INTEGRAL CALCULUS

BOOK ONE.

PART ONE

OR
METHODS FOR INVESTIGATING FUNCTIONS OF ONE VARIABLE FROM SOME GIVEN RELATION OF THE DIFFERENTIALS OF THE FIRST ORDER.

SECTION THREE

CONCERNING
THE SOLUTION OF MORE COMPLICATED DIFFERENTIAL EQUATIONS .

CONCERNING THE RESOLUTION OF DIFFERENTIAL EQUATIONS IN WHICH THE DIFFERENTIALS ARISE IN SEVERAL VARIABLES OR THUS ARE IMPLIED TRANSCENDENTALLY

PROBLEM 88

668. *On putting the relation of the differentials* $\frac{dy}{dx}=p$ *, if some equation is proposed between the two quantities x and p, to investigate the relation between the variables x and y.*

SOLUTION

Since an equation may be given between *p* and *x*, with the resolution of equations permitted, from that *p* is sought in terms of *x* and a function of *x* may be found, which will itself be equal to *p*. Hence there may be arrived at an equation of this kind $p=X$ with X being a certain function of *x* only. Whereby since there shall be $p=\frac{dy}{dx}$, then we shall have $dy = Xdx$ and thus the question has been reduced to the first section, in which an integral of the formula Xdx is required to be investigated ; with which accomplished the integral sought shall be $y=\int Xdx$.

If the equation given between *x* and *p* thus should be prepared, so that thus it is easier for *x* to be defined through *p*, *x* is sought and there is produced $x=P$ with *P* being a certain function of *p*. Therefore here from this equation with the differentials arising there shall be $dx=dP$ and hence $dy=pdx=pdP$, from which on integration there is elicited $y=\int pdP$ or $y=pP-\int Pdp$. Hence therefore both variables *x* and *y* can thus be determined by the third, so that there shall be $x=P$ and $y=pP-\int Pdp$, from which the relation between *x* and *y* has been shown.

But if neither *p* through *x* nor *x* through *p* is able to be defined conveniently, often it can be effected, so that each can be defined conveniently be a new quantity *u*; hence we may put in place to be found $x=U$ and $p=V$, so that *U* and *V* shall be functions of the same variable *u*. Hence there becomes therefore $dy=pdx=VdU$ and $y=\int VdU$ and thus *x* and y are expressed by the same new variable *u*.

COROLLARY 1

669. In a similar manner the case may be resolved, in which some equation is proposed between *p* and the other variable *y*, since it is allowed to interchange the two variables *x* and *y* between each other. Moreover then, if each *p* in terms of *y* or *y* in terms of *p* are defined by the new variable *u*, it is required to note that $dx=\frac{dy}{p}$.

COROLLARY 2

670. Since $\sqrt{(dx^2+dy^2)}$ expresses an element of the arc of a curve, the rectangular coordinates of which are x and y, if the ratio

$$\frac{\sqrt{(dx^2+dy^2)}}{dx}=\sqrt{(1+pp)} \text{ or } \frac{\sqrt{(dx^2+dy^2)}}{dy}=\frac{\sqrt{(1+pp)}}{p}$$

is equal either to a function of x or y, hence the relation between x and y can be found.
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COROLLARY 3

671. Because in this way a relation between x and y is found through integration, likewise a new constant quantity is introduced, according to that the relation for the complete integral shall be obtained.

SCHOLIUM 1

672. Thus far we have only subjected to examination differential equations of this kind in which on putting $\frac{dy}{dx}=p$ a relation of this kind is proposed between the three quantities x, y et p, from which the value of p can be expressed conveniently by x and y, thus so that $p=\frac{dy}{dx}$ is equal to some function of x and y. Therefore now relations of this kind between x, y and p come to be considered, from which the value of p is either less conveniently, or plainly not allowed, to be defined by x and y ; and here the most simple case without doubt is, when in the relation proposed either of the variables x or y clearly is absent, thus so that only a relation between p and x or p and y is proposed ; which case we shall bring out in this problem.

But the strength of the solution turns on that, as from the proposed equation between x and p without the letter p by x [present], if perhaps this can be established easily, but rather with either x by p or also with each being defined by a new variable u. Just as if this equation should be proposed :

$$xdx+ady=b\sqrt{(dx^2+dy^2)}$$

which on putting $\frac{dy}{dx}=p$ changed into $x+ap=b\sqrt{(1+pp)}$, hence less conveniently p may be define through x. But since there becomes

$$x=b\sqrt{(1+pp)}-ap\,,$$

of account of $y=\int pdx=px-\int xdp$ then

$$y=bp\sqrt{(1+pp)}-app-b\int dp\sqrt{(1+pp)}+\tfrac{1}{2}app$$

and thus a relation between *x* is *y* is agreed upon.

But if such an equation should arise:

$$x^3dx^3 + dy^3 = axdx^2dy \text{ or } x^3 + p^3 = apx,$$

hence neither is it permitted for *x* to be defined conveniently in terms of *p* nor *p* in terms of *x*; with which I put $p = ux$, and from which there becomes $x + u^3x = au$ and hence

$$x = \frac{au}{1+u^3} \text{ and } p = \frac{auu}{1+u^3}\cdot$$

Now on account of $dx = \frac{audu\left(1-2u^3\right)}{\left(1+u^3\right)^2}$ there is deduced :

$$y = aa\int \frac{uudu\left(1-2u^3\right)}{\left(1+u^3\right)^3}$$

and by reducing this form to the simpler form :

$$y = \frac{1}{6}aa\frac{2u^3-1}{\left(1+u^3\right)^2} - aa\int \frac{uudu}{\left(1+u^3\right)^2},$$

or to

$$y = \frac{1}{6}aa\frac{2u^3-1}{\left(1+u^3\right)^2} + \frac{1}{3}aa\frac{1}{1+u^3} + \text{Const.}$$

SCHOLIUM 2

673. Therefore since this case, in which an equation either is proposed between *x* and *p* or between *y* and *p*, generally is allowed to be resolved, it must be observed, in which cases the resolution succeeds, when all the three quantities *x*, *y* and *p* are in the proposed equation. And indeed I observe initially, provided the two variables *x* et *y* everywhere fulfill the same number of dimensions, just as in addition the quantity *p* enters, the resolution always can be recalled to the cases treated before ; clearly it is all right to treat such equations likewise, according to the kind that also deserved a mention, since the dimensions arising from the differentials must be equal everywhere and it should be required to demand judgement from the finite quantities *x* and *y* only. Which provided they agree on the same dimension everywhere, the equation will be taken as homogeneous, just as if it is given by $xxdy - yy\sqrt{\left(dx^2 + dy^2\right)} = 0$ or $pxx - yy\sqrt{(1+pp)} = 0$. Also equations of this kind are allowed to be solved, in which either of the variables *x* or *y* nowhere has more than a single dimension, in whatever manner besides the ratio $p=\frac{dy}{dx}$ may be present. Therefore here we shall set out these cases more carefully.

PROBLEM 89

674. *On putting* $p=\frac{dy}{dx}$*, if in the equation proposed between x, y and p, the two variables x and y satisfy the same number of dimensions, to find the relation between x and y, which shall be the complete integral of this equation.*

SOLUTION

Since in the proposed equation between *x*, *y* and *p* the two variables *x* and *y* establish everywhere the same number of dimensions, if we put $y=ux$, the quantity *x* thence by division is removed and there is obtained an equation only between quantities *u* and *p*, from which the relation of the same thus may be defined, so that it becomes possible either to determine *u* in terms of *p* or *p* in terms of *u*. Now from putting $y=ux$ there follows $dy=udx+xdu$; therefore since $dy=pdx$, then there becomes $pdx-udx=xdu$ and thus $\frac{dx}{x}=\frac{du}{p-u}$. And thus since *p* is given in terms of *u*, the differential formula $\frac{du}{p-u}$ involving a single variable can be integrated by the rules of the first section and then there shall be $lx=\int\frac{du}{p-u}$ and thus *x* is determined in terms of *u*; and since $y=ux$, both variables *x* and *y* are determined in terms of the third variable *u*, and because that integration has lead to an arbitrary constant, this relation between *x* and *y* shall be the complete integral.

COROLLARY 1

675. Since there shall be $\frac{dx}{x}=\frac{du}{p-u}$, there shall be also $lx=-l(p-u)+\int\frac{dp}{p-u}$, which formula is more convenient, if perhaps from the equation between *p* and *u*, the proposed quantity *u* can be defined easier in terms of *p*.

COROLLARY 2

676. But if it is possible to express the integral $\int\frac{du}{p-u}$ or $\int\frac{dp}{p-u}$ in terms of logarithms, so that there shall be $\int\frac{du}{p-u}=lU$, then there shall be $lx=IC+lU$ and hence $x=Cu$ and $y=CUu$; from which the relation between *x* and *y* will be given algebraically, and since there shall be $u=\frac{y}{x}$ this third variable *u* is easily elucidated.

SCHOLIUM

677. We have set out this same resolution above for ordinary homogeneous equations, which hence is not disturbed by the dimensions of the differentials ; as it also succeeds even if the ratio of the differentials $\frac{dy}{dx} = p$ is put in place transcendently. Clearly in this way the resolution leads to the integration of the separable differential equation $\frac{dx}{x} = \frac{du}{p-u}$, just as above the calculation was carried out by the first method also. Now the other method, which we have used above by requiring a factor, which rendered the differential equation itself integrable, clearly does not have a place here, since from the differentiation of a finite equation, under no circumstances are differentials of more dimensions able to come into being. Hence a finite equation between *x* and *y* cannot be found in this manner which would lead to the proposed differential equation itself, but which at any rate as with that one agreed on, and indeed without opposing that arbitrary constant, which with the integration actually undertaken returns the complete integral.

EXAMPLE 1

678. *If in the proposed differential equation neither of the variables x and y itself enters, but only the ratio of the differentials* $\frac{dy}{dx} = p$ *, to assign the complete integral. .*

Hence on putting $\frac{dy}{dx} = p$ the proposed equation only includes the variable *p* with constants, from which from the resolution of which, since it may involve several roots, there may arise $p = \alpha, p = \beta, p = \gamma$ etc . Now on account of $p = \frac{dy}{dx}$ from the individual roots the complete integrals may be elicited, which shall be

$$y = \alpha x + a, y = \beta x + b, y = \gamma x + c \quad \text{etc .,}$$

which equally satisfy the individual proposed equation. Which if we want everything to be put together in one finite equation, will be the complete integral :

$$(y - \alpha x - a)(y - \beta x - b)(y - \gamma x - c) \text{ etc.} = 0,$$

which, as it may be apparent, takes not one new constant but several *a, b, c* etc., clearly just as many as the differential equation shall have roots of several dimensions.

COROLLARIUM 1

679. Thus on account of $p = +1$ and $p = -1$ we shall have two integrals of the equations

$$dy^2 - dx^2 = 0 \text{ or } pp - 1 = 0,$$

$y = x + a$ and $y = -x + b$, which gathered into one give

$$(y - x - a)(y + x - b) = 0$$

or

$$yy - xx - (a+b)y - (a-b)x + ab = 0.$$

COROLLARY 2

680. With the proposed equation

$$dy^3 + dx^3 = 0 \text{ of } p^3 + 1 = 0$$

on account of the roots $p = -1,\ p = \frac{1+\sqrt{-3}}{2}$ and $p = \frac{1-\sqrt{-3}}{2}$ there will be either

$$y = -x + a \quad , \ y = \tfrac{1+\sqrt{-3}}{2}x + b \quad \text{or } y = \tfrac{1-\sqrt{-3}}{2}x + c,$$

which gathered together will give :

$$y^3 + x^3 - (a+b+c)yy + \left(a - \tfrac{1+\sqrt{-3}}{2}b - \tfrac{1-\sqrt{-3}}{2}c\right)xy + \left(-a + \tfrac{1-\sqrt{-3}}{2}b + \tfrac{1+\sqrt{-3}}{2}c\right)xx$$
$$+ \ (ab + ac + bc)y + \left(bc - \tfrac{1-\sqrt{-3}}{2}ac - \tfrac{1+\sqrt{-3}}{2}ab\right)x - abc = 0,$$

which equation can also be shown thus :

$$y^3 + x^3 - fyy - gxy - hxx + Ay + Bx + C = 0,$$

where the constants A, B, C thus given are to be prepared, so that the equation allows this resolution into three simple parts.

EXAMPLE 2

681. *To find the complete integral from the proposed differential equation*

$$ydx - x\sqrt{\left(dx^2 + dy^2\right)} = 0.$$

On putting $\frac{dy}{dx} = p$ there becomes $y - x\sqrt{(pp+1)} = 0$; hence let $y = ux$*;* then there shall be

$$u = \sqrt{(pp+1)} \text{ and } \frac{dx}{x} = \frac{du}{p-u}$$

from which by the other formula

$$lx = -l(p-u) + \int \frac{dp}{p - \sqrt{(pp+1)}} = -l(p-u) - \int dp\left(p + \sqrt{(pp+1)}\right),$$

but

$$\int dp\sqrt{(pp+1)} = \tfrac{1}{2}p\sqrt{(1+pp)} + \tfrac{1}{2}l\left(p+\sqrt{(1+pp)}\right),$$

from which it is deduced :

$$lx = \mathrm{C} - \tfrac{1}{2}l\left(\sqrt{(1+pp)} - p\right) - \tfrac{1}{2}p\sqrt{(1+pp)} - \tfrac{1}{2}pp$$
$$= C + \tfrac{1}{2}l\left(\sqrt{(1+pp)} + p\right) - \tfrac{1}{2}p\sqrt{(1+pp)} - \tfrac{1}{2}pp$$

and

$$y = ux = x\sqrt{(pp+1)}.$$

EXAMPLE 3

682. *To find the complete integral of this differential equation*

$$ydx - xdy = nx\sqrt{\left(dx^2+dy^2\right)}\,.$$

On account of $\frac{dy}{dx} = p$ our equation is $y - px = nx\sqrt{(1+pp)}$, which on putting $y = ux$ becomes $u - p = n\sqrt{(1+pp)}$. Since hence there shall be

$$lx = -l(p-u) + \int\frac{dp}{p-u},$$

then

$$lx = -ln\sqrt{(1+pp)} - \int\frac{dp}{n\sqrt{(1+pp)}}$$

and hence

$$lx = C - ln\sqrt{(1+pp)} - \tfrac{1}{n}l\left(p+\sqrt{(1+pp)}\right)$$

Whereby there becomes

$$x = \frac{a}{\sqrt{(1+pp)}}\left(\sqrt{(1+pp)} - p\right)^{\frac{1}{n}} \quad \text{and} \quad y = \frac{a\left(p+n\sqrt{(1+pp)}\right)}{\sqrt{(1+pp)}}\left(\sqrt{(1+pp)} - p\right)^{\frac{1}{n}}$$

Now since there shall be $uu - 2up + pp = nn + nnpp$, then becomes

$$p = \frac{u-n\sqrt{(uu+1-nn)}}{1-nn} \quad \text{and} \quad \sqrt{(1+pp)} = \frac{-nu+n\sqrt{(uu+1-nn)}}{1-nn}$$

and

$$\sqrt{(1+pp)} - p = \frac{-u+\sqrt{(uu+1-nn)}}{1-n}\,,$$

from which there becomes

$$\frac{x\left(-nu+\sqrt{(uu+1-nn)}\right)}{a(1-n)}=\left(\frac{-u+\sqrt{(uu+1-nn)}}{1-n}\right)^{\frac{1}{n}},$$

where $u=\frac{y}{x}$. But if $n=1$, then $p=\frac{uu-1}{2u}$, $\sqrt{(1+pp)}=\frac{uu+1}{2u}$ and

$$x=\frac{2au}{uu+1}\cdot\frac{1}{u}=\frac{2axx}{yy+xx} \text{ or } xx+yy=2ax.$$

If $n=-1$, there is indeed as before $p=\frac{uu-1}{2u}$ and $\sqrt{(1+pp)}=\frac{-uu-1}{2u}$, from which

$$x=\frac{a}{\sqrt{(1+pp)}}\left(\sqrt{(1+pp)}+p\right)=\frac{2a}{1+uu}=\frac{2axx}{xx+yy}.$$

Hence both $x=0$ and $xx+yy-2ax=0.$

[The first edition has a + sign rather than a – sign in this equation]

SCHOLIUM

683. This equation with both sides squared and with the root $p=\frac{dy}{dx}$ extracted is reduced to an ordinary homogeneous equation. For in the first place there becomes

$$yy-2pxy+ppxx=nnxx+nnppxx,$$

then now

$$px=\frac{xdy}{dx}=\frac{y\pm n\sqrt{(yy+xx-nnxx)}}{1-nn},$$

which on putting $y=ux$ is rendered separable . Where in the first case, in which $nn=1$, it is worth mentioning, in which it becomes

$$yy-2pxy=xx \quad\text{or}\quad p=\frac{dy}{dx}=\frac{yy-xx}{2xy},$$

and thus

$$2xydy+xxdx-yydx=0,$$

which can also be integrated by parts, since $2xydy-yydx$ becomes integrable by the factor $\frac{1}{xyy}f:\frac{yy}{x}$; from which s also the part $xxdx$ is rendered integrable, that form changed into $\frac{1}{xx}$ and thus there is obtained $\frac{2xydy-yydx}{xx}+dx=0$, the integral of which is $\frac{yy}{x}+x=2a$ as before, unless since the other solution $x=0$ hence it is not elicited. Now since the equation with that square in place on putting $n=1$ at once becomes simple, the other root is lost, which is found again on putting $n=1-\alpha$, from which there becomes

$$yy-2pxy=xx-2\alpha xx-2\alpha ppxx$$

and thus px boundless ; therefore with the terms rejected before the others vanishing there is $-2pxy = xx - 2\alpha ppxx$, which divided by x gives the other solution $x = 0$. Indeed the resolution of which succeeds , when it is allowed to elicit the value p through the roots ; but if the equation rises to more dimensions or thus becomes transcendent, we are not able to do without this method.

EXAMPLE 4

684. *With the proposed equation* $xdy^3 + ydx^3 = dydx\sqrt{xy\left(dx^2 + dy^2\right)}$ *, to find the complete integral of this.*

On putting $\frac{dy}{dx} = p$ and $y = ux$, our equation adopts this form

$$p^3 + u = p\sqrt{u(1+pp)},$$

from which there is produced

$$\frac{dx}{x} = \frac{du}{p-u} \quad \text{or} \quad lx = \int \frac{du}{p-u} = -l(p-u) + \int \frac{dp}{p-u}.$$

But from this there becomes

$$\sqrt{u} = \tfrac{1}{2}p\sqrt{(1+pp)} + \tfrac{1}{2}p\sqrt{(1-4p+pp)}$$

and on squaring

$$u = \tfrac{1}{2}pp - p^3 + \tfrac{1}{2}p^4 + \tfrac{1}{2}pp\sqrt{(1+pp)(1-4p+pp)}$$

and hence

$$p - u = \tfrac{1}{2}p(1+pp)(2-p) - \tfrac{1}{2}pp\sqrt{(1+pp)(1-4p+pp)},$$

from which we deduce

$$\frac{dp}{p-u} = \frac{dp(2-p)}{2p(1-p+pp)} + \frac{dp\sqrt{(1-4p+pp)}}{2(1-p+pp)\sqrt{(1+pp)}}.$$

In the latter members of which if there is put $\sqrt{\frac{1-4p+pp}{1+pp}} = q$, on account of

$$p = \frac{2+\sqrt{\left(4-(1-qq)^2\right)}}{1-qq}, \quad dp = \frac{4qdq\left(2+\sqrt{\left(4-(1-qq)^2\right)}\right)}{(1-qq)^2\sqrt{\left(4-(1-qq)^2\right)}}$$

and

$$1-p+pp=\frac{(3+qq)\left(2+\sqrt{\left(4-(1-qq)^2\right)}\right)}{(1-qq)^2}$$

there is obtained

$$\int\frac{dp}{p-u}=\frac{1}{2}\int\frac{dp(2-p)}{p(1-p+pp)}+2\int\frac{qqdq}{(3+qq)\sqrt{\left(4-(1-qq)^2\right)}}$$

where the latter part cannot be integrated either by logarithms or circular arcs.

EXEMPLE 5

685. *To find the relation between x and y, so that on putting $s=\int\sqrt{\left(dx^2+dy^2\right)}$ there becomes $ss=2xy$.*

Since there shall be $s=\sqrt{2xy}$, then

$$ds=\left(dx^2+dy^2\right)=\frac{xdy+ydx}{\sqrt{2xy}}$$

and hence on putting $\frac{dy}{dx}=p$ and $y=ux$ there becomes

$$\sqrt{(1+pp)}=\frac{p+u}{\sqrt{2u}}$$

or $u=\sqrt{2u(1+pp)}-p$ and with the root extracted

$$\sqrt{u}=\sqrt{\frac{1+pp}{2}}+\frac{1-p}{\sqrt{2}}=\frac{1-p+\sqrt{(1+pp)}}{\sqrt{2}},$$

whereby

$$u=1-p+pp+(1-p)\sqrt{(1+pp)}\quad\text{and}\quad p-u=-(1-p)\left(1-p+\sqrt{(1+pp)}\right).$$

Hence

$$\int\frac{dp}{p-u}=\int\frac{dp}{2p(1-p)}\left(1-p-\sqrt{(1+pp)}\right)=\frac{1}{2}lp-\frac{1}{2}\int\frac{dp\sqrt{(1+pp)}}{p(1-p)}.$$

But on putting $p=\frac{1-qq}{2q}$ there becomes

$$\int\frac{dp\sqrt{(1+pp)}}{p(1-p)}=\int\frac{-dq(1+qq)^2}{q(1-qq)(qq+2q-1)}=\int\frac{dq}{q}-2\int\frac{dq}{1-qq}-4\int\frac{dq}{(q+1)^2-2}$$

$$=lq-l\frac{1+q}{1-q}+\sqrt{2}l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$

and hence

$$\int \frac{dp}{p-u} = \frac{1}{2}lp - \frac{1}{2}lq + \frac{1}{2}l\frac{1+q}{1-q} - \frac{1}{\sqrt{2}}l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q} = l\left(\frac{1+q}{2q}\right) - \frac{1}{\sqrt{2}}l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}.$$

Now

$$p-u = \frac{(1+q)(1-2q-qq)}{2q} = \frac{(1+q)\left(2-(1+q)^2\right)}{2q}$$

and thus there is obtained

$$lx = C - l(1+q) + lq - l\left(2-(1+q)^2\right) + l\left(\frac{1+q}{q}\right) - \frac{1}{\sqrt{2}}l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$
$$= l(2a) - l\left(2-(1+q)^2\right) - \frac{1}{\sqrt{2}}l\frac{\sqrt{2}+1+q}{\sqrt{2}-1-q},$$

where there is $u = \frac{y}{x} = \frac{1}{2}(1+q)^2$ and $1+q = \sqrt{\frac{2y}{x}}$, from which

$$x = \frac{ax}{x-y}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}} \quad \text{or} \quad x-y = a\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}$$

or

$$\left(\sqrt{x}+\sqrt{y}\right)^{1+\frac{1}{\sqrt{2}}} = a\left(\sqrt{x}-\sqrt{y}\right)^{\frac{1}{\sqrt{2}}-1}.$$

Hence the equation between x and y is transcending, as it is accustomed to call.

SCHOLIUM

686. This resolution is easier sought by finding at once the value of p from the equation

$$u+p = \sqrt{2u(1+pp)} \text{ or } uu + 2up + pp = 2u + 2upp,$$

which shall become

$$p = u + \frac{\sqrt{\left(uu-4uu+2u+2u^3-uu\right)}}{2u-1} \quad \text{or} \quad p = \frac{u+(1-u)\sqrt{2u}}{2u-1}$$

and

$$p-u = \frac{(1-u)\left(2u+\sqrt{2u}\right)}{2u-1} = \frac{(1-u)\sqrt{2u}}{\sqrt{2u}-1}.$$

Whereby

$$lx = \int\frac{du}{p-u} = \int\frac{du\left(\sqrt{2u}-1\right)}{(1-u)\sqrt{2u}} = C - l(1-u) - \int\frac{du}{(1-u)\sqrt{2u}};$$

let $u = vv$ and there shall be

$$\int \frac{du}{(1-u)\sqrt{2u}} = \frac{1}{\sqrt{2}} \int \frac{2dv}{1-vv} = \frac{1}{\sqrt{2}} l \frac{1+v}{1-v}$$

and hence

$$Ix = la - l(1-u) - \frac{1}{\sqrt{2}} l \frac{1+\sqrt{u}}{1-\sqrt{u}}.$$

From which on account of $u = \frac{y}{x}$ there is found

$$x = \frac{ax}{x-y} \left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}} \right)^{\frac{1}{\sqrt{2}}}$$

as before. Whereby if the curve may be desired with the rectangular coordinates *x* and *y* to be determined, so that the arc *s* shall be $= \sqrt{2xy}$, the equation defining the nature of this shall be

$$\left(\sqrt{x}+\sqrt{y}\right)^{\frac{1}{\sqrt{2}}+1} = a\left(\sqrt{x}-\sqrt{y}\right)^{\frac{1}{\sqrt{2}}-1}.$$

Moreover it is evident that a question can be resolved in the same manner, if the arc *s* is equal to some homogeneous function of one dimension of *x* and *y*, or if some homogeneous equation is proposed between *x, y* et *s*, because that is worth showing in the following problem.

PROBLEM 90

687. *If there should be* $s = \int \sqrt{\left(dx^2 + dy^2\right)}$*, and some homogeneous equation is proposed between x, y and s, in which clearly these three variables x, y and s everywhere constitute a number of the same dimensions, to find a finite equation between x and y.*

SOLUTION

There may be put $y = ux$ and $s = vx$, so that the variable *x* can be removed from the proposed homogeneous equation by this substitution, and an equation may be obtained between the two variables *u* and *v*, from which *v* is possible to be defined in terms of *u*. Then let $dy = pdx$ and now there shall be $ds = dx\sqrt{(1+pp)}$, from which there becomes $pdx = udx + xdu$ and $dx\sqrt{(1+pp)} = vdx + xdv$, ergo

$$\frac{dx}{x} = \frac{du}{p-u} = \frac{dv}{\sqrt{(1+pp)}-v}.$$

Because now *v* is given in terms of *u*, let $dv = qdu$, so that there may be had $\sqrt{(1+pp)} = v + pq - qu$ and with the squares taken
$1 + pp = (v-qu)^2 + 2pq(v-qu) + ppqq$, from which there is elicited

$$p=\frac{q(v-qu)+\sqrt{\left((v-qu)^2-1+qq\right)}}{1-qq} \quad \text{and} \quad p-u=\frac{qv-u+\sqrt{\left((v-qu)^2-1+qq\right)}}{1-qq}.$$

From which hence we may deduce

$$\frac{dx}{x}=\frac{du(1-qq)}{qv-u+\sqrt{\left((v-qu)^2-1+qq\right)}}=\frac{du\left(qv-u-\sqrt{\left((v-qu)^2-1+qq\right)}\right)}{1+uu-vv},$$

from which, since *v* and *q* can be given in terms of *u*, *x* can be found by the same *u;* but on account of $qdu=dv$ there is produced
[note that

$$(qv-u)^2-\left[(v-qu)^2-1+qq\right]=\left(1-q^2\right)\left(u^2-v^2+1\right),$$

and that

$$qvdu-udu=vdv-udu \text{]}$$

$$lx=la-l\sqrt{(1+uu-vv)}-\int\frac{du\sqrt{\left((v-qu)^2-1+qq\right)}}{1+uu-vv},$$

then indeed there is $y=ux$, or on putting $\frac{y}{x}$ in place of u, the equation sought is obtained between *x* and *y*.

COROLLARY 1

688. Since *s* expresses the corresponding arc of the curve with the rectangular coordinates *x* et *y*, thus the curve is defined, the arc of which is equal to some function of one dimension of *x* and *y;* which hence will be algebraic if the integral

$$\int\frac{du\sqrt{\left((v-qu)^2-1+qq\right)}}{1+uu-vv}$$

can be shown by logarithms.

COROLLARY 2

689. The problem can be resolved in a similar manner, if *s* expresses the integral formula of this kind, so that there shall be $ds=Qdx$ with *Q* being some function of the quantities *p, u* and *v*. Moreover then from the equality $\frac{dx}{x}=\frac{du}{p-u}=\frac{dv}{Q-v}$ it is required to elicit the value of *p*, and because *v* is given in terms of *u*, then there will be

$$lx=\int\frac{du}{p-u}.$$

EXAMPLE 1

690. If there should be $s=\alpha x+\beta y$, then $v=\alpha+\beta u$ et $q=\frac{dv}{du}=\beta$, hence $v-qu=\alpha$, and therefore

$$lx=la-l\sqrt{\left(1+uu-(\alpha+\beta u)^2\right)}-\int\frac{du\sqrt{(\alpha\alpha+\beta\beta-1)}}{1+uu-(\alpha+\beta u)^2},$$

of which the last part is

$$-\int\frac{du\sqrt{(\alpha\alpha+\beta\beta-1)}}{1-\alpha\alpha-2\alpha\beta u+(1-\beta\beta)uu}=(\alpha\alpha+\beta\beta-1)^{\frac{1}{2}}\int\frac{du}{\alpha\alpha-1+2\alpha\beta u+(\beta\beta-1)uu},$$

which is transformed into

$$\int\frac{(\beta\beta-1)du\sqrt{(\alpha\alpha+\beta\beta-1)}}{\left(u(\beta\beta-1)+\alpha\beta-\sqrt{(\alpha\alpha+\beta\beta-1)}\right)\left(u(\beta\beta-1)+\alpha\beta+\sqrt{(\alpha\alpha+\beta\beta-1)}\right)}$$
$$=\frac{1}{2}l\frac{(\beta\beta-1)u+\alpha\beta-\sqrt{(\alpha\alpha+\beta\beta-1)}}{(\beta\beta-1)u+\alpha\beta+\sqrt{(\alpha\alpha+\beta\beta-1)}}.$$

Whereby on putting $u=\frac{y}{x}$ the equation of the integral sought becomes on taking the squares [and removing the logs.]:

$$\frac{xx+yy-(\alpha x+\beta y)^2}{aa}=\frac{(\beta\beta-1)y+\alpha\beta x-x\sqrt{(\alpha\alpha+\beta\beta-1)}}{(\beta\beta-1)y+\alpha\beta x+x\sqrt{(\alpha\alpha+\beta\beta-1)}}.$$

But on putting

$$(\beta\beta-1)y+\alpha\beta x-x\sqrt{(\alpha\alpha+\beta\beta-1)}=P,$$
$$(\beta\beta-1)y+\alpha\beta x+x\sqrt{(\alpha\alpha+\beta\beta-1)}=Q$$

there becomes

$$PQ=(\beta\beta-1)^2yy+2\alpha\beta(\beta\beta-1)xy+(\alpha\alpha-1)(\beta\beta-1)xx$$
$$=(\beta\beta-1)\left((\alpha x+\beta y)^2-xx-yy\right),$$

from which with the constant changed [from a to b where $a^2(\beta\beta-1)=b^2$] there becomes $\frac{PQ}{bb}=\frac{P}{Q}$, therefore either $P=0$ or $Q=b$; and hence in general the solution becomes :

$$(\beta\beta-1)y+\alpha\beta x\pm x\sqrt{(\alpha\alpha+\beta\beta-1)}=c,$$

which is the equation for a straight line.

EXAMPLE 2

691. If there should be $s=\frac{nyy}{x}$, then $v=nuu$ and $q=2nu$; from which

$$1+uu-vv=1+uu-nnu^4 \text{ et } v-qu=-nuu,$$

hence

$$lx=la-l\sqrt{\left(1+uu-nnu^4\right)}-\int\frac{du\sqrt{\left(nnu^4-1+4nnuu\right)}}{1+uu-nnu^4},$$

but which formula is unable to be integrated by logarithms.

EXAMPLE 3

692. If there should be $ss=xx+yy$, then there becomes $v=\sqrt{(1+uu)}$ and $q=\frac{u}{\sqrt{(1+uu)}}$,

from which there arises $1+uu-vv=0$; hence the solution agrees with the first formulas recalled, from which there becomes :

$$v-qu=\frac{1}{\sqrt{(1+uu)}},\quad qq-1=\frac{-1}{1+uu} \text{ and } qv-u=0;$$

hence

$$p-u=0 \quad\text{or}\quad \frac{dy}{dx}-\frac{y}{x}=0,$$

thus so that there is produced $y=nx$.

EXAMPLE 4

693. If there were $ss=yy+nxx$ or $v=\sqrt{(uu+n)}$ and $q=\frac{u}{\sqrt{(uu+n)}}$,

then

$$1+uu-vv=1-n,\quad v-qu=\frac{n}{\sqrt{(uu+n)}} \quad\text{and}\quad qq-1=\frac{-n}{\sqrt{(uu+n)}}.$$

From which there will be obtained :

$$lx=la-l\sqrt{(1-n)}-\frac{1}{1-n}\int\frac{du\sqrt{(nn-n)}}{\sqrt{(uu+n)}}=lb+\frac{\sqrt{n}}{\sqrt{(n-1)}}\ l\left(u+\sqrt{(uu+n)}\right)$$

and hence

$$\frac{x}{b}=\left(\frac{y+\sqrt{(yy+nxx)}}{x}\right)^{\sqrt{\frac{n}{n-1}}}.$$

Hence whenever $\frac{n}{n-1}$ is a square number, an algebraic equation will be produced between *x* and *y*.

Let $\sqrt{\frac{n}{n-1}}=m$; then $n=\frac{mm}{mm-1}$ and $ss=yy+\frac{mmxx}{mm-1}$, for this condition is satisfied by this algebraic equation :

$$x^{m+1}=b\left(y+\sqrt{\left(yy+\frac{mmxx}{mm-1}\right)}\right)^{m},$$

which is changed into

$$x^{\frac{2}{m}}-2b^{\frac{1}{m}}x^{\frac{1-m}{m}}y=\frac{mm}{mm-1}b^{\frac{2}{m}} \quad \text{or} \quad y=\frac{(mm-1)x^{\frac{2}{m}}-mmb^{\frac{2}{m}}}{2(mm-1)b^{\frac{1}{m}}x^{\frac{1-m}{m}}}.$$

COROLLARY

694. We may put $\frac{1}{m}=n$, and if there should be

$$y=\frac{b^{2n}+(nn-1)x^{2n}}{2(nn-1)b^{n}x^{n-1}},$$

then there shall be

$$ss=yy-\frac{xx}{nn-1} \quad \text{or} \quad s=\sqrt{\left(yy-\frac{xx}{nn-1}\right)}.$$

Whereby if $y=\frac{b^4+3x^4}{6bbx}$, [for n = 2] then there is : $s=\sqrt{\left(yy-\frac{xx}{3}\right)}$.

PROBLEM 91

695. *If on putting* $\frac{dy}{dx}=p$ *an equation of this kind may be given between x, y and p, in which either of the variables* [*associated*] *with y has only a single dimension, to find a relation between the two variables x and y.*

SOLUTION

Hence therefore *y* is equal to some function of *x* and *p*, from which on differentiation there becomes $dy=Pdx+Qdp$. Therefore since there shall be $dy=pdx$, this differential equation shall be obtained $(P-p)dx+Qdp=0$, [*] which it is required to integrate. Because it contains only the two variables *x* and *p* and it involves the simpler differentials, the resolution of this can be examined by the above methods of exposition.

Therefore in the first place the resolution succeeds, if there should be $P=p$ and thus $dy=pdx+Qdp$. Which comes about, if *y* can thus be determined by *x* and *p*, so that there becomes $y=px+\Pi$ with Π denoting some function of *p*. Therefore then there shall be $Q=x+\frac{d\Pi}{dp}$ [since *Q* is the partial derivative of *y* by *p*], and since the solution depends on that equation $Qdp=0$ [*], either there will be $dp=0$ and hence $p=\alpha$ or $y=\alpha x+\beta$, where either of the constants α and β is determined from that proposed equation, as on putting $p=\alpha$ there is made $\beta=\Pi$ [*i.e.* in $y=px+\Pi$, and we are dealing with a straight line]; or there shall be $Q=0$ and thus $x=-\frac{d\Pi}{dp}$ and $y=-\frac{p\Pi}{dp}+\Pi$, where therefore each solution is algebraic, but only if Π should be an algebraic function of p.

In the second place the equation $(P-p)dx+Qdp=0$ allows resolution, if the former of the variables x with its differential dx is not greater than one dimension. This comes about, if there should be $y=Px+\Pi$, provided that P and Π are functions of p only ; for then there shall be $P=P$ and $Q=\frac{xdP}{dp}+\frac{d\Pi}{dp}$ and hence this equation shall be obtained to be integrated

$$(P-p)dx+xdP+d\Pi=0 \quad \text{or} \quad dx+\frac{xdP}{P-p}=-\frac{d\Pi}{P-p},$$

which multiplied by $e^{\int\frac{dP}{P-p}}$ gives

$$e^{\int\frac{dP}{P-p}}x=-\int e^{\int\frac{dP}{P-p}}\frac{d\Pi}{P-p}.$$

If there is put $\frac{dP}{P-p}=\frac{dR}{R}$, [enabling the exponent of the integral to collapse to R]; then the equation of the integral

$$Rx=C-\int\frac{Rd\Pi}{P-p}=C-\int\frac{d\Pi dR}{dP}$$

from which there becomes

$$x=\frac{C}{R}-\frac{1}{R}\int\frac{d\Pi dR}{dP} \text{ and } y=\frac{CP}{R}+\Pi-\frac{P}{R}\int\frac{d\Pi dR}{dP}.$$

In the third place the resolution will have no difficulty, if on denoting some functions X and V of x, there should be $y=X+Vp$. Then indeed there shall be

$$dy=pdx=dX+Vdp+pdV$$

and thus

$$dp+p\frac{dV-dx}{V}=-\frac{dX}{V};$$

let $\frac{dx}{V}=\frac{dR}{R}$, so that R also shall be a function of x;

[i.e. $dp+p\left(\frac{dV}{V}-\frac{dR}{R}\right)=-\frac{dX}{V}$; or $\frac{Vdp}{R}+\frac{pdV}{R}-pV\frac{dR}{R^2}=-\frac{dX}{R}$; leading to : $d\left(\frac{pV}{R}\right)=-\frac{dX}{R}$]

there will be $\frac{V}{R}p=C-\int\frac{dX}{R}$ or

$$p=\frac{CR}{V}-\frac{R}{V}\int\frac{dX}{R} \text{ and } y=X+CR-R\int\frac{dX}{R}$$

which equation expresses the relation between x and y.

In the fourth place the equation $(P-p)dx+Qdp=0$ allows resolution, if it should be homogeneous. Therefore since the term pdx has two dimensions, this [property of homogeneity] comes about if the same dimensions are present as in the remaining terms. From which it is evident that P and Q must be homogeneous functions of one dimension of x and p. Whereby if y thus is defined by x and p, so that y is equal to a homogeneous function of the two dimensions x and p, then the resolution succeeds. But if indeed there should be $dy=Pdx+Qdp$, the equation containing the solution $(P-p)dx+Qdp=0$ will be homogeneous, and it becomes integrable by itself, if it is divided by $(P-p)x+Qp$.

COROLLARY 1

696. For the fourth case if there is put $y=zz$, the proposed equation must become homogeneous between the three variables x, z and p. From which if some homogeneous equation is proposed between x, z and p, in which these three letters x, z and p everywhere constitute a number of the same dimensions, the problem is always allowed to be solved.

COROLLARY 2

697. In a like manner with the variables inverted, if there is put $x=vv$ and $\frac{dx}{dy}=q$, so that there becomes $p=\frac{1}{q}$ and some homogeneous equation is proposed between y, v and q, likewise the problem is possible to be solved.

SCHOLIUM

698. For the fourth case, in order that the equation $(P-p)dx+Qdp=0$ becomes homogeneous, the conditions can be enlarged upon. For there may be put $x=v^{\mu}$ and $p=q^{\nu}$ and there shall be this equation with that substituted $\mu\left(P-q^{\nu}\right)v^{\mu-1}dv+\nu\ Qq^{\nu-1}dq=0$ homogeneous between v and q and P shall be a homogeneous function of dimensions ν and also Q a homogeneous function of dimensions μ. Since now there shall be

$$dy=Pdx+Qdp=\mu Pv^{\mu-1}dv+\nu Qq^{\nu-1}dq$$

then y shall be a homogeneous function of dimensions $\mu+\nu$. Whereby on putting $y=z^{\mu+\nu}$ the problem allows resolution, if a relation of this kind is proposed between x, y and p, so that on putting $y=z^{\mu+\nu}$, $x=v^{\mu}$ and $p=q^{\nu}$ a homogeneous equation is present between the three quantities z, v and q, thus in order that number of dimensions from these of that form shall be the same everywhere. And if there should be proposed a homogeneous equation of this kind between *z, v* and *q*, the solution of the problem thus can be set out.

Since there shall be $dy = pdx$, then there shall be

$$(\mu+v)z^{\mu+v-1}dz = \mu v^{\mu-1}q^{v}dv \ ;$$

now there is put $z = rq$ and $v = sq$ and the proposed equation shall contain only the two letters *r* and *s*, from which it is allowed to define each in turn ; but then from these substitutions this equation is produced :

$$(\mu+v)r^{\mu+v-1}q^{\mu+v-1}(rdq+qdr) = \mu s^{\mu-1}q^{\mu+v-1}dv(sdq+qds),$$

from which there arises

$$\frac{dq}{q} = \frac{\mu s^{\mu-1}ds-(\mu+v)r^{\mu+v-1}dr}{(\mu+v)r^{\mu+v}-\mu s^{\mu}}$$

which is a separated differential equation, since *s* is given in terms of *r*. Since also the two cases arising clearly are contained in the formulas $y = z^{\mu+v}, x = v^{\mu}$ and $p = q^{v}$, evidently the first, if $\mu = 1$ and $v = 1$, the second indeed, if $\mu = 2$ and $v = -1$.

Therefore it may be agreed upon for these cases and likewise the preceding examples to be illustrated, of which the first is notable, since by differentiation of the proposed equation $y = px + \Pi$ at once there is given the whole equation sought, neither shall there be any need for integration, if indeed we exclude the other solution arising from $dp = 0$.

EXAMPLE 1

699. *To find the integral of this proposed differential equation*

$$ydx - xdy = a\sqrt{(dx^2+dy^2)}.$$

On putting $\frac{dy}{dx} = p$ there becomes $y - px = a\sqrt{(1+pp)}$, which differential equation on account of $dy = pdx$ gives $-xdp = \frac{apdp}{\sqrt{(1+pp)}}$; which since it is divisible by *dp*, gives in the first place $p = \alpha$ and hence

$$y = \alpha x + a\sqrt{(1+\alpha\alpha)}.$$

Now from the other factor there is made available

$$x = \frac{-ap}{\sqrt{(1+pp)}} \quad \text{and hence} \quad y = \frac{-app}{\sqrt{(1+pp)}} + a\sqrt{(1+pp)} = \frac{a}{\sqrt{(1+pp)}},$$

from which there becomes

$$xx + yy = aa,$$

which also is the equation of the integral ; but because it does not involve a new constant, it cannot be taken as the complete integral. Moreover the complete integral embraces these two equations, clearly

$$y = ax + a\sqrt{(1+aa)} \text{ and } xx + yy = aa,,$$

which can be taken into this one equation :

$$(y-\alpha x)^2 - aa(1+\alpha\alpha)(xx+yy-aa) = 0.$$

SCHOLIUM

700. Unless the operation is put in place by this method, the solution of this question shall be made difficult enough. For if we may free the differential equation $ydx - xdy = a\sqrt{(dx^2+dy^2)}$ from irrationality by being squared, and thus we may define the ratio $\frac{dy}{dx}$ by the extraction of roots, there becomes

$$(xx-aa)dy - xydx = \pm adx\sqrt{(xx+yy-aa)},$$

which equation is treated with difficulty by the known methods. Indeed the multiplier can be found rendering each side integrable by itself ; for initially the part $(xx-aa)dy - xydx$ divided by $y(xx-aa)$ will become integrable with the integral arising $l\frac{y}{\sqrt{(xx-aa)}}$; from which in general that integral returning is

$$\frac{1}{y(xx-aa)}\Phi : \frac{y}{\sqrt{(xx-aa)}},$$

[Euler's notation $\Phi : x$ is equivalent to our notation $\Phi(x)$, describing a function of x.] which function thus must be determined, in order that by the same multiplier the other part too $adx\sqrt{(xx+yy-aa)}$ may be made integrable. Moreover such a multiplier is

$$\frac{1}{y(xx-aa)} \cdot \frac{y}{\sqrt{(xx+yy-aa)}} = \frac{1}{(xx-aa)\sqrt{(xx+yy-aa)}},$$

from which there is produced

$$\frac{(xx-aa)dy - xydx}{(xx-aa)\sqrt{(xx+yy-aa)}} = \frac{\pm aadx}{xx-aa}.$$

Now according to the integral of the first part to be investigated, x may be considered as a constant and the integral becomes

$$= l\big(y + (xx+yy-aa)\big) + X$$

with X denoting a certain function of x prepared thus, so that now on taking y constant there becomes :

$$\frac{xdx}{\left(y+\sqrt{(xx+yy-aa)}\right)\sqrt{(xx+yy-aa)}}+dX=\frac{-xydx}{(xx-aa)\sqrt{(xx+yy-aa)}}$$

or

$$\frac{-xdx\left(y-\sqrt{(xx+yy-aa)}\right)}{(xx-aa)\sqrt{(xx+yy-aa)}}+dX=\frac{-xydx}{(xx-aa)\sqrt{(xx+yy-aa)}},$$

from which there is made

$$dX=\frac{-xdx}{xx-aa} \quad \text{and} \quad X=l\frac{C}{\sqrt{(xx-aa)}}.$$

Whereby the integral sought is

$$l\left(y+\sqrt{(xx+yy-aa)}\right)+l\frac{C}{\sqrt{(xx-aa)}}=\pm l\frac{x+a}{x-a}$$

or

$$\frac{y+\sqrt{(xx+yy-aa)}}{\sqrt{(xx-aa)}}=\alpha\sqrt{\frac{x+a}{x-a}} \quad \text{or} \quad \alpha\sqrt{\frac{x-a}{x+a}},$$

from which there becomes

$$y+\sqrt{(xx+yy-aa)}=\alpha(x\pm a)$$

and hence

$$xx-aa=\alpha\alpha(x\pm a)^2-2\alpha(x\pm a)y$$

or

$$(x\mp a)=\alpha\alpha(x\pm a)-2\alpha y\ ;$$

but which only is one of the two equations of the integral, but the other equation of the integral $xx+yy=aa$ now as if by division has to considered removed from the calculation.

Otherwise the same solution of the equation

$$(aa-xx)dy+dxdx=\pm adx\sqrt{(aa+yy-xx)}$$

is more easily set up by putting $y=u(aa-xx)$, from which there is made

$$(aa-xx)^{\frac{3}{2}}du=\pm adx\sqrt{(aa-xx)(uu-1)} \quad \text{or} \quad \frac{du}{\sqrt{(uu-1)}}=\frac{\pm adx}{aa-xx},$$

to which indeed it suffices by putting $u=1$, nor yet is this case present in the equation of the integral, as now we have shown above [§ 562]. From which it might be permitted to doubt that other solution $xx + yy = aa$ thus being excluded, since yet otherwise it has to be taken itself, if we consider carefully that first equation $\frac{ydx-xdy}{\sqrt{(dx^2+dy^2)}} = a$.

For if x and y are rectangular coordinates of a curved line, the formula $\frac{ydx-xdy}{\sqrt{(dx^2+dy^2)}}$ expresses the perpendicular from the origin of the coordinates sent to the tangent, which hence must be constant. But this comes about with a circle set up with its centre at the origin, while the equation becomes $xx + yy = aa$, which by itself is evident. And hence the reality of these solutions, which can be considered less fitting, are confirmed, even if the accounting of these is not seen clearly enough.

EXAMPLE 2

701. *To find the integral of this proposed differential equation* $ydx - xdy = \frac{a(dx^2+dy^2)}{dx}$.

On putting $dy = pdx$ there is made $y - px = a(1 + pp)$ and on differentiating $-xdp = 2apdp$, from which is concluded that either $dp = 0$ and $p = \alpha$ and hence $y = \alpha x + a(1 + \alpha\alpha)$ or $x = -2ap$ and $y = a(1 - pp)$ and thus on account of $p = \frac{-x}{2a}$ there will be had $4ay = 4aa - xx$, which equation transferred to geometry completely fulfils that condition.

But from the proposed equation with the root extracted there is found :

$$2ady + xdx = dx\sqrt{(xx + 4ay - 4aa)},$$

which putting $y = u(4aa - xx)$ changes into

$$2adu(4aa - xx) - xdx(4au - 1) = dx\sqrt{(4aa - xx)(4au - 1)}$$

and this putting $4au - 1 = tt$ into

$$tdt(4aa - xx) - ttxdx = tdx\sqrt{(4aa - xx)};$$

which since it is divisible by t, it is permitted to conclude $t = 0$ and thus $u = \frac{1}{4a}$ and hence $4ay = 4aa - xx$.

EXAMPLE 3

702. *To assign the integral of the proposed differential equation*

$$ydx - xdy = a\sqrt[3]{\left(dx^3 + dy^3\right)}.$$

This equation is hardly able to be treated in the accustomed manner, if we wish to extract the ratio $\frac{dy}{dx}$ from this. Moreover on putting $dy = pdx$ there is made $y - px = a\sqrt[3]{\left(1 + p^3\right)}$ and by differentiation

$$xdp = \frac{-appdp}{\sqrt[3]{\left(1+p^3\right)^2}},$$

from which the twofold conclusion can be deduced : either $dp = 0$ and $p = \alpha$ and thus

$$y = \alpha x + a\sqrt[3]{\left(1 + \alpha^3\right)},$$

or

$$x = \frac{-app}{\sqrt[3]{\left(1+p^3\right)^2}} \quad \text{and} \quad y = \frac{a}{\sqrt[3]{\left(1+p^3\right)^2}},$$

from which there becomes $pp = -\frac{x}{y}$, and on account of

$$y^3\left(1 + p^3\right)^2 = a^3 \quad \text{there will be} \quad p^3 = \frac{a\sqrt{a}}{y\sqrt{y}} - 1 \quad \text{and hence}$$

$$\frac{\left(a\sqrt{a} - y\sqrt{y}\right)^2}{y^3} = -\frac{x^3}{y^3} \quad \text{or} \quad x^3 + \left(a\sqrt{a} - y\sqrt{y}\right)^2 = 0.$$

EXEMPLUM 4

703. *To find the integral of the proposed equation* $ydx - nxdy = a\sqrt{\left(dx^2 + dy^2\right)}$.

On putting $dy = pdx$ there is had $y - npx = a\sqrt{\left(1 + pp\right)}$, from which on differentiation there is elicited

$$\left(1-n\right)pdx - nxdp = \frac{apdp}{\sqrt{\left(1+pp\right)}} \quad \text{or} \quad dx - \frac{nxdp}{\left(1-n\right)p} = \frac{adp}{\left(1-n\right)\sqrt{\left(1+pp\right)}},$$

which multiplied by $p^{\frac{n}{n-1}}$ and integrated gives

$$p^{\frac{n}{n-1}}x = \frac{a}{1-n}\int \frac{p^{\frac{n}{n-1}}dp}{\sqrt{(1+pp)}}.$$

Hence we can deduce the following cases allowed to be integration [see § 118]:

$$\text{if}\quad n=\tfrac{3}{2},\quad p^3x = C - \tfrac{2}{3}a\left(pp-\tfrac{2}{1}\right)\sqrt{(1+pp)},$$

$$\text{if}\quad n=\tfrac{5}{4},\quad p^5x = C - \tfrac{4}{5}a\left(p^4-\tfrac{4}{3}p^2+\tfrac{4\cdot 2}{3\cdot 1}\right)\sqrt{(1+pp)},$$

$$\text{if}\quad n=\tfrac{7}{6},\quad p^7x = C - \tfrac{6}{7}a\left(p^6-\tfrac{6}{5}p^4+\tfrac{6\cdot 4}{5\cdot 3}p^2-\tfrac{6\cdot 4\cdot 2}{5\cdot 3\cdot 1}\right)\sqrt{(1+pp)},$$

and if $n=\frac{2\lambda+1}{2\lambda}$, there shall be

$$y = px + a\sqrt{(1+pp)} + \frac{px}{2\lambda}$$

and

$$x = \frac{C}{p^{2\lambda+1}} - \frac{2\lambda a}{(2\lambda+1)p}\left(1-\frac{2\lambda}{(2\lambda-1)pp}+\frac{2\lambda(2\lambda-2)}{(2\lambda-1)(2\lambda-3)p^4}-\text{etc.}\right)\sqrt{(1+pp)}.$$

Since hence if there is taken $\lambda=\infty$, in order that $n=1$, then there will be

$$y = px + a\sqrt{(1+pp)} \quad\text{and}\quad x = \frac{C}{p^{2\lambda+1}} - \frac{ap}{\sqrt{(1+pp)}},$$

[as the infinite series is taken as geometric with the common ratio $-1/p^2$,] from which, if the constant C shall be equal to 0, the above solution $xx+yy=aa$ follows at once [on eliminating p]. But if the constant C does not vanish, the smallest change in the quantity p leads to an infinite variation in x. Therefore by whatever amount x may be changed, the quantity p thus can be considered as constant, from which on putting $p=a$ the other solution $y=\alpha x + a\sqrt{(1+\alpha\alpha)}$ may be obtained. Hence therefore the above doubt about Example I is well illustrated.

EXAMPLE 5

704. *To investigate the integral of the proposed differential equation* $Ady^n = \left(Bx^\alpha + Cy^\beta\right)dx^n$ *with* $n=\frac{\alpha\beta}{\alpha-\beta}$ *arising..*

On putting $\frac{dy}{dx}=p$ there becomes $Ap^n = Bx^\alpha + Cy^\beta$. Now we may put,

$$p=q^{\alpha\beta},\quad x=v^{\beta n},\quad \text{et}\ \ y=z^{\alpha n}$$

in order that we can have this homogeneous equation $Aq^{\alpha\beta n} = Bv^{\alpha\beta n} + Cz^{\alpha\beta n}$, which with putting $z=rq$ and $v=sq$ changes into $A=Bs^{\alpha\beta n}+Cr^{\alpha\beta n}$. Now since there shall be :

$$dy = \alpha nz^{\alpha n-1}dz = \alpha nr^{\alpha n-1}q^{\alpha n-1}\left(rdq + qdr\right)$$

and

$$pdx = \beta nv^{\beta n-1}dv = \beta ns^{\beta n-1}q^{\alpha\beta+\beta n-1}\left(sdq + qds\right),$$

there will be

$$\alpha r^{\alpha n-1}\left(rdq + qdr\right) = \beta s^{\beta n-1}q^{\alpha\beta+\beta n-\alpha n}\left(sdq + qds\right).$$

Now by hypothesis there is $\alpha\beta + \beta n - \alpha n = 0$, from which there arises

$$\alpha r^{\alpha n}dq + \alpha r^{\alpha n-1}qdr = \beta s^{\beta n}dq + \beta s^{\beta n-1}qds$$

and hence

$$\frac{dq}{q} = \frac{\alpha r^{\alpha n-1}dr - \beta s^{\beta n-1}ds}{\beta s^{\beta n} - \alpha r^{\alpha n}}.$$

But there is $s^{\beta n} = \left(\frac{A - Cr^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}}$ and hence

$$\beta s^{\beta n-1}ds = -\frac{\beta C}{B}r^{\alpha\beta n-1}dr\left(\frac{A - Cr^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}},$$

from which there becomes

$$\frac{dq}{q} = \frac{\alpha r^{\alpha n-1}dr + \frac{\beta C}{B}r^{\alpha\beta n-1}dr\left(\frac{A - Cr^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}}}{\beta\left(\frac{A - Cr^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}} - \alpha r^{\alpha n}}.$$

But the calculation may be put in place more easily in this manner. On taking $A = 1$ there becomes

$$p = \frac{dy}{dx} = \left(Bx^{\alpha} + Cy^{\beta}\right)^{\frac{1}{n}};$$

let $y = x^{\frac{\alpha}{\beta}}u$; there is made

$$x^{\frac{\alpha}{\beta}}du + \frac{\alpha}{\beta}x^{\frac{\alpha-\beta}{\beta}}udx = x^{\frac{\alpha}{n}}dx\left(B + Cu^{\beta}\right)^{\frac{1}{n}}$$

which equation, since there shall be $\frac{\alpha}{n} = \frac{\alpha-\beta}{\beta}$ changes into this

$$\beta xdu + \alpha udx = \beta dx\left(B + Cu^{\beta}\right)^{\frac{1}{n}}$$

from which there becomes

$$\frac{dx}{x} = \frac{\beta du}{\beta\left(B + Cu^{\beta}\right)^{\frac{1}{n}} - \alpha u},$$

and thus x is determine in terms of u , and since $u = x^{-\frac{\alpha}{\beta}}y$, an equation will be obtained between x and y.

SCHOLIUM

705. Therefore in this way it will be appropriate to have established an operation, when a relation of this kind is put in place between the two variables x and y together with the ratio of the differentials $\frac{dy}{dx} = p$, from which the value of p cannot be readily elicited. Therefore the calculation is required then to be treated thus, so that on putting $dy = pdx$ or $dx = \frac{dy}{p}$, by differentiation finally a simple differential equation between only two variables is arrived at, which in the end it is necessary often to use with suitable substitutions.

And thus far it is almost allowed for Geometers to be concerned even now about the resolution of differential equations of the first degree ; for scarcely by any way are integrals are to be investigated and indeed at this stage a certain use may still be considered to be passed over here. Moreover whether or it can hoped for a much greater development of the integral calculus, I can scarcely indeed confirm, since most found [methods] stand out, which before the forces of human ingenuity had to consider in order to overcome.

Therefore since I shall divide the integral calculus into two books, of which the first is about the relation of only two variables, and the second is about three or more, and now that I have set out the first part of the first book about differentials of the first order and more, here that I have set out established with its strengths, I shall proceed to the other part, in which the relation of two variables from a given condition of the differentials of second or higher orders is required.

CALCULI INTEGRALIS
LIBER PRIOR.

PARS PRIMA
SEU
METHODUS INVESTIGANDI FUNCTIONES
UNIUS VARIABILIS EX DATA RELATIONE QUACUNQVE DIFFERENTIALIVM PRIMI GRADUS.

SECTIO TERTIA

DE
RESOLUTIONE AEQUATIONVM DIFFERENTIALUM MAGIS COMPLICATARVM.

DE RESOLUTIONE AEQUATIONUM DIFFERENTIALAUM IN QUIBUS DIFFERENTIALIA AD PLURES DIMENSIONES ASSURGUNT VEL ADEO TRANSCENDENTER IMPLICANTUR

PROBLEMA 88

668. *Posita differentialium relatione* $\frac{dy}{dx}=p$ *si proponatur aequatio quaecunque inter binas quantitates x et p, relationem inter ipsas variabiles x et y investigare.*

SOLUTIO

Cum detur aequatio inter p et x, concessa aequationum resolutione ex ea quaeratur p per x ac reperietur functio ipsius x, quae ipsi p erit aequalis. Pervenietur ergo ad huiusmodi aequationem $p=X$ existente X functione quapiam ipsius x tantum. Quare cum sit $p=\frac{dy}{dx}$, habebimus $dy=Xdx$ sicque quaestio ad sectionem primam est reducta, unde formulae Xdx integrale investigari oportet; quo facto integrale quaesitum erit $y=\int Xdx$.

Si aequatio inter x et p data ita fuerit comparata, ut inde facilius x per p definiri possit, quaeratur x prodeatque $x=P$ existente P functione quadam ipsius p. Hac igitur aequatione differentiata erit $dx=dP$ hincque $dy=pdx=pdP$, unde integrando elicitur $y=\int pdP$ seu $y=pP-\int Pdp$. Hinc ergo ambae variabiles x et y per tertiam ita determinantur, ut sit $x=P$ et $y=pP-\int Pdp$, unde relatio inter x et y est manifesta.

Si neque p commode per x neque x per p definiri queat, saepe effici potest, ut utraque commode per novam quantitatem u definiatur; ponamus ergo inveniri $x=U$ et $p=V$ ut U et V· sint functiones eiusdem variabilis u. Hinc ergo erit $dy=pdx=VdU$ et $y=\int VdU$ sicque x et y per eandem novam variabilem u exprimuntur.

COROLLARIUM 1

669. Simili modo resolvetur casus, quo aequatio quaecunque inter p et alteram variabilem y proponitur, quoniam binas variabiles x et y inter se permutare licet. Tum autem, sive p per y sive y per p sive utraque per novam variabilem u definiatur, notari oportet esse $dx=\frac{dy}{p}$.

COROLLARIUM 2

670. Cum $\sqrt{(dx^2+dy^2)}$ exprimat elementum arcus curvae, cuius coordinatae rectangulae sunt *x* et *y*, si ratio

$$\frac{\sqrt{(dx^2+dy^2)}}{dx}=\sqrt{(1+pp)} \text{ seu } \frac{\sqrt{(dx^2+dy^2)}}{dy}=\frac{\sqrt{(1+pp)}}{p}$$

aequetur functioni vel ipsius *x* vel ipsius *y*, hinc relatio inter *x* et *y* inveniri poterit.

COROLLARIUM 3

671. Quoniam hoc modo relatio inter *x* et *y* per integrationem invenitur, simul nova quantitas constans introducitur, quocirca illa relatio pro integrali completo erit habenda.

SCHOLION 1

672. Hactenus eiusmodi tantum aequationes differentiales examini subiecimus, quibus posito $\frac{dy}{dx}=p$ eiusmodi relatio inter ternas quantitates *x*, *y* et *p* proponitur, unde valor ipsius *p* commode per *x* et *y* exprimi potest, ita ut $p=\frac{dy}{dx}$ aequetur functioni cuipiam ipsarum *x* et *y*. Nunc igitur ciusmodi relationes inter *x*, *y* et *p* considerandae veniunt, ex quibus valorem ipsius *p* vel minus commode vel plane non per *x* et *y* definire liceat; atque hic simplicissimus casus sine dubio est, quando in relatione proposita altera variabilis *x* seu *y* plane deest, ita ut tantum relatio inter *p* et *x* vel *p* et *y* proponatur; quem casum in hoc problemate expedivimus.

Solutionis autem vis in eo versatur, ut proposita aequatione inter *x* et *p* non littera *p* per *x*, nisi forte hoc facile praestari queat, sed potius *x* per *p* vel etiam utraque per novam variabilem *u* definiatur. Veluti si proponatur haec aequatio

$$xdx+ady=b\sqrt{(dx^2+dy^2)}$$

quae posito $\frac{dy}{dx}=p$ abit in hanc $x+ap=b\sqrt{(1+pp)}$, hinc minus commode definiretur *p* per *x*. Cum autem sit

$$x=b\sqrt{(1+pp)}-ap,$$

ob $y=\int pdx=px-\int xdp$ erit

$$y=bp\sqrt{(1+pp)}-app-b\int dp\sqrt{(1+pp)}+\tfrac{1}{2}app$$

sicque relatio inter *x* et *y* constat.

Sin autem perventum fuerit ad talem aequationem

$$x^3dx^3+dy^3=axdx^2dy \text{ seu } x^3+p^3=apx,$$

hinc neque *x* per *p* neque *p* per *x* commode definire licet; ex quo pono $p = ux$, unde fit $x + u^3 x = au$ hincque

$$x = \frac{au}{1+u^3} \quad \text{et} \quad p = \frac{auu}{1+u^3}.$$

Iam ob $dx = \frac{audu\left(1-2u^3\right)}{\left(1+u^3\right)^2}$ colligitur

$$y = aa\int \frac{uudu\left(1-2u^3\right)}{\left(1+u^3\right)^3}$$

ac reducendo hanc formam ad simpliciorem

$$y = \frac{1}{6}aa\frac{2u^3-1}{\left(1+u^3\right)^2} - aa\int \frac{uudu}{\left(1+u^3\right)^2}$$

seu

$$y = \frac{1}{6}aa\frac{2u^3-1}{\left(1+u^3\right)^2} + \frac{1}{3}aa\frac{1}{1+u^3} + \text{Const.}$$

SCHOLION 2

673. Cum igitur hunc casum, quo aequatio vel inter *x* et *p* vel inter *x* et *p* proponitur, generatim expedire licuerit, videndum est, quibus casibus evolutio succedat, quando omnes tres quantitates x, *y* et *p* in aequatione proposita insunt. Ac primo quidem observo, dummodo binae variabiles *x* et *y* ubique eundem dimensionum numerum adimpleant, quomodocunque praeterea quantitas *p* ingrediatur, resolutionem semper ad casus ante tractatos revocari posse; tales scilicet aequationes perinde tractare licet atque aequationes homogeneas, ad quod genus etiam merito referuntur, cum dimensiones a differentialibus natae ubique debeant esse pares et iudicium ex solis quantitatibus finitis *x* et *y* peti oporteat. Quae ergo dummodo ubique eundem dimensionum numerum constituant, aequatio pro homogenea erit habenda, veluti est $xxdy - yy\sqrt{\left(dx^2 + dy^2\right)} = 0$ seu $pxx - yy\sqrt{(1+pp)} = 0$. Deinde etiam eiusmodi aequationes evolutionem admittunt, in quibus altera variabilis *x* vel *y* plus una dimensione nusquam habet, utcunque praeterea differentialium ratio $p = \frac{dy}{dx}$ ingrediatur. Hos ergo casus hic accuratius explicemus.

PROBLEMA 89

674. *Posito* $p=\frac{dy}{dx}$ *si in aequatione inter x, y et p proposita binae variabiles x et y ubique eundem dimensionum numerum compleant, invenire relationem inter x et y, quae illius aequationis sit integrale completum.*

SOLUTIO

Cum in aequatione inter *x*, *y* et *p* proposita binae variabiles *x* et *y* ubique eundem dimensionum numerum constituant, si ponamus $y=ux$, quantitas *x* inde per divisionem tolletur habebiturque aequatio inter duas tantum quantitates *u* et *p*, qua earum relatio ita definietur, ut vel *u* per *p* vel *p* per *u* determinari possit. Iam ex positione $y=ux$ sequitur $dy=udx+xdu$; cum igitur sit $dy=pdx$, erit $pdx-udx=xdu$ ideoque $\frac{dx}{x}=\frac{du}{p-u}$. Quia itaque *p* per *u* datur, formula differentialis $\frac{du}{p-u}$ unicam variabilem complectens per regulas primae sectionis integretur eritque $lx=\int\frac{du}{p-u}$ sicque *x* per *u* determinatur ; et cum sit $y=ux$, ambae variabiles *x* et *y* per eandem tertiam variabilem *u* determinantur, et quia illa integratio constantem arbitrariam inducit, haec relatio inter *x* et *y* erit integrale completum.

COROLLARIUM 1

675. Cum sit $\frac{dx}{x}=\frac{du}{p-u}$, erit etiam $lx=-l(p-u)+\int\frac{dp}{p-u}$, quae formula commodior est, si forte ex aequatione inter *p* et *u* proposita quantitas *u* facilius per *p* definitur.

COROLLARIUM 2

676. Quodsi integrale $\int\frac{du}{p-u}$ vel $\int\frac{dp}{p-u}$ per logarithmos exprimi possit, ut sit $\int\frac{du}{p-u}=lU$, erit $lx=lC+lU$ hincque $x=Cu$ et $y=CUu$; unde relatio inter *x* et *y* algebraice dabitur, et cum sit $u=\frac{y}{x}$ haec tertia variabilis *u* facile eIiditur.

SCHOLION

677. Eandem hanc resolutionem supra in aequationibus homogeneis ordinariis docuimus, quae ergo ob dimensiones differentialium non turbatur; quin etiam succedit, etiamsi ratio differentiaIium $\frac{dy}{dx}=p$ transcendenter ingrediatur. Hoc modo scilicet resolutio ad integrationem aequationis differentiaIis separatae $\frac{dx}{x}=\frac{du}{p-u}$ perducitur, quemadmodum etiam supra per priorem methodum negotium fuit expeditum. Altera vero methodus, qua supra usi sumus quaerendo factorem, qui aequationem differentialem reddat per se integrabilem, hic plane locum non habet, cum per differentiationem aequationis finitae nunquam differentiaIia ad plures dimensiones exsurgere queant. Non ergo hoc modo invenitur aequatio finita inter x et y, quae differentiata ipsam aequationem propositam reproducat, sed quae saltem cum ea conveniat et quidem non obstante arbitraria illa constante, quae per integrationem ingressa integrale completum reddit.

EXEMPLUM 1

678. *Si in aequationem propositam neutra variabilium x et y ipsa ingrediatur, sed tantum differentialium ratio* $\frac{dy}{dx}=p$ *, integrale completum assignare.*

Posito ergo $\frac{dy}{dx}=p$ aequatio proposita solam variabilem p cum constantibus complectetur, unde ex eius resolutione, prout plures involvat radices, orietur $p=\alpha$, $p=\beta$, $p=\gamma$ etc . Iam ob $p=\frac{dy}{dx}$ ex singulis radicibus integralia completa elicientur, quae erunt

$$y=\alpha x+a,\ y=\beta x+b,\ y=\gamma x+c \quad \text{etc .},$$

quae singula aequationi propositae aeque satisfaciunt. Quae si velimus omnia una aequatione finita complecti, erit integrale completum

$$(y-\alpha x-a)(y-\beta x-b)(y-\gamma x-c)\ \text{etc.}=0,$$

quae, uti apparet, non unam novam constantem, sed plures a, b, c etc. comprehendit, tot scilicet, quot aequatio differentialis plurium dimensionum habuerit radices.

COROLLARIUM 1

679. Ita aequationis differentialis

$$dy^2-dx^2=0 \ \text{ seu } \ pp-1 \ =0$$

ob $p=+1$ et $p=-1$ duo habemus integralia $y=x+a$ et $y=-x+b$, quae in unum collecta dant $(y-x-a)(y+x-b)=0$ seu

$$yy-xx-(a+b)y-(a-b)x+ab=0 .$$

COROLLARIUM 2

680. Proposita aequatione

$$dy^3+dx^3=0 \ \text{ seu } \ p^3+1=0$$

ob radices $p=-1$, $p=\frac{1+\sqrt{-3}}{2}$ et $p=\frac{1-\sqrt{-3}}{2}$ erit vel

$$y=-x+a \quad \text{vel} \quad y=\frac{1+\sqrt{-3}}{2}x+b \quad \text{vel} \quad y=\frac{1-\sqrt{-3}}{2}x+c ,$$

quae collecta praebent

$$y^3+x^3-(a+b+c)yy+\left(a-\tfrac{1+\sqrt{-3}}{2}b-\tfrac{1-\sqrt{-3}}{2}c\right)xy+\left(-a+\tfrac{1-\sqrt{-3}}{2}b+\tfrac{1+\sqrt{-3}}{2}c\right)xx$$
$$+\ (ab+ac+bc)y+\left(bc-\tfrac{1-\sqrt{-3}}{2}ac-\tfrac{1+\sqrt{-3}}{2}ab\right)x-abc=0,$$

quae aequatio etiam ita exhiberi potest

$$y^3+x^3-fyy-gxy-hxx+Ay+Bx+C=0,$$

ubi constantes A, B, C ita debent esse comparatae, ut aequatio haec resolutionem in tres simplices admittat.

EXEMPLUM 2

681. *Proposita aequatione differentiali* $ydx-x\sqrt{(dx^2+dy^2)}=0$ *eius integrale completum invenire.*

Posito $\frac{dy}{dx}=p$ fit $y-x\sqrt{(pp+1)}=0$; sit ergo $y=ux$; erit

$$u=\sqrt{(pp+1)}\quad \text{et}\quad \frac{dx}{x}=\frac{du}{p-u}$$

unde per alteram formulam

$$lx=-l(p-u)+\int\frac{dp}{p-\sqrt{(pp+1)}}=-l(p-u)-\int dp\left(p+\sqrt{(pp+1)}\right),$$

at

$$\int dp\sqrt{(pp+1)}=\tfrac{1}{2}p\sqrt{(1+pp)}+\tfrac{1}{2}l\left(p+\sqrt{(1+pp)}\right),$$

unde colligitur

$$lx=\mathrm{C}-\tfrac{1}{2}l\left(\sqrt{(1+pp)}-p\right)-\tfrac{1}{2}p\sqrt{(1+pp)}-\tfrac{1}{2}pp$$
$$=\ C+\tfrac{1}{2}l\left(\sqrt{(1+pp)}+p\right)-\tfrac{1}{2}p\sqrt{(1+pp)}-\tfrac{1}{2}pp$$

et

$$y=ux=x\sqrt{(pp+1)}.$$

EXEMPLUM 3

682. *Huius aequationis* $ydx - xdy = nx\sqrt{\left(dx^2 + dy^2\right)}$ *integrale completum invenire.*

Ob $\frac{dy}{dx} = p$ nostra aequatio est $y - px = nx\sqrt{(1+pp)}$, quae posito $y = ux$ abit in $u - p = n\sqrt{(1+pp)}$. Cum ergo sit

$$lx = -l(p-u) + \int \frac{dp}{p-u},$$

erit

$$lx = -ln\sqrt{(1+pp)} - \int \frac{dp}{n\sqrt{(1+pp)}}$$

hincque

$$lx = C - ln\sqrt{(1+pp)} - \tfrac{1}{n} l\left(p + \sqrt{(1+pp)}\right)$$

Quare habetur

$$x = \frac{a}{\sqrt{(1+pp)}}\left(\sqrt{(1+pp)} - p\right)^{\frac{1}{n}} \quad \text{et} \quad y = \frac{a\left(p+n\sqrt{(1+pp)}\right)}{\sqrt{(1+pp)}}\left(\sqrt{(1+pp)} - p\right)^{\frac{1}{n}}$$

Cum nunc sit $uu - 2up + pp = nn + nnpp$, erit

$$p = \frac{u - n\sqrt{(uu+1-nn)}}{1-nn} \quad \text{et} \quad \sqrt{(1+pp)} = \frac{-nu + n\sqrt{(uu+1-nn)}}{1-nn}$$

atque

$$\sqrt{(1+pp)} - p = \frac{-u + \sqrt{(uu+1-nn)}}{1-n},$$

unde fit

$$\frac{x\left(-nu + \sqrt{(uu+1-nn)}\right)}{a(1-n)} = \left(\frac{-u + \sqrt{(uu+1-nn)}}{1-n}\right)^{\frac{1}{n}},$$

ubi $u = \frac{y}{x}$. At si $n = 1$, erit $p = \frac{uu-1}{2u}$, $\sqrt{(1+pp)} = \frac{uu+1}{2u}$ atque

$$x = \frac{2au}{uu+1} \cdot \frac{1}{u} = \frac{2axx}{yy+xx} \quad \text{seu} \quad xx + yy = 2ax.$$

Si $n = -1$, est quidem ut ante $p = \frac{uu-1}{2u}$ et $\sqrt{(1+pp)} = \frac{-uu-1}{2u}$, unde

$$x=\frac{a}{\sqrt{(1+pp)}}\left(\sqrt{(1+pp)}+p\right)=\frac{2a}{1+uu}=\frac{2axx}{xx+yy}.$$

Ergo et $x=0$ et $xx+yy-2ax=0.$

SCHOLION

683. Haec aequatio sumendis utrinque quadratis et radice $p=\frac{dy}{dx}$ extrahenda ad aequationem homogeneam ordinariam reducitur. Fit enim primo

$$yy-2pxy+ppxx=nnxx+nnppxx\,,$$

tum vero

$$px=\frac{xdy}{dx}=\frac{y\pm n\sqrt{(yy+xx-nnxx)}}{1-nn}\,,$$

quae posito $y=ux$ separabilis redditur. Ubi imprimis casus, quo $nn=1$, notari meretur, quo fit $yy-2pxy=xx$ seu $p=\frac{dy}{dx}=\frac{yy-xx}{2xy}$ ideoque,

$$2xydy+xxdx-yydx=0\,,$$

quae etiam per partes integrari potest, cum $2xydy-yydx$ integrabile fiat per factorem $\frac{1}{xyy}f:\frac{yy}{x}$; quo ut etiam pars $xxdx$ integrabilis reddatur, illa forma abit in $\frac{1}{xx}$ sicque habebitur $\frac{2xydy-yydx}{xx}+dx=0$, cuius integrale est $\frac{yy}{x}+x=2a$ ut ante, nisi quod altera solutio $x=0$ hinc non eliciatur. Verum cum aequatio illa quadrata posito $n=1$ subito abeat in simplicem, altera radix perit, quae reperitur ponendo $n=1-\alpha$, quo fit

$$yy-2pxy=xx-2\alpha xx-2\alpha ppxx$$

ideoque px infinitum; reiectis ergo terminis prae reliquis evanescentibus est $-2pxy=xx-2\alpha ppxx$, quae divisibilis per x alteram praebet solutionem $x=0$. Talis quidem resolutio succedit, quando valorem p per radicis extractionem elicere licet; sed si aequatio ad plures dimensiones ascendat vel adeo transcendens fiat, methodo hic exposita carere non possumus

EXEMPLUM 4

684. *Proposita aequatione* $xdy^3+ydx^3=dydx\sqrt{xy\left(dx^2+dy^2\right)}$ *eius integrale completum investigare.*

Posito $\frac{dy}{dx}=p$ et $y=ux$ nostra aequatio induet hanc forma

$$p^3+u=p\sqrt{u(1+pp)},$$

unde conficitur

$$\frac{dx}{x}=\frac{du}{p-u} \quad \text{seu} \quad lx=\int\frac{du}{p-u}=-l\left(p-u\right)+\int\frac{dp}{p-u}.$$

Inde autem est

$$\sqrt{u}=\tfrac{1}{2}p\sqrt{\left(1+pp\right)}+\tfrac{1}{2}p\sqrt{\left(1-4p+pp\right)}$$

et quadrando

$$u=\tfrac{1}{2}pp-p^3+\tfrac{1}{2}p^4+\tfrac{1}{2}pp\sqrt{\left(1+pp\right)\left(1-4p+pp\right)}$$

hincque

$$p-u=\tfrac{1}{2}p\left(1+pp\right)\left(2-p\right)-\tfrac{1}{2}pp\sqrt{\left(1+pp\right)\left(1-4p+pp\right)},$$

unde colligimus

$$\frac{dp}{p-u}=\frac{dp\left(2-p\right)}{2p\left(1-p+pp\right)}+\frac{dp\sqrt{\left(1-4p+pp\right)}}{2\left(1-p+pp\right)\sqrt{\left(1+pp\right)}}.$$

In quorum membrorum posteriore si ponatur $\sqrt{\frac{1-4p+pp}{1+pp}}=q$, ob

$$p=\frac{2+\sqrt{\left(4-\left(1-qq\right)^2\right)}}{1-qq}, \quad dp=\frac{4qdq\left(2+\sqrt{\left(4-\left(1-qq\right)^2\right)}\right)}{\left(1-qq\right)^2\sqrt{\left(4-\left(1-qq\right)^2\right)}}$$

et

$$1-p+pp=\frac{\left(3+qq\right)\left(2+\sqrt{\left(4-\left(1-qq\right)^2\right)}\right)}{\left(1-qq\right)^2}$$

obtinebitur

$$\int\frac{dp}{p-u}=\frac{1}{2}\int\frac{dp\left(2-p\right)}{p\left(1-p+pp\right)}+2\int\frac{qqdq}{\left(3+qq\right)\sqrt{\left(4-\left(1-qq\right)^2\right)}}$$

ubi membrum posterius neque per logarithmos neque arcus circulares integrari potest.

EXEMPLUM 5

685. *Invenire relationem inter x et y, ut posito* $s=\int\sqrt{\left(dx^2+dy^2\right)}$ *fiat* $ss=2xy$.

Cum sit $s=\sqrt{2xy}$, erit

$$ds=\left(dx^2+dy^2\right)=\frac{xdy+ydx}{\sqrt{2xy}}$$

hincque posito $\frac{dy}{dx}=p$ et $y=ux$ fiet

$$\sqrt{(1+pp)}=\tfrac{p+u}{\sqrt{2u}}$$

seu $u=\sqrt{2u(1+pp)}-p$ et radice extracta

$$\sqrt{u}=\sqrt{\tfrac{1+pp}{2}}+\tfrac{1-p}{\sqrt{2}}=\tfrac{1-p+\sqrt{(1+pp)}}{\sqrt{2}},$$

quare

$$u=1-p+pp+(1-p)\sqrt{(1+pp)} \quad\text{et}\quad p-u=-(1-p)\left(1-p+\sqrt{(1+pp)}\right).$$

Ergo

$$\int\tfrac{dp}{p-u}=\int\tfrac{dp}{2p(1-p)}\left(1-p-\sqrt{(1+pp)}\right)=\tfrac{1}{2}lp-\tfrac{1}{2}\int\tfrac{dp\sqrt{(1+pp)}}{p(1-p)}.$$

At posito $p=\frac{1-qq}{2q}$ fit

$$\int\tfrac{dp\sqrt{(1+pp)}}{p(1-p)}=\int\tfrac{-dq(1+qq)^2}{q(1-qq)(qq+2q-1)}=\int\tfrac{dq}{q}-2\int\tfrac{dq}{1-qq}-4\int\tfrac{dq}{(q+1)^2-2}$$

$$=lq-l\tfrac{1+q}{1-q}+\sqrt{2}l\tfrac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$

hincque

$$\int\tfrac{dp}{p-u}=\tfrac{1}{2}lp-\tfrac{1}{2}lq+\tfrac{1}{2}l\tfrac{1+q}{1-q}-\tfrac{1}{\sqrt{2}}l\tfrac{\sqrt{2}+1+q}{\sqrt{2}-1-q}=l\left(\tfrac{1+q}{2q}\right)-\tfrac{1}{\sqrt{2}}l\tfrac{\sqrt{2}+1+q}{\sqrt{2}-1-q}.$$

Iam

$$p-u=\tfrac{(1+q)(1-2q-qq)}{2q}=\tfrac{(1+q)\left(2-(1+q)^2\right)}{2q}$$

sicque habetur

$$lx=C-l(1+q)+lq-l\left(2-(1+q)^2\right)+l\left(\tfrac{1+q}{q}\right)-\tfrac{1}{\sqrt{2}}l\tfrac{\sqrt{2}+1+q}{\sqrt{2}-1-q}$$
$$=l(2a)-l\left(2-(1+q)^2\right)-\tfrac{1}{\sqrt{2}}l\tfrac{\sqrt{2}+1+q}{\sqrt{2}-1-q},$$

ubi est $u=\frac{y}{x}=\frac{1}{2}(1+q)^2$ et $1+q=\sqrt{\frac{2y}{x}}$, unde

$$x=\tfrac{ax}{x-y}\left(\tfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}} \quad\text{seu}\quad x-y=a\left(\tfrac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}$$

vel

$$\left(\sqrt{x}+\sqrt{y}\right)^{1+\frac{1}{\sqrt{2}}}=a\left(\sqrt{x}-\sqrt{y}\right)^{\frac{1}{\sqrt{2}}-1}.$$

Est ergo aequatio inter x et y interscendens, uti vocari solet.

SCHOLION

686. Facilius haec resolutio absolvitur quaerendo statim ex aequatione

$$u+p=\sqrt{2u(1+pp)} \text{ seu } uu+2up+pp=2u+2upp$$

valorem ipsius p, qui fit

$$p=u+\frac{\sqrt{(uu-4uu+2u+2u^3-uu)}}{2u-1} \text{ seu } p=\frac{u+(1-u)\sqrt{2u}}{2u-1}$$

et

$$p-u=\frac{(1-u)(2u+\sqrt{2u})}{2u-1}=\frac{(1-u)\sqrt{2u}}{\sqrt{2u}-1}.$$

Quare

$$lx=\int\frac{du}{p-u}=\int\frac{du(\sqrt{2u}-1)}{(1-u)\sqrt{2u}}=C-l(1-u)-\int\frac{du}{(1-u)\sqrt{2u}};$$

sit $u=vv$ eritque

$$\int\frac{du}{(1-u)\sqrt{2u}}=\frac{1}{\sqrt{2}}\int\frac{2dv}{1-vv}=\frac{1}{\sqrt{2}}l\frac{1+v}{1-v}$$

hincque

$$Ix=la-l(1-u)-\frac{1}{\sqrt{2}}l\frac{1+\sqrt{u}}{1-\sqrt{u}}.$$

Unde ob $u=\frac{y}{x}$ reperitur

$$x=\frac{ax}{x-y}\left(\frac{\sqrt{x}-\sqrt{y}}{\sqrt{x}+\sqrt{y}}\right)^{\frac{1}{\sqrt{2}}}$$

ut ante. Quare si curva desideretur coordinatis rectangulis x et y determinanda, ut eius arcus s sit $=\sqrt{2xy}$, erit aequatio eius naturam definiens

$$\left(\sqrt{x}+\sqrt{y}\right)^{\frac{1}{\sqrt{2}}+1}=a\left(\sqrt{x}-\sqrt{y}\right)^{\frac{1}{\sqrt{2}}-1}.$$

Caeterum evidens est simili modo quaestionem resolvi posse, si arcus s functioni cuicunque homogeneae unius dimensionis ipsarum x et y aequetur, seu si proponatur aequatio quaecunque homogenea inter x, y et s, id quod sequenti problemate ostendisse operae erit pretium.

PROBLEMA 90

687. *Si fuerit* $s=\int\sqrt{\left(dx^2+dy^2\right)}$ *atque aequatio proponatur homogenea quaecunque inter x, y et s, in qua scilicet hae tres variabiles x, y et s ubique eundem dimensionum numerum constituant, invenire aequationem finitam inter x et y.*

SOLUTIO

Ponatur $y=ux$ et $s=vx$, ut hac substitutione ex aequatione homogenea proposita variabilis *x* elidatur et aequatio obtineatur inter binas *u* et *v*, unde *v* per *u* definiri possit. Tum vero sit $dy=pdx$ eritque $ds=dx\sqrt{(1+pp)}$, unde fit $pdx=udx+xdu$ et $dx\sqrt{(1+pp)}=vdx+xdv$, ergo

$$\frac{dx}{x}=\frac{du}{p-u}=\frac{dv}{\sqrt{(1+pp)}-v}.$$

Quia nunc *v* datur per *u*, sit $dv=qdu$, ut habeatur $\sqrt{(1+pp)}=v+pq-qu$ et sumtis quadratis $1+pp=(v-qu)^2+2pq(v-qu)+ppqq$, unde elicitur

$$p=\frac{q(v-qu)+\sqrt{\left((v-qu)^2-1+qq\right)}}{1-qq} \quad \text{et} \quad p-u=\frac{qv-u+\sqrt{\left((v-qu)^2-1+qq\right)}}{1-qq}.$$

Quare hinc deducimus

$$\frac{dx}{x}=\frac{du(1-qq)}{qv-u+\sqrt{\left((v-qu)^2-1+qq\right)}}=\frac{du\left(qv-u-\sqrt{\left((v-qu)^2-1+qq\right)}\right)}{1+uu-vv},$$

unde, cum *v* et *q* detur per *u*, inveniri potest *x* per eandem *u*; at ob $qdu=dv$ fiet

$$lx=la-l\sqrt{(1+uu-vv)}-\int\frac{du\sqrt{\left((v-qu)^2-1+qq\right)}}{1+uu-vv},$$

tum vero est $y=ux$ seu posito $\frac{y}{x}$ loco *u* habebitur aequatio quaesita inter *x* et *y*.

COROLLARIUM 1

688. Cum *s* exprimat arcum curvae coordinatis rectangulis *x* et *y* respondentem, sic definitur curva, cuius arcus aequatur functioni cuicunque unius dimensionis ipsarum *X* et *y;* quae ergo erit algebraica, si integrale

$$\int \frac{du\sqrt{\left((v-qu)^2-1+qq\right)}}{1+uu-vv}$$

per logarithmos exhiberi potest.

COROLLARIUM 2

689. Simili modo resolvi poterit problema, si s eiusmodi formulam integralem exprimat, ut sit $ds = Qdx$ existente Q functione quacunque quantitatum p, u et v. Tum autem ex aequalitate $\frac{dx}{x} = \frac{du}{p-u} = \frac{dv}{Q-v}$ valorem ipsius p elici oportet, et quia v per u datur, erit

$$lx = \int \frac{du}{p-u}.$$

EXEMPLUM 1

690. Si debeat esse $s = \alpha x + \beta y$, erit $v = \alpha + \beta u$ et $q = \frac{dv}{dx} = \beta$, hinc $v - qu = \alpha$, ergo

$$lx = la - l\sqrt{\left(1+uu-(\alpha+\beta u)^2\right)} - \int \frac{du\sqrt{(\alpha\alpha+\beta\beta-1)}}{1+uu-(\alpha+\beta u)^2},$$

quae postrema pars est

$$-\int \frac{du\sqrt{(\alpha\alpha+\beta\beta-1)}}{1-\alpha\alpha-2\alpha\beta u+(1-\beta\beta)uu} = (\alpha\alpha+\beta\beta-1)^{\frac{1}{2}} \int \frac{du}{\alpha\alpha-1+2\alpha\beta u+(\beta\beta-1)uu},$$

quae transformatur in

$$\int \frac{(\beta\beta-1)du\sqrt{(\alpha\alpha+\beta\beta-1)}}{\left(u(\beta\beta-1)+\alpha\beta-\sqrt{(\alpha\alpha+\beta\beta-1)}\right)\left(u(\beta\beta-1)+\alpha\beta+\sqrt{(\alpha\alpha+\beta\beta-1)}\right)}$$
$$= \frac{1}{2} l \frac{(\beta\beta-1)u+\alpha\beta-\sqrt{(\alpha\alpha+\beta\beta-1)}}{(\beta\beta-1)u+\alpha\beta+\sqrt{(\alpha\alpha+\beta\beta-1)}}.$$

Quare posito $u = \frac{y}{x}$ aequatio integralis quaesita est sumtis quadratis

$$\frac{xx+yy-(\alpha x+\beta y)^2}{aa} = \frac{(\beta\beta-1)y+\alpha\beta x-x\sqrt{(\alpha\alpha+\beta\beta-1)}}{(\beta\beta-1)y+\alpha\beta x+x\sqrt{(\alpha\alpha+\beta\beta-1)}}.$$

At posito

$$(\beta\beta-1)y+\alpha\beta x-x\sqrt{(\alpha\alpha+\beta\beta-1)} = P,$$
$$(\beta\beta-1)y+\alpha\beta x+x\sqrt{(\alpha\alpha+\beta\beta-1)} = Q$$

est

$$PQ = (\beta\beta - 1)^2 yy + 2\alpha\beta(\beta\beta - 1)xy + (\alpha\alpha - 1)(\beta\beta - 1)xx$$
$$= (\beta\beta - 1)\left((\alpha x + \beta y)^2 - xx - yy\right),$$

unde mutata constante fit $\frac{PQ}{bb} = \frac{P}{Q}$, ergo vel $P = 0$ vel $Q = b$; solutio ergo in genere est

$$(\beta\beta - 1)y + \alpha\beta x \pm x\sqrt{(\alpha\alpha + \beta\beta - 1)} = c,$$

quae est aequatio pro linea recta.

EXEMPLUM 2

691. Si debeat esse $s = \frac{nyy}{x}$, erit $v = nuu$ et $q = 2nu$; unde

$$1 + uu - vv = 1 + uu - nnu^4 \quad \text{et} \quad v - qu = -nuu,$$

ergo

$$lx = la - l\sqrt{(1 + uu - nnu^4)} - \int \frac{du\sqrt{(nnu^4 - 1 + 4nnuu)}}{1 + uu - nnu^4},$$

quae formula autem per logarithmos integrari nequit.

EXEMPLUM 3

692. Si debeat esse $ss = xx + yy$, erit $v = \sqrt{(1 + uu)}$ et $q = \frac{u}{\sqrt{(1+uu)}}$,

unde fit $1 + uu - vv = 0$; solutionem ergo ex primis formulis repeti convenit, unde fit

$$v - qu = \frac{1}{\sqrt{(1+uu)}}, \quad qq - 1 = \frac{-1}{1+uu} \quad \text{et} \quad qv - u = 0;$$

ergo

$$p - u = 0 \quad \text{seu} \quad \frac{dy}{dx} - \frac{y}{x} = 0,$$

ita ut prodeat $y = nx$.

EXEMPLUM 4

693. Si debeat esse $ss = yy + nxx$ seu $v = \sqrt{(uu + n)}$ et $q = \frac{u}{\sqrt{(uu+n)}}$,

erit

$$1 + uu - vv = 1 - n, \quad v - qu = \frac{n}{\sqrt{(uu+n)}} \quad \text{et} \quad qq - 1 = \frac{-n}{\sqrt{(uu+n)}}.$$

Quare habebitur

$$lx = la - l\sqrt{(1-n)} - \frac{1}{1-n}\int \frac{du\sqrt{(nn-n)}}{\sqrt{(uu+n)}} = lb + \frac{\sqrt{n}}{\sqrt{(n-1)}}\; l\left(u + \sqrt{(uu+n)}\right)$$

hincque

$$\frac{x}{b}=\left(\frac{y+\sqrt{(yy+nxx)}}{x}\right)^{\sqrt{\frac{n}{n-1}}}.$$

Quoties ergo $\frac{n}{n-1}$ est numerus quadratus, aequatio inter x et y prodit algebraica.

Sit $\sqrt{\frac{n}{n-1}}=m$; erit $n=\frac{mm}{mm-1}$ et $ss=yy+\frac{mmxx}{mm-1}$, cui conditioni satis fit hac aequatione algebraiea

$$x^{m+1}=b\left(y+\sqrt{\left(yy+\frac{mmxx}{mm-1}\right)}\right)^{m},$$

quae transformatur in

$$x^{\frac{2}{m}}-2b^{\frac{1}{m}}x^{\frac{1-m}{m}}y=\frac{mm}{mm-1}b^{\frac{2}{m}} \text{ or } y=\frac{(mm-1)x^{\frac{2}{m}}-mmb^{\frac{2}{m}}}{2(mm-1)b^{\frac{1}{m}}x^{\frac{1-m}{m}}}.$$

COROLLARIUM

694. Ponamus $\frac{1}{m}=n$, ac si fuerit

$$y=\frac{b^{2n}+(nn-1)x^{2n}}{2(nn-1)b^{n}x^{n-1}},$$

erit

$$ss=yy-\frac{xx}{nn-1} \quad \text{seu} \quad s=\sqrt{\left(yy-\frac{xx}{nn-1}\right)}.$$

Quare si $y=\frac{b^4+3x^4}{6bbx}$, est $s=\sqrt{\left(yy-\frac{xx}{3}\right)}$.

PROBLEMA 91

695. *Si posito $\frac{dy}{dx}=p$ eiusmodi detur aequatio inter x, y et p, in qua altera variabilis y unicam tantum habeat dimensionem, invenire relationem inter binas variabiles x et y.*

SOLUTIO

Hinc ergo y aequabitur functioni cuipiam ipsarum x et p, unde differentiando fiet $dy=Pdx+Qdp$. Cum igitur sit $dy=pdx$, habebitur haec aequatio differentialis $(P-p)dx+Qdp=0$,, quam integrari oportet. Quoniam tantum duas continet variabiles x et p et differentialia simpliciter involvit, eius resolutio per methodos supra expositas est tentanda.

Primo ergo resolutio succedet, si fuerit $P=p$ ideoque $dy=pdx+Qdp$. Quod evenit, si y per x et p ita determinetur, ut sit $y=px+\Pi$ denotante Π functionem quamcunque ipsius p. Tum ergo erit $Q=x+\frac{d\Pi}{dp}$, et cum solutio ab ista aequatione $Qdp=0$ pendeat, erit vel $dp=0$ hincque $p=\alpha$ seu $y=\alpha x+\beta$, ubi altera constantium

α et β per ipsam aequationem propositam determinatur, dum posito $p=\alpha$ fit $\beta=\Pi$; vel erit $Q=0$ ideoque $x=-\frac{d\Pi}{dp}$ et $y=-\frac{p\Pi}{dp}+\Pi$, ubi ergo utraque solutio est algebraica, si modo Π fuerit functio algebraica ipsius *p*.

Secundo aequatio $(P-p)dx+Qdp=0$ resolutionem admittet, si altera variabilis *x* cum suo differentiali *dx* unam dimensionem non superet. Evenit hoc, si fuerit $y=Px+\Pi$, dum *P* et Π sunt functiones ipsius *p* tantum; tum enim erit *P* = *P* et $Q=\frac{xdP}{dp}+\frac{d\Pi}{dp}$ hincque haec habetur aequatio integranda

$$(P-p)dx+xdP+d\Pi=0 \quad \text{seu} \quad dx+\frac{xdP}{P-p}=-\frac{d\Pi}{P-p},$$

quae per $e^{\int\frac{dP}{P-p}}$ multiplicata dat

$$e^{\int\frac{dP}{P-p}}x=-\int e^{\int\frac{dP}{P-p}}\frac{d\Pi}{P-p}$$

Sive ponatur $\frac{dP}{P-p}=\frac{dR}{R}$; erit aequatio integralis

$$Rx=C-\int\frac{Rd\Pi}{P-p}=C-\int\frac{d\Pi dR}{dP}$$

unde fit

$$x=\frac{C}{R}-\frac{1}{R}\int\frac{d\Pi dR}{dP} \quad \text{et} \quad y=\frac{CP}{R}+\Pi-\frac{P}{R}\int\frac{d\Pi dR}{dP}.$$

Tertio resolutio nullam habebit difficultatem, si denotantibus *X* et *V* functiones quascunque ipsius *x* fuerit $y=X+Vp$. Tum enim erit

$$dy=pdx=dX+Vdp+pdV$$

ideoque

$$dp+p\frac{dV-dx}{V}=-\frac{dX}{V};$$

sit $\frac{dx}{V}=\frac{dR}{R}$, ut *R* sit etiam functio ipsius *x;* erit $\frac{V}{R}p=C-\int\frac{dX}{R}$ seu

$$p=\frac{CR}{V}-\frac{R}{V}\int\frac{dX}{R} \quad \text{et} \quad y=X+CR-R\int\frac{dX}{R}$$

quae aequatio relationem inter *x* et *y* exprimit.

Quarto aequatio $(P-p)dx+Qdp=0$ resolutionem admittit, si fuerit homogenea. Cum ergo terminus *pdx* duas contineat dimensiones, hoc evenit, si totidem dimensiones et in reliquis terminis insint. Unde perspicuum est *P* et *Q* esse debere functiones homogeneas unius dimensionis ipsarum *x* et *p*.Quare si *y* ita per *x* et *p* definiatur, ut *y* aequetur functioni homogeneae duarum dimensionum ipsarum *x* et *p*, resolutio succedet. Quodsi enim fuerit $dy=Pdx+Qdp$, aequatio solutionem continens $(P-p)dx+Qdp=0$ erit homogenea fietque per se integrabilis, si dividatur per $(P-p)x+Qp$.

COROLLARIUM 1

696. Pro casu quarto si ponatur $y=zz$, aequatio proposita debet esse homogenea inter tres variabiles *x*, *z* et *p*. Unde si proponatur aequatio homogenea quaecunque inter *x*, *z* et *p*, in qua hae ternae litterae *x*, *z* et *p* ubique eundem dimensionum numerum constituant, problema semper resolutionem admittit.

COROLLARIUM 2

697. Simili modo conversis variabilibus si ponatur $x=vv$ et $\frac{dx}{dy}=q$, ut sit $p=\frac{1}{q}$ ac proponatur aequatio homogenea quaecunque inter *y*, *v* et *q*, problema itidem resolvi potest.

SCHOLION

698. Pro casu quarto, ut aequatio $(P-p)dx+Qdp=0$ fiat homogenea, conditiones magis amplificari possunt. Ponatur enim $x=v^{\mu}$ et $p=q^{\nu}$ sitque facta substitutione haec aequatio $\mu\left(P-q^{\nu}\right)v^{\mu-1}dv+\nu\ Qq^{\nu-1}dq=0$ homogenea inter *v* et *q* eritque *P* functio homogenea Π dimensionum et *Q* functio homogenea μ dimensionum. Cum iam sit

$$dy=Pdx+Qdp=\mu Pv^{\mu-1}dv+\nu Qq^{\nu-1}dq$$

erit *y* functio homogenea $\mu+\nu$ dimensionum. Quare posito $y=z^{\mu+\nu}$ problema resolutionem admittit, si inter *x*, *y* et *p* eiusmodi relatio proponatur, ut positis $y=z^{\mu+\nu}, x=v^{\mu}$ et $p=q^{\nu}$ habeatur aequatio homogenea inter ternas quantitates *z*, *v* et *q*, ita ut dimensionum ab iis formatarum numerus ubique sit idem. Ac si proposita fuerit huiusmodi aequatio homogenea inter *z*, *v* et *q*, solutio problematis ita expedietur.

Cum sit $dy=pdx$, erit

$$(\mu+\nu)z^{\mu+\nu-1}dz=\mu v^{\mu-1}q^{\nu}dv\ ;$$

ponatur iam $z=rq$ et $v=sq$ et aequatio proposita tantum duas litteras *r* et *s* continebit, ex qua alteram per alteram definire licet; tum autem per has substitutiones prodibit haec aequatio

$$(\mu+\nu)r^{\mu+\nu-1}q^{\mu+\nu-1}(rdq+qdr)=\mu s^{\mu-1}q^{\mu+\nu-1}dv(sdq+qds),$$

ex qua oritur

$$\frac{dq}{q}=\frac{\mu s^{\mu-1}ds-(\mu+\nu)r^{\mu+\nu-1}dr}{(\mu+\nu)r^{\mu+\nu}-\mu s^{\mu}}$$

quae est aequatio differentialis separata, quoniam s per r datur. Quin etiam bini casus allati manifesto continentur in formulis $y = z^{\mu+\nu}, x = v^{\mu}$ et $p = q^{\nu}$, prior scilicet, si $\mu = 1$ et $\nu = 1$, posterior vero, si $\mu = 2$ et $\nu = -1$.

Hos igitur casus perinde ac praecedentes exemplis illustrari conveniet, quorum primus praecipue est memorabilis, cum per differentiationem aequationis propositae $y = px + \Pi$ statim praebeat eaequationem integralem quaesitam neque integratione omnino sit opus, siquidem alteram solutionem ex $dp = 0$ natam excludamus.

EXEMPLUM 1

699. *Proposita aequatione differentiali* $ydx - xdy = a\sqrt{\left(dx^2 + dy^2\right)}$ *eius integrale invenire.*

Posito $\frac{dy}{dx} = p$ fit $y - px = a\sqrt{(1+pp)}$, quae aequatio differentiata ob $dy = pdx$ dat $-xdp = \frac{apdp}{\sqrt{(1+pp)}}$; quae cum sit divisibilis per dp, praebet primo $p = \alpha$ hincque

$$y = \alpha x + a\sqrt{(1+\alpha\alpha)}.$$

Alter vero factor suppeditat

$$x = \frac{-ap}{\sqrt{(1+pp)}} \quad \text{hincque} \quad y = \frac{-app}{\sqrt{(1+pp)}} + a\sqrt{(1+pp)} = \frac{a}{\sqrt{(1+pp)}},$$

unde fit

$$xx + yy = aa,$$

quae est etiam aequatio integralis; sed quia novam constantem non involvit, non pro completo integrali haberi potest. Integrale autem completum duas aequationes complectitur, scilicet

$$y = ax + a\sqrt{(1+aa)} \text{ et } xx + yy = aa,,$$

quae in hac una comprehendi possunt

$$(y - \alpha x)^2 - aa(1+\alpha\alpha)(xx + yy - aa) = 0.$$

SCHOLION

700. Nisi hoc modo operatio instituatur, solutio huius quaestionis fit satis difficilis. Si enim aequationem differentialem $ydx - xdy = a\sqrt{\left(dx^2 + dy^2\right)}$ quadrando ab irrationalitate liberemus indeque rationem $\frac{dy}{dx}$ per radicis extractionem definiamus, fit

$$(xx - aa)dy - xydx = \pm adx\sqrt{(xx + yy - aa)},$$

quae aequatio per methodos cognitas difficulter tractatur. Multiplicator quidem inveniri potest utrumque membrum per se integrabile reddens; prius enim membrum $(xx-aa)dy-xydx$ divisum per $y(xx-aa)$ fit integrabile integrali existente $l\frac{y}{\sqrt{(xx-aa)}}$;
unde in genere multiplicator id integrabile reddens est

$$\frac{1}{y(xx-aa)}\Phi:\frac{y}{\sqrt{(xx-aa)}},$$

quae functio ita determinari debet, ut eodem multiplicatore quoque alterum membrum $adx\sqrt{(xx+yy-aa)}$ fiat integrabile. Talis autem multiplicator est

$$\frac{1}{y(xx-aa)}\cdot\frac{y}{\sqrt{(xx+yy-aa)}}=\frac{1}{(xx-aa)\sqrt{(xx+yy-aa)}},$$

quo fit

$$\frac{(xx-aa)dy-xydx}{(xx-aa)\sqrt{(xx+yy-aa)}}=\frac{\pm aadx}{xx-aa}.$$

Iam ad integrale prioris membri investigandum spectetur x ut constans eritque integrale

$$=l\left(y+(xx+yy-aa)\right)+X$$

denotante X functionem quampiam ipsius x ita comparatam, ut sumta iam y constanta fiat

$$\frac{xdx}{\left(y+\sqrt{(xx+yy-aa)}\right)\sqrt{(xx+yy-aa)}}+dX=\frac{-xydx}{(xx-aa)\sqrt{(xx+yy-aa)}}$$

seu

$$\frac{-xdx\left(y-\sqrt{(xx+yy-aa)}\right)}{(xx-aa)\sqrt{(xx+yy-aa)}}+dX=\frac{-xydx}{(xx-aa)\sqrt{(xx+yy-aa)}},$$

unde fit

$$dX=\frac{-xdx}{xx-aa}\quad\text{et}\quad X=l\frac{C}{\sqrt{(xx-aa)}}.$$

Quare integrale quaesitum est

$$l\left(y+\sqrt{(xx+yy-aa)}\right)+l\frac{C}{\sqrt{(xx-aa)}}=\pm l\frac{x+a}{x-a}$$

seu

$$\frac{y+\sqrt{(xx+yy-aa)}}{\sqrt{(xx-aa)}}=\alpha\sqrt{\frac{x+a}{x-a}}\quad\text{vel}\quad\alpha\sqrt{\frac{x-a}{x+a}},$$

unde fit

$$y+\sqrt{(xx+yy-aa)}=\alpha(x\pm a)$$

hincque

$$xx-aa=\alpha\alpha(x\pm a)^2-2\alpha(x\pm a)y$$

vel

$$(x\mp a)=\alpha\alpha(x\pm a)-2\alpha y\ ;$$

quae autem tantum est altera binarum aequationum integralium, altera autem aequatio integralis $xx+yy=aa$ iam quasi per divisionem de calculo sublata est censenda.

Caeterum eadem solutio aequationis

$$(aa-xx)dy+dxdx=\pm adx\sqrt{(aa+yy-xx)}$$

facilius instituitur ponendo $y=u(aa-xx)$, unde fit

$$(aa-xx)^{\frac{3}{2}}du=\pm adx\sqrt{(aa-xx)(uu-1)} \quad \text{seu} \quad \frac{du}{\sqrt{(uu-1)}}=\frac{\pm adx}{aa-xx},$$

cui quidem satisfit sumendo $u=1$, neque tamen hic casus in aequatione integrali continetur, uti supra [§ 562] iam ostendimus. Ex quo suspicari liceret alteram solutionem $xx+yy=aa$ adeo esse excludendam, quod tamen secus se habere deprehenditur, si ipsam aequationem primariam $\frac{ydx-xdy}{\sqrt{(dx^2+dy^2)}}=a$ perpendamus.

Si enim x et y sint coordinatae rectangulae lineae curvae, formula $\frac{ydx-xdy}{\sqrt{(dx^2+dy^2)}}$ exprimit perpendiculum ex origine coordinatarum in tangentem demissum, quod ergo constans esse debet. Hoc autem evenire in circulo origine in centro constituta, dum aequatio fit $xx+yy=aa$, per se est manifestum. Atque hinc realitas harum solutionum, quae minus congruae videri poterant, confirmatur, etiamsi earum ratio haud satis clare perspiciatur.

EXEMPLUM 2

701. *Proposita aequatione differentiali* $ydx-xdy=\frac{a(dx^2+dy^2)}{dx}$ *eius integrale invenire.*

Posito $dy=pdx$ fit $y-px=a(1+pp)$ et differentiando $-xdp=2apdp$, unde concluditur vel $dp=0$ et $p=\alpha$ hincque $y=\alpha x+a(1+\alpha\alpha)$ vel $x=-2ap$ et $y=a(1-pp)$ sicque ob $p=\frac{-x}{2a}$ habebitur $4ay=4aa-xx$, quae aequatio ad geometriam translata illam conditionem omnino adimplet.

Ex aequatione autem proposita radicem extrahendo reperitur

$$2ady+xdx=dx\sqrt{(xx+4ay-4aa)},$$

quae posito $y = u(4aa - xx)$ abit in

$$2adu(4aa - xx) - xdx(4au - 1) = dx\sqrt{(4aa - xx)(4au - 1)}$$

haecque posito $4au - 1 = tt$ in

$$tdt(4aa - xx) - ttxdx = tdx\sqrt{(4aa - xx)};$$

quae cum sit divisibilis per *t*, concludere licet $t = 0$ ideoque $u = \frac{1}{4a}$ atque hinc $4ay = 4aa - xx$.

EXEMPLUM 3

702. *Proposita aequatione differentiali* $ydx - xdy = a\sqrt[3]{(dx^3 + dy^3)}$ *eius integrale assignare.*

Haec aequatio more consueto, si rationem $\frac{dy}{dx}$ inde extrahere vellemus, vix tractari posset. Posito autem $dy = pdx$ fit $y - px = a\sqrt[3]{(1 + p^3)}$ et differentiando

$$xdp = \frac{-appdp}{\sqrt[3]{(1+p^3)^2}},$$

unde duplex conclusio deducitur: vel $dp = 0$ et $p = \alpha$ sicque

$$y = \alpha x + a\sqrt[3]{(1 + \alpha^3)},$$

vel

$$x = \frac{-app}{\sqrt[3]{(1+p^3)^2}} \quad \text{et} \quad y = \frac{a}{\sqrt[3]{(1+p^3)^2}},$$

unde fit $pp = -\frac{x}{y}$, et ob $y^3(1 + p^3)^2 = a^3$ erit $p^3 = \frac{a\sqrt{a}}{y\sqrt{y}} - 1$ hincque

$$\frac{\left(a\sqrt{a} - y\sqrt{y}\right)^2}{y^3} = -\frac{x^3}{y^3} \quad \text{seu} \quad x^3 + \left(a\sqrt{a} - y\sqrt{y}\right)^2 = 0.$$

EXEMPLUM 4

703. *Proposita aequatione* $ydx - nxdy = a\sqrt{(dx^2 + dy^2)}$ *eius integrale invenire.*

Posito $dy = pdx$ habetur $y - npx = a\sqrt{(1 + pp)}$, unde differentiando elicitur $(1 - n)pdx - nxdp = \frac{apdp}{\sqrt{(1+pp)}}$ sive $dx - \frac{nxdp}{(1-n)p} = \frac{adp}{(1-n)\sqrt{(1+pp)}}$, quae per $p^{\frac{n}{n-1}}$ multiplicata et integrata praebet

$$p^{\frac{n}{n-1}}x=\frac{a}{1-n}\int\frac{p^{\frac{n}{n-1}}dp}{\sqrt{(1+pp)}}.$$

Hinc deducimus casus sequentes integrationem admittentes :

if $n=\frac{3}{2}$, $p^3x=C-\frac{2}{3}a\left(pp-\frac{2}{1}\right)\sqrt{(1+pp)}$,

if $n=\frac{5}{4}$, $p^5x=C-\frac{4}{5}a\left(p^4-\frac{4}{3}p^2+\frac{4\cdot2}{3\cdot1}\right)\sqrt{(1+pp)}$,

if $n=\frac{7}{6}$, $p^7x=C-\frac{6}{7}a\left(p^6-\frac{6}{5}p^4+\frac{6\cdot4}{5\cdot3}p^2-\frac{6\cdot4\cdot2}{5\cdot3\cdot1}\right)\sqrt{(1+pp)}$,

ac si $n=\frac{2\lambda+1}{2\lambda}$, erit

$$y=px+a\sqrt{(1+pp)}+\frac{px}{2\lambda}$$

et

$$x=\frac{C}{p^{2\lambda+1}}-\frac{2\lambda a}{(2\lambda+1)p}\left(1-\frac{2\lambda}{(2\lambda-1)pp}+\frac{2\lambda(2\lambda-2)}{(2\lambda-1)(2\lambda-3)p^4}-\text{etc.}\right)\sqrt{(1+pp)}.$$

Quodsi ergo sumatur $\lambda=\infty$, ut sit $n=1$, erit

$$y=px+a\sqrt{(1+pp)}\quad\text{and}\quad x=\frac{C}{p^{2\lambda+1}}-\frac{ap}{\sqrt{(1+pp)}},$$

unde, si constans C sit = 0, statim sequitur solutio superior $xx+yy=aa$.
At si constans C non evanescat, minimum discrimen in quantitate p infinitam varietatem ipsi x inducit. Quantumvis ergo x varietur, quantitas p ut constans spectari potest, unde posito $p=a$ altera solutio $y=\alpha x+a\sqrt{(1+\alpha\alpha)}$ obtinetur. Hinc ergo dubium supra circa Exemplum 1 natum non mediocriter illustratur.

EXEMPLUM 5

704. *Proposita aequatione differentiali* $Ady^n=\left(Bx^\alpha+Cy^\beta\right)dx^n$ *existente* $n=\frac{\alpha\beta}{\alpha-\beta}$ *eius integrale investigare.*

Posito $\frac{dy}{dx}=p$ erit $Ap^n=Bx^\alpha+Cy^\beta$. Ponamus iam,

$$p=q^{\alpha\beta},\quad x=v^{\beta n},\quad\text{et}\ \ y=z^{\alpha n}$$

ut habeamus hanc aequationem homogeneam $Aq^{\alpha\beta n}=Bv^{\alpha\beta n}+Cz^{\alpha\beta n}$, quae positis $z=rq$ et $v=sq$ abit in $A=Bs^{\alpha\beta n}+Cr^{\alpha\beta n}$. Cum vero sit

$$dy=\alpha nz^{\alpha n-1}dz=\alpha nr^{\alpha n-1}q^{\alpha n-1}\left(rdq+qdr\right)$$

et
$$pdx = \beta nv^{\beta n-1}dv = \beta ns^{\beta n-1}q^{\alpha\beta+\beta n-1}\left(sdq+qds\right),$$
erit
$$\alpha r^{\alpha n-1}\left(rdq+qdr\right) = \beta s^{\beta n-1}q^{\alpha\beta+\beta n-\alpha n}\left(sdq+qds\right).$$

Est vero per hypothesin $\alpha\beta+\beta n-\alpha n=0$, unde oritur
$$\alpha r^{\alpha n}dq+\alpha r^{\alpha n-1}qdr = \beta s^{\beta n}dq+\beta s^{\beta n-1}qds$$
hincque
$$\frac{dq}{q} = \frac{\alpha r^{\alpha n-1}dr-\beta s^{\beta n-1}ds}{\beta s^{\beta n}-\alpha r^{\alpha n}}.$$
At est $s^{\beta n} = \left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}}$ hincque
$$\beta s^{\beta n-1}ds = -\frac{\beta C}{B}r^{\alpha\beta n-1}dr\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}},$$
unde fit
$$\frac{dq}{q} = \frac{\alpha r^{\alpha n-1}dr+\frac{\beta C}{B}r^{\alpha\beta n-1}dr\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1-\alpha}{\alpha}}}{\beta\left(\frac{A-Cr^{\alpha\beta n}}{B}\right)^{\frac{1}{\alpha}}-\alpha r^{\alpha n}}.$$

Facilius autem calculus hoc modo instituetur. Sumto $A=1$ erit
$$p = \frac{dy}{dx} = \left(Bx^{\alpha}+Cy^{\beta}\right)^{\frac{1}{n}};$$

sit $y = x^{\frac{\alpha}{\beta}}u$; fiet
$$x^{\frac{\alpha}{\beta}}du+\frac{\alpha}{\beta}x^{\frac{\alpha-\beta}{\beta}}udx = x^{\frac{\alpha}{n}}dx\left(B+Cu^{\beta}\right)^{\frac{1}{n}}$$

quae aequatio, cum sit $\frac{\alpha}{n} = \frac{\alpha-\beta}{\beta}$ abit in hanc
$$\beta xdu+\alpha udx = \beta dx\left(B+Cu^{\beta}\right)^{\frac{1}{n}}$$
uude fit
$$\frac{dx}{x} = \frac{\beta du}{\beta\left(B+Cu^{\beta}\right)^{\frac{1}{n}}-\alpha u},$$
sicque *x* per *u* determinatur, et quia $u = x^{-\frac{\alpha}{\beta}}y$, habebitur aequatio inter *x* et *y*.

SCHOLION

705. Hoc igitur modo operationem institui conveniet, quando inter binas variabiles x et y una cum differentialium ratione $\frac{dy}{dx} = p$ eiusmodi relatio proponitur, ex qua valor ipsius p commode elici non potest. Tum ergo calculum ita tractari oportet, ut per ifferentiationem ponendo $dy = pdx$ vel $dx = \frac{dy}{p}$ tandem perveniatur ad aequationem differentialem simplicem inter duas tantum variabiles, quem in finem etiam saepe idoneis substitutionibus uti necesse est.

Atque hucusque fere Geometris in resolutione aequationum differentialium primi gradus etiamnum pertingere licuit ; vix enim ulla via integralia investigandi adhuc quidem adhibita hic praetermissa videtur. Num autem multo maiorem calculi integralis promotionem sperare liceat, vix equidem affirmaverim, cum plurima extent inventa, quae ante vires ingenii humani superare videbantur.

Cum igitur calculum integralem in duos libros sim partitus, quorum prior circa relationem binarum tantum variabilium, posterior vero ternarum pluriumve versatur, atque iam libri primi partem priorem in differentialibus primi ordinis constitutam hic pro viribus exposuerim, ad eius alteram partem progredior, in qua binarum variabilium relatio ex data differentialium secundi altiorisve ordinis conditione requiritur.