}{x}\int x^{\gamma}Xdx$$
$$+\frac{x^{-\gamma}}{(\alpha-\gamma)^2}\cdot\frac{3}{(\alpha-\gamma)^2}\int x^{\gamma}Xdx,$$

unde indoles harum formularum iam magis fit perspicua simulque patet partem integralis ex aliquot factoribus oriundam non pendere ab aequalitate reliquorum. Quocirca iam problema generale aggredi licebit.

PROBLEMA 171

1264. *Proposita aequatione differentiali cuiuscunque gradus huius formae*

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \frac{Ex^4d^4y}{dx^4} + \cdots + \frac{Nx^nd^ny}{dx^n},$$

ex qua forma algebraica hac lege formata

$$P = A + B(z-1) + C(z-1)(z-2) + D(z-1)(z-2)(z-3) + \cdots$$
$$+ N(z-1)(z-2)\cdots(z-n)$$

omnes factores habeat inter se inaequales, valorem ipsius y completum per formulas integrales simplices exhibere.

SOLUTIO

Sint primo formae P omnes factores simplices reales

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\cdots(\nu+z)$$

factorum numero existente $=n$ et ex antecedentibus patet ex quolibet factore nasci integralis partem. Ad has partes inveniendas eliciantur sequentes valores:

1) posito $z=-\alpha$ sit $\mathfrak{A} = \frac{P}{\alpha+z}$ seu $\mathfrak{A} = \frac{dP}{dz}$,
2) posito $z=-\beta$ sit $\mathfrak{B} = \frac{P}{\beta+z}$ seu $\mathfrak{B} = \frac{dP}{dz}$,
3) posito $z=-\gamma$ sit $\mathfrak{C} = \frac{P}{\gamma+z}$ seu $\mathfrak{C} = \frac{dP}{dz}$

etc.

Cum igitur sit

$$(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)\cdots(\nu-\alpha)=\frac{\mathfrak{A}}{N}$$

littera N ex superioribus formis per divisionem tolletur fietque integrale quaesitum

$$xy=\tfrac{1}{\mathfrak{A}}x^{-\alpha}\int x^{\alpha}Xdx+\tfrac{1}{\mathfrak{B}}x^{-\beta}\int x^{\beta}Xdx+\tfrac{1}{\mathfrak{C}}x^{-\gamma}\int x^{\gamma}Xdx+\text{etc.},$$

quoad singuli factores fuerint exhausti.

Quodsi iam forma P factores habeat imaginarios, partium inde ortarum imaginariarum ad realitatem reductio sequenti modo instituetur. Quoniam bini factores simplices imaginarii praebent factorem duplicem realem, ponamus

$$(\alpha+z)(\beta+z)=ff+2fz\cos.\theta+zz,$$

ita ut sit

$$\alpha=f\left(\cos.\theta+\sqrt{-1}\cdot\sin.\theta\right)\text{ et }\beta=f\left(\cos.\theta-\sqrt{-1}\cdot\sin.\theta\right),$$

unde primum valores literarum $\mathfrak{A}$ et $\mathfrak{B}$ definiantur; quarum cum utraque derivetur ex forma $\frac{dP}{dz}$, illa posito $z=-\alpha$, haec vero posito $z=-\beta$, in ipsa forma $\frac{dP}{dz}$ loco z ubique scribatur $-f\left(\cos.\theta\pm\sqrt{-1}\cdot\sin.\theta\right)$ prodeatque $\mathfrak{P}\pm\mathfrak{Q}\sqrt{-1}$ ac perspicuum est fore

$$\mathfrak{A}=\mathfrak{P}+\mathfrak{Q}\sqrt{-1}\quad\text{et}\quad\mathfrak{B}=\mathfrak{P}-\mathfrak{Q}\sqrt{-1},$$

ubi notandum est quantitates $\mathfrak{A}$ et $\mathfrak{B}$ esse reales. Deinde cum sit

$$x^{m+n\sqrt{-1}}=x^{m}e^{n\sqrt{-1}\cdot lx}=x^{m}\left(\cos.(nlx)+\sqrt{-1}\cdot\sin.(nlx)\right),$$

si brevitatis ergo ponamus angulum $f\sin.\theta\cdot lx=\varphi$, erit

$$x^{\alpha}=x^{f\cos.\theta}\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right),\quad x^{-\alpha}=x^{-f\cos.\theta}\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right),$$
$$x^{\beta}=x^{f\cos.\theta}\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right),\quad x^{-\beta}=x^{-f\cos.\theta}\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right),$$

Quare pro binis partibus

$$\tfrac{1}{\mathfrak{A}}x^{-\alpha}\int x^{\alpha}Xdx+\tfrac{1}{\mathfrak{A}}x^{-\beta}\int x^{\beta}Xdx$$

ob $\mathfrak{A}\mathfrak{B}=\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}$ habebimus

$$\frac{x^{-f\cos.\theta}}{\mathfrak{PP}+\mathfrak{QQ}}\left\{\begin{array}{l}\left(\mathfrak{P}-\mathfrak{Q}\sqrt{-1}\right)\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)\int x^{f\cos.\theta}\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)Xdx\\+\left(\mathfrak{P}+\mathfrak{Q}\sqrt{-1}\right)\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)\int x^{f\cos.\theta}\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)Xdx\end{array}\right\},$$

quae forma ob partes imaginarias se tollentes reducitur ad hanc

$$\frac{2x^{-f\cos.\theta}(\mathfrak{P}\cos.\varphi-\mathfrak{Q}\sin.\varphi)\int x^{f\cos.\theta}Xdx\cos.\varphi+2x^{-f\cos.\theta}(\mathfrak{P}\cos.\varphi+\mathfrak{Q}\sin.\varphi)\int x^{f\cos.\theta}Xdx\sin.\varphi}{\mathfrak{PP}+\mathfrak{QQ}}$$

Talisque forma ad integrale accedit, quoties forma *P* huiusmodi habet factorem duplicem $ff+2fz\cos.\theta+zz$.

COROLLARIUM 1

1265. Etsi autem factorum simplicium ipsius *P* quidam sunt imaginarii, eorum, qui sunt reales, evolutio inde non perturbatur, sed ex singulis partes in integrale inferendae a natura reliquorum factorum minime pendent.

COROLLARIUM 2

1266. Pars integralis ex binis factoribus imaginariis seu uno factore duplici oriunda aliquanto succinctius repraesentari potest, si ponatur

$$\mathfrak{P}=\mathfrak{O}\cos.\zeta \quad \text{et} \quad \mathfrak{Q}=\mathfrak{O}\sin.\zeta\,;$$

sic enim ea fiet

$$\frac{2}{\mathfrak{O}}x^{-f\cos.\theta}\left(\cos.(\zeta+\varphi)\int x^{f\cos.\theta}Xdx\cos.\varphi+\sin.(\zeta+\varphi)\int x^{f\cos.\theta}Xdx\sin.\varphi\right),$$

ubi ζ et θ sunt anguli constantes, φ vero variabilis ob $\varphi=f\sin.\theta\cdot lx$.

PROBLEMA 172

1267. *Si pro aequatione differentiali in praecedenti problemate proposita quantitas algebraica P inde formata duos habeat factores simplices aequales, integralis partem inde oriundam investigare.*

SOLUTIO

In forma ergo ante exhibita

$$P=N(\alpha+z)(\beta+z)(\gamma+z)(\delta+z)\ \text{etc.}$$

ponamus esse $\beta=\alpha$. Quoniam vero tum utraque integralis pars oritur infinita, altera signo +, altera signo – affecta, ita ut iunctim sumtae partem constituant finitam, ad hanc eliciendam statuamus $\beta=\alpha-\omega$ denotante ω quantitatem evanescentem eritque

$$\mathfrak{A}=-N\omega(\gamma-\alpha)(\delta-\alpha)(\varepsilon-\alpha) \text{ etc.}$$

et

$$\mathfrak{B}=+N\omega(\gamma-\beta)(\delta-\beta)(\varepsilon-\beta) \text{ etc.}$$

Ponatur iam

$$\tfrac{P}{(\alpha+z)(\beta+z)}=\tfrac{P}{(\alpha+z)^2}=Q,$$

ut sit

$$Q=N(\gamma+z)(\delta+z)(\varepsilon+z) \text{ etc.}$$

ac manifestum est fieri $\mathfrak{A}=-\omega Q$ posito $z=-\alpha$ et $\mathfrak{B}=\omega Q$ posito $z=-\beta=-\alpha+\omega$, unde intelligitur valorem ipsius Q posteriorem excedere priorem suo differentiali dQ, si fiat $z=-\alpha$ et $dz=\omega$, ita ut sit $\mathfrak{B}=\omega\left(Q+\omega\frac{dQ}{dz}\right)$ posito $z=-\alpha$ hincque

$$\tfrac{1}{\mathfrak{B}}=\tfrac{1}{\omega Q}-\tfrac{dQ}{QQdz}=\tfrac{1}{\omega Q}+\tfrac{1}{dz}d.\tfrac{1}{Q}$$

existente $\frac{1}{\mathfrak{A}}=\frac{1}{\omega Q}$. T um vero, cum sit $x^{\beta}=x^{\alpha}x^{-\omega}=x^{\alpha}(1-\omega lx)$ et $x^{-\beta}=x^{-\alpha}(1+\omega lx)$, binae partes integralis quaesitae erunt

$$-\tfrac{1}{\omega Q}x^{-\alpha}\int x^{\alpha}Xdx+\left(\tfrac{1}{\omega Q}+\tfrac{1}{dz}d.\tfrac{1}{Q}\right)x^{-\alpha}(1+\omega lx)\int x^{\alpha}Xdx(1-\omega lx),$$

ubi cum membra per ω divisa se destruant, resultat

$$\tfrac{1}{Q}x^{-\alpha}\left(lx\int x^{\alpha}Xdx-\int x^{\alpha}Xdxlx\right)+\tfrac{1}{dz}d.\tfrac{1}{Q}\cdot x^{-\alpha}\int x^{\alpha}Xdx$$

seu

$$\tfrac{1}{Q}x^{-\alpha}\int\tfrac{dx}{x}\int x^{\alpha}Xdx+\tfrac{1}{dz}d.\tfrac{1}{Q}\cdot x^{-\alpha}\int x^{\alpha}Xdx,$$

siquidem tam in valore $\frac{1}{Q}$ quam in $\frac{1}{dz}d.\frac{1}{Q}dz$ ubique loco z scribatur $-\alpha$. Cum vero sit $Q=\frac{P}{(\alpha+z)^2}$, hi valores inde facile inveniuntur.

COROLLARIUM 1

1268. Quodsi ergo quantitas algebraica *P* ex aequatione differentiali formata factorem habeat quadratum $(\alpha+z)^2$, inde in integrale transferenda est haec portio

$$\frac{(\alpha+z)^2}{P}x^{-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx+\frac{1}{dz}d.\frac{(\alpha+z)^2}{P}\cdot x^{-\alpha}\int x^{\alpha}Xdx$$

posito $z=-\alpha$; dum si hic factor $\alpha+z$ esset solitarius, integralis pars inde oriunda foret $\frac{\alpha+z}{P}x^{-\alpha}\int x^{\alpha}Xdx$ posito $z=-\alpha$.

COROLLARIUM 2

1269. Cum sit $Q=\frac{P}{(\alpha+z)^2}$, casu $z=-\alpha$ fiet $Q=\frac{ddP}{2dz^2}$; verum quia hic ipsi *z* iam valor determinatus est tributus, hinc $\frac{dQ}{dz}$ colligere non licet, sed prima [forma] est utendum, qua fit $\frac{dQ}{dz}=\frac{(\alpha+z)dP-2Pdz}{(\alpha+z)^3dz}$; cuius fractionis cum numerator et denominator casu $z=-\alpha$ evanescat, erit pro eodem casu

$$\frac{dQ}{dz}=\frac{(\alpha+z)ddP-2dzdP}{3(\alpha+z)^2dz^2}=\frac{(\alpha+z)d^3P}{6(\alpha+z)dz^3}=\frac{d^3P}{6dz^3}$$

COROLLARIUM 3

1270. Hoc valore invento, quia est eodem casu $z=-\alpha$ quantitas $Q=\frac{ddP}{2dz^2}$, erit

$$\frac{1}{dz}d.\frac{1}{Q}=-\frac{dQ}{QQdz}=-\frac{2dzd^3P}{3ddP^2}\text{ seu }\frac{1}{dz}d.\frac{1}{Q}=\frac{2dz}{3}d.\frac{1}{ddP},$$

ex quibus formulis, si factores ipsius *P* non sint evoluti, partes integralis facilius reperiuntur.

PROBLEMA 173

1271. *Si pro aequatione differentiali praecedente quantitas algebraica P inde formata factorem habeat cubicum* $(\alpha+z)^3$*, integralis partem inde oriundam investigare.*

SOLUTIO

Ponamus ergo esse $P=(\alpha+z)^2(\gamma+z)R$ existente $\gamma=\alpha-\omega$, ubi ω pro quantitate evanescente assumitur. Quod ergo ante erat *Q*, id hinc fit $Q=(\gamma+z)R$ et facto $z=-\alpha$ erit $Q=-\omega R$, si etiam in *R* ponatur $z=-\alpha$. Deinde cum sit

$$\frac{dQ}{dz}=R+\frac{(\gamma+z)dR}{dz}=R-\frac{\omega dR}{dz}$$

eodem casu erit

$$\frac{1}{dz}d.\frac{1}{Q}=-\frac{1}{\omega\omega R}+\frac{dR}{\omega RRdz}=-\frac{1}{\omega\omega}\cdot\frac{1}{R}-\frac{1}{\omega dz}d.\frac{1}{R}.$$

Quocirca ex factore quadrato $(\alpha+z)^2$ per praecedens problema haec obtinetur integralis pars

$$-\frac{1}{\omega R}x^{-\alpha}\int\frac{dx}{x}\int x^{\alpha}Xdx-\left(\frac{1}{\omega\omega R}+\frac{1}{\omega dz}d.\frac{1}{R}\right)x^{-\alpha}\int x^{\alpha}Xdx,$$

cuius ambo membra in infinitum excrescunt ob $\omega=0$. Adiiciamus autem partem ex tertio factore $\gamma+z=\alpha-\omega+z$ oriundam, quae ob $\frac{P}{\gamma+z}=(\alpha+z)^2R$ est

$$\frac{1}{(\alpha+z)^2R}x^{-\gamma}\int x^{\gamma}Xdx$$

posito $z=-\gamma=-\alpha+\omega$. Quodsi iam *R* ut ante is fuerit valor, qui oritur posito $z=-\alpha$, augendo hunc valorem particula ω loco $\frac{1}{R}$ scribi debet

$$\frac{1}{R}+\frac{\omega}{dz}d.\frac{1}{R}+\frac{\omega^2}{1\cdot 2dz^2}dd.\frac{1}{R}+\text{etc.},$$

siquidem valorem $z=-\alpha$ et hic retineamus; unde haec integralis pars ob $\alpha+z=\omega$ erit

$$\left(\frac{1}{\omega\omega R}+\frac{1}{\omega dz}d.\frac{1}{R}+\frac{1}{2dz^2}dd.\frac{1}{R}\right)x^{-\alpha+\omega}\int x^{\alpha-\omega}Xdx$$

sicque manifestum est illum ipsius $\frac{1}{R}$ valorem usque ad secundam potestatem ipsius ω continuari debuisse atque eadem lege hic alteram partem x involventem exprimi conveniet. Ad quod observo, si habeatur huiusmodi formula $x^{-\omega}\int x^{\omega}Vdx$, secundum potestates ipsius ω evolvenda, id hac ratione commodissime fieri. Posito

$$v = x^{\omega}\int x^{-\omega}Vdx,$$

ut sit

$$x^{-\omega}v = \int x^{-\omega}Vdx,$$

erit differentiando

$$dv - \tfrac{\omega v dx}{x} = Vdx,$$

quare posito

$$v = T + \omega T' + \omega^2 T'' + \omega^3 T''' + \text{etc.}$$

habebitur terminos secundum potestates ipsius ω disponendo

$$\left.\begin{array}{l} dT + \omega dT' + \omega\omega T'' + \omega^3 dT''' + \text{etc.} \\ -Vdx - \omega dT\frac{dx}{x} - \omega\omega T'\frac{dx}{x} - \omega^3 dT''\frac{dx}{x} - \text{etc.} \end{array}\right\} = 0$$

ideoque

$$T = \int Vdx,\ T' = \int\frac{dx}{x}\int Vdx,\ T'' = \int\frac{dx}{x}\int\frac{dx}{x}\int Vdx \quad \text{etc.}$$

Consequenter, cum in applicatione sit $V = x^{\alpha}X$, erit pars integralis ex factore $\gamma + z = \alpha - \omega + z$ nata

$$\left(\tfrac{1}{\omega\omega R} + \tfrac{1}{\omega dz}d.\tfrac{1}{R} + \tfrac{1}{2dz^2}dd.\tfrac{1}{R}\right)x^{-\alpha}\left(\int x^{\alpha}Xdx + \omega\int\tfrac{dx}{x}\int x^{\alpha}Xdx + \omega^2\int\tfrac{dx}{x}\int\tfrac{dx}{x}\int x^{\alpha}Xdx\right),$$

qua cum parte ex $(\alpha+z)^2$ nata iunctim sumta omnia membra infinita se mutuo destruunt, et pro quantitatis $P = (\alpha+z)^3 R$ factore cubico $(\alpha+z)^3$ in integrale ingreditur haec pars

$$\tfrac{1}{R}x^{-\alpha}\int\tfrac{dx}{x}\int\tfrac{dx}{x}\int x^{\alpha}Xdx + \tfrac{1}{dz}d.\tfrac{1}{R}\cdot x^{-\alpha}\int\tfrac{dx}{x}\int x^{\alpha}Xdx + \tfrac{1}{2dz^2}dd.\tfrac{1}{R}\cdot x^{-\alpha}\int x^{\alpha}Xdx,$$

si modo in quantitate $R = \frac{P}{(\alpha+z)^3}$ ubique scribatur $z = -\alpha$.

COROLLARIUM 1

1272. Methodus in solutione huius problematis adhibita facile ad quotcunque factores aequales extendi potest. Si enim fuerit $(\alpha + z)^m$ factor quantitatis P atque in hac fractione $\frac{(\alpha+z)^m}{P}$ suisque differentialibus, postquam fuerint evoluta, ponatur $z = -\alpha$, partes integralis inde natae ita se habebunt:

Factor quant. P	$\alpha + z$	$(\alpha + z)^2$	$(\alpha + z)^3$
Pars integralis	$\frac{\alpha+z}{P} x^{-\alpha} \int x^{\alpha} X dx$	$\frac{(\alpha+z)^2}{P} x^{-\alpha} \int \frac{dx}{x} \int x^{\alpha} X dx$	$\frac{(\alpha+z)^3}{P} x^{-\alpha} \int \frac{dx}{x} \int \frac{dx}{x} \int x^{\alpha} X dx$
		$\frac{1}{dz} d.\frac{(\alpha+z)^2}{P} \cdot x^{-\alpha} \int x^{\alpha} X dx$	$\frac{1}{dz} d.\frac{(\alpha+z)^3}{P} \cdot x^{-\alpha} \int \frac{dx}{x} \int x^{\alpha} X dx$
			$\frac{1}{2dz^2} dd.\frac{(\alpha+z)^3}{P} \cdot x^{-\alpha} \int x^{\alpha} X dx$

COROLLARIUM 2

1273. Si fuerint duo pluresve factores duplices inter se aequales, sumtis

$$\alpha = f\left(\cos.\theta + \sqrt{-1} \cdot \sin.\theta\right) \text{ et } \beta = f\left(\cos.\theta - \sqrt{-1} \cdot \sin.\theta\right)$$

partes pro $(\alpha + z)^2$ et $(\beta + z)^2$ seorsim evolutae methodo supra adhibita non difficulter coniungentur et ad realitatem reducentur.

SCHOLION

1274. Simili methodo, qua hoc caput est pertractatum, in evolutione Capitis III huius sectionis [§ 1163, 1178] uti oportebat neque tum ullum periculum in errores prolabendi fuisset pertimescendum. Superfluum autem nunc foret errores ibi commissos hic emendare, cum non solum methodus plane esset eadem, sed etiam aequatio hic tractata facile in formam ibi consideratam transmutari queat et vicissim. Quodsi enim in aequatione Capitis III

$$X = Ay + \frac{Bxdy}{dx} + \frac{Cx^2ddy}{dx^2} + \frac{Dx^3d^3y}{dx^3} + \frac{Ex^4d^4y}{dx^4} + \text{etc.}$$

statuatur $x = lv$, ut sit $dx = \frac{dv}{v}$, functio autem *X* abeat in functionem ipsius *v*, quae sit *V*, proveniet aequatio eius formae, quam hic tractavimus. Dum autem ibi elementum *dx* pro constanti est habitum, ad hanc conditionem exuendam ponamus

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \text{ etc.},$$

ut haec aequatio resultet

$$X = V = Ay + Bp + Cq + Dr + Es + \text{etc.}$$

Nunc autem ob $dx = \frac{dv}{v}$ adipiscimur elemento *dv* constanta sumto

$$p = \frac{dy}{dx} = \frac{vdy}{dv},$$
$$q = \frac{dp}{dx} = \frac{vvddy}{dv^2} + \frac{vdy}{dv},$$
$$r = \frac{dq}{dx} = \frac{v^3d^3y}{dv^3} + \frac{3vvddy}{dv^2} + \frac{vdy}{dv},$$
$$s = \frac{dr}{dx} = \frac{v^4d^4y}{dv^4} + \frac{6v^3d^3y}{dv^3} + \frac{7vvddy}{dv^2} + \frac{vdy}{dv},$$
$$t = \frac{ds}{dx} = \frac{v^5d^5y}{dv^5} + \frac{10v^4d^4y}{dv^4} + \frac{25v^3d^3y}{dv^3} + \frac{15vvddy}{dv^2} + \frac{vdy}{dv}$$
etc.

Quare aequatio inter *v* et *y* erit haec

$$\begin{aligned} V = Ay + \frac{Bvdy}{dv} &+ \frac{Cvvddy}{dv^2} + \frac{Dv^3d^3y}{dv^3} + \frac{Ev^4d^4y}{dv^4} + \text{etc.} \\ &+ C \quad + 3D \quad + 6E \quad + 10F \\ &+ D \quad + 7E \quad + 25F \\ &+ E \quad + 15F \\ &+ F \end{aligned}$$

cuius integrationem hie docuimus. Imprimis autem notandum est quantitatem algebraicam *P* hinc formandam

$$P = A + (B + C + D + E + F)(z-1) + (C + 3D + 7E + 15F)(z-1)(z-2)$$
$$+ (D + 6E + 25F)(z-1)(z-2)(z-3) + \text{etc.}$$

ad hanc formam reduci

$$P = A + B(z-1) + C(z-1)^2 + D(z-1)^3 + E(z-1)^4 + F(z-1)^5 + \text{etc.},$$

quae quantitas algebraica ab illa, qua in Capite III ad integrationem sumus usi, hoc tantum differt; quod ibi littera z id quod hic formula $z-1$ expressimus; ex quo etiam ambarum integratio facillime altera ad alteram reducitur.

CONCLUSIO LIBRI PRIMI

1275. Atque haec fere sunt, quae ad librum primum de Calculo Integrali pertinere sunt visa, ubi methodum tradere institui functiones unius variabilis ex data quacunque differentialium cuiusque ordinis relatione investigandi, quod opus mihi equidem ita pertractasse videor, ut vix quicquam eorum, quae adhuc de hoc argumento ab aliis sunt inventa et in medium allata, sit praetermissum.