CHAPTER IV

THE APPLICATION OF THE METHOD OF INTEGRATION TREATED IN THE LAST CHAPTER TO EXAMPLES

PROBLEM 156

1189. *With this differential equation proposed,*

$$X = a^n y + \frac{d^n y}{dx^n},$$

to find its complete integral.

SOLUTION

Here therefore there is $P = a^n + z^n$, where at first it may be observed that, if n shall be an odd number, then $a + z$ is a simple factor, from which there arises the part of the integral

$$\frac{1}{\mathfrak{A}} e^{-ax} \int e^{ax} X dx$$

with the value $\mathfrak{A}$ emerging from the form $\frac{P}{a+z}$, if there is put $z = -a$; which therefore since also there shall be $\frac{dP}{dz} = nz^{n-1}$, on account of the even number $n-1$, will be $\mathfrak{A} = na^{n-1}$, and thus this part of the integral

$$= \frac{1}{na^{n-1}} e^{-ax} \int e^{ax} X dx$$

All the remaining factors may be retained in this form $aa - 2az\cos.\theta + zz$ with $\theta = \frac{(2i+1)\pi}{n}$ arising, where i denotes some whole number and π equal to two right angles. On comparing this form with Problem 153 and Corollary 1, there becomes $f = -a$ and on account of $z = a\left(\cos.\theta + \sqrt{-1} \cdot \sin.\theta\right)$ from the form $\frac{dP}{dz}$ there is deduced

$$\mathfrak{P} = na^{n-1}\cos.(n-1)\theta \text{ and } \mathfrak{Q} = na^{n-1}\sin.(n-1)\theta;$$

therefore since there shall be $\cos.n\theta = -1$ and $\sin.n\theta = 0$, there will be

$$\mathfrak{P} = -na^{n-1}\cos.\theta \text{ and } \mathfrak{Q} = na^{n-1}\sin.\theta.$$

Whereby on putting $fx\sin.\theta = -ax\sin.\theta = \varphi$, the part of the integral arising from some twofold factor will be

$$\frac{2e^{ax\cos.\theta}}{na^{n-1}}\left\{\begin{array}{l}\left(-\cos.\theta\cos.\varphi - \sin.\theta\sin.\varphi\right)\int e^{-ax\cos.\theta}Xdx\cos.\varphi \\ +\left(-\cos.\theta\sin.\varphi + \sin.\theta\cos.\varphi\right)\int e^{-ax\cos.\theta}Xdx\sin.\varphi\end{array}\right\}$$

or

$$\frac{-2e^{ax\cos.\theta}}{na^{n-1}}\left(\cos.\left(\theta-\varphi\right)\int e^{-ax\cos.\theta}Xdx\cos.\varphi - \sin.\left(\theta-\varphi\right)\int e^{-ax\cos.\theta}Xdx\sin.\varphi\right)$$

and with the value for φ restored,

$$\frac{-2e^{ax\cos.\theta}}{na^{n-1}}\left\{\begin{array}{l}\cos.\left(\theta + ax\sin.\theta\right)\int e^{-ax\cos.\theta}Xdx\cos.\left(ax\sin.\theta\right) \\ +\sin.\left(\theta + ax\sin.\theta\right)\int e^{-ax\cos.\theta}Xdx\sin.\left(ax\sin.\theta\right)\end{array}\right\}.$$

Now for θ the angles are substituted successively $\frac{\pi}{n}$, $\frac{3\pi}{n}$, $\frac{5\pi}{n}$, $\frac{7\pi}{n}$ etc., as long as they shall be less than π itself, and all these forms joined together into one sum, in which case when *n* is an odd number, there is required to be added the above form first found

$$\frac{1}{na^{n-1}}e^{-ax}\int e^{ax}Xdx$$

will give the integral sought.

COROLLARY 1

1190. Indeed in the case, in which *n* is an odd number, the final value of θ becomes π, but which here we had decided to omit ; but from this on account of $ax\sin.\theta = 0$ and $\cos.\theta = -1$ the final part of the integral will be produced

$$\frac{2e^{-ax}}{na^{n-1}}\int e^{ax}Xdx$$

from the doubling of this, that it is convenient to take ; the reason for this is, because on taking $\theta = \pi$ the formula $aa + 2az + zz$ is no longer a factor, but the square root of this is $a + z$, from which it was necessary to elicit that case separately.

COROLLARY 2

1191. If there is $X=0$, the integral formulas change into arbitrary constants and from the factor

$$aa-2az\cos.\theta+zz$$

this part of the integral arises

$$\tfrac{-2e^{ax\cos.\theta}}{na^{n-1}}\left(A\cos.(\theta+ax\sin.\theta)+\mathfrak{A}\sin.(\theta+ax\sin.\theta)\right),$$

which is reduced to the form

$$Ae^{ax\cos.\theta}\cos.(\zeta+ax\sin.\theta)$$

with ζ denoting some constant angle, as we have now found above [§ 1135].

PROBLEM 157

1192. *With this differential equation proposed,*

$$X=a^n y-\tfrac{d^n y}{dx^n},$$

to find its complete integral.

SOLUTION

Hence from the algebraic form arising $P=a^n-z^n$, there may always be had the factor $a-z$, from which there arises the part of the integral $\tfrac{1}{\mathfrak{A}}e^{ax}\int e^{-ax}Xdx$ with $\mathfrak{A}=\tfrac{P}{z-a}$ present on putting $z=a$. Therefore since also there shall be $\mathfrak{A}=\tfrac{dP}{dz}=-nz^{n-1}$, there will be $\mathfrak{A}=-na^{n-1}$, and therefore this part of the integral

$$=\tfrac{-1}{na^{n-1}}e^{ax}\int e^{-ax}Xdx.$$

If then n shall be an even number and hence $n-1$ odd, also $a+z$ will be a factor, which gives this part of the integral

$$\tfrac{1}{na^{n-1}}e^{-ax}\int e^{ax}Xdx$$

All the remaining factors of P are of the twofold form $aa-2az\cos.\theta+zz$ with the angle being $\theta=\frac{2i\pi}{n}$; from which since it may be compared with the general form taken above [§ 1173] $ff+2fz\cos.\theta+zz$ there shall be $f=-a$ and from the form $\frac{dP}{dz}=-nz$ it is required to find the formula $\mathfrak{P}+\mathfrak{Q}\sqrt{-1}$ on putting $z=a\left(\cos.\theta+\sqrt{-1}\cdot\sin.\theta\right)$, from which it is deduced that

$$\mathfrak{P}=-na^{n-1}\cos.(n-1)\theta \text{ and } \mathfrak{Q}=-na^{n-1}\sin.(n-1)\theta,$$

or on account of $\cos.n\theta=1$ and $\sin.n\theta=0$ there is made

$$\mathfrak{P}=-na^{n-1}\cos.\theta \text{ and } \mathfrak{Q}=na^{n-1}\sin.\theta.$$

Now on putting the angle $-ax\sin.\theta=\varphi$; from § 1177 the part of the integral is produced

$$\frac{2e^{ax\cos.\theta}}{na^{n-1}}\left\{\begin{aligned}&\left(-\cos.\theta\cos.\varphi-\sin.\theta\sin.\varphi\right)\int e^{-\alpha x\cos.\theta}Xdx\cos.\varphi\\&+\left(-\cos.\theta\sin.\varphi+\sin.\theta\cos.\varphi\right)\int e^{-\alpha x\cos.\theta}Xdx\sin.\varphi\end{aligned}\right\}$$

which is reduced as before [§ 1189] to this form

$$\frac{-2e^{ax\cos.\theta}}{na^{n-1}}\left\{\begin{aligned}&\cos.\left(\theta+ax\sin.\theta\right)\int e^{-\alpha x\cos.\theta}Xdx\cos.\left(ax\sin.\theta\right)\\&-\sin.\left(\theta+ax\sin.\theta\right)\int e^{-\alpha x\cos.\theta}Xdx\sin.\left(ax\sin.\theta\right)\end{aligned}\right\}.$$

Now here for θ in turn there are written the angles $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, $\frac{6\pi}{n}$ etc., as long as they are less than π, and all these parts with the first found and also the other, if n should be an even number, gathered together into one sum will give the integral sought or the value of y.

COROLLARY

1193. Since the twofold factor $aa-2az\cos.\theta+zz$ in the general cases $\theta=0$ and $\theta=\pi$ may not give the simple real factors $a-z$ and $a+z$, but the squares of these, this is the reason why the part of the integral elicited from these produces the double of this, as it is required to take.

PROBLEM 158

1194. *With the proposed differential equation,*

$$X = y + \frac{dy}{dx} + \frac{ddy}{dx^2} + \frac{d^3y}{dx^3} + \ldots. + \frac{d^n y}{dx^n},$$

to find its complete integral.

SOLUTION

The algebraic form hence produced is

$$P = 1 + z + z^2 + z^3 + z^4 + \ \ldots \ + z^n,$$

of which all the factors are required to be examined carefully. Therefore since there shall be $P = \frac{1-z^{n+1}}{1-z}$, it is convenient to take the factors of the form $1 - z^{n+1}$ with the exclusion of $1 - z$, from which initially it appears, if $n+1$ should be an even number, that $1+z$ will be a simple factor, from which a part of the integral arises $\frac{1}{\mathfrak{A}} e^{-x} \int e^x X dx$ with $\mathfrak{A} = \frac{P}{1+z} = \frac{1-z^{n+1}}{1-zz}$ present on putting $z = -1$. Hence there will be also $\mathfrak{A} = \frac{(n+1)z^n}{2z}$ and thus $\mathfrak{A} = \frac{1}{2}(n+1)$, so that this part of the integral will be

$$\frac{2}{n+1} e^{-x} \int e^x X dx.$$

But the form of the twofold factors is $1 - 2z\cos.\theta + zz$ on taking the angle $\theta = \frac{2i\pi}{n}$, thus so that for §1173 there shall be $f = -1$. The form may be considered

$$\frac{dP}{dz} = \frac{1-(n+1)z^n + nz^{n+1}}{(1-z)^2},$$

which on putting $z = \cos.\theta + \sqrt{-1} \cdot \sin.\theta$ is accepted to change into $\mathfrak{P} + \mathfrak{Q}\sqrt{-1}$, and thus there will be

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{1-(n+1)\cos.n\theta + n\cos.(n+1)\theta - (n+1)\sqrt{-1}\cdot\sin.n\theta + n\sqrt{-1}\cdot\sin.(n+1)\theta}{1-2\cos.\theta + \cos.2\theta - 2\sqrt{-1}\cdot\sin.\theta + \sqrt{-1}\cdot\sin.2\theta}.$$

Now since there will be $\sin.(n+1)\theta = 0$ and $\cos.(n+1)\theta = 1$, there will be $\sin.n\theta = -\sin.\theta$ and $\cos.n\theta = \cos.\theta$ and thus

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{n+1-(n+1)\cos.\theta+(n+1)\sqrt{-1}\cdot\sin.\theta}{-2\cos.\theta+2\cos.^2\theta-2\sqrt{-1}\cdot\sin.\theta(1-\cos.\theta)}$$

or

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{n+1}{2(1-\cos.\theta)}\cdot\frac{1-\cos.\theta+\sqrt{-1}\cdot\sin.\theta}{-\cos.\theta-\sqrt{-1}\cdot\sin.\theta}.$$

The numerator and the denominator of this fraction is multiplied by $-\cos.\theta+\sqrt{-1}\cdot\sin.\theta$ and there will be produced

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \frac{-(n+1)\left(1+\cos.\theta-2\cos.^2\theta-\sqrt{-1}\cdot\sin.\theta(1-2\cos.\theta)\right)}{2(1-\cos.\theta)}$$

thus so that there will be

$$\mathfrak{P} = -\tfrac{1}{2}(n+1)(1+2\cos.\theta) \text{ et } \mathfrak{Q} = \tfrac{1}{2}(n+1)\frac{\sin.\theta(1-2\cos.\theta)}{1-\cos.\theta}$$

from which there becomes

$$\mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q} = \frac{(n+1)^2}{2(1-\cos.\theta)}.$$

Then truly on putting the angle $-x\sin.\theta = \varphi$ there is deduced

$$\mathfrak{P}\cos.\varphi - \mathfrak{Q}\sin.\varphi = \frac{-(n+1)\left(\cos.(\theta-\varphi)-\cos.(2\theta-\varphi)\right)}{2(1-\cos.\theta)},$$

$$\mathfrak{P}\sin.\varphi + \mathfrak{Q}\cos.\varphi = \frac{+(n+1)\left(\sin.(\theta-\varphi)-\sin.(2\theta-\varphi)\right)}{2(1-\cos.\theta)} \quad ;$$

since moreover there shall be

$$\cos.a - \cos.b = 2\sin.\tfrac{a+b}{2}\sin.\tfrac{a-b}{2} \quad \text{and} \quad \sin.a - \sin.b = -2\cos.\tfrac{a+b}{2}\sin.\tfrac{b-a}{2},$$

hence there becomes

$$\mathfrak{P}\cos.\varphi - \mathfrak{Q}\sin.\varphi = \frac{-(n+1)\sin.\frac{1}{2}(3\theta-2\varphi)}{2\sin.\frac{1}{2}\theta}$$

and

$$\mathfrak{P}\sin.\varphi + \mathfrak{Q}\cos.\varphi = \frac{-(n+1)\cos.\frac{1}{2}(3\theta-2\varphi)}{2\sin.\frac{1}{2}\theta},$$

from which the part of the integral sought will be

$$\tfrac{-4}{n+1}e^{x\cos.\theta}\sin.\tfrac{1}{2}\theta\left\{\begin{array}{l}\sin.\tfrac{1}{2}(3\theta+2x\sin.\theta)\int e^{-x\cos.\theta}Xdx\cos.(x\sin.\theta)\\ -\cos.\tfrac{1}{2}(3\theta+2x\sin.\theta)\int e^{-x\cos.\theta}Xdx\sin.(x\sin.\theta)\end{array}\right\}.$$

Therefore for θ the angles $\frac{2\pi}{n+1}$, $\frac{4\pi}{n+1}$, $\frac{6\pi}{n+1}$ etc. are substituted successively, as long as they are less than π, and all these parts are gathered into one sum, for which, if $n+1$ shall be an even number, the above $\frac{2}{n+1}e^{-x}\int e^{x}Xdx$ is added and thus there will be obtained the value of *y*.

COROLLARY 1

1195. If the proposed equation should be continued to infinity, so that *n* shall be an infinite number, all the former angles θ are infinitely small and thus infinite for a number, as long as the even number taken 2*i* has a finite ratio to *n* + 1; but then for θ all the finite angles are increasing in an arithmetic progression, the difference of which is $\frac{2\pi}{n+1}$ as far as π, of which the number likewise is infinite.

COROLLARY 2

1196. As long as the angle θ is infinitely small, the part of the integral arising from that adopts the form

$$\tfrac{-\theta\theta e^{x}}{n+1}\left((3+2x)\int e^{-x}Xdx-2\int e^{-x}Xxdx\right);$$

which since it shall be divided by an infinite cube, also an infinitely great number of formulas of this kind is to be considered for vanishing.

COROLLARY 3

1197. But if there should be $X=0$, so that the integral of this equation

$$X=y+\frac{dy}{dx}+\frac{ddy}{dx^2}+\frac{d^3y}{dx^3}+....+\frac{d^ny}{dx^n}$$

shall be investigated, some part of this will be

$$e^{x\cos.\theta}\left(A\sin.\tfrac{1}{2}(3\theta+2x\sin.\theta)+\mathfrak{A}\cos.\tfrac{1}{2}(3\theta+2x\sin.\theta)\right)$$

or more simply

$$Ae^{x\cos.\theta}\cos.(\zeta+x\sin.\theta).$$

Therefore since, if *n* shall be an infinite number, any angle can be taken for θ , with the particular integral of this equation being some equation

$$y = Ae^{x\cos.\theta}\cos.\left(x\sin.\theta + \zeta\right)$$

also on assuming some angle for ζ .

SCHOLIUM

1198. But now with *X* denoting some function of *x* of this differential equation extending to infinity

$$X = y + \frac{dy}{dx} + \frac{ddy}{dx^2} + \frac{d^3y}{dx^3} + \ldots + \frac{d^ny}{dx^n},$$

as the integral can be expressed more conveniently by the sum of the innumerable vanishing parts of that, the question arising requires higher powers of deduction and at this point it is considered that the bounds of analysis have not developed to this goal. Indeed in certain cases, in which *X* is a whole rational function, for example

$$X = a + bx + cx^2 + dx^3 + ex^4 + \text{etc.},$$

the thing presents no difficulty, since on taking

$$y = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.} + v$$

these coefficients α, β, γ etc. are able to be defined always, so that with the substitution made there may arise such an equation :

$$0 = v + \frac{dv}{dx} + \frac{ddv}{dx^2} + \frac{d^3v}{dx^3} + \text{etc.},$$

to which particularly the value satisfies

$$v = Ae^{x\cos.\theta}\cos.\left(x\sin.\theta + \zeta\right)$$

on taking some angles for ζ and θ. Now from a value of X of this kind given there is found

$$\alpha = a - b,\ \beta = b - 2c,\ \gamma = c - 3d,\ \delta = d - 4e,\ \varepsilon = e - 5f \quad \text{etc.}$$

Now in general since there becomes

$$\frac{dX}{dx} = \frac{dy}{dx} + \frac{ddy}{dx^2} + \frac{d^3y}{dx^3} + \text{etc.},$$

it is clear that on putting $y = X - \frac{dX}{dx} + v$, that equation always is to be transformed into this

$$0 = v + \frac{dv}{dx} + \frac{ddv}{dx^2} + \frac{d^3v}{dx^3} + \text{etc.}$$

COROLLARY

1199. Behold therefore besides the expectation of the complete integral of this differential equation extending to infinity

$$X = y + \frac{dy}{dx} + \frac{ddy}{dx^2} + \frac{d^3y}{dx^3} + \ldots + \text{ etc. to infinity},$$

for which we have learned that there is

$$y = X - \frac{dX}{dx} + Ae^{x\cos.\theta}\cos.\left(x\sin.\theta + \zeta\right),$$

since the last term on account of the arbitrary angles ζ and θ is able to be multiplied indefinitely. And this form especially is to be considered equivalent to that complicated form arising from the solution.

[The editor of the *O.O.* edition of this work, L. Schlesinger (1913) adds a note on p. 386 (Vol. 12, Series I) that the solution $y = X - \frac{dX}{dx}$ is allowed, if in the limit as *n* tends to infinity, also $\frac{d^ny}{dx^n} \to 0$; other solutions do not exist, as the series diverges. Editor's reference : G. Plana, *Nota sopra l'integratione di un' equazine differentiale data da Euler*, Memoirs Italian Soc. Sc. 18, (1824), p.44]

PROBLEM 159

1200. *With this proposed differential equation,*

$$X = y + \frac{ndy}{adx} + \frac{n(n-1)ddy}{1\cdot 2a^2dx^2} + \frac{n(n-1)(n-2)d^3y}{1\cdot 2\cdot 3a^3dx^3} + \text{etc.},$$

where n shall be a certain positive number, so that the number of terms shall be finite, to find its complete integral.

SOLUTION

The algebraic formula hence to be considered shall be

$$P = 1 + \frac{n}{1}\cdot\frac{z}{a} + \frac{n(n-1)}{1\cdot 2}\cdot\frac{z^2}{a^2} + \text{ etc.} = \left(1+\frac{z}{a}\right)^n,$$

which therefore has simple unmixed factors equal to $z + a$ amongst themselves. Therefore since there shall be $\frac{P}{(a+z)^n} = \frac{1}{a^n}$, from § 1163 the integral sought is deduced at once

$$y = a^n e^{-ax}\int dx\int dx\int dx\int ....\int e^{ax}Xdx,$$

as long as the number of integral signs is equal to the exponent *n*. But it is allowed to resolve in the following manner the form into simple integrals with the aid of the general reduction, which we know to be $\int dx\int Vdx = x\int Vdx - \int Vxdx$, from which there becomes

$$\int dx\int e^{ax}Xdx = x\int e^{ax}Xdx - \int e^{ax}Xxdx,$$

$$\int dx\int dx\int e^{ax}Xdx = \tfrac{1}{2}x^2\int e^{ax}Xdx - x\int e^{ax}Xxdx + \tfrac{1}{2}\int e^{ax}Xx^2dx,$$

$$\int dx\int dx\int dx\int e^{ax}Xdx = \frac{x^3\int e^{ax}Xdx - 3x^2\int e^{ax}Xxdx + 3x\int e^{ax}Xx^2dx - \int e^{ax}Xx^3dx}{1\cdot 2\cdot 3}$$

etc.

Therefore since the number of integral signs shall be = *n*, where we may conclude that evidently this is the complete integral, since the individual integrals involve an arbitrary constant.

COROLLARY 1

1201. Therefore if there should be $X=0$, the complete integral of the proposed differential equation becomes

$$y=e^{-ax}\left(Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+Dx^{n-4}+\ldots+Mx+N\right),$$

where the number of arbitrary constants *A*, *B*, *C* etc. everywhere is = *n*.

COROLLARY 2

1202. If the number *n* should be infinite and likewise the quantity *a* may be taken infinite, so that there shall be $a=nc$, the equation to be integrated extends to infinity and there will be

$$X=y+\frac{dy}{cdx}+\frac{ddy}{1\cdot 2c^2dx^2}+\frac{d^3y}{1\cdot 2\cdot 3c^3dx^3}+\text{etc.};$$

but the equation of the integration applied to this case provides no light.

COROLLARY 3

1203. But whatever the function *y* should be of *x*, it is agreed, if in place of *x* there should be written $x+\frac{1}{c}$, that changes into

$$y+\frac{dy}{cdx}+\frac{ddy}{1\cdot 2c^2dx^2}+\frac{d^3y}{1\cdot 2\cdot 3c^3dx^3}+\text{etc.},$$

which since it must be equal to $=X$, in turn it is apparent that *y* is equal to that function of *x*, which arises from *X*, if there in place of *x* there is written $x-\frac{1}{c}$.

SCHOLION 1

1204. Since with that it may appear easier, I note, if the proposed were some equation of this kind

$$X=Ay+B\frac{dy}{dx}+C\frac{ddy}{dx^2}+D\frac{d^3y}{dx^3}+\text{etc.},$$

always without any integration a particular integral can be found in this way by an approximation. There is put in place

$$y=\alpha X+\beta\frac{dX}{dx}+\gamma\frac{ddX}{dx^2}+\delta\frac{d^3X}{dx^3}+\text{etc.}$$

and with that substitution made there will be considered

$$\begin{aligned} X = {} & A\alpha X + A\beta \tfrac{dX}{dx} + A\gamma \tfrac{ddX}{dx^2} + A\delta \tfrac{d^3X}{dx^3} + \text{etc.} \\ & \qquad\quad +B\alpha \,'' \quad +B\beta \,'' \quad +B\gamma \,'' \\ & \qquad\qquad\qquad\quad +C\alpha \,'' \quad +C\beta \,'' \\ & \qquad\qquad\qquad\qquad\qquad\quad +D\alpha \,'' \end{aligned}$$

and thus the coefficients $\alpha, \beta, \gamma, \delta$ etc. are defined, so that there shall be $a = \frac{1}{A}$, and indeed for the rest

$$\beta = \tfrac{-B\alpha}{A} = \tfrac{-B}{A^2},$$

$$\gamma = \tfrac{C\alpha - B\beta}{A} = \tfrac{-C}{A^2} + \tfrac{BB}{A^3},$$

$$\delta = \tfrac{-D\alpha - C\beta - B\gamma}{A} = \tfrac{-D}{A^2} + \tfrac{2BC}{A^3} - \tfrac{B^3}{A^4},$$

$$\varepsilon = \tfrac{-E\alpha - D\beta - C\gamma - B\delta}{A} = \tfrac{-E}{A^2} + \tfrac{2BD + C^2}{A^3} - \tfrac{3BBC}{A^4} + \tfrac{B^4}{A^5},$$

etc.;

which if adapted to the case of the problem, there comes about

$$y = X - \tfrac{ndX}{1adx} + \tfrac{n(n+1)ddX}{1\cdot 2a^2dx^2} - \tfrac{n(n+1)(n+2)d^3X}{1\cdot 2\cdot 3a^3dx^3} + \text{etc.}$$

Hence in the case, in which $n = \infty$ and $a = nc$, there is deduced

$$y = X - \tfrac{dX}{1cdx} + \tfrac{ddX}{1\cdot 2c^2dx^2} - \tfrac{d^3X}{1\cdot 2\cdot 3c^3dx^3} + \text{etc.},$$

which expression, if on extending to infinity, evidently defines that function of *x*, which arises from *X*, if in place of x there is written $x - \frac{1}{c}$. But now if we should indicate this new variable by the sign X' and we may put $y = X' + v$, the equation of Corollary 2 will change into this :

$$0 = v + \tfrac{dv}{cdx} + \tfrac{ddv}{1\cdot 2c^2dx^2} + \tfrac{d^3v}{1\cdot 2\cdot 3c^3dx^3} + \text{etc.},$$

of which any particular integral is $v = Ae^{-ncx}x^m$ with the infinite number *n* present and with *m* a positive whole number.

[There are occasional problems with convergence ; there are such errors in § 1199 & §1204, but these errors are not incurred in similar expansions in the following paragraphs §1209, §1212, §1217 & §1224. According to the editor of the O.O. edition, these errors arise from writing $x+\frac{1}{c}$ in place of *x*, and making a Taylor expansion of the function *v*, which may not converge. We may note that such expansions can be written formally as $e^{\pm\frac{d}{cdx}}v$]

SCHOLIUM 2

1205. These lead me to the following speculation about the summation of series. Certainly let some series be

$$\overset{1}{A},\overset{2}{B},\overset{3}{C},\overset{4}{D}.....\overset{x}{T},$$

of which the term corresponding to the index x shall be *T*, some function of *x*. The sum of all of these terms is put in place

$$A+B+C+D+\ \ldots\ +T=y$$

and it is evident that *y* is a function of *x* of this kind, so that, if in that in place of x there should be written $x-1$, there will be produced that same sum *y* with the final term *T* extracted, evidently $y-T$. But on writing $x-1$ in place of x the function *y* will change into

$$y-\frac{dy}{dx}+\frac{ddy}{1\cdot 2dx^2}-\frac{d^3y}{1\cdot 2\cdot 3adx^3}+\text{etc.},$$

from which this equation arises

$$T=\frac{dy}{dx}-\frac{ddy}{1\cdot 2dx^2}+\frac{d^3y}{1\cdot 2\cdot 3dx^3}-\frac{d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}+\text{etc.}$$

which once integrated on putting $\int Tdx=X$ becomes

$$X=y-\frac{dy}{1\cdot 2dx}+\frac{ddy}{1\cdot 2\cdot 3dx^2}-\frac{d^3y}{1\cdot 2\cdot 3\cdot 4dx^3}+\text{etc.}$$

which in whatever manner we may consider it appropriate to be integrated, while we will return that a little more general.

PROBLEM 160

1206. *With this proposed differential equation,*

$$X = \frac{ny}{a} - \frac{n(n-1)dy}{1\cdot 2a^2 dx} + \frac{n(n-1)(n-2)ddy}{1\cdot 2\cdot 3a^3 dx^2} - \text{etc.},$$

to find its complete integral.

SOLUTION

Hence this algebraic quantity may be formed

$$P = \frac{n}{a} - \frac{n(n-1)z}{1\cdot 2a^2} + \frac{n(n-1)(n-2)zz}{1\cdot 2\cdot 3a^3} - \text{etc.} = \frac{1-\left(1-\frac{z}{a}\right)^n}{z}$$

or $P = \frac{a^n-(a-z)^n}{a^n z}$, any twofold factor of which will have this form

$$aa - 2a(a-z)\cos.2\zeta + (a-z)^2$$

with the angle present $2\zeta = \frac{2i\pi}{n}$. But this form will become

$$2aa(1-\cos.2\zeta) - 2az(1-\cos.2\zeta) + zz = 4aa\sin.^2\zeta - 4az\sin.^2\zeta + zz,$$

which since compared to the general form $ff + 2fz\cos.\theta + zz$ gives $f = 2a\sin.\zeta$ and $\cos.\theta = -\sin.\zeta$, from which $\theta = 90^0 + \zeta$ and $\sin.\theta = \cos.\zeta$ with $\zeta = \frac{i\pi}{n}$. Now to the part of the integral hence arising being found, the form may be considered [§ 1173–§1177]

$$\frac{dP}{dz} = \frac{-a^n + (a+(n-1)z)(a-z)^{n-1}}{a^n zz}$$

in which on putting

$$z = -f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right)$$

or

$$z = 2a\sin.\zeta\left(\sin.\zeta - \sqrt{-1}\cdot\cos.\zeta\right) = a\left(1-\cos.2\zeta - \sqrt{-1}\cdot\sin.2\zeta\right),$$

so that there shall be

$$a - z = a\left(\cos.2\zeta + \sqrt{-1}\cdot\sin.2\zeta\right),$$

there is produced

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{-1+\left(n-(n-1)\left(\cos.2\zeta+\sqrt{-1}\cdot\sin.2\zeta\right)\right)\left(\cos.2(n-1)\zeta+\sqrt{-1}\cdot\sin.2(n-1)\zeta\right)}{-4aa\sin.^{2}\zeta\left(\cos.2\zeta+\sqrt{-1}\cdot\sin.2\zeta\right)}.$$

But since there shall be $\cos.2n\zeta=1$ and $\sin.2n\zeta=0$, there will be

$$\cos.2(n-1)\zeta=\cos.2\zeta \quad\text{and}\quad \sin.2(n-1)\zeta=-\sin.2\zeta$$

and thus

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{-n+n\left(\cos.2\zeta-\sqrt{-1}\cdot\sin.2\zeta\right)}{-4aa\sin.^{2}\zeta\left(\cos.2\zeta+\sqrt{-1}\cdot\sin.2\zeta\right)},$$

which is reduced to this form

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{n}{4aa\sin.^{2}\zeta}\left(\cos.2\zeta-\sqrt{-1}\cdot\sin.2\zeta-\cos.4\zeta+\sqrt{-1}\cdot\sin.4\zeta\right),$$

from which it is concluded

$$\mathfrak{P}=\frac{n}{4aa\sin.^{2}\zeta}\left(\cos.2\zeta-\cos.4\zeta\right)=\frac{n}{4aa\sin.\zeta}\sin.3\zeta,$$

$$\mathfrak{Q}=\frac{-n}{4aa\sin.^{2}\zeta}\left(\sin.2\zeta-\sin.4\zeta\right)=\frac{n}{4aa\sin.\zeta}\cos.3\zeta,$$

and thus there shall be

$$\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}=\frac{nn}{4a^{4}\sin.^{2}\zeta}$$

and on putting $\varphi=2ax\sin.\zeta\cos.\zeta=ax\sin.2\zeta$ there is made

$$\mathfrak{P}\cos.\varphi-\mathfrak{Q}\sin.\varphi=\frac{n}{2aa\sin.\zeta}\sin.(3\zeta-\varphi)$$

and

$$\mathfrak{P}\sin.\varphi+\mathfrak{Q}\cos.\varphi=\frac{n}{2aa\sin.\zeta}\cos.(3\zeta-\varphi).$$

On which account the part of the integral hence arising will be

$$\frac{4aa\sin.\zeta}{n}e^{2ax\sin.^{2}\zeta}\left\{\begin{aligned}&\sin.(3\zeta-\varphi)\int e^{-2ax\sin.^{2}\zeta}Xdx\cos.\varphi\\&+\cos.(3\zeta-\varphi)\int e^{-2ax\sin.^{2}\zeta}Xdx\sin.\varphi\end{aligned}\right\},$$

where for ζ these angles must be written successively $\frac{\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}, \frac{4\pi}{n}$ etc., as long as they are less than a right angle; but if n should be an even number, to these parts it is required to add the above $-\frac{2aa}{n} e^{2ax}\int e^{-2ax}Xdx$ and thus the true value of y may be deduced.

COROLLARY 1

1207. If there is $X = 0$, the part of the integral arising from any angle $\zeta = \frac{i\pi}{n}$ adopts this form

$$e^{2ax\sin.^2\zeta}\left(A\sin.\left(3\zeta - ax\sin.2\zeta\right) + B\cos.\left(3\zeta - ax\sin.2\zeta\right)\right)$$

or that form

$$Ae^{2ax\sin.^2\zeta}\sin.\left(\alpha + ax\sin.2\zeta\right),$$

with α denoting some constant angle.

COROLLARY 2

1208. With any particular integral found $y = V$ because it satisfies the proposed equation, if we put then $y = V + v$, this equation arises

$$0 = \frac{nv}{a} - \frac{n(n-1)dv}{1\cdot 2a^2dx} + \frac{n(n-1)(n-2)ddv}{1\cdot 2\cdot 3a^3dx^2} - \text{etc.}$$

from which the complete integral will be

$$y = V + Ae^{2ax\sin.^2\zeta}\sin.\left(\alpha + ax\sin.2\zeta\right)$$

with this final part following all the values of ζ multiplied together.

COROLLARIUM 3

1209. If we assume $n = \infty$ and $a = n$, so that this gives rise to a differential equation extending to infinity

$$X = y - \frac{dy}{1\cdot 2dx} + \frac{ddy}{1\cdot 2\cdot 3dx^2} - \frac{d^3y}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{d^4y}{1\cdot 2\cdot 3\cdot 4\cdot 5dx^4} - \text{etc.},$$

y will be the end term of a summatory progression, of which the general term corresponding to the index x is $T = \frac{dX}{dx}$. Hence as long as the angle $\zeta = \frac{i\pi}{n}$ is infinitely small, on account of $\varphi = 2i\pi x$, any part of the integral is

$$4i\pi e^{\frac{2ii\pi\pi}{n}x}\left\{\begin{array}{l}\sin.\left(\frac{3i\pi}{n}-2i\pi x\right)\int e^{\frac{-2ii\pi\pi}{n}x}Xdx\cos.2i\pi x\\+\cos.\left(\frac{3i\pi}{n}-2i\pi x\right)\int e^{\frac{-2ii\pi\pi}{n}x}Xdx\sin.2i\pi x\end{array}\right\}$$

and with the vanishing parts omitted

$$4i\pi\left(\cos.(2i\pi x)\int Xdx\sin.(2i\pi x)-\sin.(2i\pi x)\int Xdx\cos.(2i\pi x)\right).$$

If now here for i successively all the whole numbers 1, 2, 3 etc. are substituted, the true sum will be given of all the formulas resulting in this manner, and the complete value of y.

SCHOLIUM

1210. Moreover for the proposed equation, from the method indicated before, it is allowed to find a particular integral by a series of differentials on putting

$$y=AX+\frac{BdX}{dx}+\frac{CddX}{dx^2}+\frac{Dd^3X}{dx^3}+\frac{Ed^4X}{dx^4}+\text{etc.};$$

for with the substitution made there is found

$$A=\frac{a}{n},\ B=\frac{n-1}{2n},\ C=\frac{nn-1}{12an},\ D=\frac{nn-1}{24aan},\ E=\frac{-(nn-1)(nn-19)}{720a^3n}\ \text{etc.},$$

of which series it is difficult to assign the rule of the progression in general. Now for the case $n=\infty$ and $a=n$, which is worthy of note at first in the science of progressions, these coefficients themselves may be considered :

$$A=1,\ B=\frac{1}{2},\ \mathrm{C}=\frac{1}{12},\ D=0,\ E=\frac{-1}{720}\ \text{etc.},$$

from which that form arises, as formerly I have given for the general term in the summation.

[See E25 : A General Method of Summing Progressions (translated by this writer) & E47 : Finding the Sums of any Series from the given General Term (still in Latin at present) presented in Series I, vol. 14.]

But with this term conceded in the summation, which shall be set $=V$, it is agreed to note properly the equation $y=V$ is only a particular integral of the proposed equation, now the complete integral can be shown easily, but only if all the formulas of this kind $A\sin.(\alpha+2i\pi x)$ be added to V on writing for i all the successive numbers 1, 2, 3, 4 etc., where that can be assumed for any arbitrary angle α.
But since these single values satisfy the equation

$$0=v-\frac{dv}{2dx}+\frac{ddv}{6dx^2}-\frac{d^3v}{24dx^3}+\frac{d^4v}{120dx^4}-\frac{d^5v}{720dx^5}+\text{etc.},$$

thus that is most easily shown. On putting for the sake of brevity $2in=m$, so that there shall be $v=\sin.(\alpha+mx)$, and with the substitution made there must become

$$0=\sin.(\alpha+mx)\left(1-\tfrac{mm}{6}+\tfrac{m^4}{120}-\text{etc.}\right)-\cos.(\alpha+mx)\left(-\tfrac{m}{2}+\tfrac{m^3}{24}-\tfrac{m^5}{720}+\text{etc.}\right)$$

$$=\sin.(\alpha+mx)\tfrac{1}{m}\sin.m-\cos.(\alpha+mx)\tfrac{1}{m}(\cos.m-1).$$

But since there shall be $m=2i\pi$, evidently both $\sin.m=0$ as well as $\cos.m-1=0$.

PROBLEM 161

1211. *With this proposed differential equation,*

$$X=y+\frac{n(n-1)}{1\cdot 2}\cdot\frac{ddy}{a^2dx^2}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{d^4y}{a^4dx^4}+\text{etc.},$$

to find its complete integral.

SOLUTION

The algebraic quantity requiring to be formed is

$$P=1+\frac{n(n-1)}{1\cdot 2}\cdot\frac{zz}{aa}+\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{z^4}{a^4}+\text{etc.}$$

which evidently is reduced to this form

$$P=\tfrac{1}{2}\left(1+\tfrac{z}{a}\right)^n+\tfrac{1}{2}\left(1-\tfrac{z}{a}\right)^n=\frac{(a+z)^n+(a-z)^n}{2a^n},$$

of which some trinomial factor is

$$(a+z)^2-2(aa-zz)\cos.2\zeta+(a-z)^2$$

on assuming

$$2\zeta=\frac{(2i+1)\pi}{n}\quad\text{or}\quad\zeta=\frac{(2i+1)\pi}{2n}.$$

But this form will change into

$$2aa(1-\cos.2\zeta)+2zz(1+\cos.2\zeta)=4aa\sin.^2\zeta+4zz\cos.^2\zeta,$$

which general factor may be represented in this form $aa\text{tang.}^2\zeta+zz$, and thus on comparison with the general formula $ff+2fl\cos.\theta+zz$ there is given $f=-a\text{tang.}\zeta$ and

$\theta = 90°$, from which there becomes $\varphi = -ax \text{ tang.}\zeta$ (§ 1177) and the value for z on substitution

$$-f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right) = a \text{ tang.}\zeta \cdot y\sqrt{-1},$$

from which there is agreed

$$\frac{dP}{dz} = \frac{n(a+z)^{n-1} - n(a-z)^{n-1}}{2a^n}$$

to go onto $\mathfrak{P} + \mathfrak{Q}\sqrt{-1}$, from which there becomes

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \tfrac{n}{2a}\left(\left(1 + \text{tang.}\zeta \cdot y\sqrt{-1}\right)^{n-1} - \left(1 - \text{tang.}\zeta \cdot y\sqrt{-1}\right)^{n-1}\right)$$
$$= \frac{n}{2a\cos.^{n-1}\zeta}\left(\cos.(n-1)\zeta + \sqrt{-1}\cdot\sin.(n-1)\zeta - \cos.(n-1)\zeta + \sqrt{-1}\cdot\sin.(n-1)\zeta\right)$$

and thus $\mathfrak{P} = 0$ and $\mathfrak{Q} = \frac{n\sin.(n-1)\zeta}{a\cos.^{n-1}\zeta}$. But on account of $n\zeta = \frac{2i+1}{2}\pi$ and hence $\cos.n\zeta = 0$ and $\sin.n\zeta = \pm 1$, since i should be an even or odd number, $\sin.(n-1)\zeta = \pm 1\cos.\zeta$ and thus $\mathfrak{Q} = \frac{\pm n}{a\cos.^{n-2}\zeta}$. Wherefore on account of $\cos.\theta = 0$ the part of the integral arising from this factor is

$$\pm\ \frac{2a\cos.^{n-2}\zeta}{n}\left(\cos.\varphi\int Xdx\sin.\varphi - \sin.\varphi\int Xdx\cos.\varphi\right)$$

or since $\varphi = -ax \text{ tang.}\zeta$

$$\pm\ \frac{2a\cos.^{n-2}\zeta}{n}\left\{\begin{array}{l}\sin.(ax \text{ tang.}\zeta)\int Xdx\cos.(ax \text{ tang.}\zeta)\\ -\cos.(ax \text{ tang.}\zeta)\int Xdx\sin.(ax \text{ tang.}\zeta)\end{array}\right\},$$

where for ζ there are substituted successively the angles $\frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \frac{7\pi}{2n}$ etc., as long as they are smaller than right angles, for which it is required there for the alternate signs + and – to be written ; and all these parts gathered together into one sum will give the value of the complete integral of y, while for the last part arising from the angle $\zeta = \frac{\pi}{2}$, since it comes about, if n is an odd number, only half of this is taken [§ 1190].

COROLLARIUM 1

1212. We may adapt this at once to the case $n=\infty$ and $a=nc$, so that this shall be the proposed differential equation

$$X=y+\frac{ddy}{1\cdot 2c^2dx^2}+\frac{d^4y}{1\cdot 2\cdot 3\cdot 4a^4dx^4}+\frac{d^6y}{1\cdots 6a^6dx^6}+\text{etc. to infinity.}$$

Therefore since here the values of ζ shall be indefinitely small, there will be $\cos.\zeta=1$ and $\tang.\zeta=\zeta=(4i\pm 1)cx\frac{\pi}{2}$, hence $ax\ \tang.\zeta=(4i\pm 1)cx\frac{\pi}{2}$, for which angle we may write ω. Therefore some part of the integral [is given by]

$$\pm 2c\left(\sin.\omega\int Xdx\cos.\omega-\cos.\omega\int Xdx\sin.\omega\right),$$

where the ambiguous signs themselves correspond by changing.

COROLLARY 2

1213. Only if $\frac{\pi}{2}$ is put equal to the angle φ, the whole integral will be expressed thus :

$$\begin{aligned}\frac{y}{2c}=&+\sin.\varphi\int Xdx\cos.\varphi-\cos.\varphi\int Xdx\sin.\varphi\\&-\sin.3\varphi\int Xdx\cos.3\varphi+\cos.3\varphi\int Xdx\sin.3\varphi\\&+\sin.5\varphi\int Xdx\cos.5\varphi-\cos.5\varphi\int Xdx\sin.5\varphi\\&-\sin.7\varphi\int Xdx\cos.7\varphi+\cos.7\varphi\int Xdx\sin.7\varphi\\&\text{etc.,}\end{aligned}$$

which formulas are to be continued indefinitely.

COROLLARY 3

1214. If we put $c=b\sqrt{-1}$, so that this infinite equation can be considered :

$$X=y-\frac{ddy}{1\cdot 2b^2dx^2}+\frac{d^4y}{1\cdot 2\cdot 3\cdot 4b^4dx^4}-\frac{d^6y}{1\cdots 6b^6dx^6}+\text{etc.},$$

and now we may call φ the angle $\frac{\pi}{2}bx$, there will be the complete integral

$$\begin{aligned}\tfrac{y}{b} = &+e^{-\psi}\int e^{\psi} Xdx - e^{\psi}\int e^{-\psi} Xdx\\ &-e^{-3\psi}\int e^{3\psi} Xdx + e^{3\psi}\int e^{-3\psi} Xdx\\ &+e^{-5\psi}\int e^{5\psi} Xdx - e^{5\psi}\int e^{-5\psi} Xdx\\ &\text{etc.}\end{aligned}$$

SCHOLIUM

1215. If for the equation of Corollary 1 by the method established above [§ 1204] we seek the particular integral by the differentials of *X*, in the end we come upon this expression

$$y = AX - \tfrac{BddX}{c^2dx^2} + \tfrac{Cd^4X}{c^4dx^4} - \tfrac{Dd^6X}{c^6dx^6} + \tfrac{Ed^8X}{c^8dx^8} - \text{etc.},$$

we will find these values of the coefficients

$$A = 1,\quad B = \tfrac{1}{1\cdot 2},\quad C = \tfrac{5}{1\cdots 4}\quad D = \tfrac{61}{1\cdots 6},\quad E = \tfrac{1385}{1\cdots 8},\quad F = \tfrac{50521}{1\cdots 10}\ \text{etc.}$$

And here if the value of *y* is put $=V$, on calling the angle $\tfrac{\pi}{2}cx = \varphi$, there will be the complete integral

$$y = V + A\sin.(\alpha+\varphi) + B\sin.(\beta+3\varphi) + C\sin.(\gamma+5\varphi) + D\sin.(\delta+7\varphi) + \text{etc}.$$

PROBLEM 162

1216. *With the proposed differential equation,*

$$X = y + \tfrac{n(n-1)}{1\cdot 2a^2}\cdot\tfrac{dy}{dx} + \tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4a^4}\cdot\tfrac{ddy}{dx^2} + \tfrac{n\cdots(n-5)}{1\cdots 6a^6}\cdot\tfrac{d^3y}{dx^3} + \text{etc.},$$

to find its complete integral.

SOLUTION

The algebraic quantity hence required to be formed :

$$\begin{aligned}P &= 1 + \tfrac{n(n-1)}{1\cdot 2a^2}z + \tfrac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4a^4}zz + \text{etc.}\\ &= \tfrac{1}{2}\left(1+\tfrac{\sqrt{z}}{a}\right)^n + \tfrac{1}{2}\left(1-\tfrac{\sqrt{z}}{a}\right)^n,\end{aligned}$$

since it arises from the preceding, if in place of *zz*, *z* is written there, on assuming the angle $\zeta = \tfrac{2i+1}{2n}\pi$, any factor will be $aa\,\text{tang.}^2\zeta + z$, thus so that all the simple factors of this form are real. Therefore with this factor since the formula $\alpha + z$ in comparison will be $\alpha = aa$ tang.$^2\zeta$ and on taking $\mathfrak{A} = \frac{P}{\alpha+z}$ on putting $z = -\alpha$ there will be the part of the integral arising from this factor

$$\tfrac{1}{\mathfrak{A}} e^{-\alpha x}\int e^{\alpha x} X dx .$$

Since now on P vanishing on putting $z = -\alpha$, there will be also $\mathfrak{A} = \frac{dP}{dz}$; but on differentiating this is

$$\frac{dP}{dz} = \frac{n}{4a\sqrt{z}}\left(\left(1+\frac{\sqrt{z}}{a}\right)^{n-1} - \left(1-\frac{\sqrt{z}}{a}\right)^{n-1}\right).$$

Therefore as it is required to put $\frac{\sqrt{z}}{a} = \text{tang.}\zeta \cdot \sqrt{-1}$, there will be

$$1+\frac{\sqrt{z}}{a} = \frac{\cos.\zeta + \sqrt{-1}.\sin.\zeta}{\cos.\zeta} \quad\text{and}\quad 1-\frac{\sqrt{z}}{a} = \frac{\cos.\zeta - \sqrt{-1}.\sin.\zeta}{\cos.\zeta}$$

and hence

$$\mathfrak{A} = \frac{n}{4aa\text{tang.}\zeta\sqrt{-1}} \cdot \frac{2\sqrt{-1}.\ \sin.(n-1)\zeta}{\cos.^{n-1}\zeta} = \frac{n\sin.(n-1)}{2aa\sin.\zeta\cos.^{n-2}\zeta} .$$

Now there may be noted that there is $\sin.n\zeta = \sin.(2i+1)\frac{\pi}{2} = \pm 1$ (where the upper sign prevails, if i is an even number, and the lower if odd), then truly $\cos.n\zeta = 0$, from which there becomes $\sin.(n-1)\zeta = \pm\cos.\zeta$, and from which there is constructed

$$\mathfrak{A} = \frac{\pm n}{2aa\sin.\zeta\cos.^{n-3}\zeta},$$

and the part of the integral sought will be obtained

$$\pm\ \frac{2aa\sin.\zeta\cos.^{n-3}\zeta}{n} e^{-aa\text{tang.}^2\zeta\cdot x}\int e^{aa\text{tang.}^2\zeta\cdot x} X dx.$$

Now therefore successively these values $\frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \frac{7\pi}{2n}$, etc. are attributed to ζ, provided they do not exceed a right angle, and all these parts collected into one sum will give the complete integral or the value of y.

COROLLARY 1

1217. If we put $n = \infty$ and $a = nc$, the proposed equation extends to infinity and there will be

$$X = y + \frac{dy}{1\cdot 2c^2 dx} + \frac{ddy}{1\cdot 2\cdot 3\cdot 4c^4dx^2} + \frac{d^3y}{1\cdots 6c^6dx^3} + \text{etc.}$$

and the algebraic form thus arising

$$P=1+\frac{z}{1\cdot 2c^2}+\frac{zz}{1\cdot 2\cdot 3\cdot 4c^4}+\frac{z^3}{1\cdots 6c^6}+\text{etc.}=\frac{1}{2}e^{\frac{\sqrt{z}}{c}}+\frac{1}{2}e^{\frac{-\sqrt{z}}{c}},$$

which has all simple real factors, and on account of the infinitely small ζ there will be $\text{tang.}\zeta=\zeta=\frac{2i+1}{2n}\pi$ and from this the form of the general factors

$$z+\frac{(2i+1)^2}{4}\pi\pi cc \ \text{ or } \ 1+\frac{4z}{(2i+1)^2\pi\pi cc}.$$

COROLLARY 2

1218. For the sake of brevity there is put $\frac{2i+1}{2n}\pi=\theta$; there will be $aa\text{tang.}^2\zeta=\theta\theta cc$, and then indeed $\cos.\zeta=1$ and $\frac{aa\sin.\zeta}{n}=\theta cc$, from which some part of the integral will be

$$\pm 2\theta cce^{-\theta\theta ccx}\int e^{\theta\theta ccx}Xdx,$$

where for θ successively it is required to write all these angles $\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},\frac{9\pi}{2}$ etc.

COROLLARY 3

1219. Likewise here there is, cc is taken either positive or negative; hence the integral of this infinite differential equation

$$X=y+\frac{dy}{1\cdot 2bdx}+\frac{ddy}{bbdx^2}+\frac{d^3y}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6b^3dx^3}+\text{etc.}$$

will be

$$y=2\theta be^{-\theta\theta bx}\int e^{\theta\theta bx}Xdx$$

on writing successively in place of θ all these angles with the ambiguity of the sign now removed $+\frac{\pi}{2},-\frac{3\pi}{2},+\frac{5\pi}{2},-\frac{7\pi}{2}$ etc., from which, if $X=0$, some particular integral is $y=Ae^{-\theta\theta bx}$.

PROBLEM 163

1220. *With the proposed differential equation,*

$$X=\frac{ndy}{adx}+\frac{n(n-1)(n-2)ddy}{1\cdot 2\cdot 3a^3dx^2}+\frac{n(n-1)(n-2)(n-3)(n-4)d^3y}{1\cdot 2\cdot 3\cdot 4\cdot 5a^5dx^3}+\text{etc.},$$

to find its complete integral.

SOLUTION

If this equation is taken by dx it can be integrated once straight away, yet it will be better to retain this form, from which there becomes

$$P=\frac{nz}{a}+\frac{n(n-1)(n-2)zz}{1\cdot 2\cdot 3a^3}+\frac{n(n-1)(n-2)(n-3)(n-4)z^3}{1\cdot 2\cdot 3\cdot 4\cdot 5a^5}+\text{etc.},$$

which evidently can be shown thus

$$P=\frac{\sqrt{z}}{2}\left(\left(1+\frac{\sqrt{z}}{a}\right)^n-\frac{1}{2}\left(1-\frac{\sqrt{z}}{a}\right)^n\right),$$

of which indeed one factor z presents itself, and the remainder now are contained in this form

$$\left(1+\frac{\sqrt{z}}{a}\right)^2-2\left(1-\frac{z}{aa}\right)\cos.2\zeta+\left(1-\frac{\sqrt{z}}{a}\right)^2$$

on assuming the angle $2\zeta=\frac{2i+1}{n}\pi$ or $\zeta=\frac{i\pi}{n}$, now this formula changes into

$$2(1-\cos.2\zeta)+\frac{2z}{aa}(1+\cos.2\zeta),$$

from which it is apparent that the general factor is $aa\,\text{tang.}^2\zeta+z$, which also contains that first factor z on taking $i=0$. Hence on putting $aa\,\text{tang.}^2\zeta=\alpha$ the part of the integral corresponding to this part will be

$$\frac{1}{\mathfrak{A}}e^{-ax}\int e^{ax}Xdx,$$

if on putting $z=-aa\,\text{tang.}^2\zeta$ or $\sqrt{z}=a\,\text{tang.}\zeta\sqrt{-1}$ or taking

$$\mathfrak{A}=\frac{dP}{dz}=\frac{1}{4\sqrt{z}}\left(\left(1+\frac{\sqrt{z}}{a}\right)^{n}-\frac{1}{2}\left(1-\frac{\sqrt{z}}{a}\right)^{n}\right)+\frac{n}{4a}\left(\left(1+\frac{\sqrt{z}}{a}\right)^{n-1}+\frac{1}{2}\left(1-\frac{\sqrt{z}}{a}\right)^{n-1}\right);$$

but

$$\left(1+\frac{\sqrt{z}}{a}\right)^{n}=\frac{\cos.n\zeta+\sqrt{-1}\cdot\sin.n\zeta}{\cos.^{n}\zeta}\quad\text{and}\quad\left(1-\frac{\sqrt{z}}{a}\right)^{n}=\frac{\cos.n\zeta-\sqrt{-1}\cdot\sin.n\zeta}{\cos.^{n}\zeta},$$

on account of which there becomes

$$\mathfrak{A}=\frac{\sin.n\zeta}{2a\text{tang}.\zeta\cos.^{n}\zeta}+\frac{n\cos.(n-1)\zeta}{2a\cos.^{n-1}\zeta}=\frac{\pm n}{2a\cos.^{n-2}\zeta}$$

as $\sin.n\zeta=0$ and $\cos.n\zeta=\pm1$, according as the number *i* should be either even or odd. According to which some part of the integral will be expressed thus

$$\pm\frac{2a\cos.^{n-2}\zeta}{n}e^{-\alpha x}\int e^{\alpha x}Xdx$$

with $\alpha=aa\,\text{tang}.^{2}\zeta$ arising. Now these values are attributed to the angle ζ successively $\frac{0\pi}{n},\frac{1\pi}{n},\frac{2\pi}{n},\frac{3\pi}{n}$ etc., provided they do not exceed the right angle $\frac{\pi}{2}$, and all these formulas with their own signs joined into one sum give the complete value for *y*.

COROLLARIUM 1

1221. First therefore a part of the integral arises from the angle $\zeta=0$, from which that will be $\frac{2a}{n}\int Xdx$, but in place of which on account of the reasons put up above [§ 1190] concerning the simple factors only half of this must be taken, so that this first part shall be $=\frac{a}{n}\int Xdx$, since also from that it is apparent that on putting $z=0$ clearly there becomes $\frac{P}{z}=\frac{n}{a}$.

COROLLARY 2

1222. Likewise it is to be understood from the last part, if indeed it arises from the value $\zeta=\frac{\pi}{2}$, which comes about, if *n* shall be an even number. Because now in this case there shall become $\cos.\zeta=0$, this whole part of the integral itself vanishes.

COROLLARY 3

1223. If there should be $X=0$, any part of the integral becomes $Ae^{-aa\text{tang}.^{2}\zeta\cdot x}$ with *A* denoting some arbitrary constant quantity, and so far there arises this equation $y=Ae^{-aa\text{tang}.^{2}\zeta\cdot x}$ with the particular integral of the equation, provided there is taken the angle $\zeta=\frac{i\pi}{n}$.

SCHOLIUM

1224. Hence on putting $n=\infty$ and $a=n\sqrt{b}$ this differential equation extending to infinity can be integrated :

$$\frac{X}{\sqrt{b}}=\frac{dy}{1\cdot bdx}+\frac{ddy}{1\cdot 2\cdot 3bbdx^2}+\frac{d^3y}{1\cdot 2\cdot 3\cdot 4\cdot 5b^3dx^3}+\frac{d^4y}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7b^4dx^4}+\text{etc.}$$

or also this coming from that by a single integration :

$$\sqrt{b}\cdot\int Xdx=\frac{y}{1}+\frac{dy}{1\cdot 2\cdot 3bdx}+\frac{ddy}{1\cdot 2\cdot 3\cdot 4\cdot 5b^2dx^2}+\frac{d^3y}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7b^3dx^3}+\text{etc.}$$

For since the angle $\zeta=\frac{i\pi}{n}$ shall be indefinitely small, there will be $\cos.\zeta=1$ and $a\text{tang.}\zeta=a\zeta=i\pi\sqrt{b}$ and thus $\alpha=aa\text{ tang.}^2\zeta=ii\pi\pi b$, any part of the integral will be had :

$$\pm 2\sqrt{b}\cdot e^{-ii\pi\pi bx}\int e^{ii\pi\pi bx}Xdx$$

from which the first part arising from $i=0$ is reduced by half, on account of the above reasons [§ 1190] put in place , and the complete integral will be

$$\frac{y}{\sqrt{b}}=\int Xdx-2e^{-\pi\pi bx}\int e^{\pi\pi bx}Xdx+2e^{-4\pi\pi bx}\int e^{4\pi\pi bx}Xdx$$
$$-2e^{-9\pi\pi bx}\int e^{9\pi\pi bx}Xdx+2e^{-16\pi\pi bx}\int e^{16\pi\pi bx}Xdx-\text{etc.}$$

EXEMPLE

1225. *Let* $n=6$ *and* $a=1$, *so that this equation is proposed to be integrated :*

$$X=\frac{6dy}{dx}+\frac{20ddy}{dx^2}+\frac{6d^3y}{dx^3}\quad or\quad \int Xdx=6y+20\frac{dy}{dx}+\frac{6ddy}{dx^2}.$$

Hence the values for the angle ζ and from that those depending values are

$\zeta=0$,	$30°$,	$60°$,
$\cos.\zeta=1$,	$\frac{\sqrt{3}}{2}$,	$\frac{1}{2}$,
$\alpha=0$,	$\frac{1}{3}$,	3,

from which the integral sought is deduced,

$$y=\frac{1}{6}\int Xdx-\frac{3}{16}e^{-\frac{1}{3}x}\int e^{\frac{1}{3}x}Xdx+\frac{1}{48}e^{-3x}\int e^{3x}Xdx,$$

which also will be apparent to satisfy the equation considered.

CAPUT IV

APPLICATIO METHODI INTEGRANDI IN CAPITE PRAECEDENTI TRADITAE AD EXEMPLA

PROBLEMA 156

1189. *Proposita hac aequatione differentiali*

$$X = a^n y + \frac{d^n y}{dx^n}$$

eius integrate completum invenire.

SOLUTIO

Hic ergo est $P = a^n + z^n$, ubi primo observetur, si n sit numerus impar, factorem simplicem esse $a+z$, ex quo nascitur pars integralis

$$\frac{1}{\mathfrak{A}} e^{-ax} \int e^{ax} X dx$$

existente $\mathfrak{A}$ valore ex forma $\frac{P}{a+z}$ emergente, si ponatur $z=-a$; qui ergo valor cum sit etiam $\frac{dP}{dz} = nz^{n-1}$, ob $n-1$ numerum parem erit $\mathfrak{A} = na^{n-1}$, ideoque haec integralis pars

$$= \frac{1}{na^{n-1}} e^{-ax} \int e^{ax} X dx$$

Reliqui factores omnes in hac forma continentur $aa - 2az\cos.\theta + zz$ existente $\theta = \frac{(2i+1)\pi}{n}$, ubi i denotat numerum integrum quemcunque et π angulum duobus rectis aequalem. Comparata hac forma cum Problemate 153 et Corollario 1 fit $f=-a$ et ob $z = a\left(\cos.\theta + \sqrt{-1}\cdot \sin.\theta\right)$ ex forma $\frac{dP}{dz}$ colligitur

$$\mathfrak{P} = na^{n-1}\cos.(n-1)\theta \text{ et } \mathfrak{Q} = na^{n-1}\sin.(n-1)\theta;$$

cum igitur sit $\cos.n\theta = -1$ et $\sin.n\theta = 0$, erit

$$\mathfrak{P} = -na^{n-1}\cos.\theta \text{ and } \mathfrak{Q} = na^{n-1}\sin.\theta.$$

Quare posito $fx\sin.\theta = -ax\sin.\theta = \varphi$ integralis pars ex quolibet factore duplici oriunda est

$$\frac{2e^{ax\cos.\theta}}{na^{n-1}}\left\{\begin{array}{l}\left(-\cos.\theta\cos.\varphi-\sin.\theta\sin.\varphi\right)\int e^{-ax\cos.\theta}Xdx\cos.\varphi\\+\left(-\cos.\theta\sin.\varphi+\sin.\theta\cos.\varphi\right)\int e^{-ax\cos.\theta}Xdx\sin.\varphi\end{array}\right\}$$

seu

$$\frac{-2e^{ax\cos.\theta}}{na^{n-1}}\left(\cos.(\theta-\varphi)\int e^{-ax\cos.\theta}Xdx\cos.\varphi-\sin.(\theta-\varphi)\int e^{-ax\cos.\theta}Xdx\sin.\varphi\right)$$

et pro φ vaIore restituto

$$\frac{-2e^{ax\cos.\theta}}{na^{n-1}}\left\{\begin{array}{l}\cos.(\theta+ax\sin.\theta)\int e^{-ax\cos.\theta}Xdx\cos.(ax\sin.\theta)\\+\sin.(\theta+ax\sin.\theta)\int e^{-ax\cos.\theta}Xdx\sin.(ax\sin.\theta)\end{array}\right\}.$$

Iam pro θ successive substituantur anguli $\frac{\pi}{n}$, $\frac{3\pi}{n}$, $\frac{5\pi}{n}$, $\frac{7\pi}{n}$ etc., quamdiu ipso π sunt minores, omnesque hae formae in unam summam coniectae, quibus casu, quo *n* est numerus impar, insuper addi oportet formam primo inventam

$$\frac{1}{na^{n-1}}e^{-ax}\int e^{ax}Xdx$$

dabunt integraIe quaesitum.

COROLLARIUM 1

1190. Casu quidem, quo *n* est numerus impar, ultimus valor ipsius θ foret π, quem autem hic omitti iussimus; inde autem ob $ax\sin.\theta=0$ et $\cos.\theta=-1$ prodiret ultima pars integraIis

$$\frac{2e^{-ax}}{na^{n-1}}\int e^{ax}Xdx$$

dupla eius, quam capi convenit; cuius ratio est, quod sumto $\theta=\pi$ formula $aa+2az+zz$ non amplius ipsa est factor, sed eius radix quadrata $a+z$, ex quo hunc casum seorsim erui necesse erat.

COROLLARIUM 2

1191. Si est $X=0$, formulae integrales abeunt in constantes arbitrarias et ex factore

$$aa-2az\cos.\theta+zz$$

oritur haec pars integralis

$$\frac{-2e^{ax\cos.\theta}}{na^{n-1}}\left(A\cos.\left(\theta+ax\sin.\theta\right)+\mathfrak{A}\sin.\left(\theta+ax\sin.\theta\right)\right),$$

quae reducitur ad hanc formam

$$Ae^{ax\cos.\theta}\cos.\left(\zeta+ax\sin.\theta\right)$$

denotante ζ angulum constantem quemcunque, uti iam supra [§ 1135] invenimus.

PROBLEMA 157

1192. *Proposita hac aequatione differentiali*

$$X=a^n y-\frac{d^n y}{dx^n}$$

eius integrale completum invenire.

SOLUTIO

Forma algebraica hinc nata $P=a^n-z^n$, factorem semper habet $a-z$, unde nascitur pars integralis $\frac{1}{\mathfrak{A}}e^{ax}\int e^{-ax}Xdx$ existente $\mathfrak{A}=\frac{P}{z-a}$ posito $z=a$. Cum ergo sit quoque $\mathfrak{A}=\frac{dP}{dz}=-nz^{n-1}$, erit $\mathfrak{A}=-na^{n-1}$, ideoque haec pars integralis

$$=\frac{-1}{na^{n-1}}e^{ax}\int e^{-ax}Xdx.$$

Deinde si *n* sit numerus par hincque $n-1$ impar, factor quoque erit $a+z$, qui praebet integralis partem

$$=\frac{1}{na^{n-1}}e^{-ax}\int e^{ax}Xdx$$

Reliqui factores omnes ipsius *P* sunt duplicis formae $aa-2az\cos.\theta+zz$ existente angulo $\theta=\frac{2i\pi}{n}$; qua cum generali supra [§ 1173] usurpata $ff+2fz\cos.\theta+zz$ compara fit $f=-a$ et ex forma $\frac{dP}{dz}=-nz$ quaeri oportet formulam $\mathfrak{P}+\mathfrak{Q}\sqrt{-1}$ posito $z=a\left(\cos.\theta+\sqrt{-1}\cdot\sin.\theta\right)$, unde colligitur

$$\mathfrak{P}=-na^{n-1}\cos.(n-1)\theta \text{ et } \mathfrak{Q}=-na^{n-1}\sin.(n-1)\theta,$$

seu ob $\cos.n\theta=1$ et $\sin.n\theta=0$ fit

$$\mathfrak{P} = -na^{n-1}\cos.\theta \text{ et } \mathfrak{Q} = na^{n-1}\sin.\theta .$$

Posito iam angulo $-ax\sin.\theta = \varphi$; ex § 1177 oritur pars integralis

$$\frac{2e^{ax\cos.\theta}}{na^{n-1}}\left\{\begin{array}{l}(-\cos.\theta\cos.\varphi - \sin.\theta\sin.\varphi)\int e^{-\alpha x\cos.\theta}Xdx\cos.\varphi \\ +(-\cos.\theta\sin.\varphi + \sin.\theta\cos.\varphi)\int e^{-\alpha x\cos.\theta}Xdx\sin.\varphi\end{array}\right\}$$

quae ut ante [§ 1189] reducitur ad hanc formam

$$\frac{-2e^{ax\cos.\theta}}{na^{n-1}}\left\{\begin{array}{l}\cos.(\theta + ax\sin.\theta)\int e^{-\alpha x\cos.\theta}Xdx\cos.(ax\sin.\theta) \\ -\sin.(\theta + ax\sin.\theta)\int e^{-\alpha x\cos.\theta}Xdx\sin.(ax\sin.\theta)\end{array}\right\}.$$

Hic iam pro θ successive scribantur anguli $\frac{2\pi}{n}$, $\frac{4\pi}{n}$, $\frac{6\pi}{n}$ etc., quamdiu sunt minores quam π, haeque partes omnes cum primum inventa atque etiam altera, si n fuerit numerus par, in unam summam collectae dabunt integrale quaesitum seu valorem ipsius y.

COROLLARIUM

1193. Cum factor duplex generalis $aa - 2az\cos.\theta + zz$ casibus $\theta = 0$ et $\theta = \pi$ non praebeat ipsos factores simplices reales $a - z$ et $a + z$, sed eorum quadrata, haec ratio est, cur pars integralis inde eruta prodeat dupla eius, quam capi oportet.

PROBLEMA 158

1194. *Proposita hac aequatione differentiali*

$$X = y + \frac{dy}{dx} + \frac{ddy}{dx^2} + \frac{d^3y}{dx^3} + ... + \frac{d^ny}{dx^n}$$

eius integrale completum investigare.

SOLUTIO

Forma algebraica hinc nata est

$$P = 1 + z + z^2 + z^3 + z^4 + \ ... \ + z^n ,$$

cuius omnes factores scrutari oportet. Cum igitur sit $P = \frac{1-z^{n+1}}{1-z}$ formae $1 - z^{n+1}$ factores capi convenit excluso $1 - z$, unde primo patet, si fuerit $n+1$ numerus par, factorem simplicem fore $1+z$, ex quo nascitur pars integralis $\frac{1}{\mathfrak{A}}e^{-x}\int e^{x}Xdx$ existente $\mathfrak{A}=\frac{P}{1+z}=\frac{1-z^{n+1}}{1-zz}$ posito $z=-1$. Erit ergo quoque $\mathfrak{A}=\frac{(n+1)z^{n}}{2z}$ ideoque $\mathfrak{A}=\frac{1}{2}(n+1)$, ut haec pars integralis sit

$$\frac{2}{n+1}e^{-x}\int e^{x}Xdx.$$

Factorum autem duplicium forma est $1-2z\cos.\theta+zz$ sumto angulo $\theta=\frac{2i\pi}{n}$, ita ut pro § 1173 sit $f=-1$. Consideretur forma

$$\frac{dP}{dz}=\frac{1-(n+1)z^{n}+nz^{n+1}}{(1-z)^{2}},$$

quae posito $z=\cos.\theta+\sqrt{-1}\cdot\sin.\theta$ abire sumitur in $\mathfrak{P}+\mathfrak{Q}\sqrt{-1}$, sicque erit

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{1-(n+1)\cos.n\theta+n\cos.(n+1)\theta-(n+1)\sqrt{-1}\cdot\sin.n\theta+n\sqrt{-1}\cdot\sin.(n+1)\theta}{1-2\cos.\theta+\cos.2\theta-2\sqrt{-1}\cdot\sin.\theta+\sqrt{-1}\cdot\sin.2\theta}.$$

Cum vero sit $\sin.(n+1)\theta=0$ et $\cos.(n+1)\theta=1$, erit $\sin.n\theta=-\sin.\theta$ et $\cos.n\theta=\cos.\theta$ ideoque

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{n+1-(n+1)\cos.\theta+(n+1)\sqrt{-1}\cdot\sin.\theta}{-2\cos.\theta+2\cos.^{2}\theta-2\sqrt{-1}\cdot\sin.\theta(1-\cos.\theta)}$$

seu

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{n+1}{2(1-\cos.\theta)}\cdot\frac{1-\cos.\theta+\sqrt{-1}\cdot\sin.\theta}{-\cos.\theta-\sqrt{-1}\cdot\sin.\theta}.$$

Multiplicetur huius fractionis numerator et denominator per $-\cos.\theta+\sqrt{-1}\cdot\sin.\theta$ et prodibit

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{-(n+1)\left(1+\cos.\theta-2\cos.^{2}\theta-\sqrt{-1}\cdot\sin.\theta(1-2\cos.\theta)\right)}{2(1-\cos.\theta)}$$

ita ut sit

$$\mathfrak{P}=-\frac{1}{2}(n+1)(1+2\cos.\theta)\text{ et }\mathfrak{Q}=\frac{1}{2}(n+1)\frac{\sin.\theta(1-2\cos.\theta)}{1-\cos.\theta}$$

unde fit

$$\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}=\frac{(n+1)^{2}}{2(1-\cos.\theta)}.$$

Tum vero posito angulo $-x\sin.\theta=\varphi$ colligitur

$$\mathfrak{P}\cos.\varphi-\mathfrak{Q}\sin.\varphi=\frac{-(n+1)\left(\cos.(\theta-\varphi)-\cos.(2\theta-\varphi)\right)}{2(1-\cos.\theta)},$$

$$\mathfrak{P}\sin.\varphi+\mathfrak{Q}\cos.\varphi=\frac{+(n+1)\left(\sin.(\theta-\varphi)-\sin.(2\theta-\varphi)\right)}{2(1-\cos.\theta)};$$

cum autem sit

$$\cos.a-\cos.b=2\sin.\tfrac{a+b}{2}\sin.\tfrac{a-b}{2}\quad\text{and}\quad\sin.a-\sin.b=-2\cos.\tfrac{a+b}{2}\sin.\tfrac{b-a}{2},$$

fit hinc

$$\mathfrak{P}\cos.\varphi-\mathfrak{Q}\sin.\varphi=\frac{-(n+1)\sin.\frac{1}{2}(3\theta-2\varphi)}{2\sin.\frac{1}{2}\theta}$$

et

$$\mathfrak{P}\sin.\varphi+\mathfrak{Q}\cos.\varphi=\frac{-(n+1)\cos.\frac{1}{2}(3\theta-2\varphi)}{2\sin.\frac{1}{2}\theta},$$

ex quo integralis pars quaesita erit

$$\frac{-4}{n+1}e^{x\cos.\theta}\sin.\tfrac{1}{2}\theta\left\{\begin{aligned}&\sin.\tfrac{1}{2}(3\theta+2x\sin.\theta)\int e^{-x\cos.\theta}Xdx\cos.(x\sin.\theta)\\&-\cos.\tfrac{1}{2}(3\theta+2x\sin.\theta)\int e^{-x\cos.\theta}Xdx\sin.(x\sin.\theta)\end{aligned}\right\}.$$

Pro θ ergo successive substituantur anguli $\frac{2\pi}{n+1}$, $\frac{4\pi}{n+1}$, $\frac{6\pi}{n+1}$ etc., quamdiu sunt minores quam π, haeque partes omnes in unam summam colligantur, cui, si $n+1$ sit numerus par, addatur insuper $\frac{2}{n+1}e^{-x}\int e^{x}Xdx$ sicque obtinebitur valor ipsius *y*.

COROLLARIUM 1

1195. Si aequatio proposita in infinitum progrediatur, ut sit *n* numerus infinitus, anguli θ priores omnes sunt infinite parvi ideoque numero infiniti, quoad numerus par 2*i* ad *n* + 1 rationem finitam habere incipiat; tum autem pro θ sequentur omnes anguli finiti in progressione arithmetica increscentes, cuius differentia est $\frac{2\pi}{n+1}$ usque ad π, quorum numerus itidem est infinitus.

COROLLARIUM 2

1196. Quamdiu angulus θ est infinite parvus, integralis pars ex eo oriunda hanc induit formam

$$\tfrac{-\theta\theta e^x}{n+1}\left((3+2x)\int e^{-x}Xdx-2\int e^{-x}Xxdx\right);$$

quae cum per cubum infiniti sit divisa, etiam multitudo infinita huiusmodi formularum pro evanescente est habenda.

COROLLARIUM 3

1197. Quodsi fuerit $X=0$, ut huius aequationis

$$X=y+\tfrac{dy}{dx}+\tfrac{ddy}{dx^2}+\tfrac{d^3y}{dx^3}+....+\tfrac{d^ny}{dx^n}$$

integrale sit investigandum, erit eius pars quaecunque

$$e^{x\cos.\theta}\left(A\sin.\tfrac{1}{2}(3\theta+2x\sin.\theta)+\mathfrak{A}\cos.\tfrac{1}{2}(3\theta+2x\sin.\theta)\right)$$

seu simplicius

$$Ae^{x\cos.\theta}\cos.(\zeta+x\sin.\theta).$$

Cum igitur, si n sit numerus infinitus, pro θ angulus quicunque accipi queat, erit istius aequationis integrale particulare quodcunque

$$y=Ae^{x\cos.\theta}\cos.(x\sin.\theta+\zeta)$$

sumendo pro ζ etiam angulum quemcunque.

SCHOLION

1198. Num autem huius aequationis differentialis in infinitum excurrentis

$$X=y+\tfrac{dy}{dx}+\tfrac{ddy}{dx^2}+\tfrac{d^3y}{dx^3}+....+\tfrac{d^ny}{dx^n}$$

denotante X functionem quamcunque ipsius x integrale commodius exprimi possit quam per partium illarum innumerabilium evanescentium summam, quaestio est altioris indaginis neque adhuc ad hunc scopum Analyseos fines satis videntur promoti. Casibus quidem, quibus X est functio rationalis integra, puta

$$X=a+bx+cx^2+dx^3+ex^4+\text{etc.},$$

res nullam habet difficultatem, cum sumto

$$y=\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\text{etc.}+v$$

hi coefficientes α, β, γ etc. semper ita definiri queant, ut facta substitutione prodeat talis aequatio

$$0=v+\tfrac{dv}{dx}+\tfrac{ddv}{dx^2}+\tfrac{d^3v}{dx^3}+\text{etc.},$$

cui particulariter satisfacit valor

$$v=Ae^{x\cos.\theta}\cos.(x\sin.\theta+\zeta)$$

sumtis pro ζ et θ angulis quibuscunque. Verum ex dato eiusmodi valore ipsius X invenitur

$$\alpha=a-b,\ \beta=b-2c,\ \gamma=c-3d,\ \delta=d-4e,\ \varepsilon=e-5f\quad \text{etc.}$$

Verum in genere cum fiat

$$\tfrac{dX}{dx}=\tfrac{dy}{dx}+\tfrac{ddy}{dx^2}+\tfrac{d^3y}{dx^3}+\text{etc.},$$

evidens est semper posito $y=X-\tfrac{dX}{dx}+v$ aequationem illam transformari in hanc

$$0=v+\tfrac{dv}{dx}+\tfrac{ddv}{dx^2}+\tfrac{d^3v}{dx^3}+\text{etc.}$$

COROLLARIUM

1199. En ergo praeter expectationem integrationem completam huius aequationis differentialis in infinitum excurrentis

$$X=y+\tfrac{dy}{dx}+\tfrac{ddy}{dx^2}+\tfrac{d^3y}{dx^3}+\ldots+\text{ etc. in infinitum},$$

pro qua iam novimus esse

$$y=X-\tfrac{dX}{dx}+Ae^{x\cos.\theta}\cos.(x\sin.\theta+\zeta),$$

quod postremum membrum ob angulos ζ et θ arbitrarios in infinitum multiplicari potest. Haecque forma maxime complicatae illi ex solutione oriundae aequivalere est censenda.

PROBLEMA 159

1200. *Proposita hac aequatione differentiali*

$$X = y + \frac{ndy}{adx} + \frac{n(n-1)ddy}{1\cdot 2a^2dx^2} + \frac{n(n-1)(n-2)d^3y}{1\cdot 2\cdot 3a^3dx^3} + \text{etc.},$$

ubi quidem n sit numerus integer affirmativus, ut terminorum numerus sit finitus, eius integrale completum investigare.

SOLUTIO

Formula algebraica hinc consideranda fit

$$P = 1 + \frac{n}{1}\cdot\frac{z}{a} + \frac{n(n-1)}{1\cdot 2}\cdot\frac{z^2}{a^2} + \text{ etc.} = \left(1 + \frac{z}{a}\right)^n,$$

quae ergo meros habet factores simplices inter se aequales $z + a$. Cum igjtur sit $\frac{P}{(a+z)^n} = \frac{1}{a^n}$, ex § 1163 statim colligitur integrale quaesitum

$$y = a^n e^{-ax}\int dx\int dx\int dx\int \ldots\int e^{ax}Xdx,$$

quoad signorum integralium numerus aequetur exponenti *n*. Hanc autem formam sequenti modo in integralia simplicia resolvere licet ope reductionis generalis, qua esse novimus $\int dx\int Vdx = x\int Vdx - \int Vxdx$, unde fit

$$\int dx\int e^{ax}Xdx = x\int e^{ax}Xdx - \int e^{ax}Xxdx,$$

$$\int dx\int dx\int e^{ax}Xdx = \tfrac{1}{2}x^2\int e^{ax}Xdx - x\int e^{ax}Xxdx + \tfrac{1}{2}\int e^{ax}Xx^2dx,$$

$$\int dx\int dx\int dx\int e^{ax}Xdx = \frac{x^3\int e^{ax}Xdx - 3x^2\int e^{ax}Xxdx + 3x\int e^{ax}Xx^2dx - \int e^{ax}Xx^3dx}{1\cdot 2\cdot 3}$$

etc.

Cum igitur signorum integralium numerus sit = *n*, concludimus fore ubi cum singula integralia constantem arbitrariam implicent, manifestum est hoc integrale esse completum.

COROLLARIUM 1

1201. Si ergo esset $X = 0$, aequationis differentialis propositae integrale completum foret

$$y = e^{-ax}\left(Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + Dx^{n-4} + \ldots + Mx + N\right),$$

ubi constantium arbitrariarum *A*, *B*, *C* etc. numerus utique est = *n*.

COROLLARIUM 2

1202. Si numerus *n* fuerit infinitus simulque quantitas *a* capiatur infinita, ut sit $a = nc$, aequatio integranda in infinitum excurret eritque

$$X = y + \frac{dy}{cdx} + \frac{ddy}{1\cdot 2c^2dx^2} + \frac{d^3y}{1\cdot 2\cdot 3c^3dx^3} + \text{etc.};$$

aequatio autem integralis ad hunc casum applicata nullam lucem foeneratur.

COROLLARIUM 3

1203. Quaecunque autem *y* functio fuerit ipsius *x*, constat, si loco *x* scribatur $x + \frac{1}{c}$, eam abire in

$$y + \frac{dy}{cdx} + \frac{ddy}{1\cdot 2c^2dx^2} + \frac{d^3y}{1\cdot 2\cdot 3c^3dx^3} + \text{etc.},$$

quae cum esse debeat $= X$, vicissim patet *y* aequari ei functioni ipsius *x*, quae nascitur ex *X*, si ibi loco *x* scribatur $x - \frac{1}{c}$.

SCHOLION 1

1204. Quod quo facilius appareat, observo, si proposita fuerit quaecunque eiusmodi aequatio

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \text{etc.},$$

semper sine ulla integrationa integrale particulare per approximationem hoc modo inveniri posse. Statuatur

$$y = \alpha X + \beta \frac{dX}{dx} + \gamma \frac{ddX}{dx^2} + \delta \frac{d^3X}{dx^3} + \text{etc.}$$

factaque substitutione habebitur

$$\begin{aligned} X = A\alpha X &+ A\beta \tfrac{dX}{dx} + A\gamma \tfrac{ddX}{dx^2} + A\delta \tfrac{d^3X}{dx^3} + \text{etc.} \\ &+B\alpha \quad\; +B\beta \quad\; +B\gamma \\ &\qquad\quad\;\; +C\alpha \quad\; +C\beta \\ &\qquad\qquad\qquad\quad +D\alpha \end{aligned}$$

sicque coefficientes $\alpha, \beta, \gamma, \delta$ etc. definiuntur, ut sit $a = \frac{1}{A}$, reliqui vero

$$\beta = \frac{-B\alpha}{A} = \frac{-B}{A^2},$$
$$\gamma = \frac{C\alpha - B\beta}{A} = \frac{-C}{A^2} + \frac{BB}{A^3},$$
$$\delta = \frac{-D\alpha - C\beta - B\gamma}{A} = \frac{-D}{A^2} + \frac{2BC}{A^3} - \frac{B^3}{A^4},$$
$$\varepsilon = \frac{-E\alpha - D\beta - C\gamma - B\delta}{A} = \frac{-E}{A^2} + \frac{2BD + C^2}{A^3} - \frac{3BBC}{A^4} + \frac{B^4}{A^5},$$
etc.;

quae si accommodentur ad casum problematis, fiet

$$y = X - \frac{ndX}{1adx} + \frac{n(n+1)ddX}{1\cdot 2a^2dx^2} - \frac{n(n+1)(n+2)d^3X}{1\cdot 2\cdot 3a^3dx^3} + \text{etc.}$$

Hinc casu, quo $n = \infty$ et $a = nc$, colligitur

$$y = X - \frac{dX}{1cdx} + \frac{ddX}{1\cdot 2c^2dx^2} - \frac{d^3X}{1\cdot 2\cdot 3c^3dx^3} + \text{etc.},$$

quae expressio, etsi in infinitum excurrens, manifesto definit eam ipsius x functionem, quae nascitur ex X, si loco x scribatur $x - \frac{1}{c}$. Quodsi iam hanc novam functionem signo X' indicemus ponamusque $y = X' + v$, aequatio Corollarii 2 abit in hanc

$$0 = v + \frac{dv}{cdx} + \frac{ddv}{1\cdot 2c^2dx^2} + \frac{d^3v}{1\cdot 2\cdot 3c^3dx^3} + \text{etc.},$$

cuius integrale particulare quodcunque est $v = Ae^{-ncx}x^m$ existente n numero infinito et m numero integro positivo.

SCHOLION 2

1205. Haec me deducunt ad sequentem speculationem circa serierum summationem. Sit nempe series quaecunque

$$\overset{1}{A},\overset{2}{B},\overset{3}{C},\overset{4}{D}.....\overset{x}{T},$$

cuius terminus indici *x* respondens sit *T,* functio quaecunque ipsius *x.* Statuatur summa omnium horum terminorum

$$A+B+C+D+\ \ldots\ +T=y$$

ac perspicuum est *y* fore eiusmodi functionem ipsius *x*, ut, si in ea loco *x* scribatur $x-1$, proditura sit eadem ilIa summa *y* termino ultimo *T* mulctata, scilicet $y-T$. At loco *x* scribendo $x-1$ functio *y* abit in

$$y-\frac{dy}{dx}+\frac{ddy}{1\cdot 2dx^2}-\frac{d^3y}{1\cdot 2\cdot 3adx^3}+\text{etc.},$$

unde oritur haec aequatio

$$T=\frac{dy}{dx}-\frac{ddy}{1\cdot 2dx^2}+\frac{d^3y}{1\cdot 2\cdot 3dx^3}-\frac{d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}+\text{etc.}$$

quae semel integrata posito $\int Tdx=X$ fit

$$X=y-\frac{dy}{1\cdot 2dx}+\frac{ddy}{1\cdot 2\cdot 3dx^2}-\frac{d^3y}{1\cdot 2\cdot 3\cdot 4dx^3}+\text{etc.}$$

quam quomodo integrari conveniat, videamus, dum eam aliquanto generaliorem reddemus.

PROBLEMA 160

1206. *Proposita hac aequatione differentiali*

$$X=\frac{ny}{a}-\frac{n(n-1)dy}{1\cdot 2a^2dx}+\frac{n(n-1)(n-2)ddy}{1\cdot 2\cdot 3a^3dx^2}-\text{etc.}$$

eius integrale completum investigare.

SOLUTIO

Formetur inde haec quantitas algebraica

$$P=\frac{n}{a}-\frac{n(n-1)z}{1\cdot 2a^2}+\frac{n(n-1)(n-2)zz}{1\cdot 2\cdot 3a^3}-\text{etc.}=\frac{1-\left(1-\frac{z}{a}\right)^n}{z}$$

seu $P=\frac{a^n-(a-z)^n}{a^n z}$ cuius factor duplex quicunque hanc habebit formam

$$aa-2a(a-z)\cos.2\zeta+(a-z)^2$$

existente angulo $2\zeta=\frac{2i\pi}{n}$. Abit autem haec forma in

$$2aa(1-\cos.2\zeta)-2az(1-\cos.2\zeta)+zz=4aa\sin.^2\zeta-4az\sin.^2\zeta+zz\,,$$

quae cum generali $ff+2fz\cos.\theta+zz$ comparata dat $f=2a\sin.\zeta$ et $\cos.\theta=-\sin.\zeta$, unde $\theta=90^0+\zeta$ et $\sin.\theta=\cos.\zeta$ existente $\zeta=\frac{i\pi}{n}$. Iam ad partem integralis hinc ortam inveniendam consideretur [§ 1173–§1177] forma

$$\frac{dP}{dz}=\frac{-a^n+(a+(n-1)z)(a-z)^{n-1}}{a^n zz}$$

in qua posito

$$z=-f\left(\cos.\theta+\sqrt{-1}\cdot\sin.\theta\right)$$

seu

$$z=2a\sin.\zeta\left(\sin.\zeta-\sqrt{-1}\cdot\cos.\zeta\right)=a\left(1-\cos.2\zeta-\sqrt{-1}\cdot\sin.2\zeta\right),$$

ideoque

ut sit

$$a-z=a\left(\cos.2\zeta+\sqrt{-1}\cdot\sin.2\zeta\right),$$

prodit

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{-1+\left(n-(n-1)\left(\cos.2\zeta+\sqrt{-1}\sin.2\zeta\right)\right)\left(\cos.2(n-1)\zeta+\sqrt{-1}\sin.2(n-1)\zeta\right)}{-4aa\sin.^2\zeta\left(\cos.2\zeta+\sqrt{-1}\sin.2\zeta\right)}.$$

Cum autem sit $\cos.2n\zeta=1$ et $\sin.2n\zeta=0$, erit

$$\cos.2(n-1)\zeta=\cos.2\zeta \quad \text{et} \quad \sin.2(n-1)\zeta=-\sin.2\zeta$$

ideoque

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{-n+n\left(\cos.2\zeta-\sqrt{-1}\sin.2\zeta\right)}{-4aa\sin.^2\zeta\left(\cos.2\zeta+\sqrt{-1}\sin.2\zeta\right)},$$

quae reducitur ad hanc formam

$$\mathfrak{P}+\mathfrak{Q}\sqrt{-1}=\frac{n}{4aa\sin.^2\zeta}\left(\cos.2\zeta-\sqrt{-1}\cdot\sin.2\zeta-\cos.4\zeta+\sqrt{-1}\cdot\sin.4\zeta\right),$$

unde concluditur

$$\mathfrak{P}=\frac{n}{4aa\sin.^2\zeta}\left(\cos.2\zeta-\cos.4\zeta\right)=\frac{n}{4aa\sin.\zeta}\sin.3\zeta,$$

$$\mathfrak{Q}=\frac{-n}{4aa\sin.^2\zeta}\left(\sin.2\zeta-\sin.4\zeta\right)=\frac{n}{4aa\sin.\zeta}\cos.3\zeta,$$

sicque est

$$\mathfrak{P}\mathfrak{P}+\mathfrak{Q}\mathfrak{Q}=\frac{nn}{4a^4\sin.^2\zeta}$$

et posito $\varphi=2ax\sin.\zeta\cos.\zeta=ax\sin.2\zeta$ fiet

$$\mathfrak{P}\cos.\varphi-\mathfrak{Q}\sin.\varphi=\frac{n}{2aa\sin.\zeta}\sin.(3\zeta-\varphi)$$

et

$$\mathfrak{P}\sin.\varphi+\mathfrak{Q}\cos.\varphi=\frac{n}{2aa\sin.\zeta}\cos.(3\zeta-\varphi).$$

Quocirca integralis pars hinc oriunda erit

$$\frac{4aa\sin.\zeta}{n}e^{2ax\sin.^2\zeta}\left\{\begin{aligned}&\sin.(3\zeta-\varphi)\int e^{-2ax\sin.^2\zeta}Xdx\cos.\varphi\\&+\cos.(3\zeta-\varphi)\int e^{-2ax\sin.^2\zeta}Xdx\sin.\varphi\end{aligned}\right\},$$

ubi pro ζ successive scribi debent hi anguli $\frac{\pi}{n},\frac{2\pi}{n},\frac{3\pi}{n},\frac{4\pi}{n}$ etc., quamdiu sunt angulo recto minores; at si n sit numerus par, ad has partes insuper addi oportet $-\frac{2aa}{n}e^{2ax}\int e^{-2ax}Xdx$ sicque colligetur verus valor ipsius y.

COROLLARIUM 1

1207. Si est $X=0$, pars integralis ex quolibet angulo $\zeta=\frac{i\pi}{n}$ nata induit hanc formam

$$e^{2ax\sin.^2\zeta}\left(A\sin.(3\zeta-ax\sin.2\zeta)+B\cos.(3\zeta-ax\sin.2\zeta)\right)$$

seu hanc

$$Ae^{2ax\sin.^2\zeta}\sin.(\alpha+ax\sin.2\zeta)$$

denotante α angulum quemcunque constantem.

COROLLARIUM 2

1208. Invento integrali particulari quocunque $y = V$ quod aequationi propositae satisfaciat, si ponamus deinceps $y = V + v$, orietur haec aequatio

$$0 = \frac{nv}{a} - \frac{n(n-1)dv}{1\cdot 2a^2 dx} + \frac{n(n-1)(n-2)ddv}{1\cdot 2\cdot 3a^3 dx^2} - \text{etc.}$$

ex quo integrale completum erit

$$y = V + Ae^{2ax\sin.^2\zeta}\sin.(\alpha + ax\sin.2\zeta)$$

ultima hac parte secundum omnes valores ipsius ζ multiplicata.

COROLLARIUM 3

1209. Si sumamus $n = \infty$ et $a = n$, ut haec prodeat aequatio differentialis in infinitum excurrens

$$X = y - \frac{dy}{1\cdot 2dx} + \frac{ddy}{1\cdot 2\cdot 3dx^2} - \frac{d^3y}{1\cdot 2\cdot 3\cdot 4dx^3} + \frac{d^4y}{1\cdot 2\cdot 3\cdot 4\cdot 5dx^4} - \text{etc.},$$

erit *y* terminus summatorius progressionis, cuius terminus generalis indici *x* respondens est $T = \frac{dX}{dx}$. Quamdiu ergo angulus $\zeta = \frac{i\pi}{n}$ est infinite parvus, ob $\varphi = 2i\pi x$ integralis pars quaelibet est

$$4i\pi e^{\frac{2ii\pi\pi}{n}x}\left\{\begin{aligned}&\sin.\left(\tfrac{3i\pi}{n} - 2i\pi x\right)\int e^{\frac{-2ii\pi\pi}{n}x}Xdx\cos.2i\pi x\\ &+\cos.\left(\tfrac{3i\pi}{n} - 2i\pi x\right)\int e^{\frac{-2ii\pi\pi}{n}x}Xdx\sin.2i\pi x\end{aligned}\right\}$$

et omissis evanescentibus

$$4i\pi\Big(\cos.(2i\pi x)\int Xdx\sin.(2i\pi x) - \sin.(2i\pi x)\int Xdx\cos.(2i\pi x)\Big).$$

Si iam hic pro *i* successive omnes numeri integri 1, 2, 3 etc. substituantur, omnium formularum hoc modo resultantium summa dabit verum et completum valorem ipsius *y*.

SCHOLION

1210. Pro aequatione autem proposita methodo ante indicata integrale particulare per seriem differentialium invenire licet ponendo

$$y = AX + \frac{BdX}{dx} + \frac{CddX}{dx^2} + \frac{Dd^3X}{dx^3} + \frac{Ed^4X}{dx^4} + \text{etc.};$$

facta enim substitutione reperitur

$$A=\frac{a}{n},\ B=\frac{n-1}{2n},\ C=\frac{nn-1}{12an},\ D=\frac{nn-1}{24aan},\ E=\frac{-(nn-1)(nn-19)}{720a^3n}\text{ etc.,}$$

cuius quidem seriei difficile est legem progressionis in genere assignare. Verum pro casu $n=\infty$ et $a=n$, qui imprimis in doctrina progressionum est notatu dignus, hi coefficientes ita se habent

$$A=1,\ B=\frac{1}{2},\ \mathrm{C}=\frac{1}{12},\ D=0,\ E=\frac{-1}{720}\text{ etc.,}$$

unde ea ipsa forma oritur, quam olim in genere pro termino summatorio dedi. Concesso autem hoc termino summatorio, qui fit $=V$, probe notari convenit aequationem $y=V$ tantum esse integrale particulare aequationis propositae, completum vero facile exhiberi, si modo ad V addantur omnes huiusmodi formulae $A\sin.(\alpha+2i\pi x)$ pro i scribendo successive omnes numeros 1, 2, 3, 4 etc., ubi pro quolibet angulus α pro arbitrio assumi potest.
Quod autem singuli hi valores aequationi

$$0=v-\frac{dv}{2dx}+\frac{ddv}{6dx^2}-\frac{d^3v}{24dx^3}+\frac{d^4v}{120dx^4}-\frac{d^5v}{720dx^5}+\text{etc.}$$

satisfaciant, ita facillime ostenditur. Posito brevitatis gratia $2in=m$, ut sit $v=\sin.(\alpha+mx)$, et facta substitutione fieri debet

$$0=\sin.(\alpha+mx)\left(1-\frac{mm}{6}+\frac{m^4}{120}-\text{etc.}\right)-\cos.(\alpha+mx)\left(-\frac{m}{2}+\frac{m^3}{24}-\frac{m^5}{720}+\text{etc.}\right)$$

$$\sin.(\alpha+mx)\frac{1}{m}\sin.m-\cos.(\alpha+mx)\frac{1}{m}(\cos.m-1).$$

Cum autem sit $m=2i\pi$, manifesto est tam $\sin.m=0$ quam $\cos.m-1=0$.

PROBLEMA 161

1211. *Proposita hac aequatione differentiali*

$$X=y+\frac{n(n-1)}{1\cdot2}\cdot\frac{ddy}{a^2dx^2}+\frac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4}\cdot\frac{d^4y}{a^4dx^4}+\text{etc.}$$

eius integrate completum investigare.

SOLUTIO

Quantitas algebraica hinc formanda est

$$P=1+\frac{n(n-1)}{1\cdot2}\cdot\frac{zz}{aa}+\frac{n(n-1)(n-2)(n-3)}{1\cdot2\cdot3\cdot4}\cdot\frac{z^4}{a^4}+\text{etc.}$$

quae ad hanc formam manifesto reducitur

$$P = \tfrac{1}{2}\left(1+\tfrac{z}{a}\right)^n + \tfrac{1}{2}\left(1-\tfrac{z}{a}\right)^n = \frac{(a+z)^n+(a-z)^n}{2a^n},$$

cuius factor quicunque trinomialis est

$$(a+z)^2 - 2(aa-zz)\cos.2\zeta + (a-z)^2$$

sumendo

$$2\zeta = \frac{(2i+1)\pi}{n} \quad \text{seu} \quad \zeta = \frac{(2i+1)\pi}{2n}.$$

Haec autem forma abit in

$$2aa(1-\cos.2\zeta) + 2zz(1+\cos.2\zeta) = 4aa\sin.^2\zeta + 4zz\cos.^2\zeta,$$

qui factor generalis repraesentetur hoc modo $aa\text{tang.}^2\zeta + zz$, sicque comparatio cum forma generali $ff + 2fl\cos.\theta + zz$ praebet $f = -a\text{tang.}\zeta$ et $\theta = 90°$, unde fit $\varphi = -ax\,\text{tang.}\zeta$ (§ 1177) et valor pro z substituendus

$$-f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right) = a\,\text{tang.}\zeta\cdot y\sqrt{-1},$$

quo pacto

$$\frac{dP}{dz} = \frac{n(a+z)^{n-1} - n(a-z)^{n-1}}{2a^n}$$

abire ponitur in $\mathfrak{P} + \mathfrak{Q}\sqrt{-1}$, unde fit

$$\mathfrak{P} + \mathfrak{Q}\sqrt{-1} = \tfrac{n}{2a}\left(\left(1+\text{tang.}\zeta\cdot y\sqrt{-1}\right)^{n-1} - \left(1-\text{tang.}\zeta\cdot y\sqrt{-1}\right)^{n-1}\right)$$

$$= \frac{n}{2a\cos.^{n-1}\zeta}\left(\cos.(n-1)\zeta + \sqrt{-1}\cdot\sin.(n-1)\zeta - \cos.(n-1)\zeta + \sqrt{-1}\cdot\sin.(n-1)\zeta\right)$$

ideoque $\mathfrak{P} = 0$ et $\mathfrak{Q} = \frac{n\sin.(n-1)\zeta}{a\cos.^{n-1}\zeta}$. At ob $n\zeta = \frac{2i+1}{2}\pi$ hincque $\cos.n\zeta = 0$ et $\sin.n\zeta = \pm 1$, prout i fuerit numerus par vel impar, $\sin.(n-1)\zeta = \pm 1\cos.\zeta$ ideoque $\mathfrak{Q} = \frac{\pm n}{a\cos.^{n-2}\zeta}$.
Quocirca ob $\cos.\theta = 0$ integralis pars ex hoc factore oriunda est

$$\pm\ \frac{2a\cos.^{n-2}\zeta}{n}\left(\cos.\varphi\int X dx \sin.\varphi - \sin.\varphi\int X dx\cos.\varphi\right)$$

seu ob $\varphi = -ax\,\text{tang.}\zeta$

$$\pm \frac{2a\cos.^{n-2}\zeta}{n}\left\{\begin{array}{l}\sin.(ax\,\text{tang}.\zeta)\int Xdx\cos.(ax\,\text{tang}.\zeta)\\ -\cos.(ax\,\text{tang}.\zeta)\int Xdx\sin.(ax\,\text{tang}.\zeta)\end{array}\right\},$$

ubi pro ζ successive substitantur anguli $\frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \frac{7\pi}{2n}$ etc., quamdiu sunt recto minores, pro quibus ibi alternatim + et – scribi oportet; haeque partes omnes in unam summam collectae dabunt valorem completum ipsius *y*, dummodo pro ultima parte ex angulo $\zeta = \frac{\pi}{2}$ oriunda, quod evenit, si *n* numerus impar, eius tantum semissis capiatur [§ 1190].

COROLLARIUM 1

1212. Accommodemus haec statim ad casum $n = \infty$ et $a = nc$, ut proposita sit haec aequatio differentialis

$$X = y + \frac{ddy}{1\cdot 2c^2dx^2} + \frac{d^4y}{1\cdot 2\cdot 3\cdot 4a^4dx^4} + \frac{d^6y}{1\cdots 6a^6dx^6} + \text{etc. in infinitum.}$$

Cum igitur hic valores ipsius ζ sint infinite parvi, erit $\cos.\zeta = 1$ et $\text{tang}.\zeta = \zeta = (4i \pm 1)cx\frac{\pi}{2}$, hinc $ax\,\text{tang}.\zeta = (4i \pm 1)cx\frac{\pi}{2}$, pro quo angulo scribamus ω. Ergo pars integralis quaecunque

$$\pm 2c\left(\sin.\omega\int Xdx\cos.\omega - \cos.\omega\int Xdx\sin.\omega\right),$$

ubi signa ambigua sibi mutuo respondent.

COROLLARIUM 2

1213. Si tantum angulus $\frac{\pi}{2}$ ponatur $= \varphi$, integrale universum ita erit expressum

$$\begin{aligned}\frac{y}{2c} = &+\sin.\varphi\int Xdx\cos.\varphi - \cos.\varphi\int Xdx\sin.\varphi\\ &-\sin.3\varphi\int Xdx\cos.3\varphi + \cos.3\varphi\int Xdx\sin.3\varphi\\ &+\sin.5\varphi\int Xdx\cos.5\varphi - \cos.5\varphi\int Xdx\sin.5\varphi\\ &-\sin.7\varphi\int Xdx\cos.7\varphi + \cos.7\varphi\int Xdx\sin.7\varphi\\ &\text{etc.,}\end{aligned}$$

quae formulae in infinitum sunt continuandae.

COROLLARIUM 3

1214. Si ponamus $c = b\sqrt{-1}$, ut habeatur haec aequatio infinita

$$X = y - \frac{ddy}{1\cdot 2b^2dx^2} + \frac{d^4y}{1\cdot 2\cdot 3\cdot 4b^4dx^4} - \frac{d^6y}{1\cdots 6b^6dx^6} + \text{etc.},$$

ac iam angulum $\frac{\pi}{2}bx$ vocemus φ, erit integrale completum

$$\begin{aligned}\frac{y}{b} = &+e^{-\psi}\int e^{\psi}Xdx - e^{\psi}\int e^{-\psi}Xdx\\ &-e^{-3\psi}\int e^{3\psi}Xdx + e^{3\psi}\int e^{-3\psi}Xdx\\ &+e^{-5\psi}\int e^{5\psi}Xdx - e^{5\psi}\int e^{-5\psi}Xdx\\ &\text{etc.}\end{aligned}$$

SCHOLION

1215. Si pro aequatione Corollarii 1 methodo supra [§ 1204] exposita quaeramus integrale particulare per differentialia ipsius X expressum huncque in finem ponamus

$$y = AX - \frac{BddX}{c^2dx^2} + \frac{Cd^4X}{c^4dx^4} - \frac{Dd^6X}{c^6dx^6} + \frac{Ed^8X}{c^8dx^8} - \text{etc.},$$

reperiemus hos coefficientium valores

$$A = 1,\quad B = \frac{1}{1\cdot 2},\quad C = \frac{5}{1\cdots 4}\quad D = \frac{61}{1\cdots 6},\quad E = \frac{1385}{1\cdots 8},\quad F = \frac{50521}{1\cdots 10}\ \text{etc.}$$

Hicque valor si ponatur $= V$ vocato angulo $\frac{\pi}{2}cx = \varphi$ erit integrale completum

$$y = V + A\sin.(\alpha + \varphi) + \mathrm{B}\sin.(\beta + 3\varphi) + C\sin.(\gamma + 5\varphi) + D\sin.(\delta + 7\varphi) + \text{etc}.$$

PROBLEMA 162

1216. *Proposita aequatione differentiali*

$$X = y + \frac{n(n-1)}{1\cdot 2a^2}\cdot\frac{dy}{dx} + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4a^4}\cdot\frac{ddy}{dx^2} + \frac{n\cdots(n-5)}{1\cdots 6a^6}\cdot\frac{d^3y}{dx^3} + \text{etc.}$$

eius integrale eompletum investigare.

SOLUTIO

Quantitas algebraica hinc formanda

$$P = 1 + \frac{n(n-1)}{1\cdot 2a^2}z + \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4a^4}zz + \text{etc.}$$
$$= \tfrac{1}{2}\left(1+\frac{\sqrt{z}}{a}\right)^n + \tfrac{1}{2}\left(1-\frac{\sqrt{z}}{a}\right)^n$$

cum ex praecedente nascatur, si ibi loco zz scribatur z, sumto angulo $\zeta = \frac{2i+1}{2n}\pi$ factor quicunque erit $aa\,\text{tang.}^2\zeta + z$, ita ut huius formae omnes factores simplices sint reales. Hoc ergo factore cum formula $\alpha + z$ comparato erit $\alpha = aa\,\text{tang.}^2\zeta$ et sumto $\mathfrak{A} = \frac{P}{\alpha+z}$ posito $z = -\alpha$ erit integralis pars ex hoc factore oriunda

$$\tfrac{1}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx\,.$$

Quia vero P evanescit posito $z = -\alpha$, erit quoque $\mathfrak{A} = \frac{dP}{dz}$; at est differentiando

$$\frac{dP}{dz} = \frac{n}{4a\sqrt{z}}\left(\left(1+\frac{\sqrt{z}}{a}\right)^{n-1} - \left(1-\frac{\sqrt{z}}{a}\right)^{n-1}\right).$$

Quia igitur poni oportet $\frac{\sqrt{z}}{a} = \text{tang.}\zeta\cdot\sqrt{-1}$, erit

$$1+\frac{\sqrt{z}}{a} = \frac{\cos.\zeta + \sqrt{-1}.\sin.\zeta}{\cos.\zeta} \quad \text{et} \quad 1-\frac{\sqrt{z}}{a} = \frac{\cos.\zeta - \sqrt{-1}.\sin.\zeta}{\cos.\zeta}$$

hincque

$$\mathfrak{A} = \frac{n}{4aa\text{tang.}\zeta\sqrt{-1}}\cdot\ \frac{2\sqrt{-1}.\ \sin.(n-1)\zeta}{\cos.^{n-1}\zeta} = \frac{n\sin.(n-1)}{2aa\sin.\zeta\cos.^{n-2}\zeta}.$$

Iam observetur esse $\sin.n\zeta = \sin.(2i+1)\frac{\pi}{2} = \pm 1$ (ubi signum superius valet, si i numerus par, inferius, si impar), tum vero $\cos.n\zeta = 0$, unde fit $\sin.(n-1)\zeta = \pm\cos.\zeta$, ex quo conficitur

$$\mathfrak{A} = \frac{\pm n}{2aa\sin.\zeta\cos.^{n-3}\zeta}\,,$$

et pars integralis quaesita habebitur

$$\pm\ \frac{2aa\sin.\zeta\cos.^{n-3}\zeta}{n}e^{-aa\text{tang.}^2\zeta\cdot x}\int e^{aa\text{tang.}^2\zeta\cdot x}Xdx.$$

Nunc igitur ipsi ζ successive tribuantur hi va1ores $\frac{\pi}{2n}, \frac{3\pi}{2n}, \frac{5\pi}{2n}, \frac{7\pi}{2n}$ etc., quoad angulum rectum non superent, atque omnes istae partes in unam summam collectae dabunt integrale completum seu valorem ipsius y.

COROLLARIUM 1

1217. Si ponamus $n=\infty$ et $a=nc$, aequatio proposita in infinitum excurrit eritque

$$X = y+\frac{dy}{1\cdot 2c^2dx}+\frac{ddy}{1\cdot 2\cdot 3\cdot 4c^4dx^2}+\frac{d^3y}{1\cdots 6c^6dx^3}+\text{etc.}$$

et forma algebraica inde nata

$$P=1+\frac{z}{1\cdot 2c^2}+\frac{zz}{1\cdot 2\cdot 3\cdot 4c^4}+\frac{z^3}{1\cdots 6c^6}+\text{etc.}=\tfrac{1}{2}e^{\frac{\sqrt{z}}{c}}+\tfrac{1}{2}e^{\frac{-\sqrt{z}}{c}},$$

quae omnes factores simplices habet reales, et ob ζ infinite parvum erit $\text{tang.}\zeta=\zeta=\frac{2i+1}{2n}\pi$ indeque factorum forma generalis

$$z+\frac{(2i+1)^2}{4}\pi\pi cc \quad\text{seu}\quad 1+\frac{4z}{(2i+1)^2\pi\pi cc}.$$

COROLLARIUM 2

1218. Ponatur brevitatis gratia angulus $\frac{2i+1}{2n}\pi=\theta$; erit $aa\text{tang.}^2\zeta=\theta\theta cc$, tum vero $\cos.\zeta=1$ et $\frac{aa\sin.\zeta}{n}=\theta cc$, ex quo integralis pars quaecunque erit

$$\pm 2\theta cce^{-\theta\theta ccx}\int e^{\theta\theta ccx}Xdx,$$

ubi pro θ successive omnes hos angulos scribi oportet $\frac{\pi}{2},\frac{3\pi}{2},\frac{5\pi}{2},\frac{7\pi}{2},\frac{9\pi}{2}$ etc.

COROLLARIUM 3

1219. Perinde hic est, sive *cc* negative sive positive capiatur; hinc istius aequationis differentialis infinitae

$$X = y+\frac{dy}{1\cdot 2bdx}+\frac{ddy}{bbdx^2}+\frac{d^3y}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6b^3dx^3}+\text{etc.}$$

integrale erit

$$y=2\theta be^{-\theta\theta bx}\int e^{\theta\theta bx}Xdx$$

loco θ scribendo successive omnes hos angulos ambiguitate signi iam sublata $+\frac{\pi}{2},-\frac{3\pi}{2},+\frac{5\pi}{2},-\frac{7\pi}{2}$ etc., unde, si $X=0$, integrale particulare quodvis est $y=Ae^{-\theta\theta bx}$.

PROBLEMA 163

1220. *Proposita aequatione differentiali*

$$X = \frac{ndy}{adx} + \frac{n(n-1)(n-2)ddy}{1\cdot 2\cdot 3a^3dx^2} + \frac{n(n-1)(n-2)(n-3)(n-4)d^3y}{1\cdot 2\cdot 3\cdot 4\cdot 5a^5dx^3} + \text{etc.}$$

eius integrale completum investigare.

SOLUTIO

Etsi haec aequatio in *dx* ducta sponte semel integratur, praestat tamen hanc formam retinere, unde fit

$$P = \frac{nz}{a} + \frac{n(n-1)(n-2)zz}{1\cdot 2\cdot 3a^3} + \frac{n(n-1)(n-2)(n-3)(n-4)z^3}{1\cdot 2\cdot 3\cdot 4\cdot 5a^5} + \text{etc.},$$

quae manifesto ita exhiberi potest

$$P = \frac{\sqrt{z}}{2}\left(\left(1+\frac{\sqrt{z}}{a}\right)^n - \frac{1}{2}\left(1-\frac{\sqrt{z}}{a}\right)^n\right),$$

cuius quidem statim unus factor se offert *z*, reliqui vero in hac forma continentur

$$\left(1+\frac{\sqrt{z}}{a}\right)^2 - 2\left(1-\frac{z}{aa}\right)\cos.2\zeta + \left(1-\frac{\sqrt{z}}{a}\right)^2$$

sumto angulo $2\zeta = \frac{2i+1}{n}\pi$ seu $\zeta = \frac{i\pi}{n}$, haec vero formula abit in

$$2(1-\cos.2\zeta) + \frac{2z}{aa}(1+\cos.2\zeta),$$

unde patet in genere factorem fore $aa\ \text{tang.}^2\zeta + z$, quae etiam primum illum *z* complectitur sumto $i = 0$. Hinc posito $aa\ \text{tang.}^2\zeta = \alpha$ integralis pars huic factori respondens erit

$$\frac{1}{\mathfrak{A}}e^{-ax}\int e^{ax}Xdx,$$

si posito $z = -aa\ \text{tang.}^2\zeta$ seu $\sqrt{z} = a\ \text{tang.}\zeta\sqrt{-1}$ seu capiatur

$$\mathfrak{A} = \frac{dP}{dz} = \frac{1}{4\sqrt{z}}\left(\left(1+\frac{\sqrt{z}}{a}\right)^n - \frac{1}{2}\left(1-\frac{\sqrt{z}}{a}\right)^n\right) + \frac{n}{4a}\left(\left(1+\frac{\sqrt{z}}{a}\right)^{n-1} + \frac{1}{2}\left(1-\frac{\sqrt{z}}{a}\right)^{n-1}\right);$$

at

$$\left(1+\frac{\sqrt{z}}{a}\right)^n = \frac{\cos.n\zeta+\sqrt{-1}\cdot\sin.n\zeta}{\cos.^n\zeta} \quad \text{et} \quad \left(1-\frac{\sqrt{z}}{a}\right)^n = \frac{\cos.n\zeta-\sqrt{-1}\cdot\sin.n\zeta}{\cos.^n\zeta},$$

quamobrem fiet

$$\mathfrak{A} = \frac{\sin.n\zeta}{2a\,\text{tang}.\zeta\cos.^n\zeta} + \frac{n\cos.(n-1)\zeta}{2a\cos.^{n-1}\zeta} = \frac{\pm n}{2a\cos.^{n-2}\zeta}$$

ob $\sin.n\zeta = 0$ et $\cos.n\zeta = \pm 1$, prout numerus *i* fuerit vel par vel impar. Quocirca integralis pars quaecunque ita erit expressa

$$\pm\frac{2a\cos.^{n-2}\zeta}{n} e^{-\alpha x}\int e^{\alpha x} X dx$$

existente $\alpha = aa\,\text{tang}.^2\zeta$. Iam angulo ζ successive tribuantur hi valores $\frac{0\pi}{n}, \frac{1\pi}{n}, \frac{2\pi}{n}, \frac{3\pi}{n}$ etc., quoad angulum rectum $\frac{\pi}{2}$ non excedant, haeque formulae omnes cum suis signis in unam summam coniectae dabunt valorem completum pro *y*.

COROLLARIUM 1

1221. Prima igitur integralis pars nascitur ex angulo $\zeta = 0$, unde ea erit $\frac{2a}{n}\int X dx$, cuius autem loco ob rationes supra [§ 1190] allegatas circa factores simplices eius tantum dimidium sumi debet, ut haec prima pars sit $=\frac{a}{n}\int X dx$, quod etiam inde patet, quod posito $z = 0$ fiat manifesto $\frac{P}{z} = \frac{n}{a}$.

COROLLARIUM 2

1222. Idem tenendum esset de parte ultima, siquidem ex valore $\zeta = \frac{\pi}{2}$ nascatur, quod evenit, si *n* sit numerus par. Quia vero hoc casu fit $\cos.\zeta = 0$, haec tota integralis pars per se evanescit.

COROLLARIUM 3

1223. Si esset $X = 0$, quaelibet pars integralis foret $Ae^{-aa\text{tang}.^2\zeta\cdot x}$ denotante *A* quantitatem constantem arbitrariam foretque adeo haec aequatio $y = Ae^{-aa\text{tang}.^2\zeta\cdot x}$ integrale particulare aequationis, dummodo capiatur angulus $\zeta = \frac{i\pi}{n}$.

SCHOLION

1224. Hinc posito $n=\infty$ et $a=n\sqrt{b}$ integrari potest haec aequatio differentialis in infinitum excurrens

$$\frac{X}{\sqrt{b}}=\frac{dy}{1\cdot bdx}+\frac{ddy}{1\cdot 2\cdot 3bbdx^2}+\frac{d^3y}{1\cdot 2\cdot 3\cdot 4\cdot 5b^3dx^3}+\frac{d^4y}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7b^4dx^4}+\text{etc.}$$

vel etiam haec per unam integrationem ex ista nata

$$\sqrt{b}\cdot\int Xdx=\frac{y}{1}+\frac{dy}{1\cdot 2\cdot 3bdx}+\frac{ddy}{1\cdot 2\cdot 3\cdot 4\cdot 5b^2dx^2}+\frac{d^3y}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7b^3dx^3}+\text{etc.}$$

Cum enim sit angulus $\zeta=\frac{i\pi}{n}$ infinite parvus, erit $\cos.\zeta=1$ et $a\text{tang.}\zeta=a\zeta=i\pi\sqrt{b}$ ideoque $\alpha=aa\text{ tang.}^2\zeta=ii\pi\pi b$; habebitur pars integralis quaecunque

$$\pm 2\sqrt{b}\cdot e^{-ii\pi\pi bx}\int e^{ii\pi\pi bx}Xdx$$

unde parte prima ex $i=0$ nata ad dimidium reducta ob rationes supra [§ 1190] allegatas erit integrale completum

$$\frac{y}{\sqrt{b}}=\int Xdx-2e^{-\pi\pi bx}\int e^{\pi\pi bx}Xdx+2e^{-4\pi\pi bx}\int e^{4\pi\pi bx}Xdx$$
$$-2e^{-9\pi\pi bx}\int e^{9\pi\pi bx}Xdx+2e^{-16\pi\pi bx}\int e^{16\pi\pi bx}Xdx-\text{etc.}$$

EXEMPLUM

1225. *Sit* $n=6$ *et* $a=1$, *ut integranda proponatur haec aequatio*

$$X=\frac{6dy}{dx}+\frac{20ddy}{dx^2}+\frac{6d^3y}{dx^3}\quad seu\quad \int Xdx=6y+20\frac{dy}{dx}+\frac{6ddy}{dx^2}.$$

Valores ergo pro angulo ζ et inde pendentes sunt

$\zeta=0$,	$30°$,	$60°$,
$\cos.\zeta=1$,	$\frac{\sqrt{3}}{2}$,	$\frac{1}{2}$,
$\alpha=0$,	$\frac{1}{3}$,	3,

ex quibus colligitur integrale quaesitum

$$y=\frac{1}{6}\int Xdx-\frac{3}{16}e^{-\frac{1}{3}x}\int e^{\frac{1}{3}x}Xdx+\frac{1}{48}e^{-3x}\int e^{3x}Xdx,$$

quod etiam aequationi satisfacere tentanti patebit.