## CHAPTER VII

# ON THE RESOLUTION OF THE EQUATION

$$ddy + ax^n ydx^2 = 0$$

# BY INFINITE SERIES

### PROBLEM 117

**929.** *With the element dx assumed constant, to integrate the second order differential equation* $ddy + ax^n ydx^2 = 0$ *by infinite series.*

### SOLUTION

Here we search for a series progressing following the powers of $x$, which may express the value of $y$ ; and because in the former term of our equation there is no quantity $x$ with its differential $dx$, while in the latter truly the term has dimensions $n+2$ , it is clear that the exponents of the powers of $x$ must ascend or descent with a difference $n+2$ .

I. In the first place the exponents ascend and the series is put in place

$$y = Ax^{\lambda} + Bx^{\lambda+n+2} + Cx^{\lambda+2n+4} + \text{etc}.$$

and there shall be

$$\frac{ddy}{dx^2} = \lambda(\lambda-1)Ax^{\lambda-2} + (\lambda+n+2)(\lambda+n+1)Bx^{\lambda+n} + \text{etc.},$$
$$ax^n y = \qquad aAx^{\lambda+n} + \text{etc.},$$

from which it is apparent that the first of the solitary terms must vanish, so that there shall be $\lambda(\lambda-1) = 0$ .

Whereby it is required to take either $\lambda = 0$ or $\lambda = 1$ and thus a twofold series may be obtained

$$y = A + Bx^{n+2} + Cx^{2n+4} + Dx^{3n+6} + Ex^{4n+8} + \text{ etc.}$$
$$+ \mathfrak{A}x + \mathfrak{B}x^{n+3} + \mathfrak{C}x^{2n+5} + \mathfrak{D}x^{3n+7} + \mathfrak{E}X^{4n+9} + \text{ etc.}$$

Hence with the substitution made it is required to become :

$$0=(n+2)(n+1)Bx^{n} \quad +(2n+4)(2n+3)Cx^{n+2} \quad +(3n+6)(3n+5)Dx^{8n+4}+\text{etc.,}$$
$$+\ aA \qquad\qquad +aB \qquad\qquad +aC$$
$$0=(n+3)(n+2)\mathfrak{B}x^{n+1}+(2n+5)(2n+4)\mathfrak{C}x^{2n+3}+(3n+7)(3n+6)\mathfrak{D}x^{3n+5}+\text{etc.,}$$
$$+a\mathfrak{A} \qquad\qquad +a\mathfrak{B} \qquad\qquad +\ a\mathfrak{C}$$

from which with our arbitrary $A$ and $\mathfrak{A}$, the remaining letters thus are determined in terms of these [note the assumed corresponding powers of $x$ in the lower lines]

$$B=\frac{-aA}{(n+1)(n+2)},\quad C=\frac{-aB}{2(2n+3)(n+2)},\quad D=\frac{-aC}{3(3n+5)(n+2)}+\text{etc.,}$$

$$\mathfrak{B}=\frac{-a\mathfrak{A}}{(n+3)(n+2)},\quad \mathfrak{C}=\frac{-a\mathfrak{B}}{2(2n+5)(n+2)},\quad \mathfrak{D}=\frac{-a\mathfrak{C}}{3(3n+7)(n+2)}x^{3n+5}+\text{etc.}$$

and thus the complete integral expression will thus be considered

$$y=A\ -\frac{aAx^{n+2}}{1(n+1)\cdot(n+2)}+\frac{a^{2}Ax^{2n+4}}{1\cdot 2(n+1)(2n+3)\cdot(n+2)^{2}}-\frac{a^{3}Ax^{3n+6}}{1\cdot 2\cdot 3(n+1)(2n+3)(3n+5)\cdot(n+2)^{3}}+\text{etc.}$$

$$+\mathfrak{A}x-\frac{a\mathfrak{A}x^{n+3}}{1(n+3)\cdot(n+2)}\ +\frac{a^{2}\mathfrak{A}x^{2n+5}}{1\cdot 2(n+3)(2n+5)\cdot(n+2)^{2}}-\frac{a^{3}\mathfrak{A}x^{3n+7}}{1\cdot 2\cdot 3(n+3)(2n+5)(3n+7)\cdot(n+2)^{3}}+\text{etc.}$$

II. Now the exponents descend and with the series produced

$$y=Ax^{\lambda}+Bx^{\lambda-n-2}+Cx^{\lambda-2n-4}+\text{etc}.$$

there will be considered :

$$\frac{ddy}{dx^{2}}= \qquad \lambda(\lambda-1)Ax^{\lambda-2}+(\lambda-n-2)(\lambda-n-3)Bx^{\lambda-n-4}+\text{etc.,}$$
$$ax^{n}y=aAx^{\lambda+n}\ \ + \qquad aBx^{\lambda-2}+\text{etc.,}$$

where since the term $x^{\lambda+n}$ cannot have a term similar to itself, it cannot be removed, and thus no resolution of the equation can be obtained .

### COROLLARY 1

**930.** The twin series found for *y* show the compete integral of the second order differential equation $ddy + ax^n ydx^2 = 0$ since the letters $A$ and $\mathfrak{A}$ are left to our arbitrary choice ; but by attributing given values to the letters $A$ and $\mathfrak{A}$ particular integrals may be produced.

### COROLLARY 2

**931.** If we put $n+2=m$ or $n=m-2$, the complete integral of this equation

$$ddy + ax^{m-2} ydx^2 = 0$$

thus may be expressed more conveniently

$$y = A - \frac{aAx^m}{1(m-1)\cdot m} + \frac{a^2Ax^{2m}}{1\cdot 2(m-1)(2m-1)\cdot m^2} - \frac{a^3Ax^{3m}}{1\cdot 2\cdot 3(m-1)(2m-1)(3m-1)\cdot m^3} + \text{etc.}$$

$$+\mathfrak{A}x - \frac{a\mathfrak{A}x^{m+1}}{1(m+1)\cdot m} + \frac{a^2\mathfrak{A}x^{2m+1}}{1\cdot 2(m+1)(2m+1)\cdot m^2} - \frac{a^3\mathfrak{A}x^{3m+1}}{1\cdot 2\cdot 3(m+1)(2m+1)(3m+1)\cdot m^3} + \text{etc.}$$

### COROLLARY 3

**932.** If the exponent *m* were positive and greater than one, from that these series converge more, so that a smaller value of the quantity *x* can be given ; now these series are unable to be used in practice for the other cases, unless perhaps these can be transformed into other convergent series.

### SCHOLIUM 1

**933.** Yet cases are given, in which all these series plainly fail in use, which comes about, if some factor present in the denominators vanishes and thus all the following terms become infinitely great, in which case it is agreed to change the series into other forms. Here the case occurs in the first place $m=0$ or $n=-2$, from which all the terms of each series beyond the first become infinite ; now in this case the equation, which is

$$ddy + \frac{aydx^2}{x^2} = 0,$$

since it shall be homogeneous, will allow a singular integration ; for it is possible to find the power of *x*, by which the equation will be satisfied on substitution for *y*. Evidently there may be put $y = x^\lambda$ and there is produced

$$\lambda(\lambda-1)x^{\lambda-2} + ax^{\lambda-2} = 0 \text{ or } \lambda\lambda - \lambda + 1 = 0,$$

from which there is deduced $\lambda = \frac{1}{2} \pm \sqrt{\left(\frac{1}{4} - a\right)}$, on account of which the twofold value is the complete integral

$$y = Ax^{\frac{1}{2}+\sqrt{\left(\frac{1}{4}-a\right)}} + Bx^{\frac{1}{2}-\sqrt{\left(\frac{1}{4}-a\right)}},$$

which in the case $a > \frac{1}{4}$ changes into this form

$$y = Ax^{\frac{1}{2}}\sin.\left(\left(a - \frac{1}{4}\right)^{\frac{1}{2}} lx + \alpha\right),$$

from which it is apparent to become in the case $a = \frac{1}{4}$

$$y = \left(A + Blx\right)\sqrt{x}.$$

## SCHOLIUM 2

**934.** The remaining cases leading to inconvenience are, if either $m = \frac{1}{i}$ or $m = -\frac{1}{i}$ with *i* denoting some whole number. In the case $m = \frac{1}{i}$ only the first series shall be incongruous, and truly in the case $m = -\frac{1}{i}$ only the second. Whereby in that case on putting $A = 0$, now in this case on putting $\mathfrak{A} = 0$, the series may be considered at any rate as showing a single particular integral. Now knowing a particular integral, so that it shall be $y = P$, from that the complete integral of the equation $ddy + ax^{m-2}ydx^2 = 0$ is elicited on putting $y = Pz$, from which there becomes

$$Pddz + 2dPdz + zddP + ax^{m-2}Pzdx^2 = 0;$$

by the hypothesis there is $ddP + ax^{m-2}Pdx^2 = 0$, hence there emerges

$$Pddz + 2dPdz = 0 \text{ or } PPdz = Cdx \text{ and } z = C\int \frac{dx}{PP}.$$

But since *P* shall be an infinite series, hence the value of *z* cannot be known.

But in these cases mentioned a part of the integral involves the logarithm of *x*, or which is understood from that, since $\frac{x^\theta}{\theta}$ shall be equivalent to *lx*.

[See §§ 978 below or E154 ; *O. O.* Series I, Vol. 20, initially p. 43. Part of a note added by the editor of the *O.O*. edition.]

Whereby in equation $ddy + ax^{m-2}ydx^2 = 0$ on putting $y = p + qlx$, on account of $dy = dp + \frac{q}{x} + dqlx$ there will be

$$ddp + \frac{2dxdq}{x} - \frac{qdx^2}{xx} + ddqlx + apx^{m-2}dx^2 + aqx^{m-2}dx^2lx = 0,$$

in which it is required that the parts involving *lx* cancel each other, thus in order that these two equations may be considered

$$ddq + aqx^{m-2}dx^2 = 0 \text{ and } ddp + \frac{2dxdq}{x} - \frac{qdx^2}{xx} + apx^{m-2}dx^2 = 0,$$

where of the two above series for $q$ that one must be taken, which in the case offered is without inconvenience, and that put in place from the latter equation can easily express the quantity $p$ by a series. We set up cases of this kind in the following examples ; we may observe likewise that only that operation be considered, even if in place of $lx$ there is taken $lx + \alpha$, thus so that on finding $p$ and $q$ there shall be $y = \alpha q + p + qlx$ or $y = p + ql\beta x$.

## EXAMPLE 1

**935.** *On putting* $m = 1$, *to resolve this equation* $ddy + \frac{aydx^2}{x} = 0$ *by series.*

On putting $y = p + qlx$ it is required to take

$$q = \mathfrak{A}x - \frac{a\mathfrak{A}x^2}{1\cdot 2} + \frac{a^2\mathfrak{A}x^3}{1\cdot 2\cdot 2\cdot 3} - \frac{a^3\mathfrak{A}x^4}{1\cdot 2\cdot 2\cdot 3\cdot 3\cdot 4} + \text{etc.}$$

for which we can take for the sake of brevity

$$q = \mathfrak{A}x + \mathfrak{B}x^2 + \mathfrak{C}x^3 + \mathfrak{D}x^4 + \text{etc.}$$

Then $p$ is sought from this equation

$$\frac{ddp}{dx^2} + \frac{2dq}{xdx} - \frac{q}{xx} + \frac{ap}{x} = 0$$

hence we put in place

$$p = A + Bx + Cx^2 + Dx^3 + Ex^4 + \text{etc}.$$

and with the substitution made there shall be

$$\left.\begin{array}{l} \quad 2C + 6Dx + 12Exx + 20Fx^3 + 30Gx^4 + \text{ etc.} \\ +\frac{2\mathfrak{A}}{x} + 4\mathfrak{B} + 6\mathfrak{C} + 8\mathfrak{D} + 10\mathfrak{E} + 12\mathfrak{F} \\ -\mathfrak{A} - \mathfrak{B} - \mathfrak{C} - \mathfrak{D} - \mathfrak{E} - \mathfrak{F} \\ +aA + aB + aC + aD + aE + aF \end{array}\right\} = 0.$$

Now since the coefficients $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ etc. shall be given, there will be $A = -\frac{\mathfrak{A}}{a}$ ; the quantity $B$ is not determined ; then truly [by equating coefficients : ]

$$C=\frac{-3\mathfrak{B}}{2}-\frac{aB}{1\cdot 2}=\frac{3a\mathfrak{A}}{1^3\cdot 2^2}-\frac{aB}{1\cdot 2},\quad D=\frac{-5\mathfrak{C}}{6}-\frac{aC}{2\cdot 3}=\frac{-5a^2\mathfrak{A}}{1^2\cdot 2^3\cdot 3^2}-\frac{aC}{2\cdot 3},$$
$$E=\frac{-7\mathfrak{D}}{12}-\frac{aD}{3\cdot 4}=\frac{7a^3\mathfrak{A}}{1^2\cdot 2^3\cdot 3^3\cdot 4^2}-\frac{aD}{3\cdot 4},\quad F=\frac{-9\mathfrak{E}}{20}-\frac{aE}{4\cdot 5}=\frac{-9a^4\mathfrak{A}}{1^2\cdot 2^2\cdot 3^2\cdot 4^3\cdot 5^2}-\frac{aE}{4\cdot 5}$$
etc.,

where for $B$ it is allowed to write 0, whenever in the integral $y=p+qlx$ we add the part $\alpha q$, which arises from the letter $B$, thus in order that there shall be

$$p=A+Cx^2+Dx^3+Ex^4+Fx^5+\text{etc.}$$

Hence there shall be

$$C=\frac{3a\mathfrak{A}}{1^3\cdot 2^2},\quad D=\frac{-14a^2\mathfrak{A}}{1^2\cdot 2^3\cdot 3^2},\quad E=\frac{70a^3\mathfrak{A}}{1^2\cdot 2^3\cdot 3^3\cdot 4^2},\quad F=\frac{-404a^4\mathfrak{A}}{1^2\cdot 2^2\cdot 3^2\cdot 4^3\cdot 5^2},$$

where it is to be noted that $14=3\cdot 3+5\cdot 1$, $70=4\cdot 14+7\cdot 1\cdot 2$, $404=5\cdot 70+9\cdot 1\cdot 2\cdot 3$ and for the following $2688=6\cdot 404+11\cdot 1\cdot 2\cdot 3\cdot 4$. And there shall be

$$y=p+\alpha q+qlx.$$

## EXAMPLE 2

**936.** *On putting $m=-1$ to solve this equation $ddy+\frac{aydx^2}{x^3}=0$ by series.*

On putting $y=p+\alpha q+qlx$ it is necessary to take

$$q=A-\frac{aA}{1\cdot 2x}+\frac{a^2A}{1\cdot 2^2\cdot 3x^2}-\frac{a^3A}{1\cdot 2^2\cdot 3^3\cdot 4x^3}+\text{etc.}$$

which is put for brevity :

$$q=A+\frac{B}{x}+\frac{C}{x^2}+\frac{D}{x^3}+\frac{E}{x^4}+\text{etc.};$$

then indeed the quantity $p$ must be defined from this equation

$$ddp+\frac{2dqdx}{x}-\frac{qdx^2}{xx}+\frac{apdx^2}{x^3}=0.$$

Hence we put in place

$$p=\mathfrak{A}x+\mathfrak{B}+\frac{\mathfrak{C}}{x}+\frac{\mathfrak{D}}{xx}+\frac{\mathfrak{E}}{x^3}+\frac{\mathfrak{F}}{x^4}+\text{etc.}$$

from which with the substitution made there arises

$$\left.\begin{array}{l}\frac{a\mathfrak{A}}{xx}+\frac{a\mathfrak{B}}{x^3}+\frac{a\mathfrak{C}}{x^4}+\frac{a\mathfrak{D}}{x^5}+\frac{a\mathfrak{E}}{x^6}+\frac{a\mathfrak{F}}{x^7}+\text{etc.}\\ -A-B-C-D-E-F\\ \quad -2B-4C-6D-8E-10F\\ \quad +2\mathfrak{C}+6\mathfrak{D}+12\mathfrak{E}+20\mathfrak{F}+30\mathfrak{G}\end{array}\right\}=0$$

and the coefficients $\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D},\mathfrak{E}$ etc. are determined thus, so that there shall be $\mathfrak{A}=\frac{A}{a}$; the following term $\mathfrak{B}$ is not defined; then there shall be

$$\mathfrak{C}=\frac{-3B}{1\cdot 2}-\frac{a\mathfrak{B}}{1\cdot 2}=\frac{-3aA}{1^3\cdot 2^2}-\frac{a\mathfrak{B}}{1\cdot 2},\quad \mathfrak{D}=\frac{5C}{2\cdot 3}-\frac{a\mathfrak{C}}{2\cdot 3}=\frac{5a^2A}{2^3\cdot 3^2}-\frac{a\mathfrak{C}}{2\cdot 3},$$

$$\mathfrak{E}=\frac{7D}{3\cdot 4}-\frac{a\mathfrak{D}}{3\cdot 4}=\frac{-7a^3A}{2^2\cdot 3^3\cdot 4^2}-\frac{a\mathfrak{D}}{3\cdot 4},\quad \mathfrak{F}=\frac{9E}{4\cdot 5}-\frac{a\mathfrak{E}}{4\cdot 5}=\frac{9a^4A}{2^2\cdot 3^2\cdot 4^3\cdot 5^2}-\frac{a\mathfrak{E}}{4\cdot 5}$$

etc.

If it is assumed that $\mathfrak{B}=0$, which is allowed to happen without detriment to the generality, thus so that there shall be

$$p=\mathfrak{A}x+\frac{\mathfrak{C}}{x}+\frac{\mathfrak{D}}{xx}+\frac{\mathfrak{E}}{x^3}+\frac{\mathfrak{F}}{x^4}+\text{etc.},$$

then there becomes

$$\mathfrak{C}=\frac{-3aA}{1^3\cdot 2^2},\quad \mathfrak{D}=\frac{14a^2A}{1^3\cdot 2^3\cdot 3^2},\quad \mathfrak{E}=\frac{-70a^3A}{1^3\cdot 2^3\cdot 3^3\cdot 4^2},\quad \mathfrak{F}=\frac{404a^4A}{1^3\cdot 2^3\cdot 3^3\cdot 4^3\cdot 5^2}\ \text{etc.}$$

which values are similar to the preceding.

## EXEMPLUM 3

**937.** *On putting* $m=\frac{1}{2}$ *to resolve this equation* $ddy+\frac{aydx^2}{x\sqrt{x}}=0$ *by series.*

On putting $y=p+\alpha q+qlx$ it is required to take

$$q=\mathfrak{A}x-\frac{4a\mathfrak{A}}{1\cdot 3}x^{\frac{3}{2}}+\frac{16a^2\mathfrak{A}}{1\cdot 2\cdot 3\cdot 4}x^2-\frac{64a^3\mathfrak{A}}{1\cdot 2\cdot 3\cdot 3\cdot 4\cdot 5}x^{\frac{5}{2}}+\frac{256a^4\mathfrak{A}}{1\cdot 2\cdot 3\cdot 4\cdot 3\cdot 4\cdot 5\cdot 6}x^3-\text{etc.}$$

which for brevity is written as :

$$q=\mathfrak{A}x+\mathfrak{B}x^{\frac{3}{2}}+\mathfrak{C}x^2+\mathfrak{D}x^{\frac{5}{2}}+\mathfrak{E}x^3+\mathfrak{F}x^{\frac{7}{2}}+\mathfrak{G}x^4+\text{etc.};$$

then the quantity *p* must be defined from this equation :

$$ddp + \frac{2dxdq}{x} - \frac{qdx^2}{xx} + \frac{apdx^2}{x\sqrt{x}} = 0.$$

Hence we put in place

$$p = \Delta + Ax^{\frac{1}{2}} + Bx + Cx^{\frac{3}{2}} + Dx^2 + Ex^{\frac{5}{2}} + Fx^3 + \text{etc.}$$

and there arises from the substitution made :

$$\left.\begin{array}{l} \frac{a\Delta}{x\sqrt{x}} + \frac{aA}{x} + \frac{aB}{x\sqrt{x}} + aC + aD\sqrt{x} + aEx + aFx\sqrt{x} \text{ etc.} \\ \quad -\mathfrak{A} - \mathfrak{B} - \mathfrak{C} - \mathfrak{D} - \mathfrak{E} - \mathfrak{F} \\ \quad +2\mathfrak{A} + 3\mathfrak{B} + 4\mathfrak{C} + 5\mathfrak{D} + 6\mathfrak{E} + 7\mathfrak{F} \\ \frac{-A}{4} + 0 + \frac{3}{4}C + 2D + \frac{15}{4}E + 6F + \frac{35}{4}G \end{array}\right\} = 0.$$

Hence there is deduced to be $A = \frac{-\mathfrak{A}}{a}$, $\Delta = \frac{-\mathfrak{A}}{4aa}$ ; but $B$ is not determined; again

$$C = \frac{-4aB}{1\cdot 3} - \frac{8\mathfrak{B}}{1\cdot 3} = \frac{-4aB}{1\cdot 3} + \frac{8\cdot 4a\mathfrak{A}}{1^2\cdot 3^2},$$

$$D = \frac{-4aC}{2\cdot 4} - \frac{12\mathfrak{C}}{2\cdot 4} = \frac{-4aC}{2\cdot 4} - \frac{12\cdot 16a^2\mathfrak{A}}{1\cdot 3\cdot 2^2\cdot 4^2},$$

$$E = \frac{-4aD}{3\cdot 5} - \frac{16\mathfrak{D}}{3\cdot 5} = \frac{-4aD}{3\cdot 5} + \frac{16\cdot 64a^3\mathfrak{A}}{1\cdot 3\cdot 2\cdot 4\cdot 3^2\cdot 5^2},$$

$$F = \frac{-4aE}{4\cdot 6} - \frac{12\mathfrak{E}}{4\cdot 6} = \frac{-4aE}{4\cdot 6} - \frac{20\cdot 256a^4\mathfrak{A}}{1\cdot 3\cdot 2\cdot 4\cdot 3\cdot 5\cdot 4^2\cdot 6^2},$$

etc.

But if now there is put $B = 0$, in order that there becomes

$$p = \frac{-\mathfrak{A}}{4aa} - \frac{\mathfrak{A}}{a}\sqrt{x} + * + Cx\sqrt{x} + Dx^2 + Ex^2\sqrt{x} + Fx^3 + \text{etc.}$$

there will be

$$C = \frac{8\cdot 4a\mathfrak{A}}{1^2\cdot 3^2}, \quad D = \frac{-100\cdot 16a^2\mathfrak{A}}{1^2\cdot 3^2\cdot 2^2.4^2}, \quad E = \frac{1884\cdot 64a^3\mathfrak{A}}{1^2\cdot 3^2\cdot 2^2\cdot 4^2\cdot 3^2\cdot 5^2}$$

$$F = \frac{-52416\cdot 256a^4\mathfrak{A}}{1^2\cdot 3^2\cdot 2^2\cdot 4^2\cdot 3^2\cdot 5^2 4^2\cdot 6^2} \text{ etc.},$$

where it is to be noted that

$$100 = 2\cdot 4\cdot 8 + 1\cdot 3\cdot 12, \; 1884 = 3\cdot 5\cdot 100 + 1\cdot 3\cdot 2\cdot 4\cdot 16,$$

$$52416 = 4\cdot 6\cdot 1884 + 1\cdot 3\cdot 2\cdot 4\cdot 3\cdot 5\cdot 20.$$

## EXEMPLUM 4

**938.** *On putting* $m=-\frac{1}{2}$ *to resolve this equation* $ddy+\frac{aydx^2}{xx\sqrt{x}}=0$ *by series.*

On putting $y=p+\alpha q+qlx$ it is required to take

$$q=A-\frac{4aA}{1\cdot 3}x^{-\frac{1}{2}}+\frac{16a^2A}{1\cdot 3\cdot 2\cdot 4}x^{-1}-\frac{64a^3A}{1\cdot 3\cdot 2\cdot 4\cdot 3\cdot 5}x^{-\frac{3}{2}}+\text{ etc.},$$

which for brevity we may write as :

$$q=A+Bx^{-\frac{1}{2}}+Cx^{-1}+Dx^{-\frac{3}{2}}+Ex^{-2}+Fx^{-\frac{5}{2}}+\text{ etc.},$$

and the letter $p$ must be defined from this equation

$$ddp+\frac{2dxdq}{x}-\frac{qdx^2}{xx}+\frac{apdx^2}{xx\sqrt{x}}=0.$$

We put in place

$$p=\Delta x+\mathfrak{A}\sqrt{x}+\mathfrak{B}+\mathfrak{C}x^{-\frac{1}{2}}+\mathfrak{D}x^{-1}+\mathfrak{E}x^{-\frac{3}{2}}+\mathfrak{F}x^{-2}+\mathfrak{G}x^{-\frac{5}{2}}+\text{etc.}$$

and with the substitution made there arises

$$\left.\begin{array}{ccccccc}\frac{a\Delta}{x\sqrt{x}}+\frac{a\mathfrak{A}}{xx}+ & \frac{a\mathfrak{B}}{xx\sqrt{x}} & +\frac{a\mathfrak{C}}{x^3}+ & \frac{a\mathfrak{D}}{x^3\sqrt{x}} & +\frac{a\mathfrak{E}}{x^4} & +\frac{a\mathfrak{F}}{x^4\sqrt{x}} & +\text{etc.}\\ -A\ - & B & -C\ - & D & -E\ - & F & \\ - & B & -2C- & 3D & -4E- & 5F & \\ -\frac{1}{4}\mathfrak{A}\quad + & \frac{3}{4}\mathfrak{C} & +2\mathfrak{D}+ & \frac{15}{4}\mathfrak{F} & +6\mathfrak{F}+ & \frac{35}{4}\mathfrak{G} & \end{array}\right\}=0,$$

from which the following determinations are deduced $\mathfrak{A}=\frac{A}{a}$ and $\Delta=\frac{\mathfrak{A}}{4A}=\frac{A}{4aa}$ ; but $\mathfrak{B}$ is not determined; again

$$\mathfrak{C}=\frac{-4a\mathfrak{B}}{1\cdot 3}+\frac{8\mathfrak{B}}{1\cdot 3}=\frac{-4a\mathfrak{B}}{1\cdot 3}-\frac{8\cdot 4aA}{1^2\cdot 3^2},$$

$$\mathfrak{D}=\frac{-4a\mathfrak{C}}{2\cdot 4}+\frac{12C}{2\cdot 4}=\frac{-4a\mathfrak{C}}{2\cdot 4}+\frac{12\cdot 16a^2A}{1\cdot 3\cdot 2^2\cdot 4^2},$$

$$\mathfrak{E}=\frac{-4a\mathfrak{D}}{3\cdot 5}-\frac{16D}{3\cdot 5}=\frac{-4a\mathfrak{D}}{3\cdot 5}-\frac{16\cdot 64a^3A}{1\cdot 3\cdot 2\cdot 4\cdot 3^2\cdot 5^2},$$

$$\mathfrak{F}=\frac{-4a\mathfrak{E}}{4\cdot 6}+\frac{12E}{4\cdot 6}=\frac{-4a\mathfrak{E}}{4\cdot 6}-\frac{20\cdot 256a^4A}{1\cdot 3\cdot 2\cdot 4\cdot 3\cdot 5\cdot 4^2\cdot 6^2},$$

etc.

But if now there is assumed $\mathfrak{B}=0$, there will be

$$\mathfrak{C} = \frac{-8\cdot 4aA}{1^2\cdot 3^2}, \quad \mathfrak{D} = \frac{100\cdot 16a^2 A}{1^2\cdot 3^2\cdot 2^2\cdot 4^2}, \quad \mathfrak{E} = \frac{-1884\cdot 64a^3 A}{1^2\cdot 3^2\cdot 2^2\cdot 4^2\cdot 3^2\cdot 5^2} \quad \text{etc.},$$

which numbers are progressing as before.

**SCHOLIUM**

**939.** From these examples it is observed how the series resolving the equation

$$ddy + a^{m-2}ydx^2 = 0$$

can be found in the remaining cases, in which $m = \pm\frac{1}{i}$ ; where it may be noted, if there should be $m = +\frac{1}{i}$ that for $q$ this series must be taken

$$q = \mathfrak{A}x + \mathfrak{B}x^{1+\frac{1}{i}} + \mathfrak{C}x^{1+\frac{2}{i}} + \mathfrak{D}x^{1+\frac{3}{i}} + \text{etc.},$$

then truly the form of $p$ of such a series to be expressed by :

$$p = Ax + Bx^{\frac{1}{i}} + Cx^{\frac{2}{i}} + Dx^{\frac{3}{i}} + \text{etc.},$$

the coefficients of which as before are to be defined from above. But if there should be $m = -\frac{1}{i}$ , the series is taken for $q$ :

$$q = A + Bx^{-\frac{1}{i}} + Cx^{-\frac{2}{i}} + Dx^{-\frac{3}{i}} + \text{etc.},$$

but for $p$ it is appropriate to take a form of this kind

$$p = \mathfrak{A}x + \mathfrak{B}x^{1-\frac{1}{i}} + \mathfrak{C}x^{1-\frac{2}{i}} + \mathfrak{D}x^{1-\frac{3}{i}} + \text{etc.},$$

from which equally the individual coefficients are allowed to be defined with one exception. And this trick is to be maintained in the generation , as often as in the resolution of general equations in series is to be arrived at, in which the coefficients in certain cases increase to infinity, which generally is to be the introduction of logarithms in the proof.

Also truly the same equation $ddy + a^n ydx^2 = 0$ can be resolved in series in other ways, provided that previous resolution can be changed into another form ; since when it can arise, so that the series in certain cases is terminated, to which thus an integral must be assigned, here especially we will set out such a remarkable transformation.

## PROBLEM 118

**940.** *To transfer the second order differential equation* $ddy + ax^n ydx^2 = 0$ *into another form, the resolution of which will be put in place conveniently by an infinite series.*

## SOLUTION

We may make use of the substitution $y = e^{\int pdx} z$, where *p* shall be a certain function of *x* providing a convenient resolution of the equation. Hence there shall be

$$dy = e^{\int pdx}\left(dz + pzdx\right)$$

and

$$ddy = e^{\int pdx}\left(ddz + 2pdxdz + zdxdp + ppzdx^2\right),$$

from which the proposed equation becomes

$$ddz + 2pdxdz + zdxdp + ppzdx^2 + ax^n zdx^2 = 0,$$

where *p* shall be taken thus, so that there becomes $pp + ax^n = 0$ or $p = x^{\frac{n}{2}}\sqrt{-a}$. Thus we may put $a = -cc$ and $n = 2m$, in order that the proposed shall be this equation

$$ddy - ccx^{2m} ydx^2 = 0,$$

which on putting $p = cx^m$ and $y = e^{\int pdz} z = e^{\frac{c}{m+1}x^{m+1}} z$ adopts this form

$$ddz + 2cx^m dxdz + mcx^{m-1} zdx^2 = 0;$$

in which since *x* neither has zero nor *m* +1 dimensions, we can put the value of *z* in place

$$z = Ax^{\lambda} + Bx^{\lambda+m+1} + Cx^{\lambda+2m+2} + \text{etc.},$$

with which substituted there becomes :

$$\left.\begin{array}{l} \lambda(\lambda-1)Ax^{\lambda-2} + (\lambda+m+1)(\lambda+m)Bx^{\lambda+m-1} + \text{etc.}, \\ \qquad\qquad\qquad\qquad + 2\lambda Ac \\ \qquad\qquad\qquad\qquad + mAc \end{array}\right\} = 0,$$

from which it is evident that there must be taken either $\lambda = 0$ or $\lambda = 1$. Hence we follow with a double series of this kind

$$\begin{aligned} z = {}& A + Bx^{m+1} + Cx^{2m+2} + Dx^{2m+3} + Ex^{4m+4} + \text{etc.} \\ & + \mathfrak{A}x + \mathfrak{B}x^{m+2} + \mathfrak{C}x^{2m+3} + \mathfrak{D}x^{4m+4} + \mathfrak{E}x^{4m+5} + \text{etc.}, \end{aligned}$$

with which put in place there becomes :

$$\left.\begin{array}{llll} (m+1)mBx^{m-1} & +2(m+1)(2m+1)Cx^{2m} & +3(m+1)(3m+2)Dx^{3m+1} & +\text{etc.} \\ +mAc & +2(m+1)Bc & +4(m+1)Cc & \\ & + \quad mBc & + \quad mCc & \end{array}\right\} = 0,$$

$$\left.\begin{array}{llll} (m+1)(m+2)\mathfrak{B}x^{m} & +2(m+1)(2m+3)\mathfrak{C}x^{2m+1} & +3(m+1)(3m+4)\mathfrak{D}x^{3m+2} & +\text{etc.} \\ +2\mathfrak{A}c & +2(m+2)\mathfrak{B}c & +2(2m+3)\mathfrak{C}c & \\ +m\mathfrak{A}c & +m\mathfrak{B}c & +m\mathfrak{C}c & \end{array}\right\} = 0,$$

from which all the individual coefficients are determined in the following manner :

| | |
|---|---|
| $B = \frac{-mAc}{m(m+1)}$ | $\mathfrak{B} = \frac{-(m+2)\mathfrak{A}c}{(m+2)(m+1)}$ |
| $C = \frac{-(3m+2)Bc}{2(2m+1)(m+1)}$ | $\mathfrak{C} = \frac{-(3m+4)\mathfrak{B}c}{2(2m+3)(m+1)}$ |
| $D = \frac{-(5m+4)Cc}{3(3m+2)(m+1)}$ | $\mathfrak{D} = \frac{-(5m+6)\mathfrak{C}c}{3(3m+4)(m+1)}$ |
| $E = \frac{-(7m+6)Dc}{4(4m+3)(m+1)}$ | $\mathfrak{E} = \frac{-(7m+8)\mathfrak{D}c}{4(4m+5)(m+1)}$ |
| etc. | etc. |

where the two coefficients $A$ and $\mathfrak{A}$ remain undetermined, thus in order that the complete integral shall be agreed upon.

ALTERNATIVELY

For the series assumed, in which the coefficients of $x$ decrease, there must be taken $2\lambda + m = 0$ or $m = -2\lambda$ in order that our equation shall be

$$ddy - ccx^{-4\lambda}ydx^2 = 0,$$

which on putting $p = cx^{-2\lambda}$ and $y = e^{\int pdx}z = e^{\frac{-c}{2\lambda-1}x^{-2\lambda+1}}z$ becomes

$$ddz + 2cx^{-2\lambda}dxdz - 2\lambda cx^{-2\lambda-1}zdx^2 = 0$$

Hence we may put

$$z = Ax^{\lambda} + Bx^{3\lambda-1} + Cx^{5\lambda-2} + Dx^{3\lambda-3} + \text{etc.}$$

and with the substitution made there arises

$$\begin{array}{lllll} 0 = & \lambda(\lambda-1)Ax^{\lambda-2} & +(3\lambda-1)(3\lambda-2)Bx^{3\lambda-3} & +(5\lambda-2)(5\lambda-3)CXx^{5\lambda-4} & +\text{etc.} \\ +2\lambda Acx^{-\lambda-1} & +2(3\lambda-1)Bc & +2(5\lambda-2)Cc & + & 2(7\lambda-3)Dc \\ -2\lambda Ac & - \quad 2\lambda Bc & - \quad 2\lambda Cc & - & 2\lambda Dc \end{array}$$

from which the coefficients are determined thus :

$$B = \frac{-\lambda(\lambda-1)A}{(4\lambda-2)c} = \frac{-\lambda(\lambda-1)A}{2(2\lambda-1)c},$$
$$C = \frac{-(3\lambda-1)(3\lambda-2)B}{(8\lambda-4)c} = \frac{-(3\lambda-1)(3\lambda-2)B}{4(2\lambda-1)c},$$
$$D = \frac{-(5\lambda-2)(5\lambda-3)C}{(12\lambda-6)c} = \frac{-(5\lambda-2)(5\lambda-3)C}{6(2\lambda-1)c},$$
$$E = \frac{-(7\lambda-3)(7\lambda-4)D}{(16\lambda-8)c} = \frac{-(7\lambda-3)(7\lambda-4)D}{8(2\lambda-1)c}$$
$$\text{etc.}$$

Here only the value of the letter $A$ is left to our choice, from which this series only exhibits a particular integral.

## COROLLARY 1

**941.** From the first solution it is apparent that the other series terminates [in the first table above, where the terms for *B,C, D*, etc have a factor of the following kind, and likewise for the Gothic script terms], whenever

$$(2i+1)m + 2i = 0 \text{ or } m = \tfrac{-2i}{2i+1},$$

and truly the other, whenever

$$(2i-1)m + 2i = 0 \text{ or } m = \tfrac{-2i}{2i-1}$$

with $i$ denoting some whole number. Hence at least in these cases a particular finite integral can be expressed.

**COROLLARY 2**

**942.** The alternate solution offers a finite series whenever there should be either

$$(2i+1)\lambda - i = 0 \text{ or } (2i-1)\lambda - i = 0,$$

that is $\lambda = \frac{i}{2i\pm1}$ and $m = \frac{-2i}{2i\pm1}$ as before. Truly for the remaining cases this series extends to infinity.

**COROLLARY 3**

**943.** Hence the cases, in which this equation $ddy - ccx^n dx^2 = 0$ and thus on putting $y = e^{\int udx}$ also this equation $du + uudx = ccx^n dx$, at least allows a particular integration, are $n = \frac{-4i}{2i\pm1}$ to be assumed for some whole number $i$ [§ 436-441].

**SCHOLIUM**

**944.** Moreover it suffices for a particular integral to be come upon, since from that it is possible to elicit the complete integral easily. For since the letter $c$ shall be present in the integral, provided the differential equation contains only the square $cc$, likewise there can be taken in the integral either $+c$ or $-c$. Hence, if $y = P + cQ$ is a particular integral, then also $y = P - cQ$ is a particular integral, from which the complete integral will be

$$y = \alpha(P + cQ) + \beta(P - cQ) \text{ or } y = \alpha P + \beta cQ.$$

In order that this can be made clearer, another solution of the equation

$$ddy - ccx^{-4\lambda} ydx^2 = 0$$

may be provided, for which on putting for brevity $\frac{1}{1-2\lambda} x^{1-2\lambda} = t$ we can put $y = e^{ct} z$

and we find [§ 940]

$$z = Ax^{\lambda} - \frac{\lambda(\lambda-1)A}{2(2\lambda-1)c} x^{3\lambda-1} + \frac{\lambda(\lambda-1)(3\lambda-1)(3\lambda-2)A}{2\cdot4(2\lambda-1)^2 cc} x^{5\lambda-2}$$
$$- \frac{\lambda(\lambda-1)(3\lambda-1)(3\lambda-2)(5\lambda-2)(5\lambda-3)A}{2\cdot4\cdot6(2\lambda-1)^3 c^3} x^{7\lambda-3} + \text{ etc.}$$

In order that the terms in the expression may be distinguished divided by the even powers of $c$ from those, which are divided by the odd powers, we can write $z = P - cQ$, thus in order that $P$ and $Q$ may contain only even powers of $c$, and one particular integral is $y = e^{ct}(P - cQ)$ and the other $y = e^{-ct}(P + cQ)$, from which the complete integral will be

$$y=\tfrac{1}{2}P\left(\alpha e^{ct}+\beta e^{-ct}\right)-\tfrac{1}{2}cQ\left(\alpha e^{ct}-\beta e^{-ct}\right).$$

Hence, if $c$ shall be an imaginary number or $cc=-bb$, in order that the equation shall be

$$ddy+bbx^{-4\lambda}ydx^2=0,$$

there will be $z=P-bQ\sqrt{-1}$ and $e^{ct}=e^{bt\sqrt{-1}}=\cos.bt+\sqrt{-1}\cdot\sin.bt$, hence

$$y=P\left(\tfrac{\alpha+\beta}{2}\cos.bt+\tfrac{\alpha-\beta}{2}\sqrt{-1}\cdot\sin.bt\right)-bQ\left(\tfrac{\alpha-\beta}{2}\cos.bt+\tfrac{\alpha+\beta}{2}\sqrt{-1}\cdot\sin.bt\right)\sqrt{-1};$$

let $\frac{\alpha+\beta}{2}=\gamma$ and $\frac{\alpha-\beta}{2}\sqrt{-1}=\delta$, and the complete integral thus may be expressed in this way :

$$y=P\left(\gamma\cos.bt+\delta\sin.bt\right)-bQ\left(\delta\cos.bt-\gamma\sin.bt\right)$$

or

$$y=\left(\gamma P-\delta bQ\right)cos.bt+\left(\delta P+\gamma bQ\right)\sin.bt.$$

Hence in this manner we have established the integrable quantities.

### EXEMPLE 1

**945.** *To find the integral of the equation* $ddy-ccydx^2=0$.

Here there is $\lambda=0$, $z=A$ and $t=x$, from which on account of $P=A$ and $Q=0$ the complete integral will be

$$y=\alpha e^{cx}+\beta e^{-cx}.$$

But in the case $cc=-bb$ the complete integral of the equation $ddy+bbydx^2=0$ will be

$$y=\gamma cos.bx+\delta\sin.bx.$$

### EXAMPLE 2

**946.** *To find the integral of the equation* $ddy-ccx^{-4}ydx^2=0$.

Here on account of $\lambda=1$ there is $z=Ax$ and $t=-\frac{1}{x}$, from which on account of $P=x$ and $Q=0$ there becomes :

$$y=\left(\alpha e^{ct}+\beta e^{-ct}\right)x.$$

Moreover in the case $cc=-bb$, the integral of the equation $ddy+bbx^{-4}ydx^2=0$ is

$$y = \left(\alpha \cos.bt + \beta \sin.bt\right) x$$

with $t = \frac{-1}{x}$ arising.

## EXAMPLE 3

**947.** *To find the integral of the equation* $ddy - ccx^{-\frac{4}{3}} ydx^2 = 0$.

On account of $\lambda = \frac{1}{3}$ there becomes $B = -\frac{A}{3c}$ and $z = Ax^{\frac{1}{3}} - \frac{A}{3c}$ , and $t = 3x^{\frac{1}{3}}$, from which $P = x^{\frac{1}{3}}$ and $Q = \frac{1}{3cc}$. Hence the integral will be

$$y = \left(\alpha e^{ct} + \beta e^{-ct}\right) x^{\frac{1}{3}} - \left(\alpha e^{ct} - \beta e^{-ct}\right) \frac{1}{3c}.$$

But in the case $cc = -bb$ the integral of the equation $ddy + bbx^{-\frac{4}{3}} ydx^2 = 0$ is

$$y = \left(\alpha \cos.bt + \beta \sin.bt\right) x^{\frac{1}{3}} + \frac{1}{3b}\left(\beta \cos.bt - \alpha \sin.bt\right).$$

## EXAMPLE 4

**948.** *To find the integral of the equation* $ddy - ccx^{-\frac{8}{3}} ydx^2 = 0$.

On account of $\lambda = \frac{2}{3}$ there will be made $B = \frac{A}{3c}$ and $z = Ax^{\frac{2}{3}} - \frac{A}{3c} x$, thus so that

$$P = x^{\frac{2}{3}} \text{ and } Q = \frac{-x}{3cc}.$$

Hence on putting $t = -3x^{-\frac{1}{3}}$ the integral will be expressed thus :

$$y = x^{\frac{2}{3}} \left(\alpha e^{ct} + \beta e^{-ct}\right) + \frac{x}{3c}\left(\alpha e^{ct} - \beta e^{-ct}\right).$$

But in the case $cc = -bb$ the integral of the equation $ddy + bbx^{-\frac{8}{3}} ydx^2 = 0$ is

$$y = x^{\frac{2}{3}} \left(\alpha \cos.bt + \beta \sin.bt\right) - \frac{x}{3b}\left(\beta \cos.bt - \alpha \sin.bt\right)$$

### EXAMPLE 5

**949.** *To find the integral of the equation* $ddy - ccx^{-\frac{8}{5}} ydx^2 = 0$.

On account of $\lambda = \frac{2}{5}$ there is $B = \frac{-3A}{3c}$ and $C = \frac{3A}{5^2 cc}$, hence $z = Ax^{\frac{2}{5}} - \frac{3A}{5c} x^{\frac{1}{5}} + \frac{3A}{5^2 cc}$ and thus [gathering even and odd terms] $P = x^{\frac{2}{5}} + \frac{3}{5^2 cc}$ and $Q = \frac{3}{5cc} x^{\frac{1}{5}}$. Hence on putting $t = 5x^{\frac{1}{5}}$ the integral will be

$$y = \left( x^{\frac{2}{5}} + \frac{3}{5^2 cc} \right) \left( \alpha e^{ct} + \beta e^{-ct} \right) - \frac{3}{5c} x^{\frac{1}{5}} \left( \alpha e^{ct} - \beta e^{-ct} \right).$$

But in the case $cc = -bb$ the integral of the equation $ddy + bbx^{-\frac{8}{5}} ydx^2 = 0$ is

$$y = \left( x^{\frac{2}{5}} - \frac{3}{5^2 bb} \right) \left( \alpha \cos.bt + \beta \sin.bt \right) + \frac{3}{5b} x^{\frac{1}{5}} \left( \beta \cos.bt - \alpha \sin.bt \right).$$

### PROBLEM 119

**950.** *To assign the complete integral of the second order differential equation*

$$ddy - ccx^{\frac{-4i}{2i-1}} ydx^2 = 0,$$

*with i denoting some whole number.*

### SOLUTION

For the sake of brevity let $t = -(2i-1) x^{\frac{-1}{2i-1}}$, from which there will come about $x^{\frac{1}{2i-1}} = -\frac{2i-1}{t}$, and on putting $y = e^{ct} z$ for the value of $z$ found by the series

$$z = Ax^{\frac{i}{2i-1}} + Bx^{\frac{i+1}{2i-1}} + Cx^{\frac{i+2}{2i-1}} + Dx^{\frac{i+3}{2i-1}} + \text{etc.}$$

on account of $\lambda = \frac{i}{2i-1}$ these coefficients thus are determined :

$$B = \frac{i(i-1)A}{2(2i-1)c}, \quad C = \frac{(i+1)(i-2)B}{4(2i-1)c}, \quad D = \frac{(i+2)(i-3)C}{6(2i-1)c} \quad \text{etc.},$$

with which substituted and with the value introduced : $x^{\frac{1}{2i-1}} = -\frac{2i-1}{t}$ there will be

$$z = Ax^{\frac{i}{2i-1}}\left(1 - \frac{i(i-1)}{2ct} + \frac{i(ii-1)(i-2)}{2\cdot 4cctt} - \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6c^3t^3} + \text{etc.}\right)$$

or in this way

$$z = \frac{A}{t^i}\left(1 - \frac{i(i-1)}{2ct} + \frac{i(ii-1)(i-2)}{2\cdot 4cctt} - \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6c^3t^3} + \text{etc.}\right)$$

And hence the complete integral of the proposed equation shall be expressed thus :

$$\begin{aligned} y = t^{-i}&\left(1 + \frac{i(ii-1)(i-2)}{2\cdot 4cctt} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8c^4t^4} + \text{etc.}\right)\left(\alpha e^{ct} + \beta e^{-ct}\right) \\ -t^{-i}&\left(\frac{i(i-1)}{2ct} + \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6c^3t^3} + \text{etc.}\right)\left(\alpha e^{ct} - \beta e^{-ct}\right), \end{aligned}$$

where in each progression the law of formation of the individual terms is clear.

## COROLLARY 1

**951.** Hence also the complete integral of that equation $ddy + bbx^{\frac{-4i}{2i-1}}ydx^2 = 0$ with $t = -(2i-1)x^{\frac{-1}{2i-1}}$ remaining is

$$\begin{aligned} y = t^{-i}&\left(1 - \frac{i(ii-1)(i-2)}{2\cdot 4bbtt} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8b^4t^4} - \text{etc.}\right)\left(\alpha\cos.bt + \beta\sin.bt\right) \\ +t^{-i}&\left(\frac{i(i-1)}{2bt} - \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6b^3t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2\cdot 4\cdot 6\cdot 8\cdot 10b^5t^5} - \text{etc.}\right)\left(\beta\cos.bt - \alpha\sin.bt\right), \end{aligned}$$

## COROLLARY 2

**952.** If *i* shall be a negative number, this integration likewise will succeed ; for the integral of the equation

$$ddy - ccx^{\frac{-4i}{2i+1}}ydx^2 = 0$$

on putting $t = (2i+1)x^{\frac{1}{2i+1}}$ will be

$$\begin{aligned} y = t^{i}&\left(1 + \frac{i(ii-1)(i+2)}{2\cdot 4cctt} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8c^4t^4} + \text{etc.}\right)\left(\alpha e^{ct} + \beta e^{-ct}\right) \\ -t^{i}&\left(\frac{i(i+1)}{2ct} + \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6c^3t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^5t^5} + \text{etc.}\right)\left(\alpha e^{ct} - \beta e^{-ct}\right), \end{aligned}$$

## COROLLARY 3

**953.** In a similar manner the complete integral of this equation

$$ddy + bbx^{\frac{-4i}{2i+1}}ydx^2 = 0,$$

on putting $t = (2i+1)x^{\frac{1}{2i+1}}$, will be

$$y = t^i\left(1 - \frac{i(ii-1)(i+2)}{2\cdot 4bbtt} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8b^4t^4} - \text{etc.}\right)(\alpha\cos.bt + \beta\sin.bt)$$
$$+ t^i\left(\frac{i(i+1)}{2bt} - \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6b^3t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10b^5t^5} - \text{etc.}\right)(\beta\cos.bt - \alpha\sin.bt),$$

## COROLLARY 4

**954.** In the formulas containing the sine and cosine, if there is put

$$\alpha = C\sin.\zeta \text{ and } \beta = C\cos.\zeta,$$

our expressions thus contract, so that there becomes

$$\alpha cos.bt + \beta\sin.bt = Csin.(bt + \zeta) \text{ and } \beta cos.bt - \alpha\sin.bt = Ccos.(bt + \zeta),$$

as now here $C$ and $\zeta$ shall be arbitrary constants rendering the integral complete.

## SCHOLIUM

**955.** Hence an excellent means is come upon for the case of the integration of this differential equation of the first order

$$du + uudx + ax^n dx = 0$$

and likewise the complete integral requiring to be defined being known ; for this equation $ddy + ax^n ydx^2 = 0$ arises from that on putting $y = e^{\int udx}$, and from which that in turn arises from that on putting $u = \frac{dy}{ydx}$. Therefore since the integral of this is permitted to be assigned in the cases, in which the exponent $n = \frac{4i}{2i\pm 1}$, in the same cases the integral of the differential equation of the first order is permitted to be assigned, where indeed it is agreed that two cases are to be set out, according as $a$ should be a negative number $a = -cc$ or a positive number $a = +bb$. Therefore it will be worth the effort to explore these two cases.

## PROBLEM 120

**956.** *With i denoting some whole number, either positive or negative, to find the integral of this equation*

$$du + uudx - ccx^{\frac{-4i}{2i+1}}dx = 0.$$

## SOLUTION

On putting $u = \frac{dy}{ydx}$ this equation is transformed into that :

$$ddy - ccx^{\frac{-4i}{2i+1}}ydx^2 = 0$$

[since $du + uudx = \frac{ddy}{ydx} - \frac{dydy}{y^2dx} + \left(\frac{dy}{ydx}\right)\left(\frac{dy}{ydx}\right)dx$ ]
on assuming the element $dx$ constant, the integral of which we have assigned. Clearly on putting $t = (2i+1)x^{\frac{1}{2i+1}}$ there becomes

$$y = \left(\alpha e^{ct} + \beta e^{-ct}\right)\left(t^i + \frac{i(ii-1)(i+2)}{2\cdot 4cc}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8c^4}t^{i-4} + \text{etc.}\right)$$
$$-\left(\alpha e^{ct} - \beta e^{-ct}\right)\left(\frac{i(i+1)}{2c}t^{i-1} + \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6c^3}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^5}t^{i-5} + \text{etc.}\right).$$

For the sake of brevity we put

$$y = \left(\alpha e^{ct} + \beta e^{-ct}\right)P - \left(\alpha e^{ct} - \beta e^{-ct}\right)Q$$

and since there shall be

$$dt = x^{\frac{-2i}{2i+1}}dx \quad \text{or} \quad dx = x^{\frac{2i}{2i+1}}dt,$$

there becomes

$$\frac{dy}{dx} = \frac{\left(\alpha e^{ct} + \beta e^{-ct}\right)(dP - cQdt) - \left(\alpha e^{ct} - \beta e^{-ct}\right)(cPdt - dQ)}{x^{\frac{2i}{2i+1}}dt}$$

or

$$\frac{dy}{dx} = \frac{\alpha e^{ct}(dP + cPdt - dQ - cQdt) + \beta e^{-ct}(dP - cPdt + dQ - cQdt)}{x^{\frac{2i}{2i+1}}dt}$$

But truly there is

$$\frac{dP}{dt} = it^{i-1} + \frac{i(ii-1)(i-4)}{2\cdot 4cc}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(i-16)}{2\cdot 4\cdot 6\cdot 8c^4}t^{i-5} + \text{etc.},$$
$$cQ = \frac{i(i+1)}{2}t^{i-1} + \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6c^2}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^4}t^{i-5} + \text{etc.},$$

$$cP = ct^{i} + \frac{i(ii-1)(i+2)}{2\cdot 4c}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8c^{3}}t^{i-4} + \text{etc.}$$
$$\frac{dQ}{dt} = \frac{i(ii-1)}{2c}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)}{2\cdot 4\cdot 6c^{3}}t^{i-4} + \text{etc.}$$

from which there is deduced

$$\frac{dP-cQdt}{dt} = -\frac{i(i-1)}{2}t^{i-1} - \frac{i(ii-1)(i-4)(i-3)}{2\cdot 4\cdot 6c^{2}}t^{i-3} - \frac{i(ii-1)(ii-4)(ii-9)(i-16)(i-5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^{4}}t^{i-5} - \text{etc.},$$

$$\frac{cPdt-dQ}{dt} = ct^{i} + \frac{i(ii-1)(i-2)}{2\cdot 4c}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8c^{3}}t^{i-4} + \text{etc.}$$

We may put as abbreviations

$$P = t^{i} + \frac{i(ii-1)(i+2)}{2\cdot 4cc}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8c^{4}}t^{i-4} + \text{etc.},$$

$$Q = \frac{i(i+1)}{2c}t^{i-1} + \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6c^{3}}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^{5}}t^{i-5} + \text{etc.},$$

$$R = t^{i} + \frac{i(ii-1)(i-2)}{2\cdot 4cc}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8c^{4}}t^{i-4} + \text{etc.}$$

$$S = \frac{i(i-1)}{2c}t^{i-1} + \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6c^{3}}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^{5}}t^{i-5} + \text{etc.}$$

so that there becomes

$$\frac{dy}{dx} = \frac{\left(\alpha e^{ct}+\beta e^{-ct}\right)(-cS)-\left(\alpha e^{ct}-\beta e^{-ct}\right)(cR)}{x^{\frac{2i}{2i+1}}}$$

Whereby, since there shall be $u = \frac{dy}{ydx}$, the complete integral of our equation will be

$$\frac{1}{c}x^{\frac{2i}{2i+1}}u = \frac{\left(\alpha e^{ct}-\beta e^{-ct}\right)R-\left(\alpha e^{ct}+\beta e^{-ct}\right)S}{\left(\alpha e^{ct}+\beta e^{-ct}\right)P-\left(\alpha e^{ct}-\beta e^{-ct}\right)Q}$$

or

$$\frac{1}{c}x^{\frac{2i}{2i+1}}u = \frac{\alpha e^{ct}(R-S)-\beta e^{-ct}(R+S)}{\alpha e^{ct}(P-Q)+\beta e^{-ct}(P+Q)},$$

which is complete on account of the arbitrary ratio of the constants $a:\beta$ .

### COROLLARY 1

**957.** The four formulas *P, Q, R, S,* which are terminated in the individual cases in which *i* is a whole number integer, thus depend on each other, since there shall be in the first place

$$R = P - \frac{dQ}{cdt} \text{ and } S = Q - \frac{dP}{cdt},$$

then truly

$$dP + dR = \frac{2iRdt}{t} \text{ and } dQ + dS = \frac{2iSdt}{t}.$$

### COROLLARY 2

**958.** Hence on putting either $\alpha = 0$ or $\beta = 0$ it is possible to show algebraically the particular integrals of the equation $du + uudx - ccx^{\frac{-4i}{2i+1}}dx = 0$ , which are

$$\frac{1}{c}x^{\frac{2i}{2i+1}}u = \frac{R-S}{P-Q} \text{ and } \frac{1}{c}x^{\frac{2i}{2i+1}}u = \frac{-S-R}{P+Q}$$

and thus they can be taken in a single formula

$$\frac{1}{c}x^{\frac{2i}{2i+1}}u = \frac{-S \pm R}{P \mp Q}$$

### SCHOLIUM 1

**959.** Hence for the various values of the number *i* both the quantity *t* as well as the letters *P, Q, R, S* can be considered in the following manner. Clearly initially it is apparent that if *i* = 0, then there will be $t = x$ and $P = 1,\ Q = 0,\ R = 1$ and $S = 0$; and we may present the remaining cases in the following table :

| | |
|---|---|
| $i = -1, \quad t = -\frac{1}{x}$<br>$P = \frac{1}{t}, \quad Q = 0$<br>$R = \frac{1}{t}, \quad S = \frac{1}{ctt}$ | $i = 1, \quad t = 3x^{\frac{1}{3}}$<br>$P = t, \quad Q = \frac{1}{c}$<br>$R = t, \quad S = 0$ |
| $i = -2, \quad t = -\frac{3}{x^{\frac{1}{3}}}$<br>$P = \frac{1}{tt}, \quad Q = \frac{1}{ct^3}$<br>$R = \frac{1}{tt} + \frac{3}{cct^4}, \quad S = \frac{3}{ct^3}$ | $i = 2, \quad t = 5x^{\frac{1}{5}}$<br>$P = tt + \frac{3}{cc}, \quad Q = \frac{3}{c}t$<br>$R = tt, \quad S = \frac{1}{c}t$ |

| | |
|---|---|
| $i=-3,\quad t=-\frac{5}{x^{\frac{1}{5}}}$<br>$P=\frac{1}{t^3}+\frac{1\cdot 3}{cct^5},\quad Q=\frac{3}{ct^4}$<br>$R=\frac{1}{t^3}+\frac{3\cdot 5}{cct^5},\quad S=\frac{6}{ct^4}+\frac{3\cdot 5}{c^3t^6}$ | $i=3,\quad t=7x^{\frac{1}{7}}$<br>$P=tt+\frac{3\cdot 5}{cc},\quad Q=\frac{6}{c}tt+\frac{3\cdot 5}{c^3}$<br>$R=t^3+\frac{1\cdot 3}{cc}t,\quad S=\frac{3}{c}tt$ |
| $i=-4,\quad t=-\frac{7}{x^{\frac{1}{7}}}$<br>$P=\frac{1}{t^4}+\frac{3\cdot 5}{cct^6}$<br>$Q=\frac{6}{ct^5}+\frac{3\cdot 5}{c^3t^7}$<br>$R=\frac{1}{t^4}+\frac{3\cdot 3\cdot 5}{cct^6}+\frac{3\cdot 5\cdot 7}{c^4t^8},$<br>$S=\frac{10}{ct^5}+\frac{3\cdot 5\cdot 7}{c^3t^7}$ | $i=4,\quad t=9x^{\frac{1}{9}}$<br>$P=t^4+\frac{3\cdot 3\cdot 5}{cc}tt+\frac{3\cdot 5\cdot 7}{c^4}$<br>$Q=\frac{10}{c}t^3+\frac{3\cdot 5\cdot 7}{c^3}t$<br>$R=t^4+\frac{3\cdot 5}{cc}tt$<br>$S=\frac{6}{c}t^3+\frac{3\cdot 5}{c^3}t$ |
| $i=-5,\quad t=-\frac{9}{x^{\frac{1}{9}}}$<br>$P=\frac{1}{t^5}+\frac{3\cdot 3\cdot 5}{cct^7}+\frac{3\cdot 5\cdot 7}{c^4t^9}$<br>$Q=\frac{10}{ct^6}+\frac{3\cdot 5\cdot 7}{c^3t^8}$<br>$R=\frac{1}{t^5}+\frac{3\cdot 5\cdot 7}{cct^7}+\frac{3\cdot 5\cdot 7\cdot 9}{c^4t^9}$<br>$S=\frac{15}{ct^6}+\frac{4\cdot 3\cdot 5\cdot 7}{c^3t^8}+\frac{3\cdot 5\cdot 7\cdot 9}{c^5t^{10}}$ | $i=5,\quad t=11x^{\frac{1}{11}}$<br>$P=t^5+\frac{3\cdot 5\cdot 7}{cc}t^3+\frac{3\cdot 5\cdot 7\cdot 9}{c^4}t$<br>$Q=\frac{15}{c}t^4+\frac{4\cdot 3\cdot 5\cdot 7}{c^3}tt+\frac{3\cdot 5\cdot 7\cdot 9}{c^5}$<br>$R=t^5+\frac{3\cdot 3\cdot 5}{cc}t^3+\frac{3\cdot 5\cdot 7}{c^4}t$<br>$S=\frac{10}{c}t^4+\frac{3\cdot 5\cdot 7}{c^3}tt$ |
| $i=-6,\quad t=\frac{-11}{x^{\frac{1}{11}}}$<br>$P=\frac{1}{t^6}+\frac{3\cdot 5\cdot 7}{cct^8}+\frac{3\cdot 5\cdot 7\cdot 9}{c^5t^{10}}$<br>$Q=\frac{15}{ct^7}+\frac{4\cdot 3\cdot 5\cdot 7}{c^3t^9}+\frac{3\cdot 5\cdot 7\cdot 9}{c^5t^{11}}$<br>$R=\frac{1}{t^6}+\frac{2\cdot 3\cdot 5\cdot 7}{cct^8}+\frac{5\cdot 3\cdot 5\cdot 7\cdot 9}{c^4t^{10}}+\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{c^6t^{12}}$<br>$S=\frac{21}{ct^7}+\frac{4\cdot 5\cdot 7\cdot 9}{c^3t^9}+\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{c^5t^{11}}$ | $i=6,\quad t=13x^{\frac{1}{13}}$<br>$P=t^6+\frac{2\cdot 3\cdot 5\cdot 7}{cc}t^4+\frac{5\cdot 3\cdot 5\cdot 7\cdot 9}{c^4}tt+\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{c^6}$<br>$Q=\frac{21}{c}t^5+\frac{4\cdot 5\cdot 7\cdot 9}{c^3}t^3+\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{c^5}t$<br>$R=t^6+\frac{3\cdot 5\cdot 7}{cc}t^4+\frac{3\cdot 5\cdot 7\cdot 9}{c^4}tt$<br>$S=\frac{15}{c}t^5+\frac{4\cdot 3\cdot 5\cdot 7}{c^3}t^3+\frac{3\cdot 5\cdot 7\cdot 9}{c^5}t.$ |

## SCHOLIUM 2

**960.** While these formulas are considered more closely, a new relation emerges between the values of the letters $P$, $Q$, $R$, $S$ themselves which consists of this, that there shall be always $PR-QS=t^{2i}$, the truth of which indeed in the first place may be grasped from induction, then also indeed it can be demonstrated from the relations given above. For if the values

$$R=P-\frac{dQ}{cdt} \text{ and } S=Q-\frac{dP}{cdt}$$

should be substituted into the equations

$$dP+dR=\frac{2iRdt}{t} \text{ and } dQ+dS=\frac{2iSdt}{t},$$

these two equations arise :

$$2dP-\frac{ddQ}{cdt}=\frac{2iPdt}{t}-\frac{2idQ}{ct} \text{ and } 2dQ-\frac{ddP}{cdt}=\frac{2iQdt}{t}-\frac{2idP}{ct},$$

of which the former multiplied by $P$, and the latter by $-Q$ added together give

$$2PdP-2QdQ+\frac{QddP-PddQ}{cdt}=\frac{2idt}{t}(PP-QQ)+\frac{2i}{ct}(QdP-PdQ).$$

There is put

$$PP-QQ=M \text{ and } \frac{QdP-PdQ}{cdt}=N;$$

then there will be

$$dM+dN=\frac{2itdt}{t}(M+N) \text{ or } \frac{dM+dN}{M+N}=\frac{2idt}{t}$$

and hence on integration $M+N=Ct^{2i}$. But there is

$$M+N=P\left(P-\frac{dQ}{cdt}\right)-Q\left(Q-\frac{dP}{cdt}\right)=PR-QS,$$

moreover this must be taken for the single constant $C$.

## PROBLEM 121

**961.** *With i denoting some whole number, positive or negative, to find the complete integral of this equation of this equation*

$$du+uudx+bbx^{\frac{-4i}{2i+1}}dx=0.$$

## SOLUTION

On putting $u=\frac{dy}{ydx}$ this equation is transformed into this

$$ddy+bbx^{\frac{-4i}{2i+1}}ydx^2=0$$

with the element $dx$ assumed constant, the integral of which has been assigned above. Evidently on putting $t=(2i+1)x^{\frac{1}{2i+1}}$ we come upon (§ 953, 954)

$$y=C\left(t^i-\frac{i(ii-1)(i+2)}{2\cdot 4bb}t^{i-2}+\frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8b^4}t^{i-4}-\text{etc.}\right)\sin.(bt+\zeta)$$
$$+C\left(\frac{i(i+1)}{2b}t^{i-1}-\frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6b^3}t^{i-3}+\text{etc.}\right)\cos.(bt+\zeta),$$

in place of which we may write for brevity's sake

$$y = CP\sin.(bt+\zeta) + CQ\cos.(bt+\zeta).$$

Hence on account of

$$dt = x^{\frac{-2i}{2i+1}}dx \quad \text{or} \quad dx = x^{\frac{2i}{2i+1}}dt$$

there shall be

$$\frac{dy}{dx} = \frac{C(dP-bQdt)\sin.(bt+\zeta)+C(dQ+bPdt)\cos.(bt+\zeta)}{x^{\frac{2i}{2i+1}}dt}$$

from which, since there shall be $u = \frac{dy}{ydx}$, then

$$u = \frac{(dP-bQdt)\sin.(bt+\zeta)+(dQ+bPdt)\cos.(bt+\zeta)}{x^{\frac{2i}{2i+1}}dt\left(P\sin.(bt+\zeta)+Q\cos.(bt+\zeta)\right)}.$$

We may put

$$P + \frac{dQ}{bdt} = R \quad \text{and} \quad Q - \frac{dP}{bdt} = S,$$

so that there becomes

$$\frac{1}{b}x^{\frac{2i}{2i+1}}u = \frac{R\cos.(bt+\zeta)-S\sin.(bt+\zeta)}{P\sin.(bt+\zeta)+Q\cos.(bt+\zeta)};$$

there will be

$$P = t^i - \frac{i(ii-1)(i+2)}{2\cdot 4bb}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8b^4}t^{i-4} - \text{etc.},$$

$$Q = \frac{i(i+1)}{2b}t^{i-1} - \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6b^3}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10b^5}t^{i-5} - \text{etc.},$$

$$R = t^i - \frac{i(ii-1)(i-2)}{2\cdot 4bb}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8b^4}t^{i-4} - \text{etc.}$$

$$S = \frac{i(i-1)}{2b}t^{i-1} - \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6b^3}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2\cdot 4\cdot 6\cdot 8\cdot 10b^5}t^{i-5} - \text{etc.}$$

and on account of the angle $\zeta$ introduced this will be the complete integral.

## COROLLARIUM 1

**962.** Hence the values of the four letters *P*, *Q*, *R*, *S* thus depend on each other, so that initially there shall be

$$R = P + \frac{dQ}{bdt} \text{ and } S = Q - \frac{dP}{bdt},$$

then also truly it is apparent that

$$dP + dR = \tfrac{2iRdt}{t} \text{ and } dQ + dS = \tfrac{2iSdt}{t}.$$

## COROLLARY 2

**963.** Thereupon also there is deduced that $PR + QS = t^{2i}$, which equation is deduced from the preceding formulas [§ 960] on taking $cc = -bb$, where $Q$ and $S$ are changed into into $Q\sqrt{-1}$ and $\sqrt{-1}$.

## COROLLARY 3

**964.** This case also differs from that preceding, since here no particular integral can be given algebraically [§ 958]. For whatever value is attributed to the angle $\zeta$ of the constant, the integral always involves the sine and cosine of a certain angle.

## SCHOLIUM 1

**965.** Therefore since the complete integral of the equation

$$du + uudx + bbx^{\frac{-4i}{2i+1}}dx = 0$$

on putting $t = (2i+1)x^{\frac{1}{2i+1}}$ shall be

$$\tfrac{1}{b}x^{\frac{2i}{2i+1}}u = \tfrac{R\cos.(bt+\zeta)-S\sin.(bt+\zeta)}{P\sin.(bt+\zeta)+Q\cos.(bt+\zeta)},$$

thus for the individual values of the number *i* the quantity *t* will be obtained from the letters *P, Q, R, S*. Initially if $i = 0$, then there will be

$$P = 1,\ Q = 0,\ R = 1 \text{ and } S = 0, \text{ likewise } t = x,$$

thus, so that the integral will be

$$\tfrac{1}{b}u = \tfrac{\cos.(bx+\zeta)}{\sin.(bx+\zeta)}$$

The table shows the remaining cases :

| | |
|---|---|
| $i=-1,\quad t=-\frac{1}{x}$<br>$P=\frac{1}{t},\quad Q=0$<br>$R=\frac{1}{t},\quad S=\frac{1}{btt}$ | $i=1,\quad t=3x^{\frac{1}{3}}$<br>$P=t,\quad Q=\frac{1}{b}$<br>$R=t,\quad S=0$ |
| $i=-2,\quad t=-\frac{3}{x^{\frac{1}{3}}}$<br>$P=\frac{1}{tt},\quad Q=\frac{1}{bt^3}$<br>$R=\frac{1}{tt}-\frac{3}{bbt^4},\quad S=\frac{3}{bt^3}$ | $i=2,\quad t=5x^{\frac{1}{5}}$<br>$P=tt-\frac{3}{bb},\quad Q=\frac{3}{b}t$<br>$R=tt,\quad S=\frac{1}{b}t$ |
| $i=-3,\quad t=-\frac{5}{x^{\frac{1}{5}}}$<br>$P=\frac{1}{t^3}-\frac{1\cdot 3}{bbt^5},\quad Q=\frac{3}{bt^4}$<br>$R=\frac{1}{t^3}-\frac{3\cdot 5}{bbt^5},\quad S=\frac{6}{bt^4}-\frac{3\cdot 5}{b^3t^6}$ | $i=3,\quad t=7x^{\frac{1}{7}}$<br>$P=tt-\frac{3\cdot 5}{bb},\quad Q=\frac{6}{b}tt-\frac{3\cdot 5}{b^3}$<br>$R=t^3-\frac{1\cdot 3}{bb}t,\quad S=\frac{3}{b}tt$ |
| $i=-4,\quad t=-\frac{7}{x^{\frac{1}{7}}}$<br>$P=\frac{1}{t^4}-\frac{3\cdot 5}{bbt^6}$<br>$Q=\frac{6}{bt^5}-\frac{3\cdot 5}{b^3t^7}$<br>$R=\frac{1}{t^4}-\frac{3\cdot 3\cdot 5}{bbt^6}+\frac{3\cdot 5\cdot 7}{b^4t^8},$<br>$S=\frac{10}{bt^5}-\frac{3\cdot 5\cdot 7}{b^3t^7}$ | $i=4,\quad t=9x^{\frac{1}{9}}$<br>$P=t^4-\frac{3\cdot 3\cdot 5}{bb}tt+\frac{3\cdot 5\cdot 7}{b^4}$<br>$Q=\frac{10}{b}t^3-\frac{3\cdot 5\cdot 7}{b^3}t$<br>$R=t^4-\frac{3\cdot 5}{bb}tt$<br>$S=\frac{6}{b}t^3-\frac{3\cdot 5}{b^3}t$ |
| $i=-5,\quad t=\frac{-9}{x^{\frac{1}{9}}}$<br>$P=\frac{1}{t^5}-\frac{3\cdot 3\cdot 5}{bbt^7}+\frac{3\cdot 5\cdot 7}{b^4t^9}$<br>$Q=\frac{10}{bt^6}-\frac{3\cdot 5\cdot 7}{b^3t^8}$<br>$R=\frac{1}{t^5}-\frac{3\cdot 5\cdot 7}{bbt^7}+\frac{3\cdot 5\cdot 7\cdot 9}{b^4t^9}$<br>$S=\frac{15}{bt^6}-\frac{4\cdot 3\cdot 5\cdot 7}{b^3t^8}+\frac{3\cdot 5\cdot 7\cdot 9}{b^5t^{10}}$ | $i=5,\quad t=11x^{\frac{1}{11}}$<br>$P=t^5-\frac{3\cdot 5\cdot 7}{bb}t^3+\frac{3\cdot 5\cdot 7\cdot 9}{b^4}t$<br>$Q=\frac{15}{c}t^4-\frac{4\cdot 3\cdot 5\cdot 7}{b^3}tt+\frac{3\cdot 5\cdot 7\cdot 9}{b^5}$<br>$R=t^5-\frac{3\cdot 3\cdot 5}{bb}t^3+\frac{3\cdot 5\cdot 7}{b^4}t$<br>$S=\frac{10}{b}t^4-\frac{3\cdot 5\cdot 7}{b^3}tt$ |
| $i=-6,\quad t=\frac{-11}{x^{\frac{1}{11}}}$<br>$P=\frac{1}{t^6}-\frac{3\cdot 5\cdot 7}{bbt^8}+\frac{3\cdot 5\cdot 7\cdot 9}{b^5t^{10}}$<br>$Q=\frac{15}{bt^7}-\frac{4\cdot 3\cdot 5\cdot 7}{b^3t^9}+\frac{3\cdot 5\cdot 7\cdot 9}{b^5t^{11}}$<br>$R=\frac{1}{t^6}-\frac{2\cdot 3\cdot 5\cdot 7}{bbt^8}+\frac{5\cdot 3\cdot 5\cdot 7\cdot 9}{b^4t^{10}}-\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{b^6t^{12}}$<br>$S=\frac{21}{bt^7}-\frac{4\cdot 5\cdot 7\cdot 9}{b^3t^9}+\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{b^5t^{11}}$ | $i=6,\quad t=13x^{\frac{1}{13}}$<br>$P=t^6-\frac{2\cdot 3\cdot 5\cdot 7}{bb}t^4+\frac{5\cdot 3\cdot 5\cdot 7\cdot 9}{b^4}tt-\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{b^6}$<br>$Q=\frac{21}{b}t^5-\frac{4\cdot 5\cdot 7\cdot 9}{b^3}t^3+\frac{3\cdot 5\cdot 7\cdot 9\cdot 11}{b^5}t$<br>$R=t^6-\frac{3\cdot 5\cdot 7}{bb}t^4+\frac{3\cdot 5\cdot 7\cdot 9}{b^4}tt$<br>$S=\frac{15}{b}t^5-\frac{4\cdot 3\cdot 5\cdot 7}{b^3}t^3+\frac{3\cdot 5\cdot 7\cdot 9}{b^5}t.$ |

## SCHOLIUM 2

**966.** The form of the integral found shows how the proposed equation

$$du + uudx + Ax^{\frac{-4i}{2i+1}}dx = 0$$

is to be transformed into a simpler kind. For initially there is put

$$x^{\frac{2i}{2i+1}}u = v \quad \text{or} \quad u = x^{\frac{-2i}{2i+1}}v$$

and there emerges

$$x^{\frac{-2i}{2i+1}}dv - \frac{2i}{2i+1}x^{\frac{-4i-1}{2i+1}}vdx + x^{\frac{-4i}{2i+1}}vvdx + Ax^{\frac{-4i}{2i+1}}dx = 0$$

or

$$dv - \frac{2i}{2i+1}\cdot\frac{vdx}{x} + x^{\frac{-2i}{2i+1}}vvdx + Ax^{\frac{-2i}{2i+1}}dx = 0$$

Again there is put $t = (2i+1)x^{\frac{1}{2i+1}}$; then there will be

$$dt = x^{\frac{-2i}{2i+1}}dx \quad \text{and} \quad \frac{dt}{t} = \frac{1}{2i+1}\cdot\frac{dx}{x},$$

from which there becomes

$$dv - \frac{2ivdt}{t} + vvdt + Adt = 0 .$$

In addition let there be $v = \frac{i}{t} + z$, so that there becomes

$$-\frac{idt}{tt} + dz - \frac{2iidt}{tt} - \frac{2izdt}{t} + \frac{iidt}{tt} + \frac{2izdt}{t} + zzdt + Adt = 0$$

or

$$dz + zzdt - \frac{i(i+1)dt}{tt} + Adt = 0 ,$$

which is integrable as often as *i* is a whole number.

In a like manner this equation

$$du + uudx + Ax^{n}dx = 0$$

thus can be transformed more generally : On putting $u = x^{\lambda}v$ and $v = z - \frac{1}{2}\lambda x^{-\lambda-1}$ there may be obtained

$$dz + x^{\lambda}zzdx + \tfrac{1}{4}\lambda(\lambda+2)x^{-\lambda-2}dx + Ax^{n-\lambda}dx = 0,$$

which again on putting $x^{\lambda}dx = dt$ or $x^{\lambda+1} = (\lambda+1)t$ will become

$$dz + zzdt + \frac{\lambda(\lambda+2)dt}{4(\lambda+1)tt} + A(\lambda+1)^{\frac{n-2\lambda}{\lambda+1}} t^{\frac{n-2\lambda}{\lambda+1}} dt = 0$$

which equation is integrable, whenever $n = \frac{-4i}{2i+1}$, from which the number $\lambda$ being taken as it pleases, these are able to assume innumerable forms. If there is taken $\lambda = -1$, there becomes $t = lx$ and

$$dz + zzdt - \tfrac{1}{4}dt + Ae^{(n+2)t}dt = 0 .$$

## CAPUT VII

# DE RESOLUTIONE AEQUATIONIS

$$ddy + ax^n ydx^2 = 0$$

# PER SERIES INFINITAS

### PROBLEMA 117

**929.** *Sumto elemento dx constante aequationem differentio-differentialem* $ddy + ax^n ydx^2 = 0$ *per seriem infinitam integrare.*

### SOLUTIO

Quaerimus hic seriem secundum potestates ipsius $x$ progredientem, quae valorem ipsius $y$ exprimat; et quia in altero aequationis nostrae termino quantitas $x$ cum suo differentiali $dx$ nullam, in altero vero $n+2$ dimensiones occupat, evidens est exponentes potestatum ipsius $x$ differentia $n+2$ ascendere vel descendere debere.

I. Ascendant primo exponentes et fingatur series

$$y = Ax^{\lambda} + Bx^{\lambda+n+2} + Cx^{\lambda+2n+4} + \text{etc}.$$

eritque

$$\begin{aligned}\frac{ddy}{dx^2} &= \lambda(\lambda-1)Ax^{\lambda-2} + (\lambda+n+2)(\lambda+n+1)Bx^{\lambda+n} + \text{etc.},\\ ax^n y &= \qquad\qquad\qquad\qquad\qquad\qquad\quad aAx^{\lambda+n} + \text{etc.},\end{aligned}$$

unde patet primum terminum solitarium evanescere debere, ut sit $\lambda(\lambda-1)=0$.
Quare capi oportet vel $\lambda = 0$ vel $\lambda = 1$ sicque duplex series obtinetur

$$\begin{aligned}y = A &+ Bx^{n+2} + Cx^{2n+4} + Dx^{3n+6} + Ex^{4n+8} + \text{ etc.}\\ &+ \mathfrak{A}x + \mathfrak{B}x^{n+3} + \mathfrak{C}x^{2n+5} + \mathfrak{D}x^{3n+7} + \mathfrak{E}X^{4n+9} + \text{ etc.}\end{aligned}$$

Substitutione ergo facta fieri oportet

$$0=(n+2)(n+1)Bx^{n}+(2n+4)(2n+3)Cx^{n+2}+(3n+6)(3n+5)Dx^{8n+4}+\text{etc.},$$
$$\qquad +aA \qquad\qquad +aB \qquad\qquad +aC$$
$$0=(n+3)(n+2)\mathfrak{B}x^{n+1}+(2n+5)(2n+4)\mathfrak{C}x^{2n+3}+(3n+7)(3n+6)\mathfrak{D}x^{3n+5}+\text{etc.},$$
$$\qquad +a\mathfrak{A} \qquad\qquad +a\mathfrak{B} \qquad\qquad +a\mathfrak{C}$$

unde litteris $A$ et $\mathfrak{A}$ arbitrio nostro relictis reliquae per eas ita determinantur

$$B=\frac{-aA}{(n+1)(n+2)},\quad C=\frac{-aB}{2(2n+3)(n+2)},\quad D=\frac{-aC}{3(3n+5)(n+2)}+\text{etc.},$$

$$\mathfrak{B}=\frac{-a\mathfrak{A}}{(n+3)(n+2)},\quad \mathfrak{C}=\frac{-a\mathfrak{B}}{2(2n+5)(n+2)},\quad \mathfrak{D}=\frac{-a\mathfrak{C}}{3(3n+7)(n+2)}x^{3n+5}+\text{etc.}$$

sicque habebitur integrale completum ita expressum

$$y=A-\frac{aAx^{n+2}}{1(n+1)\cdot(n+2)}+\frac{a^{2}Ax^{2n+4}}{1\cdot 2(n+1)(2n+3)\cdot(n+2)^{2}}-\frac{a^{3}Ax^{3n+6}}{1\cdot 2\cdot 3(n+1)(2n+3)(3n+5)\cdot(n+2)^{3}}+\text{etc.}$$

$$+\mathfrak{A}x-\frac{a\mathfrak{A}x^{n+3}}{1(n+3)\cdot(n+2)}+\frac{a^{2}\mathfrak{A}x^{2n+5}}{1\cdot 2(n+3)(2n+5)\cdot(n+2)^{2}}-\frac{a^{3}\mathfrak{A}x^{3n+7}}{1\cdot 2\cdot 3(n+3)(2n+5)(3n+7)\cdot(n+2)^{3}}+\text{etc.}$$

II. Descendant iam exponentes et ficta serie

$$y=Ax^{\lambda}+Bx^{\lambda-n-2}+Cx^{\lambda-2n-4}+\text{etc}.$$

habebitur

$$\frac{ddy}{dx^{2}}=\lambda(\lambda-1)Ax^{\lambda-2}+(\lambda-n-2)(\lambda-n-3)Bx^{\lambda-n-4}+\text{etc.},$$
$$ax^{n}y=aAx^{\lambda+n}+aBx^{\lambda-2}+\text{etc.},$$

ubi cum terminus $x^{\lambda+n}$ sui similem non habeat, tolli nequit, ita ut hinc nulla aequationis resolutio obtineatur.

## COROLLARIUM 1

**930.** Geminata series pro $y$ inventa, quoniam litterae $A$ et $\mathfrak{A}$ arbitrio nostro relinquuntur, integrale completum aequationis differentio-differentialis $ddy+ax^{n}ydx^{2}=0$ exhibet; tribuendo autem litteris $A$ et $\mathfrak{A}$ datos valores integralia particularia nascentur.

### COROLLARIUM 2

**931.** Si ponamus $n+2=m$ seu $n=m-2$, huius aequationis

$$ddy+ax^{m-2}ydx^2=0$$

integrale completum ita commodius exprimetur

$$y=A-\frac{aAx^m}{1(m-1)\cdot m}+\frac{a^2Ax^{2m}}{1\cdot 2(m-1)(2m-1)\cdot m^2}-\frac{a^3Ax^{3m}}{1\cdot 2\cdot 3(m-1)(2m-1)(3m-1)\cdot m^3}+\text{etc.}$$

$$+\mathfrak{A}x-\frac{a\mathfrak{A}x^{m+1}}{1(m+1)\cdot m}+\frac{a^2\mathfrak{A}x^{2m+1}}{1\cdot 2(m+1)(2m+1)\cdot m^2}-\frac{a^3\mathfrak{A}x^{3m+1}}{1\cdot 2\cdot 3(m+1)(2m+1)(3m+1)\cdot m^3}+\text{etc.}$$

### COROLLARIUM 3

**932.** Si exponens *m* fuerit positivus et unitate maior, hae series eo magis convergunt, quo minor valor quantitati *x* tribuatur; aliis vero casibus in praxi hae series adhiberi nequeunt, nisi forte eae ipsae in alias convergentes transformari possint.

### SCHOLION 1

**933.** Dantur tamen casus, quibus hae series omni plane usu destituuntur, quod evenit, si quispiam factorum denominatores constituentium evanescat sicque omnes termini sequentes in infinitum excrescant, quibus casibus series in alias formas transmutari convenit. Hic primo occurrit casus $m=0$ seu $n=-2$, quo utriusque seriei omnes termini praeter primos fiunt infiniti; hoc vero casu aequatio, quae est

$$ddy+\frac{aydx^2}{x^2}=0,$$

cum sit homogenea, singularem integrationem admittit; inveniri enim potest potestas ipsius *x*, quae pro *y* substituta aequationi satisfacit. Ponatur scilicet $y=x^\lambda$ prodibitque

$$\lambda(\lambda-1)x^{\lambda-2}+ax^{\lambda-2}=0 \text{ seu } \lambda\lambda-\lambda+1=0,$$

unde colligitur $\lambda=\frac{1}{2}\pm\sqrt{\left(\frac{1}{4}-a\right)}$, ob quem duplicem valorem est integrale completum

$$y=Ax^{\frac{1}{2}+\sqrt{\left(\frac{1}{4}-a\right)}}+Bx^{\frac{1}{2}-\sqrt{\left(\frac{1}{4}-a\right)}},$$

quae casu $a>\frac{1}{4}$ abit in hanc formam

$$y = Ax^{\frac{1}{2}}\sin.\left(\left(a-\tfrac{1}{4}\right)^{\frac{1}{2}} lx+\alpha\right),$$

unde patet casu $a=\frac{1}{4}$ fore

$$y=\left(A+Blx\right)\sqrt{x}\ .$$

**SCHOLION 2**

**934.** Reliqui casus ad incommodum ducentes sunt, si vel $m=\frac{1}{i}$ vel $m=-\frac{1}{i}$ denotante $i$ numerum quemcunque integrum. Casu $m=\frac{1}{i}$ prior tantum series fit incongrua, casu vero $m=-\frac{1}{i}$ posterior tantum. Quare illo casu ponendo $A=0$, hoc. vero $\mathfrak{A}=0$, series saltem una idonea habetur integrale particulare exhibens. Verum cognito integrali particulari, quod sit $y=P$, inde aequationis $ddy+ax^{m-2}ydx^2=0$ integrale completum eruitur ponendo $y=Pz$, unde fit

$$Pddz+2dPdz+zddP+ax^{m-2}Pzdx^2=0\,;$$

at per hypothesin est $ddP+ax^{m-2}Pdx^2=0$, ergo prodit

$$Pddz+2dPdz=\ 0 \ \text{ seu }\ PPdz=Cdx\ \text{ et }\ z=C\int\tfrac{dx}{PP}\ .$$

Cum autem $P$ sit series infinita, hinc valorem ipsius $z$, cognoscere haud licet.

At casibus illis memoratis pars integralis logarithmum ipsius $x$ involvit, quod vel inde intelligitur, quod $\frac{x^{\theta}}{\theta}$ aequivaleat ipsi $lx$. Quare in aequatione $ddy+ax^{m-2}ydx^2=0$ ponendo $y=\ p+qlx$ ob $dy=dp+\frac{q}{x}+dqlx$ erit

$$ddp+\tfrac{2dxdq}{x}-\tfrac{qdx^2}{xx}+ddqlx+apx^{m-2}dx^2+aqx^{m-2}dx^2lx=0\,,$$

in qua partes $lx$ involventes seorsim destruantur necesse est, ita ut hae binae habeantur aequationes

$$ddq+aqx^{m-2}dx^2=\ 0\ \text{ et }\ ddp+\tfrac{2dxdq}{x}-\tfrac{qdx^2}{xx}+apx^{m-2}dx^2=0\,,$$

ubi pro $q$ ea binarum superiorum serierum accipi debet, quae casu oblato incommodo caret, eaque constituta ex posteriori aequatione facile quantitas $p$ per seriem exprimetur. Huiusmodi casus in sequentibus exemplis evolvamus; tantum notemus illam operationem perinde se habere, etiamsi loco $lx$ sumatur $lx+a$, ita ut inventis $p$ et $q$ futurum sit $y=aq+\ p+qlx$ seu $y=\ p+ql\beta x$.

### EXEMPLUM 1

**935.** *Posito* $m=1$ *hanc aequationem* $ddy+\frac{aydx^2}{x}=0$ *per series resolvere.*

Posito $y=p+qlx$ capi oportet

$$q=\mathfrak{A}x-\frac{a\mathfrak{A}x^2}{1\cdot 2}+\frac{a^2\mathfrak{A}x^3}{1\cdot 2\cdot 2\cdot 3}-\frac{a^3\mathfrak{A}x^4}{1\cdot 2\cdot 2\cdot 3\cdot 3\cdot 4}+\text{etc.}$$

pro qua ponamus brevitatis gratia

$$q=\mathfrak{A}x+\mathfrak{B}x^2+\mathfrak{C}x^3+\mathfrak{D}x^4+\text{etc.}$$

Tum vero quaeratur $p$ ex hac aequatione

$$\frac{ddp}{dx^2}+\frac{2dq}{xdx}-\frac{q}{xx}+\frac{ap}{x}=0$$

fingamus ergo

$$p=A+Bx+Cx^2+Dx^3+Ex^4+\text{etc}.$$

eritque facta substitutione

$$\left.\begin{array}{l} \qquad 2C+6Dx+12Exx+20Fx^3+30Gx^4+\text{ etc.}\\ +\frac{2\mathfrak{A}}{x}+4\mathfrak{B}+6\mathfrak{C}\;+\;8\mathfrak{D}\;+\;10\mathfrak{E}\;+\;12\mathfrak{F}\\ -\mathfrak{A}\;-\mathfrak{B}-\;\mathfrak{C}\;-\;\mathfrak{D}\;-\;\mathfrak{E}\;-\;\mathfrak{F}\\ +aA\;+aB+aC\;+\;aD\;+\;aE\;+\;aF \end{array}\right\}=0.$$

Cum iam dentur coefficientes $\mathfrak{A},\mathfrak{B},\mathfrak{C},\mathfrak{D},\mathfrak{E}$ etc., erit $A=-\frac{\mathfrak{A}}{a}$; quantitas $B$ non determinatur; tum vero

$$C=\frac{-3\mathfrak{B}}{2}-\frac{aB}{1\cdot 2}=\frac{3a\mathfrak{A}}{1^3\cdot 2^2}-\frac{aB}{1\cdot 2},\quad D=\frac{-5\mathfrak{C}}{6}-\frac{aC}{2\cdot 3}=\frac{-5a^2\mathfrak{A}}{1^2\cdot 2^3\cdot 3^2}-\frac{aC}{2\cdot 3},$$

$$E=\frac{-7\mathfrak{D}}{12}-\frac{aD}{3\cdot 4}=\frac{7a^3\mathfrak{A}}{1^2\cdot 2^3\cdot 3^3\cdot 4^2}-\frac{aD}{3\cdot 4},\quad F=\frac{-9\mathfrak{E}}{20}-\frac{aE}{4\cdot 5}=\frac{-9a^4\mathfrak{A}}{1^2\cdot 2^2\cdot 3^2\cdot 4^3\cdot 5^2}-\frac{aE}{4\cdot 5}$$

etc.,

ubi pro $B$ scribere licet 0, quandoquidem in integrali $y=p+qlx$ addimus partem $\alpha q$, quae ex littera $B$ oritur, ita ut sit

$$p=A+Cx^2+Dx^3+Ex^4+Fx^5+\text{etc.}$$

Hinc erit

$$C = \frac{3a\mathfrak{A}}{1^3 \cdot 2^2}, \quad D = \frac{-14a^2\mathfrak{A}}{1^2 \cdot 2^3 \cdot 3^2}, \quad E = \frac{70a^3\mathfrak{A}}{1^2 \cdot 2^3 \cdot 3^3 \cdot 4^2}, \quad F = \frac{-404a^4\mathfrak{A}}{1^2 \cdot 2^2 \cdot 3^2 \cdot 4^3 \cdot 5^2},$$

ubi notetur esse $14 = 3 \cdot 3 + 5 \cdot 1$, $70 = 4 \cdot 14 + 7 \cdot 1 \cdot 2$, $404 = 5 \cdot 70 + 9 \cdot 1 \cdot 2 \cdot 3$ et pro sequente $2688 = 6 \cdot 404 + 11 \cdot 1 \cdot 2 \cdot 3 \cdot 4$. Estque

$$y = p + \alpha q + qlx.$$

## EXEMPLUM 2

**936.** *Posito* $m = -1$ *hanc aequationem* $ddy + \frac{aydx^2}{x^3} = 0$ *per series resolvere.*

Posito $y = p + \alpha q + qlx$ capi oportet

$$q = A - \frac{aA}{1 \cdot 2x} + \frac{a^2A}{1 \cdot 2^2 \cdot 3x^2} - \frac{a^3A}{1 \cdot 2^2 \cdot 3^3 \cdot 4x^3} + \text{etc.}$$

pro qua ponatur brevitatis gratia

$$q = A + \frac{B}{x} + \frac{C}{x^2} + \frac{D}{x^3} + \frac{E}{x^4} + \text{etc.};$$

tum vero quantitas $p$ ex hac aequatione definiri debet

$$ddp + \frac{2dqdx}{x} - \frac{qdx^2}{xx} + \frac{apdx^2}{x^3} = 0.$$

Fingamus ergo

$$p = \mathfrak{A}x + \mathfrak{B} + \frac{\mathfrak{C}}{x} + \frac{\mathfrak{D}}{xx} + \frac{\mathfrak{E}}{x^3} + \frac{\mathfrak{F}}{x^4} + \text{etc.}$$

unde facta substitutione prodit

$$\left.\begin{array}{llllll} \frac{a\mathfrak{A}}{xx} + \frac{a\mathfrak{B}}{x^3} + & \frac{a\mathfrak{C}}{x^4} & + \frac{a\mathfrak{D}}{x^5} + & \frac{a\mathfrak{E}}{x^6} & + \frac{a\mathfrak{F}}{x^7} + \text{etc.} \\ -A - B & - C & - D - & E & - F \\ \quad -2B & -4C & -6D - & 8E & -10F \\ \quad +2\mathfrak{C} & +6\mathfrak{D} & +12\mathfrak{E} & +20\mathfrak{F} & +30\mathfrak{G} \end{array}\right\} = 0$$

et coefficientes $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}, \mathfrak{E}$ etc. ita determinantur, ut sit $\mathfrak{A} = \frac{A}{a}$;
secundus $\mathfrak{B}$ non definitur; tum vero est

$$\mathfrak{C} = \frac{-3B}{1\cdot 2} - \frac{a\mathfrak{B}}{1\cdot 2} = \frac{-3aA}{1^3\cdot 2^2} - \frac{a\mathfrak{B}}{1\cdot 2}, \quad \mathfrak{D} = \frac{5C}{2\cdot 3} - \frac{a\mathfrak{C}}{2\cdot 3} = \frac{5a^2A}{2^3\cdot 3^2} - \frac{a\mathfrak{C}}{2\cdot 3},$$

$$\mathfrak{E} = \frac{7D}{3\cdot 4} - \frac{a\mathfrak{D}}{3\cdot 4} = \frac{-7a^3A}{2^2\cdot 3^3\cdot 4^2} - \frac{a\mathfrak{D}}{3\cdot 4}, \quad \mathfrak{F} = \frac{9E}{4\cdot 5} - \frac{a\mathfrak{E}}{4\cdot 5} = \frac{9a^4A}{2^2\cdot 3^2\cdot 4^3\cdot 5^2} - \frac{a\mathfrak{E}}{4\cdot 5}$$

etc.

Si sumatur, id quod sine detrimento generalitatis fieri licet, $\mathfrak{B} = 0$, ita ut sit

$$p = \mathfrak{A}x + \frac{\mathfrak{C}}{x} + \frac{\mathfrak{D}}{xx} + \frac{\mathfrak{E}}{x^3} + \frac{\mathfrak{F}}{x^4} + \text{etc.},$$

erit

$$\mathfrak{C} = \frac{-3aA}{1^3\cdot 2^2}, \quad \mathfrak{D} = \frac{14a^2A}{1^3\cdot 2^3\cdot 3^2}, \quad \mathfrak{E} = \frac{-70a^3A}{1^3\cdot 2^3\cdot 3^3\cdot 4^2}, \quad \mathfrak{F} = \frac{404a^4A}{1^3\cdot 2^3\cdot 3^3\cdot 4^3\cdot 5^2} \text{ etc.}$$

qui valores similes sunt praecedentibus.

## EXEMPLUM 3

**937.** *Posito* $m = \frac{1}{2}$ *hanc aequationem* $ddy + \frac{aydx^2}{x\sqrt{x}} = 0$ *per series resolvere.*

Posito $y = p + \alpha q + qlx$ capi oportet

$$q = \mathfrak{A}x - \frac{4a\mathfrak{A}}{1\cdot 3}x^{\frac{3}{2}} + \frac{16a^2\mathfrak{A}}{1\cdot 2\cdot 3\cdot 4}x^2 - \frac{64a^3\mathfrak{A}}{1\cdot 2\cdot 3\cdot 3\cdot 4\cdot 5}x^{\frac{5}{2}} + \frac{256a^4\mathfrak{A}}{1\cdot 2\cdot 3\cdot 4\cdot 3\cdot 4\cdot 5\cdot 6}x^3 - \text{etc.}$$

pro qua brevitatis gratia scribatur

$$q = \mathfrak{A}x + \mathfrak{B}x^{\frac{3}{2}} + \mathfrak{C}x^2 + \mathfrak{D}x^{\frac{5}{2}} + \mathfrak{E}x^3 + \mathfrak{F}x^{\frac{7}{2}} + \mathfrak{G}x^4 + \text{etc.};$$

tum vero quantitas $p$ ex hac aequatione definiri debet

$$ddp + \frac{2dxdq}{x} - \frac{qdx^2}{xx} + \frac{apdx^2}{x\sqrt{x}} = 0.$$

Fingamus ergo

$$p = \varDelta + Ax^{\frac{1}{2}} + Bx + Cx^{\frac{3}{2}} + Dx^2 + Ex^{\frac{5}{2}} + Fx^3 + \text{etc.}$$

prodibitque facta substitutione

$$\left.\begin{array}{l} \frac{a\Delta}{x\sqrt{x}}+\frac{aA}{x}+\frac{aB}{x\sqrt{x}}+aC+aD\sqrt{x}+aEx+aFx\sqrt{x} \text{ etc.} \\ -\mathfrak{A}-\mathfrak{B}-\mathfrak{C}-\mathfrak{D}-\mathfrak{E}-\mathfrak{F} \\ +2\mathfrak{A}+3\mathfrak{B}+4\mathfrak{C}+5\mathfrak{D}+6\mathfrak{E}+7\mathfrak{F} \\ \frac{-A}{4}+0+\frac{3}{4}C+2D+\frac{15}{4}E+6F+\frac{35}{4}G \end{array}\right\}=0.$$

Hinc colligitur fore $A=\frac{-\mathfrak{A}}{a}, \Delta=\frac{-\mathfrak{A}}{4aa}$ ; at $B$ non determinatur; porro

$$C=\frac{-4aB}{1\cdot 3}-\frac{8\mathfrak{B}}{1\cdot 3}=\frac{-4aB}{1\cdot 3}+\frac{8\cdot 4a\mathfrak{A}}{1^2\cdot 3^2},$$
$$D=\frac{-4aC}{2\cdot 4}-\frac{12\mathfrak{C}}{2\cdot 4}=\frac{-4aC}{2\cdot 4}-\frac{12\cdot 16a^2\mathfrak{A}}{1\cdot 3\cdot 2^2\cdot 4^2},$$
$$E=\frac{-4aD}{3\cdot 5}-\frac{16\mathfrak{D}}{3\cdot 5}=\frac{-4aD}{3\cdot 5}+\frac{16\cdot 64a^3\mathfrak{A}}{1\cdot 3\cdot 2\cdot 4\cdot 3^2\cdot 5^2},$$
$$F=\frac{-4aE}{4\cdot 6}-\frac{12\mathfrak{E}}{4\cdot 6}=\frac{-4aE}{4\cdot 6}-\frac{20\cdot 256a^4\mathfrak{A}}{1\cdot 3\cdot 2\cdot 4\cdot 3\cdot 5\cdot 4^2\cdot 6^2},$$
etc.

Quodsi iam ponatur $B=0$, ut sit

$$p=\frac{-\mathfrak{A}}{4aa}-\frac{\mathfrak{A}}{a}\sqrt{x}+*+Cx\sqrt{x}+Dx^2+Ex^2\sqrt{x}+Fx^3+\text{etc.}$$

erit

$$C=\frac{8\cdot 4a\mathfrak{A}}{1^2\cdot 3^2},\quad D=\frac{-100\cdot 16a^2\mathfrak{A}}{1^2\cdot 3^2\cdot 2^2.4^2},\quad E=\frac{1884\cdot 64a^3\mathfrak{A}}{1^2\cdot 3^2\cdot 2^2\cdot 4^2\cdot 3^2\cdot 5^2}$$
$$F=\frac{-52416\cdot 256a^4\mathfrak{A}}{1^2\cdot 3^2\cdot 2^2\cdot 4^2\cdot 3^2\cdot 5^2 4^2\cdot 6^2} \text{ etc.},$$

ubi notetur esse

$$100=2\cdot 4\cdot 8+1\cdot 3\cdot 12,\ 1884=3\cdot 5\cdot 100+1\cdot 3\cdot 2\cdot 4\cdot 16,$$
$$52416=4\cdot 6\cdot 1884+1\cdot 3\cdot 2\cdot 4\cdot 3\cdot 5\cdot 20.$$

## EXEMPLUM 4

**938.** *Posito* $m=-\frac{1}{2}$ *hanc aequationem* $ddy+\frac{aydx^2}{xx\sqrt{x}}=0$ *per series resolvere.*

Posito $y=p+\alpha q+qlx$ capi oportet

$$q=A-\frac{4aA}{1\cdot 3}x^{-\frac{1}{2}}+\frac{16a^2A}{1\cdot 3\cdot 2\cdot 4}x^{-1}-\frac{64a^3A}{1\cdot 3\cdot 2\cdot 4\cdot 3\cdot 5}x^{-\frac{3}{2}}+\text{ etc.},$$

pro qua brevitatis causa scribamus

$$q=A+Bx^{-\frac{1}{2}}+Cx^{-1}+Dx^{-\frac{3}{2}}+Ex^{-2}+Fx^{-\frac{5}{2}}+\text{ etc.},$$

et littera $p$ ex hac aequatione definiri debet

$$ddp + \frac{2dxdq}{x} - \frac{qdx^2}{xx} + \frac{apdx^2}{xx\sqrt{x}} = 0.$$

Fingamus

$$p = \Delta x + \mathfrak{A}\sqrt{x} + \mathfrak{B} + \mathfrak{C}x^{-\frac{1}{2}} + \mathfrak{D}x^{-1} + \mathfrak{E}x^{-\frac{3}{2}} + \mathfrak{F}x^{-2} + \mathfrak{G}x^{-\frac{5}{2}} + \text{etc.}$$

et facta substitutione prodit

$$\left.\begin{array}{l} \frac{a\Delta}{x\sqrt{x}} + \frac{a\mathfrak{A}}{xx} + \frac{a\mathfrak{B}}{xx\sqrt{x}} + \frac{a\mathfrak{C}}{x^3} + \frac{a\mathfrak{D}}{x^3\sqrt{x}} + \frac{a\mathfrak{E}}{x^4} + \frac{a\mathfrak{F}}{x^4\sqrt{x}} + \text{etc.} \\ \qquad -A \quad - \quad B \quad -C \quad - \quad D \quad -E \quad - \quad F \\ \qquad\qquad - \quad B \quad -2C - \quad 3D \quad -4E - 5F \\ -\frac{1}{4}\mathfrak{A} \quad + \frac{3}{4}\mathfrak{C} \quad +2\mathfrak{D} + \frac{15}{4}\mathfrak{F} \quad +6\mathfrak{F} + \frac{35}{4}\mathfrak{G} \end{array}\right\} = 0,$$

unde sequentes determinationes colliguntur $\mathfrak{A} = \frac{A}{a}$ et $\Delta = \frac{\mathfrak{A}}{4A} = \frac{A}{4aa}$; at $\mathfrak{B}$ non determinatur; porro

$$\mathfrak{C} = \frac{-4a\mathfrak{B}}{1\cdot 3} + \frac{8\mathfrak{B}}{1\cdot 3} = \frac{-4a\mathfrak{B}}{1\cdot 3} - \frac{8\cdot 4aA}{1^2\cdot 3^2},$$

$$\mathfrak{D} = \frac{-4a\mathfrak{C}}{2\cdot 4} + \frac{12C}{2\cdot 4} = \frac{-4a\mathfrak{C}}{2\cdot 4} + \frac{12\cdot 16a^2A}{1\cdot 3\cdot 2^2\cdot 4^2},$$

$$\mathfrak{E} = \frac{-4a\mathfrak{D}}{3\cdot 5} - \frac{16D}{3\cdot 5} = \frac{-4a\mathfrak{D}}{3\cdot 5} - \frac{16\cdot 64a^3A}{1\cdot 3\cdot 2\cdot 4\cdot 3^2\cdot 5^2},$$

$$\mathfrak{F} = \frac{-4a\mathfrak{E}}{4\cdot 6} + \frac{12E}{4\cdot 6} = \frac{-4a\mathfrak{E}}{4\cdot 6} - \frac{20\cdot 256a^4A}{1\cdot 3\cdot 2\cdot 4\cdot 3\cdot 5\cdot 4^2\cdot 6^2},$$

etc.

Quodsi iam sumatur $\mathfrak{B} = 0$, erit

$$\mathfrak{C} = \frac{-8\cdot 4aA}{1^2\cdot 3^2}, \quad \mathfrak{D} = \frac{100\cdot 16a^2A}{1^2\cdot 3^2\cdot 2^2\cdot 4^2}, \quad \mathfrak{E} = \frac{-1884\cdot 64a^3A}{1^2\cdot 3^2\cdot 2^2\cdot 4^2\cdot 3^2\cdot 5^2} \quad \text{etc.},$$

qui numeri ut ante progrediuntur.

## SCHOLION

**939.** Ex his exemplis perspicitur, quomodo series aequationem

$$ddy + a^{m-2}ydx^2 = 0$$

resolventes in reliquis casibus, quibus $m = \pm\frac{1}{i}$, inveniri oporteat; ubi observetur, si sit $m = +\frac{1}{i}$ pro $q$ hanc seriem accipi debere

$$q = \mathfrak{A}x + \mathfrak{B}x^{1+\frac{1}{i}} + \mathfrak{C}x^{1+\frac{2}{i}} + \mathfrak{D}x^{1+\frac{3}{i}} + \text{etc.},$$

tum vero formam ipsius $p$ tali serie exprimi

$$p = Ax + Bx^{\frac{1}{i}} + Cx^{\frac{2}{i}} + Dx^{\frac{3}{i}} + \text{etc.},$$

cuius coefficientes ex superioribus ut ante definiantur. Sin autem sit $m = -\frac{1}{i}$, pro $q$ sumatur series

$$q = A + Bx^{-\frac{1}{i}} + Cx^{-\frac{2}{i}} + Dx^{-\frac{3}{i}} + \text{etc.},$$

at pro $p$ huiusmodi formam accipi conveniet

$$p = \mathfrak{A}x + \mathfrak{B}x^{1-\frac{1}{i}} + \mathfrak{C}x^{1-\frac{2}{i}} + \mathfrak{D}x^{1-\frac{3}{i}} + \text{etc.},$$

unde pariter singulos coefficientes uno excepto determinare licebit. Atque hoc artificium in genere est tenendum, quoties in resolutione aequationis generalis ad series pervenitur, cuius coefficientes certis casibus in infinitum excrescunt, quod plerumque indicio est logarithmos esse introducendos.

Verum etiam eadem aequatio $ddy + a^n ydx^2 = 0$ aliis modis per series resolvi potest, dum ea ante resolutionem in aliam formam transmutatur; ubi cum evenire possit, ut series certis casibus abrumpatur, quibus adeo integrale revera assignari potest, talem transformationem maxime notabilem hic explicemus.

## PROBLEMA 118

**940.** *Aequationem differentio-differentialem* $ddy + ax^n ydx^2 = 0$ *in aliam formam transfundere, cuius resolutio per series infinitas commode institui possit.*

## SOLUTIO

Utamur substitutione $y = e^{\int pdx} z$, ubi $p$ sit certa functio ipsius $x$ aequationem commode resolubilem suppeditans. Erit ergo

$$dy = e^{\int pdx}\left(dz + pzdx\right)$$

et

$$ddy = e^{\int pdx}\left(ddz + 2pdxdz + zdxdp + ppzdx^2\right),$$

unde aequatio proposita abit in

$$ddz + 2pdxdz + zdxdp + ppzdx^2 + ax^n zdx^2 = 0,$$

ubi $p$ ita capiatur, ut fiat $pp+ax^{n}=0$ seu $p=x^{\frac{n}{2}}\sqrt{-a}$. Ponamus ideo $a=-cc$ et $n=2m$, ut proposita sit haec aequatio

$$ddy-ccx^{2m}ydx^{2}=0,$$

quae posito $p=cx^{m}$ et $y=e^{\int pdz}z=e^{\frac{c}{m+1}x^{m+1}}z$ induet hanc formam

$$ddz+2cx^{m}dxdz+mcx^{m-1}zdx^{2}=0;$$

in qua cum $x$ occupet vel nullam vel $m+1$ dimensiones, fingamus ipsius $z$ valorem

$$z=Ax^{\lambda}+Bx^{\lambda+m+1}+Cx^{\lambda+2m+2}+\text{etc.},$$

quo substituto fit

$$\left.\begin{array}{l}\lambda(\lambda-1)Ax^{\lambda-2}+(\lambda+m+1)(\lambda+m)Bx^{\lambda+m-1}+\text{etc.},\\ \qquad\qquad\qquad\qquad +2\lambda Ac\\ \qquad\qquad\qquad\qquad +mAc\end{array}\right\}=0,$$

unde perspicuum est sumi debere vel $\lambda=0$ vel $\lambda=1$. Consequimur ergo seriem duplicatam huiusmodi

$$\begin{array}{l}z=A+Bx^{m+1}+Cx^{2m+2}+Dx^{2m+3}+Ex^{4m+4}+\text{etc.}\\ \;+\mathfrak{A}x+\mathfrak{B}x^{m+2}+\mathfrak{C}x^{2m+3}+\mathfrak{D}x^{4m+4}+\mathfrak{E}x^{4m+5}+\text{etc.},\end{array}$$

qua substituta fit

$$\left.\begin{array}{llll}(m+1)mBx^{m-1} & +2(m+1)(2m+1)Cx^{2m} & +3(m+1)(3m+2)Dx^{3m+1} & +\text{etc.}\\ +mAc & +2(m+1)Bc & +4(m+1)Cc & \\ & +mBc & +mCc & \\ (m+1)(m+2)\mathfrak{B}x^{m} & +2(m+1)(2m+3)\mathfrak{C}x^{2m+1} & +3(m+1)(3m+4)\mathfrak{D}x^{3m+2} & +\text{etc.}\\ +2\mathfrak{A}c & +2(m+2)\mathfrak{B}c & +2(2m+3)\mathfrak{C}c & \\ +m\mathfrak{A}c & +m\mathfrak{B}c & +m\mathfrak{C}c & \end{array}\right\}=0,$$

unde utrique coefficientes sequenti modo determinantur

| | |
|---|---|
| $B = \frac{-mAc}{m(m+1)}$ | $\mathfrak{B} = \frac{-(m+2)\mathfrak{A}c}{(m+2)(m+1)}$ |
| $C = \frac{-(3m+2)Bc}{2(2m+1)(m+1)}$ | $\mathfrak{C} = \frac{-(3m+4)\mathfrak{B}c}{2(2m+3)(m+1)}$ |
| $D = \frac{-(5m+4)Cc}{3(3m+2)(m+1)}$ | $\mathfrak{D} = \frac{-(5m+6)\mathfrak{C}c}{3(3m+4)(m+1)}$ |
| $E = \frac{-(7m+6)Dc}{4(4m+3)(m+1)}$ | $\mathfrak{E} = \frac{-(7m+8)\mathfrak{D}c}{4(4m+5)(m+1)}$ |
| etc. | etc. |

ubi bini coefficientes $A$ et $\mathfrak{A}$ manent indeterminati, ita ut hoc integrale completum sit censendum.

ALITER

Sumta serie, in qua exponentes ipsius $x$ decrescant, fieri debet $2\lambda + m = 0$ seu $m = -2\lambda$ ut aequatio nostra sit

$$ddy - ccx^{-4\lambda} ydx^2 = 0,$$

quae posito $p = cx^{-2\lambda}$ et $y = e^{\int pdx} z = e^{\frac{-c}{2\lambda-1} x^{-2\lambda+1}} z$ abit in

$$ddz + 2cx^{-2\lambda} dxdz - 2\lambda cx^{-2\lambda-1} zdx^2 = 0$$

Ponamus ergo

$$z = Ax^{\lambda} + Bx^{3\lambda-1} + Cx^{5\lambda-2} + Dx^{3\lambda-3} + \text{etc.}$$

et substitutione facta prodit

$$\begin{aligned} 0 = \quad & \lambda(\lambda-1)Ax^{\lambda-2} + (3\lambda-1)(3\lambda-2)Bx^{3\lambda-3} + (5\lambda-2)(5\lambda-3)CXx^{5\lambda-4} + \text{etc.} \\ & +2\lambda Acx^{-\lambda-1} + 2(3\lambda-1)Bc \quad + 2(5\lambda-2)Cc \quad + \quad 2(7\lambda-3)Dc \\ & -2\lambda Ac \quad - \quad 2\lambda Bc \quad - \quad 2\lambda Cc \quad - \quad 2\lambda Dc \end{aligned}$$

unde coefficientes ita determinantur

$$B = \frac{-\lambda(\lambda-1)A}{(4\lambda-2)c} \qquad = \frac{-\lambda(\lambda-1)A}{2(2\lambda-1)c},$$
$$C = \frac{-(3\lambda-1)(3\lambda-2)B}{(8\lambda-4)c} = \frac{-(3\lambda-1)(3\lambda-2)B}{4(2\lambda-1)c},$$
$$D = \frac{-(5\lambda-2)(5\lambda-3)C}{(12\lambda-6)c} = \frac{-(5\lambda-2)(5\lambda-3)C}{6(2\lambda-1)c},$$
$$E = \frac{-(7\lambda-3)(7\lambda-4)D}{(16\lambda-8)c} = \frac{-(7\lambda-3)(7\lambda-4)D}{8(2\lambda-1)c}$$
etc.

Hic unius tantum litterae *A* valor arbitrio nostro relinquitur, ex quo haec series tantum integrale particulare exhibet.

**COROLLARIUM 1**

**941.** Ex solutione priori patet alteram seriem terminari, quoties

$$(2i+1)m+2i=0 \text{ seu } m=\frac{-2i}{2i+1},$$

alteram vero, quoties

$$(2i-1)m+2i=0 \text{ seu } m=\frac{-2i}{2i-1}$$

denotante *i* numerum integrum quemcunque. His ergo casibus integrale saltem particulare finite exprimi potest.

**COROLLARIUM 2**

**942.** Altera solutio praebet seriem finitam, quoties fuerit

$$\text{vel } (2i+1)\lambda-i=0 \text{ vel } (2i-1)\lambda-i=0,$$

hoc est $\lambda=\frac{i}{2i\pm1}$ et $m=\frac{-2i}{2i\pm1}$ ut ante. Reliquis vero casibus haec series in infinitum excurrit.

**COROLLARIUM 3**

**943.** Casus ergo, quibus haec aequatio $ddy-ccx^n dx^2=0$ atque adeo posito $y=e^{\int udx}$ etiam haec $du+uudx=ccx^n dx$ integrationem saltem particularem admittit, sunt $n=\frac{-4i}{2i\pm1}$ sumendo pro *i* numerum integrum quemcunque [§ 436-441].

**SCHOLION**

**944.** Sufficit autem integrale particulare invenisse, cum ex eo facile integrale completum erui possit. Cum enim in integraIi insit littera *c*, dum aequatio differentialis tantum quadratum *cc* continet, perinde est, sive in integrali sumatur +*c* sive –*c*. Hinc, si integraIe particulare sit $y=P+cQ$, erit etiam $y=P-cQ$ integrale particulare, unde integrale completum erit

$$y=\alpha\left(P+cQ\right)+\beta\left(P-cQ\right)\ \text{ seu }\ y=\alpha P+\beta cQ.$$

Quo haec clarius explicentur, ad solutionem alteram aequationis

$$ddy-ccx^{-4\lambda}ydx^2=0$$

accommodentur, pro qua ponendo brevitatis gratia $\frac{1}{1-2\lambda}x^{1-2\lambda}=t$ fecimus $y=e^{ct}z$

et invenimus [§ 940]

$$z=Ax^{\lambda}-\frac{\lambda(\lambda-1)A}{2(2\lambda-1)c}x^{3\lambda-1}+\frac{\lambda(\lambda-1)(3\lambda-1)(3\lambda-2)A}{2\cdot4(2\lambda-1)^2cc}x^{5\lambda-2}$$
$$-\frac{\lambda(\lambda-1)(3\lambda-1)(3\lambda-2)(5\lambda-2)(5\lambda-3)A}{2\cdot4\cdot6(2\lambda-1)^3c^3}x^{7\lambda-3}+\text{ etc.}$$

Pro qua expressione distinguendo terminos per potestates pares ipsius $c$ divisos ab iis, qui per potestates impares sunt divisi, scribamus $z=P-cQ$, ita ut iam $P$ et $Q$ tantum potestates pares ipsius $c$ contineant, eritque integrale particulare unum $y=e^{ct}\left(P-cQ\right)$ et alterum $y=e^{-ct}\left(P+cQ\right)$, unde completum erit

$$y=\tfrac{1}{2}P\left(\alpha e^{ct}+\beta e^{-ct}\right)-\tfrac{1}{2}cQ\left(\alpha e^{ct}-\beta e^{-ct}\right).$$

Hinc, si $c$ sit numerus imaginarius seu $cc=-bb$, ut aequatio sit

$$ddy+bbx^{-4\lambda}ydx^2=0,$$

erit $z=P-bQ\sqrt{-1}$ et $e^{ct}=e^{bt\sqrt{-1}}=\cos.bt+\sqrt{-1}\cdot\sin.bt$, ergo

$$y=P\left(\frac{\alpha+\beta}{2}\cos.bt+\frac{\alpha-\beta}{2}\sqrt{-1}\cdot\sin.bt\right)-bQ\left(\frac{\alpha-\beta}{2}\cos.bt+\frac{\alpha+\beta}{2}\sqrt{-1}\cdot\sin.bt\right)\sqrt{-1};$$

sit $\frac{\alpha+\beta}{2}=\gamma$ et $\frac{\alpha-\beta}{2}\sqrt{-1}=\delta$ atque integrale completum hoc casu ita exprimetur

$$y=P\left(\gamma\cos.bt+\delta\sin.bt\right)-bQ\left(\delta\cos.bt-\gamma\sin.bt\right)$$

seu

$$y=\left(\gamma P-\delta bQ\right)\cos.bt+\left(\delta P+\gamma bQ\right)\sin.bt.$$

Casus ergo hoc modo integrabiles evolvamus.

**EXEMPLUM 1**

**945.** *Integrale aequationis* $ddy - ccydx^2 = 0$ *invenire.*

Hic est $\lambda = 0$ et $z = A$ atque $t = x$, unde ob $P = A$ et $Q = 0$ erit integrale completum

$$y = \alpha e^{cx} + \beta e^{-cx}.$$

Casu autem $cc = -bb$ aequationis $ddy + bbydx^2 = 0$ integrale completum erit

$$y = \gamma cos.bx + \delta \sin.bx.$$

**EXEMPLUM 2**

**946.** *Integrale aequationis* $ddy - ccx^{-4}ydx^2 = 0$ *invenire.*

Hic ob $\lambda = 1$ est $z = Ax$ et $t = -\frac{1}{x}$, unde ob $P = x$ et $Q = 0$ fit

$$y = \left(\alpha e^{ct} + \beta e^{-ct}\right)x.$$

Casu autem $cc = -bb$ aequationis $ddy + bbx^{-4}ydx^2 = 0$ integrale est

$$y = (\alpha \cos.bt + \beta \sin.bt)x$$

existente $t = \frac{-1}{x}$.

**EXEMPLUM 3**

**947.** *Integrale aequationis* $ddy - ccx^{-\frac{4}{3}}ydx^2 = 0$ *invenire.*

Ob $\lambda = \frac{1}{3}$ fit $B = -\frac{A}{3c}$ et $z = Ax^{\frac{1}{3}} - \frac{A}{3c}$ atque $t = 3x^{\frac{1}{3}}$, unde $P = x^{\frac{1}{3}}$ et $Q = \frac{1}{3cc}$. Integrale ergo erit

$$y = \left(\alpha e^{ct} + \beta e^{-ct}\right)x^{\frac{1}{3}} - \left(\alpha e^{ct} - \beta e^{-ct}\right)\frac{1}{3c}.$$

Casu autem $cc = -bb$ aequationis $ddy + bbx^{-\frac{4}{3}}ydx^2 = 0$ integrale est

$$y = (\alpha \cos.bt + \beta \sin.bt)x^{\frac{1}{3}} + \frac{1}{3b}(\beta \cos.bt - \alpha \sin.bt).$$

## EXEMPLUM 4

**948.** *Integrale aequationis* $ddy - ccx^{-\frac{8}{3}}ydx^2 = 0$ *invenire.*

Ob $\lambda = \frac{2}{3}$ fit $B = \frac{A}{3c}$ et $z = Ax^{\frac{2}{3}} - \frac{A}{3c}x$, ita ut sit $P = x^{\frac{2}{3}}$ et $Q = \frac{-x}{3cc}$.
Posito ergo $t = -3x^{-\frac{1}{3}}$ integrale ita exprimitur

$$y = x^{\frac{2}{3}}\left(\alpha e^{ct} + \beta e^{-ct}\right) + \frac{x}{3c}\left(\alpha e^{ct} - \beta e^{-ct}\right).$$

Casu autem $cc = -bb$ aequationis $ddy + bbx^{-\frac{8}{3}}ydx^2 = 0$ integrale est

$$y = x^{\frac{2}{3}}\left(\alpha\cos.bt + \beta\sin.bt\right) - \frac{x}{3b}\left(\beta\cos.bt - \alpha\sin.bt\right)$$

## EXEMPLUM 5

**949.** *Integrale aequationis* $ddy - ccx^{-\frac{8}{5}}ydx^2 = 0$ *invenire.*

Ob $\lambda = \frac{2}{5}$ est $B = \frac{-3A}{3c}$ et $C = \frac{3A}{5^2cc}$, hinc $z = Ax^{\frac{2}{5}} - \frac{3A}{5c}x^{\frac{1}{5}} + \frac{3A}{5^2cc}$
ideoque $P = x^{\frac{2}{5}} + \frac{3}{5^2cc}$ et $Q = \frac{3}{5cc}x^{\frac{1}{5}}$. Posito ergo $t = 5x^{\frac{1}{5}}$ integrale erit

$$y = \left(x^{\frac{2}{5}} + \frac{3}{5^2cc}\right)\left(\alpha e^{ct} + \beta e^{-ct}\right) - \frac{3}{5c}x^{\frac{1}{5}}\left(\alpha e^{ct} - \beta e^{-ct}\right).$$

Casu autem $cc = -bb$ aequationis $ddy + bbx^{-\frac{8}{5}}ydx^2 = 0$ integrale est

$$y = \left(x^{\frac{2}{5}} - \frac{3}{5^2bb}\right)\left(\alpha\cos.bt + \beta\sin.bt\right) + \frac{3}{5b}x^{\frac{1}{5}}\left(\beta\cos.bt - \alpha\sin.bt\right).$$

## PROBLEMA 119

**950.** *Aequationis ditferentio–differentialis*

$$ddy - ccx^{\frac{-4i}{2i-1}}ydx^2 = 0$$

*integrale completum assignare denotante i numerum integrum quemcunque.*

## SOLUTIO

Sit brevitatis gratia $t = -(2i-1)x^{\frac{-1}{2i-1}}$ unde fit $x^{\frac{1}{2i-1}} = -\frac{2i-1}{t}$, ac posito $y = e^{ct}z$ pro valore ipsius $z$ per seriem invento

$$z = Ax^{\frac{i}{2i-1}} + Bx^{\frac{i+1}{2i-1}} + Cx^{\frac{i+2}{2i-1}} + Dx^{\frac{i+3}{2i-1}} + \text{etc.}$$

ob $\lambda = \frac{i}{2i-1}$ hi coefficientes ita determinantur

$$B = \frac{i(i-1)A}{2(2i-1)c},\;\; C = \frac{(i+1)(i-2)B}{4(2i-1)c},\; D = \frac{(i+2)(i-3)C}{6(2i-1)c} \;\;\text{etc.},$$

quibus substitutis et introducto valore $x^{\frac{1}{2i-1}} = -\frac{2i-1}{t}$, erit

$$z = Ax^{\frac{i}{2i-1}}\left(1 - \frac{i(i-1)}{2ct} + \frac{i(ii-1)(i-2)}{2\cdot 4cctt} - \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6c^3t^3} + \text{etc.}\right)$$

sive hoc modo

$$z = \frac{A}{t^i}\left(1 - \frac{i(i-1)}{2ct} + \frac{i(ii-1)(i-2)}{2\cdot 4cctt} - \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6c^3t^3} + \text{etc.}\right)$$

Atque hinc aequationis propositae integrale completum ita exprimetur

$$y = t^{-i}\left(1 + \frac{i(ii-1)(i-2)}{2\cdot 4cctt} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8c^4t^4} + \text{etc.}\right)\left(\alpha e^{ct} + \beta e^{-ct}\right)$$
$$-t^{-i}\left(\frac{i(i-1)}{2ct} + \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6c^3t^3} + \text{etc.}\right)\left(\alpha e^{ct} - \beta e^{-ct}\right),$$

ubi in utraque progressione lex formationis singulorum terminorum est manifesta.

## COROLLARIUM 1

**951.** Hinc quoque illius aequationis
$ddy + bbx^{\frac{-4i}{2i-1}}ydx^2 = 0$ integrale completum manente $t = -(2i-1)x^{\frac{-1}{2i-1}}$ est

$$y = t^{-i}\left(1 - \frac{i(ii-1)(i-2)}{2\cdot 4bbtt} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8b^4t^4} - \text{etc.}\right)(\alpha\cos.bt + \beta\sin.bt)$$
$$+t^{-i}\left(\frac{i(i-1)}{2bt} - \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6b^3t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2\cdot 4\cdot 6\cdot 8\cdot 10b^5t^5} - \text{etc.}\right)(\beta\cos.bt - \alpha\sin.bt),$$

### COROLLARIUM 2

**952.** Si *i* sit numerus negativus, haec integratio perinde succedit; aequationis enim

$$ddy - ccx^{\frac{-4i}{2i+1}}ydx^2 = 0$$

posito $t = (2i+1)x^{\frac{1}{2i+1}}$ integrala erit

$$y = t^i\left(1 + \frac{i(ii-1)(i+2)}{2\cdot 4cctt} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8c^4t^4} + \text{etc.}\right)\left(\alpha e^{ct} + \beta e^{-ct}\right)$$
$$-t^i\left(\frac{i(i+1)}{2ct} + \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6c^3t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^5t^5} + \text{etc.}\right)\left(\alpha e^{ct} - \beta e^{-ct}\right),$$

### COROLLARIUM 3

**953.** Simili modo huius aequationis

$$ddy + bbx^{\frac{-4i}{2i+1}}ydx^2 = 0$$

posito $t = (2i+1)x^{\frac{1}{2i+1}}$ integrale completum erit

$$y = t^i\left(1 - \frac{i(ii-1)(i+2)}{2\cdot 4bbtt} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8b^4t^4} - \text{etc.}\right)\left(\alpha\cos.bt + \beta\sin.bt\right)$$
$$+t^i\left(\frac{i(i+1)}{2bt} - \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6b^3t^3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10b^5t^5} - \text{etc.}\right)\left(\beta\cos.bt - \alpha\sin.bt\right),$$

### COROLLARIUM 4

**954.** In formulis sinus et cosinus continentibus si ponatur

$$\alpha = C\sin.\zeta \text{ et } \beta = C\cos.\zeta ,$$

expressiones nostrae ita contrahuntur, ut fiat

$$\alpha\cos.bt + \beta\sin.bt = C\sin.(bt + \zeta) \text{ et } \beta\cos.bt - \alpha\sin.bt = C\cos.(bt + \zeta),$$

ut iam hic $C$ et $\zeta$ sint constantes arbitrariae integrale completum reddentes.

### SCHOLION

**955.** Hinc egregium adipiscimur adminiculum ad casus integrabilitatis huius aequationis differentialis primi gradus

$$du + uudx + ax^n dx = 0$$

agnoscendos simulque integralia completa definienda; nascitur enim haec aequatio ex ista $ddy + ax^n ydx^2 = 0$ ponendo $y = e^{\int udx}$ unde ex illa vicissim haec oritur ponendo $u = \frac{dy}{ydx}$.
Cum igitur istius integrale assignare licuerit casibus, quibus exponens $n = \frac{4i}{2i \pm 1}$ iisdem casibus integrale aequationis differentialis primi gradus assignare licebit, ubi quidem duos casus evolvi convenit, prout $a$ fuerit vel numerus negativus $a = -cc$ vel positivus $a = +bb$. Hos igitur duos casus pertractasse operae erit pretium.

### PROBLEMA 120

**956.** *Denotante i numerum integrum sive positivum sive negativum quemcunque invenire integrale huius aequationis*

$$du + uudx - ccx^{\frac{-4i}{2i+1}} dx = 0.$$

### SOLUTIO

Posito $u = \frac{dy}{ydx}$ haec aequatio transformatur in istam

$$ddy - ccx^{\frac{-4i}{2i+1}} ydx^2 = 0$$

sumto elemento $dx$ constante, cuius integrale assignavimus. Posito scilicet $t = (2i+1)x^{\frac{1}{2i+1}}$ est

$$y = \left(\alpha e^{ct} + \beta e^{-ct}\right)\left(t^i + \frac{i(ii-1)(i+2)}{2\cdot 4cc} t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8c^4} t^{i-4} + \text{etc.}\right)$$
$$-\left(\alpha e^{ct} - \beta e^{-ct}\right)\left(\frac{i(i+1)}{2c} t^{i-1} + \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6c^3} t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^5} t^{i-5} + \text{etc.}\right).$$

Ponamus brevitatis gratia

$$y = \left(\alpha e^{ct} + \beta e^{-ct}\right)P - \left(\alpha e^{ct} - \beta e^{-ct}\right)Q$$

et cum sit

$$dt = x^{\frac{-2i}{2i+1}} dx \quad \text{seu} \quad dx = x^{\frac{2i}{2i+1}} dt,$$

erit

$$\frac{dy}{dx}=\frac{\left(\alpha e^{ct}+\beta e^{-ct}\right)\left(dP-cQdt\right)-\left(\alpha e^{ct}-\beta e^{-ct}\right)\left(cPdt-dQ\right)}{x^{\frac{2i}{2i+1}}dt}$$

sive

$$\frac{dy}{dx}=\frac{\alpha e^{ct}\left(dP+cPdt-dQ-cQdt\right)+\beta e^{-ct}\left(dP-cPdt+dQ-cQdt\right)}{x^{\frac{2i}{2i+1}}dt}$$

At vero est

$$\frac{dP}{dt}=it^{i-1}+\frac{i(ii-1)(i-4)}{2\cdot 4cc}t^{i-3}+\frac{i(ii-1)(ii-4)(ii-9)(i-16)}{2\cdot 4\cdot 6\cdot 8c^4}t^{i-5}+\text{etc.},$$

$$cQ=\frac{i(i+1)}{2}t^{i-1}+\frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6c^2}t^{i-3}+\frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^4}t^{i-5}+\text{etc.},$$

$$cP=ct^i+\frac{i(ii-1)(i+2)}{2\cdot 4c}t^{i-2}+\frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8c^3}t^{i-4}+\text{etc.}$$

$$\frac{dQ}{dt}=\frac{i(ii-1)}{2c}t^{i-2}+\frac{i(ii-1)(ii-4)(ii-9)}{2\cdot 4\cdot 6c^3}t^{i-4}+\text{etc.}$$

unde colligitur

$$\frac{dP-cQdt}{dt}=-\frac{i(i-1)}{2}t^{i-1}-\frac{i(ii-1)(i-4)(i-3)}{2\cdot 4\cdot 6c^2}t^{i-3}-\frac{i(ii-1)(ii-4)(ii-9)(i-16)(i-5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^4}t^{i-5}-\text{etc.},$$

$$\frac{cPdt-dQ}{dt}=ct^i+\frac{i(ii-1)(i-2)}{2\cdot 4c}t^{i-2}+\frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8c^3}t^{i-4}+\text{etc.}$$

Ponamus ad abbreviandum

$$P=t^i+\frac{i(ii-1)(i+2)}{2\cdot 4cc}t^{i-2}+\frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8c^4}t^{i-4}+\text{etc.},$$

$$Q=\frac{i(i+1)}{2c}t^{i-1}+\frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6c^3}t^{i-3}+\frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^5}t^{i-5}+\text{etc.},$$

$$R=t^i+\frac{i(ii-1)(i-2)}{2\cdot 4cc}t^{i-2}+\frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8c^4}t^{i-4}+\text{etc.}$$

$$S=\frac{i(i-1)}{2c}t^{i-1}+\frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6c^3}t^{i-3}+\frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2\cdot 4\cdot 6\cdot 8\cdot 10c^5}t^{i-5}+\text{etc.}$$

ut sit

$$\frac{dy}{dx}=\frac{\left(\alpha e^{ct}+\beta e^{-ct}\right)(-cS)-\left(\alpha e^{ct}-\beta e^{-ct}\right)(cR)}{x^{\frac{2i}{2i+1}}}$$

Quare, cum sit $u=\frac{dy}{ydx}$, erit nostrae aequationes integrale completum

$$\frac{1}{c}x^{\frac{2i}{2i+1}}u=\frac{\left(\alpha e^{ct}-\beta e^{-ct}\right)R-\left(\alpha e^{ct}+\beta e^{-ct}\right)S}{\left(\alpha e^{ct}+\beta e^{-ct}\right)P-\left(\alpha e^{ct}-\beta e^{-ct}\right)Q}$$

sive

$$\frac{1}{c}x^{\frac{2i}{2i+1}}u=\frac{\alpha e^{ct}(R-S)-\beta e^{-ct}(R+S)}{\alpha e^{ct}(P-Q)+\beta e^{-ct}(P+Q)},$$

quod ob rationem constantium $a:\beta$ arbitrariam est completum.

### COROLLARIUM 1

**957.** Quaternae formulae *P, Q, R, S,* quae singulae casibus, quibus *i* est numerus integer, abrumpuntur, ita a se invicem pendent, ut sit primo

$$R=P-\frac{dQ}{cdt} \quad \text{et} \quad S=Q-\frac{dP}{cdt},$$

tum vero

$$dP+dR=\frac{2iRdt}{t} \quad \text{et} \quad dQ+dS=\frac{2iSdt}{t}.$$

### COROLLARIUM 2

**958.** Posito ergo vel $\alpha=0$ vel $\beta=0$ integralia particularia algebraica aequationis $du+uudx-ccx^{\frac{-4i}{2i+1}}dx=0$ exhiberi possunt, quae sunt

$$\frac{1}{c}x^{\frac{2i}{2i+1}}u=\frac{R-S}{P-Q} \quad \text{et} \quad \frac{1}{c}x^{\frac{2i}{2i+1}}u=\frac{-S-R}{P+Q}$$

ideoque hac una formula comprehendi possunt

$$\frac{1}{c}x^{\frac{2i}{2i+1}}u=\frac{-S\pm R}{P\mp Q}$$

### SCHOLION 1

**959.** Pro variis ergo valoribus numeri $i$ tam quantitas $t$ quam litterae $P$, $Q$, $R$, $S$ sequenti modo se habebunt. Primo scilicet si $i = 0$, erit $t = x$ atque $P = 1$, $Q = 0$, $R = 1$ et $S = 0$; reliquos casus in sequenti tabella repraesentemus:

| | |
|---|---|
| $i = -1, \quad t = -\frac{1}{x}$<br>$P = \frac{1}{t}, \quad Q = 0$<br>$R = \frac{1}{t}, \quad S = \frac{1}{ctt}$ | $i = 1, \quad t = 3x^{\frac{1}{3}}$<br>$P = t, \quad Q = \frac{1}{c}$<br>$R = t, \quad S = 0$ |
| $i = -2, \quad t = -\frac{3}{x^{\frac{1}{3}}}$<br>$P = \frac{1}{tt}, \quad Q = \frac{1}{ct^3}$<br>$R = \frac{1}{tt} + \frac{3}{cct^4}, \quad S = \frac{3}{ct^3}$ | $i = 2, \quad t = 5x^{\frac{1}{5}}$<br>$P = tt + \frac{3}{cc}, \quad Q = \frac{3}{c}t$<br>$R = tt, \quad S = \frac{1}{c}t$ |
| $i = -3, \quad t = -\frac{5}{x^{\frac{1}{5}}}$<br>$P = \frac{1}{t^3} + \frac{1\cdot 3}{cct^5}, \quad Q = \frac{3}{ct^4}$<br>$R = \frac{1}{t^3} + \frac{3\cdot 5}{cct^5}, \quad S = \frac{6}{ct^4} + \frac{3\cdot 5}{c^3t^6}$ | $i = 3, \quad t = 7x^{\frac{1}{7}}$<br>$P = tt + \frac{3\cdot 5}{cc}, \quad Q = \frac{6}{c}tt + \frac{3\cdot 5}{c^3}$<br>$R = t^3 + \frac{1\cdot 3}{cc}t, \quad S = \frac{3}{c}tt$ |
| $i = -4, \quad t = -\frac{7}{x^{\frac{1}{7}}}$<br>$P = \frac{1}{t^4} + \frac{3\cdot 5}{cct^6}$<br>$Q = \frac{6}{ct^5} + \frac{3\cdot 5}{c^3t^7}$<br>$R = \frac{1}{t^4} + \frac{3\cdot 3\cdot 5}{cct^6} + \frac{3\cdot 5\cdot 7}{c^4t^8},$<br>$S = \frac{10}{ct^5} + \frac{3\cdot 5\cdot 7}{c^3t^7}$ | $i = 4, \quad t = 9x^{\frac{1}{9}}$<br>$P = t^4 + \frac{3\cdot 3\cdot 5}{cc}tt + \frac{3\cdot 5\cdot 7}{c^4}$<br>$Q = \frac{10}{c}t^3 + \frac{3\cdot 5\cdot 7}{c^3}t$<br>$R = t^4 + \frac{3\cdot 5}{cc}tt$<br>$S = \frac{6}{c}t^3 + \frac{3\cdot 5}{c^3}t$ |
| $i = -5, \quad t = -\frac{9}{x^{\frac{1}{9}}}$<br>$P = \frac{1}{t^5} + \frac{3\cdot 3\cdot 5}{cct^7} + \frac{3\cdot 5\cdot 7}{c^4t^9}$<br>$Q = \frac{10}{ct^6} + \frac{3\cdot 5\cdot 7}{c^3t^8}$<br>$R = \frac{1}{t^5} + \frac{3\cdot 5\cdot 7}{cct^7} + \frac{3\cdot 5\cdot 7\cdot 9}{c^4t^9}$<br>$S = \frac{15}{ct^6} + \frac{4\cdot 3\cdot 5\cdot 7}{c^3t^8} + \frac{3\cdot 5\cdot 7\cdot 9}{c^5t^{10}}$ | $i = 5, \quad t = 11x^{\frac{1}{11}}$<br>$P = t^5 + \frac{3\cdot 5\cdot 7}{cc}t^3 + \frac{3\cdot 5\cdot 7\cdot 9}{c^4}t$<br>$Q = \frac{15}{c}t^4 + \frac{4\cdot 3\cdot 5\cdot 7}{c^3}tt + \frac{3\cdot 5\cdot 7\cdot 9}{c^5}$<br>$R = t^5 + \frac{3\cdot 3\cdot 5}{cc}t^3 + \frac{3\cdot 5\cdot 7}{c^4}t$<br>$S = \frac{10}{c}t^4 + \frac{3\cdot 5\cdot 7}{c^3}tt$ |

| | |
|---|---|
| $i=-6,\quad t=\frac{-11}{x^{\frac{1}{11}}}$ | $i=6,\quad t=13x^{\frac{1}{13}}$ |
| $P=\frac{1}{t^6}+\frac{3\cdot5\cdot7}{cct^8}+\frac{3\cdot5\cdot7\cdot9}{c^5t^{10}}$ | $P=t^6+\frac{2\cdot3\cdot5\cdot7}{cc}t^4+\frac{5\cdot3\cdot5\cdot7\cdot9}{c^4}tt+\frac{3\cdot5\cdot7\cdot9\cdot11}{c^6}$ |
| $Q=\frac{15}{ct^7}+\frac{4\cdot3\cdot5\cdot7}{c^3t^9}+\frac{3\cdot5\cdot7\cdot9}{c^5t^{11}}$ | $Q=\frac{21}{c}t^5+\frac{4\cdot5\cdot7\cdot9}{c^3}t^3+\frac{3\cdot5\cdot7\cdot9\cdot11}{c^5}t$ |
| $R=\frac{1}{t^6}+\frac{2\cdot3\cdot5\cdot7}{cct^8}+\frac{5\cdot3\cdot5\cdot7\cdot9}{c^4t^{10}}+\frac{3\cdot5\cdot7\cdot9\cdot11}{c^6t^{12}}$ | $R=t^6+\frac{3\cdot5\cdot7}{cc}t^4+\frac{3\cdot5\cdot7\cdot9}{c^4}tt$ |
| $S=\frac{21}{ct^7}+\frac{4\cdot5\cdot7\cdot9}{c^3t^9}+\frac{3\cdot5\cdot7\cdot9\cdot11}{c^5t^{11}}$ | $S=\frac{15}{c}t^5+\frac{4\cdot3\cdot5\cdot7}{c^3}t^3+\frac{3\cdot5\cdot7\cdot9}{c^5}t.$ |

## SCHOLION 2

**960.** Dum hae formulae diligentius considerantur, nova se prodit relatio inter valores litterarum *P, Q, R, S,* quae in hoc consistit, ut perpetuo sit $PR-QS=t^{2i}$, cuius veritas primo quidem per inductionem deprehenditur, tum vero etiam per relationes supra datas demonstrari potest. Si enim valores

$$R=P-\frac{dQ}{cdt}\ \text{ et }\ S=Q-\frac{dP}{cdt}$$

in aequationibus

$$dP+dR=\frac{2iRdt}{t}\ \text{ et }\ dQ+dS=\frac{2iSdt}{t}$$

substituantur, oriuntur hae duae aequationes

$$2dP-\frac{ddQ}{cdt}=\frac{2iPdt}{t}-\frac{2idQ}{ct}\ \text{ et }\ 2dQ-\frac{ddP}{cdt}=\frac{2iQdt}{t}-\frac{2idP}{ct},$$

quarum illa per *P*, haec vero per – *Q* multiplicata iunctim dant

$$2PdP-2QdQ+\frac{QddP-PddQ}{cdt}=\frac{2idt}{t}\left(PP-QQ\right)+\frac{2i}{ct}\left(QdP-PdQ\right).$$

Ponatur

$$PP-QQ=M\ \text{ et }\ \frac{QdP-PdQ}{cdt}=N\,;$$

erit

$$dM+dN=\frac{2itdt}{t}\left(M+N\right)\ \text{ seu }\ \frac{dM+dN}{M+N}=\frac{2idt}{t}$$

hincque integrando $M+N=Ct^{2i}$. At est

$$M+N=P\left(P-\frac{dQ}{cdt}\right)-Q\left(Q-\frac{dP}{cdt}\right)=PR-QS,$$

evidens autem est pro constante *C* unitatem accipi debere.

## PROBLEMA 121

**961.** *Denotante i numerum integrum sive positivum sive negativum quemcunque invenire integrale completum huius aequationis*

$$du + uudx + bbx^{\frac{-4i}{2i+1}}dx = 0 .$$

## SOLUTIO

Posito $u = \frac{dy}{ydx}$ haec aequatio transformatur in istam

$$ddy + bbx^{\frac{-4i}{2i+1}}ydx^2 = 0$$

sumto elemento *dx* constante, cuius integrale supra est assignatum. Scilicet posito $t = (2i+1)x^{\frac{1}{2i+1}}$ invenimus (§ 953, 954)

$$y = C\left(t^i - \frac{i(ii-1)(i+2)}{2\cdot 4bb}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8b^4}t^{i-4} - \text{etc.}\right)\sin.(bt+\zeta)$$
$$+C\left(\frac{i(i+1)}{2b}t^{i-1} - \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6b^3}t^{i-3} + \text{etc.}\right)\cos.(bt+\zeta),$$

cuius loco brevitatis gratia scribamus

$$y = CP\sin.(bt+\zeta) + CQ\cos.(bt+\zeta).$$

Hinc ob

$$dt = x^{\frac{-2i}{2i+1}}dx \quad \text{seu} \quad dx = x^{\frac{2i}{2i+1}}dt$$

erit

$$\frac{dy}{dx} = \frac{C(dP-bQdt)\sin.(bt+\zeta)+C(dQ+bPdt)\cos.(bt+\zeta)}{x^{\frac{2i}{2i+1}}dt}$$

unde, cum sit $u = \frac{dy}{ydx}$, erit

$$u = \frac{(dP-bQdt)\sin.(bt+\zeta)+(dQ+bPdt)\cos.(bt+\zeta)}{x^{\frac{2i}{2i+1}}dt\left(P\sin.(bt+\zeta)+Q\cos.(bt+\zeta)\right)}$$

Ponamus

$$P + \frac{dQ}{bdt} = R \quad \text{et} \quad Q - \frac{dP}{bdt} = S,$$

ut sit

$$\frac{1}{b}x^{\frac{2i}{2i+1}}u = \frac{R\cos.(bt+\zeta)-S\sin.(bt+\zeta)}{P\sin.(bt+\zeta)+Q\cos.(bt+\zeta)}$$

erit

$$P = t^i - \frac{i(ii-1)(i+2)}{2\cdot 4bb}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i+4)}{2\cdot 4\cdot 6\cdot 8b^4}t^{i-4} - \text{etc.},$$

$$Q = \frac{i(i+1)}{2b}t^{i-1} - \frac{i(ii-1)(ii-4)(i+3)}{2\cdot 4\cdot 6b^3}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i+5)}{2\cdot 4\cdot 6\cdot 8\cdot 10b^5}t^{i-5} - \text{etc.},$$

$$R = t^i - \frac{i(ii-1)(i-2)}{2\cdot 4bb}t^{i-2} + \frac{i(ii-1)(ii-4)(ii-9)(i-4)}{2\cdot 4\cdot 6\cdot 8b^4}t^{i-4} - \text{etc.}$$

$$S = \frac{i(i-1)}{2b}t^{i-1} - \frac{i(ii-1)(ii-4)(i-3)}{2\cdot 4\cdot 6b^3}t^{i-3} + \frac{i(ii-1)(ii-4)(ii-9)(ii-16)(i-5)}{2\cdot 4\cdot 6\cdot 8\cdot 10b^5}t^{i-5} - \text{etc.}$$

atque ob angulum $\zeta$ introductum hoc integrale erit completum.

## COROLLARIUM 1

**962.** Quaternarum ergo litterarum *P, Q, R, S* valores ita a se invicem pendent, ut sit primo

$$R = P + \frac{dQ}{bdt} \text{ et } S = Q - \frac{dP}{bdt},$$

tum vero etiam patet fore

$$dP + dR = \frac{2iRdt}{t} \quad \text{et} \quad dQ + dS = \frac{2iSdt}{t}.$$

## COROLLARIUM 2

**963.** Deinde etiam colligitur fore $PR + QS = t^{2i}$, quae aequalitas ex praecedentis problematis formulis [§ 960] deducitur sumto $cc = -bb$, ubi *Q* et *S* abeunt in $Q\sqrt{-1}$ et $\sqrt{-1}$.

## COROLLARIUM 3

**964.** Hic casus a praecedente etiam hoc differt, quod hic nulla dentur integralia particularia algebraica [§ 958]. Quicunque enim valor angulo constanti $\zeta$ tribuatur, integrale semper sinum et cosinum cuiusdam anguli involvit.

## SCHOLION 1

**965.** Cum igitur aequationis

$$du + uudx + bbx^{\frac{-4i}{2i+1}}dx = 0$$

integrale completum posito $t = (2i+1)x^{\frac{1}{2i+1}}$ sit

$$\frac{1}{b}x^{\frac{2i}{2i+1}}u = \frac{R\cos.(bt+\zeta) - S\sin.(bt+\zeta)}{P\sin.(bt+\zeta) + Q\cos.(bt+\zeta)}$$

pro singulis valoribus numeri $i$ quantitas $t$ cum litteris $P, Q, R, S$ ita se habebit. Primo si $i=0$, erit

$$P=1, Q=0, R=1 \text{ et } S=0, \text{ item } t=x,$$

ita ut integrale sit

$$\frac{1}{b}u=\frac{\cos.(bx+\zeta)}{\sin.(bx+\zeta)}$$

Reliquos casus sequens tabella exhibet:

| | |
|---|---|
| $i=-1,\quad t=-\frac{1}{x}$<br>$P=\frac{1}{t},\quad Q=0$<br>$R=\frac{1}{t},\quad S=\frac{1}{btt}$ | $i=1,\quad t=3x^{\frac{1}{3}}$<br>$P=t,\quad Q=\frac{1}{b}$<br>$R=t,\quad S=0$ |
| $i=-2,\quad t=-\frac{3}{x^{\frac{1}{3}}}$<br>$P=\frac{1}{tt},\quad Q=\frac{1}{bt^3}$<br>$R=\frac{1}{tt}-\frac{3}{bbt^4},\quad S=\frac{3}{bt^3}$ | $i=2,\quad t=5x^{\frac{1}{5}}$<br>$P=tt-\frac{3}{bb},\quad Q=\frac{3}{b}t$<br>$R=tt,\quad S=\frac{1}{b}t$ |
| $i=-3,\quad t=-\frac{5}{x^{\frac{1}{5}}}$<br>$P=\frac{1}{t^3}-\frac{1\cdot 3}{bbt^5},\quad Q=\frac{3}{bt^4}$<br>$R=\frac{1}{t^3}-\frac{3\cdot 5}{bbt^5},\quad S=\frac{6}{bt^4}-\frac{3\cdot 5}{b^3t^6}$ | $i=3,\quad t=7x^{\frac{1}{7}}$<br>$P=tt-\frac{3\cdot 5}{bb},\quad Q=\frac{6}{b}tt-\frac{3\cdot 5}{b^3}$<br>$R=t^3-\frac{1\cdot 3}{bb}t,\quad S=\frac{3}{b}tt$ |
| $i=-4,\quad t=-\frac{7}{x^{\frac{1}{7}}}$<br>$P=\frac{1}{t^4}-\frac{3\cdot 5}{bbt^6}$<br>$Q=\frac{6}{bt^5}-\frac{3\cdot 5}{b^3t^7}$<br>$R=\frac{1}{t^4}-\frac{3\cdot 3\cdot 5}{bbt^6}+\frac{3\cdot 5\cdot 7}{b^4t^8},$<br>$S=\frac{10}{bt^5}-\frac{3\cdot 5\cdot 7}{b^3t^7}$ | $i=4,\quad t=9x^{\frac{1}{9}}$<br>$P=t^4-\frac{3\cdot 3\cdot 5}{bb}tt+\frac{3\cdot 5\cdot 7}{b^4}$<br>$Q=\frac{10}{b}t^3-\frac{3\cdot 5\cdot 7}{b^3}t$<br>$R=t^4-\frac{3\cdot 5}{bb}tt$<br>$S=\frac{6}{b}t^3-\frac{3\cdot 5}{b^3}t$ |
| $i=-5,\quad t=\frac{-9}{x^{\frac{1}{9}}}$<br>$P=\frac{1}{t^5}-\frac{3\cdot 3\cdot 5}{bbt^7}+\frac{3\cdot 5\cdot 7}{b^4t^9}$<br>$Q=\frac{10}{bt^6}-\frac{3\cdot 5\cdot 7}{b^3t^8}$<br>$R=\frac{1}{t^5}-\frac{3\cdot 5\cdot 7}{bbt^7}+\frac{3\cdot 5\cdot 7\cdot 9}{b^4t^9}$<br>$S=\frac{15}{bt^6}-\frac{4\cdot 3\cdot 5\cdot 7}{b^3t^8}+\frac{3\cdot 5\cdot 7\cdot 9}{b^5t^{10}}$ | $i=5,\quad t=11x^{\frac{1}{11}}$<br>$P=t^5-\frac{3\cdot 5\cdot 7}{bb}t^3+\frac{3\cdot 5\cdot 7\cdot 9}{b^4}t$<br>$Q=\frac{15}{c}t^4-\frac{4\cdot 3\cdot 5\cdot 7}{b^3}tt+\frac{3\cdot 5\cdot 7\cdot 9}{b^5}$<br>$R=t^5-\frac{3\cdot 3\cdot 5}{bb}t^3+\frac{3\cdot 5\cdot 7}{b^4}t$<br>$S=\frac{10}{b}t^4-\frac{3\cdot 5\cdot 7}{b^3}tt$ |

| | |
|---|---|
| $i=-6, \quad t=\frac{-11}{x^{\frac{1}{11}}}$ | $i=6, \quad t=13x^{\frac{1}{13}}$ |
| $P=\frac{1}{t^6}-\frac{3\cdot5\cdot7}{bbt^8}+\frac{3\cdot5\cdot7\cdot9}{b^5t^{10}}$ | $P=t^6-\frac{2\cdot3\cdot5\cdot7}{bb}t^4+\frac{5\cdot3\cdot5\cdot7\cdot9}{b^4}tt-\frac{3\cdot5\cdot7\cdot9\cdot11}{b^6}$ |
| $Q=\frac{15}{bt^7}-\frac{4\cdot3\cdot5\cdot7}{b^3t^9}+\frac{3\cdot5\cdot7\cdot9}{b^5t^{11}}$ | $Q=\frac{21}{b}t^5-\frac{4\cdot5\cdot7\cdot9}{b^3}t^3+\frac{3\cdot5\cdot7\cdot9\cdot11}{b^5}t$ |
| $R=\frac{1}{t^6}-\frac{2\cdot3\cdot5\cdot7}{bbt^8}+\frac{5\cdot3\cdot5\cdot7\cdot9}{b^4t^{10}}-\frac{3\cdot5\cdot7\cdot9\cdot11}{b^6t^{12}}$ | $R=t^6-\frac{3\cdot5\cdot7}{bb}t^4+\frac{3\cdot5\cdot7\cdot9}{b^4}tt$ |
| $S=\frac{21}{bt^7}-\frac{4\cdot5\cdot7\cdot9}{b^3t^9}+\frac{3\cdot5\cdot7\cdot9\cdot11}{b^5t^{11}}$ | $S=\frac{15}{b}t^5-\frac{4\cdot3\cdot5\cdot7}{b^3}t^3+\frac{3\cdot5\cdot7\cdot9}{b^5}t.$ |

## SCHOLIUM 2

**966.** Forma integralis inventa modum suppeditat aequationem propositam

$$du+uudx+Ax^{\frac{-4i}{2i+1}}dx=0$$

in speciem simpliciorem transformandi. Primo enim ponatur

$$x^{\frac{2i}{2i+1}}u=v \quad \text{seu} \quad u=x^{\frac{-2i}{2i+1}}v$$

ac prodibit

$$x^{\frac{-2i}{2i+1}}dv-\frac{2i}{2i+1}x^{\frac{-4i-1}{2i+1}}vdx+x^{\frac{-4i}{2i+1}}vvdx+Ax^{\frac{-4i}{2i+1}}dx=0$$

seu

$$dv-\frac{2i}{2i+1}\cdot\frac{vdx}{x}+x^{\frac{-2i}{2i+1}}vvdx+Ax^{\frac{-2i}{2i+1}}dx=0$$

Ponatur porro $t=(2i+1)x^{\frac{1}{2i+1}}$; erit

$$dt=x^{\frac{-2i}{2i+1}}dx \quad \text{et} \quad \frac{dt}{t}=\frac{1}{2i+1}\cdot\frac{dx}{x},$$

unde fit

$$dv-\frac{2ivdt}{t}+vvdt+Adt=0\,.$$

Sit insuper $v=\frac{i}{t}+z$, ut prodeat

$$-\frac{idt}{tt}+dz-\frac{2iidt}{tt}-\frac{2izdt}{t}+\frac{iidt}{tt}+\frac{2izdt}{t}+zzdt+Adt=0$$

seu

$$dz+zzdt-\frac{i(i+1)dt}{tt}+Adt=0\,,$$

quae ergo, quoties *i* est numerus integer, est integrabilis.

Simili modo haec aequatio

$$du + uudx + Ax^n dx = 0$$

generalius ita transformari potest: Posito $u = x^\lambda v$ et $v = z - \frac{1}{2}\lambda x^{-\lambda-1}$ obtinetur

$$dz + x^\lambda zzdx + \tfrac{1}{4}\lambda(\lambda+2)x^{-\lambda-2}dx + Ax^{n-\lambda}dx = 0,$$

quae porro posito $x^\lambda dx = dt$ seu $x^{\lambda+1} = (\lambda+1)t$ abit in

$$dz + zzdt + \frac{\lambda(\lambda+2)dt}{4(\lambda+1)tt} + A(\lambda+1)^{\frac{n-2\lambda}{\lambda+1}} t^{\frac{n-2\lambda}{\lambda+1}} dt = 0$$

quae aequatio est integrabilis, quoties $n = \frac{-4i}{2i+1}$, unde numerum $\lambda$ pro lubitu assumendo innumerabiles formae exhiberi possunt. Si capiatur $\lambda = -1$, fit $t = lx$ et

$$dz + zzdt - \tfrac{1}{4}dt + Ae^{(n+2)t}dt = 0 .$$