CHAPTER V

CONCERNING THE INTEGRATION BY FACTORS OF DIFFERENTIAL EQUATIONS OF THE SECOND ORDER IN WHICH THE OTHER VARIABLE [*y*] DOES NOT EXCEED ONE DIMENSION

PROBLEM 107

865. *With the element dx assumed constant, if this equation is proposed :*

$$ddy + Adxdy + Bydx^2 = Xdx^2 ,$$

where X denotes some function of x, to find a function of x, by which on multiplication this equation is made integrable.

SOLUTION

There is put $dy = pdx$, so that a form of first order differential equation may be considered :

$$dp + Apdx + Bydx = Xdx ,$$

which multiplied by a certain function of *x*, *V* , is made integrable, clearly

$$Vdp + AVpdx + BVydx = VXdx$$

where since the last term *VXdx* shall be integrable, the same by necessity in the first terms comes about. But initially it has been observed that a part of this integral shall be *Vp*, thus there is put $Vp + S$, so that there becomes $Vp + S = \int VXdx$, and there is made

$$dS = -pdV + AVpdx + BVydx \text{ or } dS = dy\left(AV - \tfrac{dV}{dx}\right) + BVydx ,$$

which form can be rendered integrable on taking $V = e^{\lambda x}$; for there is made [a total derivative of *dS*]

$$dS = e^{\lambda x}\left((A-\lambda)dy + Bydx\right) \text{ and } S = (A-\lambda)e^{\lambda x}y ,$$

where λ thus must be taken, so that there is made $A\lambda - \lambda\lambda = B$ or

$$\lambda\lambda - A\lambda + B = 0 .$$

Then therefore there shall be

$$e^{\lambda x}p+\left(A-\lambda\right)e^{\lambda x}y=\int e^{\lambda x}Xdx \text{ or } dy+\left(A-\lambda\right)ydx=e^{-\lambda x}\int e^{\lambda x}Xdx,$$

which now multiplied by $e^{(A-\lambda)x}$ shall be integrable again and gives

$$e^{(A-\lambda)x}y=\int e^{(A-2\lambda)x}dx\int e^{\lambda x}Xdx.$$

Since λ shall be one root of the equation $\lambda\lambda-A\lambda+B=0$, if we put both the roots of this f and g, so that there shall be $\lambda=f$, there will be $A-\lambda=g$ and the equation of the integral [on integrating by parts]

$$e^{gx}y=\int e^{(g-f)x}dx\int e^{fx}Xdx$$

or

$$e^{gx}y=\tfrac{1}{g-f}e^{(g-f)x}\int e^{fx}Xdx-\tfrac{1}{g-f}\int e^{gx}Xdx$$

which changes into the form found above [§ 856]

$$y=\tfrac{1}{g-f}e^{-fx}\int e^{fx}Xdx-\tfrac{1}{g-f}e^{-gx}\int e^{gx}Xdx.$$

COROLLARY 1

866. Therefore the proposed equation or the equation arising from that

$$dp+Apdx+Bydx=Xdx$$

shall be integrable, it is multiplied by $e^{\lambda x}$ with $\lambda\lambda-A\lambda+B=0$ arising, and thus a twofold factor is considered, either e^{fx} or e^{gx}.

COROLLARY 2

867. Moreover with that multiplied by the factor e^{fx}, the integral of this will be

$$dy+gydx=e^{-fx}dx\int e^{fx}Xdx$$

and thus by integration it is reduced to a differential equation of the first order, which again is rendered integrable, if it is multiplied by e^{gx}.

SCHOLIUM

868. It is thus required to determine the multiplier *V*, so that the formula

$$dy\left(AV - \tfrac{dV}{dx}\right) + BVydx$$

shall be made integrable by itself. But then, since *V* is a function of *x*, the integral will be $y\left(AV - \tfrac{dV}{dx}\right)$, from which by necessity there becomes

$$AdV - \tfrac{ddV}{dx} = BVdx \text{ or } ddV - AdxdV + BVdx^2 = 0,$$

with the integration of which equation depending on finding the factor sought *V*. But it is sufficient for the particular integral of this to be taken ; provided in fact that the proposed equation is returned integrable, and an arbitrary constant is introduced into the integration to return the complete integral.

PROBLEM 108

869. *With the element dx assumed constant, if this equation is proposed :*

$$ddy + Pdydx + Qydx^2 = Xdx^2,$$

where P, Q and X are some functions of x, to find the multiplier V, which shall be a function of x, by which that equation is returned integrable.

SOLUTION

Because the equation of the integral arising is multiplied by *V*

$$Vddy + VPdydx + VQydx^2 = VXdx^2,$$

the integral of the first part is put equal to $Vdy + Sydx$; for it cannot have any other form and it is required to become

$$VPdydx + VQydx^2 = dydV + Sdydx + ydSdx;$$

where since *S* by necessity shall be a function of *x*, there will be

$$VPdx = dV + Sdx \text{ and } VQdx = dS.$$

Moreover from this there is $S = VP - \tfrac{dV}{dx}$, whereby the multiplier *V* must be defined from this equation

$$VQdx = VdP + PdV - \tfrac{ddV}{dx} \text{ or } ddV - PdVdx + Vdx\left(Qdx - dP\right) = 0\,;$$

which therefore if it can be resolved, or if rather some particular integral of this is known, so that the multiplier V is obtained, then the integral of the proposed equation will be

$$Vdy + y\left(VPdx - dV\right) = dx\int VXdx\,,$$

[The proposed equation gives : $Vddy + VPdydx + VQydx^2 = VXdx^2$; and since $VQdx = dS$ and $VPdx = dV + Sdx$
there is found $d.\left(VP - \tfrac{dV}{dx}\right) = dS$, then $VQ = d.\left(VP - \tfrac{dV}{dx}\right)$.
Hence $Vddy + VPdydx + yd.\left(VP - \tfrac{dV}{dx}\right)dx^2 = VXdx^2$. Hence on integrating,
$Vdy - \int dVdy + \int VPdydx + ydx\left(VP - \tfrac{dV}{dx}\right) - \int dxdy\left(VP - \tfrac{dV}{dx}\right) = dx\int VXdx\,,$
or, $Vdy + y\left(VPdx - dV\right) = dx\int VXdx$ as required.]
which again is rendered integrable, if taken by $\tfrac{1}{VV}e^{\int Pdx}$; indeed the integral will be obtained

$$\tfrac{y}{V}e^{\int Pdx} = \int \tfrac{dx}{VV}e^{\int Pdx}\int VXdx \text{ or } y = e^{-\int Pdx}V\int e^{\int Pdx}\tfrac{dx}{VV}\int VXdx\,,$$

from which with the double integral sign there is introduced twin arbitrary constants constituting the complete integral.

COROLLARY 1

870. Hence finding the multiplier V also depends on the resolution of a second order differential equation, but which is considered simpler than the original, because the function X is not involved, and the quantity V with its differentials dV and ddV have a single dimension everywhere.

COROLLARY 2

871. But if hence there is put $V = e^{\int vdx}$,
[in the equation $ddV - PdVdx + Vdx\left(Qdx - dP\right) = 0$]
then the quantity v will be determined by a differential equation of the first order

$$dv + vvdx - Pvdx + Qdx - dP = 0\,,$$

of which, if at least with a particular integral established, the integration of the proposed equation can be concluded.

COROLLARY 3

872. But with the multiplier *V* given, in turn an account of the proposed equation can be defined, so that the integration comes about in this way. For there shall be either

$$Q=\frac{dP}{dx}+\frac{PdV}{Vdx}-\frac{ddV}{Vdx^2} \text{ or } dP+\frac{PdV}{V}=Qdx+\frac{ddV}{Vdx},$$

or on integrating,

$$PV=\frac{dV}{dx}+\int QVdx \text{ or } P=\frac{dV}{Vdx}+\frac{\int QVdx}{V}$$

EXAMPLE 1

873. *To define the form of the second order differential equation*

$$ddy+Pdydx+Qydx^2=Xdx^2,$$

so that it comes out integrable on being multiplied by $e^{\lambda x}$.

On putting the multiplier $V=e^{\int vdx}=e^{\lambda x}$ there will be $v=\lambda$ and it is required to satisfy this equation :

$$\lambda\lambda dx-\lambda Pdx+Qdx-dP=0,$$

from which there arises

$$Q=\lambda P-\lambda\lambda+\frac{dP}{dx}.$$

Therefore initially this eventuates, if *P* and *Q* are constant, consider $P=A$ and $Q=B$, and then *A* is required to be defined from this equation : $\lambda\lambda-A\lambda+B=0$, which is the case treated above [§ 865]. Now besides whatever function *P* should be of x, provided that there shall be the equation $Q=\lambda P-\lambda\lambda+\frac{dP}{dx}$, the [proposed] equation taken by $e^{\lambda x}$ will become integrable, with the integral arising

$$e^{\lambda x}\left(dy+ydx\left(P-\lambda\right)\right)=dx\int e^{\lambda x}Xdx$$

or

$$dy+\left(P-\lambda\right)ydx=e^{-\lambda x}dx\int e^{\lambda x}Xdx,$$

which further multiplied by $e^{\int Pdx-\lambda x}$ and integrated gives

$$y=e^{-\int Pdx+\lambda x}\int e^{\int Pdx-2\lambda x}dx\int e^{\lambda x}Xdx.$$

COROLLARY

874. Let $P = A + \alpha x$ and $Q = B + \beta x$; there will be

$$B + \beta x = A\lambda + a\lambda x - \lambda\lambda + \alpha, \text{ therefore } B = A\lambda - \lambda\lambda + \alpha \text{ and } \beta = \alpha\lambda,$$

from which on account of $\lambda = \frac{\beta}{\alpha}$, it is required to prepare the coefficients A, B, α, β thus, so that there shall be

$$B\alpha\alpha = A\alpha\beta - \beta\beta + \alpha^3 \text{ or } Baa + \beta\beta = \alpha\left(A\beta + \alpha\alpha\right).$$

EXAMPLE 2

875. *To define the form of the second order differential equation*

$$ddy + Pdydx + Qydx^2 = Xdx^2,$$

so that it is made integrable on multiplying by $e^{\int vdx}$ *with* $v = \frac{\lambda}{x} + \mu x^n$ *present.*

Since there must be [by §871]

$$dv + vvdx - Pvdx + Qdx - dP = 0,$$

there will be

$$-\tfrac{\lambda}{xx} + \mu n x^{n-1} - \tfrac{\lambda P}{x} - \mu P x^n + \tfrac{\lambda\lambda}{xx} + 2\lambda\mu x^{n-1} + \mu\mu x^{2n} + Q - \tfrac{dP}{dx} = 0,$$

ergo

$$Q = \tfrac{\lambda(1-\lambda)}{xx} - \left(2\lambda + n\right)\mu x^{n-1} - \mu\mu x^{2n} + \tfrac{\lambda P}{x} + \mu P x^n + \tfrac{dP}{dx}$$

We can put $P = \frac{\alpha}{x} + \beta x^n$; there becomes

$$Q = \tfrac{1}{xx}\left(\lambda - \lambda\lambda + \alpha\lambda - \alpha\right) + x^{n-1}\left(\beta\lambda + \alpha\mu + \beta n - 2\lambda\mu - n\mu\right) + x^{2n}\left(\beta\mu - \mu\mu\right)$$

Let $Q = \frac{\gamma}{xx} + \delta x^{n-1} + \varepsilon x^{2n}$ there is required to become

$$\lambda\lambda - \left(\alpha + 1\right)\lambda + \alpha + \gamma = 0, \quad \beta\left(\lambda + n\right) + \mu\left(\alpha - 2\lambda - n\right) = \delta \text{ and } \mu\left(\beta - \mu\right) = \varepsilon \text{ [*]},$$

from which not only the letters of the multiplier λ et μ defined, but also a certain relation between the letters $\alpha, \beta, \gamma, \delta, \varepsilon$ is defined.

Just as if there shall be $\gamma = 0$ and $\delta = 0$, then $\left(\lambda - \alpha\right)\left(\lambda - 1\right) = 0$, from which $\lambda = \alpha$; then $\left(\beta - \mu\right)\left(\alpha + n\right) = 0$, hence $\alpha = \lambda = -n$ and $\mu\mu - \beta\mu + \varepsilon = 0$. Clearly the equation

$$ddy+dxdy\left(\beta x^{n}-\tfrac{n}{x}\right)+\varepsilon x^{2n}ydx^{2}=Xdx^{2}$$

receives the multiplier $e^{\int vdx}$ with $v=-\frac{n}{x}+\mu x^{n}$ arising on taking μ thus, so that $\mu\mu-\beta\mu+\varepsilon=0$. Hence the multiplier will be

$$V=\tfrac{1}{x^{n}}e^{\frac{\mu}{n+1}x^{n+1}}\quad\text{and}\quad e^{\int Pdx}=\tfrac{1}{x^{n}}e^{\frac{\beta}{n+1}x^{n+1}}$$

Whereby if we put $\frac{1}{n+1}x^{n+1}=t$, there will be [from the final formula for y in §869]

$$y=x^{n}e^{-\beta t}\tfrac{1}{x^{n}}e^{\mu t}\int e^{\beta t-2\mu t}x^{n}dx\int\tfrac{e^{\mu t}Xdx}{x^{n}}\quad\text{or}\quad y=e^{(\mu-\beta)t}\int e^{(\beta-2\mu)t}dt\int\tfrac{e^{\mu t}Xdx}{x^{n}}.$$

COROLLARY 1

876. If there is taken $\gamma=0$ and $\varepsilon=0$, there will be [from * above]

$$\mu=\beta,\ \beta(\alpha-\lambda)=\delta\ \text{ and }\ (\lambda-\alpha)(\lambda-1)=0,$$

hence $\lambda=1$ and $\delta=(\alpha-1)\beta$ and thus $P=\frac{\alpha}{x}+\beta x^{n}$, $Q=(\alpha-1)\beta x^{n-1}$ and the multiplier of the equation

$$ddy+\left(\tfrac{\alpha}{x}+\beta x^{n}\right)dxdy+(\alpha-1)\beta x^{n-1}ydx^{2}=Xdx^{2}$$

is $V=e^{\int vdx}$: with $v=\frac{1}{x}+\mu x^{n}$, thus so that there shall be

$$V=xe^{\frac{\beta}{n+1}x^{n+1}}\quad\text{and}\quad e^{\int Pdx}=x^{\alpha}e^{\frac{\beta}{n+1}x^{n+1}}.$$

COROLLARY 2

877. Hence in this case on putting $\frac{\mu}{n+1}x^{n+1}=t$, the integral

[$y=e^{-\int Pdx}V\int e^{\int Pdx}\frac{dx}{VV}\int VXdx$] will become

$$y=e^{1-\alpha}\int e^{\alpha-2}e^{-\beta t}dx\int e^{\beta t}Xxdx,$$

which is unable to show a simpler form, because hence in general the formula $e^{-\beta t}e^{\alpha-2}dx$ cannot be integrated.

SCHOLIUM

878. Therefore as with the discovery of the multipliers, which render integrable an equation of this kind

$$ddy + Pdxdy + Qydx^2 = Xdx^2,$$

the solution of this equation is required,

$$ddV - PdVdx + Vdx\left(Qdx - dP\right) = 0,$$

which is seen to be contained in this form [*i.e.* the solution is what we now call the complementary function, where the 'force term' in dynamical equations, or right hand side is zero.]

$$ddy + Pdxdy + Qydx^2 = 0,$$

just as also this form is required to be treated with multipliers. If the multiplier of this equation is put as *V*, a certain function of *x*, then in turn the preceding form

$$ddV - PdVdx + Vdx\left(Qdx - dP\right) = 0,$$

is come upon, and setting the multiplier of this = *U*, for a function of *x*, this may be defined by this equation :

$$ddU + PdUdx + QUdx^2 = 0$$

thus so that it is sufficient for either of these two equations to be resolved. And above indeed , where we have put $y = uv$, we have come upon the latter equation. But it is no wonder that the one of these two equations depends on the other, since the first arises from the second on putting $U = e^{-\int Pdx}V$, and indeed the latter from the former on putting $V = e^{\int Pdx}U$, as it will be clear which one is easy to attempt [in trying to find a solution].

Therefore a difficulty, if it should arise, may not be possible to removed, and has to be investigated in this manner, or perhaps a multiplier of this kind completes the work, which involves each variable *x* and *y* with its differentials *dx* and *dy* or $p = \frac{dy}{dx}$. But now it is easily seen with differentials excluded that this cannot happen; for if *V* should be the multiplier, a function of *x* and *y*, from the first term *ddy* the integral part *Vdy* arises, but with which differentiated on putting $dV = Mdx + Ndy$ there may be involved in the differential the term Ndy^2 not occurring in the equation, which also cannot be removed by the remaining parts of the integral. In as much as we try to solve the problem by multipliers of this kind, which also embrace the ratio of the differentials $p = \frac{dy}{dx}$, and since the dimension of *y* with its differentials shall be a number of the same dimension everywhere, then it is necessary that the same property also be present in the multiplier ; if indeed different parts should be present, the working of the individual parts are to be completed separately.

PROBLEM 109

879. *With the element dx assumed constant to define the conditions, so that a multiplier of this form* $Mp + Ny$ *with* $p = \frac{dy}{dx}$ *arising, and with both M and N functions of* x *may return this equation integrable*

$$ddy + Pdxdy + Qydx^2 = 0,$$

where P and Q are functions of x.

SOLUTION

On account of $dy = pdx$ our equation is

$$dp + Ppdx + Qydx = 0,$$

which multiplied by $Mp + Ny$ becomes

$$Mpdp + Nydp + MPpdy + NPydy + NQyydx + MQydy = 0,$$

which is required to be integrable. On account of the terms affected by the differential *dp* a part of the integral will be $\frac{1}{2}Mpp + Nyp$, from which the integral itself is put in place $= \frac{1}{2}Mpp + Nyp + S$; of which the differential, since that equation must be provided, we will obtain therefore

$$\begin{aligned} dS = {} & MPpdy + NPydy + NQyydx \\ & + MQydy \\ & -\tfrac{1}{2}ppdM - ypdN \\ & -Npdy \end{aligned}$$

as far as the formula is required to be integrable; which since only a differential of the first order *dx* and *dy* is included, it is necessary, that the quantity *p* be removed from the calculation. Hence on putting $dM = M'dx$ and $dN = N'dx$ on account of $pdx = dy$ the first term containing *p* must be reduced to zero, so that there shall be

$$MPpdy - \tfrac{1}{2}M'pdy - Npdy = 0$$

or

$$MP-\tfrac{1}{2}M'-N=0 \text{ or } N=MP-\tfrac{dM}{2dx}.$$

Then truly there shall be

$$dS=ydy\left(NP+MQ-N'\right)+NQyydx,$$

the integral of which formula is

[*i.e.* in modern terms, we have $dS=\frac{\partial S}{\partial x}dx+\frac{\partial S}{\partial y}dy$]

$$S=\tfrac{1}{2}yy\left(NP+MQ-N'\right) \text{ or } S=yy\int NQdx,$$

which two forms are required to agree, from which there becomes

$$NP+MQ-\tfrac{dN}{dx}=2\int NQdx$$

or

$$NdP+PdN+MdQ+QdM-\tfrac{ddN}{dx}-2NQdx=0, \text{ [*]}$$

which equation joined with that $N=MP-\frac{dM}{2dx}$ determines the conditions sought ; and then the equation of the integral is produced.

COROLLARY 1

880. If the functions P and Q are given and from these it is required to define M and N, on account of $N=MP-\frac{dM}{2dx}$ there will be $dN=MdP+PdM-\frac{ddM}{2dx}$ and the function M can be defined by this equation [on substituting these two equations into [*] above] :

$$\tfrac{d^3M}{dx^3}-\tfrac{3PddM}{2dx}+\left(PP-\tfrac{5dP}{dx}+2Q\right)dM+M\left(2PdP-\tfrac{ddP}{dx}-2PQdx+dQ\right)=0,$$

which on account of the equality supports a differential of the third order .

COROLLARY 2

881. But if the multiplier $Mp+Ny$ is given, the equation itself is defined thus, so that in the first place $P=\frac{N}{M}+\frac{dM}{2Mdx}$, from which from the other [*], which is

$$dQ+\frac{QdM}{M}-\frac{2NQdx}{M}=\frac{ddN}{Mdx}-\frac{d.PN}{M}$$

and this multiplied by $Me^{-2\int\frac{Ndx}{M}}$, gives the integral

$$MQe^{-2\int\frac{Ndx}{M}}=\int e^{-2\int\frac{Ndx}{M}}\left(\frac{ddN}{Mdx}-\frac{d.PN}{M}\right).$$

COROLLARY 3

882. Let this integral $=Z$ and there becomes

$$Z=e^{-2\int\frac{Ndx}{M}}\left(\frac{dN}{dx}-PN\right)+\int e^{-2\int\frac{Ndx}{M}}\left(\frac{2NdN}{M}-\frac{2PNNdx}{M}\right),$$

which with the value substituted for P the latter term changes into

$$\int e^{-2\int\frac{Ndx}{M}}\left(\frac{2NdN}{M}-\frac{2N^3dx}{MM}-\frac{NNdM}{MM}\right),$$

the integral of which is evidently $e^{-2\int\frac{Ndx}{M}}\frac{NN}{M}$, thus so that there becomes

$$Z=e^{-2\int\frac{Ndx}{M}}\left(\frac{dN}{dx}-\frac{NdM}{2Mdx}\right)+C$$

[for on substitution, $Z=e^{-2\int\frac{Ndx}{M}}\left(\frac{dN}{dx}-\left(\frac{N}{M}+\frac{dM}{2Mdx}\right)N\right)+e^{-2\int\frac{Ndx}{M}}\frac{NN}{M}$, etc.]
and thus

$$Q=\frac{C}{M}e^{2\int\frac{Ndx}{M}}+\frac{dN}{Mdx}-\frac{NdM}{2MMdx}.$$

COROLLARY 4

883. Hence for this proposed equation :
[on subst. for P and Q into the original differential eqn.]

$$\frac{ddy}{dx}+\left(\frac{N}{M}+\frac{dM}{2Mdx}\right)dy+\left(\frac{Cdx}{M}e^{2\int\frac{Ndx}{M}}+\frac{dN}{M}-\frac{NdM}{2MM}\right)=0$$

that on multiplying by $\frac{Mdy}{dx}+Ny$ shall become

$$\frac{Mdy^2}{2dx^2}+\frac{Nydy}{dx}+\frac{1}{2}yy\left(Ce^{2\int\frac{Ndx}{M}}+\frac{NN}{M}\right)=\text{Const.}$$

SCHOLIUM

884. Therefore since it is allowed to accept any functions of x for M and N, hence we have been able to produce innumerable forms of second order differential equations, which we are able to integrate with the aid of the multiplier $\frac{Mdy}{dx}+Ny$. Evidently the general form, which are rendered integrable by this multiplier, is as we have seen,

$$\frac{ddy}{dx}+\frac{dy}{2Mdx}(dM+2Ndx)+\frac{y}{2MM}\left(2MdN-NdM+2CMe^{2\int\frac{Ndx}{M}}dx\right)$$

with the integral itself arising

$$\frac{Mdy^2}{2dx^2}+\frac{Nydy}{dx}+\frac{1}{2}yy\left(\frac{NN}{M}+Ce^{2\int\frac{Ndx}{M}}\right),$$

where is evident that each exponential part involved with the constant C can be ignored, since that shall only be a predetermined property. But if we reduce the exponential part to an algebraic part on putting $e^{2\int\frac{Ndx}{M}}=L$, then there shall be

$$2\frac{Ndx}{M}=\frac{dL}{L} \quad \text{and} \quad N=\frac{MdL}{2Ldx}$$

and hence

$$dN=\frac{MddL}{2Ldx}+\frac{dLdM}{2Ldx}-\frac{MdL^2}{2LLdx},$$

from which the [above] form itself

$$\frac{ddy}{dx}+\frac{dy}{2dx}\left(\frac{dM}{M}+\frac{dL}{L}\right)+\frac{1}{2}y\left(\frac{ddL}{Ldx}+\frac{dLdM}{2LMdx}-\frac{dL^2}{LLdx}+\frac{2CLdx}{M}\right),$$

which multiplied by $\frac{Mdy}{dx}+\frac{MydL}{2Ldx}$ gives the integral

$$\frac{Mdy^2}{2dx^2}+\frac{MydLdy}{2Ldx^2}+\frac{1}{2}yy\left(\frac{MdL^2}{4LLdx^2}+CL\right).$$

Or if we put $\frac{dM}{M}+\frac{dL}{L}=\frac{2dK}{K}$, so that there shall be $M=\frac{KK}{L}$, our second order differential equation shall be

$$\frac{ddy}{dx}+\frac{dy}{dx}\frac{dK}{K}+\frac{1}{2}y\left(d.\frac{dL}{Ldx}-\frac{dL^2}{2LLdx}+\frac{dKdL}{KLdx}+\frac{2CLLdx}{KK}\right)=0,$$

which multiplied by $\frac{KK}{L}\left(\frac{dy}{dx}+\frac{ydL}{2Ldx}\right)$ will give the integral

$$\frac{KK}{2L}\left(\frac{dy^2}{dx^2}+\frac{ydLdy}{Ldx^2}+yy\left(\frac{dL^2}{4LLdx^2}+\frac{CLL}{KK}\right)\right)=\text{Const.}$$

EXAMPLE 1

885. Let $K=x^m(a+x)^n$ and $L=x^\mu(a+x)^\nu$; there shall be

$$\frac{dK}{Kdx}=\frac{m}{x}+\frac{n}{a+x}=\frac{ma+(m+n)x}{x(a+x)},\quad \frac{dL}{Ldx}=\frac{\mu}{x}+\frac{\nu}{a+x},$$

from the coefficient of $\frac{1}{2}ydx$ shall be

$$-\frac{\mu}{xx}-\frac{\nu}{(a+x)^2}-\frac{\mu\mu}{2xx}-\frac{\mu\nu}{x(a+x)}-\frac{\nu\nu}{2(a+x)^2}+\frac{m\mu}{xx}+\frac{m\nu+n\mu}{x(a+x)}+\frac{n\nu}{(a+x)^2}$$
$$+2Cx^{2\mu-2m}(a+x)^{2\nu-2n}$$

or

$$\frac{\mu(2m-\mu-2)}{2xx}+\frac{m\nu+n\mu-\mu\nu}{x(a+x)}+\frac{\nu(2n-\nu-2)}{2(a+x)^2}+2Cx^{2\mu-2m}(a+x)^{2\nu-2n},$$

where it is pleasing to note the following cases.

I. Let $m=\mu+1$ and $n=\nu$; the coefficient of $\frac{1}{2}ydx$ shall be

$$\frac{\mu\mu+4C}{2xx}+\frac{\nu(\mu+1)}{x(a+x)}+\frac{\nu(\nu-2)}{2(a+x)^2}.$$

Hence this equation

$$\frac{ddy}{dx}+dy\left(\frac{\mu+1}{x}+\frac{\nu}{a+x}\right)+\frac{1}{4}ydx\left(\frac{\mu\mu+4C}{xx}+\frac{2\nu(\mu+1)}{x(a+x)}+\frac{\nu(\nu-2)}{(a+x)^2}\right)=0$$

multiplied by

$$x^{\mu+2}(a+x)^\nu\left(\frac{dy}{dx}+\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)\frac{y}{2}\right)$$

gives the integral

$$\tfrac{1}{2}x^{\mu+2}(a+x)^{\nu}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)+\tfrac{1}{4}yy\left(\frac{\mu\mu+4C}{xx}+\frac{2\mu\nu}{x(a+x)}+\frac{\nu\nu}{(a+x)^2}\right)\right)=\text{Const.}$$

II. Let $m=\mu+\frac{1}{2}$ and $n=\nu+\frac{1}{2}$; the coefficient of $\frac{1}{2}ydx$ itself will be

$$\frac{\mu(\mu-1)}{2xx}+\frac{2\mu\nu+\mu+\nu+4C}{2x(a+x)}+\frac{\nu(\nu-1)}{2(a+x)^2}.$$

Hence the equation

$$\frac{ddy}{dx}+dy\left(\frac{2\mu+1}{x}+\frac{2\nu+1}{2(a+x)}\right)+\tfrac{1}{4}ydx\left(\frac{\mu(\mu-1)}{xx}+\frac{2\mu\nu+\mu+\nu+4C}{x(a+x)}+\frac{\nu(\nu-1)}{(a+x)^2}\right)=0$$

multiplied by

$$x^{\mu+1}(a+x)^{\nu+1}\left(\frac{dy}{dx}+\tfrac{1}{2}y\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)\right)$$

will give the integral

$$\tfrac{1}{2}x^{\mu+1}(a+x)^{\nu+1}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)+\tfrac{1}{4}yy\left(\frac{\mu\mu}{xx}+\frac{2\mu\nu+4C}{x(a+x)}+\frac{\nu\nu}{(a+x)^2}\right)\right)=\text{Const.}$$

III. Let $m=\mu$ and $n=\nu+1$; the coefficient of $\frac{1}{2}ydx$ itself will be

$$\frac{\mu(\mu-2)}{2xx}+\frac{\mu(\mu+1)}{x(a+x)}+\frac{\nu\nu+4C}{2(a+x)^2}.$$

Hence the equation

$$\frac{ddy}{dx}+dy\left(\frac{\mu}{x}+\frac{\nu+1}{a+x}\right)+\tfrac{1}{4}ydx\left(\frac{\mu(\mu-2)}{xx}+\frac{2\mu(\nu+1)}{x(a+x)}+\frac{\nu\nu+4C}{(a+x)^2}\right)=0$$

multiplied by

$$x^{\mu}(a+x)^{\nu+2}\left(\frac{dy}{dx}+\tfrac{1}{2}y\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)\right)$$

will give the integral

$$\tfrac{1}{2}x^{\mu}(a+x)^{\nu+2}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)+\tfrac{1}{4}yy\left(\frac{\mu\mu}{xx}+\frac{2\mu\nu}{x(a+x)}+\frac{\nu\nu+4C}{(a+x)^2}\right)\right)=\text{Const.}$$

COROLLARY 1

886. In the first case let $v=2$ and $C=-\frac{1}{4}$; this equation will be obtained :

$$\frac{ddy}{dx}+\frac{(\mu+1)a+(\mu+3)x}{x(a+x)}dy+\frac{(\mu+1)ydx}{x(a+x)}=0,$$

which multiplied by

$$x^{\mu+2}(a+x)^2\left(\frac{dy}{dx}+\frac{\mu a+(\mu+2)x}{2x(a+x)}y\right)$$

gives this integral

$$\frac{1}{2}x^{\mu+2}(a+x)^2\left(\frac{dy^2}{dx^2}+\frac{\mu a+(\mu+2)x}{x(a+x)}\cdot\frac{ydy}{dx}+yy\left(\frac{\mu}{x(a+x)}+\frac{1}{(a+x)^2}\right)\right)=\text{Const.}$$

COROLLARY 2

887. In the third case let $\mu=2$ and $C=-vv$; this equation itself will be considered :

$$\frac{ddy}{dx}+dy\frac{2a+(v+3)x}{x(a+x)}+\frac{(v+1)ydx}{x(a+x)}=0,$$

which multiplied by

$$xx(a+x)^{v+2}\left(\frac{dy}{dx}+\frac{1}{2}y\left(\frac{2}{x}+\frac{v}{a+x}\right)\right)$$

will give the integral

$$\frac{1}{2}xx(a+x)^{v+2}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{2}{x}+\frac{v}{a+x}\right)+yy\left(\frac{1}{xx}+\frac{v}{x(a+x)}\right)\right)=\text{Const.}$$

EXAMPLE 2

888. Let $K=x^m(aa+xx)^n$ and $L=x^\mu(aa+xx)^v$; there will be

$$\frac{dK}{Kdx}=\frac{m}{x}+\frac{2nx}{aa+xx}\quad\text{and}\quad\frac{dL}{Ldx}=\frac{\mu}{x}+\frac{2vx}{aa+xx}$$

and the second order differential equation adopts this form

$$\frac{ddy}{dx}+dy\left(\frac{m}{x}+\frac{2nx}{aa+xx}\right)+\frac{1}{2}ydx\left(\frac{\mu(2m-\mu-2)}{2xx}+\frac{2n\mu+2v(m-\mu+1)}{aa+xx}+\frac{2v(2n-v-2)xx}{(aa+xx)^2}+\frac{2Cx^{2\mu-2m}}{(aa+xx)^{2n-2m}}\right)=0$$

of which taken by

$$x^{2m-\mu}\left(aa+xx\right)^{2n-\nu}\left(\frac{dy}{dx}+\frac{1}{2}y\left(\frac{\mu}{x}+\frac{2\nu x}{aa+xx}\right)\right)$$

the integral will be

$$\frac{1}{2}x^{2m-\mu}\left(aa+xx\right)^{2n-\nu}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{\mu}{x}+\frac{2\nu x}{a+x}\right)+\frac{1}{4}yy\left(\left(\frac{\mu}{x}+\frac{2\nu x}{aa+xx}\right)^2+\frac{4Cx^{2\mu-2m}}{\left(aa+xx\right)^{2n-2\nu}}\right)\right)=\text{Const.}$$

Here we set out the case, in which the second order differential equation maintains this form :

$$\frac{ddy}{dx}+dy\left(\frac{m}{x}+\frac{2nx}{aa+xx}\right)+\frac{1}{2}ydx\left(D+\frac{E}{2xx}+\frac{F}{aa+xx}+\frac{Gxx}{\left(aa+xx\right)^2}\right)=0\,.$$

1. There may be taken $\mu=m$ and $\nu=n$, and there will be

$$D=2C,\ E=\tfrac{1}{2}m\left(m-2\right),\ F=2n\left(m+1\right)\ \text{and}\ G=2n\left(n-2\right).$$

II. There may be taken $\mu=m-1$ and $\nu=n$, and there will be

$$D=0,\ E=2C+\tfrac{1}{2}\left(m-1\right)^2,\ F=2n\left(m+1\right)\ \text{and}\ G=2n\left(n-2\right).$$

III. There may be taken $\mu=m-1$ et $2n-2\nu=-1$ seu $\nu=n+\frac{1}{2}$; and the final term will be

$$\frac{2C\left(aa+xx\right)}{xx}=2C+\frac{2Caa}{xx}\,.$$

Hence

$$D=2C,\ E=2Caa+\tfrac{1}{2}\left(m-1\right)^2,\ F=2\left(mn+n+1\right),\ G=\tfrac{1}{2}\left(2n+1\right)\left(2n-5\right).$$

IV. There may be taken $\mu=m$ et $2n-2\nu=1$ seu $\nu=n-\frac{1}{2}$; and the final term will be

$$\frac{2C}{aa+xx}\,,$$

and thus

$$D=0,\ E=\tfrac{1}{2}m\left(m-2\right),\ F=2C+2mn+2n-1,\ G=\tfrac{1}{2}\left(2n-1\right)\left(2n-3\right).$$

V. There may be taken $\mu=m+1$ et $\nu=n-\frac{1}{2}$; and the final term will be

$$\frac{2Cxx}{aa+xx}=2C-\frac{2Caa}{aa+xx}$$

and thus

$$D=2C,\ E=\tfrac{1}{2}(m+1)(m-3),\ F=-2Caa+2n(m+1),\ G=\tfrac{1}{2}(2n-1)(2n-3).$$

VI. Let $\mu=m-1$ et $v=n-\frac{1}{2}$; the final term will be

$$\frac{2C}{xx(aa+xx)}=\frac{2C}{aaxx}-\frac{2C}{aa(aa+xx)},$$

from which there becomes

$$D=0,\ E=\tfrac{2C}{aa}+\tfrac{1}{2}(m-1)^2,\ F=\tfrac{-2C}{aa}+2mn+2n-2,\ G=\tfrac{1}{2}(2n-1)(2n-3).$$

VII. Let $\mu=m+1$ and $2n-2v=2$ or $v=n-1$; the final term will be

$$\frac{2Cxx}{(aa+xx)^2}.$$

and thus

$$D=0,\ E=\tfrac{1}{2}(m+1)(m-3),\ F=2n(m+1),\ G=2C+2(n-1)^2.$$

VIII. Let $\mu=m+2$ and $v=n-1$; the final term will be

$$\frac{2Cx^4}{(aa+xx)^2}=2C-\frac{2Caa}{aa+xx}-\frac{2Caaxx}{(aa+xx)^2},$$

from which there becomes

$$D=2C,\ E=\tfrac{1}{2}(m+2)(m-4),\ F=-2Caa+2mn+2n+2,\ G=-2Caa+2(n-1)^2.$$

IX. Let $\mu=m$ and $v=n-1$; the final term will be

$$\frac{2C}{(aa+xx)^2}=\frac{2Caa}{aa(aa+xx)}-\frac{2Cxx}{aa(aa+xx)^2},$$

and hence

$$D=0,\ E=\tfrac{1}{2}m(m-2),\ F=\tfrac{2C}{aa}+2mn+2n-2,\ G=\tfrac{-2C}{aa}+2(n-1)^2.$$

X. Let $\mu=m-1$ et $v=n-1$; and there will be the final term

$$\frac{2C}{xx(aa+xx)^2}=\frac{2C}{a^4xx}-\frac{4C}{a^4(aa+xx)}+\frac{2Cxx}{a^4(aa+xx)^2},$$

and hence

$$D=0,\ E=\tfrac{2C}{a^4}+\tfrac{1}{2}(m-1)^2,\ F=\tfrac{-4C}{a^4}+2mn+2n-4,\ \ G=\tfrac{2C}{a^4}+2(n-1)^2.$$

PROBLEM 110

889. *With the element dx assumed constant, if K and L denote some functions of x, to find the complete integral of this second order differential equation*

$$\frac{ddy}{dx}+\frac{dy}{dx}\cdot\frac{dK}{K}+\frac{1}{2}y\left(d.\frac{dL}{Ldx}-\frac{dL^2}{2LLdx}+\frac{dKdL}{KLdx}+\frac{2CLLdx}{KK}\right)=0.$$

SOLUTION

Because this equation is rendered integrable, if it should be multiplied by

$$\frac{KK}{L}\left(\frac{dy}{dx}+\frac{ydL}{2Ldx}\right)$$

the complete integral of this, as we have seen above [§ 884:], is

$$\frac{KK}{2L}\left(\left(\frac{dy}{dx}+\frac{ydL}{2Ldx}\right)^2+\frac{CLL}{KK}yy\right)=\text{Const.},$$

as a differential equation of the first order is required to be integrated at this stage. Which shall be especially difficult on account of the indefinite constant, with that ignored in the first place at any rate we may investigate the particular integral. Hence there will be from the equation :

$$\left(\frac{dy}{dx}+\frac{ydL}{2Ldx}\right)^2+\frac{CLL}{KK}yy=0$$

on extracting the root

$$\frac{dy}{dx}+\frac{ydL}{2Ldx}=\frac{Ly}{K}\sqrt{-C}\quad\text{or}\quad\frac{dy}{y}+\frac{dL}{2L}=\frac{Ldx}{K}\sqrt{-C},$$

from which there becomes

$$y\sqrt{L}=\alpha e^{\int\frac{Ldx}{K}\sqrt{-C}}$$

Therefore since these two values satisfy the proposed second order differential equation

$$y=\frac{\alpha}{\sqrt{L}}e^{\int\frac{Ldx}{K}\sqrt{-C}}\quad\text{and}\quad y=\frac{\beta}{\sqrt{L}}e^{-\int\frac{Ldx}{K}\sqrt{-C}},$$

the two together also will satisfy the equation ; from which since two arbitrary constants are introduced, the complete integral of this will be

$$y=\frac{\alpha}{\sqrt{L}}e^{\int\frac{Ldx}{K}\sqrt{-C}}+\frac{\beta}{\sqrt{L}}e^{-\int\frac{Ldx}{K}\sqrt{-C}},$$

which expression shall prevail, if the quantity $\sqrt{-C}$ were real ; but if it were imaginary, then it becomes

$$y=\frac{\gamma}{\sqrt{L}}\sin.\left(\int\frac{Ldx}{K}\sqrt{C}+\zeta\right)$$

and thus the complete integral of the proposed second order differential equation shall be complete.

COROLLARY 1

890. Hence therefore we prevail to assign the integral of the first order differential equation

$$\left(\frac{dy}{dx}+\frac{ydL}{2Ldx}\right)^2+\frac{CLLy^2}{KK}=\frac{AL}{KK},$$

which by itself is difficult enough, which is

$$y=\frac{\alpha}{\sqrt{L}}e^{\int\frac{Ldx}{K}\sqrt{-C}}+\frac{\beta}{\sqrt{L}}e^{-\int\frac{Ldx}{K}\sqrt{-C}},$$

but only if there should be a relation between the constants α and β that may be defined with respect to the constant A.

COROLLARY 2

891. But it will become, on multiplying by $\sqrt{L}$ and on differentiating

$$dy\sqrt{L}+\frac{ydL}{2\sqrt{L}}=\frac{\alpha Ldx}{K}\sqrt{-C}\cdot e^{\int\frac{Ldx}{K}\sqrt{-C}}-\frac{\beta Ldx}{K}\sqrt{-C}\cdot e^{-\int\frac{Ldx}{K}\sqrt{-C}}$$

hence

$$\frac{dy}{dx}+\frac{ydL}{2Ldx}=\frac{\sqrt{-CL}}{K}\cdot\left(\alpha e^{\int\frac{Ldx}{K}\sqrt{-C}}-\beta e^{-\int\frac{Ldx}{K}\sqrt{-C}}\right),$$

from which there becomes [from §890]

$$\frac{AL}{KK}=\frac{-CL}{KK}\left(\alpha e^{\int\frac{Ldx}{K}\sqrt{-C}}-\beta e^{-\int\frac{Ldx}{K}\sqrt{-C}}\right)^2+\frac{CL}{KK}\left(\alpha e^{\int\frac{Ldx}{K}\sqrt{-C}}+\beta e^{-\int\frac{Ldx}{K}\sqrt{-C}}\right)^2$$

and thus $A=4C\alpha\beta$ or $\beta=\frac{A}{4C\alpha}$.

SCHOLIUM 1

892. Therefore since the proposed equation is allowed to be integrated with the help of a suitable multiplier, still the other integration will be seen to be oppressed by the greatest difficulties. Yet meanwhile with the help of a substitution that differential equation of the first order is readily reduced on being treated ; for on putting $y=\frac{z}{\sqrt{L}}$ so that there becomes $dy\sqrt{L}+\frac{ydL}{2\sqrt{L}}=dz$, there arises

$$\left(\frac{dz}{dx\sqrt{L}}\right)^2+\frac{CLzz}{KK}=\frac{AL}{KK},$$

hence $\frac{dz}{dx}=\frac{L}{K}\sqrt{(A-Czz)}$ or $\frac{dz}{\sqrt{(A-Czz)}}=\frac{Ldx}{K}$, which gives the integral

$$l\left(z\sqrt{-C}+\sqrt{(A-Czz)}\right)=\int\frac{Ldx}{K}\sqrt{-C}+lB,$$

from which the preceding integral is elicited.

The remaining form of our second order differential equation can be shown a little easier in this way. If P et R shall be some functions of x and there is assumed the constant element dx, the complete integral of this equation

$$ddy-dy\left(\frac{dP}{P}+\frac{dR}{R}\right)-y\left(d.\frac{dP}{P}-\frac{dPdR}{PR}+\frac{aaRRdx^2}{PP}\right)=0$$

twice integrated is

$$y=\alpha Pe^{a\int\frac{Rdx}{P}}+\beta\alpha Pe^{-a\int\frac{Rdx}{P}},$$

if indeed a is a real quantity. But if $a=0$, then there shall be

$$y=P\left(\alpha+\beta\int\frac{Rdx}{P}\right)$$

but if there shall be $aa=-cc$, then

$$y=\alpha P\sin.\left(\beta+c\int\frac{Rdx}{P}\right).$$

Then truly that equation is rendered integrable, if it should be multiplied by

$$\frac{1}{RRdx^2}\left(dy-\frac{ydP}{P}\right),$$

and the first integral will be

$$\frac{1}{RRdx^2}\left(\left(dy-\frac{ydP}{P}\right)^2-\frac{aaRR}{PP}y^2dx^2\right)=\text{Const.}$$

Hence it is apparent that the substitution $y=Ps$ may be used conveniently in that second order differential equation, by which that is transformed into

$$ddz+dz\left(\frac{dP}{P}-\frac{dR}{R}\right)-\frac{aaRR}{PP}zdx^2=0,$$

which multiplied by $\frac{PPdz}{RRdx^2}$ shall be made integrable at once. Since also on putting $\frac{P}{R}=S$, so that there is given

$$ddz+\frac{dSdz}{S}-\frac{aazdx^2}{SS}=0,$$

the multiplier $\frac{SSdz}{dx^2}$ gives at once the integral

$$\frac{SSdz^2}{2dx^2}-\frac{1}{2}aazz=\text{Const.}$$

SCHOLIUM 2

893. Hence in turn from this most simple form

$$SSddz+SdSdz-aazdx^2=0,$$

which multiplied by dz is rendered integrable, we may be able to derive the more complicated forms on putting $z=\frac{y}{P}$ and $S=\frac{P}{R}$. Which in whatever general forms have been evident enough, yet in most examples shown this derivation has been hidden exceedingly well, as much as they are able to come to mind.

Just as in the cases set out in § 888 if we assume no. IX $m=2$ and $C=(n-1)^2aa$, there becomes $D=0, E=0, F=2n(n+1)$ and $G=0$, from which there may be considered this equation

$$\frac{ddy}{dx}+2dy\left(\frac{1}{x}+\frac{nx}{aa+xx}\right)+\frac{n(n+1)ydx}{aa+xx}=0$$

or

$$ddy+\frac{2dxdy(aa+(n+1)xx)}{x(aa+xx)}+\frac{n(n+1)ydx^2}{aa+xx}=0$$

which is rendered integrable with the aid of the multiplier

$$xx(aa+xx)^{n+1}\left(\frac{dy}{dx}+\frac{y(aa+nxx)}{x(aa+xx)}\right)$$

with the integral arising

$$\tfrac{1}{2}xx\left(aa+xx\right)^{n+1}\left(\left(\frac{dy}{dx}+\frac{y\left(aa+nxx\right)}{x\left(aa+xx\right)}\right)^{2}+\frac{\left(n-1\right)^{2}aayy}{\left(aa+xx\right)^{2}}\right)=\text{Const.}$$

Hence for the particular integral there shall be

$$\frac{dy}{y}+\frac{dx}{x}+\frac{(n-1)xdx}{aa+xx}=\pm\frac{(n-1)adx\sqrt{-1}}{aa+xx},$$

from which it is deduced

$$xy\left(aa+xx\right)^{\frac{n-1}{2}}=\alpha\left(\frac{a+x\sqrt{-1}}{a-x\sqrt{-1}}\right)^{\pm\frac{n-1}{2}}$$

Hence the twofold integrals joined together give the complete integral

$$y=\tfrac{\alpha}{x}\left(a-x\sqrt{-1}\right)^{-n+1}+\tfrac{\beta}{x}\left(a+x\sqrt{-1}\right)^{-n+1}.$$

But in this case our equation may be reduced to the simplest form with the help of the substitution $y=\frac{z}{x}\left(aa+xx\right)^{\frac{1-n}{2}}$, the account and finding of which is observed with more difficulty.

PROBLEM 111

894. *With the element dx assumed constant to investigate the conditions, by which the second order differential equation*

$$ddy+Pdxdy+Qydx^{2}=0$$

is rendered integrable with the help of a multiplier of this form

$$\frac{ydx^{2}}{Ldy^{2}+Mydydx+Nyydx^{2}},$$

with the letters L, M, N, P, Q denoting functions of x.

SOLUTION

There may be attributed to the denominator of this fraction such a form :

$$\left(dy+Rydx\right)\left(dy+Sydx\right)$$

and with a little attention summoned it is apparent that an integral of this form is to be considered

$$V + l\frac{dy+Rydx}{dy+Sydx} = \text{Const.},$$

of which hence there must be produced the differential equation proposed. But differentiation gives

$$dV + \frac{(S-R)ydxddy+(R-S)dxdy^2+ydxdy(dR-dS)+yydx^2(SdR-RdS)}{(dy+Rydx)(dy+Sydx)} = 0,$$

which reduced to the common denominator becomes

$$\begin{aligned}&(S-R)ydxddy+(R-S)dxdy^2+ydxdy(dR-dS)\\ &+yydx^2(SdR-RdS)+dVdy^2+(R+S)ydxdydV+RSyydx^2dV = 0.\end{aligned}$$

There is put in place $dV = (S-R)dx$, in order that the equation becomes divisible by *y*, and thus this equation may arise :

$$\begin{aligned}&(S-R)ddy+dy(dR-dS)\\ &+ydx(SdR-RdS)+(SS-RR)dxdy+RS(S-R)ydx^2 = 0;\end{aligned}$$

which so that it may agree with the proposed form, there is required to become

$$P = (R+S)+\frac{dR-dS}{(S-R)dx} \quad \text{and} \quad Q = RS+\frac{SdR-RdS}{(S-R)dx};$$

which values if the functions *P* and *Q* should be considered, the equation

$$ddy + Pdxdy + Qydx^2 = 0$$

multiplied by

$$\frac{(S-R)ydx}{(dy+Rydx)(dy+Sydx)}$$

will give the integral

$$\int(S-R)dx + l\frac{dy+Rydx}{dy+Syd} = \text{Const.}$$

If we put $S = M+N$ and $R = M-N$, there will be

$$P = 2M - \frac{dN}{Ndx} \text{ and } Q = MM - NN + \frac{dM}{dx} - \frac{MdN}{Ndx}.$$

COROLLARY 1

895. Hence whatever the functions of x are assumed in place of M and N from these there are defined

$$P = 2M - \frac{dN}{Ndx} \text{ and } Q = MM - NN + \frac{dM}{dx} - \frac{MdN}{Ndx},$$

of which equation

$$ddy + Pdxdy + Qydx^2 = 0,$$

and the integral will be

$$2\int Ndx + l\frac{dy+(M-N)ydx}{dy+(M+N)ydx} = \text{Const.}$$

COROLLARY 2

896. If there is put $y = e^{\int zdx}$, our differential equation of the first order becomes

$$dz + zzdx + Pzdx + Qdx = 0,$$

therefore the integral of this will be

$$2\int Ndx + l\frac{z+M-N}{z+M+N} = \text{Const.}$$

COROLLARY 3

897. If we wish, so that there shall be $P = 0$ and an equation of this kind is considered

$$ddy + Qydx^2 = 0,$$

there must be taken $2M = Ndx$ and there shall be $Q = \frac{dM}{dx} - MM - NN$, and the equation of this integral will be

$$2\int Ndx + l\frac{dy+(M-N)ydx}{dy+(M+N)ydx} = \text{Const.}$$

COROLLARY 4

898. But in general, according as the constant is taken as either $+\infty$ or $-\infty$, there will be obtained this particular integral, either

$$dy + (M+N)ydx = 0 \text{ or } dy + (M-N)ydx = 0,$$

from which there will be

$$y = \alpha e^{-\int(M+N)dx} \text{ or } y = \beta e^{-\int(M-N)dx}$$

from which there is deduced the complete integral of our equation

$$y = e^{-\int Mdx}\left(\alpha e^{-\int Ndx} + \beta e^{\int Ndx}\right).$$

EXAMPLE 1

899. Let $M = \alpha$ and $N = \beta$; there will be $P = 2\alpha$ and $Q = \alpha\alpha - \beta\beta$, from which the integral of this equation

$$ddy + 2\alpha dxdy + \left(\alpha\alpha - \beta\beta\right) ydx^2 = 0$$

will be

$$2\beta x + l\frac{dy+(\alpha-\beta)ydx}{dy+(\alpha+\beta)ydx} = \text{Const.}$$

But in finite quantities the complete integral is

$$y = e^{-\alpha x}\left(Ae^{-\beta x} + Be^{\beta x}\right).$$

But in the case, in which $\beta\beta = -\gamma\gamma$, the equation

$$ddy + 2\alpha dxdy + \left(\alpha\alpha + \gamma\gamma\right) ydx^2 = 0$$

twice integrated gives

$$y = e^{-\alpha x}\sin.\left(\gamma x + C\right);$$

but if $\gamma = 0$, the integral of the equation

$$ddy + 2\alpha dxdy + \alpha\alpha ydx^2 = 0$$

is

$$y = e^{-\alpha x}\left(A + Bx\right).$$

EXAMPLE 2

900. If $M = \frac{\alpha}{x}$ and $N = \beta x^n$, there will be

$$P = \frac{2\alpha - n}{x} \text{ and } Q = \frac{\alpha\alpha}{xx} - \beta\beta x^{2n} - \frac{\alpha}{xx} - \frac{\alpha n}{xx} = \frac{\alpha(\alpha-n-1)}{xx} - \beta\beta x^{2n}.$$

Hence the first integral of this equation

$$ddy + \frac{(2\alpha - n)dxdy}{x} + \frac{\alpha(\alpha-n-1)ydx^2}{xx} - \beta\beta x^{2n} ydx^2 = 0$$

is

$$\frac{2\beta}{n+1}x^{n+1} + l\frac{xdy+\left(\alpha-\beta x^{n+1}\right)ydx}{xdy+\left(\alpha+\beta x^{n+1}\right)ydx} = \text{Const.},$$

moreover the second integral

$$y = x^{-\alpha}\left(Ae^{\frac{-\beta x^{n+1}}{n+1}} + Be^{\frac{\beta x^{n+1}}{n+1}}\right);$$

if $\beta = 0$, that will be

$$y = x^{-\alpha}\left(A + Bx^{n+1}\right),$$

but if $\beta\beta = -\gamma\gamma$, then it will be

$$y = Ax^{-\alpha}\sin.\left(\tfrac{\gamma}{n+1}x^{n+1} + C\right).$$

COROLLARY 1

901. On taking $n = 2\alpha$, so that this equation may be considered

$$ddy - \frac{\alpha(\alpha+1)ydx^2}{xx} - \beta\beta x^{4\alpha}ydx^2 = 0,$$

the complete integral of this will be

$$y = x^{-\alpha}\left(Ae^{\frac{-\beta x^{2\alpha+1}}{2\alpha+1}} + Be^{\frac{\beta x^{2\alpha+1}}{2\alpha+1}}\right);$$

if there shall be $\beta = 0$, then that will be

$$y = x^{-\alpha}\left(A + Bx^{2\alpha+1}\right),$$

but if $\beta\beta = -\gamma\gamma$, there will be this integral

$$y = Ax^{-\alpha}\sin.\left(\tfrac{\gamma}{2\alpha+1}x^{2\alpha+1} + C\right).$$

COROLLARY 2

902. We may put $\alpha = -1$, so that we may have this equation

$$ddy - \frac{\beta\beta ydx^2}{x^4} = 0,$$

hence the integral of which will be

$$y = x\left(Ae^{\frac{\beta}{x}} + Be^{-\frac{\beta}{x}}\right),$$

where it is to be noted, if there shall be $\beta\beta = -\gamma\gamma$, the integral becomes

$$y = Ax\sin.\left(\frac{\gamma}{x} + C\right).$$

EXAMPLE 3

903. There may be put

$$N = \frac{Ax^m}{\alpha + \beta x^n},$$

so that there shall be $\frac{dN}{Ndx} = \frac{m}{x} - \frac{\beta n x^{n-1}}{\alpha + \beta x^n}$, and taking

$$M = \frac{m}{2x} - \frac{\beta n x^{n-1}}{2\left(\alpha + \beta x^n\right)},$$

so that there becomes [§ 895] $P = 0$ and

$$Q = \frac{m}{2xx} - \frac{\beta n(n-1)x^{n-2}}{2\left(\alpha + \beta x^n\right)} + \frac{\beta\beta nnx^{2n-2}}{2\left(\alpha + \beta x^n\right)^2} - \frac{m}{4xx} + \frac{\beta mnx^{n-2}}{2\left(\alpha + \beta x^n\right)} - \frac{\beta\beta nnx^{2n-2}}{4\left(\alpha + \beta x^n\right)^2} - \frac{AAx^{2m}}{\left(\alpha + \beta x^n\right)^2}$$

or

$$Q = -\frac{m(m+2)}{4xx} + \frac{n(m-n+1)\beta x^{n-2}}{2\left(\alpha + \beta x^n\right)} + \frac{\beta\beta nnx^{2n-2} - 4AAx^{2m}}{4\left(\alpha + \beta x^n\right)^2},$$

and on account of $\int Mdx = \frac{1}{2}lN$ there will be the integral

$$y = \frac{1}{\sqrt{N}}\left(Ce^{-\int Ndz} + De^{\int Ndz}\right)$$

and of this equation $ddy + Qydx^2 = 0$.

In order that the expression of Q becomes simpler, this can be done in many ways, while the numerator of the latter part is rendered divisible by $a + \beta x^n$.

I. Let $m = n - 1$ and $AA = \frac{1}{4}\beta\beta nn$ and here shall be $Q = -\frac{nn-1}{4xx}$, then indeed

$$N = \frac{\frac{1}{2}\beta nx^{n-1}}{\alpha + \beta x} \quad \text{and} \quad \int Ndx = \frac{1}{2}l\left(\alpha + \beta x\right),$$

from which the integral of the equation $ddy - \frac{(nn-1)ydx^2}{4xx} = 0$ is

$$y = \frac{1}{\sqrt{x^{n-1}}}\left(C + D\alpha + D\beta x^n\right)$$

II. Let $2m = -2$ or $m = -1$ and $4AA = \alpha\alpha nn$; there will be

$$Q=\frac{1}{4xx}-\frac{nn\beta x^{n-2}}{2\left(\alpha+\beta x^{n}\right)}+\frac{\beta nnx^{n-2}-\alpha nnx^{-2}}{4\left(\alpha+\beta x^{n}\right)},$$

or $Q=\frac{1-nn}{4xx}$ as before.

III. Let $2m=-n-2$ or $m=\frac{-n-2}{2}$ and $4AA=-\frac{\alpha^{3}nn}{\beta}$ and there is made

$$Q=-\frac{nn-4}{16xx}-\frac{3nn\beta x^{n-2}}{4\left(\alpha+\beta x^{n}\right)}+\frac{nn\left(\beta\beta x^{n-2}-\alpha\beta x^{-2}+\alpha\alpha x^{-n-2}\right)}{4\beta\left(\alpha+\beta x^{n}\right)}$$

or

$$Q=\frac{4-nn}{16xx}+\frac{nn\left(\alpha\alpha x^{-n-2}-\alpha\beta x^{-2}-2\beta\beta x^{n-2}\right)}{4\beta\left(\alpha+\beta x^{n}\right)},$$

which expression becomes

$$Q=\frac{4-nn}{16xx}+\frac{nn}{4\beta}\left(\alpha x^{-n-2}-2\beta x^{-2}\right)=\frac{4-9nn}{16xx}+\frac{nn\alpha}{4\beta x^{n+2}}.$$

Whereby, since there shall be

$$N=\frac{n\alpha\sqrt{-\alpha}}{2\sqrt{\beta}}\cdot\frac{x^{-\frac{n+1}{2}}}{\alpha+\beta x^{n}},$$

then there shall be

$$\int Ndx=\frac{n\alpha\sqrt{-\alpha}}{2\sqrt{\beta}}\cdot\int\frac{dx}{\left(\alpha+\beta x^{n}\right)x^{\frac{n+2}{2}}}.$$

There may be taken $n=\frac{2}{3}$, so that there becomes $m=-\frac{4}{3}$ and both $Q=\frac{\alpha}{9\beta x^{\frac{8}{3}}}$ and

$$N=\frac{n\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\cdot\frac{x^{-\frac{4}{3}}}{\alpha+\beta x^{\frac{2}{3}}},\text{ hence }\int Ndx=\frac{\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\cdot\int\frac{dx}{\left(\alpha+\beta x^{\frac{2}{3}}\right)x^{\frac{4}{3}}},$$

and thus the integral of the equation $ddy+\frac{\alpha}{9\beta x^{\frac{8}{3}}}ydx^{2}=0$ will be

$$y=\frac{1}{\sqrt{N}}\left(Ce^{-\int Ndz}+De^{\int Ndz}\right).$$

But if there is taken $n=-\frac{2}{3}$ so that there becomes $m=-\frac{2}{3}$ and $Q=\frac{\alpha}{9\beta x^{\frac{4}{3}}}$, there will be

$$N=\frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\cdot\frac{x^{-\frac{2}{3}}}{\alpha+\beta x^{-\frac{2}{3}}}\text{ and }\int Ndx=\frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\int\frac{x^{\frac{1}{3}}dx}{\alpha x+\beta x^{\frac{1}{3}}}=\frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\int\frac{dx}{\alpha x^{\frac{2}{3}}+\beta},$$

from which the equation $ddy + \frac{\alpha ydx^2}{9\beta x^{\frac{4}{3}}} = 0$ is integrated in a like manner.

SCHOLIUM 1

904. Hence the equation $ddy + Ax^m ydx^2 = 0$ has been allowed to be integrated in these cases $m = 0$, $m = -4$, $m = -\frac{4}{3}$, $m = -\frac{8}{3}$ and $m = -2$ or $m = -2 \pm \frac{2}{1}$, $m = -2 \pm \frac{2}{3}$. But if further we put $N = \frac{Ax^\lambda}{\alpha + \beta x^n + \gamma x^{2n}}$, in a similar manner we will obtain the integration of that case of this equation $m = -2 \pm \frac{2}{5}$, in which also the differential equation of the first order is allowed to be integrated. But the investigation of these cases of the integration is exceedingly laborious, as we pursue these more fully, since below especially [Cap. VII, Probl. 118, starting from § 943] a method occurs for establishing all these more conveniently.

SCHOLIUM 2

905. From these it is allowed to gather by how much we can expect to enjoy the proceeds from the discovery of multipliers, by which also second order differential equations are rendered integrable, even if the examples treated here only refer lightly to a specimen of this method. But anyhow, there is no doubt that some forms of the multipliers I have contemplated here, are able to call upon many other forms in like succession to be used. Again in this chapter we have treated only second order differential equations, in which the other variable *y* with its differentials *dy* and *ddy* maintain a single dimension everywhere. Now the same method too can be extended generally to other equations of this kind, which although little has been developed at this stage, yet the following application will not be without the use of this method, where the integration of other differential equations of the second order, which with the hardest considered being treated by other methods, will be taught with the aid of multipliers.

CAPUT V

DE INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM SECUNDI GRADUS IN QUIBUS ALTERA VARIABILIS UNAM DIMENSIONEM NON SUPERAT PER FACTORES

PROBLEMA 107

865. *Sumto elemento dx constante si proponatur haec aequatio*

$$ddy + Adxdy + Bydx^2 = Xdx^2,$$

ubi X denotat functionem quamcunque ipsius x, invenire functionem ipsius x, per quam haec aequatio multiplicata flat integrabilis.

SOLUTIO

Ponatur $dy = pdx$, ut habeatur forma differentialis primi gradus

$$dp + Apdx + Bydx = Xdx,$$

quae multiplicata per V functionem quandam ipsius x, fiat integrabilis, scilicet

$$Vdp + AVpdx + BVydx = VXdx$$

ubi cum posterius membrum $VXdx$ sit integrabile, idem in priori eveniat necesse est. At primo perspicuum est eius integralis partem fore Vp, unde id ponatur $Vp + S$, ut sit $Vp + S = \int VXdx$, fietque

$$dS = -pdV + AVpdx + BVydx \text{ seu } dS = dy\left(AV - \tfrac{dV}{dx}\right) + BVydx,$$

quae forma integrabilis reddi potest sumendo $V = e^{\lambda x}$; erit enim

$$dS = e^{\lambda x}\left((A-\lambda)dy + Bydx\right) \text{ et } S = (A-\lambda)e^{\lambda x}y,$$

ubi λ ita debet accipi, ut fiat $A\lambda - \lambda\lambda = B$ seu

$$\lambda\lambda - A\lambda + B = 0 .$$

Tum ergo erit

$$e^{\lambda x}p + (A-\lambda)e^{\lambda x}y = \int e^{\lambda x}Xdx \text{ seu } dy + (A-\lambda)ydx = e^{-\lambda x}\int e^{\lambda x}Xdx ,$$

quae iam per $e^{(A-\lambda)x}$ multiplicata denuo fit integrabilis datque

$$e^{(A-\lambda)x}y = \int e^{(A-2\lambda)x}dx \int e^{\lambda x}Xdx .$$

Cum λ sit una radix aequationis $\lambda\lambda - A\lambda + B = 0$, si ambas eius radices ponamus *f* et *g*, ut sit $\lambda = f$, erit $A - \lambda = g$ et aequatio integralis

$$e^{gx}y = \int e^{(g-f)x}dx \int e^{fx}Xdx$$

seu

$$e^{gx}y = \tfrac{1}{g-f}e^{(g-f)x}\int e^{fx}Xdx - \tfrac{1}{g-f}\int e^{gx}Xdx$$

quae abit in formam supra [§ 856] inventam

$$y = \tfrac{1}{g-f}e^{-fx}\int e^{fx}Xdx - \tfrac{1}{g-f}e^{-gx}\int e^{gx}Xdx .$$

COROLLARIUM 1

866. Aequatio ergo proposita seu inde nata

$$dp + Apdx + Bydx = Xdx$$

fit integrabilis, si ducatur in $e^{\lambda x}$ existente $\lambda\lambda - A\lambda + B = 0$, sicque duplex habetur factor, vel e^{fx} vel e^{gx}.

COROLLARIUM 2

867. Ea autem per factorem e^{fx} multiplicata, eius integrale erit

$$dy + gydx = e^{-fx}dx \int e^{fx}Xdx$$

sicque per integrationem ad aequationem differentialem primi gradus reducitur, quae denuo integrabilis redditur, si per e^{gx} multiplicetur.

SCHOLION

868. Multiplicatorem V ita determinari oportebat, ut formula

$$dy\left(AV - \tfrac{dV}{dx}\right) + BVydx$$

fieret per se integrabilis. Tum autem, cum V sit functio ipsius x, integrale erit $y\left(AV - \tfrac{dV}{dx}\right)$, unde fiat necesse est

$$AdV - \tfrac{ddV}{dx} = BVdx \text{ seu } ddV - AdxdV + BVdx^2 = 0,$$

a cuius aequationis integratione pendet inventio factoris quaesiti V. Sufficit autem eius integrale particulare sumsisse; dummodo enim aequatio proposita integrabilis reddatur, constans arbitraria pro integrali completo reddendo ipsa integratione introducitur.

PROBLEMA 108

869. *Sumto elemento dx constante si proponatur haec aequatio*

$$ddy + Pdydx + Qydx^2 = Xdx^2,$$

ubi P, Q et X sint functiones quaecunque ipsius x, invenire multiplicatorem V, qui sit functio ipsius x, quo illa aequatio integrabilis reddatur.

SOLUTIO

Quia aequatio per V multiplicata

$$Vddy + VPdydx + VQydx^2 = VXdx^2$$

integrabilis existit, prioris partis integrale ponatur $Vdy + Sydx$; aliam enim formam habere nequit ac fieri oportet

$$VPdydx + VQydx^2 = dydV + Sdydx + ydSdx;$$

ubi cum S sit necessario functio ipsius x, erit

$$VPdx = dV + Sdx \text{ et } VQdx = dS.$$

Inde autem est $S = VP - \tfrac{dV}{dx}$ quare multiplicator V definiri debet ex hac aequatione

$$VQdx = VdP + PdV - \tfrac{ddV}{dx} \text{ seu } ddV - PdVdx + Vdx\left(Qdx - dP\right) = 0;$$

quae ergo si resolvi poterit vel si saltem eius integrale quodpiam particulare innotescat, ut habeatur multiplicator V aequationis propositae integrale erit

$$Vdy + y(VPdx - dV) = dx\int VXdx,$$

quae porro integrabilis redditur, si ducatur in $\frac{1}{VV}e^{\int Pdx}$; obtinebitur enim integrale

$$\frac{y}{V}e^{\int Pdx} = \int \frac{dx}{VV}e^{\int Pdx}\int VXdx \quad \text{seu} \quad y = e^{-\int Pdx}V\int e^{\int Pdx}\frac{dx}{VV}\int VXdx,$$

quo duplici signo integrali gemina constans arbitraria introducitur integrale completum constituens.

COROLLARIUM 1

870. Inventio ergo multiplicatoris *V* pendet etiam a resolutione aequationis differentio-differentialis, quae autem proposita simplicior est aestimanda, quod functionem *X* non involvat et quantitas *V* cum suis differentialibus *dV* et *ddV* ubique unam dimensionem constituat.

COROLLARIUM 2

871. Quodsi ergo ponatur $V = e^{\int vdx}$, quantitas *v* determinabitur per hanc aequationem differentialem primi gradus

$$dv + vvdx - Pvdx + Qdx - dP = 0,$$

cuius si saltem integrale particulare constet, integratio aequationis propositae absolvi poterit.

COROLLARIUM 3

872. Dato autem multiplicatore *V* vicissim ratio aequationis propositae definitur, ut hoc modo integrabilis evadat. Erit enim vel

$$Q = \frac{dP}{dx} + \frac{PdV}{Vdx} - \frac{ddV}{Vdx^2} \quad \text{vel} \quad dP + \frac{PdV}{V} = Qdx + \frac{ddV}{Vdx}$$

vel integrando

$$PV = \frac{dV}{dx} + \int QVdx \quad \text{seu} \quad P = \frac{dV}{Vdx} + \frac{\int QVdx}{V}$$

EXEMPLUM 1

873. *Definire formam aequationis differentio-differentialis*

$$ddy + Pdydx + Qydx^2 = Xdx^2,$$

ut multiplicata per $e^{\lambda x}$ *integrabilis evadat.*

Posito multiplicatore $V = e^{\int vdx} = e^{\lambda x}$ erit $v = \lambda$ et satisfieri oportet huic aequationi

$$\lambda\lambda dx - \lambda Pdx + Qdx - dP = 0,$$

unde fit

$$Q = \lambda P - \lambda\lambda + \tfrac{dP}{dx}.$$

.

Primum ergo hoc evenit, si fuerint P et Q constantes; puta $P = A$ et $Q = B$, ac tum A definiri oportet ex hac aequatione $\lambda\lambda - A\lambda + B = 0$, qui est casus supra [§ 865] tractatus. Praeterea vero qualiscunque functio P fuerit ipsius x, modo sit $Q = \lambda P - \lambda\lambda + \tfrac{dP}{dx}$ aequatio in $e^{\lambda x}$ ducta erit integrabilis integrali existente

$$e^{\lambda x}\left(dy + ydx(P-\lambda)\right) = dx\int e^{\lambda x}Xdx$$

seu

$$dy + (P-\lambda)ydx = e^{-\lambda x}dx\int e^{\lambda x}Xdx,$$

quae ulterius per $e^{\int Pdx - \lambda x}$ multiplicata et integrata dat

$$y = e^{-\int Pdx + \lambda x}\int e^{\int Pdx - 2\lambda x}dx\int e^{\lambda x}Xdx.$$

COROLLARIUM

874. Sit $P = A + \alpha x$ et $Q = B + \beta x$; erit

$$B + \beta x = A\lambda + a\lambda x - \lambda\lambda + \alpha, \text{ ergo } B = A\lambda - \lambda\lambda + \alpha \text{ et } \beta = \alpha\lambda,$$

unde ob $\lambda = \frac{\beta}{\alpha}$ coefficientes A, B, α, β ita comparatos esse oportet, ut sit

$$B\alpha\alpha = A\alpha\beta - \beta\beta + \alpha^3 \text{ seu } Baa + \beta\beta = \alpha(A\beta + \alpha\alpha).$$

EXEMPLUM 2

875. *Definire formam aequationis differentio-differentiatis*

$$ddy + Pdydx + Qydx^2 = Xdx^2,$$

ut per $e^{\int vdx}$ *existente* $v = \frac{\lambda}{x} + \mu x^n$ *multiplicata fiat integrabilis.*

Cum esse debeat

$$dv + vvdx - Pvdx + Qdx - dP = 0,$$

erit

$$-\frac{\lambda}{xx} + \mu n x^{n-1} - \frac{\lambda P}{x} - \mu P x^n + \frac{\lambda\lambda}{xx} + 2\lambda\mu x^{n-1} + \mu\mu x^{2n} + Q - \frac{dP}{dx} = 0,$$

ergo

$$Q = \frac{\lambda(1-\lambda)}{xx} - (2\lambda + n)\mu x^{n-1} - \mu\mu x^{2n} + \frac{\lambda P}{x} + \mu P x^n + \frac{dP}{dx}$$

Ponamus $P = \frac{\alpha}{x} + \beta x^n$; erit

$$Q = \frac{1}{xx}(\lambda - \lambda\lambda + \alpha\lambda - \alpha) + x^{n-1}(\beta\lambda + \alpha\mu + \beta n - 2\lambda\mu - n\mu) + x^{2n}(\beta\mu - \mu\mu)$$

Sit $Q = \frac{\gamma}{xx} + \delta x^{n-1} + \varepsilon x^{2n}$ fierique oportet

$$\lambda\lambda - (\alpha + 1)\lambda + \alpha + \gamma = 0, \quad \beta(\lambda + n) + \mu(\alpha - 2\lambda - n) = \delta \text{ et } \mu(\beta - \mu) = \varepsilon,$$

unde non solum pro multiplicatore litterae λ et μ, sed etiam certa relatio inter litteras $\alpha, \beta, \gamma, \delta, \varepsilon$ definitur.

Veluti si sit $\gamma = 0$ et $\delta = 0$, erit $(\lambda - \alpha)(\lambda - 1) = 0$, unde $\lambda = \alpha$; tum $(\beta - \mu)(\alpha + n) = 0$, ergo $\alpha = \lambda = -n$ et $\mu\mu - \beta\mu + \varepsilon = 0$. Scilicet aequatio

$$ddy + dxdy\left(\beta x^n - \frac{n}{x}\right) + \varepsilon x^{2n} y dx^2 = Xdx^2$$

multiplicatorem recipit $e^{\int vdx}$ existente $v = -\frac{n}{x} + \mu x^n$ sumto μ ita, ut sit $\mu\mu - \beta\mu + \varepsilon = 0$. Erit ergo multiplicator

$$V = \frac{1}{x^n} e^{\frac{\mu}{n+1} x^{n+1}} \quad \text{et} \quad e^{\int Pdx} = \frac{1}{x^n} e^{\frac{\beta}{n+1} x^{n+1}}$$

Quare si ponamus $\frac{1}{n+1}x^{n+1}=t$, erit

$$y=x^n e^{-\beta t}\frac{1}{x^n}e^{\mu t}\int e^{\beta t-2\mu t}x^n dx\int\frac{e^{\mu t}Xdx}{x^n}\quad\text{seu}\quad y=e^{(\mu-\beta)t}\int e^{(\beta-2\mu)t}dt\int\frac{e^{\mu t}Xdx}{x^n}.$$

COROLLARIUM 1

876. Si sumatur $\gamma=0$ et $\varepsilon=0$, erit

$$\mu=\beta,\ \beta(\alpha-\lambda)=\delta\ \text{ et }\ (\lambda-\alpha)(\lambda-1)=0,$$

hinc $\lambda=1$ et $\delta=(\alpha-1)\beta$ ideoque $P=\frac{\alpha}{x}+\beta x^n$, $Q=(\alpha-1)\beta x^{n-1}$ et aequationis

$$ddy+\left(\frac{\alpha}{x}+\beta x^n\right)dxdy+(\alpha-1)\beta x^{n-1}ydx^2=Xdx^2$$

multiplicator $V=e^{\int vdx}$: existente $v=\frac{1}{x}+\mu x^n$, ita ut sit

$$V=xe^{\frac{\beta}{n+1}x^{n+1}}\quad\text{et}\quad e^{\int Pdx}=x^{\alpha}e^{\frac{\beta}{n+1}x^{n+1}}.$$

COROLLARIUM 2

877. Hoc ergo casu posito $\frac{\mu}{n+1}x^{n+1}=t$ erit integrale

$$y=e^{1-\alpha}\int e^{\alpha-2}e^{-\beta t}dx\int e^{\beta t}Xxdx,$$

quae forma simplicius exhiberi nequit, propterea quod in genere formula $e^{-\beta t}e^{\alpha-2}dx$ integrationem non admittit.

SCHOLION

878. Cum igitur inventio multiplicatorum, qui huiusmodi aequationem

$$ddy+Pdxdy+Qydx^2=Xdx^2$$

integrabilem reddunt, solutionem huius aequationis postulet

$$ddV-PdVdx+Vdx(Qdx-dP)=0,$$

quae in hac forma continetur

$$ddy+Pdxdy+Qydx^2=0,$$

videndum est, quomodo hanc formam etiam per multiplicatores tractari oporteat. Cuius multiplicaror si fingatur *V*; functio quaedam ipsius *x*, iterum ad praecedentem formam

$$ddV - PdVdx + Vdx(Qdx - dP) = 0,$$

devenitur, atque huius multiplicator statuatur = *U*, functioni ipsius *x*, hic definietur per hanc aequationem

$$ddU + PdUdx + QUdx^2 = 0$$

ita ut sufficiat alteram harum duarum aequationum resolvisse. Ac supra quidem, ubi $y = uv$ posuimus, ad hanc posteriorem aequationem pervenimus. At mirum non est harum duarum aequationum alteram ab altera pendere, cum prior ex posteriori nascatur ponendo $U = e^{-\int Pdx}V$, posterior vero ex priori ponendo $V = e^{\int Pdx}U$, uti tentanti facile patebit.

Quoniam igitur hoc modo difficultatem, si quae occurrit, tollere non licet, investigandum est, an forte eiusmodi multiplicator, qui utramque variabilem *x* et *y* cum suis differentialibus *dx* et *dy* seu $p = \frac{dy}{dx}$ involvat, negotium conficiat. At vero facile perspicitur exclusis differentialibus hoc fieri non posse; nam si multiplicator esset *V*, functio ipsarum *x* et *y*, ex primo termino *ddy* nascetur integralis pars *Vdy*, quae autem differentiata ponendo $dV = Mdx + Ndy$ involveret in differentiale partem Ndy^2 in aequatione non occurrentem, quae etiam per reliquas integralis partes tolli non posset. Quam rem tentemus eiusmodi multiplicatoribus, qui etiam rationem differentialium $p = \frac{dy}{dx}$ complectantur; et cum ipsius *y* cum suis differentialibus ubique sit idem dimensionum numerus, eadem proprietas etiam in multiplicatore insit necesse est; si enim diversae inessent, singulae seorsim negotium essent confecturae.

PROBLEMA 109

879. *Sumto elemento dx constante definire conditiones, ut multiplicator huius formae* $Mp + Ny$ *existente* $p = \frac{dy}{dx}$ *et M et N functionibus ipsius x integrabilem reddat hanc aequationem*

$$ddy + Pdxdy + Qydx^2 = 0,$$

ubi P et Q sunt functiones ipsius x.

SOLUTIO

Ob $dy = pdx$ nostra aequatio est

$$dp + Ppdx + Qydx = 0,$$

quae per $Mp + Ny$ multiplicata fit

$$Mpdp + Nydp + MPpdy + NPydy + NQyydx + MQydy = 0,$$

quam integrabilem esse oportet. Ob terminos differentiali *dp* affectos pars integralis erit $\frac{1}{2}Mpp + Nyp$, unde integrale ipsum statuatur $= \frac{1}{2}Mpp + Nyp + S$; cuius differentiale cum ipsam illam aequationem praebere debeat, habebimus

$$\begin{aligned} dS = {} & MPpdy + NPydy + NQyydx \\ & + MQydy \\ & -\tfrac{1}{2}ppdM - ypdN \\ & -Npdy \end{aligned}$$

quam ergo formulam integrabilem esset oportet; quae cum tantum differentialia primi ordinis *dx* et *dy* complectatur, necesse est, ut quantitas *p* ex calculo egrediatur. Posito ergo $dM = M'dx$ et $dN = N'dx$ ob $pdx = dy$ primus terminus continens *p* ad nihilum redigi debet, ut sit

$$MPpdy - \tfrac{1}{2}M'pdy - Npdy = 0$$

seu

$$MP - \tfrac{1}{2}M' - N = 0 \text{ vel } N = MP - \tfrac{dM}{2dx}.$$

Tum vero erit

$$dS = ydy\left(NP + MQ - N'\right) + NQyydx,$$

cuius formulae integrale est

$$S = \tfrac{1}{2}yy\left(NP + MQ - N'\right) \text{ vel } S = yy\int NQdx,$$

quas duas formas congruere oportet, unde fit

$$NP + MQ - \tfrac{dN}{dx} = 2\int NQdx$$

seu

$$NdP + PdN + MdQ + QdM - \tfrac{ddN}{dx} - 2NQdx = 0,$$

quae aequatio cum ilIa $N = MP - \frac{dM}{2dx}$ iuncta conditiones quaesitas determinat; proditque tum aequatio integralis

COROLLARIUM 1

880. Si functiones P et Q dentur indeque M et N definiri oporteat, ob $N = MP - \frac{dM}{2dx}$ erit $dN = MdP + PdM - \frac{ddM}{2dx}$ et functio M definietur per hanc aequationem

$$\frac{d^3M}{dx^3} - \frac{3PddM}{2dx} + \left(PP - \frac{5dP}{dx} + 2Q\right)dM + M\left(2PdP - \frac{ddP}{dx} - 2PQdx + dQ\right) = 0,$$

quae ob differentialia tertii ordinis parum iuvat.

COROLLARIUM 2

881. Sin autem multiplicator $Mp + Ny$ detur, ipsa aeqnatio ita definitur, ut sit primo $P = \frac{N}{M} + \frac{dM}{2Mdx}$, unde ex altera, quae est

$$dQ + \frac{QdM}{M} - \frac{2NQdx}{M} = \frac{ddN}{Mdx} - \frac{d.PN}{M}$$

haecque per $Me^{-2\int\frac{Ndx}{M}}$ multiplicata, integrale dat

$$MQe^{-2\int\frac{Ndx}{M}} = \int e^{-2\int\frac{Ndx}{M}}\left(\frac{ddN}{Mdx} - \frac{d.PN}{M}\right).$$

COROLLARIUM 3

882. Sit hoc integrale $= Z$ eritque

$$Z = e^{-2\int\frac{Ndx}{M}}\left(\frac{dN}{dx} - PN\right) + \int e^{-2\int\frac{Ndx}{M}}\left(\frac{2NdN}{M} - \frac{2PNNdx}{M}\right),$$

quod posterius membrum pro P valore substituto abit in

$$\int e^{-2\int\frac{Ndx}{M}}\left(\frac{2NdN}{M} - \frac{2N^3dx}{MM} - \frac{NNdM}{MM}\right),$$

cuius integrale est manifesto $e^{-2\int\frac{Ndx}{M}}\frac{NN}{M}$, ita ut sit

$$Z = e^{-2\int\frac{Ndx}{M}}\left(\frac{dN}{dx} - \frac{NdM}{2Mdx}\right) + C$$

ideoque

$$Q = \frac{C}{M}e^{2\int\frac{Ndx}{M}} + \frac{dN}{Mdx} - \frac{NdM}{2MMdx}.$$

COROLLARIUM 4

883. Proposita ergo hac aequatione

$$\frac{ddy}{dx}+\left(\frac{N}{M}+\frac{dM}{2Mdx}\right)dy+\left(\frac{Cdx}{M}e^{2\int\frac{Ndx}{M}}+\frac{dN}{M}-\frac{NdM}{2MM}\right)=0$$

eam per $\frac{Mdy}{dx}+Ny$ multiplicando integrale fit

$$\frac{Mdy^2}{2dx^2}+\frac{Nydy}{dx}+\frac{1}{2}yy\left(Ce^{2\int\frac{Ndx}{M}}+\frac{NN}{M}\right)=\text{Const.}$$

SCHOLION

884. Cum ergo pro M et N quascunque functiones ipsius x accipere liceat, innumerabiles hinc nacti sumus aequationum differentio–differentialium formas, quas ope multiplicatoris $\frac{Mdy}{dx}+Ny$ integrare possumus. Forma scilicet generalis, quae hoc multiplicatore integrabilis redditur, est, ut vidimus,

$$\frac{ddy}{dx}+\frac{dy}{2Mdx}(dM+2Ndx)+\frac{y}{2MM}\left(2MdN-NdM+2CMe^{2\int\frac{Ndx}{M}}dx\right)$$

ipso integrali existente

$$\frac{Mdy^2}{2dx^2}+\frac{Nydy}{dx}+\frac{1}{2}yy\left(\frac{NN}{M}+Ce^{2\int\frac{Ndx}{M}}\right),$$

ubi perspicuum est partem exponentialem constanti C affectam utrinque omitti posse, cum ea sola ista proprietate sit praedita. Quodsi partem exponentialem ad algebraicam reducamus ponendo $e^{2\int\frac{Ndx}{M}}=L$, erit

$$2\frac{Ndx}{M}=\frac{dL}{L}\quad \text{et}\quad N=\frac{MdL}{2Ldx}$$

hincque

$$dN=\frac{MddL}{2Ldx}+\frac{dLdM}{2Ldx}-\frac{MdL^2}{2LLdx},$$

unde ista forma

$$\frac{ddy}{dx}+\frac{dy}{2dx}\left(\frac{dM}{M}+\frac{dL}{L}\right)+\frac{1}{2}y\left(\frac{ddL}{Ldx}+\frac{dLdM}{2LMdx}-\frac{dL^2}{LLdx}+\frac{2CLdx}{M}\right),$$

quae per $\frac{Mdy}{dx}+\frac{MydL}{2Ldx}$ multiplicata integrale praebet

$$\frac{Mdy^2}{2dx^2}+\frac{MydLdy}{2Ldx^2}+\frac{1}{2}yy\left(\frac{MdL^2}{4LLdx^2}+CL\right).$$

Ve1 si ponamus $\frac{dM}{M}+\frac{dL}{L}=\frac{2dK}{K}$, ut sit $M=\frac{KK}{L}$, erit nostra aequation differentio-differentialis

$$\frac{ddy}{dx}+\frac{dy}{dx}\frac{dK}{K}+\frac{1}{2}y\left(d.\frac{dL}{Ldx}-\frac{dL^2}{2LLdx}+\frac{dKdL}{KLdx}+\frac{2CLLdx}{KK}\right)=0,$$

quae per $\frac{KK}{L}\left(\frac{dy}{dx}+\frac{ydL}{2Ldx}\right)$ multiplicata dabit integrale

$$\frac{KK}{2L}\left(\frac{dy^2}{dx^2}+\frac{ydLdy}{Ldx^2}+yy\left(\frac{dL^2}{4LLdx^2}+\frac{CLL}{KK}\right)\right)=\text{Const.}$$

EXEMPLUM 1

885. Sit $K=x^m(a+x)^n$ et $L=x^\mu(a+x)^\nu$; erit

$$\frac{dK}{Kdx}=\frac{m}{x}+\frac{n}{a+x}=\frac{ma+(m+n)x}{x(a+x)},\quad \frac{dL}{Ldx}=\frac{\mu}{x}+\frac{\nu}{a+x},$$

unde coefficiens ipsius $\frac{1}{2}ydx$ erit

$$-\frac{\mu}{xx}-\frac{\nu}{(a+x)^2}-\frac{\mu\mu}{2xx}-\frac{\mu\nu}{x(a+x)}-\frac{\nu\nu}{2(a+x)^2}+\frac{m\mu}{xx}+\frac{m\nu+n\mu}{x(a+x)}+\frac{n\nu}{(a+x)^2}$$
$$+2Cx^{2\mu-2m}(a+x)^{2\nu-2n}$$

seu

$$\frac{\mu(2m-\mu-2)}{2xx}+\frac{m\nu+n\mu-\mu\nu}{x(a+x)}+\frac{\nu(2n-\nu-2)}{2(a+x)^2}+2Cx^{2\mu-2m}(a+x)^{2\nu-2n},$$

ubi sequentes casus notasse iuvabit.

I. Sit $m=\mu+1$ et $n=\nu$; erit ipsius $\frac{1}{2}ydx$ coeffieiens

$$\frac{\mu\mu+4C}{2xx}+\frac{\nu(\mu+1)}{x(a+x)}+\frac{\nu(\nu-2)}{2(a+x)^2}.$$

Hinc ista aequatio

$$\frac{ddy}{dx}+dy\left(\frac{\mu+1}{x}+\frac{\nu}{a+x}\right)+\frac{1}{4}ydx\left(\frac{\mu\mu+4C}{xx}+\frac{2\nu(\mu+1)}{x(a+x)}+\frac{\nu(\nu-2)}{(a+x)^2}\right)=0$$

multiplicata per

$$x^{\mu+2}(a+x)^{\nu}\left(\frac{dy}{dx}+\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)\frac{y}{2}\right)$$

integrale dat

$$\frac{1}{2}x^{\mu+2}(a+x)^{\nu}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)+\frac{1}{4}yy\left(\frac{\mu\mu+4C}{xx}+\frac{2\mu\nu}{x(a+x)}+\frac{\nu\nu}{(a+x)^2}\right)\right)=\text{Const.}$$

II Sit $m=\mu+\frac{1}{2}$ et $n=\nu+\frac{1}{2}$; erit ipsius $\frac{1}{2}ydx$ coefficiens

$$\frac{\mu(\mu-1)}{2xx}+\frac{2\mu\nu+\mu+\nu+4C}{2x(a+x)}+\frac{\nu(\nu-1)}{2(a+x)^2}.$$

Hinc ista aequatio

$$\frac{ddy}{dx}+dy\left(\frac{2\mu+1}{x}+\frac{2\nu+1}{2(a+x)}\right)+\frac{1}{4}ydx\left(\frac{\mu(\mu-1)}{xx}+\frac{2\mu\nu+\mu+\nu+4C}{x(a+x)}+\frac{\nu(\nu-1)}{(a+x)^2}\right)=0$$

multiplicata per

$$x^{\mu+1}(a+x)^{\nu+1}\left(\frac{dy}{dx}+\frac{1}{2}y\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)\right)$$

integrale dabit

$$\frac{1}{2}x^{\mu+1}(a+x)^{\nu+1}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)+\frac{1}{4}yy\left(\frac{\mu\mu}{xx}+\frac{2\mu\nu+4C}{x(a+x)}+\frac{\nu\nu}{(a+x)^2}\right)\right)=\text{Const.}$$

III. Sit $m=\mu$ et $n=\nu+1$; erit ipsius $\frac{1}{2}ydx$ coefficiens

$$\frac{\mu(\mu-2)}{2xx}+\frac{\mu(\mu+1)}{x(a+x)}+\frac{\nu\nu+4C}{2(a+x)^2}.$$

Hinc ista aequatio

$$\frac{ddy}{dx}+dy\left(\frac{\mu}{x}+\frac{\nu+1}{a+x}\right)+\frac{1}{4}ydx\left(\frac{\mu(\mu-2)}{xx}+\frac{2\mu(\nu+1)}{x(a+x)}+\frac{\nu\nu+4C}{(a+x)^2}\right)=0$$

multiplicata per

$$x^{\mu}(a+x)^{\nu+2}\left(\frac{dy}{dx}+\frac{1}{2}y\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)\right)$$

dabit integrale

$$\frac{1}{2}x^{\mu}(a+x)^{\nu+2}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{\mu}{x}+\frac{\nu}{a+x}\right)+\frac{1}{4}yy\left(\frac{\mu\mu}{xx}+\frac{2\mu\nu}{x(a+x)}+\frac{\nu\nu+4C}{(a+x)^2}\right)\right)=\text{Const.}$$

COROLLARIUM 1

886. Sit casu primo $v=2$ et $C=-\frac{1}{4}$*;* habebitur haec aequatio

$$\frac{ddy}{dx}+\frac{(\mu+1)a+(\mu+3)x}{x(a+x)}dy+\frac{(\mu+1)ydx}{x(a+x)}=0,$$

quae per

$$x^{\mu+2}(a+x)^2\left(\frac{dy}{dx}+\frac{\mu a+(\mu+2)x}{2x(a+x)}y\right)$$

multiplicata praebet integrale

$$\tfrac{1}{2}x^{\mu+2}(a+x)^2\left(\frac{dy^2}{dx^2}+\frac{\mu a+(\mu+2)x}{x(a+x)}\cdot\frac{ydy}{dx}+yy\left(\frac{\mu}{x(a+x)}+\frac{1}{(a+x)^2}\right)\right)=\text{Const}.$$

COROLLARIUM 2

887. Sit casu tertio $\mu=2$ et $C=-vv$; habebitur ista aequatio

$$\frac{ddy}{dx}+dy\frac{2a+(v+3)x}{x(a+x)}+\frac{(v+1)ydx}{x(a+x)}=0,$$

quae multiplicata per

$$xx(a+x)^{v+2}\left(\frac{dy}{dx}+\tfrac{1}{2}y\left(\frac{2}{x}+\frac{v}{a+x}\right)\right)$$

dabit integrale

$$\tfrac{1}{2}xx(a+x)^{v+2}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{2}{x}+\frac{v}{a+x}\right)+yy\left(\frac{1}{xx}+\frac{v}{x(a+x)}\right)\right)=\text{Const}.$$

EXEMPLUM 2

888. Sit $K=x^m(aa+xx)^n$ et $L=x^\mu(aa+xx)^v$; erit

$$\frac{dK}{Kdx}=\frac{m}{x}+\frac{2nx}{aa+xx}\quad\text{et}\quad\frac{dL}{Ldx}=\frac{\mu}{x}+\frac{2vx}{aa+xx}$$

et aequatio differentio-differentialis hanc induet formam

$$\frac{ddy}{dx}+dy\left(\frac{m}{x}+\frac{2nx}{aa+xx}\right)+\tfrac{1}{2}ydx\left(\frac{\mu(2m-\mu-2)}{2xx}+\frac{2n\mu+2v(m-\mu+1)}{aa+xx}+\frac{2v(2n-v-2)xx}{(aa+xx)^2}+\frac{2Cx^{2\mu-2m}}{(aa+xx)^{2n-2m}}\right)=0$$

cuius in

$$x^{2m-\mu}(aa+xx)^{2n-v}\left(\frac{dy}{dx}+\tfrac{1}{2}y\left(\frac{\mu}{x}+\frac{2vx}{aa+xx}\right)\right)$$

ductae integrale erit

$$\tfrac{1}{2}x^{2m-\mu}\left(aa+xx\right)^{2n-\nu}\left(\frac{dy^2}{dx^2}+\frac{ydy}{dx}\left(\frac{\mu}{x}+\frac{2\nu x}{a+x}\right)+\tfrac{1}{4}yy\left(\left(\frac{\mu}{x}+\frac{2\nu x}{aa+xx}\right)^2+\frac{4Cx^{2\mu-2m}}{\left(aa+xx\right)^{2n-2\nu}}\right)\right)=\text{Const.}$$

Evolvamus hic casus, quibus aequatio differentio-differentialis hanc obtinet formam

$$\frac{ddy}{dx}+dy\left(\frac{m}{x}+\frac{2nx}{aa+xx}\right)+\tfrac{1}{2}ydx\left(D+\frac{E}{2xx}+\frac{F}{aa+xx}+\frac{Gxx}{\left(aa+xx\right)^2}\right)=0.$$

1. Sumatur $\mu=m$ et $\nu=n$, eritque

$$D=2C,\ E=\tfrac{1}{2}m\left(m-2\right),\ F=2n\left(m+1\right)\ \text{et}\ G=2n\left(n-2\right).$$

II. Sumatur $\mu=m$ et $\nu=n$, eritque

$$D=0,\ E=2C+\tfrac{1}{2}\left(m-1\right)^2,\ F=2n\left(m+1\right)\ \text{et}\ G=2n\left(n-2\right).$$

III. Sumatur $\mu=m-1$ et $2n-2\nu=-1$ seu $\nu=n+\frac{1}{2}$; eritque ultimus terminus

$$\frac{2C\left(aa+xx\right)}{xx}=2C+\frac{2Caa}{xx}.$$

Ergo

$$D=2C,\ E=2Caa+\tfrac{1}{2}\left(m-1\right)^2,\ F=2\left(mn+n+1\right),\ G=\tfrac{1}{2}\left(2n+1\right)\left(2n-5\right).$$

IV. Sumatur $\mu=m$ et $2n-2\nu=1$ seu $\nu=n-\frac{1}{2}$; eritque ultimus terminus

$$\frac{2C}{aa+xx},$$

ideoque

$$D=0,\ E=\tfrac{1}{2}m\left(m-2\right),\ F=2C+2mn+2n-1,\ G=\tfrac{1}{2}\left(2n-1\right)\left(2n-3\right).$$

V. Sumatur $\mu=m+1$ et $\nu=n-\frac{1}{2}$; eritque ultimus terminus

$$\frac{2Cxx}{aa+xx}=2C-\frac{2Caa}{aa+xx},$$

ideoque

$$D=2C,\ E=\tfrac{1}{2}\left(m+1\right)\left(m-3\right),\ F=-2Caa+2n\left(m+1\right),\ G=\tfrac{1}{2}\left(2n-1\right)\left(2n-3\right).$$

VI. Sit $\mu=m-1$ et $\nu=n-\frac{1}{2}$; erit ultimus terminus

$$\frac{2C}{xx(aa+xx)}=\frac{2C}{aaxx}-\frac{2C}{aa(aa+xx)},$$

unde fit

$$D=0,\ E=\tfrac{2C}{aa}+\tfrac{1}{2}(m-1)^2,\ F=\tfrac{-2C}{aa}+2mn+2n-2,\ \ G=\tfrac{1}{2}(2n-1)(2n-3).$$

VII. Sit $\mu=m+1$ et $2n-2v=2$ seu $v=n-1$; erit ultimus terminus

$$\frac{2Cxx}{(aa+xx)^2}.$$

ideoque

$$D=0,\ E=\tfrac{1}{2}(m+1)(m-3),\ F=2n(m+1),\ \ G=2C+2(n-1)^2.$$

VIII. Sit $\mu=m+2$ et $v=n-1$; eritque ultimus terminus

$$\frac{2Cx^4}{(aa+xx)^2}=2C-\frac{2Caa}{aa+xx}-\frac{2Caaxx}{(aa+xx)^2},$$

unde fit

$$D=2C,\ E=\tfrac{1}{2}(m+2)(m-4),\ F=-2Caa+2mn+2n+2,\ \ G=-2Caa+2(n-1)^2.$$

IX. Sit $\mu=m$ et $v=n-1$; erit ultimus terminus

$$\frac{2C}{(aa+xx)^2}=\frac{2Caa}{aa(aa+xx)}-\frac{2Cxx}{aa(aa+xx)^2},$$

hincque

$$D=0,\ E=\tfrac{1}{2}m(m-2),\ F=\tfrac{2C}{aa}+2mn+2n-2,\ \ G=\tfrac{-2C}{aa}+2(n-1)^2.$$

X. Sit $\mu=m-1$ et $v=n-1$; erit ultimus terminus

$$\frac{2C}{xx(aa+xx)^2}=\frac{2C}{a^4xx}-\frac{4C}{a^4(aa+xx)}+\frac{2Cxx}{a^4(aa+xx)^2},$$

hincque

$$D=0,\ E=\tfrac{2C}{a^4}+\tfrac{1}{2}(m-1)^2,\ F=\tfrac{-4C}{a^4}+2mn+2n-4,\ \ G=\tfrac{2C}{a^4}+2(n-1)^2.$$

PROBLEMA 110

889. *Sumto elemento dx constante si K et L denotent functiones quascunque ipsius x, invenire integrale completum huius aequationis differentio-differentialis*

$$\frac{ddy}{dx}+\frac{dy}{dx}\cdot\frac{dK}{K}+\frac{1}{2}y\left(d.\ \frac{dL}{Ldx}-\frac{dL^2}{2LLdx}+\frac{dKdL}{KLdx}+\frac{2CLLdx}{KK}\right)=0.$$

SOLUTIO

Quoniam haec aequatio integrabilis redditur, si multiplicetur per

$$\frac{KK}{L}\left(\frac{dy}{dx}+\frac{ydL}{2Ldx}\right)$$

eius integrale completum, ut supra [§ 884:] vidimus, est

$$\frac{KK}{2L}\left(\left(\frac{dy}{dx}+\frac{ydL}{2Ldx}\right)^2+\frac{CLL}{KK}yy\right)=\text{Const.},$$

quam aequationem differentialem primi gradus adhuc integrari oportet. Quod cum ob constantem indefinitam maxime sit difficile, ea neglecta primo saltem integrale particulare investigemus. Erit ergo ex aequatione

$$\left(\frac{dy}{dx}+\frac{ydL}{2Ldx}\right)^2+\frac{CLL}{KK}yy=0$$

radicem extrahendo

$$\frac{dy}{dx}+\frac{ydL}{2Ldx}=\frac{Ly}{K}\sqrt{-C}\quad \text{seu}\quad \frac{dy}{y}+\frac{dL}{2L}=\frac{Ldx}{K}\sqrt{-C}\,,$$

unde fit

$$y\sqrt{L}=\alpha e^{\int\frac{Ldx}{K}\sqrt{-C}}$$

Cum igitur aequationi differentio-differentiali propositae satisfaciant hi duo valores

$$y=\frac{\alpha}{\sqrt{L}}e^{\int\frac{Ldx}{K}\sqrt{-C}}\quad \text{et}\quad y=\frac{\beta}{\sqrt{L}}e^{-\int\frac{Ldx}{K}\sqrt{-C}}\,,$$

bini coniuncti etiam satisfacient; quibus quoniam duae constantes arbitrariae introducuntur, eius integrale completum erit

$$y=\frac{\alpha}{\sqrt{L}}e^{\int\frac{Ldx}{K}\sqrt{-C}}+\frac{\beta}{\sqrt{L}}e^{-\int\frac{Ldx}{K}\sqrt{-C}}\,,$$

quae expressio valet, si $\sqrt{-C}$ fuerit quantitas realis ; sin autem sit imaginaria, erit

$$y=\frac{\gamma}{\sqrt{L}}\sin.\left(\int\frac{Ldx}{K}\sqrt{C}+\zeta\right)$$

sicque habetur integrale completum aequationis differentio-differentialis propositae.

COROLLARIUM 1

890. Hinc igitur aequationis differentialis primi gradus

$$\left(\frac{dy}{dx}+\frac{ydL}{2Ldx}\right)^2+\frac{CLLy^2}{KK}=\frac{AL}{KK},$$

quae per se satis est difficilis, integrale assignare valemus, quod est

$$y=\frac{\alpha}{\sqrt{L}}e^{\int\frac{Ldx}{K}\sqrt{-C}}+\frac{\beta}{\sqrt{L}}e^{-\int\frac{Ldx}{K}\sqrt{-C}},$$

si modo debita relatio constantium α et β respectu constantis A definiatur.

COROLLARIUM 2

891. Erit autem per YL multiplicando et differentiando

$$dy\sqrt{L}+\frac{ydL}{2\sqrt{L}}=\frac{\alpha Ldx}{K}\sqrt{-C}\cdot e^{\int\frac{Ldx}{K}\sqrt{-C}}-\frac{\beta Ldx}{K}\sqrt{-C}\cdot e^{-\int\frac{Ldx}{K}\sqrt{-C}}$$

hinc

$$\frac{dy}{dx}+\frac{ydL}{2Ldx}=\frac{\sqrt{-C}L}{K}\cdot\left(\alpha e^{\int\frac{Ldx}{K}\sqrt{-C}}-\beta e^{-\int\frac{Ldx}{K}\sqrt{-C}}\right),$$

unde fit

$$\frac{AL}{KK}=\frac{-CL}{KK}\left(\alpha e^{\int\frac{Ldx}{K}\sqrt{-C}}-\beta e^{-\int\frac{Ldx}{K}\sqrt{-C}}\right)^2+\frac{CL}{KK}\left(\alpha e^{\int\frac{Ldx}{K}\sqrt{-C}}+\beta e^{-\int\frac{Ldx}{K}\sqrt{-C}}\right)^2$$

ideoque $A=4C\alpha\beta$ seu $\beta=\frac{A}{4C\alpha}$.

SCHOLION 1

892. Quamvis ergo aequationem propositam ope idonei multiplicatoris integrare licuerit, altera tamen integratio maximis difficultatibus premi videbatur. Interim tamen ope substitutionis aequatio illa differentialis primi gradus tractatu facilis redditur; posito enim $y=\frac{z}{\sqrt{L}}$ ut sit $dy\sqrt{L}+\frac{ydL}{2\sqrt{L}}=dz$, oritur

$$\left(\frac{dz}{dx\sqrt{L}}\right)^2+\frac{CLzz}{KK}=\frac{AL}{KK},$$

hinc $\frac{dz}{dx}=\frac{L}{K}\sqrt{(A-Czz)}$ seu $\frac{dz}{\sqrt{(A-Czz)}}=\frac{Ldx}{K}$, quae integrata dat

$$l\left(z\sqrt{-C}+\sqrt{(A-Czz)}\right)=\int\frac{Ldx}{K}\sqrt{-C}+lB,,$$

unde praecedens integrale eruitur.

Caeterum forma nostrae aequationis differentio-differentialis aliquanto commodius exhiberi potest hoc modo. Si P et R sint functiones quaecunque ipsius x sumaturque elementum dx constans, huius aequationis

$$ddy-dy\left(\frac{dP}{P}+\frac{dR}{R}\right)-y\left(d.\frac{dP}{P}-\frac{dPdR}{PR}+\frac{aaRRdx^2}{PP}\right)=0$$

bis integratae integrale completum est

$$y=\alpha Pe^{a\int\frac{Rdx}{P}}+\beta\alpha Pe^{-a\int\frac{Rdx}{P}},$$

siquidem a sit quantitas realis. At si $a=0$, erit

$$y=P\left(\alpha+\beta\int\frac{Rdx}{P}\right)$$

sin autem sit $aa=-cc$, erit

$$y=\alpha P\sin.\left(\beta+c\int\frac{Rdx}{P}\right).$$

Tum vero illa aequatio integrabilis redditur, si multiplicetur per

$$\frac{1}{RRdx^2}\left(dy-\frac{ydP}{P}\right),$$

eritque integrale primum

$$\frac{1}{RRdx^2}\left(\left(dy-\frac{ydP}{P}\right)^2-\frac{aaRR}{PP}y^2dx^2\right)=\text{Const.}$$

Hinc patet in ilIa aequatione differentio-differentiali commode hanc substitutionem adhiberi $y=Ps$, qua ea transformatur in

$$ddz+dz\left(\frac{dP}{P}-\frac{dR}{R}\right)-\frac{aaRR}{PP}zdx^2=0$$

,
quae per $\frac{PPdz}{RRdx^2}$ multiplicata sponte fit integrabilis. Quin etiam posito $\frac{P}{R}=S$, ut habeatur

$$ddz+\frac{dSdz}{S}-\frac{aazdx^2}{SS}=0,$$

multiplicator $\frac{SSdz}{dx^2}$ statim dat integra1e

$$\frac{SSdz^2}{2dx^2}-\frac{1}{2}aazz=\text{Const.}$$

SCHOLION 2

893. Vicissim ergo ex hac forma simplicissima

$$SSddz+SdSdz-aazdx^2=0,$$

quae per *dz* multiplicata integrabilis redditur, formas magis complicatas derivare potuissemus ponendo $z=\frac{y}{P}$ et $S=\frac{P}{R}$. Quae quamquam in formis generalibus satis perspicua, tamen in exemplis determinatis plerumque haec derivatio nimis est occulta, quam ut manti occurrere possit.

Veluti in casibus § 888 evolutis si *n*° IX sumamus $m=2$ et $C=(n-1)^2aa$, fiet $D=0, E=0, F=2n(n+1)$ et $G=0$, unde habetur haec aequatio

$$\frac{ddy}{dx}+2dy\left(\frac{1}{x}+\frac{nx}{aa+xx}\right)+\frac{n(n+1)ydx}{aa+xx}=0$$

seu

$$ddy+\frac{2dxdy(aa+(n+1)xx)}{x(aa+xx)}+\frac{n(n+1)ydx^2}{aa+xx}=0$$

quae integrabilis redditur ope multiplicatoris

$$xx(aa+xx)^{n+1}\left(\frac{dy}{dx}+\frac{y(aa+nxx)}{x(aa+xx)}\right)$$

integrali existente

$$\frac{1}{2}xx(aa+xx)^{n+1}\left(\left(\frac{dy}{dx}+\frac{y(aa+nxx)}{x(aa+xx)}\right)^2+\frac{(n-1)^2aayy}{(aa+xx)^2}\right)=\text{Const.}$$

Pro integrali ergo particulari erit

$$\frac{dy}{y}+\frac{dx}{x}+\frac{(n-1)xdx}{aa+xx}=\pm\frac{(n-1)adx\sqrt{-1}}{aa+xx},$$

unde colligitur

$$xy\left(aa+xx\right)^{\frac{(n-1)}{2}}=\alpha\left(\frac{a+x\sqrt{-1}}{a-x\sqrt{-1}}\right)^{\pm\frac{(n-1)}{2}}$$

Ergo bina integralia particularia coniuncta dant

$$y=\frac{\alpha}{x}\left(a-x\sqrt{-1}\right)^{-n+1}+\frac{\beta}{x}\left(a+x\sqrt{-1}\right)^{-n+1}$$

integrale completum. Hoc autem casu aequatio nostra ad formam simplicissimam reducetur ope substitutionis $y=\frac{z}{x}\left(aa+xx\right)^{\frac{1-n}{2}}$, cuius ratio et inventio difficilius perspicitur.

PROBLEMA 111

894. *Sumto elemento dx constante investigare conditiones, quibus aequatio differentio- differentialis*

$$ddy+Pdxdy+Qydx^2=0$$

integrabilis redditur ope multiplicatoris huius formae

$$\frac{ydx^2}{Ldy^2+Mydydx+Nyydx^2}$$

denotantibus litteris L, M, N, P, Q functiones ipsius x.

SOLUTIO

Tribuatur denominatori huius fractionis talis forma

$$\left(dy+Rydx\right)\left(dy+Sydx\right)$$

ac levi attentione adhibita patet integrale huiusmodi formam esse habiturum

$$V+l\frac{dy+Rydx}{dy+Sydx}=\text{Const.},$$

cuius ergo differentiale aequationem propositam producere debet. Dat autem differentiatio

$$dV+\frac{(S-R)ydxddy+(R-S)dxdy^2+ydxdy(dR-dS)+yydx^2(SdR-RdS)}{(dy+Rydx)(dy+Sydx)}=0,$$

quae ad communem denominatorem reducta abit in

$$\begin{aligned}&(S-R)ydxddy+(R-S)dxdy^2+ydxdy(dR-dS)\\&+yydx^2(SdR-RdS)+dVdy^2+(R+S)ydxdydV+RSyydx^2dV=0.\end{aligned}$$

Statuatur $dV=(S-R)dx$, ut aequatio per y divisibilis evadat, sicque orietur haec aequatio

$$\begin{aligned}&(S-R)ddy+dy(dR-dS)\\&+ydx(SdR-RdS)+(SS-RR)dxdy+RS(S-R)ydx^2=0;\end{aligned}$$

quae ut cum forma proposita conveniat, fieri oportet

$$P=(R+S)+\tfrac{dR-dS}{(S-R)dx}\quad \text{et}\quad Q=RS+\tfrac{SdR-RdS}{(S-R)dx};$$

quos valores si functiones P et Q habuerint, aequatio

$$ddy+Pdxdy+Qydx^2=0$$

per

$$\frac{(S-R)ydx}{(dy+Rydx)(dy+Sydx)}$$

multiplicata integrale dabit

$$\int(S-R)dx+l\,\tfrac{dy+Rydx}{dy+Syd}=\text{Const.}$$

Si ponamus $S=M+N$ et $R=M-N$, erit

$$P=2M-\tfrac{dN}{Ndx}\ \text{et}\ Q=MM-NN+\tfrac{dM}{dx}-\tfrac{MdN}{Ndx}.$$

COROLLARIUM 1

895. Quaecunque ergo functiones ipsius x loco M et N assumantur indeque definiantur

$$P=2M-\tfrac{dN}{Ndx}\ \text{et}\ Q=MM-NN+\tfrac{dM}{dx}-\tfrac{MdN}{Ndx},$$

huius aequationis

$$ddy+Pdxdy+Qydx^2=0$$

integrale erit

$$2\int Ndx + l\frac{dy+(M-N)ydx}{dy+(M+N)ydx} = \text{Const.}$$

COROLLARIUM 2

896. Si ponatur $y = e^{\int zdx}$, fiet nostra aequatio differentialis primi gradus

$$dz + zzdx + Pzdx + Qdx = 0,$$

cuius propterea integrale erit

$$2\int Ndx + l\frac{z+M-N}{z+M+N} = \text{Const.}$$

COROLLARlUM 3

897. Si velimus, ut sit $P = 0$ et aequatio habeatur huiusmodi

$$ddy + Qydx^2 = 0,$$

capi debet $2M = Ndx$ eritque $Q = \frac{dM}{dx} - MM - NN$ eiusque aequatio integralis

$$2\int Ndx + l\frac{dy+(M-N)ydx}{dy+(M+N)ydx} = \text{Const.}$$

COROLLARY 4

898. In genere autem, prout constans capiatur vel $+\infty$ vel $-\infty$, obtinebitur integrale particulare

$$\text{vel } dy + (M+N)ydx = 0 \text{ vel } dy + (M-N)ydx = 0,$$

unde erit

$$y = \alpha e^{-\int (M+N)dx} \text{ vel } y = \beta e^{-\int (M-N)dx}$$

ex quibus nostrae aequationis colligitur integrale completum

$$y = e^{-\int Mdx}\left(\alpha e^{-\int Ndx} + \beta e^{\int Ndx}\right).$$

EXEMPLUM 1

899. Sit $M = \alpha$ et $N = \beta$; erit $P = 2\alpha$ et $Q = \alpha\alpha - \beta\beta$, unde huius aequationis

$$ddy + 2\alpha dxdy + (\alpha\alpha - \beta\beta)ydx^2 = 0$$

$$2\beta x + l\frac{dy+(\alpha-\beta)ydx}{dy+(\alpha+\beta)ydx} = \text{Const.}$$

In quantitatibus autem finitis integrale completum

$$y = e^{-\alpha x}\left(Ae^{-\beta x} + Be^{\beta x}\right).$$

Casu autem, quo $\beta\beta = -\gamma\gamma$, aequatio

$$ddy + 2\alpha dxdy + \left(\alpha\alpha + \gamma\gamma\right)ydx^2 = 0$$

bis integrata dat

$$y = e^{-\alpha x}\sin.\left(\gamma x + C\right)$$

at si $\gamma = 0$, aequationis

$$ddy + 2\alpha dxdy + \alpha\alpha ydx^2 = 0$$

integrale est

$$y = e^{-\alpha x}\left(A + Bx\right).$$

EXEMPLUM 2

900. Si $M = \frac{\alpha}{x}$ et $N = \beta x^n$, erit

$$P = \frac{2\alpha - n}{x} \quad \text{et} \quad Q = \frac{\alpha\alpha}{xx} - \beta\beta x^{2n} - \frac{\alpha}{xx} - \frac{\alpha n}{xx} = \frac{\alpha(\alpha - n - 1)}{xx} - \beta\beta x^{2n}.$$

Ergo huius aequationis

$$ddy + \frac{(2\alpha - n)dxdy}{x} + \frac{\alpha(\alpha - n - 1)ydx^2}{xx} - \beta\beta x^{2n}ydx^2 = 0$$

integrale primum est

$$\frac{2\beta}{n+1}x^{n+1} + l\frac{xdy + \left(\alpha - \beta x^{n+1}\right)ydx}{xdy + \left(\alpha + \beta x^{n+1}\right)ydx} = \text{Const.}$$

integrale autem secundum

$$y = x^{-\alpha}\left(Ae^{\frac{-\beta x^{n+1}}{n+1}} + Be^{\frac{\beta x^{n+1}}{n+1}}\right);$$

si $\beta = 0$, erit id

$$y = x^{-\alpha}\left(A + Bx^{n+1}\right),$$

sin autem $\beta\beta = -\gamma\gamma$, erit

$$y = Ax^{-\alpha}\sin.\left(\frac{\gamma}{n+1}x^{n+1} + C\right).$$

COROLLARIUM 1

901. Sumto $n = 2\alpha$, ut habeatur haec aequatio

$$ddy - \frac{\alpha(\alpha+1)ydx^2}{xx} - \beta\beta x^{4\alpha}ydx^2 = 0,$$

erit eius integrale completum

$$y = x^{-\alpha}\left(Ae^{\frac{-\beta x^{2\alpha+1}}{2\alpha+1}} + Be^{\frac{\beta x^{2\alpha+1}}{2\alpha+1}}\right);$$

si sit $\beta = 0$, erit id

$$y = x^{-\alpha}\left(A + Bx^{2\alpha+1}\right),$$

at si $\beta\beta = -\gamma\gamma$, erit hoc integrale

$$y = Ax^{-\alpha}\sin.\left(\frac{\gamma}{2\alpha+1}x^{2\alpha+1} + C\right).$$

COROLLARIUM 2

902. Ponamus $\alpha = -1$, ut habeamus hanc aequationem

$$ddy - \frac{\beta\beta ydx^2}{x^4} = 0,$$

cuius ergo integrale erit

$$y = x\left(Ae^{\frac{\beta}{x}} + Be^{-\frac{\beta}{x}}\right),$$

ubi notandum, si sit $\beta\beta = -\gamma\gamma$, fore

$$y = Ax\sin.\left(\frac{\gamma}{x} + C\right).$$

EXEMPLUM 3

903. Ponatur

$$N = \frac{Ax^m}{\alpha+\beta x^n},$$

ut sit $\frac{dN}{Ndx} = \frac{m}{x} - \frac{\beta nx^{n-1}}{\alpha+\beta x^n}$, et sumatur

$$M = \frac{m}{2x} - \frac{\beta nx^{n-1}}{2(\alpha+\beta x^n)},$$

ut fiat [§ 895] $P = 0$ et

$$Q=\frac{m}{2xx}-\frac{\beta n(n-1)x^{n-2}}{2(\alpha+\beta x^n)}+\frac{\beta\beta nnx^{2n-2}}{2(\alpha+\beta x^n)^2}-\frac{m}{4xx}+\frac{\beta mnx^{n-2}}{2(\alpha+\beta x^n)}-\frac{\beta\beta nnx^{2n-2}}{4(\alpha+\beta x^n)^2}-\frac{AAx^{2m}}{(\alpha+\beta x^n)^2}$$

sive

$$Q=-\frac{m(m+2)}{4xx}+\frac{n(m-n+1)\beta x^{n-2}}{2(\alpha+\beta x^n)}+\frac{\beta\beta nnx^{2n-2}-4AAx^{2m}}{4(\alpha+\beta x^n)^2},$$

et ob $\int Mdx=\frac{1}{2}lN$ erit integrale

$$y=\frac{1}{\sqrt{N}}\left(Ce^{-\int Ndz}+De^{\int Ndz}\right)$$

huius aequationis $ddy+Qydx^2=0$.

Ut expressio ipsius Q fiat simplicior, hoc fieri potest pluribus modis, dum numerator partis postremae per $a+\beta x^n$ divisibilis redditur.

1. Sit $m=n-1$ et $AA=\frac{1}{4}\beta\beta nn$ eritque $Q=-\frac{nn-1}{4xx}$, tum vero

$$N=\frac{\frac{1}{2}\beta nx^{n-1}}{\alpha+\beta x}\quad \text{et}\quad \int Ndx=\frac{1}{2}l\left(\alpha+\beta x\right),$$

unde aequationis $ddy-\frac{(nn-1)ydx^2}{4xx}=0$ integra1e est

$$y=\frac{1}{\sqrt{x^{n-1}}}\left(C+D\alpha+D\beta x^n\right)$$

II. Sit $2m=-2$ seu $m=-1$ et $4AA=\alpha\alpha nn$; erit

$$Q=\frac{1}{4xx}-\frac{nn\beta x^{n-2}}{2(\alpha+\beta x^n)}+\frac{\beta nnx^{n-2}-\alpha nnx^{-2}}{4(\alpha+\beta x^n)},$$

seu $Q=\frac{1-nn}{4xx}$ ut ante.

III. Sit $2m=-n-2$ seu $m=\frac{-n-2}{2}$ et $4AA=-\frac{\alpha^3 nn}{\beta}$ fietque

$$Q=-\frac{nn-4}{16xx}-\frac{3nn\beta x^{n-2}}{4(\alpha+\beta x^n)}+\frac{nn\left(\beta\beta x^{n-2}-\alpha\beta x^{-2}+\alpha\alpha x^{-n-2}\right)}{4\beta(\alpha+\beta x^n)}$$

seu

$$Q=\frac{4-nn}{16xx}+\frac{nn\left(\alpha\alpha x^{-n-2}-\alpha\beta x^{-2}-2\beta\beta x^{n-2}\right)}{4\beta(\alpha+\beta x^n)},$$

quae expressio abit in

$$Q=\frac{4-nn}{16xx}+\frac{nn}{4\beta}\left(\alpha x^{-n-2}-2\beta x^{-2}\right)=\frac{4-9nn}{16xx}+\frac{nn\alpha}{4\beta x^{n+2}}.$$

Quare, cum sit

$$N=\frac{n\alpha\sqrt{-\alpha}}{2\sqrt{\beta}}\cdot\frac{x^{-\frac{n+1}{2}}}{\alpha+\beta x^{n}},$$

erit

$$\int Ndx=\frac{n\alpha\sqrt{-\alpha}}{2\sqrt{\beta}}\cdot\int\frac{dx}{\left(\alpha+\beta x^{n}\right)x^{\frac{n+2}{2}}}.$$

Sumatur $n=\frac{2}{3}$ ut fiat $m=-\frac{4}{3}$ et $Q=\frac{\alpha}{9\beta x^{\frac{8}{3}}}$ et

$$N=\frac{n\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\cdot\frac{x^{-\frac{4}{3}}}{\alpha+\beta x^{\frac{2}{3}}},\text{ et }\int Ndx=\frac{\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\cdot\int\frac{dx}{\left(\alpha+\beta x^{\frac{2}{3}}\right)x^{\frac{4}{3}}},$$

sicque aequationis $ddy+\frac{\alpha}{9\beta x^{\frac{8}{3}}}ydx^{2}=0$ integrale erit

$$y=\frac{1}{\sqrt{N}}\left(Ce^{-\int Ndz}+De^{\int Ndz}\right)$$

Sin autem capiatur $n=-\frac{2}{3}$ ut fiat $m=-\frac{2}{3}$ et $Q=\frac{\alpha}{9\beta x^{\frac{4}{3}}}$,erit

$$N=\frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\cdot\frac{x^{-\frac{2}{3}}}{\alpha+\beta x^{-\frac{2}{3}}},\text{ hinc }\int Ndx=\frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\int\frac{x^{\frac{1}{3}}dx}{\alpha x+\beta x^{\frac{1}{3}}}=\frac{-\alpha\sqrt{-\alpha}}{3\sqrt{\beta}}\int\frac{dx}{\alpha x^{\frac{2}{3}}+\beta},$$

unde aequatio $ddy+\frac{\alpha ydx^{2}}{9\beta x^{\frac{4}{3}}}=0$ simili modo integratur.

SCHOLION 1

904. Aequationem ergo $ddy+Ax^{m}ydx^{2}=0$ his casibus integrare licuit $m=0,\ m=-4,\ m=-\frac{4}{3},m=-\frac{8}{3}$ et $m=-2$ seu $m=-2\pm\frac{2}{1},\ m=-2\pm\frac{2}{3}$. Quodsi ulterius ponamus $N=\frac{Ax^{\lambda}}{\alpha+\beta x^{n}+\gamma x^{2n}}$, simili modo integrationem casuum istius aequationis $m=-2\pm\frac{2}{5}$ impetrabimus, quibus quoque aequatio differentialis primi gradus integrationem admittit. Haec autem casuum integrabilium investigatio nimis est operosa, quam ut eam fusius prosequamur, praesertim cum infra [Cap. VII, Probl. 118, imprimis § 943] methodus occurrat haec omnia commodius evolvendi.

SCHOLION 2

905. Ex his colligere licet, quantus fructus ex inventione multiplicatorum, quibus etiam aequationes differentio-differentiales integrabiles redduntur, expectari queat, etiamsi exempla hic tractata tantum leve huius methodi specimen referant. Aliquas autem saltem multiplicatorum formas hic sum contemplatus neque ullum est dubium, quin plures aliae formae pari successu in usum vocari queant. In hoc porro capite tantum eiusmodi aequationes differentio-differentiales tractavimus, in quibus altera variabilis y cum suis differentialibus dy et ddy ubique unicam obtinet dimensionem. Verum eadem methodus quoque ad alia huiusmodi aequationum genera extenditur, quae etsi parum adhuc est exculta, tamen usu non carebit sequens applicatio, ubi integratio aliarum aequationum differentialium secundi gradus, quae aliis methodis tractatu difficillimae videntur, ope multiplicatorum docebitur.