THE SECOND SECTION

ON
THE RESOLUTION OF DIFFERENTIAL EQUATIONS OF THE THIRD OR HIGHER ORDERS WHICH ONLY INVOLVE TWO VARIABLES

CHAPTER I

CONCERNED WITH THE INTEGRATION OF SIMPLE DIFFERENTIAL FORMULAS OF THE THIRD OR HIGHER ORDERS

PROBLEMA 140

1100. *On taking the element dx constant, to find the complete integral of the formulas* $d^3y=0,\quad d^4y=0,\quad d^5y=0$ *etc. and in general of this formula* $d^ny=0$.

SOLUTION

Since dx shall be constant, the equation $d^3y=0$ by integration gives $ddy=adx^2$ and from this on integrating again $dy=\alpha xdx+\beta dx$ and finally $y=\frac{1}{2}\alpha x^2+\beta x+\gamma$. In a like manner from the equation $d^4y=0$ by a fourfold integration there is found

$$d^3y=\alpha dx^3,$$
$$ddy=\alpha xdx^2+\beta dx^2,$$
$$dy=\tfrac{1}{2}\alpha x^2dx+\beta xdx+\gamma dx$$

and finally

$$y=\tfrac{1}{6}\alpha x^3+\tfrac{1}{2}\beta x^2+\gamma x+\delta.$$

Moreover from the equation $d^5y=0$ the integration repeated five times gives

$$y=\tfrac{1}{24}\alpha x^4+\tfrac{1}{6}\beta x^3+\tfrac{1}{2}\gamma x^2+\delta x+\varepsilon.$$

But the integral of this equation $d^6y=0$ is deduced

$$y = \frac{1}{120}\alpha x^5 + \frac{1}{24}\beta x^4 + \frac{1}{6}\gamma x^3 + \frac{1}{2}\delta x^2 + \varepsilon x + \zeta .$$

and thus it is allowed to progress to the form of the kind $d^n y = 0$, however many orders there should be, provided *n* is a positive whole number.

COROLLARY 1

1101. Hence by starting from the most simple form the integrals proceed in the following order:

Formulas	Complete integrals
$dy = 0$	$y = \alpha$
$ddy = 0$	$y = \alpha x + \beta$
$d^3 y = 0$	$y = \frac{1}{2}\alpha x^2 + \beta x + \gamma$
$d^4 y = 0$	$y = \frac{1}{6}\alpha x^3 + \frac{1}{2}\beta x^2 + \gamma x + \delta.$
etc.	etc.

COROLLARY 2

1102. Because the constants α, β, γ etc. depend on our choice, fractions can be rejected without risk:

Formulas	Integrals
$dy = 0$	$y = \alpha$
$ddy = 0$	$y = \alpha x + \beta$
$d^3 y = 0$	$y = \alpha x^2 + \beta x + \gamma$
$d^4 y = 0$	$y = \alpha x^3 + \beta x^2 + \gamma x + \delta.$
$d^5 y = 0$	$y = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon.$
etc.	etc.

COROLLARY 3

1103. Therefore whatever the order of the differential should be, just as many arbitrary constants are included in the complete integral of this, which for any case present requires the following conditions to be prescribed.

SCHOLIUM 1

1104. On putting $dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx$ etc. all the differential equations of higher grades are reduced to finite quantities, in which no further account of that element is considered, that we assumed constant. And hence the forms of all the differential equations can be represented in the following manner:

Differential Equations	General Forms
I. order	$p = f:(x \text{ and } y)$
II. order	$q = f:(x, y \text{ and } p)$
III. order	$r = f:(x, y, p \text{ and } q)$
IV. order	$s = f:(x, y, p, q \text{ and } r)$

etc.,

where the quantities y, p, q, r, s etc. thus by excluding dx from depending on each in turn, so that since there shall be

$$dx = \frac{dy}{p} = \frac{dp}{q} = \frac{dq}{r} = \frac{dr}{s} = \frac{ds}{t} \quad \text{etc.,}$$

the following relations may be treated as :

$$\begin{array}{lllll} qdy = pdp & rdy = pdq & sdy = pdr & tdy = pds & \text{etc.,} \\ rdp = qdq & sdp = qdr & tdp = qds & \text{etc.,} & \\ sdq = rdr & tdq = rds & \text{etc.,} & & \\ tdr = sds & \text{etc.,} & & & \end{array}$$

of which certain formulas are integrable by themselves, just as

$$\int qdy = \tfrac{1}{2}pp, \quad \int rdp = \tfrac{1}{2}qq, \quad \int sdq = \tfrac{1}{2}rr, \quad \int tdr = \tfrac{1}{2}ss \quad \text{etc.,}$$

from which again on account of $\int zdv = vz - \int vdz$ those are inferred :

$$\int ydq = yq - \tfrac{1}{2}pp, \quad \int pdr = pr - \tfrac{1}{2}qq, \quad \int qds = qs - \tfrac{1}{2}rr, \quad \int rdt = rt - \tfrac{1}{2}ss \quad \text{etc.,}$$

from which there is deduced with the aid of the preceding :

$$\int sdy = pr - \tfrac{1}{2}qq, \quad \text{hence} \quad \int yds = ys - pr + \tfrac{1}{2}qq,$$
$$\int tdp = qs - \tfrac{1}{2}rr, \quad \text{hence} \quad \int pdt = pt - qs + \tfrac{1}{2}rr,$$
$$\int udq = rt - \tfrac{1}{2}ss, \quad \text{hence} \quad \int qdu = qu - rt + \tfrac{1}{2}ss.$$

From this again there is defined $\int ydu = yu - \int udy$, but $\frac{dy}{p} = \frac{dt}{u}$, from which

$$\int ydu = yu - \int pdt = yu - pt + qs - \tfrac{1}{2}rr .$$

Whereby if we introduce the differentials again, we obtain the following integral formulas

$$\int ydy = \tfrac{1}{2}yy,$$
$$\int yd^3y = yddy - \tfrac{1}{2}dy^2,$$
$$\int yd^5y = yd^4y - dyd^3y + \tfrac{1}{2}ddy^2,$$
$$\int yd^7y = yd^6y - dyd^5y + ddyd^4y - \tfrac{1}{2}d^3y^2$$

etc.,

thus so that the formula $\int yd^ny$ shall be integrable, as often as n is an odd number.

SCHOLIUM 2

1105. Thus we have set up [in the last table, where it is interesting to see also just how close to modern notation Euler has now arrived] for the simpler forms of the second order differential equations, that q may be equal to a function only of x, or of y , or p, which it is thus allowed to represent by writing greater [*i. e.* capital] letters for the functions of smaller ones, so that there shall be either $q = X$, $q = Y$, or $q = P$. In a like manner from this for differential equations of the third order we are able to put in place the simpler forms

$$r = X, r = Y, r = P, r = Q,$$

thus so that only two variable quantities are involved. But for the fourth order the simpler forms shall be

$$s = X, s = Y, s = P, s = Q, s = R$$

and for the fifth

$$t = X, t = Y, t = P, t = Q, t = R, t = S;$$

and thus again for the higher orders.

Now these forms do not all admit to being integrated, while some indeed not once, others only once, others can be led through all the integrations as far as the relation between x and y, the first of this kind are in some order of another. But it is proposed always, that the relation between the two main variables x and y should be elicited.

PROBLEM 141

1106. *On putting* $dy = pdx$, $dp = qdx$, $dq = rdx$, $dr = sdx$, $ds = tdx$ *etc. for whatever order of the differentials, if a certain one of the letters p, q, r, s, t etc. is equal to a function of x, which shall be X, then to find the relation between x and y.*

SOLUTION

If at first there shall be $p = X$, on multiplying by dx there will be $pdx = dy = Xdx$ and hence

$$y = \int Xdx\,,$$

which is the case of the simple formulas of the first order.

In the second place, let $q = X$; there will be $qdx = dp = Xdx$, from this $p = \int Xdx$ and $pdx = dy = dx\int Xdx$, therefore $y = \int dx\int Xdx$, or by simple integration [by parts] :

$$y = x\int Xdx - \int Xxdx\,.$$

If in the third place $r = X$; on account of $dq = rdx$ there will be $q = \int Xdx$ and hence

$$p = \int qdx = \int dx\int Xdx = x\int Xdx - \int xXdx\,.$$

and finally

$$y = \int pdx = \int dx\int dx\int Xdx =$$

$$\left[\int dx\left(x\int Xdx - \int xXdx\right) = \tfrac{1}{2}xx\int Xdx - \tfrac{1}{2}\int xxXdx - x\int xXdx + \int xxXdx\right]$$

$$= \tfrac{1}{2}xx\int Xdx - x\int Xxdx + \tfrac{1}{2}\int Xxxdx.$$

In the forth place let $s = X$ and there is found $y = \int dx\int dx\int dx\int Xdx$, which expression is changed into this

$$y = \tfrac{1}{6}x^3\int Xdx - \tfrac{1}{2}xx\int Xxdx + \tfrac{1}{2}x\int Xxxdx - \tfrac{1}{6}\int Xx^3dx.$$

In the fifth place let $t = X$; there will be $y = \int dx \int dx \int dx \int dx \int Xdx$ or

$$y = \tfrac{1}{24} x^4 \int Xdx - \tfrac{1}{6} x^3 \int Xxdx + \tfrac{1}{4} xx \int Xxxdx - \tfrac{1}{6} x \int Xx^3 dx + \tfrac{1}{24} \int Xx^4 dx,$$

from which the rule is evident for progressing further.

COROLLARY 1

1107. Therefore however so many integral formulas there may be, the differential equation shall have just as many orders , and because whatever arbitrary constant is assumed, the same constants will be present in the integral, by which that is rendered complete ; because the same is understood from the previous form, where just as many signed integrals are involved.
[There is the germ of the inductive process present here, as in other similar results presented by Euler, which process was later to be formalized by him into the Principle of Mathematical Induction.]

COROLLARY 2

1108. On taking the element *dx* constant, thus the complete integrals themselves can be considered of the following formulas in the usual customary form of expression:

I. If $dy = Xdx$, then there is

$$y = \int Xdx.$$

II. If $ddy = Xdx^2$, then there is

$$1y = x\int Xdx - \int Xxdx.$$

Ill. If $d^3 y = Xdx^3$, then there is

$$2y = x^2 \int Xdx - 2x\int Xxdx + \int Xx^2 dx.$$

IV. If $d^4 y = Xdx^4$, then there is

$$6y = x^3 \int Xdx - 3x^2 \int Xxdx + 3x\int Xx^2 dx - \int Xx^3 dx.$$

V. If $d^5 y = Xdx^5$, then there is

$$24y = x^4 \int Xdx - 4x^3 \int Xxdx + 6x^2 \int Xx^2 dx - 4x\int Xx^3 dx + \int Xx^4 dx$$

etc.

SCHOLIUM

1109. But it is not permitted that the formulas be integrated past the second order, which we have put in place above completing the function Y in the second order. For from the third order, hence on removing p from the order $r = Y$, and if we know that $r = \frac{pdq}{dy} = \frac{qdq}{dp} = \frac{dq}{dx}$, in no manner can it be integrated, and neither from this can q be determined by y. For on taking the form $pdq = Ydy$, with $pdp = qdy$ arising on account of $p = \frac{Ydy}{dq}$, there will be

$$dp = \frac{dydY}{dq} + Yd.\frac{dy}{dq}$$

and from this on removing p

$$\frac{Ydy^2dY}{dq^2} + \frac{YYdy}{dq}d.\frac{dy}{dq} = qdy,$$

which equation is indeed of the second order, but by no means is it permitted to be resolved in general.

Generally from the fourth order the formula $s = Y$ on account of $\int sdy = pr - \frac{1}{2}qq = \int Ydy$ can be integrated once, but it is impossible to progress further from this. But for whatever simpler ultimate and penultimate orders we put in place above, these which are tractable are seized upon ; therefore we shall investigate the integration of these.

PROBLEM 142

1110. *As on putting $dy = pdx$, $dp = qdx$, $dq = rdx$ etc. at this point, the capital letters Y, P, Q, R denote functions, the lower letter of each is of the same name ; to investigate the integrals of these simple forms $p = Y$, $q = P$, $r = Q$, $s = R$, $t = S$ etc.*
[i.e. $Y = Y(y), P = P(p), Q = Q(q)$, etc. in modern notation.]

SOLUTION

The first equation on account of $p = \frac{dy}{dx}$ gives at once $dx = \frac{dy}{Y}$ and thus

$$x = \int \frac{dy}{Y}.$$

The second equation $q = P$ on account of $q = \frac{dp}{dx}$ gives $dx = \frac{dp}{P}$ and $dy = \frac{pdp}{P}$, from which, since P shall be a function of p, each of the variables x and y are determined by p in this manner

$$x=\int\frac{dp}{P}\ \text{ and }\ y=\int\frac{pdp}{P}.$$

The third equation $r=Q$ on account of $r=\frac{dq}{dx}$ gives $dx=\frac{dq}{Q}$, from this $qdx=dp=\frac{qdq}{Q}$, thus so that there shall be $x=\int\frac{dq}{Q}$ and $p=\int\frac{qdq}{Q}$, from which we deduce $pdx=dy=\frac{dq}{Q}\int\frac{qdq}{Q}$, therefore $y=\int\frac{dq}{Q}\int\frac{qdq}{Q}$. Whereby by the same variable *q* each of the variables *x* and *y* thus is determined, so that there shall be

$$x=\int\frac{dq}{Q}\ \text{ and }\ y=\int\frac{dq}{Q}\int\frac{qdq}{Q}.$$

The fourth equation $s=R=\frac{dr}{dx}$ gives $dx=\frac{dr}{R}$ from which we gather $rdx=dq=\frac{rdr}{R}$. Thus so that there shall be $q=\int\frac{rdr}{R}$. Again $qdx=dp$ gives $dp=\frac{dr}{R}\int\frac{rdr}{R}$ and hence $p=\int\frac{dr}{R}\int\frac{rdr}{R}$; and because $pdx=dy$, we will have $dy=\frac{dr}{R}\int\frac{dr}{R}\int\frac{rdr}{R}$, whereby both the main variables *x* and *y* are thus defined by *r*

$$x=\int\frac{dr}{R}\ \text{ and }\ y=\int\frac{dr}{R}\int\frac{dr}{R}\int\frac{rdr}{R}.$$

The fifth equation $t=S$ treated in a similar manner will give

$$x=\int\frac{ds}{S}\ \text{ and }\ y=\int\frac{ds}{S}\int\frac{ds}{S}\int\frac{ds}{S}\int\frac{sds}{S};$$

and thus it is allowed easily to progress further.

COROLLARY 1

1111. From the second formula it is understood, if x is equal to the function of *p*, so that $x=P$, to become $y=\int pdP=Pp-\int Pdp$, which indeed is evident by itself.

COROLLARY 2

1112. But if there shall be $x=Q$, on account of $dx=dQ$ there will be

$$qdx=dp=qdQ\ \text{ and }\ p=\int qdQ$$

and hence

$$y=\int dQ\int qdQ\ \text{ or }\ y=Q\int qdQ-\int qQdQ.$$

Or also, since there shall be $y=\int dQ\left(qQ-\int Qdq\right)$, there will be

$$y=\tfrac{1}{2}qQQ+\tfrac{1}{2}\int QQdq-Q\int Qdq$$

or in a like manner

$$2y=QQq-2Q\int Qdq+\int QQdq\,.$$

COROLLARY 3

1113. In a similar manner if $x=R$, there will be

$$q=\int rdx=\int rdR \text{ and } p=\int qdx=\int dR\int rdR$$

and

$$y=\int pdx=\int dR\int dR\int rdR$$

or

$$2y=\int dR\left(RRr-2R\int Rdr+\int RRdr\right)$$

by the preceding corollary. Therefore by similar reductions,

$$6y=R^3r-3R^2\int Rdr+3R\int RRdr-\int R^3dr\,.$$

COROLLARY 4

1114. But if there were $x=S$, by similar reductions there is found :

$$24y=S^4s-4S^3\int Sds+6S^2\int SSds-4S\int S^3ds+\int S^4ds\,,$$

therefore from this by differentiating backwards,

$$24pdS=4S^3sdS-12SSdS\int Sds+12SdS\int SSds-4dS\int S^3ds$$

or

$$6p=S^3s-3SS\int Sds+3S\int SSds-\int S^3ds$$

and

$$2q=S^2s-2S\int Sds+\int SSds,$$

then

$$r=Ss-\int Sds \text{ and } s=s\,.$$

PROBLEM 143

1115. *With the same denominations remaining, which we have used until now, to investigate the integrals of these simpler formulas* $q=Y, r=P,\ s=Q,\ t=R$ *etc.*

SOLUTION

For the first formula $q=Y$ since there shall be $q=\frac{pdp}{dy}$, there will be

$$pdp=Ydy \text{ and } pp=2\int Ydy, \text{ hence } p=\sqrt{2\int Ydy}=\frac{dy}{dx},$$

from which there is deduced

$$x=\int\frac{dy}{\sqrt{2\int Ydx}}$$

and thus *X* is determined by *y*.

For the second formula $r=P$ on account of $r=\frac{qdq}{dp}$ we will have

$$qdq=Pdp \text{ and } q=\sqrt{2\int Pdp}=\frac{pdp}{dy}=\frac{dp}{dx}$$

from which we conclude

$$x=\frac{dp}{\sqrt{2\int Pdp}} \text{ and } y=\int\frac{pdp}{\sqrt{2\int Pdp}}$$

For the third formula $s=Q$ on account of $s=\frac{rdr}{dq}$ there becomes $r=\sqrt{2\int Qdq}=\frac{qdq}{dp}$, from which there follows $p=\int\frac{qdq}{\sqrt{2\int Qdq}}$.

Now since there shall be $r=\frac{dq}{dx}$ then there will be $dx=\frac{dq}{\sqrt{2\int Qdq}}$ and on account of $pdx=dy$ we will have

$$x=\int\frac{dq}{\sqrt{2\int Qdq}} \text{ and } y=\int\frac{dq}{\sqrt{2\int Qdq}}\int\frac{dq}{\sqrt{2\int Qdq}}.$$

For the fourth formula $t=R$ on account of $t=\frac{sds}{dr}$ we arrive at $s=\sqrt{2\int Rdr}$. But there is $s=\frac{dr}{dx}$, from which there becomes $dx=\frac{dr}{\sqrt{2\int Rdr}}$. Now also there shall be $s=\frac{rdr}{dq}$ and thus $q=\int\frac{rdr}{\sqrt{2\int Rdr}}$; but since $p=\int qdx$, there will be $p=\int\frac{dr}{\sqrt{2\int Rdr}}\int\frac{rdr}{\sqrt{2\int Rdr}}$, from which there emerges $y=\int pdx$. On account of which x and y thus are determined by *r*, so that there shall be

$$x=\int\frac{dr}{\sqrt{2\int Rdr}} \text{ and } y=\int\frac{dr}{\sqrt{2\int Rdr}}\int\frac{dr}{\sqrt{2\int Rdr}}\int\frac{rdr}{\sqrt{2\int Rdr}}.$$

For the fifth formula $u=S$ on account of $u=\frac{tdt}{ds}$ we arrive at $t=\sqrt{2\int Sds}=\frac{ds}{dx}$, so that there shall be $dx=\frac{ds}{\sqrt{2\int Sds}}$.

Now there is also $t=\frac{sds}{dr}$, therefore $r=\int\frac{sds}{\sqrt{2\int Sds}}$. Then $q=\int rdx,\ p=\int qdx$ and $y=\int pdx$, from which there is assembled

$$x=\int\frac{ds}{\sqrt{2\int Sds}} \text{ and } y=\int\frac{ds}{\sqrt{2\int Sds}}\int\frac{ds}{\sqrt{2\int Sds}}\int\frac{ds}{\sqrt{2\int Sds}}\int\frac{sds}{\sqrt{2\int Sds}}.$$

SCHOLIUM

1116. These are the cases, in which it is allowed to resolve these simpler formulas examined above, and no method is apparent, by which the remainder are able to be treated. Much fewer tractable cases occur in the more composite forms, where $\frac{d^ny}{dx^n}$ is equal to a function or two or more variable quantities, on account of which it certainly supplies the reason for a missing part, that could explain in this chapter.

But of the equations which are possible to be treated by methods at this stage, this is almost the general form :

$$A+\frac{Bdy}{dx}+\frac{Cddy}{dx^2}+\frac{Dd^3y}{dx^3}+\text{etc.}=0$$

on taking the element dx constant, which also by depending on

$$dy=pdx,\ dp=qdx,\ dq=rdx \text{ etc.},$$

are possible to be represented thus

$$Ay+Bp+Cq+Dr+Es+\text{etc.}=0.$$

Then indeed the equations satisfied in this wider form apparent admit a resolution

$$Ay+Bp+Cq+Dr+Es+\text{etc.}=X$$

with X denoting some function of x. Again also the following forms, which can indeed be reduced to those, can be lead to integration

$$Ay+\frac{Bp}{x}+\frac{Cq}{xx}+\frac{Dr}{x^3}+\frac{Es}{x^4}+\text{etc.}=0$$

and

$$Ay + \frac{Bp}{x} + \frac{Cq}{xx} + \frac{Dr}{x^3} + \frac{Es}{x^4} + \text{etc.} = X ,$$

the resolution of which thus succeeds, also for whatever order of the differential arises. Therefore we turn our treatment to the development of these.

SECTIO POSTERIOR
DE
RESOLUTIONE AEQUATIONUM DIFFERENTlALUM TERTIA ALTIORUMQUE GRADUUM QUAE DUAS TANTUM VARlABILES INVOLUUNT.

CAPUT I

DE INTEGRATIONE FORMULARUM DIFFERENTIALIUM TERTII ALTIORISVE GRADUS SIMPLICIUM

PROBLEMA 140

1100. *Sumto elemento dx constante invenire integrale completum harum formularum* $d^3y=0,\quad d^4y=0,\quad d^5y=0$ *etc. atque in genere huius* $d^ny=0$.

SOLUTIO

Cum *dx* sit constans, aequatio $d^3y=0$ per integrationem dat $ddy=adx^2$ hincque porro integrando $dy=\alpha xdx+\beta dx$ et tandem $y=\frac{1}{2}\alpha x^2+\beta x+\gamma$.
Simili modo ex aequatione $d^4y=0$ per quadruplicem integrationem reperitur

$$d^3y=\alpha dx^3,$$
$$ddy=\alpha xdx^2+\beta dx^2,$$
$$dy=\tfrac{1}{2}\alpha x^2dx+\beta xdx+\gamma dx$$

et tandem

$$y=\tfrac{1}{6}\alpha x^3+\tfrac{1}{2}\beta x^2+\gamma x+\delta.$$

Ex aequatione autem $d^5y=0$ integratio quinquies repetita dat

$$y=\tfrac{1}{24}\alpha x^4+\tfrac{1}{6}\beta x^3+\tfrac{1}{2}\gamma x^2+\delta x+\varepsilon.$$

At huius aequationis $d^6y=0$ integrale colligitur

$$y=\tfrac{1}{120}\alpha x^5+\tfrac{1}{24}\beta x^4+\tfrac{1}{6}\gamma x^3+\tfrac{1}{2}\delta x^2+\varepsilon x+\zeta.$$

sicque ad huiusmodi formas $d^n y = 0$, quanticumque fuerint gradus, progredi licet, dummodo *n* fuerit numerus integer positivus.

COROLLARIUM 1

1101. A simplicissima forma ergo incipiendo integralia sequenti ordine procedunt:

Formularum	Integralia completa sunt
$dy = 0$	$y = \alpha$
$ddy = 0$	$y = \alpha x + \beta$
$d^3 y = 0$	$y = \frac{1}{2}\alpha x^2 + \beta x + \gamma$
$d^4 y = 0$	$y = \frac{1}{6}\alpha x^3 + \frac{1}{2}\beta x^2 + \gamma x + \delta.$
etc.	etc.

COROLLARIUM 2

1102. Quia constantes α, β, γ etc. ab arbitrio nostro pendent, fractiones tuto reiicere licet eritque:

Formularum	Integralia
$dy = 0$	$y = \alpha$
$ddy = 0$	$y = \alpha x + \beta$
$d^3 y = 0$	$y = \alpha x^2 + \beta x + \gamma$
$d^4 y = 0$	$y = \alpha x^3 + \beta x^2 + \gamma x + \delta.$
$d^5 y = 0$	$y = \alpha x^4 + \beta x^3 + \gamma x^2 + \delta x + \varepsilon.$
etc.	etc.

COROLLARIUM 3

1103. Quoti ergo ordinis est formula differentialis, totidem constantes arbitrarias eius integrale completum complectitur, quas pro quovis casu oblato secundum conditiones praescriptas definiri oportet.

SCHOLION 1

1104. Ponendo $dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx$ etc. omnes aequationes differentiales altiorum ordinum ad quantitates finitas reducuntur, in quibus nulla amplius ratio eius elementi, quod constans assumitur, habetur. Atque hinc formae omnium aequationum differentialium sequenti modo repraesentari possunt:

Aequationum differentialium	Forma generalis
I. gradus	$p = f:(x \text{ et } y)$
II. gradus	$q = f:(x, y \text{ et } p)$
III. gradus	$r = f:(x, y, p \text{ et } q)$
IV. gradus	$s = f:(x, y, p, q \text{ et } r)$

etc.,

ubi quantitates *y*, *p*, *q*, *r*, *s* etc. ita excludendo *dx* a se invicem pendent, ut, cum sit

$$dx = \frac{dy}{p} = \frac{dp}{q} = \frac{dq}{r} = \frac{dr}{s} = \frac{ds}{t} \quad \text{etc.},$$

sequentes relationes locum habeant

$qdy = pdp$	$rdy = pdq$	$sdy = pdr$	$tdy = pds$	etc.,
$rdp = qdq$	$sdp = qdr$	$tdp = qds$	etc.,	
$sdq = rdr$	$tdq = rds$	etc.,		
$tdr = sds$	etc.,			

quarum formularum quaedam per se sunt integrabiles, veluti

$$\int qdy = \tfrac{1}{2}pp, \quad \int rdp = \tfrac{1}{2}qq, \quad \int sdq = \tfrac{1}{2}rr, \quad \int tdr = \tfrac{1}{2}ss \quad \text{etc.},$$

ex quibus porro ob $\int zdv = vz - \int vdz$ istae concluduntur

$$\int ydq = yq - \tfrac{1}{2}pp, \quad \int pdr = pr - \tfrac{1}{2}qq, \quad \int qds = qs - \tfrac{1}{2}rr, \quad \int rdt = rt - \tfrac{1}{2}ss \quad \text{etc.},$$

quarum ope ex praecedentibus deducitur

$$\int sdy = pr - \tfrac{1}{2}qq, \quad \text{hinc} \quad \int yds = ys - pr + \tfrac{1}{2}qq,$$
$$\int tdp = qs - \tfrac{1}{2}rr, \quad \text{hinc} \quad \int pdt = pt - qs + \tfrac{1}{2}rr,$$
$$\int udq = rt - \tfrac{1}{2}ss, \quad \text{hinc} \quad \int qdu = qu - rt + \tfrac{1}{2}ss.$$

Hinc porro definitur $\int ydu = yu - \int udy$, at $\frac{dy}{p} = \frac{dt}{u}$, unde
$$\int ydu = yu - \int pdt = yu - pt + qs - \tfrac{1}{2}rr.$$

Quare si differentialia iterum introducamus, obtinebimus sequentes formulas integrales

$$\int ydy = \tfrac{1}{2}yy,$$
$$\int yd^3y = yddy - \tfrac{1}{2}dy^2,$$
$$\int yd^5y = yd^4y - dyd^3y + \tfrac{1}{2}ddy^2,$$
$$\int yd^7y = yd^6y - dyd^5y + ddyd^4y - \tfrac{1}{2}d^3y^2$$
etc.,

ita ut formula $\int yd^ny$ sit integrabilis, quoties n est numerus impar.

SCHOLION 2

1105. In aequationibus differentialibus secundi gradus formas simpliciores ita constituimus, ut q aequetur functioni vel ipsius x tantum vel ipsius y vel ipsius p, quas litteras maiusculas pro functionibus minuscularum scribendo ita repraesentare licet, ut sit vel $q = X$ vel $q = Y$ vel $q = P$. Hinc simili modo pro aequationibus differentialibus tertii gradus formas simpliciores constituere possumus

$$r = X, r = Y, r = P, r = Q,$$

ita ut tantum binas quantitates variabiles involvant. Pro quarto autem gradu essent formae simpliciores
$$s = X, s = Y, s = P, s = Q, s = R$$
et pro quinto
$$t = X, t = Y, t = P, t = Q, t = R, t = S;$$
atque ita porro pro superioribus.

Verum hae formae non omnes aeque integrationem admittunt, dum aliae ne semel quidem, aliae semel tantum, aliae per omnes integrationes usque ad relationem inter x et y

perduci possunt, cuiusmodi sunt primae quaeque in quovis gradu. Semper autem id est propositum, ut relatio inter binas variabiles principales x et y eliciatur.

PROBLEMA 141

1106. *Posito* $dy = pdx$, $dp = qdx$, $dq = rdx$, $dr = sdx$, $ds = tdx$ *etc. pro quovis differentialium gradu si litterarum p, q, r, s, t etc. quaepiam aequetur functioni ipsius x, quae sit X, invenire relationem inter x et y.*

SOLUTIO

Si primo sit $p = X$, per dx multiplicando erit $pdx = dy = Xdx$ hincque

$$y = \int Xdx,$$

qui est casus formularum differentialium primi gradus simplicium.

Sit secundo $q = X$; erit $qdx = dp = Xdx$, hinc $p = \int Xdx$ et $pdx = dy = dx\int Xdx$, ergo $y = \int dx\int Xdx$ seu per simplicia integralia

$$y = x\int Xdx - \int Xxdx.$$

Sit tertio $r = X$; ob $dq = rdx$ erit $q = \int Xdx$ hincque

$$p = \int qdx = \int dx\int Xdx = x\int Xdx - \int xXdx.$$

ac tandem

$$y = \int pdx = \int dx\int dx\int Xdx = \tfrac{1}{2}xx\int Xdx - x\int Xxdx + \tfrac{1}{2}\int Xxxdx.$$

Sit quarto $s = X$ ac reperitur $y = \int dx\int dx\int dx\int Xdx$, quae expressio evolvitur in hanc

$$y = \tfrac{1}{6}x^3\int Xdx - \tfrac{1}{2}xx\int Xxdx + \tfrac{1}{2}x\int Xxxdx - \tfrac{1}{6}\int Xx^3dx.$$

Sit quinto $t = X$; erit $y = \int dx\int dx\int dx\int dx\int Xdx$ seu

$$y = \tfrac{1}{24}x^4\int Xdx - \tfrac{1}{6}x^3\int Xxdx + \tfrac{1}{4}xx\int Xxxdx - \tfrac{1}{6}x\int Xx^3dx + \tfrac{1}{24}\int Xx^4dx.,$$

unde lex ulterius progrediendi est manifesta.

COROLLARIUM 1

1107. Tot ergo habentur formulae integrales, quoti gradus aequatio fuerit differentialis, et quia quaelibet constantem arbitrariam assumit, totidem constantes in integrale ingrediuntur, quibus id completum redditur; quod idem ex priori forma, ubi totidem signa integralia implicantur, intelligitur.

COROLLARIUM 2

1108. Sumto elemento dx constante sequentium formularum more consueto expressarum integralia completa ita se habebunt:

I. Si $dy = Xdx$, est

$$y = \int Xdx.$$

II. Si $ddy = Xdx^2$, est

$$1y = x\int Xdx - \int Xxdx.$$

Ill. Si $d^3y = Xdx^3$, est

$$2y = x^2\int Xdx - 2x\int Xxdx + \int Xx^2dx.$$

IV. Si $d^4y = Xdx^4$, est

$$6y = x^3\int Xdx - 3x^2\int Xxdx + 3x\int Xx^2dx - \int Xx^3dx.$$

V. Si $d^5y = Xdx^5$, est

$$24y = x^4\int Xdx - 4x^3\int Xxdx + 6x^2\int Xx^2dx - 4x\int Xx^3dx + \int Xx^4dx$$

etc.

SCHOLION

1109. Formulas autem, quas supra secundo loco constituimus, functionem Y complectentes post secundum gradum integrare non licet. Ex tertio enim hincque p elidendo

ordine formula $r = Y$ etsi novimus esse $r = \frac{pdq}{dy} = \frac{qdq}{dp} = \frac{dq}{dx}$ nullo modo

integrari potest neque etiam hinc q per y determinari potest. Nam sumta forma $pdq = Ydy$ existente $pdp = qdy$ ob $p = \frac{Ydy}{dq}$ erit

$$dp = \frac{dydY}{dq} + Yd.\frac{dy}{dq}$$

hincque p elidendo

$$\frac{Ydy^2dY}{dq^2}+\frac{YYdy}{dq}d.\frac{dy}{dq}=qdy,$$

quae quidem aequatio est secundi gradus, sed neutiquam in genere resolutionem admittit.

Ex quarto genere formula $s=Y$ ob $\int sdy=pr-\frac{1}{2}qq=\int Ydy$ semel integrari potest, sed hinc ulterius progredi non licet. Quas autem supra pro quovis gradu formulas simpliciores ultimo loco constituimus itemque penultimo, eae tractabiles deprehenduntur; earum ergo integrationem investigemus.

PROBLEMA 142

1110. *Posito ut hactenus* $dy=pdx$, $dp=qdx$, $dq=rdx$ *etc. litterae Y, P, Q, R denotent functiones cuiusque litterae minusculae cognominis; investigare integralia harum formularum simplicium* $p=Y$, $q=P$, $r=Q$, $s=R$, $t=S$ *etc.*

SOLUTIO

Aequatio prima ob $p=\frac{dy}{dx}$ statim dat $dx=\frac{dy}{Y}$ ideoque

$$x=\int\frac{dy}{Y}.$$

Aequatio secunda $q=P$ ob $q=\frac{dp}{dx}$ praebet $dx=\frac{dp}{P}$ et $dy=\frac{pdp}{P}$, unde, cum P sit functio ipsius p, utraque variabilis x et y per p determinatur hoc modo

$$x=\int\frac{dp}{P} \text{ et } y=\int\frac{pdp}{P}.$$

Aequatio tertia $r=Q$ ob $r=\frac{dq}{dx}$ dat $dx=\frac{dq}{Q}$, hinc $qdx=dp=\frac{qdq}{Q}$, ita ut sit $x=\int\frac{dq}{Q}$ et $p=\int\frac{qdq}{Q}$, unde colligimus $pdx=dy=\frac{dq}{Q}\int\frac{qdq}{Q}$, ergo $y=\int\frac{dq}{Q}\int\frac{qdq}{Q}$. Quare per eandem variabilem q utraque variabilis x et y ita determinatur, ut sit

$$x=\int\frac{dq}{Q} \text{ et } y=\int\frac{dq}{Q}\int\frac{qdq}{Q}.$$

Aequatio quarta $s=R=\frac{dr}{dx}$ dat $dx=\frac{dr}{R}$ unde colligimus $rdx=dq=\frac{rdr}{R}$, Ita ut sit $q=\int\frac{rdr}{R}$. Porro $qdx=dp$ dat $dp=\frac{dr}{R}\int\frac{rdr}{R}$ hincque $p=\int\frac{dr}{R}\int\frac{rdr}{R}$; et quia $pdx=dy$, habebimus $dy=\frac{dr}{R}\int\frac{dr}{R}\int\frac{rdr}{R}$, quare per r ambae variabiles principales x et y ita definiuntur

$$x=\int\frac{dr}{R} \text{ et } y=\int\frac{dr}{R}\int\frac{dr}{R}\int\frac{rdr}{R}.$$

Aequatio quinta $t=S$ simili modo tractata praebet

$$x = \int \tfrac{ds}{S} \text{ et } y = \int \tfrac{ds}{S} \int \tfrac{ds}{S} \int \tfrac{ds}{S} \int \tfrac{sds}{S};$$

sicque facile ulterius progredi licet.

COROLLARIUM 1

1111. Ex formula secunda intelligitur, si *x* aequetur functioni ipsius *p*, ut sit $x = P$, fore $y = \int pdP = Pp - \int Pdp$, quod quidem per se est manifestum.

COROLLARIUM 2

1112. Sin autem sit $x = Q$, ob $dx = dQ$ erit

$$qdx = dp = qdQ \text{ et } p = \int qdQ$$

hincque

$$y = \int dQ \int qdQ \text{ seu } y = Q\int qdQ - \int qQdQ.$$

Vel etiam, cum sit $y = \int dQ\left(qQ - \int Qdq\right)$, erit

$$y = \tfrac{1}{2} qQQ + \tfrac{1}{2}\int QQdq - Q\int Qdq$$

sive hoc modo

$$2y = QQq - 2Q\int Qdq + \int QQdq.$$

COROLLARIUM 3

1113. Simili modo si $x = R$, erit

$$q = \int rdx = \int rdR \text{ et } p = \int qdx = \int dR \int rdR$$

atque

$$y = \int pdx = \int dR \int dR \int rdR$$

seu

$$2y = \int dR\left(RRr - 2R\int Rdr + \int RRdr\right)$$

per praecedens corollarium. Ergo per similes reductiones

$$6y = R^3 r - 3R^2 \int Rdr + 3R\int RRdr - \int R^3 dr.$$

COROLLARIUM 4

1114. At si fuerit $x = S$, reperietur per similes reductiones

$$24y = S^4 s - 4S^3 \int Sds + 6S^2 \int SSds - 4S \int S^3 ds + \int S^4 ds \,,$$

hinc ergo per differentiationes retrogrediendo

$$24\,pdS = 4S^3 sdS - 12SSdS \int Sds + 12SdS \int SSds - 4dS \int S^3 ds$$

seu

$$6\,p = S^3 s - 3SS \int Sds + 3S \int SSds - \int S^3 ds$$

et

$$2q = S^2 s - 2S \int Sds + \int SSds,$$

tum

$$r = Ss - \int Sds \text{ et } s = s\,.$$

PROBLEMA 143

1115. *Iisdem manentibus denominationibus, quibus hactenus sumus usi, investigare integralia harum formularum simpliciorum* $q = Y, r = P, s = Q, t = R$ *etc.*

SOLUTIO

Pro formula prima $q = Y$ cum sit $q = \frac{pdp}{dy}$, erit

$$pdp = Ydy \text{ et } pp = 2\int Ydy, \text{ hinc } p = \sqrt{2\int Ydy} = \frac{dy}{dx}\,,$$

unde colligitur

$$x = \int \frac{dy}{\sqrt{2\int Ydx}}$$

sicque *X* per *y* determinatur.

Pro formula secunda $r = P$ ob $r = \frac{qdq}{dp}$ habebimus

$$qdq = Pdp \text{ et } q = \sqrt{2\int Pdp} = \frac{pdp}{dy} = \frac{dp}{dx}$$

unde concludimus

$$x = \frac{dp}{\sqrt{2\int Pdp}} \text{ et } y = \int \frac{pdp}{\sqrt{2\int Pdp}}$$

Pro formula tertia $s = Q$ ob $s = \frac{rdr}{dq}$ fiet $r = \sqrt{2\int Qdq} = \frac{qdq}{dp}$, unde sequitur $p = \int \frac{qdq}{\sqrt{2\int Qdq}}$.

Cum vero sit $r=\frac{dq}{dx}$ erit $dx=\frac{dq}{\sqrt{2\int Qdq}}$ et ob $pdx=dy$ habebimus

$$x=\int\frac{dq}{\sqrt{2\int Qdq}} \text{ et } y=\int\frac{dq}{\sqrt{2\int Qdq}}\int\frac{dq}{\sqrt{2\int Qdq}} \cdot$$

Pro formula quarta $t=R$ ob $t=\frac{sds}{dr}$ nanciscimur $s=\sqrt{2\int Rdr}$. At est $s=\frac{dr}{dx}$, unde fit $dx=\frac{dr}{\sqrt{2\int Rdr}}$. Est vero etiam $s=\frac{rdr}{dq}$ ideoque $q=\int\frac{rdr}{\sqrt{2\int Rdr}}$; sed quoniam $p=\int qdx$, fit $p=\int\frac{dr}{\sqrt{2\int Rdr}}\int\frac{rdr}{\sqrt{2\int Rdr}}$, ex quo prodit $y=\int pdx$. Quocirca *x* et *y* ita per *r* determinantur, ut sit

$$x=\int\frac{dr}{\sqrt{2\int Rdr}} \text{ et } y=\int\frac{dr}{\sqrt{2\int Rdr}}\int\frac{dr}{\sqrt{2\int Rdr}}\int\frac{rdr}{\sqrt{2\int Rdr}}.$$

Pro formula quinta $u=S$ ob $u=\frac{tdt}{ds}$ adipiscimur $t=\sqrt{2\int Sds}=\frac{ds}{dx}$, ut sit $dx=\frac{ds}{\sqrt{2\int Sds}}$. Est vero etiam $t=\frac{sds}{dr}$, ergo $r=\int\frac{sds}{\sqrt{2\int Sds}}$. Tum $q=\int rdx,\ p=\int qdx$ et $y=\int pdx$, ex quibus conficitur

$$x=\int\frac{ds}{\sqrt{2\int Sds}} \text{ et } y=\int\frac{ds}{\sqrt{2\int Sds}}\int\frac{ds}{\sqrt{2\int Sds}}\int\frac{ds}{\sqrt{2\int Sds}}\int\frac{sds}{\sqrt{2\int Sds}}.$$

SCHOLION

1116. Hi sunt casus, quibus formulas illas simpliciores supra recensitas resolvere licet, neque methodus patet, qua reliquae tractari queant. Multo pauciores occurrunt casus tractabiles in formis magis compositis, ubi $\frac{d^ny}{dx^n}$ aequatur functioni binarum pluriumve quantitatum variabillum, ob quam penuriam parum admodum suppetit, quod in hac sectione exponere queamus.

Aequationum autem, quae per methodos adhuc inventas tractari possunt, haec fere est forma generalis

$$Ay+\frac{Bdy}{dx}+\frac{Cddy}{dx^2}+\frac{Dd^3y}{dx^3}+\text{etc.}=0$$

sumto elemento *dx* constante, quae etiam ponendo

$$dy=pdx,\ dp=qdx,\ dq=rdx \text{ etc.}$$

ita repraesentari potest

$$Ay+Bp+Cq+Dr+Es+\text{etc.}=0.$$

Deinde vero etiam aequationes hac forma latius patente contentae resolutionem admittunt

$$Ay + Bp + Cq + Dr + Es + \text{etc.} = X$$

denotante *X* functionem quamcunque ipsius *x*. Porro quoque sequentes formae, quae quidem ad illas reduci possunt, ad integrationem perduci possunt

$$Ay + \frac{Bp}{x} + \frac{Cq}{xx} + \frac{Dr}{x^3} + \frac{Es}{x^4} + \text{etc.} = 0$$

et

$$Ay + \frac{Bp}{x} + \frac{Cq}{xx} + \frac{Dr}{x^3} + \frac{Es}{x^4} + \text{etc.} = X\ ,$$

quarum resolutio adeo succedit, ad quantumvis gradum etiam differentialitas assurgat. In harum ergo aequationum evolutione tractatio nostra versabitur.