CHAPTER VI

ON THE COMPARISON OF TRANSCENDING QUANTITIES CONTAINED IN THE FORM

$$\int \frac{Pdz}{\sqrt{\left(A+2Bz+Czz+2Dz^3+Ez^4\right)}}.$$

PROBLEM 78

606. *With this proposed relation between x and y*

$$\alpha+\gamma\left(xx+yy\right)+2\delta xy+\zeta xxyy=0,$$

thence to elicit the transcending functions of the prescribed form, which are allowed to be compared among themselves.

SOLUTION

From the proposed equation each of the variables is defined

$$y=\frac{-\delta x+\sqrt{\left(-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)xx-\gamma\zeta x^4\right)}}{\gamma+\zeta xx}$$

and

$$x=\frac{-\delta y+\sqrt{\left(-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)yy-\gamma\zeta y^4\right)}}{\gamma+\zeta yy},$$

which roots can be recalled to the prescribed form by putting [Note that *m* is a scaling factor, defined in the previous chapter]

$$-\alpha\gamma=Am,\ \delta\delta-\gamma\gamma-\alpha\zeta=Cm \quad \text{and} \quad -\gamma\zeta=Em,$$

from which there becomes

$$\alpha=-\frac{Am}{\gamma},\quad \zeta=-\frac{Em}{\gamma} \quad \text{and} \quad \delta\delta=Cm+\gamma\gamma+\frac{AEmm}{\gamma\gamma}.$$

Hence there shall be :

$$\gamma y+\delta x+\zeta xxy=\sqrt{m\left(A+Cxx+Ex^4\right)},$$
$$\gamma x+\delta y+\zeta xyy=\sqrt{m\left(A+Cyy+Ey^4\right)}.$$

But this proposed equation, if it should be differentiated, gives

$$dx(\gamma x+\delta y+\zeta xyy)+dy(\gamma y+\delta x+\zeta xxy)=0,$$

where these values substituted gives

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}}+\frac{dy}{\sqrt{(A+Cyy+Ey^4)}}=0.$$

Hence in turn from this proposed differential equation, that finite equation is satisfied :

$$-Am+\gamma\gamma(xx+yy)+2xy\sqrt{(\gamma^4+Cm\gamma\gamma+AEmm)}-Emxxyy=0$$

or on putting $\frac{\gamma\gamma}{m}=k$ this in turn :

$$-A+k(xx+yy)+2xy\sqrt{(kk+kC+AE)}-Exxyy=0,$$

which since it involves the constant *k* not present in the differential equation, this equation likewise will be the complete integral.

Hence moreover there shall become :

$$ky+x\sqrt{(kk+kC+AE)}-Exxy=\sqrt{k(A+Cxx+Ex^4)}$$

and

$$kx+y\sqrt{(kk+kC+AE)}-Exyy=\sqrt{k(A+Cyy+Ey^4)}.$$

COROLLARY 1

607. The constant *k* thus can be taken, so that on putting $x=0$ there becomes $y=b$; and moreover there arises

$$bk=\sqrt{Ak} \quad \text{and} \quad b\sqrt{(kk+kC+AE)}=\sqrt{k(A+Cbb+Eb^4)},$$

hence

$$k=\frac{A}{b} \quad \text{and} \quad \sqrt{(kk+kC+AE)}=\frac{1}{bb}\sqrt{A(A+Cbb+Eb^4)}$$

and thus we shall have :

$$Ay+x\sqrt{A(AA+Cbb+Eb^4)}-Ebbxxy=b\sqrt{A(A+Cxx+Ex^4)}$$

and

$$Ax+y\sqrt{A(AA+Cbb+Eb^4)}-Ebbxyy=b\sqrt{A(A+Cyy+Ey^4)}.$$

COROLLARY 2

608. Hence this finite relation between *x* and *y* shall be the complete integral of the differential equation :

$$\frac{dx}{\sqrt{\left(AA+Cxx+Ex^4\right)}}+\frac{dy}{\sqrt{\left(A+Cyy+Ey^4\right)}}=0,$$

which shall be the rational expression between *x* and *y* :

$$A\left(xx+yy-bb\right)+2xy\sqrt{A\left(AA+Cbb+Eb^4\right)}-Ebbxxyy=0.$$

COROLLARY 3

609. Hence *y* can therefore be expressed thus in terms of x, so that it becomes

$$y=\frac{b\sqrt{A\left(A+Cxx+Ex^4\right)}-x\sqrt{A\left(A+Cbb+Eb^4\right)}}{A-Ebbxx},$$

and from this value there is elicited

$$\sqrt{\frac{\left(A+Cyy+Ey^4\right)}{A}}=\frac{\left(A+Ebbxx\right)\sqrt{A\left(A+Cbb+Eb^4\right)\left(A+Cxx+Ex^4\right)}-2AEbx\left(bb+xx\right)-Cbx\left(A+Ebbxx\right)}{\left(A-Ebbxx\right)^2}.$$

COROLLARIUM 4

610. Hence an infinite number of particular integrals can be shown by determining the constant *b* as you wish, of which in particular there are :

1) by taking $b=0$, from which there becomes $y=-x$;

2) by taking $b=\infty$, from which there becomes $y=\frac{\sqrt{A}}{x\sqrt{E}}$;

3) if $A+Cbb+Eb^4=0$ and hence $bb=\frac{-C+\sqrt{(CC-4AE)}}{2E}$,

from which there becomes $y=\frac{b\sqrt{A\left(A+Cbb+Eb^4\right)}}{A-Ebbxx}$.

SCHOLIUM

611. Now here the use of this method, which we have arrived at by working backwards from a finite equation to a differential equation, is clearly evident. For since the integration of the formula $\frac{dx}{\sqrt{\left(A+Cxx+Ex^4\right)}}$ cannot be produced either from logarithms or the arcs of circles, it is certainly a wonder that such a differential equation thus can be integrated algebraically; which indeed in the preceding chapter have been treated with the help of this method, and also which are able to be elicited by the ordinary method, as the individual differential formulas can be expressed either by logarithms or circular arcs, the comparison of which is then reduced to an algebraic equation. Now since here by such an integration clearly no treatment can be found, clearly no other method is apparent, by which the same integral, that we have shown here, can be investigated. Whereby we shall set out this argument more carefully.

PROBLEM 79

612. *If* $\Pi: z$ *denotes a function of z of this kind, so that*

$$\Pi: z = \int \frac{Pdz}{\sqrt{(A+Czz+Ez^4)}}$$

with the integral thus taken so that it vanishes on putting $z=0$, *to investigate the comparison between functions of this kind.*

SOLUTION

With the relation between the two variables *x* and *y* we have considered defined above, there becomes :

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}} + \frac{dy}{\sqrt{(A+Cyy+Ey^4)}} = 0$$

Hence, since on putting $x=0$ there becomes $y=b$, there is elicited from the integration:

$$\Pi: x + \Pi: y = \Pi: b.$$

Since now there is no further distinction between the variables *x, y* and the constant *b* it introduces, we can put in place
$x=p$, $y=q$ and $b=-r$, so that there becomes $\Pi: b = -\Pi: r$, and this relation between the transcending functions

$$\Pi: p + \Pi: q + \Pi: r = 0$$

can be expressed by the following algebraic formulas :

$$(A-Epprr)q + p\sqrt{A(A+Crr+Er^4)} + r\sqrt{A(A+Cpp+Ep^4)} = 0$$

or

$$(A-Eppqq)r + q\sqrt{A(A+Cpp+Ep^4)} + p\sqrt{A(A+Cqq+Eq^4)} = 0$$

or

$$(A-Eqqrr)p + r\sqrt{A(A+Cqq+Eq^4)} + q\sqrt{A(A+Crr+Er^4)} = 0,$$

which arise from this equation

$$A(pp+qq-rr)-Eppqqrr+2pq\sqrt{A\left(A+Crr+Er^4\right)}=0.$$

Now this on being rationalized gives rise to :

$$AA\left(p^4+q^4+r^4-2ppqq-2pprr-2qqrr\right)-2AEppqqrr(pp+qq+rr)$$
$$-4ACppqqrr+EEp^4q^4r^4=0,$$

moreover which, on account of the plurality of the roots, satisfies all the variations of the signs in the above transcending equation.

COROLLARY 1

613. If we take *r* negative, so that there becomes

$$\Pi:r=\Pi:p+\Pi:q$$

and then there shall be

$$r=\frac{p\sqrt{A\left(A+Cqq+Eq^4\right)}+q\sqrt{A\left(A+Cpp+Ep^4\right)}}{A-Eppqq},$$

from which there may be deduced :

$$\sqrt{\frac{\left(A+Crr+Er^4\right)}{A}}=\frac{(A+Eppqq)\sqrt{A\left(A+Cpp+Ep^4\right)\left(A+Cqq+Eq^4\right)}+2AEpq(pp+qq)+Cpq(A+Eppqq)}{(A-Eppqq)^2}.$$

COROLLARY 2

614. But if hence we put $q=p$, so that there becomes

$$\Pi:r=2\Pi:p$$

then

$$r=\frac{2p\sqrt{A\left(A+Cpp+Ep^4\right)}}{A-Ep^4}$$

and

$$\sqrt{\frac{\left(A+Crr+Er^4\right)}{A}}=\frac{AA+2ACpp+6AEp^4+2CEp^6+EEp^8}{\left(A-Ep^4\right)^2}.$$

Therefore in this way a function is able to be defined equal to double a similar function.

COROLLARY 3

615. If there is put

$$q=\frac{2p\sqrt{A\left(A+Cpp+Ep^4\right)}}{A-Ep^4}$$

and

$$\sqrt{A\left(A+Cqq+Eq^4\right)}=\frac{A\left(AA+2ACpp+6AEp^4+2CEp^6+EEp^8\right)}{\left(A-Ep^4\right)^2},$$

so that there becomes $\Pi:q=2\Pi:p$, then from the first corollary there becomes

$$\Pi:r=3\Pi:p.$$

Therefore then there shall be

$$r=\frac{p\left(3AA+4ACpp+6AEp^4-EEp^8\right)}{AA-6AEp^4-4CEp^6-3EEp^8}.$$

SCHOLIUM 1

616. With a great deal of work, this multiplication of the functions can be continued further and the rule in the progression of these is much less well understood. But if for brevity we put

$$\sqrt{A\left(A+Cpp+Ep^4\right)}=AP \quad\text{and}\quad A-Ep^4=A\mathfrak{P},$$

in order that there becomes

$$Cpp=APP-A-Ep^4 \quad\text{and}\quad Ep^4=A\left(1-\mathfrak{P}\right),$$

these multiplications thus may be had as far as the fourth part : clearly if we put

$$\Pi:r=2\Pi:p,\ \Pi:s=3\Pi:p \quad\text{and}\quad \Pi:t=4\Pi:p,$$

there will be found

$$r=\frac{2Pp}{\mathfrak{P}},\quad s=\frac{p\left(4PP-\mathfrak{P}\mathfrak{P}\right)}{\mathfrak{P}\mathfrak{P}-4PP\left(1-\mathfrak{P}\right)},\quad t=\frac{4pP\mathfrak{P}\left(2PP\left(2-\mathfrak{P}\right)-\mathfrak{P}\mathfrak{P}\right)}{\mathfrak{P}^4-16P^4\left(1-\mathfrak{P}\right)}.$$

But if in a like manner we put

$$\sqrt{A\left(A+Crr+Er^4\right)}=AR \quad\text{and}\quad A-Er^4=A\mathfrak{R},$$

then there becomes

$$R=\frac{2PP\left(2-\mathfrak{P}\right)-\mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P}},\quad\text{and}\quad \mathfrak{R}=\frac{\mathfrak{P}^4-16P^4\left(1-\mathfrak{P}\right)}{\mathfrak{P}^4},$$

from which for the 4 times multiple there becomes

$$t=\frac{2Rr}{\mathfrak{R}},\quad T=\frac{2RR(2-\mathfrak{R})-\mathfrak{R}\mathfrak{R}}{\mathfrak{R}\mathfrak{R}},\quad \mathfrak{T}=\frac{\mathfrak{R}^4-16R^4(1-\mathfrak{R})}{\mathfrak{R}^4}.$$

Whereby if we put in place for the 8 times multiple $\Pi:z=8\Pi:p$, then there becomes

$$z=\frac{2Tt}{\mathfrak{T}}=\frac{4rR\mathfrak{R}\left(2RR(2-\mathfrak{R})-\mathfrak{R}\mathfrak{R}\right)}{\mathfrak{R}^4-16R^4(1-\mathfrak{R})}.$$

Hence it is understood, how in bringing about the continuous doubling required, that the law of progression still cannot be deduced. Knowing the rest of this law would be greatly wished to the advancement of analysis, in order that from these generally the relation between z and p for the equality $\Pi:z=n\Pi:p$ could be defined, just as it succeeded in doing in the preceding chapter ; hence indeed one would be allowed to know the extraordinary properties about integrals of the form $\int\frac{dz}{\sqrt{(A+Czz+Ez^4)}}$, by which the analytical science would be advanced considerably.

SCHOLIUM 2

617. An especially suitable way of enquiring into the law of the progression is considered, if we consider three terms taken in order in this way :

$$\Pi:x=(n-1)\Pi:p,\ \ \Pi:y=n\Pi:p,\ \ \Pi:z=(n+1)\Pi:p;$$

where since there shall be

$$\Pi:x=\Pi:y-\Pi:p\quad\text{et}\quad \Pi:z=\Pi:y+\Pi:p,$$

then

$$x=\frac{y\sqrt{A(A+Cpp+Ep^4)}-p\sqrt{A(A+Cyy+Ey^4)}}{A-Eppyy},$$

$$z=\frac{y\sqrt{A(A+Cpp+Ep^4)}+p\sqrt{A(A+Cyy+Ey^4)}}{A-Eppyy},$$

from which we conclude

$$(A-Eppyy)(x+z)=2y\sqrt{A\left(A+cpp+Ep^4\right)}.$$

As before we can put $\sqrt{A\left(A+cpp+Ep^4\right)}=AP$ and $A-Ep^4=A\mathfrak{P}$, and since the individual quantities x, y, z involve the factor p simply, there shall be

$$x=pX,\ \ y=pY\quad\text{and}\quad z=pZ;$$

then there becomes

$$\left(1-(1-\mathfrak{P})YY\right)(X+Z)=2PY \quad \text{or} \quad Z=\frac{2PY}{\left(1-(1-\mathfrak{P})YY\right)}-X\ ,$$

of which the formula can be found without difficulty with help from the neighbouring terms X and Y following Z. Which so that it may be more apparent, there is put $2P=Q$ et $1-\mathfrak{P}=\mathfrak{Q}$, so that there becomes $Z=\frac{QY}{1-\mathfrak{Q}YY}-X$. Now the progression sought thus shall itself be had :

$$1)\ 1, \quad 2)\ \frac{Q}{\mathfrak{P}}, \quad 3)\ \frac{QQ-\mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P}-QQ\mathfrak{Q}}, \quad 4)\ \frac{Q^3\mathfrak{P}(1+\mathfrak{Q})-2Q\mathfrak{P}^3}{\mathfrak{P}^4-Q^4\mathfrak{Q}},$$

$$(5\ \ \frac{\mathfrak{P}^6-3QQ\mathfrak{P}^4+Q^4\mathfrak{P}\mathfrak{P}(1+2\mathfrak{Q})-Q^6\mathfrak{Q}\mathfrak{Q}}{\mathfrak{P}^6-3QQ\mathfrak{P}^4\mathfrak{Q}+Q^4\mathfrak{P}\mathfrak{P}\mathfrak{Q}(1+2\mathfrak{Q})-Q^6\mathfrak{Q}} \quad \text{etc.}$$

Hence the question is returned here, so that the progression may be investigated from the given relation between the three successive terms X, Y, Z , which shall be $Z=\frac{QY}{1-\mathfrak{Q}YY}-X$, with the first term being = 1 and the second equal to $=\frac{Q}{1-\mathfrak{Q}}$.

PROBLEM 80

618. *If Π: z denotes a function of z of this kind, so that there shall be*

$$\Pi : z=\int\frac{dz\left(L+Mzz+Nz^4\right)}{\sqrt{\left(A+Czz+Ez^4\right)}}$$

thus with the integral taken so that it vanishes on putting $z=0$, *to investigate the comparison between transcending functions of this kind.*

SOLUTION

With this relation established between the two variables *x* and *y*, so that there becomes

$$Ay+\mathfrak{B}x-Ebbxxy=b\sqrt{A\left(A+Cxx+Ex^4\right)}$$

or

$$Ax+\mathfrak{B}y-Ebbxyy=b\sqrt{A\left(A+Cyy+Ey^4\right)}$$

or with the irrationality removed :

$$A\left(xx+yy-bb\right)+2\mathfrak{B}xy-Ebbxxyy=0$$

with $\mathfrak{B}=\sqrt{A\left(A+Cbb+Eb^4\right)}$ for brevity, there shall be as we have seen before :

$$\frac{dx}{\sqrt{\left(A+Cxx+Ex^4\right)}}+\frac{dy}{\sqrt{\left(A+Cyy+Ey^4\right)}}=0.$$

Therefore we may put

$$\frac{dx\left(L+Mxx+Nx^4\right)}{\sqrt{\left(A+Cxx+Ex^4\right)}}+\frac{dy\left(L+Myy+Ny^4\right)}{\sqrt{\left(A+Cyy+Ey^4\right)}}=bdV\sqrt{A},$$

as shall be indicated according to our custom :

$$\Pi:x+\Pi:y=\text{Const.}+bV\sqrt{A}\,,$$

where the constant must thus be defined, so that on putting $x=0$ there becomes $y=b$.

Hence the question is recalled in finding the function V; that in the end in place of dy with the value substituted from the former equation there shall be :

$$bdV\sqrt{A}=\frac{dx\left(M(xx-yy)+N\left(x^4-y^4\right)\right)}{\sqrt{\left(A+Cxx+Ex^4\right)}},$$

since indeed

$$b\sqrt{A\left(A+Cxx+Ex^4\right)}=Ay+\mathfrak{B}x-Ebbxxy\,,$$

we shall have

$$dV=\frac{dx(xx-yy)\left(M+N(xx+yy)\right)}{Ay+\mathfrak{B}x-Ebbxxy},$$

Now we may take the rational equation :

$$A\left(xx+yy-bb\right)+2\mathfrak{B}xy-Ebbxxyy=0$$

and we can put $xx+yy=tt$ *and* $xy=u$, so that there becomes

$$A\left(tt-bb\right)+2\mathfrak{B}u-Ebbuu=0$$

and thus

$$Atdt=-\mathfrak{B}du+Ebbudu\,.$$

Since there shall be again : $xdx+ydy=tdt$ and $xdy+ydx=du$, there shall be

$$\left(xx-yy\right)dx=xtdt-ydu$$

or

$$A\left(xx-yy\right)dx=-du\left(Ay+\mathfrak{B}x-Ebbxxy\right),$$

thus so that there becomes :

$$\frac{dx(xx-yy)}{Ay+\mathfrak{B}x-Ebbxxy}=-\frac{du}{A},$$

from which there is deduced

$$dV=-\frac{du}{A}(M+Ntt),$$

and from $tt=bb-\frac{2\mathfrak{B}u}{A}+\frac{Ebbuu}{A}$ there shall be

$$dV=-\frac{du}{AA}(AM+ANbb-2\mathfrak{B}Nu+ENbbuu),$$

from which on integrating there is elicited :

$$V=-\frac{Mu}{A}-\frac{Nbbu}{A}+\frac{\mathfrak{B}Nuu}{AA}-\frac{ENbbu^3}{3AA}$$

Hence from this value on substituting on account of $u=xy$ we shall have :

$$\Pi:x+\Pi:y=\Pi:b-\frac{Mbxy}{\sqrt{A}}-\frac{Nb^3xy}{\sqrt{A}}+\frac{\mathfrak{B}Nx^2y^2}{A\sqrt{A}}-\frac{ENb^3x^3y^3}{3A\sqrt{A}}$$

But since there shall be

$$\mathfrak{B}xy=\tfrac{1}{2}Abb-\tfrac{1}{2}A(xx+yy)+\tfrac{1}{2}Ebbxxyy,$$

then there becomes

$$\Pi:x+\Pi:y=\Pi:b-\frac{Mbxy}{\sqrt{A}}-\frac{Nbxy}{2A\sqrt{A}}\left(A(bb+xx+yy)-\tfrac{1}{3}Ebbxxyy\right),$$

which hence satisfies the equation found above by algebraic formulas, by which the relation between *x*, *y* and *b* is expressed. But if hence this equation is put in place :

$$\Pi:p+\Pi:q+\Pi:r=\frac{Mpqr}{\sqrt{A}}+\frac{Npqr}{2A\sqrt{A}}\left(A(pp+qq+rr)-\tfrac{1}{3}Ebbppqqrr\right),$$

from that the following is effected from the relation established between *p*, *q*, *r* :

$$(A-Eppqq)r+p\sqrt{A(A+Cqq+Eq^4)}+q\sqrt{A(A+Cpp+Ep^4)}=0$$

or

$$(A-Epprr)q+p\sqrt{A(A+Crr+Er^4)}+r\sqrt{A(A+Cpp+Ep^4)}=0$$

or

$$(A-Eqqrr)p+q\sqrt{A(A+Crr+Er^4)}+r\sqrt{A(A+Cqq+Eq^4)}=0$$

or through the simple irrationality

$$A(pp+qq-rr)+2pq\sqrt{A\left(A+Crr+Er^4\right)}-Eppqqrr=0$$

or

$$A(pp+rr-qq)+2pr\sqrt{A\left(A+Cqq+Eq^4\right)}-Eppqqrr=0$$

or

$$A(qq+rr-pp)+2qr\sqrt{A\left(A+Cpp+Ep^4\right)}-Eppqqrr=0$$

and with the inner irrationality removed :

$$EEp^4q^4r^4-2AEppqqrr(pp+qq+rr)-4ACppqqrr$$

$$+AA\left(p^4+q^4+r^4-2ppqq-2pprr-2qqrr\right)=0.$$

COROLLARY 1

619. Let $q=r=s$, so that we may have this equation :

$$\Pi: p+2\Pi: s=\tfrac{Mpss}{\sqrt{A}}+\tfrac{Npss}{2A\sqrt{A}}\left(A(pp+2ss)-\tfrac{1}{3}Epps^4\right),$$

from which this equation is satisfied :

$$\left(A-Es^4\right)p+2s\sqrt{A\left(A+Css+Es^4\right)}=0.$$

COROLLARIUM 2

620. We may take *s* negative and in place of p we may substitute this value here, so that we shall have :

$$2\Pi: s+\Pi: q+\Pi: r+\tfrac{Mpss}{\sqrt{A}}+\tfrac{Npss}{2A\sqrt{A}}\left(A(pp+2ss)-\tfrac{1}{3}Epps^4\right)$$
$$=\tfrac{Mpqr}{\sqrt{A}}+\tfrac{Npqr}{2A\sqrt{A}}\left(A(pp+qq+rr)-\tfrac{1}{3}Eppqqrr\right)$$

on setting

$$p=\frac{2s\sqrt{A\left(A+Css+Es^4\right)}}{A-Es^4},$$

from which there becomes [§ 615]

$$\sqrt{A\left(A+Cpp+Ep^4\right)}=\frac{A\left(A+Css+Es^4\right)^2+A(4AE-CC)s^4}{\left(A-Es^4\right)^2},$$

which values must be substituted in the above formulas.

COROLLARIUM 3

621. This can be done in this way, so that the algebraic parts vanish and the transcendental functions alone can be compared between themselves. Just as if there should be $N=0$, it is required to put in place $ss=qr$, so that there becomes :

$$2\Pi:s+\Pi:q+\Pi:r=0.$$

But on putting $ss=qr$ there becomes

$$p=\frac{2\sqrt{Aqr(A+Cqr+Eqqrr)}}{A-Eqqrr}.$$

Now also there shall be

$$p=\frac{-q\sqrt{A\left(A+Crr+Er^4\right)}-r\sqrt{A\left(A+Cqq+Eq^4\right)}}{A-Eqqrr}$$

from which values equated this equation arises :

$$\left(AA+EEq^4r^4\right)(qq-6qr+rr)-8Cqqrr(A+Eqqrr)$$
$$-2AEqqrr(qq+10qr+rr)=0.$$

SCHOLIUM

622. If $\Pi:z$ should express the arc of a certain line of a curve corresponding to an abscissa or string [length] *z*, hence several arcs of the same curve can be compared between themselves, so that the difference of two arcs becomes either algebraic or the arcs are shown to maintain a given ratio between each other. In this way the conspicuous properties of the curves are elicited, the ratio of which can scarcely be found in any other way. Indeed the comparison of circular arcs from the elements noted in the previous chapter, as we have seen, is easily put in place from which also the comparison of parabolic arcs is derived. But from this chapter a comparison of elliptic or hyperbolic arcs can be put in place in a similar manner ; for since in general the arcs of conic sections are expressed by a formula such as

$$\int dx\sqrt{\frac{a+bxx}{c+exx}},$$

and this transformed into that

$$\int \frac{dx(a+bxx)}{\sqrt{(ac+(ae+bc)xx+bex^4)}}$$

by the discussed precepts, can be treated by putting $A=ac,\ C=ae+bc,\ E=be$, and $L=a,\ M=b$ and $N=0$. But this investigation of the formulas, of which the denominator is $\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}$, can be extended and it is similar to the preceding, which on that account I have set out here, from which likewise it will be apparent hence to be the last of the terms, as far as which it is allowed to progress. For the more complicated integral formulas, where within the root sign the higher powers of z occur or the root itself involves a higher power, in this way the forms will not be able to be compared among themselves except for a very few cases, which are able to be reduced to a form of this kind by a certain substitution.

PROBLEM 81

623. *If $\Pi: z$ denotes a function of this kind of z, so that there shall be*

$$\Pi: z=\int \frac{dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}},$$

to compare functions of this kind amongst themselves.

SOLUTION

Between the two variables x and y the relation expressed by this equation is put in place :

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy+2\varepsilon xy(x+y)+\zeta xxyy=0;$$

from which since there becomes

$$yy=\frac{-2y(\beta+\delta x+\varepsilon xx)-\alpha-2\beta x-\gamma xx}{\gamma+2\varepsilon x+\zeta xx},$$

there will be with the root [of the quadratic in y] extracted :

$$y=\frac{-\beta-\delta x-\varepsilon xx+\sqrt{((\beta+\delta x+\varepsilon xx)^2-(\alpha+2\beta x+\gamma xx)(\gamma+2\varepsilon x+\zeta xx))}}{\gamma+2\varepsilon x+\zeta xx}.$$

The expression under the square root is reduced to the proposed form on putting

$$\beta\beta-\alpha\gamma=Am,\quad \beta\delta-\alpha\varepsilon-\beta\gamma=Bm,$$
$$\delta\delta-2\beta\varepsilon-\alpha\zeta-\gamma\gamma=Cm,\quad \delta\varepsilon-\beta\zeta-\gamma\varepsilon=Dm,$$
$$\varepsilon\varepsilon-\gamma\zeta=Em,$$

from which from the six coefficients $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ five are defined, and to the sixth above there is given the letter *m*, thus so that the assumed equation involves an arbitrary constant at this stage. From which therefore, if for brevity we put

$$\sqrt{\left(A+2Bx+ Cxx+2Dx^3+Ex^4\right)}=X$$

and

$$\sqrt{\left(A+2By+ Cyy+2Dy^3+Ey^4\right)}=Y,$$

we will have

$$\beta+\gamma y+\delta x+\varepsilon xx+2\varepsilon xy+\zeta xxy=X\sqrt{m}$$

and

$$\beta+\gamma x+\delta y+\varepsilon yy+2\varepsilon xy+\zeta xyy=Y\sqrt{m}.$$

But the assumed equation on differentiation gives

$$dx\left(\beta+\gamma x+\delta y+2\varepsilon xy+\varepsilon yy+\zeta xyy\right)+$$
$$dy\left(\beta+\gamma y+\delta x+\varepsilon xx+2\varepsilon xy+\zeta xxy\right)=0;$$

which expressions since they agree with the above, give

$$Ydx\sqrt{m}+Xdy\sqrt{m}=0 \quad\text{or}\quad \tfrac{dx}{X}+\tfrac{dy}{Y}=0,$$

from which on integration we deduce :

$$\Pi: x+\Pi: y=\text{Const.},$$

which constant, if on putting $x=0$ makes $y=b$, then it is $=\Pi: 0+\Pi: b$, or in general if on putting $x=a$ there is made $y=b$, that becomes $\Pi: a+\Pi: b$. But if hence the letters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ are defined by the above conditions, the assumed algebraic equation between *x* and *y* will be the complete integral of this differential equation :

$$\frac{dx}{\sqrt{\left(A+2Bx+ Cxx+2Dx^3+Ex^4\right)}}+\frac{dy}{\sqrt{\left(A+2By+ Cyy+2Dy^3+Ey^4\right)}}=0.$$

COROLLARY 1

624. Towards defining these letters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ first there are taken the two equations placed on the right, which are

$$(\delta-\gamma)\beta-\alpha\varepsilon=Bm \quad \text{and} \quad (\delta-\gamma)\varepsilon-\zeta\beta=Dm,$$

from which the two β and ε are sought, and there is found

$$\beta=\frac{(\delta-\gamma)\mathrm{B}+\alpha\mathrm{D}}{(\delta-\gamma)^2-\alpha\zeta}m \quad \text{and} \quad \varepsilon=\frac{(\delta-\gamma)\mathrm{D}+\zeta\mathrm{B}}{(\delta-\gamma)^2-\alpha\zeta}m.$$

COROLLARY 2

625. For brevity let $\delta-\gamma=\lambda$ or $\delta=\gamma+\lambda$; then there shall be

$$\beta=\frac{\mathrm{D}\alpha+\mathrm{B}\lambda}{\lambda\lambda-\alpha\zeta}m \quad \text{and} \quad \varepsilon=\frac{\mathrm{B}\zeta+\mathrm{D}\lambda}{\lambda\lambda-\alpha\zeta}m.$$

Now from the first and final condition there arises :

$$\beta\beta\zeta-\alpha\varepsilon\varepsilon=(A\zeta-E\alpha)m,$$

where these values substituted produce $\frac{\beta\beta\zeta-DD\alpha}{\lambda\lambda-\alpha\zeta}m=A\zeta-E\alpha$, from which there becomes

$$m=\frac{(\lambda\lambda-\alpha\zeta)(A\zeta-E\alpha)}{\beta\beta\zeta-DD\alpha}$$

But from the first and last it follows that

$$DD\beta-BB\varepsilon\varepsilon+\gamma(\mathrm{BB}\zeta-DD\alpha)=(ADD-BBE)m,$$

from which it is deduced:

$$\gamma=\frac{(A\zeta-Ea)\big((ADD-BBE)\lambda\lambda+2BD(A\zeta-Ea)\lambda+ABB\zeta\zeta-DDE\alpha\alpha\big)}{(BB\zeta-DD\alpha)^2}.$$

COROLLARY 3

626. The third equation remains :

$$2\gamma\lambda + \lambda\lambda - 2\beta\varepsilon - \alpha\zeta = Cm,$$

and since with the value for *m* substituted there shall be

$$\beta = \frac{(A\zeta - E\alpha)(D\alpha + B\lambda)}{BB\zeta - DD\alpha} \quad \text{and} \quad \varepsilon = \frac{(A\zeta - E\alpha)(B\zeta + D\lambda)}{BB\zeta - DD\alpha},$$

and if these values are substituted, from which there is deduced conveniently

$$\lambda = \frac{C(A\zeta - E\alpha)(BB\zeta - DD\alpha) - 2BD(A\zeta - E\alpha)^2 - (BB\zeta - DD\alpha)^2}{2(A\zeta - E\alpha)(ADD - BBE)}.$$

SCHOLION 1

627. Because it is not permitted for these values to be used whenever there should be the condition $ADD - BBE = 0$, I give another resolution to avoid this inconvenience.

On putting $\delta = \gamma + \lambda$ there shall be above $\lambda\lambda = \alpha\zeta + \mu$, so that the initial formulas become

$$\beta = \tfrac{m}{\mu}(D\alpha + B\lambda) \quad \text{and} \quad \varepsilon = \tfrac{m}{\mu}(B\zeta + D\lambda).$$

Now with the first and last taken together there is produced

$$A\zeta - E\alpha = m(BB\zeta - DD\alpha),$$

from which equation the ratio between α and ζ is defined; which since it should suffice, there will be

$$\alpha = \mu A - BBm \quad \text{and} \quad \zeta = \mu E - DDm$$

and hence

$$\lambda\lambda = \mu + (\mu A - BBm)(\mu E - DDm),$$

from which we deduce

$$\gamma = \tfrac{mm}{\mu\mu}\left(2BD\lambda + (ADD + BBE)\mu\right) - \tfrac{2BBDDm^3}{\mu\mu} - \tfrac{m}{\mu}.$$

The values α and ζ substituted into the formula of Corollary 3 give

$$\lambda = \tfrac{\mu\mu}{2m} + BDm - \tfrac{1}{2}C\mu,$$

the square of which equated to that value $\alpha\zeta + \mu$ leads to this equation :

$$\mu(\mu - Cm)^2 + 4(BD - AE)mm\mu + 4(ADD - BCD + BBE)m^3 = 4mm;$$

to the resolution of which there is put $\mu = Mm$ and there becomes

$$m = \frac{4}{M(M-C)^2 + 4M(BD-AE) + 4(ADD-BCD+BBE)}$$

and here M is a constant, with that required arbitrary for the complete integral. In this manner all the letters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ are produced to be affected by the same denominator, with which omitted we shall have

$$\alpha = 4(AM - BB),\quad \beta = 2B(M - C) + 4AD,\quad \gamma = 4AE - (M - C)^2,$$
$$\zeta = 4(\mathrm{EM} - \mathrm{DD}),\quad \varepsilon = 2D(M - C) + 4BE,\quad \delta = MM - CC + 4(AE + BD),$$

from which found our canonical equation

$$0 = \alpha + 2\beta(x + y) + \gamma(xx + yy) + 2\delta xy + 2\varepsilon xy(x + y) + \zeta xxyy,$$

if for brevity we put

$$M(M - \mathrm{C})^2 + 4M(BD - AE) + 4(ADD - BCD + BBE) = \varDelta,$$

resolved will give

$$\beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx) = \pm 2\sqrt{\varDelta(A + 2Bx + Cxx + 2Dx^3 + Ex^4)},$$
$$\beta + \delta y + \varepsilon yy + x(\gamma + 2\varepsilon y + \zeta yy) = \pm 2\sqrt{\varDelta(A + 2By + Cyy + 2Dy^3 + Ey^4)},$$

which hence is the complete integral of this differential equation :

$$0 = \frac{dx}{\pm\sqrt{\varDelta(A+2Bx+Cxx+2Dx^3+Ex^4)}} + \frac{dy}{\pm\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}$$

SCHOLIUM 2

628. Since here the whole investigation depends on the determination of a suitable coefficient, it will be worth the effort to explain that more clearly. Therefore on putting at once $\delta = \gamma + \lambda$ and $\lambda\lambda - \alpha\zeta = Mm$ five conditions are being implemented :

$$\text{I}.\beta\beta - a\gamma = Am, \quad \text{II}.\varepsilon\varepsilon - \gamma\zeta = Em,$$
$$\text{III}.\beta\lambda - \alpha\varepsilon = Bm, \quad \text{IV}.\varepsilon\lambda - \beta\zeta = Dm,$$
$$\text{V}.Mm + 2\gamma\lambda - 2\beta\varepsilon = Cm.$$

Hence from three and four combined there is deduced :

$$m(B\lambda + D\alpha) = \beta(\lambda\lambda - \alpha\zeta) = \beta Mm, \quad \text{hence} \quad \beta = \tfrac{B\lambda + Da}{M},$$
$$m(D\lambda + B\zeta) = \varepsilon(\lambda\lambda - \alpha\zeta) = \varepsilon Mm, \quad \text{hence} \quad \varepsilon = \tfrac{D\lambda + B\zeta}{M}.$$

Now on eliciting γ from the first and second, there arises :

$$m(A\zeta - Ea) = \beta\beta\zeta - \varepsilon\varepsilon\alpha = \tfrac{\beta\beta\zeta - DD\alpha}{M} m$$

and hence

$$\zeta(\text{AM} - BB) = \alpha(EM - DD),$$

whereby there is put in place:

$$\alpha = n(AM - BB) \quad \text{and} \quad \zeta = n(EM - DD).$$

Now from the same source there is

$$E\beta\beta - E\alpha\gamma = A\varepsilon\varepsilon - A\gamma\alpha \quad \text{or} \quad \gamma(A\zeta - E\alpha) = A\varepsilon\varepsilon - E\beta\beta\,;$$

from which on being treated, since there shall be for α and ζ, with the values substituted

$$\beta = nAD + \tfrac{B}{M}(\lambda - nBD) \ \text{ and } \ \varepsilon = nBE + \tfrac{D}{M}(\lambda - nBD),$$

hence for brevity there shall be $\lambda - nBD = nMN$, so that we may have

$$\beta = n(AD + BN) \ \text{ and } \ \varepsilon = n(BE + DN),$$

and because

$$A\zeta - E\alpha = n(BBE - ADD)$$

and

$$A\varepsilon\varepsilon - E\beta\beta = nn\left(ABBEE + ADDNN - AADDE - BBENN\right)$$

or

$$A\varepsilon\varepsilon - E\beta\beta = nn\left(BBE - ADD\right)\left(AE - NN\right),$$

there becomes

$$\gamma = n\left(AE - NN\right).$$

But since there shall be

$$\lambda = n\left(BD + MN\right) \text{ and } \lambda\lambda = nn\left(AM - BB\right)\left(EM - DD\right) + Mm,$$

then there becomes

$$Mm = nn\left(2BDMN + MMNN - AEMM + M\left(ADD + BBE\right)\right)$$

or

$$m = nn(2BDN + MNN - AEM + ADD + BBE).$$

Finally the fifth equation set out $\beta\varepsilon - \gamma\lambda = \frac{1}{2}m\left(M - C\right)$ gives :

$$\beta\varepsilon - \gamma\lambda = nn(AD + BN)(BE + DN) - (AE - NN)(BD + MN))$$
$$= nnN(2BDN + MNN - AEM + ADD + BBE) = Nm,$$

from which there becomes $N = \frac{1}{2}\left(M - C\right)$, and therefore

$$m = nn(BD(M - C) + \tfrac{1}{4}M(M - C)^2 - AEM + ADD + BBE).$$

And hence on taking $n = 4$ the above values can be obtained.

EXAMPLE 1

629. *To find the complete integral of this differential equation*

$$\frac{dp}{\pm\sqrt{(a+bp)}} + \frac{dq}{\pm\sqrt{(a+bq)}} = 0.$$

Here there is $x = p,\ y = q,\ A = a,\ B = \frac{1}{2}b,\ C = 0, D = 0, E = 0,$ from which the coefficients become :

$$\alpha = 4aM - bb,\ \beta = bM,\ \gamma = -MM,\ \zeta = 0, \varepsilon = 0, \delta = MM$$

and

$$\varDelta = M^3,$$

from which the complete integral will be

$$bM + MMp - MMq = \pm 2M\sqrt{M\left(a+bp\right)}$$

or

$$b + M\left(p-q\right) = \pm 2\sqrt{M\left(a+bp\right)} \quad \text{or} \quad b + M\left(q-p\right) = \pm 2\sqrt{M\left(a+bq\right)},$$

which ambiguous signs of the roots are required to agree with the signs in the differential equation.

EXAMPLE 2

630. *To find the complete integral of this differential equation*

$$\frac{dp}{\pm\sqrt{\left(a+bp^2\right)}} + \frac{dq}{\pm\sqrt{\left(a+bq^2\right)}} = 0.$$

On taking $x = p$ and $y = q$ there shall be $A = a,\ B = 0, C = b,\ D = 0, E = 0$, hence

$$\alpha = 4aM, \quad \beta = 0, \quad \gamma = -\left(M-b\right)^2, \quad \zeta = 0,\ \varepsilon = 0, \delta = MM - bb$$

and

$$\varDelta = M\left(M-b\right)^2$$

from which the complete integral shall be contained in these equations :

$$(MM-bb)p - (M-b)^2 q = \pm 2(M-b)\sqrt{M\left(a+bpp\right)}$$

or

$$(M+b)p - (M-b)q = \pm 2\sqrt{M\left(a+bpp\right)}$$

and

$$(M+b)q - (M-b)p = \pm 2\sqrt{M\left(a+bqq\right)}.$$

EXAMPLE 3

631. *To find the complete integral of this differential equation*

$$\frac{dp}{\pm\sqrt{\left(a+bp^3\right)}} + \frac{dq}{\pm\sqrt{\left(a+bq^3\right)}} = 0.$$

On assuming $x = p,\ y = q$ there shall be $A = a,\ B = 0,\ C = 0,\ D = \frac{1}{2}b,\ E = 0$, and hence $\alpha = 4aM,\ \beta = 2ab,\ \gamma = -MM, \zeta = -bb, B = bM, \delta = MM$

and

$$\varDelta = M^3 + abb,$$

from which the complete integral shall be

$$2ab + MMp + bMpp + q(-MM + 2bMp - bbpp) = \pm 2\sqrt{\left(M^3 + abb\right)\left(a + bp^3\right)}$$

or

$$2ab + Mp(M + bp) - q\left(M - bp\right)^2 = \pm 2\sqrt{\left(M^3 + abb\right)\left(a + bp^3\right)}$$

and

$$2ab + Mq\left(M + bq\right) - p\left(M - bq\right)^2 = \pm 2\sqrt{\left(M^3 + abb\right)\left(a + bq^3\right)}.$$

EXAMPLE 4

632. *To find the complete integral of this differential equation*

$$\frac{dp}{\pm\sqrt{\left(a+bp^4\right)}} + \frac{dq}{\pm\sqrt{\left(a+bq^4\right)}} = 0.$$

On putting $x = p,\ y = q$ there shall be $A = a,\ B = 0,\ C = 0,\ D = 0,\ E = b$, and hence

$$\alpha = 4aM,\ \beta = 0, \gamma = 4ab - MM, \zeta = 4bM,\ \varepsilon = 0, \delta = MM + 4ab$$

and

$$\varDelta = M^3 - 4abM,$$

from which the complete integral becomes

$$\left(MM + 4ab\right)p + q\left(4ab - MM + 4bMpp\right) = \pm 2\sqrt{M\left(MM - 4ab\right)\left(a + bp^4\right)},$$

$$\left(MM + 4ab\right))q + p(\left(4ab - MM + 4bMqq\right) = \pm 2\sqrt{M\left(MM - 4ab\right)\left(a + bq^4\right)}.$$

EXAMPLE 5

633. *To find the complete integral of this differential equation*

$$\frac{dp}{\pm\sqrt{\left(a+bp^6\right)}} + \frac{dq}{\pm\sqrt{\left(a+bq^6\right)}} = 0.$$

There is put $x = pp$ and $y = qq$ and our general equation adopts this form on putting $A = 0$

$$\frac{dp}{\pm\sqrt{\left(2B+Cpp+2Dp^4+Ep^6\right)}}+\frac{dq}{\pm\sqrt{\left(2B+Cqq+2Dq^4+Eq^6\right)}}=0.$$

Hence it is required to become $B=\frac{1}{2}a$, $C=0, D=0$ and $E=b$, from which the coefficients thus are determined :

$$\alpha=-aa,\quad \beta=aM,\quad \gamma=-MM,\quad \zeta=4bM,\quad \varepsilon=2ab,\quad \delta=MM$$

and

$$\Delta=M^3+aab,$$

hence the complete integral shall be

$$aM+MMpp+2abp^4+qq\left(-MM+4abpp+4bMp^4\right)$$
$$=\pm2p\sqrt{\left(M^3+aab\right)\left(a+bp^6\right)}$$

or

$$aM+MMqq+2abq^4+pp\left(-MM+4abqq+4bMq^4\right)$$
$$=\pm2q\sqrt{\left(M^3+aab\right)\left(a+bq^6\right)}.$$

COROLLARY

634. If the constant is assumed $M=-\sqrt[3]{aab}$, so that there becomes $M^3+aab=0$, there will be produced the particular integral, which thus itself shall have the form

$$pp=\frac{qq\sqrt[3]{b}+\sqrt[3]{a}}{2qq\sqrt[3]{b}-\sqrt[3]{a}}\cdot\sqrt[3]{\frac{a}{b}}\quad\text{or}\quad qq=\frac{pp\sqrt[3]{b}+\sqrt[3]{a}}{2pp\sqrt[3]{b}-\sqrt[3]{a}}\cdot\sqrt[3]{\frac{a}{b}}.$$

which certainly satisfies the differential equation.

PROBLEM 82

635. *From this proposed differential equation*

$$\frac{dp}{\pm\sqrt{\left(a+bpp+cp^4+ep^6\right)}}+\frac{dq}{\pm\sqrt{\left(a+bqq+cq^4+eq^6\right)}}=0$$

to assign the complete integral of this algebraically.

SOLUTION

The preceding differential equation integrated algebraically is reduced to this form on putting $x = pp$ and $y = qq$ and $A = 0$; indeed it gives

$$\frac{dp}{\pm\sqrt{\left(2B+Cpp+2Dp^4+Ep^6\right)}}+\frac{dq}{\pm\sqrt{\left(2B+Cqq+2Dq^4+Eq^6\right)}}=0.$$

Whereby it is needed only, that there be made

$$A=0,\ B=\tfrac{1}{2}a,\ C=b,\ D=\tfrac{1}{2}c,\ E=e,$$

from which the coefficients $\alpha,\ \beta,\ \gamma,\delta,\ \varepsilon,\zeta$ thus may be defined :

$$\alpha=-aa,\quad \beta=a\left(M-b\right),\ \gamma=-\left(M-b\right)^2,$$
$$\zeta=4eM-cc,\quad \varepsilon=c\left(M-b\right)+2ae,\quad \delta=MM-bb+ac,$$
$$\Delta=M(M-b)^2+acM-abc+aae=(M-\text{b})^3+b\left(M-b\right)^2+ac\left(M-b\right)+aae,$$

and hence the complete integral on account of the constant M depending on our choice will be

$$\beta+\delta pp+\varepsilon p^4+qq(\gamma+2\varepsilon pp+\zeta p^4)=\pm 2p\sqrt{\Delta(a+bpp+cp^4+ep^6)},$$
$$\beta+\delta qq+\varepsilon q^4+pp(\gamma+2\varepsilon qq+\zeta q^4)=\pm 2q\sqrt{\Delta(a+bqq+cq^4+eq^6)},$$

which two equations indeed agree between themselves, but on account of the ambiguity of the signs in the differential equation itself both must be noted with the ambiguity thus removed. But each gives this rational equation :

$$0=\alpha+2\beta(pp+qq)+\gamma(p^4+q^4)+2\delta ppqq+2\varepsilon ppqq(pp+qq)+\zeta p^4q^4.$$

COROLLARY 1

636. If the constant M is assumed thus, so that there is made $\Delta = 0$, there is obtained a particular integral of this form

$$qq = \frac{E+Fpp}{G+Hpp},$$

which also from before it is allowed to know. In order that indeed it is satisfied, there must be taken

$$aG^3 + bEGG + cEEG + eE^3 = 0,$$

from which the ratio $E : G$ is defined ; then indeed there is found $F = -G$ and finally

$$H = \frac{-cEG-2eEE}{aG} = \frac{2aGG+2bEG+cEE}{aE}.$$

COROLLARY 2

637. The constant M may be changed thus, so that there shall be $M - b = \frac{a}{ff}$, and there becomes

$$\alpha = -aa, \quad \beta = \frac{aa}{ff}, \quad \gamma = \frac{aa}{f^4},$$

$$\zeta = 4be - cc + \frac{4ae}{ff}, \quad \varepsilon = \frac{ac}{ff} + 2ae, \quad \delta = \frac{aa}{f^4} + \frac{2ab}{ff} + ac$$

and

$$\Delta = \frac{aa}{f^6}\left(a + bff + cf^4 + ef^6\right)$$

and the equation of the integral shall be

$$aaff + a\left(a + 2bff + cf^4\right)pp + aff\left(c + 2eff\right)p^4$$
$$-qq\left(aa - 2aff\left(c + 2eff\right)pp + ff\left(ccff - 4beff - 4ae\right)p^4\right)$$
$$= \pm 2afp\sqrt{\left(a + bff + cf^4 + ef^6\right)\left(a + bpp + cp^4 + ep^6\right)},$$

from which it is apparent on putting $p = 0$ to become $qq = ff$.

COROLLARY 3

638. This equation can be easily transformed into this form :

$$\begin{aligned}&aff\left(a+bpp+Cp^4+ep^6\right)+app\left(a+bff+cf^4+ef^6\right)\\&-qq(a-cffpp)^2-aeffpp(ff-pp)^2+4effppqq(aff+app+bffpp)\\&=\pm 2fp\sqrt{a\left(a+bff+cf^4+ef^6\right)a\left(a+bpp+cp^4+ep^6\right)},\end{aligned}$$

from which it is at once apparent, if there shall be $e=0$, to become this equation with the root being extracted :

$$f\sqrt{a\left(a+bpp+cp^4\right)}\mp p\sqrt{a\left(a+bff+cf^4\right)}=q(a-cffpp),$$

which is the complete integral of this differential

$$\frac{dp}{\pm\sqrt{\left(a+bpp+cp^4\right)}}+\frac{dq}{\pm\sqrt{\left(a+bqq+cq^4\right)}}=0,$$

entirely as we have now found above [§ 607].

COROLLARY 4

639. In a similar manner it is apparent in general, when *e* does not vanish, the complete integral thus can be expressed more conveniently :

$$\begin{aligned}&\left(f\sqrt{a\left(a+bpp+Cp^4+ep^6\right)}\mp p\sqrt{a\left(a+bff+cf^4+ef^6\right)}\right)^2\\&=qq(a-cffpp)^2+aeffpp(ff-pp)^2-4effppqq(aff+app+bffpp),\end{aligned}$$

which therefore, since on putting $p=0$ there is made $q=f$, corresponds to this relation of the transcending functions :

$$\pm\,\Pi:p\pm\,\Pi:q=\pm\,\Pi:0\pm\,\Pi:f.$$

SCHOLION 1

640. Hence the kinds of transcendental functions, which likewise in this way and which among themselves are allowed to be compared with the arcs of circles, and which are contained in these two integral formulas

$$\int \frac{dz}{\sqrt{\left(A+2Bz+Czz+2Dz^3+Ez^4\right)}} \quad \text{et} \quad \int \frac{dz}{\sqrt{\left(a+bzz+cz^4+ez^6\right)}}$$

nor this method is seen to be extended to other more complex forms. Nor also are odd powers of z to be admitted into the denominator, unless perhaps is should suffice by a simple substitution to be reduced to another form

$$\int \frac{dz}{\sqrt{\left(A+2Bz+Czz+2Dz^3+Ez^4+2Fz^5+Gz^6\right)}}.$$

But it is readily apparent that it is certainly not possible for another form to be treated by this method. For if the coefficients are to be compared thus, so that the extraction of the roots should succeed, such a formula

$$\int \frac{dz}{a+bz+czz+ez^3}$$

would be produced; since the integration of which involves both logarithms and circular arcs, cannot be made altogether, as several functions of this kind are to be compared algebraically between each other. Moreover the former formula appears wider than the latter, since this arises from that on putting $\mathrm{A}=0$, if zz is written in place of z. But from the first it is noteworthy, that the same formula is preserved, even if it may be transformed by this substitution

$$z=\frac{a+\beta y}{\gamma+\delta y};$$

for there is produced

$$\int \frac{(\beta\gamma-\alpha\delta)dy}{\sqrt{\left(A(\gamma+\delta y)^4+2B(\alpha+\beta y)(\gamma+\delta y)^3+C(\alpha+\beta y)^2(\gamma+\delta y)^2+2D(\alpha+\beta y)^3(\gamma+\delta y)+E(\alpha+\beta y)^4\right)}},$$

from which it is understood that the quantities $\alpha,\ \beta,\ \gamma,\ \delta$ thus can be taken, so that the odd powers vanish. Or also thus they are able to be defined, so that the first term and the last vanish ; for then on putting $y=uu$ again a form exempt from the odd powers arises.

SCHOLIUM 2

641. Moreover the removal of the odd power as thus can conveniently be put in place. Since the formula

$$A+2Bz+Czz+2Dz^3+Ez^4$$

certainly always has twp real factors, thus the formula of the integral may be shown :

$$\int\frac{dz}{\sqrt{(a+2bz+czz)(f+2gz+hzz)}},$$

which on putting $z=\frac{\alpha+\beta y}{\gamma+\delta y}$ changes into

$$\int\frac{(\beta\gamma-\alpha\delta)dy}{\sqrt{\left(a(\gamma+\delta y)^4+2b(\alpha+\beta y)(\gamma+\delta y)+c(\alpha+\beta y)^2\right)\left(f(\gamma+\delta y)^2+2g(\alpha+\beta y)(\gamma+\delta y)+E(\alpha+\beta y)^2\right)}}$$

where with the denominators expanded out the factors are :

$$(a\gamma\gamma+2b\alpha\gamma+c\alpha\alpha)+2(a\gamma\delta+b\alpha\delta+b\beta\gamma+c\alpha\beta)y+(a\delta\delta+2b\beta\delta+c\beta\beta)yy,$$
$$(f\gamma\gamma+2g\alpha\gamma+h\alpha\alpha)+2(f\gamma\delta+g\alpha\delta+g\beta\gamma+h\alpha\beta)y+(f\delta\delta+2g\beta\delta+h\beta\beta)yy;$$

and if now each middle term is reduced vanishing, there shall be made,

$$\frac{\delta}{\beta}=\frac{-b\gamma-c\alpha}{a\gamma+b\alpha}=\frac{-g\gamma-h\alpha}{f\gamma+g\alpha}$$

and hence

$$bf\gamma\gamma+(bg+cf)\alpha\gamma+cg\alpha\alpha=ag\gamma\gamma+(ah+bg)\alpha\gamma+bh\alpha\alpha$$

or

$$\gamma\gamma=\frac{(ah-cf)\alpha\gamma+(bh-cg)\alpha\alpha}{bf-ag},$$

from which there becomes

$$\frac{\gamma}{\alpha}=\frac{ah-cf+\sqrt{\left((ah-cf)^2+4(bf-ag)(bh-cg)\right)}}{2(bf-ag)}..$$

Hence it is possible to suffice, that only formulas are to be treated, in which odd powers are absent, which we have put in place at the start of this chapter, but if in addition a numerator is accepted, then this reduction can no longer be put in place.

PROBLEM 83

642. *With n denoting some whole number to find the complete integral algebraically of this differential equation*

$$\frac{dy}{\sqrt{\left(A+2By+Cyy+2Dy^3+Ey^4\right)}}=\frac{ndx}{\sqrt{\left(A+2Bx+Cxx+2Dx^3+Ex^4\right)}}.$$

SOLUTION

Through transcending functions the complete integral is

$$\Pi:y=n\Pi:x+\text{Const}.$$

But as we have elicited the same expression algebraically, on putting $M-C=L$ by the formulas found above (§ 627), there shall be:

$$\alpha=4\left(AC-BB+AL\right),\quad \beta=4AD+2BL,\quad \gamma=4AE-LL,$$
$$\zeta=4\left(CE-DD+EL\right),\quad \varepsilon=4BE+2DL,\quad \delta=4AE+4BD+2CL+LL$$

and

$$\Delta=L^3+CLL+4\left(BD-AE\right)L+4(ADD+BBE-ACE).$$

With which put in place if there should be

$$\beta+\delta p+\varepsilon pp+q(\gamma+2\varepsilon p+\zeta pp)=2\sqrt{\Delta\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)},$$
$$\beta+\delta q+\varepsilon qq+p(\gamma+2\varepsilon q+\zeta qq)=-2\sqrt{\Delta\left(A+2Bq+Cqq+2Dq^3+Eq^4\right)},$$

then there shall be $\Pi:q=\Pi:p+\text{Const}$.

But since these two equations agree between themselves and in these on being rationalized they are contained by :

$$\alpha+2\beta(p+q)+\gamma(pp+qq)+2\delta pq+2\varepsilon pq(p+q)+\zeta ppqq=0,$$

if we assume on putting $p=a$ that there becomes $q=b$, that constant L thus must be defined, so that there shall be :

$$\alpha+2\beta(a+b)+\gamma(aa+bb)+2\delta ab+2\varepsilon ab(a+b)+\zeta aabb=0,$$

and there shall be

$$\Pi: q = \Pi: p + \Pi: b - \Pi: a \,,$$

where now there is no difference between the constants and the variables. Hence we may put $b = p$, so that there shall be

$$\Pi: q = 2\Pi: p - \Pi: a \,,$$

and the above algebraic equations agree with this, only if the quantity *L* is defined thus, so that there shall be :

$$\alpha + 2\beta(a+p) + \gamma(aa+pp) + 2\delta ap + 2\varepsilon ap(a+p) + \zeta aapp = 0,$$

from which there is deduced :

$$\begin{aligned}&\tfrac{1}{2}L\left(p-a\right)^2 = A + B\left(a+p\right) + Cap + Dap\left(a+p\right) + Eaapp\\ &\pm\sqrt{\left(A+2Ba+Caa+2Da^3+Ea^4\right)\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)}.\end{aligned}$$

Hence with this value for *L* put in place and finally with the letters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ duly defined by the above formulas, if now *p* and *q* as the variables, and we now consider *a* as constant, then this equation

$$\alpha + 2\beta(p+q) + \gamma(pp+qq) + 2\delta pq + 2\varepsilon pq(p+q) + \zeta ppqq = 0,$$

shall be the complete integral of this differential equation

$$\frac{dq}{\sqrt{\left(A+2Bq+Cqq+2Dq^3+Eq^4\right)}} = \frac{2dp}{\sqrt{\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)}}.$$

After we have divided *q* by *p* in this manner, *r* may be determined by this equation :

$$\alpha + 2\beta(q+r) + \gamma(qq+rr) + 2\delta qr + 2\varepsilon qr(q+r) + \zeta qqrr = 0,$$

then there shall be

$$\Pi: r - \Pi: q = \Pi: p - \Pi: a,$$

because on putting $q = a$ and $r = p$ the letter *L*, which is present in terms of the values $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$, finally is defined as before. Whereby since there shall be $\Pi: q = 2\Pi : p - \Pi : a$, then there shall be

$$\Pi: r = 3\Pi: p - 2\Pi: a,$$

from which on assuming *a* constant that algebraic equation between *q* and *r*, while *q* is defined from *p* by the preceding equation, will be the complete integral of this differential equation

$$\frac{dr}{\sqrt{(A+2Br+Crr+2Dr^3+Er^4)}} = \frac{3dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

With this value of *r* found through *p*, *s* is sought from this equation :

$$\alpha + 2\beta(r+s) + \gamma(rr+ss) + 2\delta rs + 2\varepsilon rs(r+s) + \zeta rrss = 0,$$

with *L* always keeping the first value assigned and there shall be :

$$\Pi: s - \Pi: r = \Pi: p - \Pi: a \quad \text{or} \quad \Pi: s = 4\Pi: p - 3\Pi: a,$$

from which this algebraic equation shall be the complete integral of this differential equation

$$\frac{ds}{\sqrt{(A+2Bs+Css+2Ds^3+Es^4)}} = \frac{4dp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}}.$$

Since in this way, as far as it should be wished, this is allowed to progress and it is evident that for finding the complete integral of this differential equation

$$\frac{dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}} = \frac{ndp}{\sqrt{(A+2Bp+Cpp+2Dp^3+Ep^4)}},$$

the following operations are required to be put in place :

1) The quantity *L* is sought, so that there shall be

$$\tfrac{1}{2}L(p-a)^2 = A + B(a+p) + Cap + Dap(a+p) + Eaapp$$
$$\pm\sqrt{(A+2Ba+Caa+2Da^3+Ea^4)(A+2Bp+Cpp+2Dp^3+Ep^4)}.$$

2) Hence the letters $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ are determined by these formulas :

$$\alpha = 4(AC - BB + AL), \quad \beta = 4AD + 2BL, \quad \gamma = 4AE - LL,$$
$$\zeta = 4(CE - DD + EL), \quad \varepsilon = 4BE + 2DL, \quad \delta = 4AE + 4BD + 2CL + LL$$

3) The series of quantities *p, q, r, s, t, ... z,* shall be formed, of which the first shall be *p*, the second *q*, the third *r* etc., and indeed the final with order *n* shall be *z*, which terms are determined successively by these equations

$$\alpha+2\beta(p+q)+\gamma(pp+qq)+2\delta pq+2\varepsilon pq(p+q)+\zeta ppqq=0,$$
$$\alpha+2\beta(q+r)+\gamma(qq+rr)+2\delta qr+2\varepsilon qr(q+r)+\zeta qqrr=0,$$
$$\alpha+2\beta(r+s)+\gamma(rr+ss)+2\delta rs+2\varepsilon rs(r+s)+\zeta rrss=0,$$
etc.,

while in the end *z* is reached.

4) The relation, which hence is concluded between *p* and *z*, is the complete integral of the proposed differential equation, and of the arbitrary constant entering into the integration which in turn is borne by the letter *a* .

COROLLARY

643. Hence also the complete can be found of this differential equation

$$\frac{mdy}{\sqrt{\left(A+2By+Cyy+2Dy^3+Ey^4\right)}}=\frac{ndx}{\sqrt{\left(A+2Bx+Cxx+2Dx^3+Ex^4\right)}}$$

with *m* and *n* designating whole numbers. For each member is put in place equal to

$$\frac{mdu}{\sqrt{\left(A+2Bu+Cuu+2Du^3+Eu^4\right)}}$$

and the relation both between *x* and *u* as well as between *y* and *u;* from which with *u* removed there arises an algebraic equation between *x* and *y*.

SCHOLIUM

644. Lest this extraction of the root in the individual equations creates an ambiguity on repetition, in place of any one it will be agreed to now use two elicited per extraction. Clearly as from the first value *q* found duly that by *p* is defined, indeed for the first we shall have

$$q=\frac{-\beta-\delta p-\varepsilon pp+2\sqrt{\Delta\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)}}{\gamma+2\varepsilon p+\zeta pp},$$

then there must be taken

$$2\sqrt{\Delta\left(A+2Bq+Cqq+2Dq^3+Eq^4\right)}$$
$$=-\beta-\delta q-\varepsilon qq-p\left(\gamma+2\varepsilon q+\zeta qq\right)$$

and in a like manner in the relation between two following quantities being investigated will be the proceeding.

Moreover at this point it is agreed that the whole numbers *m* and *n* must be positive and neither is this investigation to be extended to negative numbers, therefore because the formula of the differential

$$\frac{dz}{\sqrt{\left(A+2Bz+Czz+2Dz^3+Ez^4\right)}}$$

changes its nature on putting *z*. Yet meanwhile since this equation $\varPi : x+\varPi : y=$ Const. we could have expressed above algebraically, with the help of this too these cases are able to be resolved, where either *m* or *n* is a negative number ; for if there should be $\varPi : z=n\varPi : p+C$, *y* is sought, in order that there shall be $\varPi : y+\varPi : z=\text{Const.}$, and then there shall become

$$\varPi : y=-n\varPi : p+\text{Const.}$$

PROBLEM 84

645. *If $\varPi : z$ denotes a transcending function of z of this kind, so that there shall be*

$$\varPi : z=\int\frac{dz\left(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\mathfrak{E}z^4\right)}{\sqrt{\left(A+2Bz+Czz+2Dz^3+Ez^4\right)}},$$

to investigate the comparison between functions of this kind.

SOLUTION

From the coefficients *A*, *B*, *C*, *D*, *E* with one arbitrary constant *L* the following values may be determined

$$\alpha=4\left(AC-BB+AL\right),\ \ \beta=4AD+2BL,\quad \gamma=4AE-LL,$$
$$\zeta=4\left(CE-DD+EL\right),\quad \varepsilon=4BE+2DL,\quad \delta=4AE+4BD+2CL+LL$$

and between the two variables *x* and *y* this relation is set up :

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy+2\varepsilon xy(x+y)+\zeta xxyy=0$$

and then there shall be

$$\frac{dx}{\sqrt{\left(A+2Bx+Cxx+2Dx^3+Ex^4\right)}}+\frac{dy}{\sqrt{\left(A+2By+Cyy+2Dy^3+Ey^4\right)}}=0,$$

or

from which without ambiguity there may be had

$$\beta+\delta x+\varepsilon xx+y\left(\gamma+2\varepsilon x+\zeta xx\right)=2\sqrt{\Delta\left(A+2Bx+Cxx+2Dx^3+Ex^4\right)},$$
$$\beta+\delta y+\varepsilon yy+x\left(\gamma+2\varepsilon y+\zeta yy\right)=2\sqrt{\Delta\left(A+2By+Cyy+2Dy^3+Ey^4\right)}$$

with the relation present

$$\Delta=L^3+CL^2+4\left(BD-AE\right)L+4\left(ADD+BBE-ACE\right).$$

Whereby if we put

$$\frac{dx\left(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}xx+\mathfrak{D}x^3+\mathfrak{E}x^4\right)}{\sqrt{\left(A+2Bx+Cxx+2Dx^3+Ex^4\right)}}+\frac{dy\left(\mathfrak{A}+\mathfrak{B}y+\mathfrak{C}yy+\mathfrak{D}y^3+\mathfrak{E}y^4\right)}{\sqrt{\left(A+2By+Cyy+2Dy^3+Ey^4\right)}}=2dV\sqrt{\Delta},$$

in order that there becomes

$$\Pi:x+\Pi:y=\text{Const.}+2V\sqrt{\Delta},$$

then there shall be

$$\frac{dx\left(\mathfrak{B}(x-y)+\mathfrak{C}\left(x^2-y^2\right)+\mathfrak{D}\left(x^3-y^3\right)+\mathfrak{E}\left(x^4-y^4\right)\right)}{\sqrt{\left(A+2Bx+Cxx+2Dx^3+Ex^4\right)}}=2dV\sqrt{\Delta}$$

or

$$dV=\frac{dx\left(\mathfrak{B}(x-y)+\mathfrak{C}\left(x^2-y^2\right)+\mathfrak{D}\left(x^3-y^3\right)+\mathfrak{E}\left(x^4-y^4\right)\right)}{\beta+\delta x+\varepsilon xx+y\left(\gamma+2\varepsilon x+\zeta xx\right)}.$$

Now there is put $x+y=t$ and $xy=u$, and because $dx+dy=dt$ and $xdy+ydx=du$, there shall be $dx=\frac{xdt-du}{x-y}$ or $\left(x-y\right)dx=xdt-du$, then there shall now become $x=\frac{1}{2}t+\sqrt{\left(\frac{1}{4}tt-u\right)}$. But with these in place the assumed equation adopts this form :

$$\alpha+2\beta t+\gamma tt+2\left(\delta-\gamma\right)u+2\varepsilon tu+\zeta uu=0,$$

from which there becomes on differentiation

$$dt\left(\beta+\gamma t+\varepsilon u\right)+du\left(\delta-\gamma+\varepsilon t+\zeta u\right)=0$$

hence $dt=\frac{-du\left(\delta-\gamma+\varepsilon t+\zeta u\right)}{\beta+\gamma t+\varepsilon u}$ and

$$xdt-du=\frac{-du\left(\beta+\gamma t+\varepsilon u+\left(\delta-\gamma\right)x+\varepsilon tx+\zeta ux\right)}{\beta+\gamma t+\varepsilon u}$$

or

$$xdt - du = \frac{-du\left(\beta+\delta x+\varepsilon xx+y\left(\gamma+2\varepsilon x+\zeta xx\right)\right)}{\beta+\gamma t+\varepsilon u},$$

and thus we shall have

$$\frac{dx(x-y)}{\beta+\delta x+\varepsilon xx+y\left(\gamma+2\varepsilon x+\zeta xx\right)} = \frac{-du}{\beta+\gamma t+\varepsilon u},$$

hence

$$dV = \frac{-du\left(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)\right)}{\beta+\gamma t+\varepsilon u}$$

or

$$dV = \frac{+dt\left(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)\right)}{\delta-\gamma+\varepsilon t+\zeta u}.$$

Now with that equation resolved

$$t = \frac{-\beta-\varepsilon u+\sqrt{\left(\beta\beta-\alpha\gamma+2(\gamma\gamma+\beta\varepsilon-\gamma\delta)u+(\varepsilon\varepsilon-\gamma\zeta)uu\right)}}{\gamma}$$

or [§ 628]

$$t = \frac{-\beta-\varepsilon u+2\sqrt{\varDelta(A+Lu+Euu)}}{\gamma},$$

with which completed

$$dV = \frac{-du\left(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)\right)}{2\sqrt{\varDelta(A+Lu+Euu)}}$$

and thus

$$\varPi : x + \varPi : y = \text{Const.} - \int \frac{du\left(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)\right)}{2\sqrt{(A+Lu+Euu)}}.$$

Or since there is found :

$$u = \frac{-(\delta-\gamma)-\varepsilon t+\sqrt{\left((\delta-\gamma)^2-\alpha\zeta+2\left((\delta-\gamma)\varepsilon-\beta\zeta\right)t+(\varepsilon\varepsilon-\gamma\zeta)tt\right)}}{\zeta},$$

which expression changes into this [§ 628] :

$$u = \frac{-(\delta-\gamma)-\varepsilon t+\sqrt{\varDelta(L+C+2Dt+Ett)}}{\zeta},$$

from which there becomes

$$dV = \frac{dt\left(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)\right)}{2\sqrt{\varDelta(L+C+2Dt+Ett)}}$$

and thus we shall have through t

$$\varPi : x + \varPi : y = \text{Const.} + \int \frac{dt\left(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)\right)}{\sqrt{(L+C+2Dt+Ett)}},$$

which expression, unless it shall be algebraic, certainly can be shown either by logarithms or by the arcs of circles. Then truly after the integration there is only the need, that in place of *t* the value of $x+y$ is restored.

COROLLARIUM 1

646. If we wish, so that on putting $x=a$ there becomes $y=b$, the constant *L* must be defined thus, so that there shall be :

$$\tfrac{1}{2}L(a-b)^2 = A+B(a+b)+Cab+Dab(a+b)+Eaabb$$
$$\pm\sqrt{\left(A+2Ba+Caa+2Da^3+Ea^4\right)\left(A+2Bb+Cbb+2Db^2+Eb^4\right)};$$

then hence our constant becomes equal to $\Pi: a+\text{II}: b$ with the integral next taken, so that it vanishes on putting $t=a+b$.

COROLLARY 2

647. In the same manner also the difference of the functions $\Pi: x-\Pi: y$ can be expressed by changing the signs of each of the formulas for the square roots, with which agreed upon the sign of each of the formulas of the differentials is changed.

COROLLARIUM 3

648. The quantity *V* serving for the comparison of these formulas shall be algebraic, if this differential formula

$$\frac{dt\left(\mathfrak{B}\zeta+\mathfrak{C}\zeta t+\mathfrak{D}(\delta-\gamma+\varepsilon t+\zeta tt)+\mathfrak{E}\left(2(\delta-\gamma)+2\varepsilon t+\zeta tt\right)t\right)}{\zeta\sqrt{(L+C+2Dt+Ett)}}$$

is allowed to be integrated, as the other part $\frac{-2dt\sqrt{\Delta}}{\zeta}(\mathfrak{D}+2\mathfrak{E}\mathrm{t})$ is integrable by itself.

SCHOLIUM

649. Hence this clearly is a new argument concerning the comparison of transcending functions of this kind which we have explored so fully, as the present arrangement is seen to set out. But when the application of this is to be made to the comparison of circular arcs, of which the length are expressed by functions of this kind, the work will be enriched by the excellent use that can be brought, when individual properties are considered, and which are elicited in this way. Moreover this argument conveniently is seen to refer to the doctrine of the resolution of differential equations, if indeed thence the integrals of equations of this kind are able to be shown completely and indeed algebraically, which are investigated by other methods in vain. Therefore for the end of this section a general method shall now be made for the integrals of all differential equations to be determined approximately.

CAPUT VI

DE COMPARATIONE QUANTITATUM TRANSCENDENTIUM IN FORMA $\int \frac{Pdz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}$ CONTENTARUM.

PROBLEMA 78

606. *Proposita relatione inter x et y hac*

$$\alpha+\gamma(xx+yy)+2\delta xy+\zeta xxyy=0$$

inde elicere functiones transcendentes formae praescriptae, quas inter se comparare liceat.

SOLUTIO

Ex proposita aequatione definiatur utraque variabilis

$$y=\frac{-\delta x+\sqrt{(-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)xx-\gamma\zeta x^4)}}{\gamma+\zeta xx}$$

et

$$x=\frac{-\delta y+\sqrt{(-\alpha\gamma+(\delta\delta-\gamma\gamma-\alpha\zeta)yy-\gamma\zeta y^4)}}{\gamma+\zeta yy},$$

quae radicalia ad formam praescriptam revocentur ponendo

$$-\alpha\gamma=Am,\ \delta\delta-\gamma\gamma-\alpha\zeta=Cm \quad \text{et} \quad -\gamma\zeta=Em,$$

unde fit

$$\alpha=-\frac{Am}{\gamma}, \quad \zeta=-\frac{Em}{\gamma} \quad \text{et} \quad \delta\delta=Cm+\gamma\gamma+\frac{AEmm}{\gamma\gamma}.$$

Erit ergo

$$\gamma y+\delta x+\zeta xxy=\sqrt{m\left(A+Cxx+Ex^4\right)},$$
$$\gamma x+\delta y+\zeta xyy=\sqrt{m\left(A+Cyy+Ey^4\right)}.$$

Ipsa autem aequatio proposita, si differentietur, dat

$$dx(\gamma x+\delta y+\zeta xyy)+dy(\gamma y+\delta x+\zeta xxy)=0,$$

ubi illi valores substituti praebent

$$\frac{dx}{\sqrt{(A+Cxx+Ex^4)}}+\frac{dy}{\sqrt{(A+Cyy+Ey^4)}}=0 .$$

Vicissim ergo proposita hac aequatione differentiali ei satisfaciet haec aequatio finita

$$-Am+\gamma\gamma(xx+yy)+2xy\sqrt{(\gamma^4+Cm\gamma\gamma+AEmm)}-Emxxyy=0$$

seu ponendo $\frac{\gamma\gamma}{m}=k$ haec

$$-A+k(xx+yy)+2xy\sqrt{(kk+kC+AE)}-Exxyy=0 ,$$

quae cum involvat constantem k in aequatione differentiali non contentam, simul erit integrale completum.

Hinc autem fit

$$ky+x\sqrt{(kk+kC+AE)}-Exxy=\sqrt{k\left(A+Cxx+Ex^4\right)}$$

et

$$kx+y\sqrt{(kk+kC+AE)}-Exyy=\sqrt{k\left(A+Cyy+Ey^4\right)} .$$

COROLLARIUM 1

607. Constans k ita assumi potest, ut posito $x=0$ fiat $y=b$; oritur autem

$$bk=\sqrt{Ak} \quad \text{et} \quad b\sqrt{(kk+kC+AE)}=\sqrt{k\left(A+Cbb+Eb^4\right)} ,$$

ergo

$$k=\frac{A}{b} \quad \text{et} \quad \sqrt{(kk+kC+AE)}=\frac{1}{bb}\sqrt{A\left(A+Cbb+Eb^4\right)}$$

ideoque habebimus

$$Ay+x\sqrt{A\left(AA+Cbb+Eb^4\right)}-Ebbxxy=b\sqrt{A\left(A+Cxx+Ex^4\right)}$$

et

$$Ax+y\sqrt{A\left(AA+Cbb+Eb^4\right)}-Ebbxyy=b\sqrt{A\left(A+Cyy+Ey^4\right)} .$$

COROLLARIUM 2

608. Haec igitur relatio finita inter x et y erit integrale completum aequationis differentialis

$$\frac{dx}{\sqrt{(AA+Cxx+Ex^4)}}+\frac{dy}{\sqrt{(A+Cyy+Ey^4)}}=0,$$

quod rationaliter inter x et y expressum erit

$$A\left(xx+yy-bb\right)+2xy\sqrt{A\left(AA+Cbb+Eb^4\right)}-Ebbxxyy=0.$$

COROLLARIUM 3

609. Hinc ergo *y* ita per *x* exprimetur, ut sit

$$y=\frac{b\sqrt{A\left(A+Cxx+Ex^4\right)}-x\sqrt{A\left(A+Cbb+Eb^4\right)}}{A-Ebbxx},$$

atque ex hoc valore elicitur

$$\sqrt{\frac{\left(A+Cyy+Ey^4\right)}{A}}=\frac{\left(A+Ebbxx\right)\sqrt{A\left(A+Cbb+Eb^4\right)\left(A+Cxx+Ex^4\right)}-2AEbx\left(bb+xx\right)-Cbx\left(A+Ebbxx\right)}{\left(A-Ebbxx\right)^2}.$$

COROLLARIUM 4

610. Hinc constantem *b* pro lubitu determinando infinita integralia particularia exhiberi possunt, quorum praecipua sunt:

1) sumendo $b=0$, unde fit $y=-x$;

2) sumendo $b=\infty$, unde fit $y=\frac{\sqrt{A}}{x\sqrt{E}}$;

3) si $A+Cbb+Eb^4=0$ hincque $bb=\frac{-C+\sqrt{(CC-4AE)}}{2E}$, unde fit

$$y=\frac{b\sqrt{A\left(A+Cbb+Eb^4\right)}}{A-Ebbxx}.$$

SCHOLION

611. Hic iam usus istius methodi, qua retrogrediendo ab aequatione finita ad aequationem differentialem pervenimus, luculenter perspicitur. Cum enim integratio formulae $\frac{dx}{\sqrt{\left(A+Cxx+Ex^4\right)}}$ nullo modo neque per logarithmos neque arcus circulares perfici posset, mirum sane est talem aequationem differentialem adeo algebraice integrari posse; quae quidem in praecedente capite ope eiusdem methodi sunt tradita, etiam methodo ordinaria erui possunt, dum singulae formulae differentiales vel per logarithmos vel arcus circulares exprimuntur, quorum deinceps comparatio ad aequationem algebraicam reducitur. Verum quia hic talis integratio plane non locum invenit, nulla certe alia methodus patet, qua idem integrale, quod hic exhibuimus, investigari posset. Quare hoc argumentum diligentius evolvamus.

PROBLEMA 79

612. *Si* $\Pi:z$ *denotet eiusmodi functionem ipsius z, ut sit*

$$\varPi : z = \int \frac{Pdz}{\sqrt{\left(A + Czz + Ez^4\right)}}$$

integrali ita sumto, ut evanescat posito $z = 0$, *comparationem inter huiusmodi functiones investigare.*

SOLUTIO

Posita inter binas variabiles *x* et *y* relatione supra definita vidimus fore

$$\frac{dx}{\sqrt{\left(A+Cxx+Ex^4\right)}} + \frac{dy}{\sqrt{\left(A+Cyy+Ey^4\right)}} = 0$$

Hinc, cum posito $x = 0$ fiat $y = b$, elicitur integrando

$$\varPi : x + \varPi : y = \varPi : b .$$

Cum iam nullum amplius discrimen inter variabiles *x*, *y* et constantem *b* intercedat, statuamus $x = p$, $y = q$ et $b = -r$, ut sit $\varPi : b = -\varPi : r$, atque haec relatio inter functiones transcendentes

$$\varPi : p + \varPi : q + \varPi : r = 0$$

per sequentes formulas algebraicas exprimetur

$$\left(A - Epprr\right)q + p\sqrt{A\left(A + Crr + Er^4\right)} + r\sqrt{A\left(A + Cpp + Ep^4\right)} = 0$$

seu

$$\left(A - Eppqq\right)r + q\sqrt{A\left(A + Cpp + Ep^4\right)} + p\sqrt{A\left(A + Cqq + Eq^4\right)} = 0$$

seu

$$\left(A - Eqqrr\right)p + r\sqrt{A\left(A + Cqq + Eq^4\right)} + q\sqrt{A\left(A + Crr + Er^4\right)} = 0 ,$$

quae oriuntur ex hac aequatione

$$A\left(pp + qq - rr\right) - Eppqqrr + 2pq\sqrt{A\left(A + Crr + Er^4\right)} = 0 .$$

Haec vero ad rationalitatem perducta fit

$$AA\left(p^4 + q^4 + r^4 - 2ppqq - 2pprr - 2qqrr\right) - 2AEppqqrr\left(pp + qq + rr\right)$$
$$- 4ACppqqrr + EEp^4q^4r^4 = 0,$$

quae autem ob pluralitatem radicum satisfacit omnibus signorum variationibus in superiori aequatione transcendente.

COROLLARIUM 1

613. Sumamus *r* negative, ut fiat

$$\Pi : r = \Pi : p + \Pi : q$$

eritque

$$r = \frac{p\sqrt{A\left(A+Cqq+Eq^4\right)}+q\sqrt{A\left(A+Cpp+Ep^4\right)}}{A-Eppqq}$$

unde colligitur fore

$$\sqrt{\frac{\left(A+Crr+Er^4\right)}{A}} = \frac{(A+Eppqq)\sqrt{A\left(A+Cpp+Ep^4\right)\left(A+Cqq+Eq^4\right)}+2AEpq(pp+qq)+Cpq(A+Eppqq)}{(A-Eppqq)^2}.$$

COROLLARIUM 2

614. Quodsi ergo ponamus $q = p$, ut sit

$$\Pi : r = 2\Pi : p$$

erit

$$r = \frac{2p\sqrt{A\left(A+Cpp+Ep^4\right)}}{A-Ep^4}$$

atque

$$\sqrt{\frac{\left(A+Crr+Er^4\right)}{A}} = \frac{AA+2ACpp+6AEp^4+2CEp^6+EEp^8}{\left(A-Ep^4\right)^2}.$$

Hoc igitur modo functio assignari potest aequalis duplo similis functionis.

COROLLARIUM 3

615. Si ponatur

$$q = \frac{2p\sqrt{A\left(A+Cpp+Ep^4\right)}}{A-Ep^4}$$

et

$$\sqrt{A\left(A+Cqq+Eq^4\right)} = \frac{A\left(AA+2ACpp+6AEp^4+2CEp^6+EEp^8\right)}{\left(A-Ep^4\right)^2},$$

ut sit $\Pi : q = 2\Pi : p$, fiet ex primo corollario

$$\Pi : r = 3\Pi : p.$$

Tum igitur erit

$$r=\frac{p\left(3AA+4ACpp+6AEp^4-EEp^8\right)}{AA-6AEp^4-4CEp^6-3EEp^8}.$$

SCHOLION 1

616. Nimis operosum est hanc functionum multiplicationem ulterius continuare multoque minus legem in earum progressione deprehendere licet. Quodsi ponamus brevitatis gratia

$$\sqrt{A\left(A+Cpp+Ep^4\right)}=AP \quad \text{et} \quad A-Ep^4=A\mathfrak{P},$$

ut sit

$$Cpp=APP-A-Ep^4 \quad \text{et} \quad Ep^4=A\left(1-\mathfrak{P}\right),$$

hae multiplicationes usque ad quadruplum ita se habebunt; scilicet si statuamus

$$\varPi:r=2\varPi:p,\ \varPi:s=3\varPi:p \quad \text{et} \quad \varPi:t=4\varPi:p,$$

reperietur

$$r=\frac{2Pp}{\mathfrak{P}},\ \ s=\frac{p\left(4PP-\mathfrak{P}\mathfrak{P}\right)}{\mathfrak{P}\mathfrak{P}-4PP\left(1-\mathfrak{P}\right)},\quad t=\frac{4pP\mathfrak{P}\left(2PP\left(2-\mathfrak{P}\right)-\mathfrak{P}\mathfrak{P}\right)}{\mathfrak{P}^4-16P^4\left(1-\mathfrak{P}\right)}.$$

Quodsi simili modo ponamus

$$\sqrt{A\left(A+Crr+Er^4\right)}=AR \quad \text{et} \quad A-Er^4=A\mathfrak{R},$$

erit

$$R=\frac{2PP\left(2-\mathfrak{P}\right)-\mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P}},\quad \text{et} \quad \mathfrak{R}=\frac{\mathfrak{P}^4-16P^4\left(1-\mathfrak{P}\right)}{\mathfrak{P}^4},$$

unde pro quadruplicatione fit

$$t=\frac{2Rr}{\mathfrak{R}},\quad T=\frac{2RR\left(2-\mathfrak{R}\right)-\mathfrak{R}\mathfrak{R}}{\mathfrak{R}\mathfrak{R}},\quad \mathfrak{T}=\frac{\mathfrak{R}^4-16R^4\left(1-\mathfrak{R}\right)}{\mathfrak{R}^4}.$$

Quare si pro octuplicatione statuamus $\varPi:z=8\varPi:p$, erit

$$z=\frac{2Tt}{\mathfrak{T}}=\frac{4rR\mathfrak{R}\left(2RR\left(2-\mathfrak{R}\right)-\mathfrak{R}\mathfrak{R}\right)}{\mathfrak{R}^4-16R^4\left(1-\mathfrak{R}\right)}.$$

Hinc intelligitur, quomodo in continua duplicatione versari oporteat, neque tamen legem progressionis observare licet. Caeterum cognitio huius legis ad incrementum Analyseos maxime esset optanda, ut inde generatim relatio inter z et p pro aequalitate $\varPi:z=n\varPi:p$ definiri posset, quemadmodum hoc in capite praecedente successit; hinc enim eximias proprietates circa integralia formae $\int \frac{dz}{\sqrt{(A+Czz+Ez^4)}}$ cognoscere liceret, quibus scientia analytica haud mediocriter promoveretur.

SCHOLION 2

617. Modus maxime idoneus in legem progressionis inquirendi videtur, si ternos terminos se ordine excipientes contemplemur hoc modo

$$\Pi : x = (n-1)\Pi : p,\ \ \Pi : y = n\Pi : p,\ \ \Pi : z = (n+1)\Pi : p;$$

ubi cum sit

$$\Pi : x = \Pi : y - \Pi : p \quad \text{et} \quad \Pi : z = \Pi : y + \Pi : p,$$

erit

$$x = \frac{y\sqrt{A(A+Cpp+Ep^4)} - p\sqrt{A(A+Cyy+Ey^4)}}{A-Eppyy},$$

$$z = \frac{y\sqrt{A(A+Cpp+Ep^4)} + p\sqrt{A(A+Cyy+Ey^4)}}{A-Eppyy},$$

unde concludimus

$$(A-Eppyy)(x+z) = 2y\sqrt{A(A+cpp+Ep^4)}.$$

Ponamus ut ante $\sqrt{A(A+cpp+Ep^4)} = AP$ et $A - Ep^4 = A\mathfrak{P}$, et quia singulae quantitates x, y, z factorem p simpliciter involvunt, sit

$$x = pX,\ \ y = pY \quad \text{et} \quad z = pZ;$$

erit

$$(1-(1-\mathfrak{P})YY)(X+Z) = 2PY \quad \text{seu} \quad Z = \frac{2PY}{(1-(1-\mathfrak{P})YY)} - X,$$

cuius formulae ope ex binis terminis contiguis X et Y sequens Z haud difficulter invenitur. Quod quo facilius appareat, ponatur $2P = Q$ et $1-\mathfrak{P} = \mathfrak{Q}$, ut sit $Z = \frac{QY}{1-\mathfrak{Q}YY} - X$. Iam progressio quaesita ita se habebit

$$1)\ 1, \quad 2)\ \frac{Q}{\mathfrak{P}}, \quad 3)\ \frac{QQ-\mathfrak{P}\mathfrak{P}}{\mathfrak{P}\mathfrak{P}-QQ\mathfrak{Q}}, \quad 4)\ \frac{Q^3\mathfrak{P}(1+\mathfrak{Q})-2Q\mathfrak{P}^3}{\mathfrak{P}^4-Q^4\mathfrak{Q}},$$

$$(5\ \ \frac{\mathfrak{P}^6-3QQ\mathfrak{P}^4+Q^4\mathfrak{P}\mathfrak{P}(1+2\mathfrak{Q})-Q^6\mathfrak{Q}\mathfrak{Q}}{\mathfrak{P}^6-3QQ\mathfrak{P}^4\mathfrak{Q}+Q^4\mathfrak{P}\mathfrak{P}\mathfrak{Q}(1+2\mathfrak{Q})-Q^6\mathfrak{Q}} \ \text{etc.}$$

Quaestio ergo huc redit, ut investigetur progressio ex data relatione inter ternos terminos successivos X, Y, Z, quae sit $Z = \frac{QY}{1-\mathfrak{Q}YY} - X$, existente termino primo = 1 et secundo $= \frac{Q}{1-\mathfrak{Q}}$.

PROBLEMA 80

618. *Si* $\Pi : z$ *eiusmodi denotet functionem ipsius z, ut sit*

$$\Pi : z = \int \frac{dz(L + Mzz + Nz^4)}{\sqrt{(A + Czz + Ez^4)}}$$

integrali ita sumto, ut evanescat posito z = 0, *comparationem inter huiusmodi functiones transcendentes investigare.*

SOLUTIO

Stabilita inter binas variabiles *x* et *y* hac relatione, ut sit

$$Ay + \mathfrak{B}x - Ebbxxy = b\sqrt{A(A + Cxx + Ex^4)}$$

seu

$$Ax + \mathfrak{B}y - Ebbxyy = b\sqrt{A(A + Cyy + Ey^4)}$$

sive sublata irrationalitate

$$A(xx + yy - bb) + 2\mathfrak{B}xy - Ebbxxyy = 0$$

existente brevitatis gratia $\mathfrak{B} = \sqrt{A(A + Cbb + Eb^4)}$, erit, uti ante vidimus,

$$\frac{dx}{\sqrt{(A + Cxx + Ex^4)}} + \frac{dy}{\sqrt{(A + Cyy + Ey^4)}} = 0.$$

Ponamus igitur

$$\frac{dx(L + Mxx + Nx^4)}{\sqrt{(A + Cxx + Ex^4)}} + \frac{dy(L + Myy + Ny^4)}{\sqrt{(A + Cyy + Ey^4)}} = bdV\sqrt{A},$$

ut sit nostro signandi more

$$\Pi : x + \Pi : y = \text{Const.} + bV\sqrt{A},$$

ubi constans ita definiri debet, ut posito $x = 0$ fiat $y = b$.

Quaestio ergo ad inventionem functionis *V* revocatur; quem in finem loco *dy* valore ex priori aequatione substituto erit

$$bdV\sqrt{A} = \frac{dx\left(M(xx - yy) + N(x^4 - y^4)\right)}{\sqrt{(A + Cxx + Ex^4)}},$$

verum quia

$$b\sqrt{A\left(A+Cxx+Ex^4\right)}=Ay+\mathfrak{B}x-Ebbxxy\,,$$

habebimus

$$dV=\frac{dx\left(xx-yy\right)\left(M+N\left(xx+yy\right)\right)}{Ay+\mathfrak{B}x-Ebbxxy}\,,$$

Sumamus iam aequationem rationalem

$$A\left(xx+yy-bb\right)+2\mathfrak{B}xy-Ebbxxyy=0$$

et ponamus $xx+yy=tt$ et $xy=u$, ut sit

$$A\left(tt-bb\right)+2\mathfrak{B}u-Ebbuu=0$$

ideoque

$$Atdt=-\mathfrak{B}du+Ebbudu\,.$$

Cum porro sit $xdx+ydy=tdt$ et $xdy+ydx=du$, erit

$$\left(xx-yy\right)dx=xtdt-ydu$$

seu

$$A\left(xx-yy\right)dx=-du\left(Ay+\mathfrak{B}x-Ebbxxy\right),$$

ita ut sit

$$\frac{dx\left(xx-yy\right)}{Ay+\mathfrak{B}x-Ebbxxy}=-\frac{du}{A}\,,$$

ex quo deducitur

$$dV=-\frac{du}{A}\left(M+Ntt\right),$$

et ob $tt=bb-\frac{2\mathfrak{B}u}{A}+\frac{Ebbuu}{A}$ erit

$$dV=-\frac{du}{AA}\left(AM+ANbb-2\mathfrak{B}Nu+ENbbuu\right),$$

unde integrando elicitur

$$V=-\frac{Mu}{A}-\frac{Nbbu}{A}+\frac{\mathfrak{B}Nuu}{AA}-\frac{ENbbu^3}{3AA}$$

Hoc ergo valore substituto ob $u=xy$ habebimus

$$\varPi:x+\varPi:y=\varPi:b-\frac{Mbxy}{\sqrt{A}}-\frac{Nb^3xy}{\sqrt{A}}+\frac{\mathfrak{B}Nx^2y^2}{A\sqrt{A}}-\frac{ENb^3x^3y^3}{3A\sqrt{A}}$$

Cum autem sit

$$\mathfrak{B}xy=\tfrac{1}{2}Abb-\tfrac{1}{2}A\left(xx+yy\right)+\tfrac{1}{2}Ebbxxyy\,,$$

erit

$$\Pi: x + \Pi: y = \Pi: b - \frac{Mbxy}{\sqrt{A}} - \frac{Nbxy}{2A\sqrt{A}}\left(A\left(bb + xx + yy\right) - \tfrac{1}{3}Ebbxxyy\right),$$

cui ergo aequationi satisfit per formulas algebraicas supra exhibitas, quibus relatio inter *x*, *y* et *b* exprimitur. Quodsi ergo statuatur haec aequatio

$$\Pi: p + \Pi: q + \Pi: r = \frac{Mpqr}{\sqrt{A}} + \frac{Npqr}{2A\sqrt{A}}\left(A\left(pp + qq + rr\right) - \tfrac{1}{3}Ebbppqqrr\right),$$

ea efficitur sequenti relatione inter *p*, *q*, *r* constituta

$$\left(A - Eppqq\right)r + p\sqrt{A\left(A + Cqq + Eq^4\right)} + q\sqrt{A\left(A + Cpp + Ep^4\right)} = 0$$

seu

$$\left(A - Epprr\right)q + p\sqrt{A\left(A + Crr + Er^4\right)} + r\sqrt{A\left(A + Cpp + Ep^4\right)} = 0$$

seu

$$\left(A - Eqqrr\right)p + q\sqrt{A\left(A + Crr + Er^4\right)} + r\sqrt{A\left(A + Cqq + Eq^4\right)} = 0,$$

sive per simplicem irrationalitatem

$$A\left(pp + qq - rr\right) + 2pq\sqrt{A\left(A + Crr + Er^4\right)} - Eppqqrr = 0$$

seu

$$A\left(pp + rr - qq\right) + 2pr\sqrt{A\left(A + Cqq + Eq^4\right)} - Eppqqrr = 0$$

seu

$$A\left(qq + rr - pp\right) + 2qr\sqrt{A\left(A + Cpp + Ep^4\right)} - Eppqqrr = 0$$

penitusque irrationalitate sublata

$$EEp^4q^4r^4 - 2AEppqqrr\left(pp + qq + rr\right) - 4ACppqqrr$$
$$+ AA\left(p^4 + q^4 + r^4 - 2ppqq - 2pprr - 2qqrr\right) = 0.$$

COROLLARIUM 1

619. Sit $q = r = s$, ut habeamus hanc aequationem

$$\Pi: p + 2\Pi: s = \frac{Mpss}{\sqrt{A}} + \frac{Npss}{2A\sqrt{A}}\left(A\left(pp + 2ss\right) - \tfrac{1}{3}Epps^4\right),$$

cui satisfacit haec relatio

$$\left(A-Es^4\right)p+2s\sqrt{A\left(A+Css+Es^4\right)}=0.$$

COROLLARY 2

620. Sumamus s negative et loco p substituamus ibi hunc valorem, ut habeamus

$$2\Pi:s+\Pi:q+\Pi:r+\tfrac{Mpss}{\sqrt{A}}+\tfrac{Npss}{2A\sqrt{A}}\left(A\left(pp+2ss\right)-\tfrac{1}{3}Epps^4\right)$$
$$=\tfrac{Mpqr}{\sqrt{A}}+\tfrac{Npqr}{2A\sqrt{A}}\left(A\left(pp+qq+rr\right)-\tfrac{1}{3}Eppqqrr\right)$$

existente

$$p=\frac{2s\sqrt{A\left(A+Css+Es^4\right)}}{A-Es^4},$$

unde fit [§ 615]

$$\sqrt{A\left(A+Cpp+Ep^4\right)}=\frac{A\left(A+Css+Es^4\right)^2+A\left(4AE-CC\right)s^4}{\left(A-Es^4\right)^2},$$

qui valores in superioribus formulis substitui debent.

COROLLARIUM 3

621. Hoc modo effici poterit, ut partes algebraicae evanescant atque functiones transcendentes solae inter se comparentur. Veluti si esset $N=0$, statui oporteret $ss=qr$, ut fieret

$$2\Pi:s+\Pi:q+\Pi:r=0.$$

At posito $ss=qr$ fit

$$p=\frac{2\sqrt{Aqr\left(A+Cqr+Eqqrr\right)}}{A-Eqqrr}.$$

Est vero etiam

$$p=\frac{-q\sqrt{A\left(A+Crr+Er^4\right)}-r\sqrt{A\left(A+Cqq+Eq^4\right)}}{A-Eqqrr}$$

quibus valoribus aequatis oritur haec aequatio

$$\left(AA+EEq^4r^4\right)\left(qq-6qr+rr\right)-8Cqqrr\left(A+Eqqrr\right)$$
$$-2AEqqrr\left(qq+10qr+rr\right)=0.$$

SCHOLION

622. Si $\Pi:z$ exprimat arcum cuiuspiam lineae curvae respondentem abscissae vel cordae *z*, hinc pIures arcus eiusdem curvae inter se comparare licet, ut vel differentia binorum arcuum fiat algebraica vel arcus exhibeantur datam rationem inter se tenentes. Hoc modo eiusmodi insignes curvarum proprietates eruuntur, quarum ratio aliunde vix perspici queat. Comparatio quidem arcuum circularium ex elementis nota per caput praecedens, ut vidimus, facile expeditur, unde etiam comparatio arcuum parabolicorum derivatur. Ex hoc autem capite comparatio arcuum ellipticorum et hyperbolicorum simili modo institui potest; cum enim in genere arcus sectionis conicae tali formula exprimatur

$$\int dx\sqrt{\frac{a+bxx}{c+exx}},$$

haec transformata in istam

$$\int \frac{dx(a+bxx)}{\sqrt{(ac+(ae+bc)xx+bex^4)}}$$

per praecepta tradita tractari potest ponendo $A=ac$, $C=ae+bc$, $E=be$ et $L=a$, $M=b$ atque $N=0$. Haec autem investigatio ad formulas, quarum denominator est $\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}$, extendi potest similisque est praecedenti, quam idcirco hic sum expositurus, unde simul patebit hunc esse ultimum terminum, quousque progredi liceat. Formulae enim integrales magis complicatae, ubi post signum radicale altiores potestates ipsius *z* occurrunt vel ipsum signum radicale altiorem dignitatem involvit, hoc modo non videntur inter se eomparari posse paucissimis casibus exceptis, qui per quampiam substitutionem ad huiusmodi formam reduci queant.

PROBLEMA 81

623. *Si* $\Pi : z$ *eiusmodi functionem ipsius z denotet, ut sit*

$$\Pi : z = \int \frac{dz}{\sqrt{(A+2Bz+Czz+2Dz^3+Ez^4)}}$$

huiusmodi functiones inter se comparare.

SOLUTIO

Inter binas variabiles *x* et *y* statuatur relatio hac aequatione expressa

$$\alpha+2\beta(x+y)+\gamma(xx+yy)+2\delta xy+2\varepsilon xy(x+y)+\zeta xxyy=0;$$

unde cum fiat

$$yy=\frac{-2y(\beta+\delta x+\varepsilon xx)-\alpha-2\beta x-\gamma xx}{\gamma+2\varepsilon x+\zeta xx},$$

erit radice extracta

$$y=\frac{-\beta-\delta x-\varepsilon xx+\sqrt{((\beta+\delta x+\varepsilon xx)^2-(\alpha+2\beta x+\gamma xx)(\gamma+2\varepsilon x+\zeta xx))}}{\gamma+2\varepsilon x+\zeta xx}.$$

Reducatur signum radicale ad formam propositam ponendo

$$\beta\beta-\alpha\gamma=Am,\quad \beta\delta-\alpha\varepsilon-\beta\gamma=Bm,$$
$$\delta\delta-2\beta\varepsilon-\alpha\zeta-\gamma\gamma=Cm,\quad \delta\varepsilon-\beta\zeta-\gamma\varepsilon=Dm,$$
$$\varepsilon\varepsilon-\gamma\zeta=Em,$$

unde ex sex coefficientibus $\alpha,\ \beta,\ \gamma,\delta,\ \varepsilon,\zeta$ quinque definiuntur, atque ad sextum insuper accedit littera *m*, ita ut aequatio assumta adhuc constantem arbitrariam involvat. Inde ergo, si brevitatis gratia ponamus

$$\sqrt{\left(A+2Bx+Cxx+2Dx^3+Ex^4\right)}=X$$

et

$$\sqrt{\left(A+2By+Cyy+2Dy^3+Ey^4\right)}=Y,$$

habebimus

$$\beta+\gamma y+\delta x+\varepsilon xx+2\varepsilon xy+\zeta xxy=X\sqrt{m}$$

et

$$\beta+\gamma x+\delta y+\varepsilon yy+2\varepsilon xy+\zeta xyy=Y\sqrt{m}.$$

At aequatio assumta per differentiationem dat

$$dx\left(\beta+\gamma x+\delta y+2\varepsilon xy+\varepsilon yy+\zeta xyy\right)+$$
$$dy\left(\beta+\gamma y+\delta x+\varepsilon xx+2\varepsilon xy+\zeta xxy\right)=0;$$

quae expressiones quia cum superioribus conveniunt, dant

$$Ydx\sqrt{m}+Xdy\sqrt{m}=0\ \text{ seu }\ \frac{dx}{X}+\frac{dy}{Y}=0,$$

unde integrando colligimus

$$\Pi:x+\Pi:y=\text{Const.},$$

quae constans, si posito $x=0$ fiat $y=b$, erit $=\Pi:0+\Pi:b$, vel in genere si posito $x=a$ fiat $y=b$, ea erit $\Pi:a+\Pi:b$. Quodsi ergo litterae $\alpha,\ \beta,\ \gamma,\delta,\ \varepsilon,\zeta$ per conditiones superiores definiantur, aequatio assumta algebraica inter *x* et *y* erit integrale completum huius aequationis differentialis

$$\frac{dx}{\sqrt{\left(A+2Bx+Cxx+2Dx^3+Ex^4\right)}}+\frac{dy}{\sqrt{\left(A+2By+Cyy+2Dy^3+Ey^4\right)}}=0.$$

COROLLARIUM 1

624. Ad has litteras $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ definiendas sumantur primo aequationes binae ad dextram positae, quae sunt

$$(\delta-\gamma)\beta-\alpha\varepsilon=Bm \quad \text{et} \quad (\delta-\gamma)\varepsilon-\zeta\beta=Dm,$$

unde quaerantur binae β et ε, reperieturque

$$\beta=\frac{(\delta-\gamma)\mathrm{B}+\alpha\mathrm{D}}{(\delta-\gamma)^2-\alpha\zeta}m \quad \text{et} \quad \varepsilon=\frac{(\delta-\gamma)\mathrm{D}+\zeta\mathrm{B}}{(\delta-\gamma)^2-\alpha\zeta}m.$$

COROLLARIUM 2

625. Sit brevitatis gratia $\delta-\gamma=\lambda$ seu $\delta=\gamma+\lambda$; erit

$$\beta=\frac{\mathrm{D}\alpha+\mathrm{B}\lambda}{\lambda\lambda-\alpha\zeta}m \quad \text{et} \quad \varepsilon=\frac{\mathrm{B}\zeta+\mathrm{D}\lambda}{\lambda\lambda-\alpha\zeta}m.$$

Iam ex conditione prima et ultima oritur

$$\beta\beta\zeta-\alpha\varepsilon\varepsilon=(A\zeta-E\alpha)m,$$

ubi illi valores substituti praebent $\frac{\beta\beta\zeta-DD\alpha}{\lambda\lambda-\alpha\zeta}m=A\zeta-E\alpha$, unde fit

$$m=\frac{(\lambda\lambda-\alpha\zeta)(A\zeta-E\alpha)}{\beta\beta\zeta-DD\alpha}$$

At ex prima et ultima sequitur

$$DD\beta-BB\varepsilon\varepsilon+\gamma(\mathrm{BB}\zeta-DD\alpha)=(ADD-BBE)m,$$

unde colligitur

$$\gamma=\frac{(A\zeta-Ea)((ADD-BBE)\lambda\lambda+2BD(A\zeta-Ea)\lambda+ABB\zeta\zeta-DDE\alpha\alpha)}{(BB\zeta-DD\alpha)^2}.$$

COROLLARIUM 3

626. Superest tertia aequatio

$$2\gamma\lambda+\lambda\lambda-2\beta\varepsilon-\alpha\zeta=Cm,$$

et cum pro *m* substituto valore sit

$$\beta = \frac{(A\zeta - E\alpha)(D\alpha + B\lambda)}{BB\zeta - DD\alpha} \quad \text{et} \quad \varepsilon = \frac{(A\zeta - E\alpha)(B\zeta + D\lambda)}{BB\zeta - DD\alpha},$$

si isti valores substituantur, commode inde colligitur

$$\lambda = \frac{C(A\zeta - E\alpha)(BB\zeta - DD\alpha) - 2BD(A\zeta - E\alpha)^2 - (BB\zeta - DD\alpha)^2}{2(A\zeta - E\alpha)(ADD - BBE)}.$$

SCHOLION 1

627. Quia his valoribus uti non licet, quoties fuerit $ADD - BBE = 0$, aliam resolutionem huic incommodo non obnoxiam tradam.

Posito $\delta = \gamma + \lambda$ sit insuper $\lambda\lambda = \alpha\zeta + \mu$, ut primae formulae fiant

$$\beta = \frac{m}{\mu}(D\alpha + B\lambda) \quad \text{et} \quad \varepsilon = \frac{m}{\mu}(B\zeta + D\lambda).$$

Iam prima et ultima iunctis prodit

$$A\zeta - E\alpha = m(BB\zeta - DD\alpha),$$

qua aequatione ratio inter α et ζ definitur; quae cum sufficiat, erit

$$\alpha = \mu A - BBm \quad \text{et} \quad \zeta = \mu E - DDm$$

hincque

$$\lambda\lambda = \mu + (\mu A - BBm)(\mu E - DDm),$$

unde colligimus

$$\gamma = \frac{mm}{\mu\mu}\left(2BD\lambda + (ADD + BBE)\mu\right) - \frac{2BBDDm^3}{\mu\mu} - \frac{m}{\mu}.$$

Valores α et ζ in formula Corollarii 3 substituti dant

$$\lambda = \frac{\mu\mu}{2m} + BDm - \frac{1}{2}C\mu,$$

cuius quadratum illi valori $\alpha\zeta + \mu$ aequatum perducit ad hanc aequationem

$$\mu(\mu - Cm)^2 + 4(BD - AE)mm\mu + 4(ADD - BCD + BBE)m^3 = 4mm;$$

ad quam resolvendam ponatur $\mu = Mm$ fietque

$$m = \frac{4}{M(M-C)^2 + 4M(BD-AE) + 4(ADD-BCD+BBE)}$$

atque hic est M constans ilIa arbitraria pro integrali completo requisita. Hoc modo omnes litterae $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ eodem denominatore affecti prodibunt, quo omisso habebimus

$$\alpha = 4(AM - BB), \quad \beta = 2B(M-C) + 4AD, \quad \gamma = 4AE - (M-C)^2,$$
$$\zeta = 4(\mathrm{EM} - \mathrm{DD}), \quad \varepsilon = 2D(M-C) + 4BE, \quad \delta = MM - CC + 4(AE + BD),$$

quibus inventis aequatio nostra canonica

$$0 = \alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy,$$

si brevitatis gratia ponamus

$$M(M-\mathrm{C})^2 + 4M(BD-AE) + 4(ADD-BCD+BBE) = \Delta,,$$

resoluta dabit

$$\beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx) = \pm 2\sqrt{\Delta(A + 2Bx + Cxx + 2Dx^3 + Ex^4)},$$
$$\beta + \delta y + \varepsilon yy + x(\gamma + 2\varepsilon y + \zeta yy) = \pm 2\sqrt{\Delta(A + 2By + Cyy + 2Dy^3 + Ey^4)},$$

quae ergo est integrale completum huius aequationis differentialis

$$0 = \frac{dx}{\pm\sqrt{\Delta(A+2Bx+Cxx+2Dx^3+Ex^4)}} + \frac{dy}{\pm\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}}$$

SCHOLION 2

628. Cum hic ab idonea coefficientium determinatione totum negotium pendeat, operae pretium erit eam luculentius exponere. Posito igitur statim $\delta = \gamma + \lambda$ et $\lambda\lambda - \alpha\zeta = Mm$ quinque conditiones adimplendae sunt

$$\text{1.}\beta\beta - a\gamma = Am, \quad \text{II.}\varepsilon\varepsilon - \gamma\zeta = Em,$$
$$\text{III.}\beta\lambda - \alpha\varepsilon = Bm, \quad \text{IV.}\varepsilon\lambda - \beta\zeta = Dm,$$
$$\text{V.}Mm + 2\gamma\lambda - 2\beta\varepsilon = Cm.$$

Hinc ex tertia et quarta combinando deducitur

$$m(B\lambda + D\alpha) = \beta(\lambda\lambda - \alpha\zeta) = \beta Mm, \quad \text{hence} \quad \beta = \frac{B\lambda + Da}{M},$$
$$m(D\lambda + B\zeta) = \varepsilon(\lambda\lambda - \alpha\zeta) = \varepsilon Mm, \quad \text{hence} \quad \varepsilon = \frac{D\lambda + B\zeta}{M}.$$

Iam ex prima et secunda elidendo γ oritur

$$m(A\zeta - Ea) = \beta\beta\zeta - \varepsilon\varepsilon\alpha = \tfrac{\beta\beta\zeta - DD\alpha}{M}m$$

hincque

$$\zeta(\mathrm{AM} - BB) = \alpha(EM - DD),$$

quare statuatur

$$\alpha = n(AM - BB) \quad \text{et} \quad \zeta = n(EM - DD).$$

Tum vero indidem est

$$E\beta\beta - E\alpha\gamma = A\varepsilon\varepsilon - A\gamma\alpha \quad \text{seu} \quad \gamma(A\zeta - E\alpha) = A\varepsilon\varepsilon - E\beta\beta\,;$$

pro qua tractanda cum sit pro α et ζ substitutis valoribus

$$\beta = nAD + \tfrac{B}{M}(\lambda - nBD) \text{ and } \varepsilon = nBE + \tfrac{D}{M}(\lambda - nBD),$$

sit brevitatis ergo $\lambda - nBD = nMN$, ut habeamus

$$\beta = n(AD + BN) \text{ et } \varepsilon = n(BE + DN),$$

et quia

$$A\zeta - E\beta\beta = n(BBE - ADD)$$

atque

$$A\varepsilon\varepsilon - E\beta\beta = nn(ABBEE + ADDNN - AADDE - BBENN)$$

seu

$$A\varepsilon\varepsilon - E\beta\beta = nn(BBE - ADD)(AE - NN),$$

fiet

$$\gamma = n(AE - NN).$$

Cum autem sit

$$\lambda = n(BD + MN) \text{ et } \lambda\lambda = nn(AM - BB)(EM - DD) + Mm,$$

erit

$$Mm = nn(2BDMN + MMNN - AEMM + M(ADD + BBE))$$

seu

$$m = nn(2BDN + MNN - AEM + ADD + BBE).$$

Denique aequatio quinta $\beta\varepsilon - \gamma\lambda = \tfrac{1}{2}m(M - C)$ evoluta praebet

$$\beta\varepsilon - \gamma\lambda = nn(AD + BN)(BE + DN) - (AE - NN)(BD + MN))$$
$$= nnN(2BDN + MNN - AEM + ADD + BBE) = Nm,$$

unde fit $N = \frac{1}{2}(M - C)$, ac propterea

$$m = nn(BD(M - C) + \tfrac{1}{4}M(M - C)^2 - AEM + ADD + BBE).$$

Hincque sumendo $n = 4$ superiores valores obtinentur.

EXEMPLUM 1

629. *Invenire integrale completum huius aequationis differentialis*

$$\frac{dp}{\pm\sqrt{(a+bp)}} + \frac{dq}{\pm\sqrt{(a+bq)}} = 0.$$

Hic est $x = p,\ y = q,\ A = a,\ B = \frac{1}{2}b,\ C = 0, D = 0, E = 0,$ unde fiunt coefficientes

$$\alpha = 4aM - bb,\ \beta = bM,\ \gamma = -MM,\ \zeta = 0, \varepsilon = 0, \delta = MM$$

et

$$\varDelta = M^3$$

unde integrale completum erit

$$bM + MMp - MMq = \pm 2M\sqrt{M(a + bp)}$$

seu

$$b + M(p - q) = \pm 2\sqrt{M(a + bp)} \quad \text{vel} \quad b + M(q - p) = \pm 2\sqrt{M(a + bq)},$$

quae signa ambigua radicalium cum signis in aequatione differentiali convenire debent.

EXEMPLUM 2

630. *Invenire integrale completum huius aequationis differentialis*

$$\frac{dp}{\pm\sqrt{(a+bp^2)}} + \frac{dq}{\pm\sqrt{(a+bq^2)}} = 0.$$

Sumto $x = p$ et $y = q$ erit $A = a,\ B = 0, C = b,\ D = 0, E = 0$, ergo

$$\alpha = 4aM, \quad \beta = 0, \quad \gamma = -(M - b)^2, \quad \zeta = 0,\ \varepsilon = 0, \delta = MM - bb$$

atque

$$\varDelta = M(M - b)^2$$

unde integrale completum in his aequationibus continebitur

$$(MM-bb)p-(M-b)^2q=\pm 2(M-b)\sqrt{M\left(a+bpp\right)}$$

seu

$$(M+b)p-(M-b)q=\pm 2\sqrt{M\left(a+bpp\right)}$$

et

$$(M+b)q-(M-b)p=\pm 2\sqrt{M\left(a+bqq\right)}.$$

EXEMPLUM 3

631. *Invenire integrale completum huius aequationis differentialis*

$$\frac{dp}{\pm\sqrt{\left(a+bp^3\right)}}+\frac{dq}{\pm\sqrt{\left(a+bq^3\right)}}=0.$$

Sumto $x=p,\ y=q$ erit $A=a,\ B=0,\ C=0,\ D=\frac{1}{2}b,\ E=0$, ergo
$\alpha=4aM$, $\beta=2ab$, $\gamma=-MM$, $\zeta=-bb, B=bM$, $\delta=MM$
et

$$\varDelta=M^3+abb,$$

unde integrale completum

$$2ab+MMp+bMpp+q(-MM+2bMp-bbpp)=\pm 2\sqrt{\left(M^3+abb\right)\left(a+bp^3\right)}$$

sive

$$2ab+Mp(M+bp)-q\left(M-bp\right)^2=\pm 2\sqrt{\left(M^3+abb\right)\left(a+bp^3\right)}$$

et

$$2ab+Mq\left(M+bq\right)-p\left(M-bq\right)^2=\pm 2\sqrt{\left(M^3+abb\right)\left(a+bq^3\right)}.$$

EXEMPLUM 4

632. *Invenire integrale completum huius aequationis differentialis*

$$\frac{dp}{\pm\sqrt{\left(a+bp^4\right)}}+\frac{dq}{\pm\sqrt{\left(a+bq^4\right)}}=0.$$

Posito $x=p,\ y=q$ erit $A=a,\ B=0,\ C=0,\ D=0,\ E=b$, ergo

$$\alpha=4aM,\ \beta=0, \gamma=4ab-MM, \zeta=4bM,\ \varepsilon=0, \delta=MM+4ab$$

et

$$\varDelta=M^3-4abM,$$

unde integrale completum

$$(MM+4ab)p+q(4ab-MM+4bMpp)=\pm 2\sqrt{M(MM-4ab)(a+bp^4)},$$

$$(MM+4ab))q+p((4ab-MM+4bMqq)=\pm 2\sqrt{M(MM-4ab)(a+bq^4)}.$$

EXEMPLUM 5

633. *Invenire integrale completum huius aequationis differentialis*

$$\frac{dp}{\pm\sqrt{(a+bp^6)}}+\frac{dq}{\pm\sqrt{(a+bq^6)}}=0.$$

Ponatur $x=pp$ et $y=qq$ atque aequatio nostra generalis induet posito $A=0$ hanc formam

$$\frac{dp}{\pm\sqrt{(2B+Cpp+2Dp^4+Ep^6)}}+\frac{dq}{\pm\sqrt{(2B+Cqq+2Dq^4+Eq^6)}}=0.$$

Fieri ergo oportet $B=\frac{1}{2}a$, $C=0, D=0$ et $E=b$, unde coefficientes ita determinantur

$$\alpha=-aa,\quad \beta=aM,\quad \gamma=-MM,\quad \zeta=4bM,\quad \varepsilon=2ab,\quad \delta=MM$$

et

$$\varDelta=M^3+aab,$$

ergo integrale completum

$$aM+MMpp+2abp^4+qq\left(-MM+4abpp+4bMp^4\right)$$
$$=\pm 2p\sqrt{\left(M^3+aab\right)\left(a+bp^6\right)}$$

sive

$$aM+MMqq+2abq^4+pp\left(-MM+4abqq+4bMq^4\right)$$
$$=\pm 2q\sqrt{\left(M^3+aab\right)\left(a+bq^6\right)}$$

COROLLARIUM

634. Si sumatur constans $M=-\sqrt[3]{aab}$, ut sit $\mathrm{M}^3+aab=0$, prodibit integrale particulare, quod ita se habebit

$$pp=\frac{qq\sqrt[3]{b}+\sqrt[3]{a}}{2qq\sqrt[3]{b}-\sqrt[3]{a}}\cdot\sqrt[3]{\frac{a}{b}}\quad \text{seu}\quad qq=\frac{pp\sqrt[3]{b}+\sqrt[3]{a}}{2pp\sqrt[3]{b}-\sqrt[3]{a}}\cdot\sqrt[3]{\frac{a}{b}}.$$

quod aequationi differentiali utique satisfacit.

PROBLEMA 82

635. *Proposita hac aequatione differentiali*

$$\frac{dp}{\pm\sqrt{\left(a+bpp+cp^4+ep^6\right)}}+\frac{dq}{\pm\sqrt{\left(a+bqq+cq^4+eq^6\right)}}=0$$

eius integrale completum algebraice assignare.

SOLUTIO

Aequatio praecedens differentialis algebraice integrata ad hanc formam reducitur ponendo $x=pp$ et $y=qq$ atque $A=0$; prodibit enim

$$\frac{dp}{\pm\sqrt{\left(2B+Cpp+2Dp^4+Ep^6\right)}}+\frac{dq}{\pm\sqrt{\left(2B+Cqq+2Dq^4+Eq^6\right)}}=0.$$

Quare tantum opus est, ut fiat

$$A=0,\ B=\tfrac{1}{2}a,\ C=b,\ D=\tfrac{1}{2}c,\ E=e,$$

unde coefficientes $\alpha,\ \beta,\ \gamma,\delta,\ \varepsilon,\zeta$ ita definientur

$$\alpha=-aa,\quad \beta=a\left(M-b\right),\ \gamma=-\left(M-b\right)^2,$$
$$\zeta=4eM-cc,\quad \varepsilon=c\left(M-b\right)+2ae,\quad \delta=MM-bb+ac,$$
$$\varDelta=M(M-b)^2+acM-abc+aae=(M-\mathrm{b})^3+b\left(M-b\right)^2+ac\left(M-b\right)+aae,$$

hincque integrale completum ob constantem M ab arbitrio nostro pendentem erit

$$\beta+\delta pp+\varepsilon p^4+qq(\gamma+2\varepsilon pp+\zeta p^4)=\pm 2p\sqrt{\varDelta(a+bpp+cp^4+ep^6)},$$
$$\beta+\delta qq+\varepsilon q^4+pp(\gamma+2\varepsilon qq+\zeta q^4)=\pm 2q\sqrt{\varDelta(a+bqq+cq^4+eq^6)},$$

quae binae quidem aequationes inter se conveniunt, sed ob ambiguitatem signorum in ipsa aequatione differentiali ambae notari debent ambiguitate inde sublata. Utrinque autem haec aequatio rationalis resultat

$$0=\alpha+2\beta(pp+qq)+\gamma(p^4+q^4)+2\delta ppqq+2\varepsilon ppqq(pp+qq)+\zeta p^4q^4.$$

COROLLARIUM 1

636. Si constans M ita sumatur, ut fiat $\varDelta=0$, obtinetur integrale particulare huius formae

$$qq=\frac{E+Fpp}{G+Hpp},$$

quod etiam a posteriori cognoscere licet. Ut enim satisfaciat, sumi debet

$$aG^3 + bEGG + cEEG + eE^3 = 0,$$

unde ratio $E : G$ definitur; tum vero invenitur $F = -G$ et denique

$$H = \frac{-cEG-2eEE}{aG} = \frac{2aGG+2bEG+cEE}{aE}.$$

COROLLARIUM 2

637. Constans M ita mutetur, ut sit $M - b = \frac{a}{ff}$, fietque

$$\alpha = -aa, \quad \beta = \frac{aa}{ff}, \quad \gamma = \frac{aa}{f^4},$$

$$\zeta = 4be - cc + \frac{4ae}{ff}, \quad \varepsilon = \frac{ac}{ff} + 2ae, \quad \delta = \frac{aa}{f^4} + \frac{2ab}{ff} + ac$$

et

$$\Delta = \frac{aa}{f^6}\left(a + bff + cf^4 + ef^6\right)$$

et aequatio integralis erit

$$\begin{aligned} &aaff + a\left(a + 2bff + cf^4\right)pp + aff\left(c + 2eff\right)p^4 \\ &- qq\left(aa - 2aff\left(c + 2eff\right)pp + ff\left(ccff - 4beff - 4ae\right)p^4\right) \\ &= \pm 2afp\sqrt{\left(a + bff + cf^4 + ef^6\right)\left(a + bpp + cp^4 + ep^6\right)}, \end{aligned}$$

unde patet posito $p = 0$ fore $qq = ff$.

COROLLARIUM 3

638. Haec aequatio facile in hanc formam transmutatur

$$\begin{aligned} &aff\left(a + bpp + Cp^4 + ep^6\right) + app\left(a + bff + cf^4 + ef^6\right) \\ &- qq\left(a - cffpp\right)^2 - aeffpp\left(ff - pp\right)^2 + 4effppqq\left(aff + app + bffpp\right) \\ &= \pm 2fp\sqrt{a\left(a + bff + cf^4 + ef^6\right)a\left(a + bpp + cp^4 + ep^6\right)}, \end{aligned}$$

unde statim patet, si sit $e=0$, fore hanc aequationem radicem extrahendo

$$f\sqrt{a\left(a+bpp+cp^4\right)}\mp p\sqrt{a\left(a+bff+cf^4\right)}=q\left(a-cffpp\right),$$

quae est integralis completa huius differentialis

$$\frac{dp}{\pm\sqrt{\left(a+bpp+cp^4\right)}}+\frac{dq}{\pm\sqrt{\left(a+bqq+cq^4\right)}}=0,$$

prorsus ut supra [§ 607] iam invenimus.

COROLLARIUM 4

639. Simili modo patet in genere, quando *e* non evanescit, integrale completum ita commodius exprimi posse

$$\left(f\sqrt{a\left(a+bpp+Cp^4+ep^6\right)}\mp p\sqrt{a\left(a+bff+cf^4+ef^6\right)}\right)^2$$
$$=qq\left(a-cffpp\right)^2+aeffpp\left(ff-pp\right)^2-4effppqq\left(aff+app+bffpp\right),$$

quae ergo, cum posito $p=0$ fiat $q=f$, respondet huic functionum transcendentium relationi

$$\pm\,\Pi:p\pm\,\Pi:q=\pm\,\Pi:0\pm\,\Pi:f.$$

SCHOLION 1

640. Genera igitur functionum transcendentium, quas hoc modo perinde atque arcus circulares inter se comparare licet, in his binis formulis integralibus continentur

$$\int\frac{dz}{\sqrt{\left(A+2Bz+Czz+2Dz^3+Ez^4\right)}}\quad\text{et}\quad\int\frac{dz}{\sqrt{\left(a+bzz+cz^4+ez^6\right)}}$$

neque haec methodus ad alias formas magis complexas extendi posse videtur. Neque etiam posterior in denominatore potestates impares ipsius *z* admittit, nisi forte simplex substitutio reductioni ad illam formam

$$\int\frac{dz}{\sqrt{\left(A+2Bz+Czz+2Dz^3+Ez^4+2Fz^5+Gz^6\right)}}$$

sufficiat. Facile autem patet huiusmodi formam hac methodo tractari certe non posse. Si enim coefficientes ita essent comparati, ut radicis extractio succederet, talis formula

$$\int \frac{dz}{a+bz+czz+ez^3}$$

prodiret; cuius integratio cum tam logarithmos quam arcus circulares involvat, fieri omnino nequit, ut plures huiusmodi functiones algebraice inter se comparentur. Caeterum prior formula latius patet quam posterior, cum haec ex illa nascatur posito $\mathrm{A}=0$, si zz loco z scribatur. De priori autem notari meretur, quod eandem formam servet, etiamsi transformetur hac substitutione

$$z=\frac{a+\beta y}{\gamma+\delta y};$$

prodit enim

$$\int \frac{(\beta\gamma-\alpha\delta)dy}{\sqrt{\left(A(\gamma+\delta y)^4+2B(\alpha+\beta y)(\gamma+\delta y)^3+C(\alpha+\beta y)^2(\gamma+\delta y)^2+2D(\alpha+\beta y)^3(\gamma+\delta y)+E(\alpha+\beta y)^4\right)}},$$

ex quo intelligitur quantitates $\alpha,\ \beta,\ \gamma,\ \delta$ ita accipi posse, ut potestates impares evanescant. Vel etiam ita definiri poterunt, ut terminus primus et ultimus evanescat; tum enim posito $y=uu$ iterum forma a potestatibus imparibus immunis nascitur.

SCHOLION 2

641. Sublatio autem potestatum imparium ita commodissime instituitur.
Cum formula

$$A+2Bz+Czz+2Dz^3+Ez^4$$

certe semper habeat duos factores reales, ita exhibeatur formula integralis

$$\int \frac{dz}{\sqrt{(a+2bz+czz)(f+2gz+hzz)}},$$

quae posito $z=\frac{a+\beta y}{\gamma+\delta y}$ abit in

$$\int \frac{(\beta\gamma-\alpha\delta)dy}{\sqrt{\left(a(\gamma+\delta y)^4+2b(\alpha+\beta y)(\gamma+\delta y)+c(\alpha+\beta y)^2\right)\left(f(\gamma+\delta y)^2+2g(\alpha+\beta y)(\gamma+\delta y)+E(\alpha+\beta y)^2\right)}}$$

ubi denominatoris factores evoluti sunt

$$(a\gamma\gamma+2b\alpha\gamma+c\alpha\alpha)+2(a\gamma\delta+b\alpha\delta+b\beta\gamma+c\alpha\beta)y+(a\delta\delta+2b\beta\delta+c\beta\beta)yy,$$
$$(f\gamma\gamma+2g\alpha\gamma+h\alpha\alpha)+2(f\gamma\delta+g\alpha\delta+g\beta\gamma+h\alpha\beta)y+(f\delta\delta+2g\beta\delta+h\beta\beta)yy;$$

quodsi iam utroque terminus medius evanescens reddatur, fit

$$\frac{\delta}{\beta}=\frac{-b\gamma-c\alpha}{a\gamma+b\alpha}=\frac{-g\gamma-h\alpha}{f\gamma+g\alpha}$$

hincque

$$bf\gamma\gamma+(bg+cf)\alpha\gamma+cg\alpha\alpha=ag\gamma\gamma+(ah+bg)\alpha\gamma+bh\alpha\alpha$$

seu

$$\gamma\gamma=\frac{(ah-cf)\alpha\gamma+(bh-cg)\alpha\alpha}{bf-ag},$$

unde fit

$$\frac{\gamma}{\alpha}=\frac{ah-cf+\sqrt{\left((ah-cf)^2+4(bf-ag)(bh-cg)\right)}}{2(bf-ag)}..$$

Hinc sufficere posset eas tantum formulas, in quibus potestates impares desunt, tractasse, id quod initio huius capitis fecimus, sed si insuper numerator accedat, haec reductio non amplius locum habet.

PROBLEMA 83

642. *Denotante n numerum integrum quemcunque invenire integrale completum algebraice expressum huius aequationis differentialis*

$$\frac{dy}{\sqrt{\left(A+2By+Cyy+2Dy^3+Ey^4\right)}}=\frac{ndx}{\sqrt{\left(A+2Bx+Cxx+2Dx^3+Ex^4\right)}}.$$

SOLUTIO

Per functiones transcendentes integrale completum est

$$\varPi:y=n\varPi:x+\text{Const.}$$

At ut idem algebraice expressum eruamus, posito $M-C=L$ sit per formulas supra (§ 627) inventas

$$\alpha=4(AC-BB+AL),\ \beta=4AD+2BL,\quad \gamma=4AE-LL,$$
$$\zeta=4(CE-DD+EL),\quad \varepsilon=4BE+2DL,\quad \delta=4AE+4BD+2CL+LL$$

et

$$\varDelta=L^3+CLL+4(BD-AE)L+4(ADD+BBE-ACE).$$

Quibus positis si fuerit

$$\beta+\delta p+\varepsilon pp+q(\gamma+2\varepsilon p+\zeta pp)=2\sqrt{\Delta\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)},$$
$$\beta+\delta q+\varepsilon qq+p(\gamma+2\varepsilon q+\zeta qq)=-2\sqrt{\Delta\left(A+2Bq+Cqq+2Dq^3+Eq^4\right)},$$

erit $\Pi:q=\Pi:p+\text{Const}$.

Cum autem hae duae aequationes inter se conveniant et in hac rationali contineantur

$$\alpha+2\beta(p+q)+\gamma(pp+qq)+2\delta pq+2\varepsilon pq(p+q)+\zeta ppqq=0,$$

si sumamus posito $p=a$ fieri $q=b$, constans ilIa L ita definiri debet, ut sit

$$\alpha+2\beta(a+b)+\gamma(aa+bb)+2\delta ab+2\varepsilon ab(a+b)+\zeta aabb=0,$$

eritque

$$\Pi:q=\Pi:p+\Pi:b-\Pi:a,$$

ubi iam nullum inest discrimen inter constantes et variabiles. Ponamus ergo $b=p$, ut sit

$$\Pi:q=2\Pi:p-\Pi:a,$$

atque huic aequationi superiores aequationes algebraicae conveniunt, si modo quantitas L ita definiatur, ut sit

$$\alpha+2\beta(a+p)+\gamma(aa+pp)+2\delta ap+2\varepsilon ap(a+p)+\zeta aapp=0,$$

unde deducitur

$$\tfrac{1}{2}L(p-a)^2=A+B(a+p)+Cap+Dap(a+p)+Eaapp$$
$$\pm\sqrt{\left(A+2Ba+Caa+2Da^3+Ea^4\right)\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)}.$$

Hoc ergo valore pro L constituto indeque litteris $\alpha,\beta,\gamma,\delta,\ \varepsilon,\zeta$ per superiores formulas rite definitis, si iam p et q ut variabiles, a vero ut constantem spectemus, erit haec aequatio

$$\alpha+2\beta(p+q)+\gamma(pp+qq)+2\delta pq+2\varepsilon pq(p+q)+\zeta ppqq=0,$$

integrale completum huius aequationis differentialis

$$\frac{dq}{\sqrt{\left(A+2Bq+Cqq+2Dq^3+Eq^4\right)}}=\frac{2dp}{\sqrt{\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)}}.$$

Postquam hoc modo *q* per *p* definivimus, determinetur *r* per hanc aequationem

$$\alpha+2\beta(q+r)+\gamma(qq+rr)+2\delta qr+2\varepsilon qr(q+r)+\zeta qqrr=0,$$

erit

$$\Pi: r-\Pi: q=\Pi:p-\Pi: a,$$

quoniam posito $q=a$ et $r=p$ littera *L*, quae in valores $\alpha,\beta,\gamma,\delta,\ \varepsilon,\zeta$ ingreditur, perinde definitur ut ante. Quare cum sit $\Pi: q=2\Pi: p-\Pi: a$, erit

$$\Pi: r=3\Pi: p-2\Pi: a,$$

unde sumto *a* constante illa aequatio algebraica inter *q* et *r*, dum *q* per praecedentem aequationem ex *p* definitur, erit integrale completum huius aequationis differentialis

$$\frac{dr}{\sqrt{\left(A+2Br+Crr+2Dr^3+Er^4\right)}}=\frac{3dp}{\sqrt{\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)}}.$$

Hoc valore ipsius *r* per *p* invento quaeratur *s* per hanc aequationem

$$\alpha+2\beta(r+s)+\gamma(rr+ss)+2\delta rs+2\varepsilon rs(r+s)+\zeta rrss=0,$$

retinente *L* semper valorem primo assignatum eritque

$$\Pi: s-\Pi: r=\Pi: p-\Pi: a \quad \text{seu} \quad \Pi: s=4\Pi: p-3\Pi: a,$$

unde ista aequatio algebraica erit integrale completum huius aequationis differentialis

$$\frac{ds}{\sqrt{\left(A+2Bs+Css+2Ds^3+Es^4\right)}}=\frac{4dp}{\sqrt{\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)}}.$$

Cum hoc modo, quousque libuerit, progredi liceat, perspieuum est ad integrale completum huius aequationis differentialis inveniendum

$$\frac{dz}{\sqrt{\left(A+2Bz+Czz+2Dz^3+Ez^4\right)}}=\frac{ndp}{\sqrt{\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)}}$$

sequentes operationes institui oportere:

1) Quaeratur quantitas *L*, ut sit

$$\tfrac{1}{2}L(p-a)^2 = A + B(a+p) + Cap + Dap(a+p) + Eaapp$$
$$\pm\sqrt{\left(A+2Ba+Caa+2Da^3+Ea^4\right)\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)}.$$

2) Hinc determinentur litterae $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ per has formulas

$$\alpha = 4(AC-BB+AL),\quad \beta = 4AD+2BL,\quad \gamma = 4AE-LL,$$
$$\zeta = 4(CE-DD+EL),\quad \varepsilon = 4BE+2DL,\quad \delta = 4AE+4BD+2CL+LL$$

3) Formetur series quantitatum *p, q, r, s, t, ... z,* quarum prima sit *p*,secunda *q*, tertia *r* etc., ultima vero ordine *n* sit *z*, quae successive per has aequationes determinentur

$$\alpha+2\beta(p+q)+\gamma(pp+qq)+2\delta pq+2\varepsilon pq(p+q)+\zeta ppqq=0,$$
$$\alpha+2\beta(q+r)+\gamma(qq+rr)+2\delta qr+2\varepsilon qr(q+r)+\zeta qqrr=0,$$
$$\alpha+2\beta(r+s)+\gamma(rr+ss)+2\delta rs+2\varepsilon rs(r+s)+\zeta rrss=0,$$
etc.,

donec ad ultimam *z* perveniatur.

4) Relatio, quae hinc concluditur inter *p* et *z*, erit integrale completum aequationis differentialis propositae et littera *a* vicem gerit constantis arbitrariae per integrationem ingressae.

COROLLARIUM

643. Hinc etiam integrale completum inveniri potest huius aequationis differentialis

$$\frac{mdy}{\sqrt{\left(A+2By+Cyy+2Dy^3+Ey^4\right)}} = \frac{ndx}{\sqrt{\left(A+2Bx+Cxx+2Dx^3+Ex^4\right)}}$$

designantibus *m* et *n* numeros integros. Statuatur enim utrumque membrum

$$\frac{mdu}{\sqrt{\left(A+2Bu+Cuu+2Du^3+Eu^4\right)}}$$

et quaeratur relatio tam inter *x* et *u* quam inter *y* et *u;* unde elisa *u* orietur aequatio algebraica inter *x* et *y.*

SCHOLION

644. Ne hic extractio radicis in singulis aequationibus repetenda ambiguitatem creet, loco uniuscuiusque uti conveniet binis per extractionem iam erutis. Scilicet ut ex prima valor q rite per p definiatur, primo quidem habemus

$$q=\frac{-\beta-\delta p-\varepsilon pp+2\sqrt{\Delta\left(A+2Bp+Cpp+2Dp^3+Ep^4\right)}}{\gamma+2\varepsilon p+\zeta pp},$$

tum vero capi debet

$$\begin{aligned}&2\sqrt{\Delta\left(A+2Bq+Cqq+2Dq^3+Eq^4\right)}\\&=-\beta-\delta q-\varepsilon qq-p\left(\gamma+2\varepsilon q+\zeta qq\right)\end{aligned}$$

similique modo in relatione inter binas sequentes quantitates investiganda erit procedendum.

Caeterum adhue notari convenit numeros integros m et n positivos esse debere neque hanc investigationem ad negativos extendi, propterea quod formula differentialis

$$\frac{dz}{\sqrt{\left(A+2Bz+Czz+2Dz^3+Ez^4\right)}}$$

posito z negativo naturam suam mutat. Interim tamen cum hanc aequalitatem $\Pi: x+\Pi: y=$ Const. supra algebraice expresserimus, eius ope quoque ii casus resolvi possunt, ubi est m vel n numerus negativus; si enim fuerit $\Pi: z=n\Pi: p+C$, quaeratur y, ut sit $\Pi: y+\Pi: z=\text{Const.}$, eritque

$$\Pi: y=-n\Pi: p+\text{Const.}$$

PROBLEMA 84

645. *Si $\Pi: z$ eiusmodi functionem transcendentem ipsius z denotet, ut sit*

$$\Pi: z=\int\frac{dz\left(\mathfrak{A}+\mathfrak{B}z+\mathfrak{C}zz+\mathfrak{D}z^3+\mathfrak{E}z^4\right)}{\sqrt{\left(A+2Bz+Czz+2Dz^3+Ez^4\right)}},$$

comparationem inter huiusmodi functiones investigare.

SOLUTIO

Ex coefficientibus A, B, C, D, E una cum constante arbitraria L determinentur sequentes valores

$$\alpha=4\left(AC-BB+AL\right),\quad \beta=4AD+2BL,\quad \gamma=4AE-LL,$$
$$\zeta=4\left(CE-DD+EL\right),\quad \varepsilon=4BE+2DL,\quad \delta=4AE+4BD+2CL+LL$$

et inter binas variabiles x et y haec constituatur relatio

$$\alpha + 2\beta(x+y) + \gamma(xx+yy) + 2\delta xy + 2\varepsilon xy(x+y) + \zeta xxyy = 0$$

eritque

$$\frac{dx}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} + \frac{dy}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}} = 0,$$

seu

pro qua sine ambiguitate habetur

$$\beta + \delta x + \varepsilon xx + y(\gamma + 2\varepsilon x + \zeta xx) = 2\sqrt{\Delta(A+2Bx+Cxx+2Dx^3+Ex^4)},$$

$$\beta + \delta y + \varepsilon yy + x(\gamma + 2\varepsilon y + \zeta yy) = 2\sqrt{\Delta(A+2By+Cyy+2Dy^3+Ey^4)}$$

existente

$$\Delta = L^3 + CL^2 + 4(BD - AE)L + 4(ADD + BBE - ACE).$$

Quare si ponamus

$$\frac{dx(\mathfrak{A}+\mathfrak{B}x+\mathfrak{C}xx+\mathfrak{D}x^3+\mathfrak{E}x^4)}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} + \frac{dy(\mathfrak{A}+\mathfrak{B}y+\mathfrak{C}yy+\mathfrak{D}y^3+\mathfrak{E}y^4)}{\sqrt{(A+2By+Cyy+2Dy^3+Ey^4)}} = 2dV\sqrt{\Delta},$$

ut sit

$$\Pi : x + \Pi : y = \text{Const.} + 2V\sqrt{\Delta},$$

erit

$$\frac{dx\left(\mathfrak{B}(x-y)+\mathfrak{C}(x^2-y^2)+\mathfrak{D}(x^3-y^3)+\mathfrak{E}(x^4-y^4)\right)}{\sqrt{(A+2Bx+Cxx+2Dx^3+Ex^4)}} = 2dV\sqrt{\Delta}$$

seu

$$dV = \frac{dx\left(\mathfrak{B}(x-y)+\mathfrak{C}(x^2-y^2)+\mathfrak{D}(x^3-y^3)+\mathfrak{E}(x^4-y^4)\right)}{\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)}.$$

Ponatur nunc $x + y = t$ et $xy = u$, et quia $dx + dy = dt$ et $xdy + ydx = du$, erit $dx = \frac{xdt-du}{x-y}$ seu $(x-y)dx = xdt - du$, tum vero est $x = \frac{1}{2}t + \sqrt{\left(\frac{1}{4}tt - u\right)}$. At his positionibus aequatio assumta induit hanc formam

$$\alpha + 2\beta t + \gamma tt + 2(\delta - \gamma)u + 2\varepsilon tu + \zeta uu = 0,$$

unde fit differentiando

$$dt(\beta + \gamma t + \varepsilon u) + du(\delta - \gamma + \varepsilon t + \zeta u) = 0$$

ergo $dt = \frac{-du(\delta-\gamma+\varepsilon t+\zeta u)}{\beta+\gamma t+\varepsilon u}$ et

$$xdt - du = \frac{-du\left(\beta+\gamma t+\varepsilon u+(\delta-\gamma)x+\varepsilon tx+\zeta ux\right)}{\beta+\gamma t+\varepsilon u}$$

sive

$$xdt - du = \frac{-du\left(\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)\right)}{\beta+\gamma t+\varepsilon u},$$

sicque habebimus

$$\frac{dx(x-y)}{\beta+\delta x+\varepsilon xx+y(\gamma+2\varepsilon x+\zeta xx)} = \frac{-du}{\beta+\gamma t+\varepsilon u},$$

ergo

$$dV = \frac{-du\left(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)\right)}{\beta+\gamma t+\varepsilon u}$$

seu

$$dV = \frac{+dt\left(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)\right)}{\delta-\gamma+\varepsilon t+\zeta u}.$$

Est vero aequatione illa resoluta

$$t = \frac{-\beta-\varepsilon u+\sqrt{\left(\beta\beta-\alpha\gamma+2(\gamma\gamma+\beta\varepsilon-\gamma\delta)u+(\varepsilon\varepsilon-\gamma\zeta)uu\right)}}{\gamma}$$

seu [§ 628]

$$t = \frac{-\beta-\varepsilon u+2\sqrt{\Delta(A+Lu+Euu)}}{\gamma},$$

unde conficitur

$$dV = \frac{-du\left(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)\right)}{2\sqrt{\Delta(A+Lu+Euu)}}$$

ideoque

$$\Pi : x + \Pi : y = \text{Const.} - \int \frac{du\left(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)\right)}{2\sqrt{(A+Lu+Euu)}}.$$

Vel cum reperiatur

$$u = \frac{-(\delta-\gamma)-\varepsilon t+\sqrt{\left((\delta-\gamma)^2-\alpha\zeta+2\left((\delta-\gamma)\varepsilon-\beta\zeta\right)t+(\varepsilon\varepsilon-\gamma\zeta)tt\right)}}{\zeta},$$

quae expressio abit in hanc [§ 628]

$$u = \frac{-(\delta-\gamma)-\varepsilon t+\sqrt{\Delta(L+C+2Dt+Ett)}}{\zeta},$$

inde fit

$$dV = \frac{dt\left(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)\right)}{2\sqrt{\Delta(L+C+2Dt+Ett)}}$$

sicque habebimus per t

$$\Pi: x + \Pi: y = \text{Const.} + \int \frac{dt\left(\mathfrak{B}+\mathfrak{C}t+\mathfrak{D}(tt-u)+\mathfrak{E}t(tt-2u)\right)}{\sqrt{(L+C+2Dt+Ett)}},$$

quae expressio, nisi sit algebraica, certe vel per logarithmos vel arcus circulares exhiberi potest. Tum vero post integrationem tantum opus est, ut loco t restituatur eius valor $x+y$.

COROLLARIUM 1

646. Si velimus, ut posito $x = a$ fiat $y = b$, constans L ita debet definiri, ut sit

$$\tfrac{1}{2}L(a-b)^2 = A + B(a+b) + Cab + Dab(a+b) + Eaabb$$
$$\pm\sqrt{\left(A+2Ba+Caa+2Da^3+Ea^4\right)\left(A+2Bb+Cbb+2Db^2+Eb^4\right)};$$

tum igitur constans nostra erit $= \Pi: a + \text{II}: b$ integrali postremo ita sumto, ut evanescat posito $t = a+b$.

COROLLARIUM 2

647. Eodem modo etiam differentia functionum $\Pi: x - \Pi: y$ exprimi potest mutando alterutrius formulae radicalis signum, quo pacto formularum differentialium signum alterius convertetur.

COROLLARIUM 3

648. Quantitas V comparationi harum functionum inserviens erit algebraica, si haec formula differentialis

$$\frac{dt\left(\mathfrak{B}\zeta+\mathfrak{C}\zeta t+\mathfrak{D}(\delta-\gamma+\varepsilon t+\zeta tt)+\mathfrak{E}\left(2(\delta-\gamma)+2\varepsilon t+\zeta tt\right)t\right)}{\zeta\sqrt{(L+C+2Dt+Ett)}}$$

integrationem admittat, quia altera pars $\frac{-2dt\sqrt{\Delta}}{\zeta}\left(\mathfrak{D}+2\mathfrak{E}\text{t}\right)$ per se est integrabilis.

SCHOLION

649. Hoc ergo argumentum plane novum de comparatione huiusmodi functionum transcendentium tam copiose pertractavimus, quam praesens institutum postulare videbatur. Quando autem eius applicatio ad comparationem arcuum curvarum, quorum longitudo huiusmodi functionibus exprimitur, erit facienda, uberiori evolutione erit opus, ubi contemplatio singularium proprietatum, quae hoc modo eruuntur, eximium usum afferre poterit. Commode autem hoc argumentum ad doctrinam de resolutione aequationum differentialium referri videtur, siquidem inde eiusmodi aequationum integralia completa et quidem algebraice exhiberi possunt, quae allis methodis frustra indagantur. Nunc igitur huic sectioni finem faciet methodus generalis omnium equationum differentialium integralia proxime determinandi.