CHAPTER IV

CONCERNING SECOND ORDER DIFFERENTIAL EQUATIONS IN WHICH THE OTHER VARIABLE [*y*] HAS A SINGLE DIMENSION

PROBLEM 101

83l. *With the element dx assumed constant, if an equation of this form is proposed :*

$$ddy + Pdxdy + Qydx^2 = 0,$$

where P and Q shall be some functions of x, to recall that to a differential equation of the first order.

SOLUTION

On putting $dy = pdx$ and $dp = qdx$ the proposed equation adopts this form $q + Pp + Qy = 0$; in which if by the method presented before we put in place $p = uy$ and $q = vy$, then we will obtain this equation $v + Pu + Q = 0$ between *x*, *u* and *v*, and hence $v = -Pu - Q$. Then there shall become now $dy = uydx$ and $udy + ydu = vydx$, thus so that $\frac{dy}{y} = udx = \frac{vdx-du}{u}$ and thus with the value for *v* substituted :

$$du + uudx + Pudx + Qdx = 0,$$

and with which equation resolved there becomes $ly = \int udx$.

Or without these substitutions, we may put at once $y = e^{\int udx}$ into the proposed equation, from which there is produced $dy = e^{\int udx}udx$ and $ddy = e^{\int udx}\left(dudx + uudx^2\right)$, and since with the substitution of the exponential $e^{\int udx}$ accomplished it may be removed from the calculation, and the proceeding equation of the first order will be obtained

$$du + uudx + Pudx + Qdx = 0,$$

on the resolution of which the integration of the second order differential equation depends.

COROLLARY 1

832. This differential equation of the first order can be changed in many ways into other forms almost similar to each other. Just as if we put $u = Mz$, then there is produced :

$$Mdz + z\left(dM + PMdx\right) + MMzzdx + Qdx = 0,$$

where for *M* it is permitted to take a function of this kind of *x*, so that the [second] term affected by the letter *z* vanishes, which comes about, if

$$dM + MPdx = 0 \text{ or } M = Ce^{-\int Pdx}$$

COROLLARY 2

833. This may be produced in a similar form by putting $u = \frac{K}{z}$; for there becomes

$$-\frac{Kdz}{zz} + \frac{dK}{z} + \frac{KKdx}{zz} + \frac{KPdx}{z} + Qdx = 0$$

or

$$Kdz - z(dK + KPdx) - Qzzdx - KKdx = 0,$$

where the second term vanished equally on assuming $K = Ce^{-\int Pdx}$.

COROLLARY 3

834. Again a more general transformation can be set up on putting $u = K + Mz$;
for there is produced :

$$dK + Mdz + zdM + KKdx + 2KMzdx + MMzzdx^2$$
$$+ KPdx + MPzdx + Qdx = 0,$$

which gives in order

$$Mdz + z(dM + 2KMdx + MPdx) + MMzzdx^2$$
$$+dK + KKdx + KPdx + Qdx = 0,$$

from which the second term is removed on assuming

$$M = Ce^{-\int dx(2K+P)} \text{ or } K = \frac{-dM - MPdx}{2Mdx}.$$

COROLLARY 4

835. At this stage a more general form of the kind arises, if there is put $u = \frac{K+Mz}{L+Nz}$, from which there is produced

$$dz(LM - KN) + LdK - KdL + z(LdM - MdL + NdK - KdN)$$
$$+zz(NdM - MdN) + (K + Mz)^2 dx$$
$$+P(K + Mz)(L + Nz)dx + Q(L + Nz)^2 dx = 0,$$

which is reduced to this form

$$0 = dz\left(LM - KN\right) + z\left(LdM - MdL + NdK - KdN + 2KMdx + P\left(KN + LM\right)dx + 2LNQdx\right)$$
$$+ zz\left(NdM - MdN + MMdx + MNPdx + NNQdx\right)$$
$$+ LdK - KdL + KKdx + KLPdx + LLQdx,$$

where for K, L, M and N functions of this kind of x are allowed to be taken, so that the form produced can be treated easily.

SCHOLIUM

836. Because second order differential equations of this kind, in which the variable y has a single dimension, are accustomed to occur most often, then so much more deserved the work and zeal of the geometer in establishing the solution of the equation

$$du + uudx + Pudx + Qdx = 0\,,$$

which also thus can be represented in the more general form, of which indeed the exemplary case $dz + zzdx = ax^n dx$ which the count Riccatti proposed at one time, should not to be scorned in the advancement of analysis. [See Acta Erud. for 1723, p. 509, and Supp. Book VIII, 1724, p. 66; and §441 of this work.] But among the transformations of this case there is merit to observe particularly

$$x = t^{\frac{2}{n+2}},$$

which gives

$$dz + \tfrac{2}{n+2}zzt^{\frac{-2}{n+2}}dt = \tfrac{2a}{n+2}t^{\frac{2}{n+2}}dt,$$

from which on putting $z = Ct^{\frac{n}{n+2}}v$ there is produced

$$\left(n+2\right)Ct^{\frac{n}{n+2}}dv + Cnt^{\frac{-2}{n+2}}vdt + 2CCt^{\frac{n}{n+2}}vvdt = 2at^{\frac{n}{n+2}}dt$$

or

$$\left(n+2\right)Cdv + \tfrac{Cnvdt}{t} + 2CCvvdt = 2adt\,,$$

thus as here no indefinite power of t occurs. If here there is put $v = \frac{\alpha}{t} + s$, there becomes

$$\left.\begin{array}{lll} -\frac{(n+2)C\alpha dt}{tt} & +\left(n+2\right)Cds + \frac{nCsdt}{t} & + 2CCssdt \\ + \frac{nC\alpha dt}{tt} & + \frac{4CC\alpha sdt}{t} & \\ + \frac{2CC\alpha\alpha dt}{tt} & & \end{array}\right\} = 2adt;$$

where if there is taken $\alpha = \frac{-n}{4C}$, then there arises

$$(n+2)Cds+2CCssdt=2adt-\tfrac{n(n+4)dt}{8tt},$$

which is considered with the simplest form.

THEOREM

837. *On assuming the element dx constant, if* $y=M$ *and* $y=N$ *satisfy being particular integrals of the second order differential equation*

$$ddy+Pdxdy+Qydx^2=0,$$

thus so that the ratio $M:N$ *is not constant, then the complete integral of this equation will be*

$$y=\alpha M+\beta N.$$

DEMONSTRATION

Since the values $y=M$ and $y=N$ satisfy the proposed equation, then there will be

$$ddM+PdxdM+QMdx^2=0 \text{ and } ddN+PdxdN+QNdx^2=0,$$

from which it is apparent on putting $y=\alpha M+\beta N$ for the equation also to be satisfied , since there becomes

$$\alpha\left(ddM+PdxdM+QMdx^2\right)+\beta\left(ddN+PdxdN+QNdx^2\right)=0.$$

Now because this integral $y=\alpha M+\beta N$ embraces the two constants α and β, which it is permitted to define as it pleases, that by necessity shall be complete, unless perhaps N may be a multiple of M.

COROLLARY 1

838. Therefore from two given particular integrals of this kind of equation, the complete integral of this can be formed, if indeed these two integrals should be different from each other.

COROLLARY 2

839. Since on putting $y=e^{\int udx}$ or $u=\frac{dy}{ydx}$ there is produced

$$du+uudx+Pudx+Qdx=0,$$

if the values $u=\frac{dM}{Mdx}$ and $u=\frac{dN}{Ndx}$ satisfy this equation, then the value $u=\frac{\alpha dM+\beta dN}{(aN+\beta M)dx}$ also shall satisfy the same.

COROLLARY 3

840. Therefore if there are had two particular integrals of the equation $du+uudx+Pudx+Qdx=0$, $u=R$ and $u=S$, on account of $M=e^{\int Rdx}$ and $N=e^{\int Sdx}$ the complete integral will be

$$u=\frac{\alpha e^{\int Rdx}R+\beta e^{\int Sdx}S}{\alpha e^{\int Rdx}+\beta e^{\int Sdx}} \quad \text{or} \quad u=R+\frac{\beta e^{\int Sdx}(S-R)}{\alpha e^{\int Rdx}+\beta e^{\int Sdx}}$$

SCHOLIUM

841. This observation is of the greatest interest, because in equations of this kind from two known particular integrals it is possible to assign the complete integral. But generally with one of the known particular integrals in that the sign is accustomed to be present with the root, on account of which ambiguity two like particular integrals are known. Thus if the value $u=T+\sqrt{V}$ satisfies the equation $du+uudx+Pudx+Qdx=0$, $u=T-\sqrt{V}$ will satisfy the same equation, from which the complete integral will be

$$u=T+\sqrt{V}-\frac{2\beta\sqrt{V}}{\alpha e^{2\int dx\sqrt{V}}+\beta} \quad \text{or} \quad u=T+\frac{\alpha e^{2\int dx\sqrt{V}}\sqrt{V}-\beta\sqrt{V}}{\alpha e^{2\int dx\sqrt{V}}+\beta}.$$

And if perhaps $\sqrt{V}$ should be imaginary, on putting $\sqrt{V}=X\sqrt{-1}$, on account of

$$e^{\pm\int dx\sqrt{V}}=cos.\int Xdx\pm\sqrt{-1}\cdot sin.\int Xdx$$

then there shall be

$$u=T+\frac{(\alpha-\beta)cos.\int Xdx+(\alpha+\beta)sin.\int Xdx\cdot\sqrt{-1}}{(\alpha+\beta)cos.\int Xdx+(\alpha-\beta)sin.\int Xdx\cdot\sqrt{-1}}\cdot X\sqrt{-1}$$

or on putting $(\alpha-\beta)\sqrt{-1}=\gamma$ and $(\alpha+\beta)=\delta$

$$u=T+\frac{\gamma cos.\int Xdx-\delta sin.\int Xdx}{\delta cos.\int Xdx+\gamma sin.\int Xdx}\cdot X$$

or also

$$u=T+X\ \text{tang.}\left(\int Xdx+\zeta\right).$$

PROBLEM 102

842. *With the element dx assumed constant, to find the complete integral of this second order differential equation*

$$ddy + Adydx + Bydx^2 = 0.$$

SOLUTION

On putting $y = e^{\int udx}$ this equation is produced

$$du + uudx + Audx + Bdx = 0$$

or

$$dx = \frac{-du}{uu + Au + B},$$

which is satisfied by attributing a constant value to *u* of this kind [where *du/dx* = 0], so that it may become $uu + Au + B = 0$, which [constant values] are

$$u = -\tfrac{1}{2}A \pm \sqrt{\left(\tfrac{1}{4}AA - B\right)}$$

Hence therefore since there shall be had two particular integrals $y = e^{\int udx}$, on putting $\sqrt{\left(\tfrac{1}{4}AA - B\right)} = n$ the complete integral will be

$$y = e^{-\frac{1}{2}Ax}\left(\alpha e^{nx} + \beta e^{-nx}\right),$$

and if *n* shall be an imaginary number, on putting $n = m\sqrt{-1}$, there shall be

$$y = e^{-\frac{1}{2}Ax}\left(\alpha \cos .mx + \beta \sin .mx\right) = Ce^{-\frac{1}{2}Ax} \sin .\left(mx + \gamma\right).$$

But if there should be sit *n* = 0, then there is produced

$$y = e^{-\frac{1}{2}Ax}\left(\alpha + \beta x\right).$$

COROLLARY 1

843. Hence in finding the integral of the proposed equation it is required to resolve the algebraic equation

$$uu + Au + B = 0,$$

which arises from the proposed equation $ddy + Adydx + Bydx^2 = 0$, if in place of *y, dy, ddy* there is written u^0, u^1, u^2 and the element *dx* is removed ; for then the two roots of this equation will give the complete integral.

COROLLARY 2

844. Clearly if the factors of the equation $uu + Au + B = 0$ shall be $(u+f)(u+g)$, on account of the values $u = -f$ and $u = -g$ the complete integral will be
$y = ae^{-fx} + \beta e^{-gx}$.
But if there shall be $g = f$, then there will be $y = e^{-fx}(a + \beta x)$.

COROLLARY 3

845. If the equation $uu + Au + B = 0$ should have imaginary factors, in which case a form of this kind will be had $uu + 2fu\cos.\zeta + ff = 0$, then there becomes

$$u = -f\cos.\zeta \pm f\sqrt{-1}.\sin.\zeta$$

and the complete integral will be

$$y = e^{-fx\cos.\zeta}\left(\alpha\cos.(fx\sin.\zeta) + \beta\sin.(fx\sin.\zeta)\right)$$

or

$$y = Ce^{-f\cos.\zeta}\left(fx\sin.\zeta + \gamma\right).$$

SCHOLIUM

846. The same complete integral can be found by the usual method from the equation $dx = \frac{-du}{uu+Au+B}$; for on putting $uu + Au + B = (u+f)(u+g)$ there will be

$$(g-f)dx = \frac{du}{u+g} - \frac{du}{u+f} \quad \text{and} \quad Ce^{(g-f)x} = \frac{u+g}{u+f},$$

from which there becomes

$$u = \frac{g - Cfe^{(g-f)x}}{Cfe^{(g-f)x} - 1} \text{ or } u = \frac{-\alpha fe^{gx} + \beta ge^{fx}}{\alpha e^{gx} - \beta e^{fx}}.$$

Then truly there shall be

$$\int udx = -\int \frac{\alpha fe^{(g-f)x} - \beta g}{\alpha e^{(g-f)x} - \beta}dx = -\int \frac{udu}{(u+f)(u+g)}$$

and thus [from taking partial fractions]

$$\int udx = \frac{f}{g-f}l(u+f) - \frac{g}{g-f}l(u+g).$$

Hence

$$y = e^{\int udx} = C(u+f)^{\frac{f}{g-f}}(u+g)^{\frac{-g}{g-f}}.$$

But there is

$$u + f = \frac{\beta(g-f)e^{fx}}{\alpha e^{gx} - \beta e^{fx}} \quad \text{and} \quad u + g = \frac{\alpha(g-f)e^{gx}}{\alpha e^{gx} - \beta e^{fx}},$$

from which there is deduced on changing the constant C

$$y=\frac{Ce^{\frac{ffx}{g-f}}e^{\frac{-ggx}{g-f}}}{\left(\alpha e^{gx}-\beta e^{fx}\right)^{\frac{f-g}{g-f}}}=Ce^{-(f+g)x}\left(\alpha e^{gx}-\beta e^{fx}\right)$$

or $y=ae^{-fx}+\beta e^{-gx}$ as before. Hence it is therefore apparent, how much help is brought to the formation of the complete integral from the two particular integrals.

PROBLEM 103

847. *On assuming the element dx constant, if this second order differential equation is put in place*

$$ddy+\frac{Adydx}{x}+\frac{Bydx^2}{xx}=0,$$

to find the complete integral of this.

SOLUTION

There is put $dy=pdx$ and $dp=qdx$, so that we may have :

$$q+\frac{Ap}{x}+\frac{By}{xx}=0 \quad \text{or} \quad q=-\frac{Ap}{x}-\frac{By}{xx}.$$

Now let there be $p=\frac{uy}{x}$; then there shall be $dy=\frac{uydx}{x}$ and

$$dp=\frac{udy+ydu}{x}-\frac{uydx}{xx}=-\frac{Ady}{x}-\frac{Bydx}{xx}$$

from which we gather

$$\frac{dy}{y}=\frac{udx}{x}=\frac{udx-Bdx-xdu}{xu+Ax}$$

or

$$xdu+Bdx+uudx+(A-1)udx=0$$

and hence

$$\frac{dx}{x}=\frac{-du}{uu+(A-1)u+B},$$

to which in particular this is satisfied on putting $uu+(A-1)u+B=0$.

In the first place let $uu+(A-1)u+B=(u+f)(u+g)$; in particular there will be $ly=-flx$ and $y=x^{-f}$, and in a similar manner $y=x^{-g}$, from which the complete integral will be

$$y = \alpha x^{-f} + \beta x^{-g}$$

If there may be put $g = f$, then on putting in place $g = f - \omega$; with ω vanishing there will be $x^{-g} = x^{-f} \cdot x^{-\omega} = x^{-f}(1 + \omega lx)$,
hence in this case there becomes

$$y = x^{-f}(\alpha + \beta lx).$$

Finally let $uu + (A-1)u + B = uu + 2fu\cos.\zeta + ff$; then there will be

$$u = -f\left(\cos.\zeta \pm \sqrt{-1}\cdot\sin.\zeta\right),$$

hence in particular

$$y = x^{-f\cos.\zeta} \cdot x^{\pm f\sqrt{-1}.\sin.\zeta} = x^{-f\cos.\zeta}\left(\cos.\left(f\sqrt{-1}.\sin.\zeta \cdot lx\right) \pm \sqrt{-1}\cdot\sin.\left(f\sin.\zeta.lx\right)\right),$$

whereby the complete integral will be

$$y = Cx^{-f\cos.\zeta}\sin.\left(f\sin.\zeta \cdot lx + \gamma\right).$$

COROLLARY 1

848. Hence the complete integral of this equation

$$ddy + (f + g + 1)\frac{dydx}{x} + \frac{fgydx^2}{xx} = 0$$

is

$$y = \alpha x^{-f} + \beta x^{-g}.$$

But the complete integral of this equation

$$ddy + (2f + 1)\frac{dydx}{x} + \frac{ffydx^2}{xx} = 0$$

is

$$y = x^{-f}(\alpha + \beta lx).$$

COROLLARIUM 2

849. But if the proposed equation should have a form of this kind

$$ddy + (1 + 2f\cos.\zeta)\frac{dydx}{x} + \frac{ffydx^2}{xx} = 0,$$

then the complete integral of this will be

$$y = Cx^{-f\cos.\zeta}\sin.\left(f\sin.\zeta\cdot lx+\gamma\right).$$

SCHOLIUM

850. This second order differential equation also admits a resolution :

$$ddy - \tfrac{ndydx}{x} + Ax^n dydx + Bx^{2n}ydx^2 = 0.$$

For on putting $dy = x^n yudx$, and since there shall be

$$ddy = x^n ydxdu + nx^{n-1}yudx^2 + x^{2n}yuudx^2,$$

there becomes on dividing by y :

$$x^n dxdu + nx^{n-1}udx^2 + x^{2n}uudx^2 - nx^{n-1}udx^2 + Ax^{2n}udx^2 + Bx^{2n}dx^2 = 0$$

hence

$$du + x^n uudx + Ax^2 udx + Bx^2 dx = 0$$

and thus

$$x^n dx = \tfrac{-du}{uu+Au+B},$$

to which in particular it is satisfied on putting $uu + Au + B = 0$, from which u the two values of the constant follows, of which one shall be $u = -f$, and the other $u = -g$. On account of which the particular integrals will be

$$y = e^{\frac{-f}{n+1}x^{n+1}} \quad \text{and} \quad y = e^{\frac{-g}{n+1}x^{n+1}},$$

For the sake of brevity let $\dfrac{x^{n+1}}{n+1} = t$; the complete integral will be

$$y = \alpha e^{-ft} + \beta e^{-gt},$$

clearly for the case $uu + Au + B = (u+f)(u+g)$.

But for the case $uu + Au + B = (u+f)^2$ there will be

$$y = e^{-ft}(\alpha + \beta t).$$

Moreover for the case, in which $uu + Au + B = uu + 2fu\cos.\zeta + ff$, there will be

$$y = Ce^{-ft\cos.\zeta}\sin.\left(ft\sin.\zeta+\gamma\right).$$

This integration thus can be extended to this form on taking for X some function of x

$$ddy - \frac{dxdy}{X} + AXdydx + BXXydx^2 = 0.$$

For on putting $dy = Xuydx$ or $\frac{dy}{y} = Xudx$ there shall be

$$Xdx = \frac{-du}{uu+Au+B},$$

from which on putting $\int Xdx = t$, the complete integral will be had as before. Clearly

1) if $A = f + g$ and $B = fg$, the integral will be

$$y = \alpha e^{-ft} + \beta e^{-gt},$$

2) if $A = 2f$ and $B = ff$, the integral will be

$$y = e^{-ft}(\alpha + \beta t)$$

3) if $A = 2f\cos.\zeta$ and $B = ff$, the integral will be

$$y = Ce^{-ft\cos.\zeta}\sin.(ft\sin.\zeta + \gamma).$$

PROBLEM 104

851. *On assuming the element dx constant, if P, Q and X denote some functions of x, to reduce the integration of this second order differential equation*

$$ddy + Pdydx + Qydx^2 = Xdx^2$$

to a differential equation of the first order.

SOLUTION

Here we may proceed in the singular manner by introducing two new unknowns in place of y.
Clearly there can be put in place $y = uv$, and since there shall be

$$dy = udv + vdu \quad \text{and} \quad ddy = uddv + 2dudv + vddu,$$

our equation adopts this form

$$uddv + 2dudv + vddu + Pudxdv + Pvdxdu + Quvdx^2 = Xdx^2.$$

Now the one variable v thus may be found, so that so that the terms affected by the letter u cancel out, which shall happen, if

$$ddv + Pdvdx + Qvdx^2 = 0, \ [*]$$

from which by the above [§ 831] v can be determined in terms of x ; with which accomplished there remains this equation :

$$2dudv + vddu + Pvdxdu = Xdx^2,$$

from which, since v now can be given in terms of x, the quantity u must be defined. Putting $du = sdx$, there shall be

$$vds + 2sdv + Psvdx = Xdx,$$

which multiplied by $ve^{\int Pdx}$ is rendered integrable; indeed there is produced

$$vvse^{\int Pdx} = \int e^{\int Pdx} Xvdx$$

and thus

$$s = \frac{e^{-\int Pdx}}{vv} \int e^{\int Pdx} Xvdx \text{ and } u = \int \frac{e^{-\int Pdx} dx}{vv} \int e^{\int Pdx} Xvdx$$

Whereby since the unknown v was to be determined from the equation

$$ddv + Pdxdv + Qvdx^2 = 0,,$$

the integral of the proposed equation will be

$$y = v \int \frac{e^{-\int Pdx} dx}{vv} \int e^{\int Pdx} Xvdx.$$

COROLLARY 1

852. So that the integration of the differential equation of the first order can be recalled [*], there is put $v = e^{\int tdx}$ and the quantity t can be defined by this equation :

$$dt + ttdx + Ptdx + Qdx = 0,$$

from which done the integral sought will be [on replacing v everywhere] :

$$y = e^{\int tdx}\int e^{-\int(P+2t)dx}dx\int e^{\int(P+t)dx}Xdx.$$

COROLLARY 2

853. Since there shall be $(P+t)dx = -\frac{dt}{t} - \frac{Qdx}{t}$ then $e^{\int(P+t)dx} = \frac{1}{t}e^{-\int\frac{Qdx}{t}}$, and hence

$$y = e^{\int tdx}\int e^{\int\frac{Qdx}{t} - \int tdx}tdx\int e^{-\int\frac{Qdx}{t}}X\,\frac{dx}{t},$$

where the twofold integration leads to the complete integral.

SCHOLIUM 1

854. By another method, which approaches closer to the previous usage, the same integration can be put in place. Evidently [from the above solution] there can be put in place for the proposed equation $dy = tydx + vdx$, where v designates a certain function of x being determined from the function X.
Therefore since there shall be

$$ddy = ydtdx + tdx(tydx + vdx) + dvdx,$$

there becomes with the substitution made :

$$ydtdx + ttydx^2 + Ptydx^2 + Qydx^2 + tvdx^2 + dvdx + Pvdx^2 - Xdx^2 = 0,$$

each part of which equation, both that which is multiplied by y, as well as that free from y separately may be equated to zero, from which we obtain these two equations

$$dt + ttdx + Ptdx + Qdx = 0 \text{ and } dv + tvdx + Pvdx = Xdx,$$

from which that t must be defined in terms of x as before ; then truly there will be from the second equation :

$$e^{\int(P+t)dx}v = \int e^{\int(P+t)dx}Xdx.$$

Now from the assumed equation $dy - tydx = vdx$ there is deduced

$$e^{-\int tdx}y = \int e^{-\int tdx}vdx,$$

where if in place of v the value found in this way may be substituted, there will be obtained the preceding form of the integral.

SCHOLIUM 2

855. From this operation it is seen to follow that the integration of the proposed equation

$$ddy + Pdydx + Qydx^2 = Xdx^2$$

necessarily depends on the integration of this equation :

$$ddv + Pdvdx + Qvdx^2 = 0,$$

since with that conceded the other can be shown. Yet this cannot be deduced at all in turn, if the resolution of the latter exceeds our resources, also the first cannot be integrated in any way ; yet it is rather easy to show the infinite case, in which the first integration is allowed, since yet the latter remains unresolved. For on letting $P =$ 0 and $Q = \alpha x$ then it is certain that at this point the equation

$$ddv + \alpha xvdx^2 = 0$$

cannot be resolved by this method, since on putting $v = e^{\int tdx}$ it changes into $dt + ttdx + \alpha xdx = 0$; yet neither yet does it follow that the first equation

$$ddy + \alpha xydx^2 = Xdx^2$$

is always intractable. Indeed there can be assigned an infinity of cases to X , in which the integration succeeds. For with some function of *x* taken for *y* a function of this kind is found for *X*, so that the assumed value of *y* satisfies the equation. Just as on putting $y = \frac{\beta x}{\alpha}$ on account of $ddy = 0$ there becomes $X = \beta xx$ and for the equation

$$ddy + \alpha xydx^2 = \beta xxdx^2$$

to be satisfied everywhere by the integral $y = \frac{\beta x}{\alpha}$. Yet meanwhile this is only a particular integral and doubt is relinquished at this point, or it may be possible to show the complete integral. But on putting $y = \frac{\beta x}{\alpha} + z$ for finding the complete integral there is produced $ddz + \alpha xzdx^2 = 0$, since the resolution may be rejected, it is evident that the complete integral also in general cannot be shown, and likewise the integration of the other equation is allowed.

PROBLEM 105

856. *On assuming the element dx constant, to find the complete integral of this second order differential equation*

$$ddy + Adydx + Bydx^2 = Xdx^2$$

with X denoting some function of x.

SOLUTION

On putting $y = uv$ the proposed equation is resolved into the two following equations:

$$ddv + Advdx + Bvdx^2 = 0 \text{ and } vddu + 2dvdu + Avdxdu = Xdx^2.$$

But if from the first the value of v is defined in terms of x, the complete integral thus will be had from the second, so that on putting $P = A$ there shall be

$$y = v\int \frac{e^{-Ax}dx}{vv}\int e^{Ax}Xvdx,$$

where since the twofold integration produces the complete integral, it suffices for v to be assumed from the particular integral of the first equation, that which also becomes apparent from the general solution. Therefore since for the resolution of the first equation this quadratic equation shall be formed :

$$tt + At + B = 0,$$

for the nature of this it is agreed to set out three cases.

I. If $tt + At + B = (t+f)(t+g)$, so that there shall be $A = f + g$ and $B = fg$, then there will be

$$v = \alpha e^{-fx} + \beta e^{-gx}.$$

Yet first on taking the particular integral $v = e^{-fx}$ and on account of $A = f + g$ there becomes

$$y = e^{-fx}\int e^{(f-g)x}dx\int e^{gx}Xdx;$$

let

$$e^{(f-g)x}dx = dR \quad \text{and} \quad \int e^{gx}Xdx = S,$$

so that there becomes

$$y = e^{-fx}\int SdR = e^{-fx}\left(RS - \int RdS\right);$$

but there is $R=\frac{1}{f-g}e^{(f-g)x}$, from which there is deduced

$$y=\frac{1}{f-g}e^{-gx}S-\frac{1}{f-g}e^{-fx}\int e^{fx}Xdx$$

or

$$(f-g)y=e^{-gx}\int e^{gx}Xdx-e^{-fx}\int e^{fx}Xdx\,,$$

because the same integral shall be produced, if we should be using the other particular integral $v=e^{-gx}$.

But in general on taking $v=\alpha e^{-fx}+\beta e^{-gx}$ there is put in place as before

$$\frac{e^{-(f+g)x}}{vv}=dR \quad \text{and} \quad \int e^{(f+g)x}Xvdx=S\,,$$

so that equally there shall be

$$y=v\int SdR=v\left(RS-\int RdS\right).$$

We may put in place $R=C\frac{e^{\lambda x}}{v}$ and on account of $dv=-dx\left(\alpha fe^{-fx}+\beta ge^{-gx}\right)$ there will be

$$dR=\frac{Ce^{\lambda x}dx\left(\alpha\lambda e^{-fx}+\beta\lambda ge^{-gx}+\alpha fe^{-fx}+\beta ge^{-gx}\right)}{vv}.$$

Now let $\lambda=-g$ and $C\alpha(f-g)=1$, so that the given value dR is come upon; on account of $C=\frac{1}{\alpha(f-g)}$ there will be

$$R=\frac{e^{-gx}}{\alpha(f-g)v} \quad \text{and} \quad RdS=\frac{1}{\alpha(f-g)}e^{fx}Xdx\,,$$

then indeed

$$S=\alpha\int e^{gx}Xdx+\beta\int e^{fx}Xdx\,,$$

from which there is made

$$y=v\left(\frac{e^{-gx}}{(f-g)v}\int e^{gx}Xdx+\frac{\beta e^{-gx}}{\alpha(f-g)v}\int e^{fx}Xdx-\frac{1}{\alpha(f-g)}\int e^{fx}Xdx\right)$$

or therefore as before :

$$(f-g)y = e^{-gx}\int e^{gx}Xdx - e^{-fx}\int e^{fx}Xdx$$

II. If $tt + At + B = (t+f)^2$ or $A = 2f$ et $B = ff$, from the first equation there will be $v = e^{-fx}(\alpha + \beta x)$. Putting as before :

$$\frac{e^{-2fx}dx}{vv} = dR \quad \text{and} \quad \int e^{2fx}Xvdx = S,$$

so that there is had $y = v\left(RS - \int RdS\right)$. Hence since there shall be $dR = \frac{dx}{(\alpha+\beta x)^2}$ then

$$R = -\frac{1}{\beta(\alpha+\beta x)} = -\frac{e^{-fx}}{\beta v} \quad \text{and} \quad S = \alpha\int e^{fx}Xdx + \beta\int e^{fx}Xxdx = \int e^{fx}Xdx(\alpha+\beta x),$$

whereby

$$vRS = -\frac{\alpha}{\beta}e^{-fx}\int e^{fx}Xdx - e^{-fx}\int e^{fx}Xxdx \quad \text{and} \quad \int RdS = -\frac{1}{\beta}\int e^{fx}Xdx,$$

from which there is completed

$$y = e^{-fx}x\int e^{fx}Xdx - e^{-fx}\int e^{fx}Xxdx$$

or since there shall be

$$d.e^{fx}y = dx\int e^{fx}Xdx,$$

that becomes more succinctly

$$y = e^{-fx}\int dx\int e^{fx}Xdx.$$

III. If $tt + At + B = tt + 2ft\cos.\zeta + ff$ or $A = 2f\cos.\zeta$ and $B = ff$, then

$$v = e^{-fx\cos.\zeta}\sin.(fx\sin.\zeta + \gamma).$$

Putting

$$\frac{e^{-2fx\cos.\zeta}dx}{vv} = \frac{dx}{\sin.(fx\sin.\zeta+\gamma)^2} = dR$$

and

$$e^{2fx\cos.\zeta}Xvdx = e^{fx\cos.\zeta}Xdx\sin.(fx\sin.\zeta+\gamma) = dS,$$

so that there is obtained $y = vRS - v\int RdS$. But there is

$$R = -\tfrac{1}{f\,sin.\zeta}\cdot\tfrac{cos.(fx\,sin.\zeta+\gamma)}{sin.(fx\,sin.\zeta+\gamma)},$$

hence

$$vRS = -\tfrac{1}{f\,sin.\zeta}e^{-fx\,cos.\zeta}\,cos.(fx\,sin.\zeta+\gamma)\int e^{fx\,cos.\zeta}Xdx\,sin.(fx\,sin.\zeta+\gamma)$$

and

$$\int RdS = -\tfrac{1}{f\,sin.\zeta}\int e^{fx\,cos.\zeta}Xdx\,cos.(fx\,sin.\zeta+\gamma).$$

On account of which there is obtained

$$\begin{aligned} fy\,sin.\zeta = &+e^{-fx\,cos.\zeta}\,sin.(fx\,sin.\zeta+\gamma)\int e^{fx\,cos.\zeta}Xdx\,cos.(fx\,sin.\zeta+\gamma)\\ &-e^{-fx\,cos.\zeta}\,cos.(fx\,sin.\zeta+\gamma)\int e^{fx\,cos.\zeta}Xdx\,sin.(fx\,sin.\zeta+\gamma).\end{aligned}$$

COROLLARY 1

857. In this last integral if we put $fx\,sin.\zeta=\varphi$, there will be

$$\begin{aligned} fe^{fx\,cos.\zeta}y\,sin.\zeta = &(sin.\gamma\,cos.\varphi+cos.\gamma\,sin.\varphi)\int e^{fx\,cos.\zeta}Xdx(cos.\gamma\,cos.\varphi-sin.\gamma\,sin.\varphi)\\ &+(sin.\gamma\,sin.\varphi-cos.\gamma\,cos.\varphi)\int e^{fx\,cos.\zeta}Xdx(sin.\gamma\,cos.\varphi+cos.\gamma\,sin.\varphi)\end{aligned}$$

or

$$\begin{aligned} &fe^{fx\,cos.\zeta}y\,sin.\zeta =\\ &+sin.\gamma\,cos.\gamma\,cos.\varphi\int e^{fx\,cos.\zeta}Xdx\,cos.\varphi - sin.\gamma^2\,cos.\varphi\int e^{fx\,cos.\zeta}Xdx\,sin.\varphi\\ &+cos.\gamma^2\,sin.\varphi\int e^{fx\,cos.\zeta}Xdx\,cos.\varphi - sin.\gamma\,cos.\gamma\,sin.\varphi\int e^{fx\,cos.\zeta}Xdx\,sin.\varphi\\ &+sin.\gamma^2\,sin.\varphi\int e^{fx\,cos.\zeta}Xdx\,cos.\varphi + sin.\gamma\,cos.\gamma\,sin.\varphi\int e^{fx\,cos.\zeta}Xdx\,sin.\varphi\\ &-sin.\gamma\,cos.\gamma\,cos.\varphi\int e^{fx\,cos.\zeta}Xdx\,cos.\varphi - cos.\gamma^2\,cos.\varphi\int e^{fx\,cos.\zeta}Xdx\,sin.\varphi,\end{aligned}$$

from which it is apparent that the angle γ is completely excluded from the calculation ; indeed there becomes :

$$fe^{fx\,cos.\zeta}y\,sin.\zeta = sin.\varphi\int e^{fx\,cos.\zeta}Xdx\,cos.\varphi - cos.\varphi\int e^{fx\,cos.\zeta}Xdx\,sin.\varphi.$$

COROLLARY 2

858. Therefore since in place of a single equation we have formed two integrands, we see that it suffices if perhaps one integral should be considered as particular. For in both the preceding cases the constants α et β giving the complete integral vanish from the calculation, and in the third case the constant γ likewise has vanished.

EXAMPLE

859. *With the element dx assumed constant to find the integral of this equation :*

$$ddy + Adydx + Bydx^2 = dx^2\left(n(n-1)x^{n-2} + nAx^{n-1} + Bx^n\right).$$

This example has been prepared thus, so that for that to be evidently satisfied by the value $y = x^n$, which constitutes a particular integral of this. Therefore towards finding the complete integral let $A = f + g$ and $B = fg$, and since there shall be :

$$X = fgx^n + n(f+g)x^{n-1} + n(n-1)x^{n-2},$$

then

$$\int e^{gx}Xdx = fe^{gx}x^n + ne^{gx}x^{n-1} + \alpha \quad \text{and} \quad \int e^{fx}Xdx = ge^{fx}x^n + ne^{fx}x^{n-1} + \beta,$$

from which from the form found the complete integral emerges

$$(f-g)y = fx^n + nx^{n-1} + \alpha e^{-gx} - gx^n - nx^{n-1} - \beta e^{-fx}$$

or

$$y = x^n + \tfrac{\alpha}{f-g}e^{-gx} - \tfrac{\beta}{f-g}e^{-fx}$$

or with the form of the constants changed

$$y = x^n + \alpha e^{-fx} + \beta e^{-gx}.$$

If there should be $g = f$, putting $g = f + w$ with $\omega = 0$ arising and on account of

$$e^{-gx} = e^{-fx}\cdot e^{-\omega x} = e^{-fx}(1-\omega x)$$

for $\alpha + \beta$ and $-\beta\omega$ on writing α and β there will be

$$y = x^n + e^{-fx}(\alpha + \beta x).$$

But if there should be $f = a + b\sqrt{-1}$ and $g = a - b\sqrt{-1}$, since there becomes

$$y = x^n + e^{-ax}\left(\alpha e^{-bx\sqrt{-1}} + \beta e^{bx\sqrt{-1}}\right),$$

on account of $e^{\pm bx\sqrt{-1}} = cos.bx \pm \sqrt{-1} \cdot sin.bx$ with the form of the constants changed, we will have :

$$y = x^n + e^{-ax}\left(\alpha \cos.bx + \beta \sin.bx\right).$$

SCHOLIUM

860. But in general if a particular integral of this kind of equation

$$ddy + Adydx + Bydx^2 = Xdx^2$$

may be agreed upon or the value of this satisfying $y = t$, then the complete integral can be found easily on putting $y = t + z$. For since by hypothesis there shall be

$$ddt + Adtdx + Btdx^2 = Xdx^2,$$

with this substitution made there shall arise

$$ddz + Adzdx + Bzdx^2 = 0,$$

from which, if $A = f + g$ and $B = fg$, there is deduced $z = \alpha e^{-fx} + \beta e^{-gx}$ and thus the complete integral will be

$$y = t + \alpha e^{-fx} + \beta e^{-gx}$$

just as also we have found in the example reported on.

[We now of course say that the general integral is the sum of the complementary function and the particular integral.]

PROBLEM 106

861. *With the element dx assumed constant, if this second order differential equation is put in place*

$$ddy - \frac{ndydx}{x} + Ax^n dydx + Bx^{2n} ydx^2 = Xdx^2$$

with some function X of x arising, to find the complete integral of this.

SOLUTION

The resolution of this equation as above can be derived from putting in place $y = uv$, but here by using another method brought to light we can reduce that by substitution to the form of the preceding problem.

Clearly we can put $x^n dx = dt$, so that there shall be $x^{n+1} = (n+1)t$, with which substituted the function X changes into a certain function T of t. But lest the assumption of the constant element dx may be disturbed, we remove this condition by putting $dy = pdx$ and $dp = qdx$ and we will have

$$q - \frac{np}{x} + Ax^n p + Bx^{2n} y = X$$

Now since there shall be $dx = \frac{dt}{x^n}$ there will be $p = x^n \frac{dy}{dt}$ and hence now on assuming the element dt constant

$$dp = \frac{nx^{n-1}dxdy}{dt} + \frac{x^n ddy}{dt} = qdx = \frac{qdt}{x^n}, \quad \text{hence} \quad q = \frac{nx^{n-1}dy}{dt} + \frac{x^{2n}ddy}{dt^2}$$

and thus our equation will be

$$\frac{x^{2n}ddy}{dt^2} + \frac{nx^{n-1}dy}{dt} - \frac{nx^{n-1}dy}{dt} + \frac{Ax^{2n}dy}{dt} + Bx^{2n} y = X.$$

Let $Xx^{-2n} = \Theta$, which quantity on putting $x^{n+1} = (n+1)t$ can be considered as a function of t, and thus there becomes

$$ddy + Adydt + Bydt^2 = \Theta dt^2,$$

in which equation the element dt has been assumed constant, hence the integral of this by the above is given [§ 856].

1. If $A = f + g$ and $B = fg$, the integral will be

$$(f-g)\,y=e^{-gt}\int e^{gt}\Theta dt-e^{-ft}\int e^{ft}\Theta dt,$$

where with the values restored $dt=x^{n}dx$ et $\Theta=Xx^{-2n}$, on retaining $t=\frac{1}{(n+1)}x^{n+1}$ for the sake of brevity, the value of y thus can be expressed in terms of x

$$(f-g)\,y=e^{-gt}\int e^{gt}x^{-n}Xdx-e^{-ft}\int e^{ft}x^{-n}Xdx.$$

II. If $A=2f$ and $B=ff$, the integral will be

$$y=e^{-ft}t\int e^{ft}\Theta dt-e^{-ft}\int e^{ft}\Theta tdt \quad \text{or} \quad y=e^{-ft}\int dt\int e^{ft}\Theta dt,$$

which therefore can be expressed in terms of x

$$y=e^{-ft}\int x^{n}dx\int e^{ft}x^{-n}Xdx.$$

III. If finally $A=2f\cos.\zeta$ et $B=ff$, the integral will be

$$fe^{ft\cos.\zeta}y\sin.\zeta=\sin.\varphi\int e^{ft\cos.\zeta}\Theta dt\cos.\varphi-\cos.\varphi\int e^{ft\cos.\zeta}\Theta dt\sin.\varphi$$

with $\varphi=ft\sin.\zeta$ arising, or $\varphi=\frac{f\sin.\zeta}{n+1}x^{n+1}$ on account of $t=\frac{1}{n+1}x^{n+1}$.

Whereby the integral of the proposed equation will be

$$fe^{ft\cos.\zeta}y\sin.\zeta=\sin.\varphi\int e^{ft\cos.\zeta}x^{-n}Xdx\cos.\varphi-\cos.\varphi\int e^{ft\cos.\zeta}x^{-n}Xdx\sin.\varphi.$$

COROLLARY 1

862. If $n=0$, the proposed equation will change into that itself, that we treated in the preceding problem, and there becomes $t=x$, from which also the same integral is returned.

COROLLARY 2

863. But if there shall be $n=-1$, our equation becomes

$$ddy+(A+1)\frac{dydx}{x}+\frac{Bydx^{2}}{xx}=Xdx^{2},$$

where hence there will be $t=lx$ and $e^{\lambda t}=x^{\lambda}$, then truly for the third case the angle $\varphi=f\sin.\zeta\cdot lx$.

SCHOLIUM

864. The method, which we have used here, is considered not to integrate enough second order differential equations of this kind. Because in finding factors in differential equations of the first order, in which these are rendered themselves integrable, a conspicuous reward was seen to be offered, hence we might try also to show the use of this in second order differential equations. Here indeed nothing so complete is allowed to be expected, which extends generally to all forms of equations, but also we will be able to excel a little, and that should not be despised but considered to advance analysis. But by this method these second order differential equations can be treated well enough, in which mainly the other variable y with its differentials never exceeds one dimension, and thus a way may be seen conveniently, how that may be improved upon more.

CAPUT IV

DE AEQUATIONIBUS DIFFERENTIO-DIFFERENTIALIBUS IN QUIBUS ALTERA VARIABILIS UNICAM HABET DIMENSIONEM

PROBLEMA 101

83l. *Sumto elemento dx constante si proponatur aequatio huius formae*

$$ddy + Pdxdy + Qydx^2 = 0,$$

ubi P et Q sint functiones quaecunque ipsius x, eam ad aequationem differentialem primi gradus revocare.

SOLUTIO

Ponendo $dy = pdx$ et $dp = qdx$ aequatio proposita induit hanc formam $q + Pp + Qy = 0$; in qua si methodo ante exposita statuamus $p = uy$ et $q = vy$, obtinebimus hanc inter *x, u* et *v* aequationem $v + Pu + Q = 0$ hincque $v = -Pu - Q$. Tum vero fit $dy = uydx$ et $udy + ydu = vydx$, ita ut $\frac{dy}{y} = udx = \frac{vdx-du}{u}$ ideoque

$$du + uudx + Pudx + Qdx = 0$$

substituto pro *v* valore. Qua aequatione resoluta erit $ly = \int udx$.

Vel sine his substitutionibus statim in ipsa aequatione proposita ponamus $y = e^{\int udx}$, unde fit $dy = e^{\int udx}udx$ et $ddy = e^{\int udx}\left(dudx + uudx^2\right)$, et cum facta substitutione quantitas exponentialis $e^{\int udx}$ ex calculo tollatur, obtinebitur praecedens aequatio differentialis primi gradus

$$du + uudx + Pudx + Qdx = 0,$$

a cuius resolutione integratio aequationis differentio–differentialis propositae pendet.

COROLLARIUM 1

832. Haec aequatio differentialis primi gradus pluribus modis in alias formas sibi fere similes transmutari potest. Veluti si ponamus $u = Mz$, prodit

$$Mdz + z\left(dM + PMdx\right) + MMzzdx + Qdx = 0,$$

ubi pro *M* eiusmodi functionem ipsius *x* accipere licet, ut terminus ipsa littera *z* affectus evanescat, quod fit, si

$$dM + MPdx = 0 \text{ seu } M = Ce^{-\int Pdx}$$

COROLLARIUM 2

833. Similis forma prodit ponendo $u = \frac{K}{z}$; fit enim

$$-\tfrac{Kdz}{zz} + \tfrac{dK}{z} + \tfrac{KKdx}{zz} + \tfrac{KPdx}{z} + Qdx = 0$$

seu

$$Kdz - z\left(dK + KPdx\right) - Qzzdx - KKdx = 0,$$

ubi secundus terminus sumendo $K = Ce^{-\int Pdx}$ pariter evanescit.

COROLLARIUM 3

834. Similis transformatio generalius instituitur ponendo $u = K + Mz$;
prodit enim

$$dK + Mdz + zdM + KKdx + 2KMzdx + MMzzdx^2$$
$$+ KPdx + MPzdx + Qdx = 0,$$

quae ordinata praebet

$$Mdz + z\left(dM + 2KMdx + MPdx\right) + MMzzdx^2$$
$$+dK + KKdx + KPdx + Qdx = 0,$$

unde secundus terminus tollitur sumendo

$$M = Ce^{-\int dx(2K+P)} \text{ seu } K = \tfrac{-dM - MPdx}{2Mdx}.$$

COROLLARIUM 4

835. Adhuc generalius similis forma oritur, si ponatur $u = \frac{K+Mz}{L+Nz}$, unde prodit

$$dz\left(LM - KN\right) + LdK - KdL + z\left(LdM - MdL + NdK - KdN\right)$$
$$+zz\left(NdM - MdN\right) + \left(K + Mz\right)^2 dx$$
$$+P\left(K + Mz\right)\left(L + Nz\right)dx + Q\left(L + Nz\right)^2 dx = 0,$$

quae reducitur ad hanc formam

$$0 = dz\left(LM - KN\right) + z\left(LdM - MdL + NdK - KdN + 2KMdx + P\left(KN + LM\right)dx + 2LNQdx\right)$$
$$+ zz\left(NdM - MdN + MMdx + MNPdx + NNQdx\right)$$
$$+ LdK - KdL + KKdx + KLPdx + LLQdx,$$

ubi pro K, L, M et N functiones eiusmodi ipsius x accipere licet, ut forma prodeat tractatu facillima.

SCHOLION

836. Quoniam huiusmodi aequationes differentio–differentiales, in quibus variabilis y unicam habet dimensionem, frequentissime occurrere solent, merito geometrae tantopere in resolvenda aequatione

$$du + uudx + Pudx + Qdx = 0$$

studium et operam collocarunt, quae etiam forma generaliori ita repraesentan potest cuius quidem casum eximium $dl + zldx = axfldx$ olim Comes RICCATI in haud spernendum Analyseos incrementum proposuerat. Inter transformationes autem huius casus praecipue notari meretur positio

$$x = t^{\frac{2}{n+2}},$$

quae dat

$$dz + \tfrac{2}{n+2}zzt^{\frac{-2}{n+2}}dt = \tfrac{2a}{n+2}t^{\frac{2}{n+2}}dt,$$

unde ponendo $z = Ct^{\frac{n}{n+2}}v$ prodit

$$\left(n+2\right)Ct^{\frac{n}{n+2}}dv + Cnt^{\frac{-2}{n+2}}vdt + 2CCt^{\frac{n}{n+2}}vvdt = 2at^{\frac{n}{n+2}}dt$$

seu

$$\left(n+2\right)Cdv + \tfrac{Cnvdt}{t} + 2CCvvdt = 2adt,$$

ita ut hic nulla potestas indefinita ipsius t occurrat. Si hic porro ponatur $v = \frac{\alpha}{t} + s$, fiet

$$\left.\begin{array}{ll} -\frac{(n+2)C\alpha dt}{tt} + \left(n+2\right)Cds + \frac{nCsdt}{t} + 2CCssdt & \\ +\ \frac{nC\alpha dt}{tt} \qquad\qquad + \frac{4CC\alpha sdt}{t} & \\ +\ \frac{2CC\alpha\alpha dt}{tt} & \end{array}\right\} = 2adt;$$

ubi si capiatur $\alpha = \frac{-n}{4C}$, orietur

$$\left(n+2\right)Cds + 2CCssdt = 2adt - \frac{n(n+4)dt}{8tt},$$

quae forma simplicissima videtur.

THEOREMA

837. *Sumto elemento dx constante si aequationi differentio–differentiali*

$$ddy + Pdxdy + Qydx^2 = 0$$

satisfaciant integralia particularia $y = M$ *et* $y = N$ *, ita ut ratio* $M : N$ *non sit constans, erit eius integrale completum* $y = \alpha M + \beta N$.

DEMONSTRATIO

Cum valores $y = M$ et $y = N$ satisfaciant aequationi propositae, erit

$$ddM + PdxdM + QMdx^2 = 0 \text{ et } ddN + PdxdN + QNdx^2 = 0,$$

unde patet ponendo $y = \alpha M + \beta N$ aequationi quoque satisfieri, cum fiat

$$\alpha\left(ddM + PdxdM + QMdx^2\right) + \beta\left(ddN + PdxdN + QNdx^2\right) = 0.$$

Quoniam vero hoc integrale $y = \alpha M + \beta N$ duas constantes α et β complectitur, quas pro lubitu definire licet, id completum sit necesse est, nisi forte *N* sit multiplum ipsius *M*.

COROLLARIUM 1

838. Ex datis ergo duobus integralibus particularibus huiusmodi aequationis, eius integrale completum formari potest, siquidem illa duo integralia sint inter se diversa.

COROLLARIUM 2

839. Cum posito $y = e^{\int udx}$ seu $u = \frac{dy}{ydx}$ prodeat

$$du + uudx + Pudx + Qdx = 0,$$

si huic aequationi satisfaciant valores $u = \frac{dM}{Mdx}$ et $u = \frac{dN}{Ndx}$, eidem quoque satisfaciet valor $u = \frac{\alpha dM + \beta dN}{(aN + \beta M)dx}$.

COROLLARIUM 3

840. Si ergo aequationis $du + uudx + Pudx + Qdx = 0$ habeantur duo integralia particularia $u = R$ et $u = S$, ob $M = e^{\int Rdx}$ et $N = e^{\int Sdx}$ integrale completum erit

$$u = \frac{\alpha e^{\int Rdx} R + \beta e^{\int Sdx} S}{\alpha e^{\int Rdx} + \beta e^{\int Sdx}} \quad \text{sive} \quad u = R + \frac{\beta e^{\int Sdx}(S - R)}{\alpha e^{\int Rdx} + \beta e^{\int Sdx}}$$

SCHOLION

841. Maximi momenti est haec observatio, quod in huiusmodi aequationibus ex cognitis binis integralibus particularibus integrale completum assignari possit. Plerumque autem cognito uno integrali particulari in eo signum radicale inesse solet, ob cuius ambiguitatem duo simul integralia particularia innotescunt. Ita si aequationi $du+uudx+Pudx+Qdx=0$ satisfaciat valor $u=T+\sqrt{V}$, eidem satisfaciet $u=T-\sqrt{V}$, unde integrale completum erit

$$u=T+\sqrt{V}-\frac{2\beta\sqrt{V}}{\alpha e^{2\int dx\sqrt{V}}+\beta} \text{ seu } u=T+\frac{\alpha e^{2\int dx\sqrt{V}}\sqrt{V}-\beta\sqrt{V}}{\alpha e^{2\int dx\sqrt{V}}+\beta}.$$

Ac si forte sit $\sqrt{V}$ imaginarium, puta $\sqrt{V}=X\sqrt{-1}$, ob

$$e^{\pm\int dx\sqrt{V}}=cos.\int Xdx\pm\sqrt{-1}\cdot sin.\int Xdx$$

erit

$$u=T+\frac{(\alpha-\beta)cos.\int Xdx+(\alpha+\beta)sin.\int Xdx\cdot\sqrt{-1}}{(\alpha+\beta)cos.\int Xdx+(\alpha-\beta)sin.\int Xdx\cdot\sqrt{-1}}\cdot X\sqrt{-1}$$

seu posito $(\alpha-\beta)\sqrt{-1}=\gamma$ et $(\alpha+\beta)=\delta$

$$u=T+\frac{\gamma cos.\int Xdx-\delta sin.\int Xdx}{\delta cos.\int Xdx+\gamma sin.\int Xdx}\cdot X$$

vel etiam

$$u=T+X\ \text{tang.}\left(\int Xdx+\zeta\right).$$

PROBLEMA 102

842. *Sumto elemento dx constante invenire integrale completum huius aequationis differentio- differentialis*

$$ddy+Adydx+Bydx^2=0.$$

SOLUTIO

Posito $y=e^{\int udx}$ prodit haec aequatio

$$du+uudx+Audx+Bdx=0$$

sive

$$dx=\frac{-du}{uu+Au+B},$$

cui satisfit tribuendo ipsi *u* eiusmodi valorem constantem, ut evadat $uu+Au+B=0$, qui sunt

$$u=-\tfrac{1}{2}A\pm\sqrt{\left(\tfrac{1}{4}AA-B\right)}$$

Hinc ergo cum habeantur bina integralia particularia $y = e^{\int udx}$, posito $\sqrt{\left(\frac{1}{4}AA - B\right)} = n$ erit integrale completum

$$y = e^{-\frac{1}{2}Ax}\left(\alpha e^{nx} + \beta e^{-nx}\right),$$

ac si sit n numerus imaginarius, puta $n = m\sqrt{-1}$, erit

$$y = e^{-\frac{1}{2}Ax}\left(\alpha \cos.mx + \beta \sin.mx\right) = Ce^{-\frac{1}{2}Ax}\sin.\left(mx + \gamma\right).$$

Sin autem sit n = 0, prodibit

$$y = e^{-\frac{1}{2}Ax}\left(\alpha + \beta x\right).$$

COROLLARIUM 1

843. Ad integrale ergo aequationis propositae inveniendum resolvi oportet aequationem algebraicam

$$uu + Au + B = 0,$$

quae oritur ex proposita $ddy + Adydx + Bydx^2 = 0$, si loco y, dy, ddy scribatur u^0, u^1, u^2 et elementum dx reiiciatur; tum enim binae radices illius aequationis dabunt integrale completum.

COROLLARIUM 2

844. Scilicet si aequationis $uu + Au + B = 0$ factores sint $\left(u + f\right)\left(u + g\right)$, ob valores $u = -f$ et $u = -g$ integrale completum erit $y = ae^{-fx} + \beta e^{-gx}$.
At si sit $g = f$, erit $y = e^{-fx}\left(a + \beta x\right)$.

COROLLARIUM 3

845. Si aequatio $uu + Au + B = 0$ habeat factores imaginarios, quo casu huiusmodi formam habebit $uu + 2fu\cos.\zeta + ff = 0$, erit

$$u = -f\cos.\zeta \pm f\sqrt{-1}.\sin.\zeta$$

hincque integrale completum erit

$$y = e^{-fx\cos.\zeta}\left(\alpha\cos.\left(fx\sin.\zeta\right) + \beta\sin.\left(fx\sin.\zeta\right)\right)$$

sive

$$y = Ce^{-f\cos.\zeta}\left(fx\sin.\zeta + \gamma\right).$$

SCHOLION

846. Idem integrale completum reperitur methodo consueta ex aequatione $dx = \frac{-du}{uu+Au+B}$; posito enim $uu + Au + B = (u+f)(u+g)$ erit

$$(g-f)dx = \frac{du}{u+g} - \frac{du}{u+f} \text{ et } Ce^{(g-f)x} = \frac{u+g}{u+f},$$

unde fit

$$u = \frac{g-Cfe^{(g-f)x}}{Cfe^{(g-f)x}-1} \text{ seu } u = \frac{-\alpha fe^{gx}+\beta ge^{fx}}{\alpha e^{gx}-\beta e^{fx}}.$$

Tum vero fit

$$\int udx = -\int \frac{\alpha fe^{(g-f)x}-\beta g}{\alpha e^{(g-f)x}-\beta}dx = -\int \frac{udu}{(u+f)(u+g)}$$

ideoque

$$\int udx = \frac{f}{g-f}l(u+f) - \frac{g}{g-f}l(u+g).$$

Hinc

$$y = e^{\int udx} = C(u+f)^{\frac{f}{g-f}}(u+g)^{\frac{-g}{g-f}}.$$

At est

$$u+f = \frac{\beta(g-f)e^{fx}}{\alpha e^{gx}-\beta e^{fx}} \quad \text{et} \quad u+g = \frac{\alpha(g-f)e^{gx}}{\alpha e^{gx}-\beta e^{fx}},$$

unde colligitur mutando constantem C

$$y = \frac{Ce^{\frac{ffx}{g-f}}e^{\frac{-ggx}{g-f}}}{\left(\alpha e^{gx}-\beta e^{fx}\right)^{\frac{f-g}{g-f}}} = Ce^{-(f+g)x}\left(\alpha e^{gx}-\beta e^{fx}\right)$$

seu $y = ae^{-fx} + \beta e^{-gx}$ ut ante. Hinc ergo patet, quantum subsidium afferat formatio integralis completi ex binis particularibus.

PROBLEMA 103

847. *Sumto elemento dx constante si proponatur haec aequatio differentio-differentialis*

$$ddy + \frac{Adydx}{x} + \frac{Bydx^2}{xx} = 0,$$

eius integrale completum invenire.

SOLUTIO

Ponatur $dy = pdx$ et $dp = qdx$, ut habeamus

$$q+\frac{Ap}{x}+\frac{By}{xx}=0 \quad \text{seu} \quad q=-\frac{Ap}{x}-\frac{By}{xx}.$$

Sit nunc $p=\frac{uy}{x}$; erit $dy=\frac{uydx}{x}$ et

$$dp=\frac{udy+ydu}{x}-\frac{uydx}{xx}=-\frac{Ady}{x}-\frac{Bydx}{xx}$$

unde colligimus

$$\frac{dy}{y}=\frac{udx}{x}=\frac{udx-Bdx-xdu}{xu+Ax}$$

seu

$$xdu+Bdx+uudx+(A-1)udx=0$$

hincque

$$\frac{dx}{x}=\frac{-du}{uu+(A-1)u+B},$$

cui particulariter satisfit ponendo $uu+(A-1)u+B=0$.

Sit primo $uu+(A-1)u+B=(u+f)(u+g)$; erit particulariter $ly=-flx$ et $y=x^{-f}$ similique modo $y=x^{-g}$, unde integrale completum erit

$$y=\alpha x^{-f}+\beta x^{-g}$$

Si sit $g=f$, statuatur $g=f-\omega$; evanescente ω erit $x^{-g}=x^{-f}\cdot x^{-\omega}=x^{-f}(1+\omega lx)$, ergo hoc casu fit

$$y=x^{-f}(\alpha+\beta lx).$$

Sit denique $uu+(A-1)u+B=uu+2fu\cos.\zeta+ff$; erit

$$u=-f\left(\cos.\zeta\pm\sqrt{-1}\cdot\sin.\zeta\right),$$

ergo particulariter

$$y=x^{-f\cos.\zeta}\cdot x^{\pm f\sqrt{-1}.\sin.\zeta}=x^{-f\cos.\zeta}\left(\cos.\left(f\sqrt{-1}.\sin.\zeta\cdot lx\right)\pm\sqrt{-1}\cdot\sin.\left(f\sin.\zeta.lx\right)\right),$$

quare integrale completum erit

$$y=Cx^{-f\cos.\zeta}\sin.\left(f\sin.\zeta\cdot lx+\gamma\right)..$$

COROLLARIUM 1

848. Cuius ergo aequationis

$$ddy+(f+g+1)\frac{dydx}{x}+\frac{fgydx^2}{xx}=0$$

integrale completum est

$$y=\alpha x^{-f}+\beta x^{-g}.$$

Huius autem

$$ddy+(2f+1)\frac{dydx}{x}+\frac{ffydx^2}{xx}=0$$

integrale completum est

$$y = x^{-f}\left(\alpha + \beta lx\right).$$

COROLLARIUM 2

849. At si aequatio proposita huiusmodi formam habuerit

$$ddy + \left(1 + 2f\cos.\zeta\right)\frac{dydx}{x} + \frac{ffydx^2}{xx} = 0,$$

tum eius integrale completum erit

$$y = Cx^{-f\cos.\zeta}\sin.\left(f\sin.\zeta \cdot lx + \gamma\right).$$

SCHOLION

850. Similem resolutionem quoque admittit haec aequatio differentio–differentialis

$$ddy - \frac{ndydx}{x} + Ax^n dydx + Bx^{2n}ydx^2 = 0.$$

Ponatur enim $dy = x^n yudx$, et cum sit

$$ddy = x^n ydxdu + nx^{n-1}yudx^2 + x^{2n}yuudx^2,$$

erit per y dividendo

$$x^n dxdu + nx^{n-1}udx^2 + x^{2n}uudx^2 - nx^{n-1}udx^2 + Ax^{2n}udx^2 + Bx^{2n}dx^2 = 0$$

hinc

$$du + x^n uudx + Ax^2 udx + Bx^2 dx = 0$$

ideoque

$$x^n dx = \frac{-du}{uu + Au + B},$$

cui particulariter satisfit ponendo $uu + Au + B = 0$, unde u duplicem consequitur valorem constantem, quorum alter sit $u = -f$, alter $u = -g$. Quocirca integralia particularia erunt

$$y = e^{\frac{-f}{n+1}x^{n+1}} \quad \text{et} \quad y = e^{\frac{-g}{n+1}x^{n+1}},$$

Sit brevitatis gratia $\frac{x^{n+1}}{n+1} = t$; erit integrale completum

$$y = \alpha e^{-ft} + \beta e^{-gt},$$

pro casu scilicet $uu + Au + B = \left(u + f\right)\left(u + g\right)$.

At pro casu $uu + Au + B = (u + f)^2$ erit

$$y = e^{-ft}(\alpha + \beta t).$$

Casu autem, quo $uu + Au + B = uu + 2fu\cos.\zeta + ff$, erit

$$y = Ce^{-ft\cos.\zeta}\sin.(ft\sin.\zeta + \gamma).$$

Haec integratio adeo ad hanc formam extendi potest sumendo pro X functionem quamcunque ipsius x

$$ddy - \frac{dxdy}{X} + AXdydx + BXXydx^2 = 0.$$

Posito enim $dy = Xuydx$ seu $\frac{dy}{y} = Xudx$ fit y

$$Xdx = \frac{-du}{uu+Au+B},$$

unde posito $\int Xdx = t$ integrale completum se habebit ut ante. Scilicet
1) si $A = f + g$ et $B = fg$, erit integrale

$$y = \alpha e^{-ft} + \beta e^{-gt},$$

2) si $A = 2f$ et $B = ff$, erit integrale

$$y = e^{-ft}(\alpha + \beta t)$$

3) si $A = 2f\cos.\zeta$ et $B = ff$, erit integrale

$$y = Ce^{-ft\cos.\zeta}\sin.(ft\sin.\zeta + \gamma).$$

PROBLEMA 104

851. *Sumto elemento dx constante si P, Q et X denotent functiones quascunque ipsius x, integrationem huius aequationis differentio-differentialis*

$$ddy + Pdydx + Qydx^2 = Xdx^2$$

ad aequationem differentialem primi ordinis reducere.

SOLUTIO

Hic singulari modo procedamus loco *y* binas novas incognitas introducendo. Statuatur scilicet $y = uv$, et cum sit

$$dy = udv + vdu \quad \text{et} \quad ddy = uddv + 2dudv + vddu,$$

aequatio nostra induet hanc formam

$$uddv + 2dudv + vddu + Pudxdv + Pvdxdu + Quvdx^2 = Xdx^2.$$

Iam altera *v* ita determinetur, ut termini ipsa littera *u* affecti destruantur, quod fit, si

$$ddv + Pdvdx + Qvdx^2 = 0,$$

unde per superiora [§ 831] *v* per *x* determinetur; quo facto superest haec aequatio

$$2dudv + vddu + Pvdxdu = Xdx^2,$$

unde, cum *v* iam detur per *x*, quantitas *u* definiri debet. Ponatur $du = sdx$ eritque

$$vds + 2sdv + Psvdx = Xdx,$$

quae multiplicata per $ve^{\int Pdx}$ integrabilis redditur; prodit enim

$$vvse^{\int Pdx} = \int e^{\int Pdx} Xvdx$$

ideoque

$$s = \frac{e^{-\int Pdx}}{vv} \int e^{\int Pdx} Xvdx \text{ and } u = \int \frac{e^{-\int Pdx}dx}{vv} \int e^{\int Pdx} Xvdx$$

Quare cum incognita *v* fuerit determinata ex aequatione

$$ddv + Pdxdv + Qvdx^2 = 0,,$$

integrale aequationis propositae erit

$$y = v\int \frac{e^{-\int Pdx}dx}{vv} \int e^{\int Pdx} Xvdx.$$

COROLLARIUM 1

852. Sit integratio ad aequationem differentialem primi ordinis revocetur, ponatur $v = e^{\int tdx}$ et quantitas *t* definietur per hanc aequationem

$$dt + ttdx + Ptdx + Qdx = 0,$$

quo facto integrale quaesitum erit

$$y = e^{\int tdx}\int e^{-\int(P+2t)dx}dx\int e^{\int(P+t)dx}Xdx.$$

COROLLARIUM 2

853. Cum sit $(P+t)dx = -\frac{dt}{t} - \frac{Qdx}{t}$ erit $e^{\int(P+t)dx} = \frac{1}{t}e^{-\int\frac{Qdx}{t}}$ hincque

$$y = e^{\int tdx}\int e^{\int\frac{Qdx}{t}-\int tdx}tdx\int e^{-\int\frac{Qdx}{t}}X\,\frac{dx}{t},$$

ubi duplex integratio ad integrale completum perducit.

SCHOLION 1

854. Alio modo, qui propius ad ante usitatum accedat, eadem integratio institui potest. Ponatur scilicet pro aequatione proposita $dy = tydx + vdx$, ubi v certam ipsius x functionem designet ex functione X determinandam.
Cum igitur sit

$$ddy = ydtdx + tdx(tydx + vdx) + dvdx,$$

erit facta substitutione

$$ydtdx + ttydx^2 + Ptydx^2 + Qydx^2 + tvdx^2 + dvdx + Pvdx^2 - Xdx^2 = 0,$$

cuius aequationis utraque pars, tam ea, quae per y multiplicatur, quam altera ab y libera seorsim nihilo aequetur, unde has duas aequationes nanciscimur

$$dt + ttdx + Ptdx + Qdx = 0 \quad \text{et} \quad dv + tvdx + Pvdx = Xdx,$$

ex quarum illa t per x ut ante definiri debet; tum vero erit ex ista

$$e^{\int(P+t)dx}v = \int e^{\int(P+t)dx}Xdx.$$

Iam vero ex aequatione assumta $dy - tydx = vdx$ colligitur

$$e^{-\int tdx}y = \int e^{-\int tdx}vdx,$$

ubi si loco v valor modo inventus substituatur, praecedens integralis forma obtinetur.

SCHOLION 2

855. Ex hac operatione sequi videtur aequationis propositae

$$ddy + Pdydx + Qydx^2 = Xdx^2$$

integrationem necessario pendere ab integratione huius

$$ddv + Pdvdx + Qvdx^2 = 0,$$

quandoquidem hac concessa illa exhiberi potest. Minime tamen hinc vicissim colligere licet, si posterioris resolutio vires nostras superet, etiam priorem nullo modo integrari posse; quin potius facile est infinitos casus exhibere, quibus prior integrationem admittat, cum tamen posterior irresolubilis existat. Sit enim $P = 0$ et $Q = \alpha x$ atque certum est aequationem posteriorem

$$ddv + \alpha xvdx^2 = 0$$

nulla adhuc methodo resolvi posse, cum posito $v = e^{\int tdx}$ abeat in $dt + ttdx + \alpha xdx = 0$; neque tamen hinc sequitur aequationem priorem

$$ddy + \alpha xydx^2 = Xdx^2$$

semper esse intractabilem. Infiniti enim casus pro X assignari possunt, quibus integratio succedat. Sumta enim pro y functione quacunque ipsius x reperietur pro X eiusmodi functio, ut aequationi valor pro y assumtus satisfaciat. Veluti posito $y = \frac{\beta x}{\alpha}$ ob $ddy = 0$ fit $X = \beta xx$ atque aequationi

$$ddy + \alpha xydx^2 = \beta xxdx^2$$

utique satisfacit integrale $y = \frac{\beta x}{\alpha}$. Interim tamen hoc integrale tantum est particulare ac dubium adhuc relinquitur, an etiam integrale completum exhiberi possit. At posito $y = \frac{\beta x}{\alpha} + z$ pro integrali completo inveniendo prodit $ddz + \alpha xzdx^2 = 0$, quae cum resolutionem respuat, evidens est integrale completum etiam in genere exhiberi non posse, nisi simul altera aequatio integrationem admittat.

PROBLEMA 105

856. *Sumto elemento dx constante invenire integrale completum huius aequationis differentio-differentialis*

$$ddy + Adydx + Bydx^2 = Xdx^2$$

denotante X functionem quamcunque ipsius x.

SOLUTIO

Posito $y = uv$ aequatio proposita in duas sequentes resolvitur

$$ddv + Advdx + Bvdx^2 = 0 \text{ et } vddu + 2dvdu + Avdxdu = Xdx^2.$$

Quodsi ergo ex priore valor ipsius v per x definiatur, integrale completum ex posteriore ita se habebit, ut ob $P = A$ sit

$$y = v\int \frac{e^{-Ax}dx}{vv}\int e^{Ax}Xvdx,$$

ubi cum duplex integratio integrale completum producat, sufficiet pro v integrale particulare prioris aequationis assumsisse, id quod etiam ex solutione generali patebit. Cum igitur pro resolutione prioris aequationis formanda sit haec aequatio quadratica

$$tt + At + B = 0,$$

pro eius indole tres casus evolvi conveniet.

I. Si $tt + At + B = (t+f)(t+g)$, ut sit $A = f + g$ et $B = fg$, erit

$$v = \alpha e^{-fx} + \beta e^{-gx}.$$

Sumatur primo tantum integrale particulare $v = e^{-fx}$ et ob $A = f + g$ fiet

$$y = e^{-fx}\int e^{(f-g)x}dx\int e^{gx}Xdx;$$

sit

$$\frac{e^{-(f+g)x}}{vv} = dR \quad \text{et} \quad \int e^{(f+g)x}Xvdx = S,$$

ut fiat

$$y = e^{-fx}\int SdR = e^{-fx}\left(RS - \int RdS\right);$$

at est $R=\frac{1}{f-g}e^{(f-g)x}$ unde colligitur

$$y=\frac{1}{f-g}e^{-gx}S-\frac{1}{f-g}e^{-fx}\int e^{fx}Xdx$$

sive

$$(f-g)y=e^{-gx}\int e^{gx}Xdx-e^{-fx}\int e^{fx}Xdx,$$

quod idem integrale prodiisset, si altero particulari $v=e^{-gx}$ usi essemus.

In genere autem sumto $v=\alpha e^{-fx}+\beta e^{-gx}$ statuatur ut ante

$$\frac{e^{(f-g)x}}{vv}=dR \quad \text{et} \quad \int e^{(f+g)x}Xvdx=S,$$

ut sit pariter

$$y=v\int SdR=v\left(RS-\int RdS\right).$$

Fingamus $R=C\frac{e^{\lambda x}}{v}$ et ob $dv=-dx\left(\alpha fe^{-fx}+\beta ge^{-gx}\right)$ erit

$$dR=\frac{Ce^{\lambda x}dx\left(\alpha\lambda e^{-fx}+\beta\lambda ge^{-gx}+\alpha fe^{-fx}+\beta ge^{-gx}\right)}{vv}.$$

Sit iam $\lambda=-g$ et $C\alpha(f-g)=1$, ut dR datum adipiscatur valorem; ob $C=\frac{1}{\alpha(f-g)}$ erit

$$R=\frac{e^{-gx}}{\alpha(f-g)v} \quad \text{et} \quad RdS=\frac{1}{\alpha(f-g)}e^{fx}Xdx,$$

tum vero

$$S=\alpha\int e^{gx}Xdx+\beta\int e^{fx}Xdx,$$

unde conficitur

$$y=v\left(\frac{e^{-gx}}{(f-g)v}\int e^{gx}Xdx+\frac{\beta e^{-gx}}{\alpha(f-g)v}\int e^{fx}Xdx-\frac{1}{\alpha(f-g)}\int e^{fx}Xdx\right)$$

seu prorsus ut ante

$$(f-g)y=e^{-gx}\int e^{gx}Xdx-e^{-fx}\int e^{fx}Xdx$$

II. Si $tt+At+B=(t+f)^2$ seu $A=2f$ et $B=ff$, erit ex priore equatione $v=e^{-fx}(\alpha+\beta x)$. Ponatur ut ante

$$\frac{e^{-2fx}dx}{vv}=dR \quad \text{et} \quad \int e^{2fx}Xvdx=S,$$

ut habeatur $y=v\left(RS-\int RdS\right)$. Cum ergo sit $dR=\frac{dx}{(\alpha+\beta x)^2}$ erit

$$R=-\frac{1}{\beta(\alpha+\beta x)}=-\frac{e^{-fx}}{\beta v} \quad \text{et} \quad S=\alpha\int e^{fx}Xdx+\beta\int e^{fx}Xxdx=\int e^{fx}Xdx(\alpha+\beta x),$$

quare

$$vRS=-\frac{\alpha}{\beta}e^{-fx}\int e^{fx}Xdx-e^{-fx}\int e^{fx}Xxdx \quad \text{et} \quad \int RdS=-\frac{1}{\beta}\int e^{fx}Xdx,$$

unde conficitur

$$y=e^{-fx}x\int e^{fx}Xdx-e^{-fx}\int e^{fx}Xxdx$$

seu cum sit

$$d.e^{fx}y=dx\int e^{fx}Xdx,$$

erit succinctius

$$y=e^{-fx}\int dx\int e^{fx}Xdx.$$

III. Si $tt+At+B=tt+2ft\cos.\zeta+ff$ seu $A=2f\cos.\zeta$ et $B=ff$, erit

$$v=e^{-fx\cos.\zeta}\sin.(fx\sin.\zeta+\gamma).$$

Ponatur

$$\frac{e^{-2fx\cos.\zeta}dx}{vv}=\frac{dx}{\sin.(fx\sin.\zeta+\gamma)^2}=dR$$

et

$$e^{2fx\cos.\zeta}Xvdx=e^{fx\cos.\zeta}Xdx\sin.(fx\sin.\zeta+\gamma)=dS,$$

ut obtineatur $y=vRS-v\int RdS$. At est

$$R=-\frac{1}{f\sin.\zeta}\cdot\frac{\cos.(fx\sin.\zeta+\gamma)}{\sin.(fx\sin.\zeta+\gamma)}$$

hincque

$$vRS = -\tfrac{1}{f\,sin.\zeta} e^{-fx\,cos.\zeta} cos.\left(fx\,sin.\zeta+\gamma\right)\int e^{fx\,cos.\zeta} Xdx\,sin.\left(fx\,sin.\zeta+\gamma\right)$$

et

$$\int RdS = -\tfrac{1}{f\,sin.\zeta}\int e^{fx\,cos.\zeta} Xdx\,cos.\left(fx\,sin.\zeta+\gamma\right).$$

Quocirca obtinebitur

$$\begin{aligned} fy\,sin.\zeta = &+e^{-fx\,cos.\zeta} sin.\left(fx\,sin.\zeta+\gamma\right)\int e^{fx\,cos.\zeta} Xdx\,cos.\left(fx\,sin.\zeta+\gamma\right)\\ &-e^{-fx\,cos.\zeta} cos.\left(fx\,sin.\zeta+\gamma\right)\int e^{fx\,cos.\zeta} Xdx\,sin.\left(fx\,sin.\zeta+\gamma\right). \end{aligned}$$

COROLLARIUM 1

857. In hoc postremo integrali si ponamus $fx\,sin.\zeta=\varphi$, erit

$$\begin{aligned} fe^{fx\,cos.\zeta} y\,sin.\zeta = &\left(sin.\gamma\,cos.\varphi+cos.\gamma\,sin.\varphi\right)\int e^{fx\,cos.\zeta} Xdx\left(cos.\gamma\,cos.\varphi-sin.\gamma\,sin.\varphi\right)\\ &+\left(sin.\gamma\,sin.\varphi-cos.\gamma\,cos.\varphi\right)\int e^{fx\,cos.\zeta} Xdx\left(sin.\gamma\,cos.\varphi+cos.\gamma\,sin.\varphi\right) \end{aligned}$$

seu

$$\begin{aligned} &fe^{fx\,cos.\zeta} y\,sin.\zeta =\\ &+sin.\gamma\,cos.\gamma\,cos.\varphi\int e^{fx\,cos.\zeta} Xdx\,cos.\varphi - sin.\gamma^2\,cos.\varphi\int e^{fx\,cos.\zeta} Xdx\,sin.\varphi\\ &+cos.\gamma^2\,sin.\varphi\int e^{fx\,cos.\zeta} Xdx\,cos.\varphi - sin.\gamma\,cos.\gamma\,sin.\varphi\int e^{fx\,cos.\zeta} Xdx\,sin.\varphi\\ &+sin.\gamma^2\,sin.\varphi\int e^{fx\,cos.\zeta} Xdx\,cos.\varphi + sin.\gamma\,cos.\gamma\,sin.\varphi\int e^{fx\,cos.\zeta} Xdx\,sin.\varphi\\ &-sin.\gamma\,cos.\gamma\,cos.\varphi\int e^{fx\,cos.\zeta} Xdx\,cos.\varphi - cos.\gamma^2\,cos.\varphi\int e^{fx\,cos.\zeta} Xdx\,sin.\varphi, \end{aligned}$$

unde patet angulum γ prorsus ex calculo excedere; fit enim

$$fe^{fx\,cos.\zeta} y\,sin.\zeta = sin.\varphi\int e^{fx\,cos.\zeta} Xdx\,cos.\varphi - cos.\varphi\int e^{fx\,cos.\zeta} Xdx\,sin.\varphi.$$

COROLLARIUM 2

858. Cum igitur loco unius aequationis duas formaverimus integrandas, vidimus sufficere, si alterius integrale saltem particulare fuerit cognitum. In ambobus enim praecedentibus casibus constantes α et β integrale completum praebentes ex calculo sponte evanuerunt et casu tertio constans γ itidem excessit.

EXEMPLUM

859. *Sumto elemento dx constante invenire integrale huius aequationis*

$$ddy + Adydx + Bydx^2 = dx^2\left(n(n-1)x^{n-2} + nAx^{n-1} + Bx^n\right).$$

Hoc exemplum ita est comparatum, ut ei manifesto satisfaciat valor $y = x^n$, qui eius integrale particulare constituit. Ad completum ergo inveniendum sit $A = f + g$ et $B = fg$, et cum sit

$$X = fgx^n + n(f+g)x^{n-1} + n(n-1)x^{n-2},$$

erit

$$\int e^{gx}Xdx = fe^{gx}x^n + ne^{gx}x^{n-1} + \alpha \quad \text{et} \quad \int e^{fx}Xdx = ge^{fx}x^n + ne^{fx}x^{n-1} + \beta\,,$$

unde ex forma inventa prodit integrale completum

$$(f-g)y = fx^n + nx^{n-1} + \alpha e^{-gx} - gx^n - nx^{n-1} - \beta e^{-fx}$$

seu

$$y = x^n + \tfrac{\alpha}{f-g}e^{-gx} - \tfrac{\beta}{f-g}e^{-fx}$$

vel mutata constantium forma

$$y = x^n + \alpha e^{-fx} + \beta e^{-gx}.$$

Si sit $g = f$, ponatur $g = f + w$ existente $\omega = 0$ et ob

$$e^{-gx} = e^{-fx}\cdot e^{-\omega x} = e^{-fx}(1-\omega x)$$

pro $\alpha + \beta$ et $-\beta\omega$ scribendo α et β erit

$$y = x^n + e^{-fx}(\alpha + \beta x).$$

Sin autem sit $f = a + b\sqrt{-1}$ et $g = a - b\sqrt{-1}$, cum fiat

$$y = x^n + e^{-ax}\left(\alpha e^{-bx\sqrt{-1}} + \beta e^{bx\sqrt{-1}}\right),$$

ob $e^{\pm bx\sqrt{-1}} = cos.bx \pm \sqrt{-1}\cdot sin.bx$ mutata forma constantium habebimus

$$y = x^n + e^{-ax}(\alpha\, cos.bx + \beta\, sin.bx).$$

SCHOLION

860. In genere autem si huiusmodi aequationis

$$ddy + Adydx + Bydx^2 = Xdx^2$$

constet integrale particulare seu valor ipsi satisfaciens $y = t$, integrale completum facile reperitur ponendo $y = t + z$. Cum enim per hypothesin sit

$$ddt + Adtdx + Btdx^2 = Xdx^2$$

facta hac substitutione orietur

$$ddz + Adzdx + Bzdx^2 = 0,$$

unde, si $A = f + g$ et $B = fg$, colligitur $z = \alpha e^{-fx} + \beta e^{-gx}$ sicque integrale completum erit

$$y = t + \alpha e^{-fx} + \beta e^{-gx}$$

quemadmodum etiam in exemplo allato invenimus.

PROBLEMA 106

861. *Sumto elemento dx constante si proponatur haec aequatio differentio–differentialis*

$$ddy - \tfrac{ndydx}{x} + Ax^n dydx + Bx^{2n} ydx^2 = Xdx^2$$

existente X functione quacunque ipsius x, eius integrale completum investigare.

SOLUTIO

Resolutio huius aequationis ut supra ex positione $y = uv$ derivari posset, sed alia methodo hic utentes levi substitutione eam ad formam problematis praecedentis reducamus.

Scilicet ponamus $x^n dx = dt$, ut sit $x^{n+1} = (n+1)t$, qua substitutione functio *X* abeat in *T* functionem quandam ipsius *t*. Ne autem assumtio elementi *dx* constantis turbet, hanc conditionem tollamus ponendo $dy = pdx$ et $dp = qdx$ habebimusque

$$q - \tfrac{np}{x} + Ax^n p + Bx^{2n} y = X$$

Cum nunc sit $dx = \frac{dt}{x^n}$ erit $p = x^n \frac{dy}{dt}$ hincque sumto iam elemento *dt* constante

$$dp = \frac{nx^{n-1}dxdy}{dt} + \frac{x^n ddy}{dt} = qdx = \frac{qdt}{x^n}, \quad \text{ergo} \quad q = \frac{nx^{n-1}dy}{dt} + \frac{x^{2n}ddy}{dt^2}$$

sicque nostra aequatio erit

$$\frac{x^{2n}ddy}{dt^2} + \frac{nx^{n-1}dy}{dt} - \frac{nx^{n-1}dy}{dt} + \frac{Ax^{2n}dy}{dt} + Bx^{2n}y = X.$$

Sit $Xx^{-2n} = \Theta$, quae quantitas posito $x^{n+1} = (n+1)t$ ut functio ipsius t spectari potest, sicque fiet

$$ddy + Adydt + Bydt^2 = \Theta dt^2,$$

in qua aequatione elementum dt constans est assumtum, cuius ergo integrale per superiora [§ 856] datur.

1. Si $A = f + g$ et $B = fg$, erit integrale

$$(f-g)\, y = e^{-gt}\int e^{gt}\Theta dt - e^{-ft}\int e^{ft}\Theta dt,$$

ubi restitutis valoribus $dt = x^n dx$ et $\Theta = Xx^{-2n}$, retento brevitatis gratia $t = \frac{1}{(n+1)}x^{n+1}$, valor ipsius y ita per x exprimetur

$$(f-g)\, y = e^{-gt}\int e^{gt}x^{-n}Xdx - e^{-ft}\int e^{ft}x^{-n}Xdx.$$

II. Si $A = 2f$ et $B = ff$, erit integrale

$$y = e^{-ft}t\int e^{ft}\Theta dt - e^{-ft}\int e^{ft}\Theta tdt \quad \text{seu} \quad y = e^{-ft}\int dt\int e^{ft}\Theta dt,$$

quod ergo per x ita exprimetur

$$y = e^{-ft}\int x^n dx\int e^{ft}x^{-n}Xdx.$$

III. Si denique $A = 2f\cos.\zeta$ et $B = ff$, erit integrale

$$fe^{ft\cos.\zeta}y\sin.\zeta = \sin.\varphi\int e^{ft\cos.\zeta}\Theta dt\cos.\varphi - \cos.\varphi\int e^{ft\cos.\zeta}\Theta dt\sin.\varphi$$

existente $\varphi = ft\sin.\zeta$ seu $\varphi = \frac{f\sin.\zeta}{n+1}x^{n+1}$ ob $t = \frac{1}{n+1}x^{n+1}$.

Quare aequationis propositae integrale erit

$$fe^{ft\cos.\zeta}y\sin.\zeta=\sin.\varphi\int e^{ft\cos.\zeta}x^{-n}Xdx\cos.\varphi-\cos.\varphi\int e^{ft\cos.\zeta}x^{-n}Xdx\sin.\varphi.$$

COROLLARIUM 1

862. Si $n=0$, aequatio proposita abit in eam ipsam, quam problemate praecedente tractavimus, fitque $t=x$, unde etiam integrale eodem redit.

COROLLARIUM 2

863. Sin autem sit $n=-1$, aequatio nostra fit

$$ddy+(A+1)\frac{dydx}{x}+\frac{Bydx^2}{xx}=Xdx^2,$$

ubi ergo erit $t=lx$ et $e^{\lambda t}=x^{\lambda}$, tum vero pro casu tertio angulus $\varphi=f\sin.\zeta\cdot lx$.

SCHOLION

864. Methodus, qua hic usi sumus, huiusmodi aequationes differentio–differentiales integrandi haud satis naturalis videtur, cum ad has quasi solas formas sit adstricta. Quoniam igitur in aequationibus differentialibus primi gradus inventio factorum, quibus eae per se integrabiles reddantur, insignem fructum polliceri videbatur, eius quoque usum in aequationibus differentialibus secundi gradus ostendere conemur. Hic quidem nihil tam absolutum expectare licet, quod ad omnes omnino aequationum forrnas pateat, sed, quantillum etiam praestare potuerimus, id haud contemnendum Analyseos incrementum spectari debebit. Hac autem methodo eas potissimum aequationes differentio–differentiales, in quibus altera variabilis y cum suis differentialibus unam dimensionem nusquam transgreditur, satis commode tractare licet hincque via perspicietur, quomodo eam magis excoli oporteat.