## CHAPTER VII

# CONCERNING THE APPROXIMATE INTEGRATION OF DIFFERENTIAL EQUATIONS

### PROBLEM 85

**650.** *To assign an approximate value to the complete integral of any differential equation.*

### SOLUTION

Let $x$ and $y$ be two variables, between which the differential equation is proposed, and this equation shall have a form of this kind, so that $\frac{dy}{dx} = V$ with $V$ being some function of $x$ and $y$. Now since the complete integral is desired, this has to be interpreted thus, so that while $x$ is given a certain value, for example $x = a$, the other variable $y$ is given a certain value, for example $y = b$. Hence in the first place we are to treat the question, so that we can find the value of $y$, when the value of $x$ is attributed a value differing a little from $a$, o on putting $x = a + \omega$ so that we may find $y$. But since $\omega$ shall be the smallest possible amount, the value of $y$ will differ minimally from $b$; from which, while $x$ only is changed from $a$ as far as to $a + \omega$, the quantity $V$ is allowed to be looked on as being constant. Whereby on putting $x = a$ and $y = b$ there is made $V = A$ and from this very small change we will have $\frac{dy}{dx} = A$ and thus on integrating, $y = b + A(x - a)$, clearly with a constant of this kind to be added so that on putting $x = a$ there becomes $y = b$. Hence we may put in place $x = a + \omega$ and there becomes $y = b + A\omega$.

Hence just as here from the values given initially $x = a$ and $y = b$ we find approximately the following $x = a + \omega$ and $y = b + A\omega$, thus from these in a like manner it is allowed to progress through another very short interval, as long as it arrives finally at values however far from the starting value. Which operations are put in place so that they appear clearer, are put in place in the following manner successively.

The values of successive

| | |
|---|---|
| $x$ | $a,\ a',\ a'',a''',a^{IV}.\ \ .\ \ .\ \ 'x,\ x$ |
| $y$ | $b,\ b',\ b'',b''',b^{IV}.\ \ .\ \ .\ \ 'y,\ y\ b,$ |
| $V$ | $A,\ A',\ A'',A''',A^{IV}.\ \ .\ \ .\ \ 'V,\ V.$ |

Clearly from the first given $x = a$ and $y = b$ there is had $V = A$, then truly with the following there will be $b' = b + A(a' - a)$ with the smallest difference $a' - a$ assumed as you wish. Hence on putting $x = a'$ and $y = b'$ there is deduced $V = A'$ and from which for the third there will be obtained $b'' = b' + A'(a'' - a')$, where on putting $x = a''$ and $y = b''$

there is found $V = A''$. Now for the fourth we shall have $b''' = b'' + A''(a''' - a'')$ and hence on putting $x = a'''$ and $y = b'''$ we may deduce that $V = A'''$ and thus it is permitted to progress to values however distant from the start. Moreover the series showing the successive first values of $x$ as you please can be taken, while in the interval it may either increase or decrease.

### COROLLARY 1

**651.** Hence for the individual smallest intervals, the calculation is put in place in the same manner and thus the values, upon which the series depends , may be obtained. Hence in this way with the individual values assumed for $x$ the corresponding values of $y$ are able to be assigned.

### COROLLARY 2

**652.** Because with smaller intervals taken, through which the values of $x$ are assumed to progress, from that more accurate values for the individual points are elicited. Yet meanwhile the errors in the individual points are joined together, and even if they are much smaller, on account of the multitude of these they add up.

### COROLLARY 3

**653.** But the errors in this calculation thence arise, because in the individual intervals we regard both the quantities $x$ and $y$ as constants and thus the function $V$ may be had as constant. From which hence the more the value of $V$ in some interval following is unchanged, from that the greater the errors are to become worried about.

### SCHOLIUM 1

**654.** This inconvenience first occurs, when the value of $V$ either vanishes or increases to infinity, even if the changes happening to $x$ and $y$ should be small enough. But from these cases anyhow to avoid falling into immense errors proceed in this manner. For the start of this interval let there be $x = a$ and $y = b$, then there is put $x = a + \omega$ and $y = b + \psi$ into the proposed equation, so that there becomes $\frac{d\psi}{d\omega} = V$, moreover on $V$ becoming thus on substituting $x = a + \omega$ and $y = b + \psi$, in order that the quantities $\omega$ and $\psi$ are considered as very small, clearly on rejecting the higher powers before the lower ones; for in this way generally an integration for these intervals actually can be put in place. But there will scarcely be a need for this improvement, unless the terms arising from the values of $a$ and $b$ cancel each other. Just as if there should be this equation :

$$\frac{dy}{dx} = \frac{aa}{xx - \mathrm{yy}}$$

and for the initial term there must be $x = a$ and $y = a$; now for the start of this interval there is put $x = a + \omega$ and $y = b + \psi$ and there will be found :

$$\frac{d\psi}{d\omega} = \frac{aa}{2a\omega - 2\mathrm{a}\psi},$$

or $2\omega d\psi - 2\psi d\psi = ad\omega$ or $d\omega - \frac{2\omega d\psi}{a} = -\frac{2\psi d\psi}{a}$, which multiplied by $e^{\frac{-2\psi}{a}} = 1 - \frac{2\psi}{a}$ and integrated gives :

$$\left(1-\tfrac{2\psi}{a}\right)\omega = \tfrac{-2}{a}\int\left(1-\tfrac{2\psi}{a}\right)\psi d\psi = -\tfrac{\psi\psi}{a},$$

since on putting $\omega = 0$ there must become $\psi = 0$. Hence there is had therefore $\omega = \frac{-\psi\psi}{a-2\psi} = \frac{-\psi\psi}{a}$ or $a(a'-a) = -(b'-b)^2$ with the condition $b = a$, from which here is deduced for the following sequence $b' = b + \sqrt{-a(a'-a)}$, from which case it is apparent that the value *x* cannot increase beyond *a*, because *y* would become imaginary.

## SCHOLIUM 2

**655.** Here and there the rules are examined expressing the integration of differential equations by infinite series, but which generally are troubled by this difficulty, as only particular integrals may be shown, besides which since these series may converge only in certain cases, hence with other cases nothing useful will be given.
Just as if the proposed equation may be

$$dy + ydx = ax^n dx,$$

we appoint a series of this kind in general to be devised :

$$y = Ax^{\alpha} + Bx^{\alpha+1} + Cx^{\alpha+2} + Dx^{\alpha+3} + Ex^{\alpha+4} + \text{etc.},$$

with which in place there becomes

$$\left.\begin{array}{l} \alpha Ax^{\alpha-1} + (\alpha+1)Bx^{\alpha} + (\alpha+2)Cx^{\alpha+1} + (\alpha+3)Dx^{\alpha+2} + \text{etc.} \\ \qquad\quad + \qquad\ A \qquad\qquad\qquad B \qquad\qquad\qquad C \\ -ax^n \end{array}\right\} = 0.$$

Hence *n* is put in place of $\alpha - 1$, or $a = n+1$ and there becomes $A = \frac{\alpha}{n+1}$, then with the remaining terms reduced to zero : $B = \frac{-A}{n+2}$, $C = \frac{-B}{n+3}$, $D = \frac{-C}{n+4}$, etc. and thus this series is found :

$$y = \frac{ax^{n+1}}{n+1} - \frac{ax^{n+2}}{(n+1)(n+2)} + \frac{ax^{n+3}}{(n+1)(n+2)(n+3)} - \frac{ax^{n+4}}{(n+1)(n+2)(n+3)(n+4)} + \text{etc.}$$

Now this is only a particular integral, since with x vanishing so likewise *y*, unless *n* +1 shall be a negative number ; then this series does not now converge, unless *x* is taken very small. On account of which it is permitted to know very small values of *y*, which correspond to certain values of *x*. But the method is not troubled by this fault, which we have outlined here, since in the first place the complete integral will be given, while clearly for a given value of $x$ the value of $y$ is attributed, while now proceeding by the smallest interval always agreeing approximately with the true value and that can progress as far as it is desired. Moreover, in the following problem, this method will be perfected more.

### PROBLEM 86

**656.** *The preceding method for approximating the integral is to be perfected more, so that it differs less from the true value.*

### SOLUTION

In the proposed equation to be integrated $\frac{dy}{dx}=V$ the error of the method set out above arises thus, since through the individual intervals the function $V$ is considered as constant, as yet a change actually enters, unless the intervals are established especially small. But the variation of $V$ in whatever interval in a similar manner can be lead into the computation, which we have used in the preceding section § 321. Clearly if now $x$ agrees with $y$, then from the nature of the differentiation we see that $x-ndx$ agrees with

$$y-ndy+\frac{n(n+1)}{1\cdot 2}ddy-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}d^3y+\text{etc.},$$

which value with $n$ assumed infinite becomes

$$y-ndy+\frac{nnddy}{1\cdot 2}-\frac{n^3d^3y}{1\cdot 2\cdot 3}+\frac{n^4d^4y}{1\cdot 2\cdot 3\cdot 4}-\text{etc.},$$

[The modern reader may raise an eyebrow at times over some of Euler's derivations; remember however that the idea of a limit had not been set out clearly at that stage, and it appears to this translator at least that Euler always had in mind for *dx* and *dy* exceedingly small or vanishing quantities, as he occasionally informed his readers, but which bear usually a finite ratio to each other; hence he felt justified in multiplying a vanishing amount such as *dx* by an infinite number such as *n*, to obtain a desired finite quantity, such as $x - a$ here; in other words, an integration has taken place, but which falls outside the accepted form of writing an integral. Part of the trouble with the Calculus is the fact that the notation, still in use today, was developed by Leibniz before a full understanding of the limiting process had been thought out. Thus, what we understand by *dy/dx*, usually represented now at least in elementary texts, as the limiting value of the ratio of two continuous quantities, is not necessarily exactly the same as what Euler considered at the time of writing. Thus we might set $dx = 1$ and $dy = 3$ as long as they can appear in the form $dy/dx = 3/1$ in some equation, while Euler would have stuck with infinitesimal quantities in the same ratio. Do people still argue over this sort of thing?]

Now there is put in place $x-ndx=a$ and $y-ndy+\frac{nnddy}{1\cdot 2}-\frac{n^3d^3y}{1\cdot 2\cdot 3}+\frac{n^4d^4y}{1\cdot 2\cdot 3\cdot 4}-\text{etc.}=b$

and these values may be considered as the first in some interval, while the end values are indicated by *x* and *y*. Therefore since there shall be $n = \frac{x-a}{dx}$, there arises

$$y = b + \frac{(x-a)dy}{dx} - \frac{(x-a)^2 ddy}{1\cdot 2dx^2} + \frac{(x-a)^3 d^3y}{1\cdot 2\cdot 3dx^3} - \frac{(x-a)^4 d^4y}{1\cdot 2\cdot 3\cdot 4dx^4} + \text{etc.},$$

which expression, if *x* is not much greater than *a*, definitely converges and thus certainly is suitable for finding an approximate value for *y*. Now concerning the individual terms of this series to be set out, it is required to be noted that $\frac{dy}{dx} = V$ and hence $\frac{ddy}{dx^2} = \frac{dV}{dx}$. But since *V* shall be a function of *x* and *y*, if we put $dV = Mdx + Ndy$, on account of $\frac{dy}{dx} = V$ there becomes $\frac{ddy}{dx^2} = M + NV$ or by expressing in the way now set out above :

$$\frac{ddy}{dx^2} = \left(\frac{dV}{dx}\right) + V\left(\frac{dV}{dy}\right);$$

which expression to be used has arisen from the preceding $\frac{dy}{dx} = V$, thus from this there arises the following :

$$\frac{d^3y}{dx^3} = \left(\frac{ddV}{dx^2}\right) + \left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right) + 2V\left(\frac{ddV}{dxdy}\right) + V\left(\frac{dV}{dy}\right)^2 + VV\left(\frac{ddV}{dy^2}\right).$$

[Note that

$$\frac{d^3y}{dx^3} = \frac{d}{dx}\left(\left(\frac{dV}{dx}\right) + V\left(\frac{dV}{dy}\right)\right) + \frac{d}{dy}\left(\left(\frac{dV}{dx}\right) + V\left(\frac{dV}{dy}\right)\right)V \text{ ]}$$

Because as now the value of *y* is not yet known, in this way at least the equation can be obtained algebraically, from which the relation between *x* and *y* can be expressed, unless perhaps it suffices for the smallest terms to put $y = b$.

But the other operation in § 322 [see Part I, Ch. 7] set out the value of *y*, which corresponds to *x* at the end of some interval, will be determined explicitly, since at the start of the same interval there should be $x = a$ and $y = b$. For since hence on putting $x = a + nda$, if indeed we regard *a* and *b* as variables, there becomes

$$y = b + ndb + \frac{n(n-1)}{1\cdot 2}ddb + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}d^3b + \text{etc.},$$

then because $n = \frac{x-a}{da}$ and thus is an infinite number, there shall be

$$y = b + \frac{(x-a)db}{da} + \frac{(x-a)^2 ddb}{1\cdot 2\, da^2} + \frac{(x-a)^3 d^3b}{1\cdot 2\cdot 3\, da^3} + \text{etc.}$$

Now there shall be $\frac{db}{da} = V$, if indeed with the function *V* there is written $x = a$ and $y = b$; then with the same values substituted for *x* and *y* there will be

$$\frac{ddb}{da^2}=\left(\frac{dV}{dx}\right)+V\left(\frac{dV}{dy}\right)$$

and

$$\frac{d^3b}{da^3}=\left(\frac{ddV}{da^2}\right)+2V\left(\frac{ddV}{dxdy}\right)+VV\left(\frac{ddV}{dy^3}\right)+\left(\frac{dV}{dy}\right)\left(\left(\frac{dV}{dx}\right)+V\left(\frac{dV}{dy}\right)\right),$$

from which it is required that the following are to be formed in the same manner. Therefore let there be, after we have written :

$$x=a \text{ and } y=b\,,\ \frac{dy}{dx}=A,\quad \frac{ddy}{dx^2}=B,\ \frac{d^3y}{dx^3}=C,\ \frac{d^4y}{dx^4}=D \quad \text{etc.}$$

and this value will be agreed on for the value $x=a+\omega$ :

$$y=b+A\omega+\tfrac{1}{2}B\omega^2+\tfrac{1}{6}C\omega^3+\tfrac{1}{24}D\omega^4+\text{etc.},$$

which two values shall now be the initial values for the following interval, from which in a like manner it is required to elicit the final values.

### COROLLARY 1

**657.** Because here we have had an account of the variability of the function *V*, now greater intervals are allowed to be put in place, and if we wish these formulas *A*, *B*, *C*, *D* etc. to continue to infinity, intervals with a certain size are able to be assumed ; moreover then there may arise this infinite series for $y$.

### COROLLARY 2

**658.** If we take only the first two terms of the series found, so that there shall be $y=b+A\omega$, the preceding determination will be in place, from which likewise it is apparent that the whole error here to be equal to the sum of the following terms together.

### COROLLARY 3

**659.** But even if we take several terms of the series found, it is still agreed that it will not constitute a very large interval, as $\omega$ is given a small value, especially if the quantities *B*, *C*, *D* etc. are not very great.

### SCHOLIUM

**660.** These operations are disturbed by the greatest inconvenience, when certain of these coefficients *A*, *B*, *C*, *D* etc. increase to infinity. But this comes about only in certain intervals, where the quantity *V* either goes to zero or to infinity, and just as soon as we acknowledge the inconvenience occurring we may show a more accurate result. The remaining calculation for the individual intervals is to be put in place in a like manner, thus in order that, since the ration for the first interval should be found, which starts for arguments sake from the values assumed $x = a$ and $y = b$, the same shall prevail for the subsequent intervals. Since indeed for the end of the first interval there is made $x = a + \omega = a'$ and $y = b + A\omega + \frac{1}{2}B\omega^2 + \frac{1}{6}C\omega^3 + \frac{1}{24}D\omega^4 + \text{etc.} = b'$, these shall be the initial values for the second interval, from which in a like manner the final is required to be elicited ; clearly this calculation likewise is depending on the letters $a'$ and $b'$ and on the first letters $a$ et $b$, which becomes more apparent from the adjoining examples.

## EXAMPLE 1

**661.** *To investigate approximately the complete integral of the differential equation*

$$dy = dx\left(x^n + cy\right) .$$

Since here there shall be $V = \frac{dy}{dx} = x^n + cy$, then on differentiation there becomes

$$\frac{ddy}{dx^2} = nx^{n-1} + cx^n + ccy$$

and thus again

$$\frac{d^3y}{dx^3} = n(n-1)x^{n-2} + ncx^{n-1} + ccx^n + c^3y,$$

$$\frac{d^4y}{dx^4} = n(n-1)(n-2)x^{n-3} + n(n-1)cx^{n-2} + nccx^{n-1} + c^3x^n + c^4y,$$

etc.

But if now we put for the value $x = a$ to agree with $y = b$, for some other value $x = a + \omega$ there shall be in agreement

$$\begin{aligned} y = b &+ \omega\left(a^n + cb\right) + \tfrac{1}{2}\omega^2\left(ccb + ca^n + na^{n-1}\right) \\ &+ \tfrac{1}{6}\omega^3\left(c^3b + cca^n + nca^{n-1} + n(n-1)a^{n-2}\right) \\ &+ \tfrac{1}{24}\omega^4\left(c^4b + c^3a^n + ncca^{n-1} + n(n-1)ca^{n-2} + n(n-1)(n-2)a^{n-3}\right), \end{aligned}$$

etc.,

which series with the quantity $\omega$ of a small enough size as you wish converges at once, and thus on putting $a + \omega = a'$ and with the corresponding value of $y = b'$, hence in a similar manner we may come upon the next values, which operation is allowed to be continued as far as it pleases.

### EXAMPLE 2

**662.** *To investigate approximately the complete integral of the differential equation*

$$dy = dx(xx + yy).$$

Since here there shall be $\frac{dy}{dx} = V = xx + yy$, then on continually differentiating,

$$\frac{ddy}{dx^2} = 2x + 2xxy + 2y^3$$

and

$$\frac{d^3y}{dx^3} = 2 + 4xy + 2x^4 + 8xxyy + 6y^4,$$

$$\frac{d^4y}{dx^4} = 4y + 12x^3 + 20xyy + 16x^4y + 40xxy^3 + 24y^5,$$

$$\frac{d^5y}{dx^5} = 40x^2 + 24y^2 + 104x^3y + 120xy^3 + 16x^6 + 136x^4y^2 + 240x^2y^4 + 120y^6,$$

etc.

Whereby if initially there should be $x = a$ and $y = b$, then there shall be

$$A = aa + bb,$$

$$B = 2a + 2aab + 2b^3,$$

$$C = 2 + 4ab + 2a^4 + 8aabb + 6b^4,$$

$$D = 4b + 12a^3 + 20abb + 16a^4b + 40aab^3 + 24b^5,$$

$$E = 40a^2 + 24b^2 + 104a^3b + 120ab^3 + 16a^6 + 136a^4b^4 + 240a^2b^4 + 120b^6,$$

from which for whatever value $x = a + \omega$ for the other there shall be in agreement :

$$y = b + A\omega + \tfrac{1}{2}B\omega^2 + \tfrac{1}{6}C\omega^3 + \tfrac{1}{24}D\omega^4 + \tfrac{1}{120}E\omega^5 + \text{etc.},$$

and from two such values, which shall be $x = a'$ and $y = b'$, anew the next values are able to be elicited.

### SCHOLIUM

**663.** Because the whole procedure has been reduced to the discovery of the coefficients $A, B, C, D$ etc., I note that the same can be found without differentiation, thus that which was in this last example $\frac{dx}{dy} = xx + yy$ will be performed. Since we have decided by putting $x = a$ for $y = b$ to be established, in general we can put $x = a + \omega$ and $y = b + \psi$, and then our equation adopts this form :

$$\frac{d\psi}{d\omega} = aa + bb + 2a\omega + \omega\omega + 2b\psi + \psi\psi \ ,$$

and because with $\omega$ vanishing likewise $\psi$ vanishes, we may assume

$$\psi = \alpha\omega + \beta\omega^2 + \gamma\omega^3 + \delta\omega^4 + \varepsilon\omega^5 + \text{etc} .$$

and with this value substituted there is produced :

$$\begin{aligned} &\alpha + 2\beta\omega + 3\gamma\omega^2 + 4\delta\omega^3 + 5\varepsilon\omega^4 + \text{etc.} \\ = \ &aa + bb + 2a\omega + \omega^2 \\ &+ 2\alpha b\omega + 2\beta b\omega^2 + 2\gamma b\omega^3 + 2\delta b\omega^4 \\ &+ \alpha^2\omega^2 + 2\alpha\beta\omega^3 + 2\alpha\gamma\omega^4 \\ &+ \beta\beta\omega^4 \end{aligned}$$

Hence with the individual terms reduced to zero there becomes :

$$\alpha = aa + bb, \quad \beta = 2\alpha b + 2a, \ 3\gamma = 2\beta b + \alpha\alpha + 1, \ 4\delta = 2\gamma b + 2\alpha\beta,$$
$$5\varepsilon = 2\delta b + 2\alpha\gamma + \beta\beta, \quad 6\zeta = 2\varepsilon b + 2\alpha\delta + 2\beta\gamma \quad \text{etc.},$$

from which the same values which above by differentiation were elucidated. As this method is simpler than the preceding, thus also this excels that, since it can always be called into use, while that one occasionally may be applied in vain, just as in the examples reported it comes about, if the initial values *a* and *b* vanish, where several coefficients become zero. Because we have noticed the same inconvenience above now, where thus it came about that all the coefficients either vanish or become infinite. Now this comes to be used only in certain intervals, for which hence it is convenient for a special calculation to be put in place ; but for the remaining intervals the method here set out of proceeding by differentiation is seen to be the more convenient to be adhered to, certainly which is often easier to be put in place than by the substitution , and surely from the rules there is also always a place held for treating transcending equations. Whereby for these individual intervals it will be required for the treatment of these to be examined.

## PROBLEM 87

**664.** *If in the integration of the equation $\frac{dy}{dx} = V$ for some interval it may come about, that the quantity V either vanishes or becomes infinite, then to put in place the integration for such an interval.*

## SOLUTION

Let there be for the first interval that we are considering, $x = a$ and $y = b$; in which case since $V$ either vanishes or becomes infinite, we may put $\frac{dy}{dx} = \frac{P}{Q}$ thus so that on putting $x = a$ and $y = b$ either $P$ or $Q$ or each vanishes. Hence we are to put in place, so that from these terms we may progress further, $x = a + \omega$ and $y = b + \psi$ and there becomes $\frac{dy}{dx} = \frac{d\psi}{d\omega}$ and as $P$ as well as $Q$ will be a function of $\omega$ and $\psi$, either of which perhaps vanishes on putting $\omega = 0$ and $\psi = 0$. Now towards investigating the relation between $\omega$ and $\psi$ approximately in any case there is put $\psi = m\omega^n$; then there shall be $\frac{d\psi}{d\omega} = mn\omega^{n-1}$ and hence $mnQ\omega^{n-1} = P$, where $P$ and $Q$ on account of $\psi = m\omega^n$ will contain pure powers of $\omega$, of which it suffices to retain only the least in the calculation, since higher ones are considered to vanish before these. Hence the lowest powers of $\omega$ are returned equal to each other and likewise may be reduced to nothing ; from which so the exponent $n$ as well as the coefficient $m$ will be determined. If then we wish to know the relation between $\omega$ and $\psi$ more precisely, we may rise to higher powers of $m$ and $n$ by putting $\psi = m\omega^n + M\omega^{n+\mu} + N\omega^{n+v} + \text{etc.}$ and hence in a like manner the following terms may be defined, and as far as it should appear necessary on account of the magnitude of the interval or of the minuteness of $\omega$.

## COROLLARY 1

**665.** If on putting $x = a$ and $y = b$ neither $P$ nor $Q$ vanishes, with the substitution used there may be found $\frac{d\psi}{d\omega} = \frac{A+\text{etc.}}{\alpha+\text{etc.}}$ and hence $\alpha d\psi = Ad\omega$ approximately and $\psi = \frac{A}{\alpha}\omega$, which is the first term of the preceding approximation, from which the discovery of the rest may themselves be had as before.

## COROLLARY 2

**666.** If only $\alpha$ vanishes, there will be had approximately $\frac{d\psi}{d\omega}\left(M\omega^\mu + N\psi^v\right) = A$, from which on putting $\psi = m\omega^n$ there is made $A = mn\omega^{n-1}\left(M\omega^\mu + Nm^v\omega^{nv}\right)$; where if $nv > \mu$, then there must become $n = 1 - \mu$ and $mnM = A$; but that it will not be the case, unless there shall be $v(1-\mu) > \mu$ or $v > \frac{\mu}{1-\mu}$. But if there should be $v < \frac{\mu}{1-\mu}$, then there must be put in place $n - 1 + nv = 0$ or $n = \frac{1}{1+v}$ with the other term observed as the lowest power. But if there should be $\nu = \frac{\mu}{1-\mu}$ both terms are to be had with equal powers, and there becomes $n = 1 - \mu$ and $A = mn\left(M + Nm^{\nu}\right)$, from which $m$ must be defined.

## SCHOLIUM

**667.** In general scarcely any preparatory work is allowed here, but generally it is evident that not in all cases presented is a solution found with difficulty. If indeed all the exponents are to be whole numbers, from those Newtonian rules, by which with the aid of a parallelogram it was shown how to resolve an equation, this may be put to use; then the reduction of fractional exponents to integers had been observed well enough. [The reader may recall that Newton was able to establish the general shape of most cubic curves, from their approximate shape at certain points, such as axis intercepts, turning points, nodes, etc., by discarding higher order terms. See if you can that excellent little book, now long out of print, one presumes : *Topics in Recreational Mathematics* by J. H. Cadwell, CUP (1960); Curve Tracing] Now cases of this kind occur so rarely, that it would be a useless exercise in excessive instruction, since in whatever case they are readily put together from the exercise. Just as if that equation should be come upon : $\frac{d\psi}{d\omega}\left(\alpha\sqrt{\omega} + \beta\psi\right) = \gamma$, from above it is apparent that the first operation gives $\psi = m\sqrt{\omega}$, from which there is made $\frac{1}{2}m\left(\alpha + \beta m\right) = \gamma$, from which $m$ becomes known and that in the square root manner. Also since this equation is reduced to homogeneity on putting $\sqrt{\omega} = p$, and thus can actually be integrated. Now this will scarcely have any use and I shall not pursue it further, but, since at this point this remains to be treated, I shall explain, how differential equations of this kind are to be resolved, in which the ratio of the differentials, for example $\frac{dy}{dx} = p$, or higher powers may be found, or thus to be undertaken in a transcendental manner ; to which the following part is completely devoted, and in which differentials of higher orders are present.

# CAPUT VII
# DE INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM PER APPROXIMATIONEM

### PROBLEMA 85

**650.** *Proposita aequatione differentiali quacunque eius integrale completum vero proxime assignare.*

### SOLUTIO

Sint $x$ et $y$ binae variabiles, inter quas aequatio differentialis proponitur, atque haec aequatio huiusmodi habebit formam, ut sit $\frac{dy}{dx}=V$ existente $V$ functione quacunque ipsarum $x$ et $y$. Iam cum integrale completum desideretur, hoc ita est interpretandum, ut, dum ipsi $x$ certus quidem valor, puta $x=a$, tribuitur, altera variabilis $y$ datum quemdam valorem, puta $y=b$, adipiscatur. Quaestionem ergo primo ita tractemus, ut investigemus valorem ipsius $y$, quando ipsi $x$ valor paulisper ab $a$ discrepans tribuitur, seu posito $x=a+\omega$ ut quaeramus $y$. Cum autem $\omega$ sit particula minima, etiam valor ipsius $y$ minime a $b$ discrepabit; unde, dum $x$ ab $a$ usque ad $a+\omega$ tantum mutatur, quantitatem $V$ interea tanquam constantem spectare licet. Quare posito $x=a$ et $y=b$ fiat $V=A$ et pro hac exigua mutatione habebimus $\frac{dy}{dx}=A$ ideoque integrando $y=b+A(x-a)$, eiusmodi scilicet constante adiecta, ut posito $x=a$ fiat $y=b$. Statuamus ergo $x=a+\omega$ fietque $y=b+A\omega$.

Quemadmodum ergo hic ex valoribus initio datis $x=a$ et $y=b$ proxime sequentes $x=a+\omega$ et $y=b+A\omega$ invenimus, ita ab his simili modo per intervalla minima ulterius progredi licet, quoad tandem ad valores a primitivis quantumvis remotos perveniatur. Quae operationes quo clarius ob oculos ponantur, sequenti modo successive instituantur.

| Ipsius | valores successivi |
|---|---|
| $x$ | $a,\ a',\ a'',a''',a^{IV}\ .\ \ .\ \ .\ \ 'x,\ x$ |
| $y$ | $b,\ b',\ b'',b''',b^{IV}\ .\ \ .\ \ .\ \ 'y,\ y\ b,$ |
| $V$ | $A,\ A',\ A'',A''',A^{IV}\ .\ \ .\ \ .\ \ 'V,\ V.$ |

Scilicet ex primis $x=a$ et $y=b$ datis habetur $V=A$, tum vero pro secundis erit $b'=b+A(a'-a)$ differentia $a'-a$ minima pro lubitu assumta. Hinc ponendo $x=a'$ et $y=b'$ colligitur $V=A'$ indeque pro tertiis obtinebitur $b''=b'+A'(a''-a')$, ubi posito $x=a''$ et $y=b''$ invenitur $V=A''$. Iam pro quartis habebimus $b'''=b''+A''(a'''-a'')$ hincque ponendo $x=a'''$ et $y=b'''$ colligemus $V=A'''$ sicque ad valores a primitivis quantumvis remotos progredi licebit. Series autem prima valores ipsius *x* successivos exhibens pro lubitu accipi potest, dummodo per intervalla minima ascendat vel etiam descendat.

### COROLLARIUM 1

**651.** Pro singulis ergo intervallis minimis calculus eodem modo instituitur sicque valores, a quibus sequentia pendent, obtinentur. Hoc ergo modo singulis pro *x* assumtis valoribus valores respondentes ipsius *y* assignari possunt.

### COROLLARIUM 2

**652.** Quo minora accipiuntur intervalla, per quae valores ipsius *x* progredi assumuntur, eo accuratius valores pro singulis eliciuntur. Interim tamen errores in singulis commissi, etiamsi sint multo minores, ob multitudinem coacervantur.

### COROLLARIUM 3

**653.** Errores autem in hoc calculo inde oriuntur, quod in singulis intervallis ambas quantitates *x* et *y* ut constantes spectemus sicque functio *V* pro constante habeatur. Quo magis ergo valor ipsius *V* a quovis intervallo ad sequens immutatur, eo maiores errores sunt pertimescendi.

### SCHOLION 1

**654.** Hoc incommodum imprimis occurrit, ubi valor ipsius *V* vel evanescit vel in infinitum excrescit, etiamsi mutationes ipsis *x* et *y* accidentes sint satis parvae. His autem casibus errores saltim enormes sequenti modo evitabuntur. Sit pro initio huiusmodi intervalli $x=a$ et $y=b$, tum vero in ipsa aequatione proposita ponatur $x=a+\omega$ et $y=b+\psi$, ut sit $\frac{d\psi}{d\omega}=V$, in *V* autem ita fiat substitutio $x=a+\omega$ et $y=b+\psi$, ut quantitates $\omega$ et $\psi$ tanquam minimae spectentur, reiiciendo scilicet altiores potestates prae inferioribus; hoc enim modo plerumque integratio pro his intervallis actu institui poterit. Hac autem emendatione vix unquam erit opus, nisi termini ex ipsis valoribus *a* et *b* nati se destruant. Veluti si habeatur haec aequatio

$$\frac{dy}{dx}=\frac{aa}{xx-yy}$$

ac pro initio debeat esse $x=a$ et $y=a$; iam pro intervallo hinc incipiente ponatur $x=a+\omega$ et $y=b+\psi$ habebiturque

$$\frac{d\psi}{d\omega}=\frac{aa}{2a\omega-2a\psi}$$

seu $2\omega d\psi-2\psi d\psi=ad\omega$ seu $d\omega-\frac{2\omega d\psi}{a}=-\frac{2\omega d\psi}{a}$, quae per $e^{\frac{-2\psi}{a}}=1-\frac{2\psi}{a}$ multiplicata et integrata praebet

$$\left(1-\frac{2\psi}{a}\right)\omega=\frac{-2}{a}\int\left(1-\frac{2\psi}{a}\right)\psi d\psi=-\frac{\psi\psi}{a},$$

quia posito $\omega = 0$ fieri debet $\psi = 0$. Hinc ergo habetur $\omega = \frac{-\psi\psi}{a-2\psi} = \frac{-\psi\psi}{a}$ seu $a\left(a'-a\right) = -\left(b'-b\right)^2$ existente $b = a$, unde colligitur pro sequente intervallo $b' = b + \sqrt{-a\left(a'-a\right)}$, quo casu patet valorem $x$ non ultra $a$ augeri posse, quia $y$ fieret imaginarium.

**SCHOLION 2**

**655.** Passim traduntur regulae aequationum differentialium integralia per series infinitas exprimendi, quae autem plerumque hoc vitio laborant, ut integralia tantum particularia exhibeant, praeterquam quod series illae certo tantum casu convergant neque ergo aliis casibus ullum usum praestent.
Veluti si proposita sit aequatio

$$dy + ydx = ax^n dx,$$

iubemur huiusmodi seriem in genere fingere

$$y = Ax^{\alpha} + Bx^{\alpha+1} + Cx^{\alpha+2} + Dx^{\alpha+3} + Ex^{\alpha+4} + \text{etc}.,$$

qua substituta fit

$$\left.\begin{array}{l} \alpha Ax^{\alpha-1} + (\alpha+1)Bx^{\alpha} + (\alpha+2)Cx^{\alpha+1} + (\alpha+3)Dx^{\alpha+2} + \text{etc.} \\ \qquad\quad + \qquad\quad A \qquad\qquad\qquad B \qquad\qquad\qquad C \\ -ax^n \end{array}\right\} = 0.$$

Statuatur ergo $\alpha - 1 \quad n$ seu $a = n+1$ eritque $A = \frac{\alpha}{n+1}$ tum vero reliquis terminis ad nihilum reductis $B = \frac{-A}{n+2}$, $C = \frac{-B}{n+3}$, $D = \frac{-C}{n+4}$, etc. sicque habebitur haec series

$$y = \frac{ax^{n+1}}{n+1} - \frac{ax^{n+2}}{(n+1)(n+2)} + \frac{ax^{n+3}}{(n+1)(n+2)(n+3)} - \frac{ax^{n+4}}{(n+1)(n+2)(n+3)(n+4)} + \text{etc.}$$

Verum hoc integrale tantum est particulare, quoniam evanescente $x$ simul $y$ evanescit, nisi $n$ +1 sit numerus negativus; tum vero haec series non eonvergit, nisi $x$ capiatur valde parvum. Quamobrem hinc minime cognoscere licet valores ipsius $y$, qui respondeant valoribus quibuscunque ipsius $x$. Hoc autem vitio non laborat methodus, quam hic adumbravimus, cum primo integrale completum praebeat, dum scilicet pro dato ipsius $x$ valore datum ipsi $y$ valorem tribuit, tum vero per intervalla minima procedens semper proxime ad veritatem accedat et, quousque libuerit, progredi liceat. Sequenti autem modo haec methodus magis perfici poterit.

## PROBLEMA 86

**656.** *Methodum praecedentem aequationes differentiales proxime integrandi magis perficere, ut minus a veritate aberret.*

## SOLUTIO

Proposita aequatione integranda $\frac{dy}{dx}=V$ error methodi supra expositae inde oritur, quod per singula intervalla functio $V$ ut constans spectetur, cum tamen revera mutationem subeat, praecipue nisi intervalla statuantur minima. Variabilitas autem ipsius $V$ per quodvis intervallum simili modo in computum duci potest, quo in sectione praecedente § 321 usi sumus. Scilicet si iam ipsi $x$ conveniat $y$, ex natura differentialium ipsi $x-ndx$ vidimus convenire

$$y-ndy+\frac{n(n+1)}{1\cdot 2}ddy-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}d^3y+\text{etc.},$$

qui valor sumto $n$ infinito erit

$$y-ndy+\frac{nnddy}{1\cdot 2}-\frac{n^3d^3y}{1\cdot 2\cdot 3}+\frac{n^4d^4y}{1\cdot 2\cdot 3\cdot 4}-\text{etc.},$$

Statuatur iam $x-ndx=a$ et $y-ndy+\frac{nnddy}{1\cdot 2}-\frac{n^3d^3y}{1\cdot 2\cdot 3}+\frac{n^4d^4y}{1\cdot 2\cdot 3\cdot 4}-\text{etc.}=b$ hique valores in quovis intervallo ut primi spectentur, dum extremi per $x$ et $y$ indicantur. Cum igitur sit $n=\frac{x-a}{dx}$, fiet

$$y=b+\frac{(x-a)dy}{dx}-\frac{(x-a)^2ddy}{1\cdot 2dx^2}+\frac{(x-a)^3d^3y}{1\cdot 2\cdot 3dx^3}-\frac{(x-a)^4d^4y}{1\cdot 2\cdot 3\cdot 4dx^4}+\text{etc.},$$

quae expressio, si $x$ non multum superat $a$, valde convergit ideoque admodum est idonea ad valorem $y$ proxime inveniendum. Verum ad singulos terminos huius seriei evolvendos notari oportet esse $\frac{dy}{dx}=V$ hincque $\frac{ddy}{dx^2}=\frac{dV}{dx}$. Cum autem $V$ sit functio ipsarum $x$ et $y$, si ponamus $dV=Mdx+Ndy$, ob $\frac{dy}{dx}=V$ erit $\frac{ddy}{dx^2}=M+NV$ seu exprimendi modo iam supra exposito

$$\frac{ddy}{dx^2}=\left(\frac{dV}{dx}\right)+V\left(\frac{dV}{dy}\right);$$

quae expressio uti nata est ex praecedente $\frac{dy}{dx}=V$ ita ex ea nascetur sequens

$$\frac{d^3y}{dx^3}=\left(\frac{ddV}{dx^2}\right)+\left(\frac{dV}{dx}\right)\left(\frac{dV}{dy}\right)+2V\left(\frac{ddV}{dxdy}\right)+V\left(\frac{dV}{dy}\right)^2+VV\left(\frac{ddV}{dy^2}\right).$$

Quoniam vero ipse valor ipsius *y* nondum est cognitus, hoc modo saltem obtinetur aequatio algebraica, qua relatio inter *x* et *y* exprimitur, nisi forte sufficiat in terminis minimis posuisse $y=b$.

Altera autem operatio § 322 exposita valorem ipsius *y*, qui ipsi *x* in fine cuiusque intervalli respondet, explicite determinabit, cum in initio ciusdem intervalli fuerit $x=a$ et $y=b$. Cum enim hinc posito $x=a+nda$, si quidem *a* et *b* ut variabiles spectemus, fiat

$$y=b+ndb+\frac{n(n-1)}{1\cdot 2}ddb+\frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}d^3b+\text{etc.},$$

quia est $n=\frac{x-a}{da}$ ideoque numerus infinitus, erit

$$y=b+\frac{(x-a)db}{da}+\frac{(x-a)^2ddb}{1\cdot 2\,da^2}+\frac{(x-a)^3d^3b}{1\cdot 2\cdot 3\,da^3}+\text{etc.}$$

Est vero $\frac{db}{da}=V$, siquidem in functione *V* scribatur $x=a$ et $y=b$; tum vero iisdem pro *x* et *y* valoribus substitutis erit

$$\frac{ddb}{da^2}=\left(\frac{dV}{dx}\right)+V\left(\frac{dV}{dy}\right)$$

et

$$\frac{d^3b}{da^3}=\left(\frac{ddV}{da^2}\right)+2V\left(\frac{ddV}{dxdy}\right)+VV\left(\frac{ddV}{dy^3}\right)+\left(\frac{dV}{dy}\right)\left(\left(\frac{dV}{dx}\right)+V\left(\frac{dV}{dy}\right)\right),$$

unde sequentes simili modo formari oportet. Sit igitur, postquam scripserimus

$$x=a \text{ et } y=b,\ \frac{dy}{dx}=A,\quad \frac{ddy}{dx^2}=B,\ \frac{d^3y}{dx^3}=C,\ \frac{d^4y}{dx^4}=D \quad \text{etc.}$$

ac valori $x=a+\omega$ conveniet iste valor

$$y=b+A\omega+\frac{1}{2}B\omega^2+\frac{1}{6}C\omega^3+\frac{1}{24}D\omega^4+\text{etc.},$$

qui duo valores iam pro sequente intervallo erunt initiales, ex quibus simili modo finales erui oportet.

### COROLLARIUM 1

**657.** Quoniam hic variabilitatis functionis *V* rationem habuimus, intervalla iam maiora statuere licet, ac si illas formulas *A, B, C, D* etc. in infinitum continuare vellemus, intervalla quantumvis magna assumi possent; tum autem pro *y* oriretur series infinita.

### COROLLARIUM 2

**658.** Si seriei inventae tantnm binos terminos primos sumamus, ut sit $y = b + A\omega$, habebitur determinatio praecedens, unde simul patet errorem ibi commissum sequentibus terminis iunctim sumtis aequari.

### COROLLARIUM 3

**659.** Etiamsi autem seriei inventae plures terminos capiamus, consultum tamen non erit intervalla nimis magna constitui, ut $\omega$ valorem modicum obtineat, praecipue si quantitates *B, C, D* etc. evadant valde magnae.

### SCHOLION

**660.** Maximo incommodo hae operationes turbantur, si quando horum coefficientium *A, B, C, D* etc. quidam in infinitum excrescant. Evenit autem hoc tantum in certis intervallis, ubi ipsa quantitas *V* vel in nihilum abit vel in infinitum, cui incommodo quemadmodum sit occurrendum, iam innuimus et mox accuratius ostendemus. Caeterum calculus pro singulis intervallis pari modo instituitur, ita ut, cum eius ratio pro intervallo primo fuerit inventa, quod incipit a valoribus pro lubitu assumtis $x = a$ et $y = b$, eadem pro sequentibus intervallis sit valitura. Cum enim pro fine intervalli primi fiat $x = a + \omega = a'$ et $y = b + A\omega + \frac{1}{2}B\omega^2 + \frac{1}{6}C\omega^3 + \frac{1}{24}D\omega^4 + \text{etc.} = b'$, hi erunt valores initiales pro intervallo secundo, ex quibus simili modo finales elici oportet; hic scilicet calculus innitetur perinde litteris *a'* et *b'* ac prior litteris *a* et *b*, id quod clarius ex exemplis subiunctis patebit.

### EXEMPLUM 1

**661.** *Aequationis differentialis* $dy = dx\left(x^n + cy\right)$ *integrale completum proxime investigare.*

Cum hic sit $V = \frac{dy}{dx} = x^n + cy$, erit differentiando $\frac{ddy}{dx^2} = nx^{n-1} + cx^n + ccy$ sicque porro

$$\frac{d^3y}{dx^3} = n\left(n-1\right)x^{n-2} + ncx^{n-1} + ccx^n + c^3y,$$
$$\frac{d^4y}{dx^4} = n\left(n-1\right)\left(n-2\right)x^{n-3} + n\left(n-1\right)cx^{n-2} + nccx^{n-1} + c^3x^n + c^4y,$$
etc.

Quodsi ergo ponamus valori $x = a$ convenire $y = b$, alii cuicunque valori $x = a + \omega$ conveniet

$$y = b + \omega\left(a^n + cb\right) + \tfrac{1}{2}\omega^2\left(ccb + ca^n + na^{n-1}\right)$$
$$+ \tfrac{1}{6}\omega^3\left(c^3b + cca^n + nca^{n-1} + n(n-1)a^{n-2}\right)$$
$$+ \tfrac{1}{24}\omega^4\left(c^4b + c^3a^n + ncca^{n-1} + n(n-1)ca^{n-2} + n(n-1)(n-2)a^{n-3}\right),$$
$$\text{etc.},$$

quae series sumta quantitate $\omega$ satis parva quantumvis promte convergit, sicque posito $a + \omega = a'$ et respondente valore ipsius $y = b'$ hinc simili modo ad sequentes perveniemus, quam operationem, quousque lubuerit, continuare licet.

## EXEMPLUM 2

**662.** *Aequationis differentialis* $dy = dx(xx + yy)$ *integrale completum proxime investigare.*

Cum hic sit $\frac{dy}{dx} = V = xx + yy$, erit continuo differentiando

$$\frac{ddy}{dx^2} = 2x + 2xxy + 2y^3$$

et

$$\frac{d^3y}{dx^3} = 2 + 4xy + 2x^4 + 8xxyy + 6y^4,$$
$$\frac{d^4y}{dx^4} = 4y + 12x^3 + 20xyy + 16x^4y + 40xxy^3 + 24y^5,$$
$$\frac{d^5y}{dx^5} = 40x^2 + 24y^2 + 104x^3y + 120xy^3 + 16x^6 + 136x^4y^2 + 240x^2y^4 + 120y^6,$$
$$\text{etc.}$$

Quare si initio sit $x = a$ et $y = b$, erit

$$A = aa + bb,$$
$$B = 2a + 2aab + 2b^3,$$
$$C = 2 + 4ab + 2a^4 + 8aabb + 6b^4,$$
$$D = 4b + 12a^3 + 20abb + 16a^4b + 40aab^3 + 24b^5,$$
$$E = 40a^2 + 24b^2 + 104a^3b + 120ab^3 + 16a^6 + 136a^4b^4 + 240a^2b^4 + 120b^6,$$

unde valori cuicunque alii $x = a + \omega$ conveniet

$$y = b + A\omega + \tfrac{1}{2}B\omega^2 + \tfrac{1}{6}C\omega^3 + \tfrac{1}{24}D\omega^4 + \tfrac{1}{120}E\omega^5 + \text{etc.},$$

atque ex talibus binis valoribus, qui sint $x = a'$ et $y = b'$, denuo sequentes elici possunt.

### SCHOLION

**663.** Quoniam totum negotium ad inventionem horum coefficientium *A, B, C, D* etc. redit, observo eosdem sine differentiatione inveniri posse, id quod in hoc postremo exemplo $\frac{dx}{dy} = xx + yy$ ita praestabitur. Cum statuamus posito $x = a$ fieri $y = b$, ponamus in genere $x = a + \omega$ et $y = b + \psi$ et nostra aequatio induet hanc formam

$$\frac{d\psi}{d\omega} = aa + bb + 2a\omega + \omega\omega + 2b\psi + \psi\psi ,$$

et quia evanescente $\omega$ simul evanescit $\psi$, sumamus

$$\psi = \alpha\omega + \beta\omega^2 + \gamma\omega^3 + \delta\omega^4 + \varepsilon\omega^5 + \text{etc}.$$

hocque valore substituto prodibit

$$\begin{aligned} &\alpha + 2\beta\omega + 3\gamma\omega^2 + 4\delta\omega^3 + 5\varepsilon\omega^4 + \text{etc.} \\ = {} & aa + bb + 2a\omega + \omega^2 \\ &+ 2\alpha b\omega + 2\beta b\omega^2 + 2\gamma b\omega^3 + 2\delta b\omega^4 \\ &+ \alpha^2\omega^2 + 2\alpha\beta\omega^3 + 2\alpha\gamma\omega^4 \\ &+ \beta\beta\omega^4 \end{aligned}$$

Singulis ergo terminis ad nihilum reductis fiat

$$\begin{aligned} &\alpha = aa + bb, \quad \beta = 2\alpha b + 2a, \; 3\gamma = 2\beta b + \alpha\alpha + 1, \; 4\delta = 2\gamma b + 2\alpha\beta, \\ &5\varepsilon = 2\delta b + 2\alpha\gamma + \beta\beta, \quad 6\zeta = 2\varepsilon b + 2\alpha\delta + 2\beta\gamma \quad \text{etc.}, \end{aligned}$$

unde iidem valores qui supra per differentiationem eliciuntur. Ut haec methodus simplicior est praecedente, ita etiam hoc illi praestat, quod semper in usum vocari possit, cum illa interdum frustra applicetur, veluti in exemplis allatis evenit, si valores initiales *a* et *b* evanesecant, ubi plerique coefficientes in nihilum abirent. Quod idem incommodum iam supra animadvertimus, cum adeo evenire possit, ut omnes coefficientes vel evaneseant vel in infinitum abeant. Verum hoc nonnisi in certis intervallis usu venit, pro quibus ergo calculum peculiari modo institui conveniet; reliquis autem intervallis methodus hic exposita per differentiationem procedens commodius adhiberi videtur, quippe quae saepe facilius instituitur quam substitutio certisque regulis continetur semper locum habentibus etiam in aequationibus transcendentibus. Quare pro singularibus illis intervallis praecepta tradere oportebit.

### PROBLEMA 87

**664.** *Si in integratione aequationis $\frac{dy}{dx}=V$ pro quopiam intervallo eveniat, ut quantitas $V$ vel evanescat vel fiat infinita, integrationem pro isto intervallo instituere.*

### SOLUTIO

Sit pro initio intervalli, quod contemplamur, $x=a$ et $y=b$; quo casu cum $V$ vel evanescat vel in infinitum abeat, ponamus $\frac{dy}{dx}=\frac{P}{Q}$ ita ut posito $x=a$ et $y=b$ vel $P$ vel $Q$ vel utrumque evanescat. Statuamus ergo, ut ab his terminis ulterius progrediamur, $x=a+\omega$ et $y=b+\psi$ fietque $\frac{dy}{dx}=\frac{d\psi}{d\omega}$ atque tam $P$ quam $Q$ erit functio ipsarum $\omega$ et $\psi$, quarum altera saltem evanescat facto $\omega=0$ et $\psi=0$. Iam ad rationem inter $\omega$ et $\psi$ proxime saltem investigandam ponatur $\psi=m\omega^{n}$; erit $\frac{d\psi}{d\omega}=mn\omega^{n-1}$ hincque $mnQ\omega^{n-1}=P$, ubi $P$ et $Q$ ob $\psi=m\omega^{n}$ meras potestates ipsius $\omega$ continebunt, quarum tantum minimas in calculo retinuisse sufficit, cum altiores prae his ut evanescentes spectari queant. Infimae ergo potestates ipsius $\omega$ inter se aequales reddantur simulque ad nihilum redigantur; unde tam exponens $n$ quam coefficiens $m$ determinabitur. Si deinde relationem inter $\omega$ et $\psi$ exactius cognoscere velimus, inventis $m$ et $n$ ad altiores potestates ascendamus ponendo $\psi=m\omega^{n}+M\omega^{n+\mu}+N\omega^{n+\nu}+\text{etc.}$ hincque simili modo sequentes partes definientur, quousque ob magnitudinem intervalli sen particulae $\omega$ necessarium visum fuerit.

### COROLLARIUM 1

**665.** Si posito $x=a$ et $y=b$ neque $P$ neque $Q$ evanescat, substitutione adhibita reperietur $\frac{d\psi}{d\omega}=\frac{A+\text{etc.}}{\alpha+\text{etc.}}$ hincque proxime $\alpha d\psi=Ad\omega$ et $\psi=\frac{A}{\alpha}\omega$, qui est primus terminus praecedentis approximationis, quo invento reliqui ut ante se habebunt.

### COROLLARIUM 2

**666.** Si $\alpha$ tantum evanescat, habebitur $\frac{d\psi}{d\omega}\left(M\omega^{\mu}+N\psi^{\nu}\right)=A$ proxime, unde posito $\psi=m\omega^{n}$ fit $A=mn\omega^{n-1}\left(M\omega^{\mu}+Nm^{\nu}\omega^{n\nu}\right)$; ubi si $n\nu>\mu$, debet esse $n=1-\mu$ et $mnM=A$; quod autem non valet, nisi sit $\nu(1-\mu)>\mu$ seu $\nu>\frac{\mu}{1-\mu}$. Sin autem sit $\nu<\frac{\mu}{1-\mu}$, statui debet $n-1+n\nu=0$ seu $n=\frac{1}{1+\nu}$ altero termino ut infima potestate spectato. At si fuerit $\nu=\frac{\mu}{1-\mu}$ ambo termini pro paribus potestatibus erunt habendi fietque $n=1-\mu$ et $A=mn\left(M+Nm^{\nu}\right)$, unde $m$ definiri debet.

### SCHOLION

**667.** In genere hic vix quicquam praecipere licet, sed quovis casu oblato haud difficile est omnia, quae ad solutionem perducunt, perspicere. Siquidem omnes exponentes essent integri, regula ilIa NEUTONIANA, qua ope parallelogrammi resolutio aequationum instruitur, hic in usum vocari posset; tum vero exponentium fractorum ad integros reductio satis est nota. Verum huiusmodi casus tam raro occurrunt, ut inutile foret in praeceptis prolixum esse, quae quovis casu ab exercitato facile conduntur. Veluti si perveniatur ad hanc aequationem $\frac{d\psi}{d\omega}\left(\alpha\sqrt{\omega}+\beta\psi\right)=\gamma$ , ex superioribus patet primam operationem dare $\psi=m\sqrt{\omega}$ , unde fit $\frac{1}{2}m\left(\alpha+\beta m\right)=\gamma$ , unde $m$ innotescit idque duplici modo. Quin etiam haec aequatio posito $\sqrt{\omega}=p$ ad homogeneitatem reducitur ideoque revera integrari potest. Verum haec vix unquam usum habitura fusius non prosequor, sed, quod adhuc in hac parte pertractandum restat, exponam, quomodo eiusmodi aequationes differentiales resolvi oporteat, in quibus differentialium ratio, puta $\frac{dy}{dx}=p$ , vel plures obtinet dimensiones vel adeo transcendenter ingreditur; quo absoluto partem secundam, in qua differentialia altiorum graduum occurrunt, aggrediar.