CHAPTER III

CONCERNING THE INTEGRATION OF DIFFERENTIAL EQUATIONS OF THIS FORM

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \text{etc.}$$

PROBLEM 147

1138. *For the proposed differential equation*

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + ... + \frac{Nd^ny}{dx^n}$$

on assuming the element dx constant and with X signifying some function of x, to find the function of x, being multiplied by which the equation becomes integrable.

SOLUTION

Let Pdx be this multiplier, which we seek, and since the first member X is returned integrable by that, there is required to give an account of defining P from the other member. But the form of this multiplier P is easily understood to be of this kind $e^{\lambda x}$, thus so that the quantity λ must be defined. Therefore in the first place let $e^{\lambda x}dx$ be the multiplier and it is required that this form

$$e^{\lambda x}dx\left(Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + ... + \frac{Nd^ny}{dx^n}\right)$$

be integrabile, and therefore an integral of this form may be put in place

$$e^{\lambda x}\left(A'y + \frac{B'dy}{dx} + \frac{C'ddy}{dx^2} + ... + \frac{M'd^{n-1}y}{dx^{n-1}}\right),$$

thus so that the differential of this must agree with that form ; which since there shall be

$$e^{\lambda x}dx\left(\lambda A'y + \frac{\lambda B'dy}{dx} + \frac{\lambda C'ddy}{dx^2} + ... + \frac{\lambda M'd^{n-1}y}{dx^{n-1}} + \frac{A'dy}{dx} + \frac{B'ddy}{dx^2} + ... + \frac{M'd^ny}{dx^n}\right),$$

it is necessary that [,on equating the coefficients of the derivatives,]

$$A' = \frac{A}{\lambda},\ B' = \frac{B-A'}{\lambda},\ C' = \frac{C-B'}{\lambda},\ D' = \frac{D-C'}{\lambda}, \ldots\ M' = \frac{M-L'}{\lambda}$$

and $M' = N$. Hence there will be

$$\begin{aligned} A' &= \frac{A}{\lambda},\\ B' &= \frac{B}{\lambda} - \frac{A}{\lambda^2},\\ C' &= \frac{C}{\lambda} - \frac{B}{\lambda^2} + \frac{A}{\lambda^3},\\ D' &= \frac{D}{\lambda} - \frac{C}{\lambda^2} + \frac{B}{\lambda^3} - \frac{A}{\lambda^4},\\ &\vdots\\ M' &= \frac{M}{\lambda} - \frac{L}{\lambda^2} + \frac{K}{\lambda^3} - \ldots \pm \frac{A}{\lambda^n}, \end{aligned}$$

and

$$0 = \frac{N}{\lambda} - \frac{M}{\lambda^2} + \frac{L}{\lambda^3} - \ldots \mp \frac{A}{\lambda^{n+1}},$$

where the quantity λ must be elicited from the final equation, which equation adopts this form

$$A - B\lambda + C\lambda^2 - D\lambda^3 + E\lambda^4 - \ldots \pm N\lambda^n = 0;$$

from which since λ can be obtained with *n* values, all the multipliers also can be found.

We may consider how these determinations for the individual values of the exponent *n* may themselves be put in place.

I. If *n* = 1, there will be $A - B\lambda = 0$, and then $A' = \frac{A}{\lambda} = B$.

II. If *n* = 2, there will be $A - B\lambda + C\lambda^2 = 0$, and then

$$A' = \frac{A}{\lambda} = B - C\lambda \text{ and } B' = \frac{B\lambda - A}{\lambda^2} = C.$$

III. If *n* = 3, there will be $A - B\lambda + C\lambda^2 - D\lambda^3 = 0$, and then

$$A' = \frac{A}{\lambda} = B - C\lambda + D\lambda^2, \quad B' = \frac{B\lambda - A}{\lambda^2} = C - D\lambda$$

and

$$C' = \frac{C\lambda^2 - B\lambda + A}{\lambda^3} = D.$$

IV. If *n* = 4, there will be $A - B\lambda + C\lambda^2 - D\lambda^3 + E\lambda^4 = 0$, then indeed

$$\begin{aligned} A' &= \frac{A}{\lambda} = B - C\lambda + D\lambda^2 - E\lambda^3, \quad & B' &= \frac{B\lambda - A}{\lambda^2} = C - D\lambda + E\lambda^2,\\ C' &= \frac{C\lambda^2 - B\lambda + A}{\lambda^3} = D - E\lambda, & D' &= \frac{D\lambda^3 - C\lambda^2 + B\lambda - A}{\lambda^4} = E. \end{aligned}$$

V. If $n=5$, there will be $A-B\lambda+C\lambda^2-D\lambda^3+E\lambda^4-F\lambda^5=0$, then truly

$$A'=\tfrac{A}{\lambda}=B-C\lambda+D\lambda^2-E\lambda^3+F\lambda^4,$$
$$B'=\tfrac{B\lambda-A}{\lambda^2}=C-D\lambda+E\lambda^2-F\lambda^3,$$
$$C'=\tfrac{C\lambda^2-B\lambda+A}{\lambda^3}=D-E\lambda+F\lambda^2,$$
$$D'=\tfrac{D\lambda^3-C\lambda^2+B\lambda-A}{\lambda^4}=E-F\lambda,$$
$$E'=\tfrac{E\lambda^4-D\lambda^3+C\lambda^2-B\lambda+A}{\lambda^5}=F$$

and thus so forth.

But with this multiplier $e^{\lambda x}dx$ found , the previous member of the equation shall be $\int e^{\lambda x}Xdx$ and the proposed equation, which is of a differential of order *n*, by integration is reduced by one order simpler to

$$\int e^{\lambda x}Xdx=e^{\lambda x}\left(A'y+\frac{B'dy}{dx}+\frac{C'ddy}{dx^2}+....+\frac{M'd^{n-1}y}{dx^{n-1}}\right).$$

COROLLARY 1

1139. Therefore by integration this first proposed equation established is reduced by a single order and with the coefficients defined *A'*, *B'*, *C'* etc. from the above principles the equation of the integral can be shown by this formula

$$e^{-\lambda x}\int e^{\lambda x}Xdx=A'y+\frac{B'dy}{dx}+\frac{C'ddy}{dx^2}+....+\frac{M'd^{n-1}y}{dx^{n-1}}.$$

COROLLARY 2

1140. Since the first member $e^{-\lambda x}\int e^{\lambda x}Xdx$ shall be a function of *x* involving an arbitrary constant, if in place of this there is put *X'*, then this equation has a similar form and and that proposition likewise by the same method integrated again can be reduced to a differential equation of order $n-2$, which will be considered to be of this form

$$X''=A''y+B''\frac{dy}{dx}+C''\frac{ddy}{dx^2}+....+L''\frac{d^{n-2}y}{dx^{n-2}}.$$

COROLLARY 3

1141. By progressing further in this manner finally a differential equation of the first order may be come upon :

$$X^{(n-1)} = A^{(n-1)}y + B^{(n-1)}\tfrac{dy}{dx},$$

which is reduced in a similar manner to a finite equation $X^{(n)} = A^{(n)}y$, from which the relation between the variables *x* and *y* is expressed.

SCHOLIUM

1142. This therefore is the method by which differential equations of this kind of higher orders are to be integrated through the orders successively , where just as many integrations are needed as the proposed equation has orders of the differential. Hence the whole matter rests on the finding of the successive coefficients, which are required to be defined with the aid of the preceding multipliers. In general indeed the rule, by which these are to be determined continually from the preceding, thus has not been made clear, so that in this manner the form of the furthest integral should become evident; indeed since we have learned from the case in the above chapter, in which the first member *X* vanishes, also that the final integral can be established extremely well by a simple rule, which here likewise we come to distrust deservedly on using that simplest rule, if gradually we progress from the lower orders to the higher orders. [Difficulties arise on making roots of the algebraic equation equal due to infinities, as we shall discover.] And indeed in the first case, in which the equation is a differential of the first order, $X = Ay + B\tfrac{dy}{dx}$, the multiplier will be $e^{\lambda x}dx$ on putting $A - \lambda B = 0$, as there becomes $A' = \tfrac{A}{\lambda}$, and since there shall be $A' = \tfrac{A}{\lambda} = B$, the integral will be

$$\int e^{\lambda x}Xdx = Be^{\lambda x}y \quad \text{or} \quad e^{-\lambda x}\int e^{\lambda x}Xdx = By.$$

According to this similarity, we work towards the equations of higher orders and we investigate the form of the integral of the highest order.

PROBLEM 148

1143. *With the proposed differential equation of the second order*

$$X = Ay + B\tfrac{dy}{dx} + C\tfrac{ddy}{dx^2}$$

to investigate the relation between x and y by a twofold integration..

SOLUTION

Let $e^{\lambda x}dx$ be the multiplier returning this equation integrable itself and there will be $A - B\lambda + C\lambda^2 = 0$; then there is taken $A' = \tfrac{A}{\lambda} = B - C\lambda$ and $B' = \tfrac{B\lambda - A}{\lambda^2} = C$

and on putting

$$e^{-\lambda x}\int e^{\lambda x}Xdx = X'$$

the equation integrated once is

$$X' = A'y + B'\tfrac{dy}{dx}.$$

Now the multiplier of this shall be $e^{\mu x}dx$ and there will be $A' - B'\mu = 0$ and there is put $A'' = \tfrac{A'}{\mu} = B'$ and on putting

$$e^{-\mu x}\int e^{\mu x}X'dx = X'',$$

we will have

$$X'' = A''y,$$

which is the equation twice integrated expressing the relation sought between *x* and *y*. [The reader may note that Euler has tranferred his attention from exponent $n = 2$ to exponent $n = 1$ in iterating the solution, using an extra dash on the coefficients, a device that he continued to use to indicate the coefficents of an equation reduced by an order: the use of indices to represent such orders and changes succinctly had not yet been introduced.]

Therefore since here there shall be $A'' = B'$ and $B' = C$, there will be $A'' = C$. From that on putting in place the values A' and B' the equation $A' - B'\mu = 0$ adopts the form

$$B - C\lambda - C\mu = 0 \text{ or } B - C(\lambda + \mu) = 0; \text{[since } A' = \tfrac{A}{\lambda} = B - C\lambda = \mu B' = \mu\tfrac{B\lambda - A}{\lambda^2} = \mu C\text{]}$$

from which since there shall be $\lambda + \mu = \tfrac{B}{C}$, it is apparent that $\lambda + \mu$ is equal to the sum of the two roots of the equation $A - B\lambda + C\lambda^2 = 0$. Therefore since λ is one root of this, μ by necessity denotes the other root of this. Whereby if from the equation proposed, as we have done in the preceding chapter, we may form this equation $A + Bz + Czz = 0$, the roots of this will $z = -\lambda$ and $z = -\mu$. Or if we put in place the factors of this $C(\alpha + z)(\beta + z)$, the letters α and β will give the values λ and μ. Hence since there shall be

$$X' = e^{-\alpha x}\int e^{\alpha x}Xdx,$$

then there will be

$$X'' = e^{-\beta x}\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx.$$

But

$$\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx = \tfrac{1}{\beta-\alpha}e^{(\beta-\alpha)x}\int e^{\alpha x}Xdx - \tfrac{1}{\beta-\alpha}\int e^{\beta x}Xdx,$$

from which there is concluded

$$X'' = \frac{1}{\beta-\alpha}e^{-\alpha x}\int e^{\alpha x}Xdx + \frac{1}{\alpha-\beta}e^{-\beta x}\int e^{\beta x}Xdx.$$

On account of which the complete integral of the proposed equation is

$$Cy = \frac{1}{\beta-\alpha}e^{-\alpha x}\int e^{\alpha x}Xdx + \frac{1}{\alpha-\beta}e^{-\beta x}\int e^{\beta x}Xdx.$$

where the letters α and β thus are to be taken, so that there shall be

$$A + Bz + Czz = C(\alpha + z)(\beta + z).$$

COROLLARY 1

1144. If these two factors shall be equal or $\beta = \alpha$, there will be

$$X'' = e^{-\alpha x}\int dx\int e^{\alpha x}Xdx = e^{-\alpha x}x\int e^{\alpha x}Xdx - e^{-\alpha x}\int e^{\alpha x}Xxdx$$

and thus in this case

$$A + Bz + Czz = C(\alpha + z)^2$$

and the integral of our equation is

$$Cy = e^{-\alpha x}\left(x\int e^{\alpha x}Xdx - \int e^{\alpha x}Xxdx\right).$$

COROLLARY 2

1145. If the two factors shall be imaginary, which comes about if

$$A + Bz + Czz = C(ff + 2fzcos.\theta + zz),$$

there will be

$$\alpha = f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right) \text{ and } \beta = f\left(\cos.\theta - \sqrt{-1}\cdot\sin.\theta\right),$$

hence

$$e^{\alpha x} = e^{fx\cos.\theta}\left(\cos.(fx\sin.\theta) + \sqrt{-1}\cdot\sin.(fx\sin.\theta)\right)$$

and

$$e^{\beta x} = e^{fx\cos.\theta}\left(\cos.(fx\sin.\theta) - \sqrt{-1}\cdot\sin.(fx\sin.\theta)\right)$$

and

$$\beta - \alpha = -2\sqrt{-1}\cdot f\sin.\theta.$$

COROLLARY 3

1146. Where we may be able to substitute these easier, if for the sake of brevity,

$$e^{fx\cos.\theta}=m,\quad \cos.(fx\sin.\theta)=p \quad \text{and} \quad \sin.(fx\sin.\theta)=q,$$

so that there shall be

$$e^{\alpha x}=mp+mq\sqrt{-1} \ \text{ and } \ e^{\beta x}=mp-mq\sqrt{-1}.$$

Hence there will become

$$\int e^{\alpha x}Xdx=\int mpXdx+\int mqXdx\sqrt{-1}$$

and

$$\int e^{\beta x}Xdx=\int mpXdx-\int mqXdx\sqrt{-1}$$

Then indeed there shall be

$$e^{-\alpha x}=\tfrac{p-q\sqrt{-1}}{m} \ \text{ and } e^{-\beta x}=\tfrac{p+q\sqrt{-1}}{m}$$

COROLLARY 4

1147. From these we deduce

$$e^{-\alpha x}\int e^{\alpha x}Xdx=\tfrac{p}{m}\int mpXdx-\tfrac{q\sqrt{-1}}{m}\int mpXdx+\tfrac{p\sqrt{-1}}{m}\int mqXdx+\tfrac{q}{m}\int mqXdx$$

and on taking $\sqrt{-1}$ negative there arises $e^{-\beta x}\int e^{\beta x}Xdx$, which form thus subtracted leaves

$$-\tfrac{2q\sqrt{-1}}{m}\int mpXdx+\tfrac{2p\sqrt{-1}}{m}\int mqXdx,$$

and this remainder must be divided by $\beta-\alpha=-2\sqrt{-1}\cdot f\sin.\theta$. From which we deduce the integral

$$Cy=\tfrac{q}{mf\sin.\theta}\int mpXdx-\tfrac{p}{mf\sin.\theta}\int mqXdx.$$

COROLLARY 5

1148. With the assumed values restored for *m, p, q* and the integral of our equation, if it should be

$$A + Bz + Czz = C\left(ff + 2fz\cos.\theta + zz \right),$$

will be

$$Cy = e^{-fx\cos.\theta}\left\{ \begin{array}{l} \frac{\sin.(fx\sin.\theta)}{f\sin.\theta}\int e^{fx\cos.\theta}Xdx\cos.(fx\sin.\theta) \\ -\frac{\cos.(fx\sin.\theta)}{f\sin.\theta}\int e^{fx\cos.\theta}Xdx\sin.(fx\sin.\theta) \end{array} \right\},$$

which expression therefore is equivalent to that, if α and β may prevail with imaginary values.

PROBLEM 149

1149. *With the proposed equation of the third order*

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3}$$

to find the complete integral by a threefold integration.

SOLUTION

On putting the multiplier $e^{\lambda x}dx$ in place there must be $A - B\lambda + C\lambda^2 - D\lambda^3 = 0$; then there is taken $A' = B - C\lambda + D\lambda^2$, $B' = C - D\lambda$ and $C' = D$ and on putting

$$e^{-\lambda x}\int e^{\lambda x}Xdx = X'$$

the equation integrated once is given

$$X' = A'y + B'\frac{dy}{dx} + C'\frac{ddy}{dx^2}.$$

Again the multiplier of this is put in place $e^{\mu x}dx$, so that there shall be $A' - B'\mu + C'\mu^2 = 0$, and there is taken $A'' = B' - C'\mu$ and $B'' = C'$ and on putting

$$e^{-\mu x}\int e^{\mu x}X'dx = X''$$

the second equation of the integral is

$$X'' = A''y + B''\tfrac{dy}{dx},$$

the multiplier of this will be $e^{vx}dx$ on taking $A'' - B''v = 0$; but on putting $A''' = B''$ there will be the third equation of the integral

$$e^{-vx}\int e^{vx}X''dx = A'''y = Dy;$$

therefore the quantities λ, μ, v are required to be found. Now in the first place there is

$$A - B\lambda + C\lambda^2 - D\lambda^3 = 0,$$

then

$$B - C(\lambda + \mu) + D(\lambda\lambda + \lambda\mu + \mu\mu) = 0$$

[For $A' = B - C\lambda + D\lambda^2$, $B' = C - D\lambda$ and $C' = D$; $A' - B'\mu + C'\mu^2 = 0$, or $B - C\lambda + D\lambda^2 - (C - D\lambda)\mu + D\mu^2 = 0$, giving $B - C(\lambda + \mu) + D(\lambda\lambda + \lambda\mu + \mu\mu) = 0$.]
and on account of

$$A'' = C - D(\lambda + \mu) \text{ and } B'' = D$$

there will be in the third place

$$C - D(\lambda + \mu + v) = 0,$$

from which latter equality it is apparent that $\lambda + \mu + v$ is equal to the sum of the roots of the first equation, of which λ is a single root. But that μ and v shall be the remaining roots, is shown in this manner. The equation may be examined

$$A + Bz + Czz + Dz^3 = 0;$$

of which if one root shall be $z = -\lambda$ or $\lambda + z$ one factor, the equation is divided by that and there will be produced

$$Dz^2 + (C - D\lambda)z + B - C\lambda + D\lambda\lambda = 0,$$

[We know that $A - B\lambda + C\lambda\lambda - D\lambda^3 = 0$, or $\lambda\left(B - C\lambda + D\lambda^2\right) + Bz + Czz + Dz^3 = 0$

giving, $\left(Dz^2 + (C - D\lambda)z + B - C\lambda + D\lambda\lambda\right)(z + \lambda) = 0$.]

which is the second equation itself $C'zz + B'z + A' = 0$, the roots of which are $z = -\mu$ and $z = -v$ or the factors $(\mu + z)(v + z)$, as we have shown in the preceding problem. Whereby if the factors of the formula $A + Bz + Cz^2 + Dz^3$ shall be

$$D(\alpha + z)(\beta + z)(v + z),$$

there is put in place on finding the final integral

$$e^{-\alpha x}\int e^{\alpha x}Xdx = X', \quad e^{-\beta x}\int e^{\beta x}X'\,dx = X'' \quad \text{and} \quad e^{-\gamma x}\int e^{\gamma x}X''\,dx = X'''$$

and there will be

$$Dy = X'''.$$

Now by the reduction of the integration there is, as we have seen above,

$$X'' = \tfrac{1}{\beta-\alpha}e^{-\alpha x}\int e^{\alpha x}Xdx + \tfrac{1}{\alpha-\beta}e^{-\beta x}\int e^{\beta x}Xdx$$

and hence again

$$\int e^{\gamma x}X''\,dx = \tfrac{1}{(\beta-\alpha)(\gamma-\alpha)}e^{(\gamma-\alpha)x}\int e^{\alpha x}Xdx - \tfrac{1}{(\beta-\alpha)(\gamma-\alpha)}\int e^{\gamma x}Xdx$$
$$+\tfrac{1}{(\alpha-\beta)(\gamma-\beta)}e^{(\gamma-\beta)x}\int e^{\beta x}Xdx - \tfrac{1}{(\alpha-\beta)(\gamma-\beta)}\int e^{\gamma x}Xdx,$$

where the two final terms taken together give

$$\tfrac{1}{(\alpha-\gamma)(\beta-\gamma)}\int e^{\gamma x}Xdx.$$

On account of which the integral sought is

$$Dy = \tfrac{e^{-\alpha x}\int e^{\alpha x}Xdx}{(\beta-\alpha)(\gamma-\alpha)} + \tfrac{e^{-\beta x}\int e^{\beta x}Xdx}{(\alpha-\beta)(\gamma-\beta)} + \tfrac{e^{-\gamma x}\int e^{\gamma x}Xdx}{(\alpha-\gamma)(\beta-\gamma)}.$$

COROLLARIUM 1

1150. If two factors of the formula $A+Bz+Cz^2+Dz^3$ were equal, for example $\gamma=\beta$, there will be

$$\int e^{\beta x}X''\,dx = \tfrac{1}{(\beta-\alpha)^2}e^{(\beta-\alpha)x}\int e^{\alpha x}Xdx - \tfrac{1}{(\beta-\alpha)^2}\int e^{\beta x}Xdx$$
$$+\tfrac{1}{(\alpha-\beta)}x\int e^{\beta x}Xdx - \tfrac{1}{(\alpha-\beta)}\int e^{\beta x}Xxdx$$

and thus the integral in this case will be

$$Dy = \frac{e^{-\alpha x}\int e^{\alpha x}Xdx - e^{-\beta x}\int e^{\beta x}Xdx}{(\beta-\alpha)^2} + \frac{e^{-\beta x}x\int e^{\beta x}Xdx - e^{-\beta x}\int e^{\beta x}Xxdx}{\alpha-\beta}$$

COROLLARY 2

1151. If all the three factors should be equal, or $\alpha = \beta = \gamma$, then there will be

$$e^{\alpha x} X'' = \int dx \int e^{\alpha x} X dx = x \int e^{\alpha x} X dx - \int e^{\alpha x} X x dx$$

and

$$e^{\alpha x} X''' = \int e^{\alpha x} X'' dx = \int dx \int dx \int e^{\alpha x} X dx$$

or

$$e^{\alpha x} X''' = \tfrac{1}{2} xx \int e^{\alpha x} X dx - x \int e^{\alpha x} X x dx + \tfrac{1}{2} \int e^{\alpha x} X x x dx,$$

from which the integral in this case will be

$$Dy = \tfrac{1}{2} e^{-\alpha x} \left(xx \int e^{\alpha x} X dx - 2x \int e^{\alpha x} X x dx + \int e^{\alpha x} X x x dx \right)$$

or

$$Dy = e^{-\alpha x} \int dx \int dx \int e^{\alpha x} X dx.$$

SCHOLION

1152. In general also by no reduction of the integrals summoned can the integral of our equation be expressed thus, so that there shall be

$$Dy = e^{-\gamma x} \int e^{(\gamma - \beta) x} dx \int e^{(\beta - \alpha) x} dx \int e^{\alpha x} X dx$$

on putting

$$A + Bz + Cz^2 + Dz^3 = D(\alpha + z)(\beta + z)(v + z),$$

where it is especially worthy of note to have occured, that the three letters α, β, γ are permitted to be interchanged between themselves in any manner, thus so that the expression of the integral can be varied in six ways. Also in the preceding problem [§1143], where only two factors occur

$$C(\alpha + z)(\beta + z) = A + Bz + Czz ,$$

the complete integral of the equation

$$X = Ay + B\frac{dy}{dx} + \frac{Cddy}{dx^2}$$

thus can be shown

$$Cy = e^{-\beta x} \int e^{(\beta - \alpha) x} dx \int e^{\alpha x} X dx$$

and on permuting the letters α and β in this manner also

$$Cy = e^{-\alpha x}\int e^{(\alpha-\beta)x}dx\int e^{\beta x}Xdx.$$

If the equality of which formulas should be evident, for that to be attempted is readily apparent, that also is indicated in the preceding variation. Indeed let $e^{-\alpha x}\int e^{\alpha x}Xdx = X'$; for the above the formula will be

$$Dy = e^{-\gamma x}\int e^{(\gamma-\beta)x}dx\int e^{\beta x}X'dx \ ;$$

since the equal of that shall be this

$$Dy = e^{-\beta x}\int e^{(\beta-\gamma)x}dx\int e^{\gamma x}X'dx,$$

and also for X' with the value restored

$$Dy = e^{-\beta x}\int e^{(\beta-\gamma)x}dx\int e^{(\gamma-\alpha)x}dx\int e^{\alpha x}Xdx,$$

this only differs from the first, because the letters β and γ are interchanged. But also the letters β and γ with α are able to be permuted, moreover this is more difficult to show, but however it is evident from the reduction used in the solution, and thus from the nature of the solution.

PROBLEM 150

1153. *With the proposed differential equation of the fourth order*

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + E\frac{d^4y}{dx^4}$$

on taking the element dx constant, and on denoting some function X of x, to investigate the integral of this.

SOLUTION

An algebraic formula that may be called in aid being easily formed from the proposed equation

$$A + Bz + Czz + Dz^3 + Ez^4 = P,$$

which may be resolved into its own simple factors, so that there shall be

$$P = E(\alpha+z)(\beta+z)(\gamma+z)(\delta+z),$$

and the multiplier returning our equation integrable will be $e^{\lambda x}dx$ on taking λ to be equal to one one the letters $\alpha,\beta,\gamma,\delta$. Therefore there is taken $\lambda=\alpha$, so that the multiplier will be $e^{\alpha x}dx$, and on putting

$$e^{-\alpha x}\int e^{\alpha x}Xdx = X'$$

the equation integrated once will be

$$X' = A'y + B'\tfrac{dy}{dx} + C'\tfrac{ddy}{dx^2} + D'\tfrac{d^3y}{dx^3}$$

where A', B', C', D' thus are to be determined, so that there shall be

$$A' = \tfrac{A}{\alpha}, \quad B' = \tfrac{B\alpha - A}{\alpha^2}, \quad C' = \tfrac{C\alpha^2 - B\alpha + A}{\alpha^3}, \quad D' = \tfrac{D\alpha^3 - C\alpha^2 + B\alpha - A}{\alpha^4}$$

or

$$A' = \tfrac{A}{\alpha}, \quad B' = \tfrac{B - A'}{\alpha}, \quad C' = \tfrac{C - B'}{\alpha}, \quad D' = \tfrac{D - C'}{\alpha}$$

or also

$$A = A'\alpha,\ B = B'\alpha + A',\ C = C'\alpha + B',\ D = D'\alpha + C'.$$

From which determinations it will be evident, if there is put

$$A' + B'z + C'z^2 + D'z^3 = Q,$$

that this formula Q comes from the formula P, if this is divided by $\alpha+z$, thus so that there will be

$$Q = \tfrac{P}{\alpha+z} = E(\beta+z)(\gamma+z)(\delta+z).$$

Therefore in the same manner we may put in place the second differentiation with the aid of the multiplier $e^{\beta x}dx$ and on putting

$$e^{-\beta x}\int e^{\beta x}X'dx = X''$$

the equation of the integral will be

$$X'' = A''y + B''\tfrac{dy}{dx} + C''\tfrac{ddy}{dx^2}$$

with the coefficients A'', B'', C'' taken thus, so that there shall be

$$A''+B''z+C''z^2=\frac{P}{(\alpha+z)(\beta+z)}=E(\gamma+z)(\delta+z).$$

Hence again with the aid of the multiplier $e^{\gamma x}dx$ we will find for the integration if we put

$$e^{-\gamma x}\int e^{\gamma x}X''dx=X''',$$

that

$$X''=A'''y+B'''\frac{dy}{dx}$$

with the equation arising

$$A'''+B'''z=\frac{P}{(\alpha+z)(\beta+z)(\gamma+z)}=E(\delta+z).$$

And at last with the aid of the multiplier $e^{\delta x}dx$ on putting the form

$$e^{-\delta x}\int e^{\delta x}X'''dx=X''''$$

the final integral is found

$$X''''=A''''y$$

on setting $A''''=E$. Therefore on gathering everything together, the integral sought will be

$$Ey=e^{-\delta x}\int e^{(\delta-\gamma)x}dx\int e^{(\gamma-\beta)x}dx\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx,$$

which expression now can be put together without any ambiguity from the resolution of the principal form

$$P=A+Bz+Cz^2+Dz^3+Ez^4$$

evidently in the factored form

$$P=E(\alpha+z)(\beta+z)(\gamma+z)(\delta+z),$$

where it is to be observed, whatever the order of the letters $\lambda, \mu, \nu, \delta$ interchanged, there must always be the same value produced for Ey.

COROLLARY 1

1154. Since there shall be $X' = e^{-\alpha x}\int e^{\alpha x}Xdx$, there will be, as we have seen now,

$$X'' = e^{-\beta x}\int e^{\beta x}X'dx = e^{-\beta x}\left(\frac{e^{(\beta-\alpha)x}}{\beta-\alpha}\int e^{\alpha x}Xdx - \frac{1}{\beta-\alpha}\int e^{\beta x}Xdx\right)$$

or

$$X'' = \frac{e^{-\alpha x}\int e^{\alpha x}Xdx}{\beta-\alpha} + \frac{e^{-\beta x}\int e^{\beta x}Xdx}{\alpha-\beta}.$$

COROLLARY 2

1155. Again on account of $X''' = e^{-\gamma x}\int e^{\gamma x}X''dx$ in a like manner the reduction may be put in place

$$X''' = \frac{e^{-\alpha x}\int e^{\alpha x}Xdx}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\gamma x}\int e^{\gamma x}Xdx}{(\beta-\alpha)(\alpha-\gamma)} + \frac{e^{-\beta x}\int e^{\beta x}Xdx}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x}Xdx}{(\alpha-\beta)(\beta-\gamma)},$$

which is reduced to this form

$$X''' = \frac{e^{-\alpha x}\int e^{\alpha x}Xdx}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x}Xdx}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x}Xdx}{(\alpha-\gamma)(\beta-\gamma)}.$$

COROLLARIUM 3

1156. Hence in a similar manner the value X'''' is found, where indeed it suffices for the first member to be elicited, clearly from which on account of permutations the remainder may be formed at once. In this manner the integral of our equation may be found expressed in this form

$$Ey = \frac{e^{-\alpha x}\int e^{\alpha x}Xdx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x}Xdx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}$$
$$+ \frac{e^{-\gamma x}\int e^{\gamma x}Xdx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} + \frac{e^{-\delta x}\int e^{\delta x}Xdx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}.$$

SCHOLIUM

1157. If two or more roots shall be equal or imaginary, the integrals found must be transformed, which we will investigate in what follows – and this may be seen to be more suitable with the latter form, by which the transformations may be repeated.

Thus for the equality of the factors if there shall be $\delta = \gamma$, the two latter members only require a reduction ; according to finding which there is put $\delta = \gamma - \omega$ and the penultimate member will be $\frac{e^{-\gamma x}\int e^{\gamma x}Xdx}{\omega(\alpha-\gamma)(\beta-\gamma)}$, but for the final one it is to be noted that

$$\frac{1}{\alpha-\delta}=\frac{1}{\alpha-\gamma+\omega}=\frac{1}{\alpha-\gamma}-\frac{\omega}{(\alpha-\gamma)^2} \quad \text{and} \quad \frac{1}{\beta-\delta}=\frac{1}{\beta-\gamma}-\frac{\omega}{(\beta-\gamma)^2}$$

and hence

$$\frac{1}{(\alpha-\delta)(\beta-\delta)}=\frac{1}{(\alpha-\gamma)(\beta-\gamma)}+\frac{\omega(2\gamma-\alpha-\beta)}{(\alpha-\gamma)^2(\beta-\gamma)^2},$$

from which

$$\frac{1}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}=\frac{1}{\omega(\alpha-\gamma)(\beta-\gamma)}+\frac{2\gamma-\alpha-\beta}{(\alpha-\gamma)^2(\beta-\gamma)^2}.$$

Then truly for the numerator there will be

$$e^{-\delta x}=e^{-\gamma x}(1+\omega x) \quad \text{et} \quad e^{\delta x}=e^{\gamma x}(1-\omega x)$$

and thus

$$e^{-\delta x}\int e^{\delta x}Xdx=e^{-\gamma x}\int e^{\gamma x}Xdx+e^{-\gamma x}\omega x\int e^{\gamma x}Xdx-\omega e^{-\gamma x}\int e^{\gamma x}Xxdx$$

and hence the two final members on account the terms divided by ω cancel each other out and change into this form

$$\frac{(2\gamma-\alpha-\beta)e^{-\gamma x}\int e^{\gamma x}Xdx}{(\alpha-\gamma)^2(\beta-\gamma)^2}+\frac{e^{-\gamma x}x\int e^{\gamma x}Xdx-e^{-\gamma x}\int e^{\gamma x}Xxdx}{(\alpha-\gamma)(\beta-\gamma)},$$

which expression also was elicited previously. The general problem can be solved in the same manner.

PROBLEMA 151

1158. *With the proposed differential equation of any order*

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + ... + N\frac{d^ny}{dx^n},$$

on taking the element dx constant and with X denoting some function of x, to investigate the integral of this.

SOLUTION

The algebraic formula

$$A + Bz + Cz^2 + Dz^3 + ... + Nz^n = P,$$

may be formed from this equation which can be resolved into simple factors, so that there shall be

$$P = N(\alpha + z)(\beta + z)(\gamma + z)...(\nu + z),$$

the number of which is *n*. Since if now we may progress continually by individual integrations, we arrive finally at that outermost integral equation

$$Ny = e^{-\nu x}\int e^{(\nu-\mu)x}dx\int e^{(\mu-\lambda)x}dx\int ......\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx,$$

or since the factors are allowed to be permuted among themselves, there will be also

$$Ny = e^{-\alpha x}\int e^{(\alpha-\beta)x}dx\int e^{(\beta-\gamma)x}dx\int ......\int e^{(\mu-\nu)x}dx\int e^{\nu x}Xdx.$$

Now this expression can be resolved into the following parts by similar reductions to those which we have used above; towards representing these more suitably, there shall be put in place for the sake of brevity :

$$(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)....(\nu-\alpha) = \alpha',$$
$$(\alpha-\beta)(\gamma-\beta)(\delta-\beta)....(\nu-\beta) = \beta',$$
$$(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)......(\nu-\gamma) = \gamma',$$
$$\vdots$$
$$(\alpha-\nu)(\beta-\nu)(\gamma-\nu)......(\mu-\nu) = \nu'$$

and hence

$$Ny = \frac{1}{\alpha'} e^{-\alpha x} \int e^{\alpha x} X dx + \frac{1}{\beta'} e^{-\beta x} \int e^{\beta x} X dx + \frac{1}{\gamma'} e^{-\gamma x} \int e^{\gamma x} X dx + \ldots + \frac{1}{\nu'} e^{-\nu x} \int e^{\nu x} X dx.$$

But lest there should be a need according to the values α', β', γ' etc. found, that all the factors in turn be multiplied together, since there shall be

$$\frac{P}{N(\alpha+z)} = (\beta+z)(\gamma+z)(\delta+z).....(\nu+z),$$

it is clear that this formula gives the value α', if in that there is put $z=-\alpha$; but in this case both the numerator and the denominator of the fraction $\frac{P}{N(\alpha+z)}$ vanish, from which the value of this is $\frac{dP}{Ndz}$. Whereby since there is

$$P = A + Bz + Cz^2 + Dz^3 + \ldots + Nz^n,$$

then there will be

$$\frac{dP}{dz} = B + 2Cz + 3Dz^2 + \ldots + nNz^{n-1},$$

which expression may be called Q, from which it is apparent that

$$\alpha' = \frac{Q}{N} \text{ on putting } z=-\alpha, \beta' = \frac{Q}{N} \text{ on putting } z=-\beta, \gamma' = \frac{Q}{N} \text{ on putting } z=-\gamma, \text{etc.}$$

Or since with these values put in place the equation of the integal can be divided by N [see Cor. I], the following values are collected together

$$B - 2C\alpha + 3D\alpha^2 - 4E\alpha^3 + ... \pm nN\alpha^{n-1} = \mathfrak{A},$$
$$B - 2C\beta + 3D\beta^2 - 4E\beta^3 + ... \pm nN\beta^{n-1} = \mathfrak{B},$$
$$B - 2C\gamma + 3D\gamma^2 - 4E\gamma^3 + ... \pm nN\gamma^{n-1} = \mathfrak{C},$$
$$\vdots$$
$$B - 2C\nu + 3D\nu^2 - 4E\nu^3 + ... \pm nN\nu^{n-1} = \mathfrak{N},$$

with which put in place the integral sought will be

$$y = \frac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X dx + \frac{1}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X dx + \frac{1}{\mathfrak{C}} e^{-\gamma x} \int e^{\gamma x} X dx + \text{etc.},$$

since all the factors have been introduced into the calculation.

COROLLARY 1

1159. Since there shall be

$$\alpha' = \frac{\mathfrak{A}}{N}, \quad \beta' = \frac{\mathfrak{B}}{N}, \quad \gamma' = \frac{\mathfrak{C}}{N}, \quad \text{etc.}$$

then

$$\mathfrak{A} = N\alpha', \quad \mathfrak{B} = N\beta', \quad \mathfrak{C} = N\gamma', \quad \text{etc.}$$

Hence on account of

$$P = A + Bz + Cz^2 + Dz^3 + ... + Nz^n = N(\alpha + z)(\beta + z)(\gamma + z)....(\gamma + z)$$

there will be

$$\mathfrak{A} = \frac{P}{\alpha+z} \text{ on putting } z = -\alpha, \mathfrak{B} = \frac{P}{\beta+z} \text{ on putting } z = -\beta, \mathfrak{C} = \frac{P}{\gamma+z} \text{ on putting } z = -\gamma$$

and so on.

COROLLARY 2

1160. Therefore the rule itself for finding the complete integral of this proposed equation thus may be had :

The algebraic formula may be formed from this

$$A + Bz + Cz^2 + Dz^3 + ... + Nz^n = P,$$

all the factors of which are simple, which shall be

$$\alpha + z, \ \beta + z, \ \gamma + z, \ \delta + z \quad \text{etc.},$$

of which the number n is equal to the total, then with the individual factors from these the following constant quantities are defined $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc., so that there shall be

$$\mathfrak{A} = \frac{P}{\alpha+z} \text{ on putting } z = -\alpha, \text{or } \mathfrak{A} = B - 2C\alpha + 3D\alpha^2 - 4E\alpha^3 + ... \pm nN\alpha^{n-1},$$

$$\mathfrak{B} = \frac{P}{\beta+z} \text{ on putting } z = -\beta, \text{or } \mathfrak{B} = B - 2C\beta + 3D\beta^2 - 4E\beta^3 + ... \pm nN\beta^{n-1},$$

$$\mathfrak{B} = \frac{P}{\gamma+z} \text{ on putting } z = -\gamma, \text{ or } \mathfrak{B} = B - 2C\gamma + 3D\gamma^2 - 4E\gamma^3 + ... \pm nN\gamma^{n-1},$$

etc.;

with these found, the integral sought will be

$$y = \frac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X dx + \frac{1}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X dx + \frac{1}{\mathfrak{C}} e^{-\gamma x} \int e^{\gamma x} X dx + \text{etc.},$$

which form depends on as many parts as there are simple factors in the formula P.

COROLLARY 3

1161. Since in this manner the integral depends on as many parts as there are orders of the proposed differential equation, and each single part by integration brings in an arbitrary constant, it is evident that with the aid of this rule, the integral is complete.

SCHOLIUM

1162. Hence the integration of differential equations of this kind is troubled with no further difficulty, but only if all the factors of this algebraic formula *P* are simple, or what amounts to the same thing, if all the roots of this algebraic equation

$$A+Bz+Cz^2+Dz^3+...+Nz^n=0,$$

can be assigned by the number *n*. Now here cases of two kinds occur, in which the integration is strongly impeded, when evidently either two or more of the simple factors become equal to each other or become imaginery, from which indeed in the latter case only this inconvenience arises, because some parts of the integral involve imaginary quantities, but which with the reduction performed cancel each other out. Now in the first case the parts arising from equal factors thus becomes infinite, moreover thus affected by different signs, so that jointly they do not refer to any finite quantity, the value of which could be found by many circuitous methods, where it is to be noted properly that the part of the integral found for the rest of the integral, which agrees with nonequal factors, hence by no means is disturbed. Moreover I will explain a method applied to this end in the following problem.

PROBLEM 152

1163. *With the proposed equation of any order*

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + E\frac{d^4y}{dx^4} + ... + N\frac{d^ny}{dx^n}$$

if the algebraic form thus made

$$P = A + Bz + Cz^2 + Dz^3 + ... + Nz^n$$

should have two or more simple factors are equal to each other, to investigate the part of the integral arising from this.

SOLUTION

In the first place let there be two factors $\alpha + z$ and $\beta + z$ equal to each other, or $\beta = \alpha$, the remaining of the form *P* shall now be a factor $= Q$, so that there may be considered

$$P = (\alpha + z)(\beta + z)Q = (\alpha + z)^2 Q;$$

moreover on putting $z = -\alpha$, Q becomes $\mathfrak{C}$. Now initially at least the letters α and β may be considered as different only for the quantity $\mathfrak{C}$, which shall be the same for both, and with the two parts of the integral, we will have from these two factors arising

$$\mathfrak{A} = (\beta - \alpha)\mathfrak{C} \text{ and } \mathfrak{B} = (\alpha - \beta)\mathfrak{C}.$$

But the parts of the integral arising from this may be described by the letter *v*, [this is just an abbreviation] so that there shall be

$$(\beta - \alpha)\mathfrak{C}v = e^{-\alpha x}\int e^{\alpha x} Xdx - e^{-\beta x}\int e^{\beta x} Xdx,$$

from which on differentiating we gather

$$(\beta - \alpha)\mathfrak{C}dv = -\alpha e^{-\alpha x}dx\int e^{\alpha x} Xdx + \beta e^{-\beta x}dx\int e^{\beta x} Xdx,$$

to that there is added the former multiplied by βdx and there becomes

$$(\beta - \alpha)\mathfrak{C}dv + (\beta - \alpha)\mathfrak{C}\beta vdx = (\beta - \alpha)e^{-\alpha x}dx\int e^{\alpha x} Xdx,$$

which divided by $\beta - \alpha$ and multiplied by e^{ax} on account of $\beta = \alpha$ gives the integral

$$\mathfrak{C}e^{\alpha x}v = \int dx \int e^{\alpha x} X dx .$$

On account of which, in place of the two parts arising from the equality of the factors $\alpha + z$ and $\beta + z$ it is necessary for this formula to be written

$$v = \tfrac{1}{\mathfrak{C}} e^{-\alpha x} \int dx \int e^{\alpha x} X dx ,$$

where $\mathfrak{A}$ arises from the form $\frac{P}{(\alpha+z)^2}$ on putting $z = -\alpha$.

Now we may put in place the formula P having three equal factors, so that there shall be $\alpha + z = \beta + z = \gamma + z$, which indeed initially we may consider as different. Hence we put

$$P = (\alpha + z)(\beta + z)(\gamma + z)Q$$

and Q becomes $\mathfrak{M}$ on putting $z = -\alpha$ and for the parts of the integral we will have

$$\mathfrak{A} = (\beta - \alpha)(\gamma - \alpha)\mathfrak{M}, \quad \mathfrak{B} = (\alpha - \beta)(\gamma - \beta)\mathfrak{M}, \quad \mathfrak{C} = (\alpha - \gamma)(\beta - \gamma)\mathfrak{M} .$$

Hence, if the sum of the three parts of the integral, which we seek, we may denote by the letter v, there will be

$$\mathfrak{M}v = \frac{e^{-\alpha x}\int e^{\alpha x} X dx}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x} X dx}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} X dx}{(\alpha-\gamma)(\beta-\gamma)}$$

Now since there will be

$$\frac{1}{(\beta-\alpha)(\gamma-\alpha)} + \frac{1}{(\alpha-\beta)(\gamma-\beta)} + \frac{1}{(\alpha-\gamma)(\beta-\gamma)} = 0,$$

on differentiating there will be

$$\frac{\mathfrak{M}dv}{dx} = -\frac{\alpha e^{-\alpha x}\int e^{\alpha x} X dx}{(\beta-\alpha)(\gamma-\alpha)} - \frac{\beta e^{-\beta x}\int e^{\beta x} X dx}{(\alpha-\beta)(\gamma-\beta)} - \frac{\beta e^{-\gamma x}\int e^{\gamma x} X dx}{(\alpha-\gamma)(\beta-\gamma)} ;$$

to which if the first multiplied by α is added, there becomes

$$\mathfrak{M}\left(\frac{dv}{dx} + \alpha v\right) = \frac{e^{-\beta x}\int e^{\beta x} X dx}{\gamma-\beta} + \frac{e^{-\gamma x}\int e^{\gamma x} X dx}{\beta-\gamma(\alpha-\gamma)} .$$

This equation can be differentiated again, so that there is produced

$$\mathfrak{M}\left(\frac{ddv}{dx^2} + \frac{\alpha dv}{dx}\right) = -\frac{\beta e^{-\beta x}\int e^{\beta x} X dx}{\gamma-\beta} - \frac{\gamma e^{-\gamma x}\int e^{\gamma x} X dx}{\beta-\gamma(\alpha-\gamma)} ;$$

to which if that multiplied by β is added, there arises

$$\mathfrak{M}\left(\tfrac{ddv}{dx^2}+\tfrac{2\alpha dv}{dx}+\alpha\alpha v\right)=e^{-\gamma x}\int e^{\gamma x}Xdx=e^{-\alpha x}\int e^{\alpha x}Xdx,$$

from which now all the troublesome terms have been removed. Now it is multiplied by $e^{\alpha x}dx$ and an integration will give

$$\mathfrak{M}e^{\alpha x}\left(\tfrac{dv}{dx}+\alpha v\right)=\int dx\int e^{\alpha x}Xdx,$$

which on multiplying again by dx shall become integrable, and there is produced

$$\mathfrak{M}e^{\alpha x}v=\int dx\int dx\int e^{\alpha x}Xdx.$$

On account of which if the form P should have a cubic factor $(\alpha+z)^3$, the quantity $\mathfrak{A}$ is sought, so that there shall be $\mathfrak{M}=\frac{P}{(\alpha+z)^3}$ on putting $z=-\alpha$, and the part of the integral hence arising shall be

$$\tfrac{1}{\mathfrak{M}}e^{-\alpha x}\int dx\int dx\int e^{\alpha x}Xdx,$$

In a similar manner if the formula P should have four equal factors, so that there shall be $P=(\alpha+z)^4Q$, there may be taken $\mathfrak{N}=\frac{P}{(\alpha+z)^4}$ or on putting $z=-\alpha$ into $\mathfrak{N}=Q$ and the part of the integral arising from this will be

$$\tfrac{1}{\mathfrak{N}}e^{-\alpha x}\int dx\int dx\int dx\int e^{\alpha x}Xdx,$$

and we may designate the signs and thus also the cases in which the formula P, now having many equal factors, may be resolved easily.

Note. This whole solution is faulty, as evidently the quantities α,β,γ etc., which are put equal, may be considered as different, yet for the individual members the quantity $\mathfrak{M}$ is assumed to retain the same value. But if indeed the letters α,β,γ etc. are considered to disagree in turn from each other by an infinitely small amount, it is required to acknowledge also an infinitely small amount in the values indicated for the letter $\mathfrak{M}$, from which, since the individual parts of the integral become infinite, and with these described the infinite members mutually cancel each other, and from the infinitely small differences of the parts of the letter $\mathfrak{M}$, that also emerge finite. The correction of these errors clearly is desired from the following Probl. 154, while the equal factors in the particular equation are taken together. But I have preferred to leave this labour of correction to the industry of readers in order that this work be freed from such an error ; indeed often it is a benefit to have more errors, in which also it touches on exercises come upon to be preserved, from which the studious may add more to their knowledge of these things, [yet] there needs to be a warning from so much careful consideration, lest our minds wander from too much reasoning.

[Thus, Euler had a problem at this stage concerning the two methods, and he was always keen to show the trials gone through in his investigations before he reached a final acceptable form ; in this case, the first generates a series of integrals in a straight forwards manner involving X, X', X'', etc. in turn, or in terms of a product of integrals via the exponentials of the roots and their differences, and which eventually gives the solution required, and into which one can substitute formally equal roots without any trouble ; on the other hand, if one integrates by parts on performing each iteration, there accumulates a number of difference of root terms in the denominator which become infinite if some of these roots are made equal. Clearly there should be some way of removing such infinities and obtaining the same result as the first method, so that the analysis is consistent by both methods. Euler applied his thinking cap and found the solution on realizing that some basic arithmetical property was present, and upon establishing a certain Arithemetic Theorem in what follows he was able to lay to rest some of his uncertainties, but not those for the multiple roots. A simpler and perhaps more elegant method appeared when Cauchy showed how such problems could be solved using complex integration, in which inverse zeros or simple poles could be integrated around along the real axis in the complex plane, by means of his celebrated Residue Theorem. This approach is referred to in the *O.O*. edition, but which I have not thought fit to inserted here, as it takes us out of the historical sequence. The reference for this is work is : *Oeuvres complètes d'Augustin Cauchy*, II^{e} série, t. VI. Paris 1888, p.252, and the original work *Exércises de Mathematiques* (1826)].

COROLLARY 1

1164. Here everything is noteworthy, since these formulas

$$dv+\alpha vdx,\ ddv+2\alpha dxdv+\alpha^2vdx^2,\ d^3v+3\alpha dxddv+3\alpha^2dx^2dv+\alpha^3vdx^2$$

and this in general :

$$d^nv+\tfrac{n}{1}\alpha d^{n-1}v+\tfrac{n(n-1)}{1\cdot 2}\alpha^2dx^2d^{n-2}v+\tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\alpha^3dx^3d^{n-3}v+\text{etc.},$$

if it should be multiplied once by e^{ax}, successively permit as many integrations as there are units contained in the index *n*, thus so that the last integral shall be $e^{ax}v$.

COROLLARY 2

1165. But the reason for this phenomenon is clear from this, because, if the formula $e^{ax}v$ should be continually differentiated on taking the element dx constant, these formulas are produced multiplied by e^{ax}, so that there shall be

$$d^n.e^{\alpha x}v=e^{\alpha x}\left(d^nv+\tfrac{n}{1}\alpha d^{n-1}v+\tfrac{n(n-1)}{1\cdot 2}\alpha^2dx^2d^{n-2}v+\text{etc.}\right).$$

COROLLARY 3

1166. And equally worthy of note is the other phenomenon, because this solution offers to us, evidently on taking any of the numbers $\alpha, \beta, \gamma, \delta$ etc. the following equations always to be present, so that there shall be

$$\frac{1}{\alpha-\beta}+\frac{1}{\beta-\alpha}=0,$$

$$\frac{1}{(\alpha-\beta)(\alpha-\gamma)}+\frac{1}{(\beta-\alpha)(\beta-\gamma)}+\frac{1}{(\gamma-\alpha)(\gamma-\beta)}=0,$$

$$\frac{1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)}+\frac{1}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)}+\frac{1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)}+\frac{1}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)}=0$$

etc.,

for however many numbers may be taken in this manner.

COROLLARY 4

1167. If the formula P is put in place resolved into simple factor

$$P=N(\alpha+z)(\beta+z)(\gamma+z)\ldots(\mu+z)(\nu+z),$$

the expression of the integral first found (§ 1158), which was

$$Ny=e^{-\alpha x}\int e^{(\alpha-\beta)x}dx\int e^{(\beta-\gamma)x}dx\int \ldots\ldots\int e^{(\mu-\nu)x}dx\int e^{\nu x}Xdx,$$

on account of equal factors does not imply any difficulty. but the latter form [§1160], in which the integral may be shown distributed in parts arising from the individual factors and which may be considered much more convenient to use, with that a more difficult presentation was required .

SCHOLIUM

1168. The phenomenon observed in Corollary 3 therefore merits more attention, because it can be transferred also to ordinary arithmetic, where it may be observed that it is not without a significent use, especially because the demonstration of this shall not be at all easy, but must be returned from the inner places of deeper analysis ; for I do not consider myself alienated, if I concede a place here to this conspicuous arithmetical theorem and therefore even more, because the solution of this problem set out here without the demonstration of this theorem would be less than perfect.

AN ARITHMETICAL THEOREM

1169. *If whatever number of numbers a, b, c, d etc. should be considered, while from any individual number the rest are subtracted, and from these the extended sequence may be formed*

$$(a-b)(a-c)(a-d)(a-e) \text{ etc.} = \mathfrak{A},$$
$$(b-a)(b-c)(b-d)(b-e) \text{ etc.} = \mathfrak{B},$$
$$(c-a)(c-b)(c-d)(c-e) \text{ etc.} = \mathfrak{C},$$
$$(d-a)(d-b)(d-c)(d-e) \text{etc.} = \mathfrak{D}$$
$$\text{etc.},$$

then there will be had always

$$\tfrac{1}{\mathfrak{A}} + \tfrac{1}{\mathfrak{B}} + \tfrac{1}{\mathfrak{C}} + \tfrac{1}{\mathfrak{D}} + \text{etc.} = 0.$$

DEMONSTRATION

This fraction may be considered following the principles treated in the *Introductione ad Analysin* [*i. e.* Eulers' *Introduction to Analysis*, Book I, Ch. II, which provides a simple introduction to partial fractions]:

$$\frac{Z}{(z-a)(z-b)(z-c)(z-d) \text{ etc.}},$$

where *Z* denotes an integral rational function of *z* of this kind, in which the highest power of *z* shall be less than the number of factors of the denominator; and this fraction can be resolved into these simple fractions, to which it shall be equal with these taken jointly, evidently to

$$\frac{A}{z-a} + \frac{B}{z-b} + \frac{C}{z-c} + \frac{D}{z-d} + \text{ etc.}$$

For which resolution, we may suppose that the numerator $Z = z^n$ with the whole number *n* present being smaller than the number of factors contained in the denominator,

[Thus, we may put in place $\frac{A(z-b)(z-c)...+B(z-a)(z-c)(z-d)+\text{etc.}}{(z-a)(z-b)(z-c)(z-d)\text{etc.}} = \frac{z^n}{(z-a)(z-b)(z-c)(z-d)\text{etc.}}$, for some choice of *A*, *B*, *C*, etc., and where the order of *n* for *z* in the numerator is less than in the denominator; for on cancelling the *z* – *a* term in the denominator, and then setting *z* = *a*, we find the following, which might be reconciled by a limiting approach :]

and these numerators may be defined thus, so that there shall be :

$$A = \frac{a^n}{(a-b)(a-c)(a-d)\text{ etc.}},$$
$$B = \frac{b^n}{(b-a)(b-c)(b-d)\text{ etc}},$$
$$C = \frac{c^n}{(c-a)(c-b)(c-d)\text{etc.}}$$
etc.

Therefore since these fractions taken negative, clearly

$$\frac{A}{a-z} + \frac{B}{b-z} + \frac{C}{c-z} + \frac{D}{d-z} + \text{etc.}$$

on being added to the proposed fraction become zero, if z shall be the last of the proposed numbers a, b, c, d etc., of which therefore the amount [incl. zero as a power in the numerator] is greater than $n+1$, there is put

$$(a-b)(a-c)(a-d).....(a-z) = \mathfrak{A},$$
$$(b-a)(b-c)(b-d).....(b-z) = \mathfrak{B},$$
$$(c-a)(c-b)(c-d)......(c-z) = \mathfrak{C},$$
$$(d-a)(d-b)(d-c).....(d-z) = \mathfrak{D}$$
$$\vdots$$
$$(z-a)(z-b)(z-c)......(z-y) = \mathfrak{Z},$$

so that the [final] proposed fraction shall be $\frac{z^n}{\mathfrak{Z}}$. And hence it is evident that the sum of all these fractions shall be

$$\frac{a^n}{\mathfrak{A}} + \frac{b^n}{\mathfrak{B}} + \frac{c^n}{\mathfrak{C}} + \frac{d^n}{\mathfrak{D}} + ... + \frac{z^n}{\mathfrak{Z}} = 0,$$

while $n+1$shall be less than the number of terms. Therefore on taking $n = 0$ the case of the Theorem arises.

COROLLARY 1

1170. These definitions if transferred to the numbers $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. above (§ 1160), where some small distinction in the constitution of the factors is to be noted properly [$a \to \alpha, b \to \beta,$ etc., and n = 0, 1, 2, etc.], there is understood to be

$$+\tfrac{1}{\mathfrak{A}}+\tfrac{1}{\mathfrak{B}}+\tfrac{1}{\mathfrak{C}}+\tfrac{1}{\mathfrak{D}}+\text{etc.}=0,$$
$$-\tfrac{\alpha}{\mathfrak{A}}-\tfrac{\beta}{\mathfrak{B}}-\tfrac{\gamma}{\mathfrak{C}}-\tfrac{\delta}{\mathfrak{D}}-\text{etc.}=0,$$
$$+\tfrac{\alpha^2}{\mathfrak{A}}+\tfrac{\beta^2}{\mathfrak{B}}+\tfrac{\gamma^2}{\mathfrak{C}}+\tfrac{\delta^2}{\mathfrak{D}}+\text{etc.}=0,$$
$$-\tfrac{\alpha^3}{\mathfrak{A}}-\tfrac{\beta^3}{\mathfrak{B}}-\tfrac{\gamma^3}{\mathfrak{C}}-\tfrac{\delta^3}{\mathfrak{D}}-\text{etc.}=0,$$
etc.,

while this form is come upon

$$\pm\tfrac{\alpha^{n-1}}{\mathfrak{A}}\pm\tfrac{\beta^{n-1}}{\mathfrak{B}}\pm\tfrac{\gamma^{n-1}}{\mathfrak{C}}\pm\tfrac{\delta^{n-1}}{\mathfrak{D}}\pm\text{etc.},$$

the sum of which no longer vanishes [as it lies outside the bounds of the arithemtical theorem, instead it is calculated from the ongoing analysis], but is equal to the fraction $\frac{1}{N}$.

COROLLARIUM 2

1171. This also can be deduced from the presentation of the form of the theorem summoned. And indeed if this should be put in place

$$\frac{z^{n-1}}{(z-a)(z-b)(z-c).....(z-y)}$$

with the number of all the letters present *a, b, c* etc. $=n$, because here z^{n-1} has as many dimensions, as there are factors in the denominator, the integral part contained in this fraction is unity ; which also is conserved in the resolution made and in the application to the case mentioned will change into $\frac{1}{N}$.

SCHOLION

1172. Finally, after the demonstration of this theorem from the last part it can be shown, how the above integral (§ 1160) of the differential equation shown is able to satisfy the same proposition. For with the expressions noted in §1170 since above we have found the integral

$$y=\tfrac{1}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx+\tfrac{1}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx+\text{etc.},$$

there will by on differentiating continually :

$$\frac{dy}{dx} = -\frac{\alpha}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx + \frac{-\beta}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx - \text{etc.},$$
$$\frac{ddy}{dx^2} = +\frac{\alpha^2}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx + \frac{\beta^2}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx + \text{etc.},$$
$$\frac{d^3y}{dx^3} = -\frac{\alpha^3}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx - \frac{\beta^3}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx - \text{etc.}$$
etc.

as far as
$$\frac{d^{n-1}y}{dx^{n-1}} = \pm\frac{\alpha^{n-1}}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx \pm \frac{\beta^{n-1}}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx \pm \text{etc.},$$

from which the following form of the differential results

$$\frac{d^ny}{dx^n} = \mp\frac{\alpha^n}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx \mp \frac{\beta^n}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx \mp \text{etc.}$$
$$\pm\left(\frac{\alpha^{n-1}}{\mathfrak{A}} + \frac{\beta^{n-1}}{\mathfrak{B}} + \frac{\gamma^{n-1}}{\mathfrak{C}} + \text{etc}\right)X,$$

because the last member will change into $\frac{1}{N}X$.
If now all these forms may be multiplied by the quantities $A, B, C, D, ...N$, because there is
$$A - B\alpha + C\alpha^2 - D\alpha^3 + ... \mp N\alpha^n = 0,$$
$$A - B\beta + C\beta^2 - D\beta^3 + ... \mp N\beta^n = 0,$$
etc.,

because therefore $\alpha + z,\ \beta + z,\ \gamma + z$ etc. are the factors of the form

$$A + Bz + Cz^2 + Dz^3 + ... + Nz^n,$$
clearly we will obtain

$$Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + ... + N\frac{d^ny}{dx^n} = X,$$

which is the initial differential equation proposed itself.

PROBLEM 153

1173. *With the proposed differential equation of any order*

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \ldots + N\frac{d^ny}{dx^n}$$

if hence the algebraic expression formed

$$P = A + Bz + Cz^2 + Dz^3 + \ldots + Nz^n$$

may have two simple imaginary factors contained by the square factor $ff + 2fz\cos.\theta + zz$ *, to find the integral of the parts hence arising.*

SOLUTION

Let $\alpha + z$ and $\beta + z$ be these two imaginary factors, so that there shall be

$$\alpha = f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right) \text{ and } \beta = f\left(\cos.\theta - \sqrt{-1}\cdot\sin.\theta\right)$$

on account of $(\alpha + z)(\beta + z) = ff + 2fz\cos.\theta + zz$, and there is put in place

$$P = (ff + 2fz\cos.\theta + zz)Q,$$

with Q being

$$Q = A' + B'z + C'z^2 + \ldots + N'z^{n-2}.$$

Therefore since the parts of the integral shall have come from these simple imaginary factors

$$\tfrac{1}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx + \tfrac{1}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx = v,$$

these imaginary values are required to lead to real values. But the imaginary quantities $\mathfrak{A}$ and $\mathfrak{B}$ shall be resulting from the forms $\left(f\cos.\theta \mp \sqrt{-1}\cdot f\sin.\theta + z\right)Q$, if in place of z there should be written $-f\cos.\theta \mp \sqrt{-1}\cdot f\sin.\theta$. But with this substitution made there becomes

$$Q = A' - B'f\cos.\theta + C'ff\cos.2\theta - D'f^3\cos.3\theta + \text{etc.}$$
$$\mp\sqrt{-1}\cdot B'f\sin.\theta \pm \sqrt{-1}\cdot C'ff\sin.2\theta \mp \sqrt{-1}\cdot D'f^3\sin.3\theta \pm \text{etc.}$$

For the sake of brevity we may put

$$A' - B'\, f\cos.\theta + C'\, ff\cos.2\theta - D'\, f^{3}\cos.3\theta + \text{etc.} = \mathfrak{M}$$

and

$$-B'\, f\sin.\theta + C'\, ff\sin.2\theta - D'\, f^{3}\sin.3\theta + \text{etc.} = \mathfrak{N},$$

so that there shall be $Q = \mathfrak{M} \pm \mathfrak{N}\sqrt{-1}$, where the upper ambiguity of the signs appears for the letters α and $\mathfrak{A}$, the lower for the letters β and $\mathfrak{B}$. Hence therefore there will be

$$\mathfrak{A} = -2\sqrt{-1}\cdot f\sin.\theta\left(\mathfrak{M} + \mathfrak{N}\sqrt{-1}\right) \text{ and } \mathfrak{B} = +2\sqrt{-1}\cdot f\sin.\theta\left(\mathfrak{M} - \mathfrak{N}\sqrt{-1}\right)$$

and thus

$$2v\sqrt{-1}\cdot f\sin.\theta = -\frac{e^{-\alpha x}\int e^{\alpha x}Xdx}{\mathfrak{M}+\mathfrak{N}\sqrt{-1}} + \frac{e^{-\beta x}\int e^{\beta x}Xdx}{\mathfrak{M}-\mathfrak{N}\sqrt{-1}}.$$

Truly there is

$$e^{\alpha x} = e^{f\cos.\theta}\left(\cos.(fx\sin.\theta) + \sqrt{-1}\cdot\sin.(fx\sin.\theta)\right)$$

and

$$e^{\beta x} = e^{f\cos.\theta}\left(\cos.(fx\sin.\theta) - \sqrt{-1}\cdot\sin.(fx\sin.\theta)\right).$$

For brevity's sake let the angle be $fx\sin.\theta = \varphi$; there will be

$$\begin{aligned}
&-2\sqrt{-1}\cdot v\left(\mathfrak{M}\mathfrak{M} + \mathfrak{N}\mathfrak{N}\right) f\sin.\theta \\
&= -\left(\mathfrak{M} - \mathfrak{N}\sqrt{-1}\right)e^{-fx\cos.\theta}\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right)\int e^{fx\cos.\theta}Xdx\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right) \\
&+\left(\mathfrak{M} + \mathfrak{N}\sqrt{-1}\right)e^{-fx\cos.\theta}\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right)\int e^{fx\cos.\theta}Xdx\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right) \\
&= e^{-fx\cos.\theta}\,2\sqrt{-1}\left(\mathfrak{M}\sin.\varphi + \mathfrak{N}\cos.\varphi\right)\int e^{fx\cos.\theta}Xdx\cos.\varphi \\
&-e^{-fx\cos.\theta}\,2\sqrt{-1}\left(\mathfrak{M}\cos.\varphi - \mathfrak{N}\sin.\varphi\right)\int e^{fx\cos.\theta}Xdx\sin.\varphi.
\end{aligned}$$

On account of which we will have the part of the integral sought

$$v = \frac{1}{\left(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N}\right) f\sin.\theta}\left\{\begin{aligned}&+e^{-fx\cos.\theta}\left(\mathfrak{M}\sin.\varphi + \mathfrak{N}\cos.\varphi\right)\int e^{fx\cos.\theta}Xdx\cos.\varphi \\ &-e^{-fx\cos.\theta}\left(\mathfrak{M}\cos.\varphi - \mathfrak{N}\sin.\varphi\right)\int e^{fx\cos.\theta}Xdx\sin.\varphi\end{aligned}\right\}$$

with $\varphi = fx\sin.\theta$ being present.

COROLLARY 1

1174. Therefore the particular aid here consists in the finding of the imaginary formula $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$, which must be deduced from the quantity Q, while in place of z the imaginary value $f\left(\cos.\theta+\sqrt{-1}\cdot f\sin.\theta\right)$ was written, from which there arises this convenience, so that in place of z^n it is required to write $(-f)^n\left(\cos.n\theta+\sqrt{-1}\cdot\sin.n\theta\right)$.

COROLLARY 2

1175. Since there shall be $Q=\frac{P}{ff+2fz\cos.\theta+zz}$ also it possible to find from this form, by the same substitution, the imaginary formula $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$, but where it is to be noted that by this substitution both the numerator P as well as the denominator vanish. From which it is evident that the value of this formula duly can be obtained from this fraction

$$\frac{dP}{2(f\cos.\theta+z)dz}=\frac{dP}{-2\sqrt{-1}\cdot f\sin.\theta dz}.$$

COROLLARY 3

1176. Therefore since there shall be

$$\frac{dP}{dz}=B+2Cz+3Dz^2+4Ez^3+\ldots+nNz^{n-1},$$

if we put in place

$$\mathfrak{P}=B-2Cf\cos.\theta+3Df^2\cos.2\theta-4Ef^3\cos.3\theta+\ldots\pm nNf^{3n-1}\cos.(n-1)\theta,$$
$$\mathfrak{Q}=\quad-2Cf\sin.\theta+3Df^2\sin.2\theta-4Ef^3\sin.3\theta+\ldots\pm nNf^{3n-1}\sin.(n-1)\theta,$$

as with the substitution made there becomes $\frac{dP}{dz}=\mathfrak{P}+\sqrt{-1}\cdot\mathfrak{Q}$, we will have

$$\mathfrak{M}+\mathfrak{N}\sqrt{-1}=\frac{\mathfrak{P}+\sqrt{-1}\cdot\mathfrak{Q}}{-2\sqrt{-1}\cdot f\sin.\theta}=\frac{-\mathfrak{Q}+\sqrt{-1}\cdot\mathfrak{P}}{2f\sin.\theta}$$

and thus

$$\mathfrak{M}=\frac{-\mathfrak{Q}}{2f\sin.\theta}\text{ et }\mathfrak{N}=\frac{\mathfrak{P}}{2f\sin.\theta}.$$

COROLLARY 4

1177. Immediately therefore from the quantity P and from this with the derivatives $\mathfrak{P}$ and $\mathfrak{Q}$ on putting $fx\sin.\theta = \varphi$ the part of the integral arising from the double factor $ff + 2fz cos.\theta + zz$ will be expressed

$$v = \frac{2e^{-fx\cos.\theta}}{\mathfrak{P}\mathfrak{P}+\mathfrak{N}\mathfrak{N}}\left\{\begin{array}{l}(\mathfrak{P}\cos.\varphi - \mathfrak{Q}\sin.\varphi)\int e^{fx\cos.\theta}Xdx\cos.\varphi\\ +(\mathfrak{P}\sin.\varphi + \mathfrak{Q}\cos.\varphi)\int e^{fx\cos.\theta}Xdx\sin.\varphi\end{array}\right\}.$$

SCHOLIUM

1178. Therefore however many double factors the form

$$P = A + Bz + Cz^2 + Dz^3 + \text{etc.}$$

may have had, for the individual ones the parts of the integral are defined easily with the aid of these precepts, and because hence the finding of the parts which agree with simple factors, these shall be either unequal or equal, shall not be disturbed, with all the parts taken together in a sum the complete integral of the equation proposed shall be had. Yet truly these precepts are not sufficient, if two or more of the double factors are equal to each other, for cases of this kind are to be examined by a special procedure similar to that, which I have used for the case of two or more simple factors equal to each other. But lest I prolong this discussion, the case will suffice for two double factors equal to each other to be set out, with that method easily extended to more.

PROBLEM 154

1179. *With the proposed differential equation of any grade*

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + ... + N\frac{d^ny}{dx^n}$$

if the algebraic expression formed from this

$$P = A + Bz + Cz^2 + Dz^3 + ... + Nz^n$$

should have a double factor squared $\left(ff + 2fz\cos.\theta + zz\right)^2$*, to find the part of the integral in agreement with that.*

SOLUTION

Therefore we may put $P = A + Bz + Cz^2 + Dz^3 + ... + Nz^n$ and there shall be

$$Q = A' + B'z + C'zz + ... + N'z^{n-4}$$

and in the first place we put in place the imaginary roots not being attended to

$$\alpha = f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right) \text{ and } \beta = f\left(\cos.\theta - \sqrt{-1}\cdot\sin.\theta\right)$$

so that there shall be

$$P = (\alpha + z)^2(\beta + z)^2 Q.$$

Now from these, which we have shown above (§ 1163) for two simple equal factors, we may put in place the form $\frac{P}{(\alpha+z)^2} = (\beta + z)^2 Q$ on putting $z = -\alpha$ changing into $\mathfrak{A}$, but this form $\frac{P}{(\beta+z)^2} = (\alpha + z)^2 Q$ on putting $z = -\beta$ changes into $\mathfrak{B}$; from which the quantities $\mathfrak{A}$ and $\mathfrak{B}$ found there to be shown to be the parts of the integral thus arising

$$\tfrac{1}{\mathfrak{A}}e^{-\alpha x}\int dx\int e^{\alpha x}Xdx + \tfrac{1}{\mathfrak{B}}e^{-\beta x}\int dx\int e^{\beta x}Xdx = v,$$

which, since now they involve imaginary [terms], it is required to reduce to real [terms]. We may accomplish this as in the preceding problem :

$$\mathfrak{M} = A' - B'\, f\cos.\theta + C'\, ff\cos.2\theta - D'\, f^3\cos.3\theta + \text{etc.}$$
$$\mathfrak{N} = \quad -\; B'\, f\sin.\theta + C'\, ff\sin.2\theta\; - D'\, f^3\sin.3\theta + \text{etc.},$$

so that the quantity Q on putting $z = -\alpha = -f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right)$, it can become $\mathfrak{M} + \mathfrak{N}\sqrt{-1}$, but on putting $z = -\beta = -f\left(\cos.\theta - \sqrt{-1}\cdot\sin.\theta\right)$, it becomes $\mathfrak{M} - \mathfrak{N}\sqrt{-1}$. Since now there shall be

$$(\beta-\alpha)^2 = \left(-2\sqrt{-1}\cdot f\sin.\theta\right)^2 = -4\, ff\sin.^2\theta,$$

to which also $(\alpha-\beta)^2$ is equal, there will be

$$\mathfrak{A} = -4\, ff\sin.^2\theta\left(\mathfrak{M} + \mathfrak{N}\sqrt{-1}\right) \text{ and } \mathfrak{B} = -4\, ff\sin.^2\theta\left(\mathfrak{M} - \mathfrak{N}\sqrt{-1}\right)$$

and thus

$$-4\, ff\sin.^2\theta(\mathfrak{M}\mathfrak{M} + \mathfrak{N}\mathfrak{N})v$$
$$= \left(\mathfrak{M} - \mathfrak{N}\sqrt{-1}\right)e^{-\alpha x}\int dx\int e^{\alpha x}Xdx + \left(\mathfrak{M} + \mathfrak{N}\sqrt{-1}\right)e^{-\beta x}\int dx\int e^{\beta x}Xdx.$$

But on putting $fx\sin.\theta = \varphi$ there is, as we have seen [§ 1173],

$$e^{\alpha x} = e^{fx\cos.\theta}\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right), \quad e^{-\alpha x} = e^{-fx\cos.\theta}\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right)$$

and

$$e^{\beta x} = e^{fx\cos.\theta}\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right), \quad e^{-\beta x} = e^{-fx\cos.\theta}\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right)$$

From where the one member of these equations becomes

$$e^{-fx\cos.\theta}\left(\mathfrak{M}\cos.\varphi - \mathfrak{N}\sin.\varphi - \mathfrak{N}\sqrt{-1}\cdot\cos.\varphi - \mathfrak{M}\sqrt{-1}\cdot\sin.\varphi\right)$$
$$\times\int dx\int e^{-fx\cos.\theta}Xdx\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right)$$
$$+e^{-fx\cos.\theta}\left(\mathfrak{M}\cos.\varphi - \mathfrak{N}\sin.\varphi + \mathfrak{N}\sqrt{-1}\cdot\cos.\varphi + \mathfrak{M}\sqrt{-1}\cdot\sin.\varphi\right)$$
$$\times\int dx\int e^{-fx\cos.\theta}Xdx\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right),$$

where the imaginary parts at once cancel each other, thus so that there may be obtained

$$v = \frac{1}{2(\mathfrak{MM}+\mathfrak{NN})ff\sin.^2\theta}\left\{\begin{array}{l} -e^{-fx\cos.\theta}\left(\mathfrak{M}\cos.\varphi-\mathfrak{N}\sin.\varphi\right)\int dx\int e^{fx\cos.\theta}Xdx\cos.\varphi \\ -e^{-fx\cos.\theta}\left(\mathfrak{M}\cos.\varphi+\mathfrak{N}\sin.\varphi\right)\int dx\int e^{fx\cos.\theta}Xdx\sin.\varphi \end{array}\right\}$$

or in this manner

$$v = \frac{-e^{-fx\cos.\theta}}{2(\mathfrak{MM}+\mathfrak{NN})ff\sin.^2\theta}\left\{\begin{array}{l} \left(\mathfrak{M}\cos.\varphi-\mathfrak{N}\sin.\varphi\right)\int dx\int e^{fx\cos.\theta}Xdx\cos.\varphi \\ +\left(\mathfrak{M}\sin.\varphi+\mathfrak{N}\cos.\varphi\right)\int dx\int e^{fx\cos.\theta}Xdx\sin.\varphi \end{array}\right\},$$

which expression has been freed completely from imaginary parts.

COROLLARY 1

1180. Because the imaginary formula $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$ arises from the quantity Q, if in place of z there is written $-f\left(\cos.\theta+\sqrt{-1}\cdot\sin.\theta\right)$, with the same in place it may be found also from the form $\frac{P}{(ff+2fz\cos.\theta+zz)^2}$, truly here both the numerator as well as the denominator arising vanish.

COROLLARY 2

1181. Therefore the same value may arise also from the formula

$$\frac{dP}{4dz\left(f^3\cos.\theta+ffz\left(1+2\cos.^2\theta\right)+3fzz\cos.\theta+z^3\right)};$$

where since the same inconvenience occurs again, it arises also from this formula

$$\frac{ddP}{4dz^2\left(ff\left(1+2\cos.^2\theta\right)+6fz\cos.\theta+3zz\right)}.$$

COROLLARY 3

1182. Initially here there is put in place in the denominator $z=-f\left(\cos.\theta+\sqrt{-1}\cdot\sin.\theta\right)$ and this formula is produced

$$\frac{-ddP}{8ffdz^2\sin.^2\theta}.$$

Then since there shall be

$$\frac{ddP}{2dz^2}=C+3Dz+6Ezz+\ldots+\frac{n(n-1)}{1\cdot 2}Nz^{n-2},$$

we may put for the sake of brevity

$$\mathfrak{P} = C - 3Df\cos.\theta + 6Eff\cos.2\theta - .... \pm \frac{n(n-1)}{1\cdot 2} Nf^{n-2}\cos.(n-2)\theta,$$
$$\mathfrak{Q} = \quad -3Df\sin.\theta + 6Eff\sin.2\theta \; - .... \pm \frac{n(n-1)}{1\cdot 2} Nf^{n-2}\sin.(n-2)\theta,$$

so that there shall be made on substitution

$$\frac{ddP}{2dz^2} = \left(\mathfrak{P} + \mathfrak{Q}\sqrt{-1}\right) \text{ and thus } \mathfrak{M} + \mathfrak{N}\sqrt{-1} = \frac{-\mathfrak{P} - \mathfrak{Q}\sqrt{-1}}{4\,ff\sin.^2\theta}$$

and consequently

$$\mathfrak{M} = \frac{-\mathfrak{P}}{4\,ff\sin.^2\theta} \quad \text{and} \quad \mathfrak{N} = \frac{-\mathfrak{Q}}{4\,ff\sin.^2\theta}.$$

Which values therefore are allowed to be substituted into the part of the integral found.

COROLLARY 4

1183. Moreover with the substitution made the double quadratic factor

$$(ff + 2fz\cos.\theta + zz)^2$$

gives this part of the integral

$$v = \frac{2e^{-fx\cos.\theta}}{\mathfrak{P}\mathfrak{P} + \mathfrak{Q}\mathfrak{Q}} \left\{ \begin{array}{l} (\mathfrak{P}\cos.\varphi - \mathfrak{Q}\sin.\varphi)\int dx \int e^{fx\cos.\theta} Xdx \cos.\varphi \\ +(\mathfrak{P}\sin.\varphi + \mathfrak{Q}\cos.\varphi)\int dx \int e^{fx\cos.\theta} Xdx \sin.\varphi \end{array} \right\},$$

where φ denotes the angle $fx\sin.\theta$.

SCHOLIUM

1184. If we should compare this expression with that, which we found in the previous problem, scarcely from the actual similarity brought out will there be a need for more complicated cases. Thus if the quantity

$$P = A + Bz + Cz^2 + Dz^3 + ... + Nz^n$$

should have a double cubic

$$(ff + 2fz\cos.\theta + zz)^3,$$

the quantities $\mathfrak{P}$ and $\mathfrak{Q}$, are defined thus, so that there shall be

$$\mathfrak{P} = D - 4Ef\cos.\theta + 10Fff\cos.2\theta - 20Gf^3\cos.3\theta + \ldots \pm \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3} Nf^{n-3}\cos.(n-3)\theta,$$

$$\mathfrak{Q} = -4Ef\sin.\theta + 10Fff\sin.2\theta - 20Gf^3\sin.3\theta + \ldots \pm \tfrac{n(n-1)(n-2)}{1\cdot 2\cdot 3} Nf^{n-3}\sin.(n-3)\theta,$$

with which found, the part of the integral hence arising

$$v = \tfrac{2e^{-fx\cos.\theta}}{\mathfrak{PP}+\mathfrak{QQ}} \left\{ \begin{array}{l} (\mathfrak{P}\cos.\varphi - \mathfrak{Q}\sin.\varphi)\int dx \int dx \int e^{fx\cos.\theta} X dx \cos.\varphi \\ +(\mathfrak{P}\sin.\varphi + \mathfrak{Q}\cos.\varphi)\int dx \int dx \int e^{fx\cos.\theta} X dx \sin.\varphi \end{array} \right\}$$

and now further progress is not liable to have any greater difficulty. On account of which equation proposed in this chapter thus I consider indeed to have been solved neatly by me , so that nothing further could be desired. Meanwhile this argument will be greatly illustrated, if we apply these precepts to particular examples ; the following chapter is set up with this in mind. But before that I will put in place a conspicuous property about general equations of this kind, which is considered to have a remarkable use in analysis.

PROBLEM 155

1185. *With a proposed differential equation of any order*

$$X = Ay + B\tfrac{dy}{dx} + C\tfrac{ddy}{dx^2} + D\tfrac{d^3y}{dx^3} + \ldots + N\tfrac{d^{m+n}y}{dx^{m+n}}$$

if the algebraic formula arising from this

$$P = A + Bz + Cz^2 + Dz^3 + \ldots + Nz^{m+n}$$

depends on the two factors $P = QR$ *, so that there shall be*

$$Q = \mathfrak{A} + \mathfrak{B}z + \mathfrak{C}z^2 + \ldots + \mathfrak{N}z^m \quad \textit{and} \quad R = \mathfrak{a} + \mathfrak{b}z + \mathfrak{c}z^2 + \ldots + \mathfrak{n}z^n,$$

then the integration of this equation is recalled to the integration of the two simple equations.

SOLUTION

If at first we carefully assess the integral form (§ 1158), from this with hardly any difficulty we may deduce from this, after we have integrated this equation

$$X = \mathfrak{A}v + \mathfrak{B}\tfrac{dv}{dx} + \mathfrak{C}\tfrac{ddv}{dx^2} + \ldots + \mathfrak{N}\tfrac{d^mv}{dx^m}$$

from which we have defined the value of v itself by x and X , the value of y for the proposed equation is going to be deduced from this equation

$$v = \mathfrak{a}y + \mathfrak{b}\frac{dy}{dx} + \mathfrak{c}\frac{ddy}{dx^2} + \ldots . + \mathfrak{n}\frac{d^n y}{dx^n},$$

an account of this thus has been brought into view, while from this equation the values for v and of this for the differentials are substituted. Indeed there will be produced

$$\begin{aligned} X = \mathfrak{A}\mathfrak{a}y &+ \mathfrak{A}\mathfrak{b}\frac{dy}{dx} + \mathfrak{A}\mathfrak{c}\frac{ddy}{dx^2} + \mathfrak{A}\mathfrak{d}\frac{d^3y}{dx^3} + \text{etc.} \\ &+ \mathfrak{B}\mathfrak{a} \quad + \mathfrak{B}\mathfrak{b} \qquad + \mathfrak{B}\mathfrak{c} \\ &\qquad\quad + \mathfrak{C}\mathfrak{a} \qquad + \mathfrak{C}\mathfrak{b} \\ &\qquad\qquad\qquad\quad + \mathfrak{D}\mathfrak{a} \end{aligned}$$

But since by hypothesis there shall be $P = QR$, with the series Q and R multiplied by each other by necessity there becomes

$$A = \mathfrak{A}a, \quad B = \mathfrak{A}\mathfrak{b} + \mathfrak{B}\mathfrak{a}, \quad C = \mathfrak{A}c + \mathfrak{B}\mathfrak{b} + \mathfrak{C}\mathfrak{a} \text{ etc.}$$

and thus this last equation is reduced to that itself proposed.

COROLLARY 1

1186. If only we consider simple factors, the integral of the first equation is expressed by terms of this kind

$$v = \Gamma e^{-\alpha x}\int e^{\alpha x} X dx \quad \text{etc .,}$$

now the integral of the latter equation by terms of this kind

$$y = \Delta e^{-\beta x}\int e^{\beta x} v dx \quad \text{etc}$$

COROLLARY 2

1187. But if now in the individual terms of the latter integral we substitute the individual terms of the first, there becomes

$$y = \Gamma\Delta e^{-\beta x}\int e^{(\beta-\alpha)x} dx \int e^{\alpha x} X dx,$$

which form is reduced to this

$$y = \tfrac{\Gamma\Delta}{\beta-\alpha}\left(e^{-\alpha x}\int e^{\alpha x} X dx - e^{-\beta x}\int e^{\beta x} X dx\right),$$

and the terms of this kind are found immediately by the integration of the proposed equation.

COROLLARY 3

1188. If here there should be $\beta = \alpha$, the form may be produced without any reduction

$$y = \Gamma\Delta e^{-\alpha x}\int dx\int e^{\alpha x}Xdx$$

for the above form for the case of two simple equal factors found. Meanwhile since the total business corresponds to the resolution into either simple or double real factors, the proposed equation itself by the way proposed before is easily obtained.

CAPUT III

DE INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM HUIUS FORMAE

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \text{etc.}$$

PROBLEMA 147

1138. *Proposita aequatione differentiali*

$$X = Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \dots + \frac{Nd^ny}{dx^n}$$

sumto elemento dx constante et significante X functionem quamcunque ipsius x invenire functionem ipsius x, per quam haec aequatio multiplicata integrabilis evadat.

SOLUTIO

Sit Pdx iste multiplicator, quem quaerimus, et cum prius, membrum X eo integrabile reddatur, eius rationem ex altero membro definiri oportet. Facile autem intelligitur formam huius multiplicatoris P eiusmodi fore $e^{\lambda x}$, ita ut quantitas λ definiri debeat. Sit ergo $e^{\lambda x}dx$ multiplicator atque hanc formam

$$e^{\lambda x}dx\left(Ay + \frac{Bdy}{dx} + \frac{Cddy}{dx^2} + \frac{Dd^3y}{dx^3} + \dots + \frac{Nd^ny}{dx^n}\right)$$

integrabilem esse oportet, euius integrale propterea statuatur

$$e^{\lambda x}\left(A'y + \frac{B'dy}{dx} + \frac{C'ddy}{dx^2} + \dots + \frac{M'd^{n-1}y}{dx^{n-1}}\right),$$

ita ut huius differentiale cum illa forma congruere debeat; quod cum sit

$$e^{\lambda x}dx\left(\lambda A'y + \frac{\lambda B'dy}{dx} + \frac{\lambda C'ddy}{dx^2} + \dots + \frac{\lambda M'd^{n-1}y}{dx^{n-1}} + \frac{A'dy}{dx} + \frac{B'ddy}{dx^2} + \dots + \frac{M'd^ny}{dx^n}\right),$$

necesse est sit

$$A' = \frac{A}{\lambda},\ B' = \frac{B-A'}{\lambda},\ C' = \frac{C-B'}{\lambda},\ D' = \frac{D-C'}{\lambda}, \ldots\ M' = \frac{M-L'}{\lambda}$$

atque $M' = N$. Hinc erit

$$\begin{aligned} A' &= \tfrac{A}{\lambda}, \\ B' &= \tfrac{B}{\lambda} - \tfrac{A}{\lambda^2}, \\ C' &= \tfrac{C}{\lambda} - \tfrac{B}{\lambda^2} + \tfrac{A}{\lambda^3}, \\ D' &= \tfrac{D}{\lambda} - \tfrac{C}{\lambda^2} + \tfrac{B}{\lambda^3} - \tfrac{A}{\lambda^4}, \\ &\vdots \\ M' &= \tfrac{M}{\lambda} - \tfrac{L}{\lambda^2} + \tfrac{K}{\lambda^3} - \ldots \pm \tfrac{A}{\lambda^n}, \end{aligned}$$

et

$$0 = \frac{N}{\lambda} - \frac{M}{\lambda^2} + \frac{L}{\lambda^3} - \ldots \mp \frac{A}{\lambda^{n+1}},$$

ubi ex ultima aequatione quantitas λ erui debet, quae aequatio induit hanc formam

$$A - B\lambda + C\lambda^2 - D\lambda^3 + E\lambda^4 - \ldots \pm N\lambda^n = 0;$$

unde cum λ sortiatur *n* valores, totidem quoque multiplicatores inveniuntur.

Videamus, quomodo hae determinationes pro singulis valoribus exponentis *n* se habeant.

I. Si $n = 1$, erit $A - B\lambda = 0$, tum vero $A' = \frac{A}{\lambda} = B$.

II. Si $n = 2$, erit $A - B\lambda + C\lambda^2 = 0$, tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda \ \text{ et } \ B' = \frac{B\lambda - A}{\lambda^2} = C.$$

III. Si $n = 3$, erit $A - B\lambda + C\lambda^2 - D\lambda^3 = 0$, tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda + D\lambda^2, \quad B' = \frac{B\lambda - A}{\lambda^2} = C - D\lambda$$

et

$$C' = \frac{C\lambda^2 - B\lambda + A}{\lambda^3} = D.$$

IV. Si $n = 4$, erit $A - B\lambda + C\lambda^2 - D\lambda^3 + E\lambda^4 = 0$, tum vero

$$A' = \frac{A}{\lambda} = B - C\lambda + D\lambda^2 - E\lambda^3, \quad B' = \frac{B\lambda - A}{\lambda^2} = C - D\lambda + E\lambda^2,$$
$$C' = \frac{C\lambda^2 - B\lambda + A}{\lambda^3} = D - E\lambda, \qquad D' = \frac{D\lambda^3 - C\lambda^2 + B\lambda - A}{\lambda^4} = E.$$

V. Si $n=5$, erit $A-B\lambda+C\lambda^2-D\lambda^3+E\lambda^4-F\lambda^5=0$, tum vero

$$A'=\tfrac{A}{\lambda}=B-C\lambda+D\lambda^2-E\lambda^3+F\lambda^4,$$
$$B'=\tfrac{B\lambda-A}{\lambda^2}=C-D\lambda+E\lambda^2-F\lambda^3,$$
$$C'=\tfrac{C\lambda^2-B\lambda+A}{\lambda^3}=D-E\lambda+F\lambda^2,$$
$$D'=\tfrac{D\lambda^3-C\lambda^2+B\lambda-A}{\lambda^4}=E-F\lambda,$$
$$E'=\tfrac{E\lambda^4-D\lambda^3+C\lambda^2-B\lambda+A}{\lambda^5}=F$$

atque ita porro.

Invento autem hoc multiplicatore $e^{\lambda x}dx$ prius aequationis membrum fit $\int e^{\lambda x}Xdx$ et aequatio proposita, quae est differentialis gradus *n*, per integrationem reducitur ad hanc uno gradu simpliciorem

$$\int e^{\lambda x}Xdx=e^{\lambda x}\left(A'y+\frac{B'dy}{dx}+\frac{C'ddy}{dx^2}+....+\frac{M'd^{n-1}y}{dx^{n-1}}\right).$$

COROLLARIUM 1

1139. Integratione ergo hac prima instituta aequatio proposita uno gradu deprimitur et definitis coefficientibus *A'*, *B'*, *C'* etc. ex superioribus formulis aequatio integralis hac forma exhiberi potest

$$e^{-\lambda x}\int e^{\lambda x}Xdx=A'y+\frac{B'dy}{dx}+\frac{C'ddy}{dx^2}+....+\frac{M'd^{n-1}y}{dx^{n-1}}.$$

COROLLARIUM 2

1140. Cum prius membrum $e^{-\lambda x}\int e^{\lambda x}Xdx$ sit functio ipsius *x* constantem arbitrariam involvens, si eius loco ponatur *X'*, haec aequatio similem formam habet atque ipsa proposita ideoque eadem methodo iterum integrari et ad gradum differentialitatis $n-2$ reduci potest, quae huiusmodi formam habebit

$$X''=A''y+B''\frac{dy}{dx}+C''\frac{ddy}{dx^2}+....+L''\frac{d^{n-2}y}{dx^{n-2}}.$$

COROLLARIUM 3

1141. Hoc modo ulterius progrediendo tandem ad aequationem differentialem primi gradus pervenietur

$$X^{(n-1)}=A^{(n-1)}y+B^{(n-1)}\frac{dy}{dx},$$

quae simili modo ad aequationem finitam $X^{(n)}=A^{(n)}y$ reducitur, qua relatio inter ipsas variabiles *x* et *y* exprimitur.

SCHOLION

1142. Haec igitur est methodus huiusmodi aequationes differentiales altiorum graduum successive per gradus integrandi, ubi tot opus est integrationibus, quoti gradus differentialis fuerit ipsa aequatio proposita. Totum ergo negotium situm est in inventione successiva coefficientium, quos ex praecedentibus ope multiplicatoris definiri oportet. In genere quidem lex, qua ii continuo ex antecedentibus determinantur, non ita est perspicua, ut inde forma integralis extremi perspici possit; verum quia ex capite superiori novimus casu, quo primum membrum X evanescit, etiam ultimum integrale lege satis simplici contineri, idem hic usu venire merito suspicamur eamque legem facillime agnoscemus, si pedetentim a gradibus inferioribus ad altiores progrediamur.

Ac primo quidem casu, quo aequatio est differentialis primi gradus $X = Ay + B\frac{dy}{dx}$, multiplicator erit $e^{\lambda x}dx$ posito $A - \lambda B = 0$, ut sit $A' = \frac{A}{\lambda}$, et cum sit $A' = \frac{A}{\lambda} = B$, integrale erit $\int e^{\lambda x}Xdx = Be^{\lambda x}y$ seu $e^{-\lambda x}\int e^{\lambda x}Xdx = By$.

Ad hanc similitudinem aequationes graduum altiorum evolvamus ac formam integralis ultimi investigemus.

PROBLEMA 148

1143. *Proposita aequatione differentiali secundi gradus*

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2}$$

per duplicem integrationem relationem inter x et y investigare.

SOLUTIO

Sit $e^{\lambda x}dx$ multiplicator hanc aequationem per se integrabilem reddens eritque $A - B\lambda + C\lambda^2 = 0$; tum sumatur $A' = \frac{A}{\lambda} = B - C\lambda$ et $B' = \frac{B\lambda - A}{\lambda^2} = C$ positoque

$$e^{-\lambda x}\int e^{\lambda x}Xdx = X'$$

aequatio semel integrata est

$$X' = A'y + B'\frac{dy}{dx}.$$

Huius iam multiplicator sit $e^{\mu x}dx$ eritque $A' - B'\mu = 0$ ac statuatur $A'' = \frac{A'}{\mu} = B'$ positoque

$$e^{-\mu x}\int e^{\mu x}X'dx = X'',$$

habebimus

$$X'' = A''y,$$

quae est aequatio bis integrata relationem quaesitam inter x et y exprimens.

Cum igitur hic sit $A'' = B'$ et $B' = C$, erit $A'' = C$. Deinde loco A' et B' substitutis valoribus aequatio $A' - B'\mu = 0$ induit hanc formam

$$B - C\lambda - C\mu = 0 \text{ seu } B - C(\lambda + \mu) = 0;$$

ex qua cum sit $\lambda + \mu = \frac{B}{C}$, patet $\lambda + \mu$ aequari summae binarum radicum aequationis $A - B\lambda + C\lambda^2 = 0$. Quoniam igitur λ eius una est radix, μ necessaria eius alteram radicem denotat. Quare si ex aequatione proposita, uti in capite praecedente fecimus, hanc formemus aequationem $A + Bz + Czz = 0$, eius radices erunt $z = -\lambda$ et $z = -\mu$. Seu si factores eius statuamus $C(\alpha + z)(\beta + z)$, litterae α et β praebebunt valores λ et μ. Hinc cum sit

$$X' = e^{-\alpha x}\int e^{\alpha x}Xdx,$$

erit

$$X'' = e^{-\beta x}\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx.$$

At

$$\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx = \tfrac{1}{\beta-\alpha}e^{(\beta-\alpha)x}\int e^{\alpha x}Xdx - \tfrac{1}{\beta-\alpha}\int e^{\beta x}Xdx,$$

unde concluditur

$$X'' = \tfrac{1}{\beta-\alpha}e^{-\alpha x}\int e^{\alpha x}Xdx + \tfrac{1}{\alpha-\beta}e^{-\beta x}\int e^{\beta x}Xdx.$$

Quocirca aequationis propositae integrale completum est

$$Cy = \tfrac{1}{\beta-\alpha}e^{-\alpha x}\int e^{\alpha x}Xdx + \tfrac{1}{\alpha-\beta}e^{-\beta x}\int e^{\beta x}Xdx.$$

ubi litterae α et β ita sunt capiendae, ut sit

$$A + Bz + Czz = C(\alpha + z)(\beta + z).$$

COROLLARIUM 1

1144. Si bini hi factores sint aequales seu $\beta = \alpha$, erit

$$X'' = e^{-\alpha x}\int dx\int e^{\alpha x}Xdx = e^{-\alpha x}x\int e^{\alpha x}Xdx - e^{-\alpha x}\int e^{\alpha x}Xxdx$$

ideoque casu

$$A+Bz+Czz=C(\alpha+z)^2$$

aequationis nostrae integrale est

$$Cy=e^{-\alpha x}\left(x\int e^{\alpha x}Xdx-\int e^{\alpha x}Xxdx\right).$$

COROLLARIUM 2

1145. Si bini factores sint imaginarii, quod evenit, si

$$A+Bz+Czz=C(ff+2fz\cos.\theta+zz),$$

erit

$$\alpha=f\left(\cos.\theta+\sqrt{-1}\cdot\sin.\theta\right) \text{ et } \beta=f\left(\cos.\theta-\sqrt{-1}\cdot\sin.\theta\right),$$

hinc

$$e^{\alpha x}=e^{fx\cos.\theta}\left(\cos.(fx\sin.\theta)+\sqrt{-1}\cdot\sin.(fx\sin.\theta)\right)$$

et

$$e^{\beta x}=e^{fx\cos.\theta}\left(\cos.(fx\sin.\theta)-\sqrt{-1}\cdot\sin.(fx\sin.\theta)\right)$$

atque

$$\beta-\alpha=-2\sqrt{-1}\cdot f\sin.\theta.$$

COROLLARIUM 3

1146. Quo haec facilius substituere queamus, sit brevitatis gratia

$$e^{fx\cos.\theta}=m, \quad \cos.(fx\sin.\theta)=p \quad \text{et} \quad \sin.(fx\sin.\theta)=q,$$

ut sit

$$e^{\alpha x}=mp+mq\sqrt{-1} \text{ et } e^{\beta x}=mp-mq\sqrt{-1}.$$

Hinc fit

$$\int e^{\alpha x}Xdx=\int mpXdx+\int mqXdx\sqrt{-1}$$

et

$$\int e^{\beta x}Xdx=\int mpXdx-\int mqXdx\sqrt{-1}$$

Tum vero est

$$e^{-ax}=\frac{p-q\sqrt{-1}}{m} \text{ et } e^{-\beta x}=\frac{p+q\sqrt{-1}}{m}$$

COROLLARIUM 4

1147. Ex his colligimus

$$e^{-\alpha x}\int e^{\alpha x}Xdx=\tfrac{p}{m}\int mpXdx-\tfrac{q\sqrt{-1}}{m}\int mpXdx+\tfrac{p\sqrt{-1}}{m}\int mqXdx+\tfrac{q}{m}\int mqXdx$$

et sumto $\sqrt{-1}$ negativo prodit $e^{-\beta x}\int e^{\beta x}Xdx$, quae forma inde subtracta relinquit

$$-\tfrac{2q\sqrt{-1}}{m}\int mpXdx+\tfrac{2p\sqrt{-1}}{m}\int mqXdx,$$

hocque residuum dividi debet per $\beta-\alpha=-2\sqrt{-1}\cdot f\sin.\theta$. Unde integrale colligitur

$$Cy=\tfrac{q}{mf\sin.\theta}\int mpXdx-\tfrac{p}{mf\sin.\theta}\int mqXdx.$$

COROLLARIUM 5

1148. Restituantur pro *m, p, q* valores assumti atque aequationis nostrae, si fuerit

$$A+Bz+Czz=C\left(ff+2fzcos.\theta+zz\right),$$

integrale erit

$$Cy=e^{-fx\cos.\theta}\begin{Bmatrix}\frac{\sin.(fx\sin.\theta)}{f\sin.\theta}\int e^{fx\cos.\theta}Xdx\cos.(fx\sin.\theta)\\ -\frac{\cos.(fx\sin.\theta)}{f\sin.\theta}\int e^{fx\cos.\theta}Xdx\sin.(fx\sin.\theta)\end{Bmatrix},$$

quae ergo expressio aequivalet illi, si α et β valores imaginarios obtineant.

PROBLEMA 149

1149. *Proposita aequatione differentiali tertii gradus*

$$X=Ay+B\frac{dy}{dx}+C\frac{ddy}{dx^2}+D\frac{d^3y}{dx^3}$$

per triplicem integrationem eius integrale completum invenire.

SOLUTIO

Posito multiplicatore $e^{\lambda x}dx$ debet esse $A-B\lambda+C\lambda^2-D\lambda^3=0$; tum sumatur $A'=B-C\lambda+D\lambda^2,\quad B'=C-D\lambda$ et $C'=D$ positoque

$$e^{-\lambda x}\int e^{\lambda x}Xdx = X'$$

aequatio semel integrata praebet

$$X' = A'y + B'\frac{dy}{dx} + C'\frac{ddy}{dx^2}.$$

Huius porro multiplicator statuatur $e^{\mu x}dx$, ut sit $A' - B'\mu + C'\mu^2 = 0$, sumaturque $A'' = B' - C'\mu$ et $B'' = C'$ et posito

$$e^{-\mu x}\int e^{\mu x}X'dx = X''$$

aequatio secunda integralis est

$$X'' = A''y + B''\frac{dy}{dx},$$

cuius multiplicator erit $e^{\nu x}dx$ sumendo $A'' - B''\nu = 0$; at posito $A''' = B''$ erit aequatio integralis tertia

$$e^{-\nu x}\int e^{\nu x}X''dx = A'''y = Dy;$$

quaeri ergo oportet quantitates λ, μ, ν. Est vero primo

$$A - B\lambda + C\lambda^2 - D\lambda^3 = 0,$$

tum

$$B - C(\lambda+\mu) + D(\lambda\lambda + \lambda\mu + \mu\mu) = 0$$

et ob

$$A'' = C - D(\lambda+\mu) \text{ et } B'' = D$$

erit tertio

$$C - D(\lambda+\mu+\nu) = 0,$$

ex qua postrema aequalitate patet $\lambda+\mu+\nu$ aequari summae radicum aequationis primae, cuius λ est una radix. Quod autem μ et ν sint reliquae radices, hoc modo ostenditur. Consideretur aequatio

$$A + Bz + Czz + Dz^3 = 0;$$

cuius si una radix sit $z = -\lambda$ seu $\lambda + z$ unus factor, dividatur per eum aequatio ac prodibit

$$Dz^2 + (C - D\lambda)z + B - C\lambda + D\lambda\lambda = 0,$$

quae est ipsa aequatio secunda $C'zz + B'z + A' = 0$, cuius radices sunt

$z=-\mu$ et $z=-\nu$ seu factores $(\mu+z)(\nu+z)$, uti in problemate praecedente ostendimus. Quare si formulae $A+Bz+Cz^2+Dz^3$ factores sint

$$D(\alpha+z)(\beta+z)(\nu+z),$$

pro integrali ultimo inveniendo ponatur

$$e^{-\alpha x}\int e^{\alpha x}Xdx=X', \quad e^{-\beta x}\int e^{\beta x}X'\,dx=X'' \quad \text{et} \quad e^{-\gamma x}\int e^{\gamma x}X''\,dx=X'''$$

eritque

$$Dy=X'''.$$

Verum per reductionem integralium est, uti supra vidimus,

$$X''=\tfrac{1}{\beta-\alpha}e^{-\alpha x}\int e^{\alpha x}Xdx+\tfrac{1}{\alpha-\beta}e^{-\beta x}\int e^{\beta x}Xdx$$

hincque porro

$$\int e^{\gamma x}X''\,dx=\tfrac{1}{(\beta-\alpha)(\gamma-\alpha)}e^{(\gamma-\alpha)x}\int e^{\alpha x}Xdx-\tfrac{1}{(\beta-\alpha)(\gamma-\alpha)}\int e^{\gamma x}Xdx$$
$$+\tfrac{1}{(\alpha-\beta)(\gamma-\beta)}e^{(\gamma-\beta)x}\int e^{\beta x}Xdx-\tfrac{1}{(\alpha-\beta)(\gamma-\beta)}\int e^{\gamma x}Xdx,$$

ubi bini postremi termini contrahuntur in

$$\tfrac{1}{(\alpha-\gamma)(\beta-\gamma)}\int e^{\gamma x}Xdx.$$

Quamobrem integrale quaesitum est

$$Dy=\tfrac{e^{-\alpha x}\int e^{\alpha x}Xdx}{(\beta-\alpha)(\gamma-\alpha)}+\tfrac{e^{-\beta x}\int e^{\beta x}Xdx}{(\alpha-\beta)(\gamma-\beta)}+\tfrac{e^{-\gamma x}\int e^{\gamma x}Xdx}{(\alpha-\gamma)(\beta-\gamma)}.$$

COROLLARIUM 1

1150. Si formulae $A+Bz+Cz^2+Dz^3$ duo factores fuerint aequales, puta $\gamma=\beta$, erit

$$\int e^{\beta x}X''\,dx=\tfrac{1}{(\beta-\alpha)^2}e^{(\beta-\alpha)x}\int e^{\alpha x}Xdx-\tfrac{1}{(\beta-\alpha)^2}\int e^{\beta x}Xdx$$
$$+\tfrac{1}{(\alpha-\beta)}x\int e^{\beta x}Xdx-\tfrac{1}{(\alpha-\beta)}\int e^{\beta x}Xxdx$$

ideoque integrale hoc casu erit

$$Dy = \frac{e^{-\alpha x}\int e^{\alpha x}Xdx - e^{-\beta x}\int e^{\beta x}Xdx}{(\beta-\alpha)^2} + \frac{e^{-\beta x}x\int e^{\beta x}Xdx - e^{-\beta x}\int e^{\beta x}Xxdx}{\alpha-\beta}$$

COROLLARIUM 2

1151. Si omnes tres factores sint aequales seu $\alpha=\beta=\gamma$, erit

$$e^{\alpha x}X'' = \int dx\int e^{\alpha x}Xdx = x\int e^{\alpha x}Xdx - \int e^{\alpha x}Xxdx$$

et

$$e^{\alpha x}X''' = \int e^{\alpha x}X''dx = \int dx\int dx\int e^{\alpha x}Xdx$$

seu

$$e^{\alpha x}X''' = \tfrac{1}{2}xx\int e^{\alpha x}Xdx - x\int e^{\alpha x}Xxdx + \tfrac{1}{2}\int e^{\alpha x}Xxxdx,$$

unde integrale hoc casu erit

$$Dy = \tfrac{1}{2}e^{-\alpha x}\left(xx\int e^{\alpha x}Xdx - 2x\int e^{\alpha x}Xxdx + \int e^{\alpha x}Xxxdx\right)$$

seu

$$Dy = e^{-\alpha x}\int dx\int dx\int e^{\alpha x}Xdx.$$

SCHOLION

1152. In genere etiam nulla integralium reductione adhibita integrale nostrae aequationis ita exprimi potest, ut sit

$$Dy = e^{-\gamma x}\int e^{(\gamma-\beta)x}dx\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx$$

posito

$$A + Bz + Cz^2 + Dz^3 = D(\alpha+z)(\beta+z)(v+z)$$

ubi imprimis notatu dignum occurrit, quod ternas litteras α, β, γ quomodocunque inter se permutare licet, ita ut haec integralis expressio sex modis variari possit. In problemate etiam praecedente [§ 1143], ubi duo tantum factores occurrunt

$$C(\alpha+z)(\beta+z) = A + Bz + Czz\,,$$

aequationis

$$X = Ay + B\frac{dy}{dx} + \frac{Cddy}{dx^2}$$

integrale completum ita exhiberi potest

$$Cy = e^{-\beta x}\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx$$

ac permutatis litteris α et β etiam hoc modo

$$Cy = e^{-\alpha x}\int e^{(\alpha-\beta)x}dx\int e^{\beta x}Xdx$$

Quarum formularum aequalitas si fuerit perspecta, id quod tentanti facile patebit, praecedentium quoque variationem declarat. Sit enim $e^{-\alpha x}\int e^{\alpha x}Xdx = X'$; erit pro superiori formula

$$Dy = e^{-\gamma x}\int e^{(\gamma-\beta)x}dx\int e^{\beta x}X'dx\ ;$$

cui cum aequalis sit ista

$$Dy = e^{-\beta x}\int e^{(\beta-\gamma)x}dx\int e^{\gamma x}X'dx\,,$$

erit etiam pro X' valore restituto

$$Dy = e^{-\beta x}\int e^{(\beta-\gamma)x}dx\int e^{(\gamma-\alpha)x}dx\int e^{\alpha x}Xdx,$$

quae a prima hoc tantum differt, quod litterae β et γ sunt permutatae. Quod autem etiam litterae β et γ cum α permutari queant, hoc modo difficilius ostenditur, ex reductione autem in solutione adhibita atque adeo ex ipsa solutionis indole per se est manifestum.

PROBLEMA 150

1153. *Proposita aequatione differentiali quarti gradus*

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + E\frac{d^4y}{dx^4}$$

sumto elemento dx constante et denotante X functionem quamcunque ipsius x eius integrale investigare.

SOLUTIO

In subsidium vocetur formula algebraica ex aequatione proposita facile formanda

$$A + Bz + Czz + Dz^3 + Ez^4 = P\,,$$

quae in factores suos simplices resolvatur, ut sit

$$P = E(\alpha+z)(\beta+z)(\gamma+z)(\delta+z),$$

et multiplicator aequationem nostram integrabilem reddens erit $e^{\lambda x}dx$ sumendo λ aequali uni litterarum $\alpha,\beta,\gamma,\delta$. Sumatur ergo $\lambda=\alpha$, ut sit multiplicator $e^{\alpha x}dx$, atque posito

$$e^{-\alpha x}\int e^{\alpha x}Xdx = X'$$

aequatio semel integrata erit

$$X' = A'y + B'\frac{dy}{dx} + C'\frac{ddy}{dx^2} + D'\frac{d^3y}{dx^3}$$

ubi A', B', C', D' ita determinantur, ut sit

$$A' = \frac{A}{\alpha}, \quad B' = \frac{B\alpha-A}{\alpha^2}, \quad C' = \frac{C\alpha^2-B\alpha+A}{\alpha^3}, \quad D' = \frac{D\alpha^3-C\alpha^2+B\alpha-A}{\lambda^4}$$

seu

$$A' = \frac{A}{\alpha}, \quad B' = \frac{B-A'}{\alpha}, \quad C' = \frac{C-B'}{\alpha}, \quad D' = \frac{D-C'}{\alpha}$$

vel etiam

$$A = A'\alpha,\ B = B'\alpha + A',\ C = C'\alpha + B',\ D = D'\alpha + C'.$$

Ex quibus determinationibus liquet, si ponatur

$$A' + B'z + C'z^2 + D'z^3 = Q,$$

hanc formulam Q nasci ex formula P, si haec per $\alpha+z$ dividatur, ita ut sit

$$Q = \frac{P}{\alpha+z} = E(\beta+z)(\gamma+z)(\delta+z).$$

Eodem ergo modo secundam integrationem instituamus ope multiplicatoris $e^{\beta x}dx$ et posito

$$e^{-\beta x}\int e^{\beta x}X'dx = X''$$

erit aequatio integralis

$$X'' = A''y + B''\frac{dy}{dx} + C''\frac{ddy}{dx^2}$$

coefficientibus A'', B'', C'' ita sumtis, ut sit

$$A''+B''z+C''z^2=\frac{P}{(\alpha+z)(\beta+z)}=E(\gamma+z)(\delta+z).$$

Hinc porro ope multiplicatoris $e^{\gamma x}dx$ integrando si ponamus

$$e^{-\gamma x}\int e^{\gamma x}X''dx=X''',$$

inveniemus

$$X''=A'''y+B'''\frac{dy}{dx}$$

existente

$$A'''+B'''z=\frac{P}{(\alpha+z)(\beta+z)(\gamma+z)}=E(\delta+z).$$

Ac tandem ope multiplicatoris $e^{\delta x}dx$ posita forma

$$e^{-\delta x}\int e^{\delta x}X'''dx=X''''$$

integrale ultimum reperitur

$$X''''=A''''y$$

existente $A''''=E$. Haec igitur omnia colligendo integrale quaesitum erit

$$Ey=e^{-\delta x}\int e^{(\delta-\gamma)x}dx\int e^{(\gamma-\beta)x}dx\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx,$$

quae expressio iam sine ullis ambagibus ex resolutione formae principalis

$$P=A+Bz+Cz^2+Dz^3+Ez^4$$

in factores, scilicet

$$P=E(\alpha+z)(\beta+z)(\gamma+z)(\delta+z),$$

confici potest, ubi notandum, quomodocunque ordo litterarum $\lambda,\ \mu,\nu,\delta$ permutetur, pro *Ey* semper eundem valorem prodire debere.

COROLLARIUM 1

1154. Cum sit $X'=e^{-\alpha x}\int e^{\alpha x}Xdx$, erit, uti iam vidimus,

$$X''=e^{-\beta x}\int e^{\beta x}X'dx=e^{-\beta x}\left(\frac{e^{(\beta-\alpha)x}}{\beta-\alpha}\int e^{\alpha x}Xdx-\frac{1}{\beta-\alpha}\int e^{\beta x}Xdx\right)$$

seu

$$X'' = \frac{e^{-\alpha x}\int e^{\alpha x} Xdx}{\beta-\alpha} + \frac{e^{-\beta x}\int e^{\beta x} Xdx}{\alpha-\beta}.$$

COROLLARIUM 2

1155. Porro ob $X''' = e^{-\gamma x}\int e^{\gamma x} X''dx$ erit simili modo reductionem instituendo

$$X''' = \frac{e^{-\alpha x}\int e^{\alpha x} Xdx}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\gamma x}\int e^{\gamma x} Xdx}{(\beta-\alpha)(\alpha-\gamma)} + \frac{e^{-\beta x}\int e^{\beta x} Xdx}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} Xdx}{(\alpha-\beta)(\beta-\gamma)},$$

quae reducitur ad hanc formam

$$X''' = \frac{e^{-\alpha x}\int e^{\alpha x} Xdx}{(\beta-\alpha)(\gamma-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x} Xdx}{(\alpha-\beta)(\gamma-\beta)} + \frac{e^{-\gamma x}\int e^{\gamma x} Xdx}{(\alpha-\gamma)(\beta-\gamma)}.$$

COROLLARY 3

1156. Hinc simili modo evolvitur valor X'''', ubi quidem sufficeret primum membrum eruisse, quippe ex quo ob permutabilitatem reliqua sponte formantur. Hoc modo integrale nostrae aequationis reperietur hac forma expressum

$$Ey = \frac{e^{-\alpha x}\int e^{\alpha x} Xdx}{(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)} + \frac{e^{-\beta x}\int e^{\beta x} Xdx}{(\alpha-\beta)(\gamma-\beta)(\delta-\beta)}$$
$$+ \frac{e^{-\gamma x}\int e^{\gamma x} Xdx}{(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)} + \frac{e^{-\delta x}\int e^{\delta x} Xdx}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)}.$$

SCHOLION

1157. Si duae pluresve radices sint aequales vel imaginariae, integralia inventa transformationem postulant, quam deinceps investigabimus. Atque haec postrema quidem forma magis apta videtur, unde transformationes repetantur.

Ita pro factorum aequalitate si sit $\delta = \gamma$, bina postrema membra tantum reductionem postulant; ad quam inveniendam ponatur $\delta = \gamma - \omega$ et penultimum membrum erit $\frac{e^{-\gamma x}\int e^{\gamma x} Xdx}{\omega(\alpha-\gamma)(\beta-\gamma)}$ pro ultimo autem notandum est esse

$$\frac{1}{\alpha-\delta} = \frac{1}{\alpha-\gamma+\omega} = \frac{1}{\alpha-\gamma} - \frac{\omega}{(\alpha-\gamma)^2} \quad \text{et} \quad \frac{1}{\beta-\delta} = \frac{1}{\beta-\gamma} - \frac{\omega}{(\beta-\gamma)^2}$$

hincque

$$\frac{1}{(\alpha-\delta)(\beta-\delta)} = \frac{1}{(\alpha-\gamma)(\beta-\gamma)} + \frac{\omega(2\gamma-\alpha-\beta)}{(\alpha-\gamma)^2(\beta-\gamma)^2},$$

unde

$$\frac{1}{(\alpha-\delta)(\beta-\delta)(\gamma-\delta)} = \frac{1}{\omega(\alpha-\gamma)(\beta-\gamma)} + \frac{2\gamma-\alpha-\beta}{(\alpha-\gamma)^2(\beta-\gamma)^2}.$$

Tum vero pro numeratore erit

$$e^{-\delta x} = e^{-\gamma x}(1+\omega x) \text{ et } e^{\delta x} = e^{\gamma x}(1-\omega x)$$

ideoque

$$e^{-\delta x}\int e^{\delta x}Xdx = e^{-\gamma x}\int e^{\gamma x}Xdx + e^{-\gamma x}\omega x\int e^{\gamma x}Xdx - \omega e^{-\gamma x}\int e^{\gamma x}Xxdx$$

atque hinc bina ultima membra ob terminos per ω divisos se destruentes abeunt in hanc formam

$$\frac{(2\gamma-\alpha-\beta)e^{-\gamma x}\int e^{\gamma x}Xdx}{(\alpha-\gamma)^2(\beta-\gamma)^2} + \frac{e^{-\gamma x}x\int e^{\gamma x}Xdx - e^{-\gamma x}\int e^{\gamma x}Xxdx}{(\alpha-\gamma)(\beta-\gamma)},$$

quae expressio etiam ex priori elicitur. Eodem modo problema in genere resolvi potest.

PROBLEMA 151

1158. *Proposita aequatione differentiali cuiuscunque gradus*

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \ldots + N\frac{d^ny}{dx^n}$$

sumto elemento dx constante et denotate X functionem quamcunque ipsius x eius integrale investigare.

SOLUTIO

Formetur ex hac aequatione formula algebraica

$$A + Bz + Cz^2 + Dz^3 + \ldots + Nz^n = P,$$

quae in factores simplices resolvatur, ut sit

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\ldots(\nu+z),$$

quorum numerus est *n.* Quodsi iam simili modo per singulas integrationes continuo progrediamur, tandem ad hanc aequationem integralem extremam perveniemus

$$Ny = e^{-\nu x}\int e^{(\nu-\mu)x}dx\int e^{(\mu-\lambda)x}dx\int \ldots\ldots\int e^{(\beta-\alpha)x}dx\int e^{\alpha x}Xdx,$$

seu cum factores inter se permutare liceat, erit etiam

$$Ny = e^{-\alpha x}\int e^{(\alpha-\beta)x}dx\int e^{(\beta-\gamma)x}dx\int \ldots\ldots\int e^{(\mu-\nu)x}dx\int e^{\nu x}Xdx.$$

Haec vero expressio per similes reductiones, quibus supra sumus usi, in sequentes partes resolvi potest; ad quas commodius repraesentandas sit brevitatis gratia

$$(\beta-\alpha)(\gamma-\alpha)(\delta-\alpha)....(v-\alpha)=\alpha',$$
$$(\alpha-\beta)(\gamma-\beta)(\delta-\beta)....(v-\beta)=\beta',$$
$$(\alpha-\gamma)(\beta-\gamma)(\delta-\gamma)......(v-\gamma)=\gamma',$$
$$\vdots$$
$$(\alpha-v)(\beta-v)(\gamma-v)......(\mu-v)=v'$$

hincque erit

$$Ny=\tfrac{1}{\alpha'}e^{-\alpha x}\int e^{\alpha x}Xdx+\tfrac{1}{\beta'}e^{-\beta x}\int e^{\beta x}Xdx+\tfrac{1}{\gamma'}e^{-\gamma x}\int e^{\gamma x}Xdx+...+\tfrac{1}{v'}e^{-vx}\int e^{vx}Xdx.$$

Ne autem opus sit ad valores α',β',γ' etc. inveniendos tot factores in se invicem multiplicare, cum sit

$$\tfrac{P}{N(\alpha+z)}=(\beta+z)(\gamma+z)(\delta+z).....(v+z),$$

evidens est hanc formulam praebere valorem α', si in ea statuatur $z=-\alpha$; hoc autem casu fractionis $\frac{P}{N(\alpha+z)}$ tam numerator quam denominator evanescit, ex quo eius valor $\frac{dP}{Ndz}$. Quare cum sit

$$P=A+Bz+Cz^2+Dz^3+...+Nz^n,$$

erit

$$\tfrac{dP}{dz}=B+2Cz+3Dz^2+...+nNz^{n-1},$$

quae expressio vocetur Q, unde patet fore

$$\alpha'=\tfrac{Q}{N}\text{ posito }z=-\alpha,\ \beta'=\tfrac{Q}{N}\text{ posito }z=-\beta,\ \gamma'=\tfrac{Q}{N}\text{ posito }z=-\gamma,\text{etc.}$$

Vel cum his valoribus substitutis aequatio integralis per N dividi queat, sequentes valores colligantur

$$B-2C\alpha+3D\alpha^2-4E\alpha^3+....\pm nN\alpha^{n-1}=\mathfrak{A},$$
$$B-2C\beta+3D\beta^2-4E\beta^3+....\pm nN\beta^{n-1}=\mathfrak{B},$$
$$B-2C\gamma+3D\gamma^2-4E\gamma^3+.......\pm nN\gamma^{n-1}=\mathfrak{C},$$
$$\vdots$$
$$B-2Cv+3Dv^2-4Ev^3+........\pm nNv^{n-1}=\mathfrak{N},$$

quibus constitutis erit integrale quaesitum

$$y=\frac{1}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx+\frac{1}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx+\frac{1}{\mathfrak{C}}e^{-\gamma x}\int e^{\gamma x}Xdx+\text{etc.},$$

quoad omnes factores fuerint in computum ducti.

COROLLARIUM 1

1159. Cum sit

$$\alpha'=\frac{\mathfrak{A}}{N},\quad \beta'=\frac{\mathfrak{B}}{N},\quad \gamma'=\frac{\mathfrak{C}}{N},\quad \text{etc.}$$

erit

$$\mathfrak{A}=N\alpha',\quad \mathfrak{B}=N\beta',\quad \mathfrak{C}=N\gamma',\quad \text{etc.}$$

Hinc ob

$$P=A+Bz+Cz^2+Dz^3+...+Nz^n=N(\alpha+z)(\beta+z)(\gamma+z)....(\gamma+z)$$

erit

$$\mathfrak{A}=\frac{P}{\alpha+z}\text{ posito }z=-\alpha,\mathfrak{B}=\frac{P}{\beta+z}\text{ posito }z=-\beta,\mathfrak{C}=\frac{P}{\gamma+z}\text{ posito }z=-\gamma$$

et ita porro.

COROLLARIUM 2

1160. Regula ergo huius aequationis propositae integrale completum inveniendi ita se habet:

Formetur inde formula algebraica haec

$$A+Bz+Cz^2+Dz^3+...+Nz^n=P,$$

cuius quaerantur omnes factores simplices, qui sint

$$\alpha+z,\ \beta+z,\ \gamma+z,\ \delta+z\quad \text{etc.},$$

quorum multitudo numero n est aequalis, tum pro singulis his factoribus sequentes quantitates constantes definiantur $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$, $\mathfrak{D}$ etc., ut sit

$$\mathfrak{A}=\frac{P}{\alpha+z}\text{ posito }z=-\alpha,\text{seu }\mathfrak{A}=B-2C\alpha+3D\alpha^2-4E\alpha^3+....\pm nN\alpha^{n-1},$$

$$\mathfrak{B}=\frac{P}{\beta+z}\text{ posito }z=-\beta,\text{seu }\mathfrak{B}=B-2C\beta+3D\beta^2-4E\beta^3+....\pm nN\beta^{n-1},$$

$$\mathfrak{B}=\frac{P}{\gamma+z}\text{ posito }z=-\gamma,\text{ seu }\mathfrak{B}=B-2C\gamma+3D\gamma^2-4E\gamma^3+....\pm nN\gamma^{n-1},$$

etc.;

his omnibus inventis erit integrale quaesitum

$$y = \frac{1}{\mathfrak{A}} e^{-\alpha x} \int e^{\alpha x} X dx + \frac{1}{\mathfrak{B}} e^{-\beta x} \int e^{\beta x} X dx + \frac{1}{\mathfrak{C}} e^{-\gamma x} \int e^{\gamma x} X dx + \text{etc.},$$

quae forma tot constat partibus, quot fuerint factores simplices in formula P.

COROLLARIUM 3

1161. Cum hoc modo integrale tot constet partibus, quoti ordinis est aequatio differentialis proposita, et unaquaeque pars per integrationem unam invehat constantem arbitrariam, manifestum est integrale ope huius regulae inventum fore completum.

SCHOLION

1162. Integratio ergo huiusmodi aequationum differentialium nulla amplius laborat difficultate, si modo formulae illius algebraicae P omnes factores simplices seu, quod eodem redit, huius aequationis algebraicae

$$A + Bz + Cz^2 + Dz^3 + \ldots + Nz^n = 0,$$

omnes radices numero n assignari queant. Hic vero duplicis generis casus occurrunt, quibus haec integratio vehementer impeditur, quando scilicet vel duo pluresve eorum factorum simplicium inter se fiunt aequales vel imaginarii, quo quidem posteriori casu hoc tantum incommodi accedit, quod partes quaepiam integralis inventi imaginaria involvant, quae autem facta reductione se mutuo destruunt. Priori vero casu partes ex factoribus aequalibus oriundae adeo fiunt infinitae, sed ita diversis signis affectae, ut coniunctim nihilominus quantitatem finitam referant, cuins valorem nonnisi per plures ambages elicere licet, ubi probe notandum est utroque casu inventionem reliquarum integralis partium, quae factoribus inaequalibus conveniunt, neutiquam hinc turbari. Methodum autem huic fini accommodatam in sequenti problemate explicabo.

PROBLEMA 152

1163. *Proposita aequatione differentiali cuiuscunque gradus*

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + E\frac{d^4y}{dx^4} + \ldots. + N\frac{d^ny}{dx^n}$$

si forma algebraica inde facta

$$P = A + Bz + Cz^2 + Dz^3 + \ldots + Nz^n$$

duos pluresve factores simplices inter se habeat aequales, partem integralis inde oriundam investigare.

SOLUTIO

Sint primo duo factores $\alpha+z$ et $\beta+z$ inter se aequales seu $\beta=\alpha$, reliquus vero factor formae P sit $=Q$, ut habeatur

$$P=(\alpha+z)(\beta+z)Q=(\alpha+z)^2 Q;$$

posito autem $z=-\alpha$ abeat Q in $\mathfrak{C}$. Iam initio saltem litterae α et β ut diversae spectentur excepta quantitate $\mathfrak{C}$, quae utrinque sit eadem, atque pro binis integralis partibus ex his binis factoribus oriundis habebimus

$$\mathfrak{A}=(\beta-\alpha)\mathfrak{C} \text{ et } \mathfrak{B}=(\alpha-\beta)\mathfrak{C}.$$

Partes autem integralis inde oriundae littera v designentur, ut sit

$$(\beta-\alpha)\mathfrak{C}v=e^{-\alpha x}\int e^{\alpha x}Xdx-e^{-\beta x}\int e^{\beta x}Xdx,$$

unde differentiando colligimus

$$(\beta-\alpha)\mathfrak{C}dv=-\alpha e^{-\alpha x}dx\int e^{\alpha x}Xdx+\beta e^{-\beta x}dx\int e^{\beta x}Xdx,$$

ad hanc addatur prior per βdx multiplicata fietque

$$(\beta-\alpha)\mathfrak{C}dv+(\beta-\alpha)\mathfrak{C}\beta vdx=(\beta-\alpha)e^{-\alpha x}dx\int e^{\alpha x}Xdx,$$

quae per $\beta-\alpha$ divisa et per e^{ax} multiplicata ob, $\beta=\alpha$ integrale praebet

$$\mathfrak{C}e^{\alpha x}v=\int dx\int e^{\alpha x}Xdx.$$

Quocirca loco binarum partium ex factoribus aequalibus $\alpha+z$ et $\beta+z$ oriundarum scribi oportet hanc formulam

$$v=\frac{1}{\mathfrak{C}}e^{-\alpha x}\int dx\int e^{\alpha x}Xdx,$$

ubi $\mathfrak{A}$ oritur ex forma $\frac{P}{(\alpha+z)^2}$ posito $z=-\alpha$.

Ponamus iam formulam P tres habere factores simplices aequales, ut sit $\alpha+z=\beta+z=\gamma+z$, quos quidem initio ut diversos spectemus. Ponamus ergo

$$P=(\alpha+z)(\beta+z)(\gamma+z)Q$$

abeatque Q in $\mathfrak{M}$ posito $z=-\alpha$ ac pro integralis partibus habebimus

$$\mathfrak{A}=(\beta-\alpha)(\gamma-\alpha)\mathfrak{M},\quad \mathfrak{B}=(\alpha-\beta)(\gamma-\beta)\mathfrak{M},\quad \mathfrak{C}=(\alpha-\gamma)(\beta-\gamma)\mathfrak{M}\,.$$

Hinc, si summam trium integralis partium, quam quaerimus, littera v denotemus, erit

$$\mathfrak{M}v=\frac{e^{-\alpha x}\int e^{\alpha x}Xdx}{(\beta-\alpha)(\gamma-\alpha)}+\frac{e^{-\beta x}\int e^{\beta x}Xdx}{(\alpha-\beta)(\gamma-\beta)}+\frac{e^{-\gamma x}\int e^{\gamma x}Xdx}{(\alpha-\gamma)(\beta-\gamma)}$$

Cum nunc sit

$$\frac{1}{(\beta-\alpha)(\gamma-\alpha)}+\frac{1}{(\alpha-\beta)(\gamma-\beta)}+\frac{1}{(\alpha-\gamma)(\beta-\gamma)}=0,$$

erit differentiando

$$\frac{\mathfrak{M}dv}{dx}=-\frac{\alpha e^{-\alpha x}\int e^{\alpha x}Xdx}{(\beta-\alpha)(\gamma-\alpha)}-\frac{\beta e^{-\beta x}\int e^{\beta x}Xdx}{(\alpha-\beta)(\gamma-\beta)}-\frac{\beta e^{-\gamma x}\int e^{\gamma x}Xdx}{(\alpha-\gamma)(\beta-\gamma)}\,;$$

ad quam si prima per α multiplicata addatur, fit

$$\mathfrak{M}\left(\frac{dv}{dx}+\alpha v\right)=\frac{e^{-\beta x}\int e^{\beta x}Xdx}{\gamma-\beta}+\frac{e^{-\gamma x}\int e^{\gamma x}Xdx}{\beta-\gamma(\alpha-\gamma)}.$$

Haec aequatio denuo differentietur, ut prodeat

$$\mathfrak{M}\left(\frac{ddv}{dx^2}+\frac{\alpha dv}{dx}\right)=-\frac{\beta e^{-\beta x}\int e^{\beta x}Xdx}{\gamma-\beta}-\frac{\gamma e^{-\gamma x}\int e^{\gamma x}Xdx}{\beta-\gamma(\alpha-\gamma)}\,;$$

ad quam illa per β multiplicata si addatur, oritur

$$\mathfrak{M}\left(\frac{ddv}{dx^2}+\frac{2\alpha dv}{dx}+\alpha\alpha v\right)=e^{-\gamma x}\int e^{\gamma x}Xdx=e^{-\alpha x}\int e^{\alpha x}Xdx,$$

unde iam omnia incommoda sunt sublata. Multiplicetur nunc per $e^{\alpha x}dx$ et integratio dabit

$$\mathfrak{M}e^{\alpha x}\left(\frac{dv}{dx}+\alpha v\right)=\int dx\int e^{\alpha x}Xdx,$$

quae per dx multiplicata denuo fit integrabilis, proditque

$$\mathfrak{M}e^{\alpha x}v=\int dx\int dx\int e^{\alpha x}Xdx.$$

Quocirca si forma *P* factorem habeat cubicum $(\alpha+z)^3$, quaeratur quantitas $\mathfrak{A}$, ut sit $\mathfrak{M}=\frac{P}{(\alpha+z)^3}$ posito $z=-\alpha$, et integralis pars hinc oriunda erit

$$\frac{1}{\mathfrak{M}}e^{-\alpha x}\int dx\int dx\int e^{\alpha x}Xdx,$$

Simili modo si formula *P* quatuor habeat factores aequales, ut sit $P=(\alpha+z)^4 Q$, capiatur $\mathfrak{N}=\frac{P}{(\alpha+z)^4}$ seu in $\mathfrak{N}=Q$ posito $z=-\alpha$ et integralis pars inde nata erit

$$\frac{1}{\mathfrak{N}}e^{-\alpha x}\int dx\int dx\int dx\int e^{\alpha x}Xdx,$$

ac designemus signa sicque etiam casus, quibus formula *P* adhuc plures habet factores aequales, facile resolventur.

Nota. Tota haec solutio est vitiosa, propterea quod, licet quantitates α,β,γ etc., quae ponuntur aequales, ut diversae spectentur, tamen pro singulis membris quantitas $\mathfrak{M}$ eundem valorem retinere assumitur. Quodsi enim litterae α,β,γ etc. infinite parum a se invicem discrepare concipiantur, etiam in valoribus littera $\mathfrak{M}$ indicatis differentiam infinite parvam agnoscere oportet, unde, cum singulae partes integralis fiant infinitae iisque evolutis membra infinita se mutuo tollant, ex differentiis infinite parvis litterae $\mathfrak{M}$ partes quoque finitae emergunt. Correctionem horum errorum petere licet ex seq. Probl. 154, dum factores aequales in aequationem peculiarem coniiciuntur. Malui autem hunc correctionis laborem industriae lectorum relinquere quam hoc opus a tali errore liberare; saepe enim plus prodest errores, in quos etiam exercitatis incidere contingit, conservari, quo melius harum rerum studiosi addiscant, quanta circumspectione cavendum sit, ne in ratiocinando hallucinemur.

COROLLARIUM 1

1164. Notatu hic omnino est dignum, quod hae formulae

$$dv+\alpha vdx,\ ddv+2\alpha dxdv+\alpha^2vdx^2,\ d^3v+3\alpha dxddv+3\alpha^2dx^2dv+\alpha^3vdx^2$$

et in genere haec

$$d^nv+\frac{n}{1}\alpha d^{n-1}v+\frac{n(n-1)}{1\cdot 2}\alpha^2dx^2d^{n-2}v+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}\alpha^3dx^3d^{n-3}v+\text{etc.},$$

si semel per e^{ax} multiplicentur, successive toties integrationem admittant, quot unitates continet index *n*, ita ut postremum integrale sit $e^{ax}v$.

COROLLARIUM 2

1165. Ratio autem huius phaenomeni inde est manifesta, quod, si formula $e^{ax}v$ continuo differentietur sumto elemento dx constante, formulae illae differentiales per e^{ax} multiplicatae prodeant, ita ut sit

$$d^n.e^{\alpha x}v = e^{\alpha x}\left(d^n v + \tfrac{n}{1}\alpha d^{n-1}v + \tfrac{n(n-1)}{1\cdot 2}\alpha^2 dx^2 d^{n-2}v + \text{etc.}\right).$$

COROLLARIUM 3

1166. Aeque memoratu dignum est alterum phaenomenum, quod solutio ista nobis offert, sumtis scilicet numeris quibuscunque $\alpha, \beta, \gamma, \delta$ etc. sequentes aequalitates semper locum habere, ut sit

$$\tfrac{1}{\alpha-\beta} + \tfrac{1}{\beta-\alpha} = 0,$$

$$\frac{1}{(\alpha-\beta)(\alpha-\gamma)} + \frac{1}{(\beta-\alpha)(\beta-\gamma)} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)} = 0,$$

$$\frac{1}{(\alpha-\beta)(\alpha-\gamma)(\alpha-\delta)} + \frac{1}{(\beta-\alpha)(\beta-\gamma)(\beta-\delta)} + \frac{1}{(\gamma-\alpha)(\gamma-\beta)(\gamma-\delta)} + \frac{1}{(\delta-\alpha)(\delta-\beta)(\delta-\gamma)} = 0$$

etc.,

quodcunque numeri hoc modo capiantur.

COROLLARIUM 4

1167. Si formula P in factores simplices resoluta ponatur

$$P = N(\alpha+z)(\beta+z)(\gamma+z)\ldots(\mu+z)(\nu+z),$$

expressio integralis prius inventa (§ 1158), quae erat

$$Ny = e^{-\alpha x}\int e^{(\alpha-\beta)x}dx\int e^{(\beta-\gamma)x}dx\int \ldots\ldots \int e^{(\mu-\nu)x}dx\int e^{\nu x}Xdx,$$

ob factores aequales nulla implicatur difficultate, forma autem posterior [§1160], qua integrale in partes ex singulis factoribus ortas distributum exhibetur et quae ad usum multo magis accommodata videtur, eo difficiliori egebat evolutione.

SCHOLION

1168. Phaenomenum Corollario 3 observatum eo maiorem attentionem meretur, quod etiam ad Arithmeticam vulgarem transferri potest, ubi usu adeo insigni non cariturum videtur, praecipue quod eius demonstratio minime sit obvia, sed ex profundioribus Analyseos penetralibus repeti debeat; ex quo haud alienum fore arbitror, si huic insigni theoremati arithmetico hic locum concedam idque eo magis, quod solutio problematis hic exposita sine demonstratione istius theorematis minime foret perfecta.

THEOREMA ARITHMETICUM

1169. *Si habeantur numeri quotcunque a, b, c, d etc. ex iisque, dum a quolibet singuli reliqui subtrahantur, formentur sequentia produota*

$$(a-b)(a-c)(a-d)(a-e)\text{ etc.}=\mathfrak{A},$$
$$(b-a)(b-c)(b-d)(b-e)\text{ etc.}=\mathfrak{B},$$
$$(c-a)(c-b)(c-d)(c-e)\text{ etc.}=\mathfrak{C},$$
$$(d-a)(d-b)(d-c)(d-e)\text{etc.}=\mathfrak{D}$$
$$\text{etc.}$$

semper habehitur

$$\tfrac{1}{\mathfrak{A}}+\tfrac{1}{\mathfrak{B}}+\tfrac{1}{\mathfrak{C}}+\tfrac{1}{\mathfrak{D}}+\text{etc.}=0.$$

DEMONSTRATIO

Consideretur secundum principia in *Introductione ad Analysin* tradita haec fractio

$$\frac{Z}{(z-a)(z-b)(z-c)(z-d)\text{ etc.}},$$

ubi Z denotet eiusmodi functionem rationalem integram ipsius z, in qua summa potestas ipsius z minor sit numero factorum denominatoris; haecque fractio resolvi poterit in has fractiones simplices, quibus ea iunctim sumtis sit aequalis, scilicet

$$\frac{A}{z-a}+\frac{B}{z-b}+\frac{C}{z-c}+\frac{D}{z-d}+\text{ etc.}$$

Ad quam resolutionem sumamus illum numeratorem $Z=z^n$ existente n numero integro minore, quam denominator continet factores, atque hi numeratores ita definiuntur, ut sit

$$A = \frac{a^n}{(a-b)(a-c)(a-d)\text{ etc.}},$$

$$B = \frac{b^n}{(b-a)(b-c)(b-d)\text{ etc}},$$

$$C = \frac{c^n}{(c-a)(c-b)(c-d)\text{etc.}}$$

etc.

Cum igitur istae fractiones negative sumtae, nempe

$$\frac{A}{a-z} + \frac{B}{b-z} + \frac{C}{c-z} + \frac{D}{d-z} + \text{etc.}$$

ad fractionem propositam adiectae in nihilum abeant, si z sit numerorum propositorum a, b, c, d etc. ultimus, quorum adeo multitudo maior est quam $n+1$, ponatur

$$(a-b)(a-c)(a-d).....(a-z) = \mathfrak{A},$$
$$(b-a)(b-c)(b-d).....(b-z) = \mathfrak{B},$$
$$(c-a)(c-b)(c-d)......(c-z) = \mathfrak{C},$$
$$(d-a)(d-b)(d-c).....(d-z) = \mathfrak{D}$$

etc.

$$(z-a)(z-b)(z-c)......(z-y) = \mathfrak{Z},$$

ut fractio proposita sit $\frac{z^n}{\mathfrak{Z}}$. Atque hinc perspicuum est summam omnium harum fractionum esse

$$\frac{a^n}{\mathfrak{A}} + \frac{b^n}{\mathfrak{B}} + \frac{c^n}{\mathfrak{C}} + \frac{d^n}{\mathfrak{D}} + ... + \frac{z^n}{\mathfrak{Z}} = 0,$$

dum sit $n+1$ minor numero terminorum. Sumto ergo $n = 0$ oritur casus Theorematis.

COROLLARIUM 1

1170. Haec si transferantur ad numeros $\mathfrak{A}$, $\mathfrak{B}$, $\mathfrak{C}$ etc. supra (§ 1160) definitos, ubi aliquod leve discrimen in factorum constitutione probe est notandum, intelligetur esse

$$+\frac{1}{\mathfrak{A}}+\frac{1}{\mathfrak{B}}+\frac{1}{\mathfrak{C}}+\frac{1}{\mathfrak{D}}+\text{etc.}=0,$$
$$-\frac{\alpha}{\mathfrak{A}}-\frac{\beta}{\mathfrak{B}}-\frac{\gamma}{\mathfrak{C}}-\frac{\delta}{\mathfrak{D}}-\text{etc.}=0,$$
$$+\frac{\alpha^2}{\mathfrak{A}}+\frac{\beta^2}{\mathfrak{B}}+\frac{\gamma^2}{\mathfrak{C}}+\frac{\delta^2}{\mathfrak{D}}+\text{etc.}=0,$$
$$-\frac{\alpha^3}{\mathfrak{A}}-\frac{\beta^3}{\mathfrak{B}}-\frac{\gamma^3}{\mathfrak{C}}-\frac{\delta^3}{\mathfrak{D}}-\text{etc.}=0,$$
etc.,

donec perveniatur ad hanc formam

$$\pm\frac{\alpha^{n-1}}{\mathfrak{A}}\pm\frac{\beta^{n-1}}{\mathfrak{B}}\pm\frac{\gamma^{n-1}}{\mathfrak{C}}\pm\frac{\delta^{n-1}}{\mathfrak{D}}\pm\text{etc.},$$

cuius summa non amplius est evanescens, sed aequalis fractioni $\frac{1}{N}$.

COROLLARIUM 2

1171. Hoc etiam ex evolutione formae in Theoremate adhibitae colligere licet. Etenim si ea statuatur

$$\frac{z^{n-1}}{(z-a)(z-b)(z-c).....(z-y)}$$

existente omnium litterarum *a, b, c* etc. numero $=n$, quia hic numerator z^{n-1} tot habet dimensiones, quot sunt factores in denominatore, pars integra in hac fractione contenta est unitas; quae etiam facta resolutione conservatur et in applicatione ad casum memoratum abit in $\frac{1}{N}$.

SCHOLION

1172. Post huius Theorematis demonstrationem demum clare a posteriori ostendi potest, quemadmodum integrale supra (§ 1160) exhibitum aequationi differentiali ibidem propositae satisfaciat. Notatis enim expressionibus §1170 cum supra invenerimus integrale

$$y=\frac{1}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx+\frac{1}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx+\text{etc.},$$

erit continuo differentiando

$$\frac{dy}{dx}=-\frac{\alpha}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx+\frac{-\beta}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx-\text{etc.},$$
$$\frac{ddy}{dx^2}=+\frac{\alpha^2}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx+\frac{\beta^2}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx+\text{etc.},$$
$$\frac{d^3y}{dx^3}=-\frac{\alpha^3}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx-\frac{\beta^3}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx-\text{etc.}$$
etc.

usque ad

$$\frac{d^{n-1}y}{dx^{n-1}} = \pm\frac{\alpha^{n-1}}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx \pm \frac{\beta^{n-1}}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx \pm \text{etc.},$$

unde sequens forma differentialis resultat

$$\frac{d^{n}y}{dx^{n}} = \mp\frac{\alpha^{n}}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx \mp \frac{\beta^{n}}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx \mp \text{etc.}$$
$$\pm\left(\frac{\alpha^{n-1}}{\mathfrak{A}} + \frac{\beta^{n-1}}{\mathfrak{B}} + \frac{\gamma^{n-1}}{\mathfrak{C}} + \text{etc}\right)X,$$

quod postremum membrum abit in $\frac{1}{N}X$.
Si iam omnes hae formae singulatim multiplicentur per quantitates $A, B, C, D, \ldots N$, quoniam est

$$A - B\alpha + C\alpha^2 - D\alpha^3 + \ldots . \mp N\alpha^n = 0,$$
$$A - B\beta + C\beta^2 - D\beta^3 + \ldots . \mp N\beta^n = 0,$$
$$\text{etc.},$$

propterea quod $\alpha + z,\ \beta + z,\ \gamma + z$ etc. sunt factores formae

$$A + Bz + Cz^2 + Dz^3 + \ldots . + Nz^n,$$

manifesto obtinebimus

$$Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \ldots . + N\frac{d^ny}{dx^n} = X,$$

quae est ipsa aequatio differentialis initio proposita.

PROBLEMA 153

1173. *Proposita aequatione differentiali cuiuscunque gradus*

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \ldots . + N\frac{d^ny}{dx^n}$$

si expressio algebraica h//inc formata

$$P = A + Bz + Cz^2 + Dz^3 + \ldots . + Nz^n$$

duos habeat factores simplices imaginarios factore duplici $ff + 2fz\cos.\theta + zz$ *contentos, investigare partes integralis hinc oriundas.*

SOLUTIO

Sint $\alpha + z$ et $\beta + z$ hi duo factores imaginarii, ut sit

$$\alpha = f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right) \text{ et } \beta = f\left(\cos.\theta - \sqrt{-1}\cdot\sin.\theta\right)$$

ob $(\alpha + z)(\beta + z) = ff + 2fz\cos.\theta + zz$, ac statuatur

$$P = (ff + 2fz\cos.\theta + zz)Q$$

existente

$$Q = A' + B'z + C'z^2 + ... + N'z^{n-2}.$$

Cum igitur integralis partes ex binis illis factoribus simplicibus imaginariis ortae sint

$$\tfrac{1}{\mathfrak{A}}e^{-\alpha x}\int e^{\alpha x}Xdx + \tfrac{1}{\mathfrak{B}}e^{-\beta x}\int e^{\beta x}Xdx = v,$$

hos valores imaginarios ad realitatem perduci oportet. Erunt autem $\mathfrak{A}$ et $\mathfrak{B}$ quantitates imaginariae resultantes ex forma $\left(f\cos.\theta \mp \sqrt{-1}\cdot f\sin.\theta + z\right)Q$, si loco z scribatur $-f\cos.\theta \mp \sqrt{-1}\cdot f\sin.\theta$. At facta hac substitutione fit

$$\begin{aligned} Q = {} & A' - B'f\cos.\theta + C'ff\cos.2\theta - D'f^3\cos.3\theta + \text{etc.} \\ & \mp\sqrt{-1}\cdot B'f\sin.\theta \pm \sqrt{-1}\cdot C'ff\sin.2\theta \mp \sqrt{-1}\cdot D'f^3\sin.3\theta \pm \text{etc.} \end{aligned}$$

Ponamus brevitatis gratia

$$A' - B'f\cos.\theta + C'ff\cos.2\theta - D'f^3\cos.3\theta + \text{etc.} = \mathfrak{M}$$

et

$$-B'f\sin.\theta + C'ff\sin.2\theta - D'f^3\sin.3\theta + \text{etc.} = \mathfrak{N},$$

ut sit $Q = \mathfrak{M} \pm \mathfrak{N}\sqrt{-1}$, ubi signorum ambiguorum superius valet pro litteris α et $\mathfrak{A}$, inferius pro litteris β et $\mathfrak{B}$. Hinc ergo erit

$$\mathfrak{A} = -2\sqrt{-1}\cdot f\sin.\theta\left(\mathfrak{M} + \mathfrak{N}\sqrt{-1}\right) \text{ et } \mathfrak{B} = +2\sqrt{-1}\cdot f\sin.\theta\left(\mathfrak{M} - \mathfrak{N}\sqrt{-1}\right)$$

ideoque

$$2v\sqrt{-1}\cdot f\sin.\theta=-\frac{e^{-\alpha x}\int e^{\alpha x}Xdx}{\mathfrak{M}+\mathfrak{N}\sqrt{-1}}+\frac{e^{-\beta x}\int e^{\beta x}Xdx}{\mathfrak{M}-\mathfrak{N}\sqrt{-1}}.$$

Est vero

$$e^{\alpha x}=e^{f\cos.\theta}\left(\cos.\left(fx\sin.\theta\right)+\sqrt{-1}\cdot\sin.\left(fx\sin.\theta\right)\right)$$

et

$$e^{\beta x}=e^{f\cos.\theta}\left(\cos.\left(fx\sin.\theta\right)-\sqrt{-1}\cdot\sin.\left(fx\sin.\theta\right)\right).$$

Sit brevitatis gratia angulus $fx\sin.\theta=\varphi$; erit

$$\begin{aligned}&-2\sqrt{-1}\cdot v\left(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N}\right)f\sin.\theta\\
&=-\left(\mathfrak{M}-\mathfrak{N}\sqrt{-1}\right)e^{-fx\cos.\theta}\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)\int e^{fx\cos.\theta}Xdx\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)\\
&+\left(\mathfrak{M}+\mathfrak{N}\sqrt{-1}\right)e^{-fx\cos.\theta}\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)\int e^{fx\cos.\theta}Xdx\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)\\
&=e^{-fx\cos.\theta}2\sqrt{-1}\left(\mathfrak{M}\sin.\varphi+\mathfrak{N}\cos.\varphi\right)\int e^{fx\cos.\theta}Xdx\cos.\varphi\\
&-e^{-fx\cos.\theta}2\sqrt{-1}\left(\mathfrak{M}\cos.\varphi-\mathfrak{N}\sin.\varphi\right)\int e^{fx\cos.\theta}Xdx\sin.\varphi.\end{aligned}$$

Quocirca habebimus integralis partem quaesitam

$$v=\frac{1}{\left(\mathfrak{M}\mathfrak{M}+\mathfrak{N}\mathfrak{N}\right)f\sin.\theta}\left\{\begin{aligned}&+e^{-fx\cos.\theta}\left(\mathfrak{M}\sin.\varphi+\mathfrak{N}\cos.\varphi\right)\int e^{fx\cos.\theta}Xdx\cos.\varphi\\&-e^{-fx\cos.\theta}\left(\mathfrak{M}\cos.\varphi-\mathfrak{N}\sin.\varphi\right)\int e^{fx\cos.\theta}Xdx\sin.\varphi\end{aligned}\right\}$$

existente $\varphi=fx\sin.\theta$.

COROLLARIUM 1

1174. Praecipuum igitur opus hic consistit in inventione formulae imaginariae $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$, quae colligi debet ex quantitate Q, dum loco z scribitur valor imaginarius $f\left(\cos.\theta+\sqrt{-1}\cdot f\sin.\theta\right)$, unde hoc commodi nascitur, ut loco z^n scribi oporteat $\left(-f\right)^n\left(\cos.n\theta+\sqrt{-1}\cdot\sin.n\theta\right)$.

COROLLARIUM 2

1175. Cum sit $Q=\frac{P}{ff+2fz\cos.\theta+zz}$ etiam ex hac forma per eandem substitutionem formula imaginaria $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$ inveniri potest, ubi autem notandum est hac substitutione tam numeratorem P quam denominatorem evanescere. Ex quo manifestum est valorem istius formulae rite obtineri ex hac fractione

$$\frac{dP}{2(f\cos.\theta+z)dz} = \frac{dP}{-2\sqrt{-1}\cdot f\sin.\theta dz}.$$

COROLLARIUM 3

1176. Quoniam igitur est

$$\tfrac{dP}{dz} = B + 2Cz + 3Dz^2 + 4Ez^3 + \ldots\ + nNz^{n-1},$$

si statuamus

$$\mathfrak{P} = B - 2Cf\cos.\theta + 3Df^2\cos.2\theta - 4Ef^3\cos.3\theta + \ldots\ \pm nNf^{3n-1}\cos.(n-1)\theta,$$
$$\mathfrak{Q} = \quad -2Cf\sin.\theta + 3Df^2\sin.2\theta - 4Ef^3\sin.3\theta + \ldots\ \pm nNf^{3n-1}\sin.(n-1)\theta,$$

ut facta substitutione fiat $\tfrac{dP}{dz} = \mathfrak{P} + \sqrt{-1}\cdot\mathfrak{Q}$, habebimus

$$\mathfrak{M} + \mathfrak{N}\sqrt{-1} = \tfrac{\mathfrak{P}+\sqrt{-1}\cdot\mathfrak{Q}}{-2\sqrt{-1}\cdot f\sin.\theta} = \tfrac{-\mathfrak{Q}+\sqrt{-1}\cdot\mathfrak{P}}{2f\sin.\theta}$$

ideoque

$$\mathfrak{M} = \tfrac{-\mathfrak{Q}}{2f\sin.\theta} \text{ et } \mathfrak{N} = \tfrac{\mathfrak{P}}{2f\sin.\theta}.$$

COROLLARIUM 4

1177. Immediate ergo ex quantitate P indeque derivatis $\mathfrak{P}$ et $\mathfrak{Q}$ posito $fx\sin.\theta = \varphi$ *;* integralis pars ex factore duplici $ff + 2fzcos.\theta + zz$ nata erit expressa

$$v = \tfrac{2e^{-fx\cos.\theta}}{\mathfrak{PP}+\mathfrak{NN}}\left\{\begin{matrix}(\mathfrak{P}\cos.\varphi - \mathfrak{Q}\sin.\varphi)\int e^{fx\cos.\theta}Xdx\cos.\varphi\\ +(\mathfrak{P}\sin.\varphi + \mathfrak{Q}\cos.\varphi)\int e^{fx\cos.\theta}Xdx\sin.\varphi\end{matrix}\right\}.$$

SCHOLION

1178. Quotcunque ergo forma

$$P = A + Bz + Cz^2 + Dz^3 + \text{etc.}$$

habuerit factores duplices, pro singulis ope horum praeceptorum partes integralis facile definiuntur, et quia hinc inventio partium, quae factoribus simplicibus conveniunt, sive ii sint inaequales sive aequales, non turbatur, omnibus partibus in unam summam coniectis habebitur integrale completum aequationis differentialis propositae. Verum tamen haec praecepta non sufficiunt, si factorum duplicium bini pluresve inter se fuerint aequales; huiusmodi enim casus peculiarem exigunt evolutionem similem eius, qua pro casu duorum pluriumve factorum simplicium inter se aequalium sum usus. Ne autem hanc tractationem nimis protraham, sufficiet casum pro duobus factoribus duplicibus inter se aequalibus evolvisse, eum inde methodus ad plures facile extendatur.

PROBLEMA 154

1179. *Proposita aequatione differentiali cuiuscunque gradus*

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \ldots + N\frac{d^ny}{dx^n}$$

si expressio algebraica inde formata

$$P = A + Bz + Cz^2 + Dz^3 + \ldots + Nz^n$$

habeat factorem duplicem quadratum $\left(ff + 2fz\cos.\theta + zz\right)^2$*, partem integralis ei convenientem investigare.*

SOLUTIO

Ponamus ergo $P = A + Bz + Cz^2 + Dz^3 + \ldots + Nz^n$ sitque

$$Q = A' + B'z + C'zz + \ldots + N'z^{n-4}$$

ac primo imaginaria non curantes statuamus

$$\alpha = f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right) \text{ et } \beta = f\left(\cos.\theta - \sqrt{-1}\cdot\sin.\theta\right) \text{ut sit}$$

$$P = (\alpha + z)^2(\beta + z)^2 Q.$$

Iam ex iis, quae supra (§ 1163) de binis factoribus simplicibus aequalibus docuimus, ponamus formam $\frac{P}{(\alpha+z)^2} = (\beta + z)^2 Q$ posito $z = -\alpha$ abire in $\mathfrak{A}$, at hanc formam $\frac{P}{(\beta+z)^2} = (\alpha + z)^2 Q$ posito $z = -\beta$ in $\mathfrak{B}$; quibus quantitatibus $\mathfrak{A}$ et $\mathfrak{B}$ inventis ibi ostendi fore integralis partes hinc oriundas

$$\frac{1}{\mathfrak{A}}e^{-\alpha x}\int dx\int e^{\alpha x}Xdx + \frac{1}{\mathfrak{B}}e^{-\beta x}\int dx\int e^{\beta x}Xdx = v,$$

quas, cum iam imaginaria involvant, ad realitatem reduci oportet. Faciamus ut in problemate praecedente

$$\mathfrak{M} = A' - B' f\cos.\theta + C' ff\cos.2\theta - D' f^3\cos.3\theta + \text{etc.}$$
$$\mathfrak{N} = \quad - B' f\sin.\theta + C' ff\sin.2\theta - D' f^3\sin.3\theta + \text{etc.},$$

ut quantitas Q posito $z = -\alpha = -f\left(\cos.\theta + \sqrt{-1}\cdot\sin.\theta\right)$ abeat in $\mathfrak{M} + \mathfrak{N}\sqrt{-1}$, at posito $z = -\beta = -f\left(\cos.\theta - \sqrt{-1}\cdot\sin.\theta\right)$ in $\mathfrak{M} - \mathfrak{N}\sqrt{-1}$.

Cum iam sit

$$(\beta - \alpha)^2 = \left(-2\sqrt{-1}\cdot f\sin.\theta\right)^2 = -4ff\sin.^2\theta,$$

cui quoque $(\alpha - \beta)^2$ aequatur, erit

$$\mathfrak{A} = -4ff\sin.^2\theta\left(\mathfrak{M} + \mathfrak{N}\sqrt{-1}\right) \text{ et } \mathfrak{B} = -4ff\sin.^2\theta\left(\mathfrak{M} - \mathfrak{N}\sqrt{-1}\right)$$

ideoque

$$-4ff\sin.^2\theta(\mathfrak{MM} + \mathfrak{NN})v$$
$$= \left(\mathfrak{M} - \mathfrak{N}\sqrt{-1}\right)e^{-\alpha x}\int dx\int e^{\alpha x}Xdx + \left(\mathfrak{M} + \mathfrak{N}\sqrt{-1}\right)e^{-\beta x}\int dx\int e^{\beta x}Xdx.$$

At posito $fx\sin.\theta = \varphi$ est, ut vidimus [§ 1173],

$$e^{\alpha x} = e^{fx\cos.\theta}\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right), \quad e^{-\alpha x} = e^{-fx\cos.\theta}\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right)$$

et

$$e^{\beta x} = e^{fx\cos.\theta}\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right), \quad e^{-\beta x} = e^{-fx\cos.\theta}\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right)$$

Unde illius aequationis alterum membrum abit in

$$e^{-fx\cos.\theta}\left(\mathfrak{M}\cos.\varphi - \mathfrak{N}\sin.\varphi - \mathfrak{N}\sqrt{-1}\cdot\cos.\varphi - \mathfrak{M}\sqrt{-1}\cdot\sin.\varphi\right)$$
$$\times\int dx\int e^{-fx\cos.\theta}Xdx\left(\cos.\varphi + \sqrt{-1}\cdot\sin.\varphi\right)$$
$$+e^{-fx\cos.\theta}\left(\mathfrak{M}\cos.\varphi - \mathfrak{N}\sin.\varphi + \mathfrak{N}\sqrt{-1}\cdot\cos.\varphi + \mathfrak{M}\sqrt{-1}\cdot\sin.\varphi\right)$$
$$\times\int dx\int e^{-fx\cos.\theta}Xdx\left(\cos.\varphi - \sqrt{-1}\cdot\sin.\varphi\right),$$

ubi partes imaginariae sponte se destruunt, ita ut obtineatur

$$v = \frac{1}{2(\mathfrak{MM} + \mathfrak{NN})ff\sin.^2\theta}\left\{\begin{array}{l} -e^{-fx\cos.\theta}\left(\mathfrak{M}\cos.\varphi - \mathfrak{N}\sin.\varphi\right)\int dx\int e^{fx\cos.\theta}Xdx\cos.\varphi \\ -e^{-fx\cos.\theta}\left(\mathfrak{M}\cos.\varphi + \mathfrak{N}\sin.\varphi\right)\int dx\int e^{fx\cos.\theta}Xdx\sin.\varphi \end{array}\right\}$$

seu hoc modo

$$v=\frac{-e^{-fx\cos.\theta}}{2(\mathfrak{MM}+\mathfrak{NN})ff\sin.^2\theta}\left\{\begin{array}{l}(\mathfrak{M}\cos.\varphi-\mathfrak{N}\sin.\varphi)\int dx\int e^{fx\cos.\theta}Xdx\cos.\varphi\\+(\mathfrak{M}\sin.\varphi+\mathfrak{N}\cos.\varphi)\int dx\int e^{fx\cos.\theta}Xdx\sin.\varphi\end{array}\right\},$$

quae expressio ab imaginariis penitus est liberata.

Nota. Etiam haec solutio insigni correctione indiget diligentiae lectorum relicta.

COROLLARIUM 1

1180. Quoniam formula imaginaria $\mathfrak{M}+\mathfrak{N}\sqrt{-1}$ nascitur ex quantitate *Q*, si loco *z* scribatur $-f\left(\cos.\theta+\sqrt{-1}\cdot\sin.\theta\right)$, eadem positione quoque reperietur ex forma $\frac{P}{(ff+2fz\cos.\theta+zz)^2}$, verum hic tam numerator quam denominator prodit evanescens.

COROLLARIUM 2

1181. Orietur ergo quoque idem valor ex formula

$$\frac{dP}{4dz\left(f^3\cos.\theta+ffz\left(1+2\cos.^2\theta\right)+3fzz\cos.\theta+z^3\right)};$$

ubi cum idem incommodum denuo recurrat, orietur quoque ex hac formula

$$\frac{ddP}{4dz^2\left(ff\left(1+2\cos.^2\theta\right)+6fz\cos.\theta+3zz\right)}.$$

COROLLARIUM 3

1182. Statuatur hic primo in denominatore $z=-f\left(\cos.\theta+\sqrt{-1}\cdot\sin.\theta\right)$ fietque haec formula

$$\frac{-ddP}{8ffdz^2\sin.^2\theta}.$$

Deinde cum sit

$$\frac{ddP}{2dz^2}=C+3Dz+6Ezz+\ldots+\frac{n(n-1)}{1\cdot2}Nz^{n-2},$$

ponamus brevitatis gratia

$$\mathfrak{P} = C - 3Df\cos.\theta + 6Eff\cos.2\theta - .... \pm \frac{n(n-1)}{1\cdot 2} Nf^{n-2}\cos.(n-2)\theta,$$
$$\mathfrak{Q} = \quad -3Df\sin.\theta + 6Eff\sin.2\theta - .... \pm \frac{n(n-1)}{1\cdot 2} Nf^{n-2}\sin.(n-2)\theta,$$

ut sit facta substitutione

$$\frac{ddP}{2dz^2} = \left(\mathfrak{P} + \mathfrak{Q}\sqrt{-1}\right) \text{ ideoque } \mathfrak{M} + \mathfrak{N}\sqrt{-1} = \frac{-\mathfrak{P} - \mathfrak{Q}\sqrt{-1}}{4ff\sin.^2\theta}$$

et consequenter

$$\mathfrak{M} = \frac{-\mathfrak{P}}{4ff\sin.^2\theta} \quad \text{et} \quad \mathfrak{N} = \frac{-\mathfrak{Q}}{4ff\sin.^2\theta}.$$

Quos ergo valores in parte integralis inventa substituere licet.

COROLLARIUM 4

1183. Facta autem substitutione factor duplex quadratus

$$(ff + 2fz\cos.\theta + zz)^2$$

hanc praebet integralis partem

$$v = \frac{2e^{-fx\cos.\theta}}{\mathfrak{PP} + \mathfrak{QQ}} \left\{ \begin{array}{l} (\mathfrak{P}\cos.\varphi - \mathfrak{Q}\sin.\varphi)\int dx \int e^{fx\cos.\theta} Xdx\cos.\varphi \\ +(\mathfrak{P}\sin.\varphi + \mathfrak{Q}\cos.\varphi)\int dx \int e^{fx\cos.\theta} Xdx\sin.\varphi \end{array} \right\},$$

ubi φ denotat angulum $fx\sin.\theta$.

SCHOLION

1184. Si hanc expressionem cum ea, quam problemate praecedente invenimus, comparemus, vix actuali simili evolutione erit opus pro casibus magis complicatis. Ita si quantitas

$$P = A + Bz + Cz^2 + Dz^3 + ... + Nz^n$$

factorem habeat duplicem cubicum

$$(ff + 2fz\cos.\theta + zz)^3,$$

quantitates $\mathfrak{P}$ et $\mathfrak{Q}$, ita definiuntur, ut sit

$$\mathfrak{P} = D - 4Ef\cos.\theta + 10Fff\cos.2\theta - 20Gf^3\cos.3\theta + ... \pm \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} Nf^{n-3}\cos.(n-3)\theta,$$
$$\mathfrak{Q} = -4Ef\sin.\theta + 10Fff\sin.2\theta - 20Gf^3\sin.3\theta + ... \pm \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3} Nf^{n-3}\sin.(n-3)\theta,$$

quibus inventis erit integralis pars hinc nata

$$v = \frac{2e^{-fx\cos.\theta}}{\mathfrak{PP}+\mathfrak{QQ}}\left\{\begin{aligned}&\left(\mathfrak{P}\cos.\varphi-\mathfrak{Q}\sin.\varphi\right)\int dx\int dx\int e^{fx\cos.\theta}Xdx\cos.\varphi\\&+\left(\mathfrak{P}\sin.\varphi+\mathfrak{Q}\cos.\varphi\right)\int dx\int dx\int e^{fx\cos.\theta}Xdx\sin.\varphi\end{aligned}\right\}$$

neque iam ulterior progressio ulli amplius difficultati est obnoxia. Quocirca aequationis hoc capite propositae resolutionem ita concinne mihi equidem absolvisse videor, ut nihil ampIius desiderari possit. Interim hoc argumentum maxime illustrabitur, si haec praecepta ad exempla particularia accommodabimus; cui instituto sequens caput est destinatum. Ante autem insignem proprietatem circa huiusmodi aequationes generales proponam, quae in Analysi ingentem usum habitura videtur.

PROBLEMA 155

1185. *Proposita aequatione differentiali cuiuscunque gradus*

$$X = Ay + B\frac{dy}{dx} + C\frac{ddy}{dx^2} + D\frac{d^3y}{dx^3} + \ldots + N\frac{d^{m+n}y}{dx^{m+n}}$$

si formula algebraica inde nata

$$P = A + Bz + Cz^2 + Dz^3 + \ldots + Nz^{m+n}$$

duobus factoribus constet $P = QR$ *, ut sit*

$$Q = \mathfrak{A} + \mathfrak{B}z + \mathfrak{C}z^2 + \ldots + \mathfrak{N}z^m \quad et \quad R = \mathfrak{a} + \mathfrak{b}z + \mathfrak{c}z^2 + \ldots + \mathfrak{n}z^n,$$

integrationem illius aequationis ad integrationem binarum aequationum simpliciorum revocare.

SOLUTIO

Si formam integralem primo (§ 1158) perpendamus, haud difficulter inde colligimus, postquam hanc aequationem integraverimus

$$X = \mathfrak{A}v + \mathfrak{B}\frac{dv}{dx} + \mathfrak{C}\frac{ddv}{dx^2} + \ldots + \mathfrak{N}\frac{d^mv}{dx^m}$$

indeque valorem ipsius v per x et X definiverimus, valorem ipsius y pro aequatione proposita ex hac aequatione erutum iri

$$v = \mathfrak{a}y + \mathfrak{b}\frac{dy}{dx} + \mathfrak{c}\frac{ddy}{dx^2} + \ldots + \mathfrak{n}\frac{d^ny}{dx^n},$$

cuius ratio adeo in promtu est posita, dum ex hac aequatione valores pro v eiusque differentialibus substituantur. Prodibit enim

$$\begin{aligned} X = \mathfrak{A}\mathfrak{a}y &+ \mathfrak{A}\mathfrak{b}\frac{dy}{dx} + \mathfrak{A}\mathfrak{c}\frac{ddy}{dx^2} + \mathfrak{A}\mathfrak{d}\frac{d^3y}{dx^3} + \text{etc.} \\ &+ \mathfrak{B}\mathfrak{a} \quad + \mathfrak{B}\mathfrak{b} \quad\ + \mathfrak{B}\mathfrak{c} \\ &\qquad\quad\ + \mathfrak{C}\mathfrak{a} \quad\ + \mathfrak{C}\mathfrak{b} \\ &\qquad\qquad\qquad\quad + \mathfrak{D}\mathfrak{a} \end{aligned}$$

Cum autem per hypothesin sit $P = QR$, seriebus Q et R in se multiplicatis necesse est fieri

$$A = \mathfrak{A}a, \quad B = \mathfrak{A}\mathfrak{b} + \mathfrak{B}\mathfrak{a}, \quad C = \mathfrak{A}c + \mathfrak{B}\mathfrak{b} + \mathfrak{C}\mathfrak{a} \text{ etc.}$$

sicque haec postrema aequatio ad ipsam propositam reducitur.

COROLLARIUM 1

1186. Si tantum ad factores simplices respiciamus, prioris aequationis integrale per huiusmodi terminos exprimitur

$$v = \Gamma e^{-\alpha x}\int e^{\alpha x} X dx \quad \text{etc.,}$$

posterioris vero aequationis integrale per huiusmodi

$$y = \Delta e^{-\beta x}\int e^{\beta x} v dx \quad \text{etc}$$

COROLLARIUM 2

1187. Quodsi iam in singulis terminis posterioris integralis substituamus singulos prioris, fiet

$$y = \Gamma\Delta e^{-\beta x}\int e^{(\beta-\alpha)x} dx \int e^{\alpha x} X dx,$$

quae forma ad hanc reducitur

$$y = \tfrac{\Gamma\Delta}{\beta-\alpha}\left(e^{-\alpha x}\int e^{\alpha x} X dx - e^{-\beta x}\int e^{\beta x} X dx\right),$$

euiusmodi termini per integrationem aequationis propositae immediate inveniuntur.

COROLLARIUM 3

1188. Si hic fuisset $\beta = \alpha$, sine ulla reductione statim prodiisset forma

$$y = \Gamma\Delta e^{-\alpha x}\int dx \int e^{\alpha x} X dx$$

supra pro casu duorum factorum simplicium aequalium inventa. Interim cum totum negotium ad resolutionem in factores vel simplices vel duplices reales redeat, ipsa aequatio proposita modo ante exposito facillime expeditur.