CHAPTER II

CONCERNING THE INTEGRATION OF DIFFERENTIAL EQUATIONS WITH THE AID OF MULTIPLIERS

PROBLEM 58

443. *To examine a proposed differential equation, to see whether or not it shall be integrable by itself.* [*i.e. whither or not it is an exact differential.*]

SOLUTIO

With all the indicated terms of the equation set out in the same part of the equation, so that a form of this kind is obtained $Pdx + Qdy = 0$, the equation is integrable by itself, if $Pdx + Qdy$ is a true differential of some function of the two variables x and y, [*i.e.* there is no need to look for an extra multiplying factor, or the total derivative is exact]. But this comes about, as we have shown in the *Calculo Differentiali* [See, Part I, Ch. 7 : § 231 and §240 ; see later in this chapter], if with the differential of P taken only of the variable y to dy and then the same ratio is had with the differential of Q taken of the variable x to dx, or by being designated in this way, which we have used in the *Calculo Differentiali*, where should be the relation

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right).$$

Now if Z is that function, the differential of which is $Pdx + Qdy$, it is hence to be designated in this manner

$$P = \left(\frac{dZ}{dx}\right) \text{ and } Q = \left(\frac{dZ}{dy}\right);$$

and hence there follows

$$\left(\frac{dP}{dy}\right) = \left(\frac{ddZ}{dxdy}\right) \text{ and } \left(\frac{dQ}{dx}\right) = \left(\frac{ddZ}{dydx}\right).$$

But there is

$$\left(\frac{ddZ}{dxdy}\right) = \left(\frac{ddZ}{dydx}\right),$$

from which it is deduced that

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right).$$

Whereby it will be discerned in this manner whether the proposed equation of the differential $Pdx + Qdy = 0$ shall be integrable or not. The values $\left(\frac{dP}{dy}\right)$ and $\left(\frac{dQ}{dx}\right)$ are sought by differentiation and, if they are equal to each other, the equation by itself is integrable; but if these values are not equal then the equation is not integrable by itself.

COROLLARY 1

444. Therefore all the differential equations, in which the variables are separated from each other in turn, are integrable by themselves ; for they have a form of this kind $Xdx + Ydy = 0$, so that X is a function of x only and Y only of y, and therefore there will be $\left(\frac{dX}{dy}\right) = 0$ and $\left(\frac{dY}{dx}\right) = 0$.

COROLLARY 2

445. Therefore in turn, if for the proposed differential equation $Pdx + Qdy = 0$ there should be $\left(\frac{dP}{dy}\right) = 0$ and $\left(\frac{dQ}{dx}\right) = 0$, then the variables are separable in that ; for the letter P is a function of x only and Q only of y. As if the separated equations constitute the first kind of equations integrable by themselves.

COROLLARY 3

446. But it is apparent that it is possible for $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, even if neither of these values should be equal to zero. Hence there are given equations integrable by themselves, and it is permitted for the variables in these not to separated.

SCHOLIUM

447. This criterion, by which we acknowledge differential equations to be integrable by themselves, is of the greatest importance in this method of integration which we undertake to relate. For if the equation taken is integrable by itself, the integral of this can be found by the rules now established ; but if the equation should not be integrable by itself, there will always be given a quantity by which, if that is multiplied, it is made integrable by itself; from which the whole calculation will be returned from that, so that for any proposed equation not integrable by itself there may be found a suitable multiplier, which reduces that to an equation integrable by itself ; which if it is always possible to be found, nothing greater could be wished for in this method of integration. Now this investigation rarely succeeds and scarcely extends further at this stage than to these equations, which with the help of the separation of variables we have shown now how to treat ; yet meanwhile without doubt this method is by far preferred to the preceding, since it is considered more adapted to the nature of the equation and also it extends to differential equations of higher grades, in which the separation of the variables is of no use.

PROBLEM 59

448. *To find the integral of a differential equation, that it is agreed to be integrable by itself.*

SOLUTION

Let the equation of the differential $Pdx + Qdy = 0$; in which since there shall be $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, then $Pdx + Qdy$ will be the differential of some function of the two variables *x* and *y*, which shall be *Z*, in order that $dZ = Pdx + Qdy$. Hence since we may have this equation $dZ = 0$, the integral sought will be $Z = C$. Hence the whole calculation has been reduced to this, so that this function *Z* may be elicited, since we know this to be $dZ = Pdx + Qdy$, which can be performed without difficulty. For since by taking *x* alone for the variable and with the other variable *y* kept as a constant, it is seen that $dZ = Pdx$, here we have a simple differential formula involving a single variable *x*, which by the rules from the above sections gives the integral $Z = \int Pdx + \text{Const.}$, but where it is to be noted that the constant here involves some *y*, which is written as *Y* in place of this, so that there becomes

$$Z = \int Pdx + Y .$$

Then in a similar manner *x* it taken for the constant with only *y* considered to be variable, and since there arises $dZ = Qdy$, then also $Z = \int Qdy + \text{Const.}$, but which constant quantity involves *x*, thus so that it shall be a function of *x*, which on putting *X* then

$$Z = \int Qdy + X .$$

But though here neither the function *X* nor there the function *Y* is determined, yet, because there must be $\int Pdx + Y = \int Qdy + \mathrm{X}$, hence each can be determined. For since there shall be

$$\int Pdx - \int Qdy = \mathrm{X} - Y$$

this quantity $\int Pdx - \int Qdy$ can always be separated into two parts of this kind, the first of which is a function of *x* only and the other only of *y*, from which the values of *X* and *Y* are recognised at once.

COROLLARY 1

449. Since $Q = \left(\frac{dZ}{dy}\right)$, there is no need indeed for a double integration. For on finding the integral $\int Pdx$, that can be differentiated again on taking *y* alone to be variable and there is produced *Vdy*, [i.e. the terms of *Z* which are functions of *x* alone vanish], from which it is necessary that there arises $Vdy + dY = Qdy$ and thus

$$dY = Qdy - Vdy = \left(Q - V\right)dy.$$

COROLLARY 2

450. Thus the integration of the integrals by themselves $Pdx + Qdy = 0$ can be completed. The integral $\int Pdx$ is sought with y considered constant and that again can be differentiated again with y alone considered to be variable, from which there emerges Vdy ; then $Q - V$ is a function of y only, from which there is sought $Y = \int (Q - V) dy$, and the equation of the integral is $\int Pdx + Y = \text{Const}$.

COROLLARIUM 3

451. Or $\int Qdy$ is sought with x considered constant, which integral may again be differentiated on taking x to be variable, but with y constant, from which there emerges *Udx;* then clearly $P - U$ is a function of x only, from which there is sought $\mathrm{X} = \int (p - U) dx$, and the equation of the integral sought is $\int Qdy + X = \text{Const}$.

COROLLARIUM 4

452. From the nature of the proposition it is likewise apparent, whichever way it should proceed ; for it is necessary that the same integral equation is arrived at, if indeed the proposed differential equation should be integrable by itself. Moreover then it certainly comes about, so that in the first case $Q - V$ shall be a function only of *y,* but in the latter case $P - U$ a function only of *x.*

SCHOLION

453. This method of integrating is also possible to be tested, before there should be an exploration, whether the equation of integrability is present ; for if either in the manner of Corollary 2 it should arise, that $Q - V$ should be a function of y only, or in the manner of Corollary 3, that $P - U$ should be a function of x only, with this indication the equation is to be integrable by itself. Now it is better than all the other tests to scrutinize whether or not the equation is integrable, or whether or not $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, because this test is completed by differentiation alone. Therefore we offer some examples of equations integrable by themselves, by which not only the method of integration, but also these notable properties that we have mentioned, are clearer understood.

EXAMPLE 1

454. *To integrate the equation*

$$dx(\alpha x+\beta y+\gamma)+dy(\beta x+\delta y+\varepsilon)=0$$

by itself.

Since here $P=\alpha x+\beta y+\gamma$ and $Q=\beta x+\delta y+\varepsilon$, then $\left(\frac{dP}{dy}\right)=\beta$ and $\left(\frac{dQ}{dx}\right)=\beta$, from which equality the integrability by itself is confirmed. Hence by Corollary 2 there is sought, with y considered as constant

$$\int Pdx=\tfrac{1}{2}\alpha xx+\beta yx+\gamma x\ ;$$

then there is $Vdy=\beta xdy$ and

$$(Q-V)dy=dy(\delta y+\varepsilon)=dY$$

and thus

$$Y=\tfrac{1}{2}\delta yy+\varepsilon y,$$

from which the integral will be

$$\tfrac{1}{2}axx+\beta yx+\gamma x+\tfrac{1}{2}\delta yy+\varepsilon y=C.$$

But in the manner of Corollary 3 with x considered constant there shall be

$$\int Qdy=\beta xy+\tfrac{1}{2}\delta yy+\varepsilon y,$$

which with y considered constant there is produced $Udx=\beta ydx$ and hence

$$(P-U)dx=(\alpha x+\gamma)dx \text{ and } X=\tfrac{1}{2}axx+\gamma x,$$

from which $\int Qdy+X=C$ gives the integral as before. Hence likewise also it is understood that

$$\int Pdx-\int Qdy=\tfrac{1}{2}\alpha xx+\gamma x-\tfrac{1}{2}\delta yy-\varepsilon y\ ,$$

which can at once be separated into the two functions $X-Y$.

EXAMPLE 2

455. *To integrate the equation*

$$\frac{dy}{y} = \frac{xdy - ydx}{y\sqrt{(xx+yy)}} \quad or \quad \frac{dx}{\sqrt{(xx+yy)}} + \frac{dy}{y}\left(1 - \frac{x}{\sqrt{(xx+yy)}}\right) = 0$$

itself integrable.

Since here there shall be $P = \frac{1}{\sqrt{(xx+yy)}}$ and $Q = \frac{1}{y} - \frac{x}{y\sqrt{(xx+yy)}}$,

for the character of integrability by itself it is to be recognised that

$$\left(\frac{dP}{dy}\right) = \frac{-y}{(xx+yy)^{\frac{3}{2}}} \quad \text{and} \quad \left(\frac{dQ}{dx}\right) = \frac{-y}{(xx+yy)^{\frac{3}{2}}}$$

which two values are equal to each other everywhere. Now in order that the integral may be found we use the rule of Corollary 2 and we have

$$\int Pdx = l\left(x + \sqrt{(xx+yy)}\right) \quad \text{and} \quad Vdy = \frac{ydy}{\left(x+\sqrt{(xx+yy)}\right)\sqrt{(xx+yy)}}$$

or on multiplying above and below by $\sqrt{(xx+yy)} - x$

$$V = \frac{\sqrt{(xx+yy)} - x}{y\sqrt{(xx+yy)}} = \frac{1}{y} - \frac{x}{y\sqrt{(xx+yy)}}$$

from which

$$Q - V = 0 \quad \text{and} \quad Y = \int (Q - V)dy = 0,$$

and thus the integral sought $l\left(x + \sqrt{(xx+yy)}\right) = \text{Const.}$

By the rule of Corollary 3 we have

$$\int Qdy = ly - x\int \frac{dy}{y\sqrt{(xx+yy)}},$$

but on putting $y = \frac{1}{z}$ there becomes

$$\int \frac{dy}{y\sqrt{(xx+yy)}} = -\int \frac{dz}{y\sqrt{(xxzz+1)}} = -\frac{1}{x}\, l\left(xz + \sqrt{(xxzz+1)}\right)$$

hence

$$\int Qdy = ly + l\,\frac{x+\sqrt{(xx+yy)}}{y} = l\left(x + \sqrt{(xx+yy)}\right),$$

from which

$$Udx = \frac{dx}{\sqrt{(xx+yy)}},$$

and hence $(P-U)dx = 0$.

EXAMPLE 3

456. *To integrate the equation*

$$(xx+yy-aa)dy + (aa+2xy+xx)dx = 0$$

integrable by itself.

Here therefore there is $P = aa+2xy+xx$ and $Q = xx+yy-aa$, from which $\left(\frac{dP}{dy}\right) = 2x$ and $\left(\frac{dQ}{dx}\right) = 2x$, which equation satisfies the condition of integrability by itself [*lit*., 'gives the nod to']. Then there now is

$$\int Pdx = aax + xxy + \tfrac{1}{3}x^3 \text{ and } Vdy = xxdy,$$

from which

$$(Q-V)dy = (yy-aa)dy \text{ and } Y = \tfrac{1}{3}y^3 - aay.$$

Hence the integral

$$aax + xxy + \tfrac{1}{3}x^3 + \tfrac{1}{3}y^3 - aay = \text{Const.}$$

By the other way there is

$$\int Qdy = xxy + \tfrac{1}{3}y^3 - aay \text{ and hence } Udx = 2xydx,$$

hence

$$(P-U)dx = (aa+xx)dx \text{ and } X = aax + \tfrac{1}{3}x^3,$$

from which the integral arises as before.

SCHOLIUM

457. In these examples it has been clear how the integral $\int Pdx$ actually is to be shown and then with the differential of this taken, Vdy is assigned to be a variable of y alone. But if this integral $\int Pdx$ is unable to be set out explicitly, it may not be clear how it can be elicited from the differential Vdy, since there is included some constant considered in the formula $\int Pdx$, which also involves y in itself. Therefore then we may consider how one should proceed.

We may put

$$Z=\int Pdx+Y\ ,$$

and since

$$\left(\frac{d.\int Pdx}{dy}\right)=V,\text{ is sought},\quad \text{on account of } \int Pdx=Z-Y \text{ there will be } V=\left(\frac{dZ}{dy}\right)-\frac{dY}{dy}.$$

But there is

$$\left(\frac{dZ}{dx}\right)=P,\text{ hence }\left(\frac{ddZ}{dxdy}\right)=\left(\frac{dP}{dy}\right)=\left(\frac{dV}{dx}\right)\text{ on account of }\left(\frac{dZ}{dy}\right)=V+\frac{dY}{dy}.$$

Hence there shall be

$$V=\int dx\left(\frac{dP}{dy}\right);$$

whereby the quantity V is found by the integration of this formula $\int dx\left(\frac{dP}{dy}\right)$, in which y is considered as a constant, after finding the value $\frac{dP}{dy}$, y must be assumed to be the variable. Now since here anew the constant may involve y, hence that function Y, which we sought, is not determined. The reason for this inconvenience is clearly situated in the ambiguity of the integrals $\int Pdx$ and $\int dx\left(\frac{dP}{dy}\right)$, while each receives an arbitrary function of y. Hence a remedy will be produced, if each integral can be determined by a certain condition. Thus when the integral $\int Pdx$ we put in place to be accepted thus, so that it vanishes on putting $x=f$, where it is permitted to accept a certain constant f for argument's sake, then the other integral $\int dx\left(\frac{dP}{dy}\right)$ is taken by the same rule. With which done then $Q-\int dx\left(\frac{dP}{dy}\right)$ will be a function of y only and the integral of the equation $Pdx+Qdy=0$ will become

$$\int Pdx+\int dy\left(Q-\int dx\left(\frac{dP}{dy}\right)\right)=\text{Const.}$$

provided both the integrals $\int Pdx$ and $\int dx\left(\frac{dP}{dy}\right)$, in which y is treated as a constant, thus may be determined, so that they vanish, while the same value f of x is attributed to each. Whereby hence we have deduced this rule :

RULES FOR THE INTEGRATION OF EQUATIONS INTEGRABLE BY THEMSELVES [*i.e. exact differentials,*] $Pdx+Qdy=0$ IN WHICH $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$

458. *The integrals* $\int Pdx$ *and* $\int dx\left(\frac{dP}{dy}\right)$ *are sought, by considering* y *as a constant thus, so that both vanish while a certain value of* x *is attributed, for example* $x = f$. *Then* $Q-\int dx\left(\frac{dP}{dy}\right)$ *becomes a function of* y *only, which shall be* $= Y$, *and with the integral sought there shall be* $\int Pdx+\int Ydy=\text{Const.}$ *Or, which returns the same, the integrals* $\int Qdy$ *and* $\int dy\left(\frac{dQ}{dx}\right)$ *are sought by considering* x *as a constant thus, so that both vanish, while a certain value, for example* $y=g$ *is attributed to* y *; while* $P-\int dy\left(\frac{dP}{dx}\right)$ *is a function of* x *only, which on putting* $= X$ *becomes the integral sought* $\int Qdy+\int Xdx=\text{Const}$.

DEMONSTRATION

The truth of this rule can be observed from the preceding, if perhaps with the precaution to be assumed for that, that both formulas $\int Pdx$ and $\int dx\left(\frac{dP}{dy}\right)$ must be determined from the same rule, so that, while a certain value is attributed to x such as $x = f$, both vanish. But lest it might be thought perhaps that the rule could be determined equally from another integration, I add this demonstration. Indeed in the first place the integration depends on our choice, as hence we have assumed thus, that the integral $\int Pdx$ vanished on putting $x = f$; with which in place I say that it is necessary to determine the other integral $\int dx\left(\frac{dP}{dy}\right)$, from the same condition. For let $\int Pdx=Z$ and Z is a function of this kind of x and y, which vanished on putting $x = f$; hence there will be had the factor $f-x$ or a certain positive power of this $(f-x)^{\lambda}$ present, thus so that there can be $Z=(f-x)^{\lambda}T$. Now because $\int dx\left(\frac{dP}{dy}\right)$ expresses the value of $\left(\frac{dZ}{dy}\right)$, then $\int dx\left(\frac{dP}{dy}\right)=(f-x)^{\lambda}\left(\frac{dT}{dy}\right)$, from which it is clear that this integral also vanishes on putting $x = f$, thus so that the determination of this integral is no longer left to our choice. With this in place certainly it will be the integral of the differential equation $Pdx+Qdy=0$, integrable by itself,

$$\int Pdx+\int Ydy=\text{Const}.,$$

with the function present

$$Y = Q - \int dx\left(\tfrac{dP}{dy}\right);$$

for on putting $\int Pdx = Z$, clearly as in this integration *y* it taken as constant, this equation is obtained $Z + \int Ydy = \text{Const.}$, which is to be the integral sought or as becomes apparent from that by differentiation. Since indeed there shall be

$$dZ = Pdx + dy\left(\tfrac{dZ}{dy}\right) = Pdx + dy\int dx\left(\tfrac{dP}{dy}\right),$$

and it will be with the differential

$$Pdx + dy\int dx\left(\tfrac{dP}{dy}\right) + Ydy = 0$$

of the equation found,
but $Y = Q - \int dx\left(\tfrac{dP}{dy}\right)$, from which there is produced $Pdx + Qdy = 0$, which is itself the equation of the differential proposed. But since $Q - \int dx\left(\tfrac{dP}{dy}\right)$ shall be a function of *y* only, from which it follows, because the equation of the differential is integrable by itself.

THEOREM

459. *For each equation, which is not integrable by itself, a quantity can always be given, by which on multiplying by that, it is rendered integrable.*

DEMONSTRATIO

Let $Pdx + Qdy = 0$ be the differential equation [which is not exact] and we may consider a complete integral of this, which will be a certain equation between *x* and *y*, in which an arbitrary constant quantity may be introduced. From this equation this arbitrary constant may itself be elicited, in order that an equation of this kind may be produced :

The constant is equal to a certain function of *x* and *y* themselves, which differentiated gives rise to

$$0 = Mdx + Ndy;$$

which equation now is free from that arbitrary constant introduced by integration and thus it is necessary, in order that this differential equation shall agree with that proposed, otherwise the supposed integral should not be true. Hence it is required, in order the relation produces the same on both sides, from which there shall be

$$\tfrac{P}{Q} = \tfrac{M}{N}$$

and thus

$$M = LP \quad \text{and} \quad N = LQ.$$

But since $Mdx + Ndy$ is the true differential arising from the differentiation of a certain function of *x* et *y*, there is $\left(\frac{dM}{dy}\right) = \left(\frac{dN}{dx}\right)$. Whereby for the equation $Pdx + Qdy = 0$ a certain multiplier *L* will be given, in order that

$$\left(\frac{d.LP}{dy}\right) = \left(\frac{d.LQ}{dx}\right),$$

or so that the equation multiplied by *L* is made integrable by itself [*i.e.* and exact or total derivative].

COROLLARY 1

460. Hence for each equation $Pdx + Qdy = 0$ a function *L* is given of this kind, in order that $\left(\frac{d.LP}{dy}\right) = \left(\frac{d.LQ}{dx}\right)$ and thus on expanding out

$$L\left(\frac{dP}{dy}\right) + P\left(\frac{dL}{dy}\right) = L\left(\frac{dQ}{dx}\right) + Q\left(\frac{dL}{dx}\right)$$

or

$$L\left(\left(\frac{dP}{dy}\right) - \left(\frac{dQ}{dx}\right)\right) = Q\left(\frac{dL}{dx}\right) - P\left(\frac{dL}{dy}\right);$$

which function *L* if it should have been found, the differential equation $LPdx + LQdy = 0$ by itself shall become integrable.

COROLLARY 2

461. In the proposed equation it is allowed without risk to write one in place of *Q*, since every equation can be represented by this form $Pdx + dy = 0$. Hence the discovery of the multiplier *L*, which reduces that to be integrable by itself, depends on the resolution of this equation

$$L\left(\frac{dP}{dy}\right) = \left(\frac{dL}{dx}\right) - P\left(\frac{dL}{dy}\right)$$

where it is to be observed that

$$\mathrm{d}L = \left(\frac{dL}{dx}\right)dx + dy\left(\frac{dL}{dy}\right).$$

SCHOLIUM

462. Because here a function of the two variables *x* and *y* is sought, the mutual relation of which is seen to be the smallest, since it involves the equation $Pdx + Qdy = 0$, this investigation is met with in our second book, where a function of this kind has to be investigated from a certain given relation of this kind. For in this investigation we have not attended to the proposed equation, by which the formula $Pdx + Qdy$ must be returned equal to zero, but a multiplier *L* is sought unconditionally, the formula $Pdx + Qdy$ multiplied by which may be changed into a true differential of a certain finite function, which shall be *Z*, thus in order that there may be had $dZ = LPdx + LQdy$. With which multiplier L found then at last the equality $Pdx + Qdy = 0$ is considered and thus it is concluded that the function Z is required to be equal to a constant quantity. Therefore since little can be expected , as we examine the method multipliers of this kind are to be found for whatever differential equation, we run across these cases in which such a multiplier exists, however it may be found. Yet meanwhile to the fuller use of this method it will be helpful to note, that we will have known at once a certain multiplier for some differential equation, from that easily innumerable others can be deduced, which equally render the proposed differential equation integrable by itself.

PROBLEM 60

463. *From a single multiplier L, which renders the equation* $Pdx + Qdy = 0$ *integrable by itself, to find innumerable other multipliers, which perform the same duty.*

SOLUTION

Therefore since $L\left(Pdx + Qdy\right)$ shall be a true differential of some function Z, this function Z is sought from the above precepts, thus so that it becomes

$$L\left(Pdx + Qdy\right) = dZ ,$$

and now it is evident that this formula dZ also must be permitted to be integrated, if it is to be multiplied by some function of Z, that we may indicate thus by $\varphi : Z$.
[We would now perhaps write this in the form $\varphi\left(Z\right)$]
Therefore since also this formula shall be integrable

$$\left(Pdx + Qdy\right)L\varphi : Z ,$$

then also $L\varphi : Z$ shall be a multiplier of the proposed equation $Pdx + Qdy = 0$, which renders that integrable. Whereby from the single multiplier L there is found by integration $Z = \int L\left(Pdx + Qdy\right)$ and then the expression $L\varphi : Z$, where for some function $\varphi : Z$ of Z able to be assumed, there will be given an infinite number of other multipliers performing the same duty.

SCHOLIUM

464. Even if it is sufficient that a single multiplier be known for some differential equation, yet cases occur, in which furthermore it is extremely useful, as indeed an infinite number of multipliers can be made apparent. Just as if the proposed equation can be conveniently split into two parts of this kind

$$\left(Pdx + Qdy\right) + \left(Rdx + Sdy\right) = 0$$

and all the multipliers may exist, in which each part separately $Pdx + Qdy$

and $Rdx+Sdy$ can be returned integrable, from which sometimes a common multiplier can be included returning each integrable part. For let $L\varphi : Z$ be the general expression for all multipliers of the formula $Pdx+Qdy$ and $M\varphi : V$ be the general expression for all multipliers of the formula $Rdx+Sdy$, and because $\varphi : Z$ and $\varphi : V$ specify some functions of the quantities Z and V, if it is permitted to take these, in order that $L\varphi : Z = M\varphi : V$, then there will be had a suitable multiplier for the equation

$$Pdx+Qdy+Rdx+Sdy=0.$$

But it is understood that this can be performed only in these cases, in which the multiplier for the whole equation also is the multiplier of the individual parts of this taken separately, and which is returned integrable. Whereby it is to be stipulated, lest too much is attributed to this method and, when that does not succeed, the equation may be taken as insoluble ; for it is certainly possible to eventuate, so that the whole equation has a multiplier, which does not agree with the individual parts of this. Thus with the proposed equation $Pdx+Qdy=0$ the multiplier returning the separate integrable part Pdx clearly is $\frac{X}{P}$; with X denoting some function of x, and the multiplier returning the other part integrable Qdy is $\frac{Y}{Q}$; but even if it is by no means possible that $\frac{X}{P}=\frac{Y}{Q}$ or $\frac{P}{Q}=\frac{X}{Y}$, except in easy cases, yet the whole formula $Pdx+Qdy$ certainly always has a multiplier, by which that is returned integrable.

EXAMPLE 1

465. *To find all the multipliers, by which the formula $\alpha ydx+\beta xdy$ is returned integrable.*

First the multiplier $\frac{1}{xy}$ presents itself , which provides $\frac{\alpha dx}{x}+\frac{\beta dy}{y}$, of which the integral is $\alpha lx+\beta ly = lx^{\alpha}y^{\beta}$. Hence some function of this $\varphi{:}x^{\alpha}y^{\beta}$ multiplied by $\frac{1}{xy}$ will give a suitable multiplier, and thus the general form is $\frac{1}{xy}\varphi{:}x^{\alpha}y^{\beta}$.

For a function of the quantity $x^{\alpha}y^{\beta}$ also is a function of the logarithm of the same quantity. For if P should be a function of p and Π a function of P, also Π is a function of p and vice versa.

COROLLARY

466. If some power $x^{n\alpha}y^{n\beta}$ is taken for the function, the formula $\alpha ydx+\beta xdy$ is returned integrable, if it is multiplied by $x^{n\alpha-1}y^{n\beta-1}$, indeed in which case it is apparent at once ; for it is $\frac{1}{n}x^{n\alpha}y^{n\beta}$.

EXAMPLE 2

467. *To find all the multipliers, which return this formula* $Xydx + dy$ *integrable.*

First the multiplier presents itself, from which since there shall be

$$\int\left(Xdx+\frac{dy}{y}\right)=\int Xdx+ly \text{ or } le^{\int Xdx}y,$$

all the functions of this quantity or of this $e^{\int Xdx}y$ divided by y give suitable multipliers. From which the general expression for all the multipliers will be equal to $\frac{1}{y}\varphi : e^{\int Xdx}y$.

COROLLARY

468. Hence for the formula $Xydx + dy$, $e^{\int Xdx}$ is also a multiplier, which is a function of x; from which hence also the formula $\mathfrak{X}dx$ with $\mathfrak{X}$ [the Gothic capital x] some function of x is returned integrable, that multiplier also will be appropriate for this formula $dy + Xydx + \mathfrak{X}dx$.

PROBLEM 61

469. *For the proposed equation* $dy + Xydx = \mathfrak{X}dx$, *in which X and* $\mathfrak{X}$ *are some functions of* x, *to find a suitable multiplier and to integrate that equation.*

SOLUTION

Since the other term $\mathfrak{X}dx$ becomes integrable multiplied by some function of x, it may be considered whether also the first term $dy + Xydx$ can be rendered integrable by a multiplication of this kind. Since the multiplier $e^{\int Xdx}$ may prevail, on using this the equation may be found

$$e^{\int Xdx}y=\int e^{\int Xdx}\mathfrak{X}dx$$

or

$$y=e^{-\int Xdx}\int e^{\int Xdx}\mathfrak{X}dx,$$

as we have now found above [§ 420].

COROLLARY 1

470. Also it is apparent, if in place of y some function of y should be present, as this equation may be had $dy + Xydx = \mathfrak{X}dx$, that through the multiplier $e^{\int Xdx}$ is returned integrable and the integral becomes

$$e^{\int Xdx}Y=\int e^{\int Xdx}\mathfrak{X}dx.$$

COROLLARY 2

471. Whereby also this equation $dy + yXdx = y^n \mathfrak{X}dx$ because on division by y^n it becomes

$$\frac{dy}{y^n} + \frac{Xdx}{y^{n-1}} = \mathfrak{X}dx,$$

where on putting $\frac{1}{y^{n-1}} = Y$ on account of $-\frac{(n-1)dy}{y^n} = dY$ or $\frac{dy}{y^n} = -\frac{dY}{(n-1)}$ there emerges

$$-\frac{dY}{(n-1)} + YXdx = \mathfrak{X}dx \text{ or } dY - (n-1)YXdx = -(n-1)\mathfrak{X}dx,$$

which through the multiplier $e^{-(n-1)\int Xdx}$ shall become integrable : and the integral of this shall be

$$e^{-(n-1)\int Xdx}Y = -(n-1)\int e^{-(n-1)\int Xdx}\mathfrak{X}dx$$

or [§ 429]

$$\frac{1}{y^{n-1}} = -(n-1)e^{(n-1)\int Xdx}\int e^{-(n-1)\int Xdx}\mathfrak{X}dx.$$

SCHOLIUM

472. Since the general multiplier for the term $dy + yXdx$ shall be $\frac{1}{y}\varphi{:}e^{\int Xdx}y$, with a suitable power selected in place of the function the multiplier will be $e^{m\int Xdx}y^{m-1}$ providing $\frac{1}{m}e^{m\int Xdx}y^m$. Hence it must be brought about, that also the same multiplier returns the other term $y^n\mathfrak{X}dx$ integrable; which comes about on taking $m-1=-n$ or $m=1-n$, from which the integral of this other term shall be $\int e^{m\int Xdx}\mathfrak{X}dx$, thus in order that the integral sought may be obtained

$$\frac{1}{1-n}e^{(1-n)\int Xdx}Y^{1-n} = \int e^{(1-n)\int Xdx}\mathfrak{X}dx,$$

which agrees absolutely with that found in this the manner.

PROBLEM 62

473. *With the proposed differential equation*

$$\alpha ydx+\beta xdy=x^m y^n\left(\gamma ydx+\delta xdy\right)$$

to find a suitable multiplier, which returns the same integrable, and to assign the integral of this.

SOLUTION

Each term should be considered separately ; and for the first $\alpha ydx+\beta xdy$ we have seen that all the suitable multipliers are to be contained in this form

$$\tfrac{1}{xy}\varphi:x^\alpha y^\beta .$$

For the other part $x^m y^n\left(\gamma ydx+\delta xdy\right)$ the first multiplier is

$$\tfrac{1}{x^{m+1}y^{n+1}},$$

from which there is produced $\frac{\gamma dx}{x}+\frac{\delta dy}{y}$, the integral of which is $lx^\gamma y^\delta$; hence the general form for the multipliers of this part is :

$$\tfrac{1}{x^{m+1}y^{n+1}}\varphi:x^\gamma y^\delta .$$

From which now these two multipliers may be returned equal, in place of the functions powers are taken and there becomes

$$x^{\mu\alpha-1}y^{\mu\beta-1}=x^{v\gamma-m-1}y^{v\delta-n-1},$$

from which it is necessary to put in place

$$\mu\alpha=v\gamma-m \text{ and } \mu\beta=v\delta-n$$

and hence it is deduced:

$$\mu=\tfrac{\gamma n-\delta m}{\alpha\delta-\beta\gamma} \text{ and } v=\tfrac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}$$

On account of which the multiplier will be :

$$x^{\mu\alpha-1}y^{\mu\beta-1}=x^{v\gamma-m-1}y^{v\delta-n-1},$$

from which our equation adopts this form

$$x^{\mu\alpha-1}y^{\mu\beta-1}\left(\alpha ydx+\beta xdy\right)=x^{v\gamma-1}y^{v\delta-1}\left(\gamma ydx+\delta xdy\right)$$

where each term by itself is integrable and thus the integral sought

$$\tfrac{1}{\mu}x^{\mu\alpha}y^{\mu\beta}=\tfrac{1}{\nu}x^{\nu\gamma}y^{\nu\delta}+\text{Const.},$$

which agrees with that which was found in the preceding chapter [§ 431].

COROLLARY 1

474. Hence on putting for the sake of brevity $\mu=\frac{\gamma n-\delta m}{\alpha\delta-\beta\gamma}$ and $\nu=\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}$; the completed integral of the differential equation

$$aydx+\beta xdy=x^m y^n\left(\gamma ydx+\delta xdy\right)$$

is

$$\tfrac{1}{\mu}x^{\mu\alpha}y^{\mu\beta}=\tfrac{1}{\nu}x^{\nu\gamma}y^{\nu\delta}+\text{Const.}$$

COROLLARY 2

475. If it arises that $\mu=0$ or $\gamma n=\delta m$, then the integral is reduced to logarithms and it shall be :

$$lx^{\alpha}y^{\beta}=\tfrac{1}{\nu}x^{\nu\gamma}y^{\nu\delta}+\text{Const.};$$

but if $\nu=0$ or $\alpha n=\beta m$, then the integral shall be

$$\tfrac{1}{\mu}x^{\mu\alpha}y^{\mu\beta}=lx^{\gamma}y^{\delta}+\text{Const.}$$

SCHOLIUM

476. But hence the exceptional case may be seen, in which $\alpha\delta=\beta\gamma$, because then both the numbers μ and ν become infinite. Now if $\delta=\frac{\beta\gamma}{\alpha}$, our equation adopts that form

$$aydx+\beta xdy=\tfrac{\gamma}{\alpha}x^m y^n\left(aydx+\beta xdy\right)\quad\text{or}\quad\left(aydx+\beta xdy\right)\left(1-\tfrac{\gamma}{\alpha}x^m y^n\right)=0,$$

which because it has two factors, a two–fold solution from each separately reduced to nothing is derived. Clearly the first arises from $aydx+\beta xdy=0$, of which the integral is $x^{\alpha}y^{\beta}=\text{Const.}$, now the other factor by itself gives the finite equation $1-\frac{\gamma}{\alpha}x^m y^n=0$, of which the solution satisfies each equally. And this in general is to be understood for all differential equations, which it is permitted to be resolved into factors, and where likewise in finite equations the individual factors provide solutions. But generally finite factors, before the integration is undertaken, are accustomed to be removed at once by division, since not by the nature of the calculation, but by the operations put in place they are agreed to be added at last, thus so that , as in algebra it is often the custom to happen, that useless solutions are to be lead through the solution.

PROBLEM 63

477. *To find a suitable multiplier with a proposed homogeneous differential equation, which returns that integrable, and thus to elicit the integral of this.*

SOLUTION

Let $Pdx+Qdy=0$ be the proposed equation, in which *P* and *Q* shall be homogeneous functions of *n* of *x* and *y*, and we seek a multiplier *L*, which shall also be a homogeneous function, the dimension of which shall be the number λ. Now since the formula $L(Pdx+Qdy)$ shall be integrable, the integral will be a function of dimensions $\lambda+n+1$ of x and *y*, which function if there is put *Z*, will be from the nature of homogeneous functions [§ 481]

$$LPx+LQy=(\lambda+n+1)Z.$$

Whereby if λ it taken equal to $-n-1$, the quantity $LPx+LQy$ either is equal to 0 or is constant, from which we may obtain $L=\frac{1}{Px+Qy}$, which therefore is a suitable multiplier for our equation.

Likewise also this is deduced from the separation of the variables, for on putting ; for on putting $y=ux$ there becomes $P=x^nU$ and $Q=x^nV$ with the functions *U* and *V* of *u* only and on account of $dy=udx+xdu$ then there becomes

$$Pdx+Qdy=x^nUdx+x^nVudx+x^nVxdu$$

or

$$Pdx+Qdy=x^n(U+Vu)dx+x^{n+1}Vdu$$

But this formula divided by $x^{n+1}(U+Vu)$ becomes integrable and thus our formula $Pdx+Qdy$ divided by $x^{n+1}(U+Vu)=Px+Qy$, with the values restored $U=\frac{P}{x^n}$, $V=\frac{Q}{x^n}$, and $u=\frac{y}{x}$, becomes integrable; or a suitable multiplier is $\frac{1}{Px+Qy}$, from which this equation $\frac{Pdx+Qdy}{Px+Qy}=0$ always is integrable by itself [*i.e.* it becomes an exact differential].

Now for the integral of this to be found the formula $\int\frac{Pdx}{Px+Qy}$ is integrated by regarding *y* as a constant and it may be determined by a certain reason, that it vanishes on putting $x=f$. Then on putting for the cause of brevity $\frac{P}{Px+Qy}=R$ the value $\left(\frac{dR}{dy}\right)$ is taken and by the same rule the integral $\int dx\left(\frac{dR}{dy}\right)$ again is sought by regarding *y* as constant. Then $\frac{Q}{Px+Qy}-\int dx\left(\frac{dR}{dy}\right)$ shall be a function of y only, or $\frac{Q}{Px+Qy}-\int dx\left(\frac{dR}{dy}\right)=Y$ and hence this shall be the integral sought :

$$\int \frac{Pdx}{Px+Qy} + \int Ydy = \text{Const.}$$

COROLLARY 1

478. Therefore since the formula $Pdx + Qdy$ shall be integrable by itself, if for the sake of brevity we put

$$\frac{P}{Px+Qy} = R \quad \frac{Q}{Px+Qy} = S\ ,$$

by necessity there shall be $\left(\frac{dR}{dy}\right) = \left(\frac{dS}{dx}\right)$. But

$$\left(\frac{dR}{dy}\right) = \left(Qy\left(\frac{dP}{dy}\right) - Py\left(\frac{dQ}{dy}\right) - PQ\right) : \left(Px + Qy\right)^2$$

and

$$\left(\frac{dS}{dx}\right) = \left(Px\left(\frac{dQ}{dx}\right) - Qx\left(\frac{dP}{dy}\right) - PQ\right) : \left(Px + Qy\right)^2 .$$

On account of which there shall be had

$$Qy\left(\frac{dP}{dy}\right) - Py\left(\frac{dQ}{dy}\right) = Px\left(\frac{dQ}{dx}\right) - Qx\left(\frac{dP}{dx}\right).$$

COROLLARY 2

479. This equation can also be deduced from the nature of homogeneous functions. For since P and Q shall be functions of n dimensions of x and y, on account of

$$dP = dx\left(\frac{dP}{dx}\right) + dy\left(\frac{dP}{dy}\right) \text{ and } dQ = dx\left(\frac{dQ}{dx}\right) + dy\left(\frac{dQ}{dy}\right)$$

there will be

$$nP = x\left(\frac{dP}{dx}\right) + y\left(\frac{dP}{dy}\right) \text{ and } nQ = x\left(\frac{dQ}{dx}\right) + y\left(\frac{dQ}{dy}\right).$$

But the equation has been found

$$Q\left(x\left(\frac{dP}{dx}\right) + y\left(\frac{dP}{dy}\right)\right) = P\left(x\left(\frac{dQ}{dx}\right) + y\left(\frac{dQ}{dy}\right)\right),$$

which hence changes into the identity $nPQ = nPQ$.

COROLLARY 3

480. If the homogeneous equation $Pdx+Qdy=0$ should be integrable by itself and P and Q shall be functions of dimension -1, then $Px+Qy$ shall be a constant number. Just as because $\frac{xdx+ydy}{xx+yy}=0$ shall be an equation of this kind, if in place of dx and dy there are written x and y there is produced $\frac{xx+yy}{xx+yy}=1$.

SCHOLIUM

481. In the *Calculo Differentiali* we have shown [Part I, §222], if V should be a homogeneous function of n dimensions of x and y and there is put $dV=Pdx+Qdy$, there becomes

$$Px+Qy=nV.$$

Whereby if $Pdx + Qdy$ should be an integrable formula and P and Q homogeneous functions of dimension $n-1$, the integral may be had at once ; for it shall be $V=\frac{1}{n}\left(Px+Qy\right)$ neither in this is there a need for any integration. Yet meanwhile we should consider hence an exceptional case, in which $n = 0$, as shall be made in our equation to be integrable through multiplication with $\frac{Pdx+Qdy}{Px+Qy}=0$ returned, where dx and dy are multiplied by functions of dimension -1; and indeed this integral cannot be returned without an integration here. Moreover the reason for this exception is contained in this, because of the formula of integrability $Pdx+Qdy$, in which P and Q are homogeneous functions of dimension $n-1$, then the integral shall be a homogeneous function of degree n, when n is not equal to 0; for in this case alone it can happen, that the integral shall not be a function of no dimensions, just as it becomes in this formula of the differential $\frac{xdx+ydy}{xx+yy}$, clearly the integral of which is $\frac{1}{2}l\left(xx+yy\right)$. On account of which , since the formula $\frac{Pdx+Qdy}{Px+Qy}$ shall be integrable, we have shown in this singular way by deducing by reason of separability. Yet meanwhile without any respect, from which we might understand this, that in the present circumstances is most worthy of note that all homogeneous equations $Pdx+Qdy=0$ through the multiplier $\frac{1}{Px+Qy}$ are made integrable by themselves. Therefore the method may be desired, from the benefit of which this multiplier is allowed to be found in the first place ; from which method certainly there might be brought about a great advance in analysis. But as long as the theory is not able to be extended this far, for the most part it will be the concern for many cases to be noted properly; which now we have made available in two kinds of equation, and for the remaining equations which we have shown how to integrate above, we may investigate the multipliers ; moreover that reduction to the separation may itself reveal to us these multipliers, as we shall show in the following problem.

Extract from *Calculo Differentiali* Part I, Ch. 7 :

In the previous sections, functions of zero dimensions of the variables x and y have been considered, and it has been demonstrated that for which the derivative with dx and dy replaced by x and y , there results zero. For example, if $V=\frac{x}{y}$, then $dV=\frac{ydx-xdy}{yy}$, and on putting x in place of dx and y in place of dy, here becomes $\frac{yx-xy}{yy}=0.$ Euler extends his results to homogeneous functions of non–zero degrees.

222. We may now consider other homogeneous functions and let V be a function of n dimensions of x and y. Whereby if there is put $y=tx$, then V adopts a form of this kind Tx^n with T being a function of t and there becomes $dT=\varTheta dt$; then

$$dV=x^n\varTheta dt+nTx^{n-1}dx.$$

Hence if we put in place $dV=Pdx+Qdy,$ on account of $dy=tdx+xdt$, there becomes

$$dV=Pdx+Qtdx+Qxdt;$$

which form, since it must agree with that , will be

$$P+Qt=nTx^{n-1}=\frac{nV}{x}$$

on account of $V=Tx^n$. Hence as $t=\frac{y}{x}$ there becomes

$$Px+Qy=nV\ ,$$

which equation thus defines the relation between P and Q, so that, if one shall be known, then the other can be easily found. Because again there is the relation $Qx=x^n\varTheta$, Qx and thus also Qy and Px will be a function of n dimensions of x and y.

223. Hence if in the differential of any function of x and y in place of dx and dy there may be put x et y, a quantity arises equal to the function of which the differential is proposed, multiplied by the number of dimensions.

1. If there shall be $V=\sqrt{(xx+yy)},$ then n = 1 and because $dV=\frac{xdx+ydy}{\sqrt{(xx+yy)}}$, then there becomes

$$\frac{xx+yy}{\sqrt{(xx+yy)}}=V=\sqrt{(xx+yy)}.$$

II. If there shall be $V = \frac{y^3+x^3}{y-x}$, then $n = 2$ and

$$dV = \frac{2y^3dy - 3yyxdy + 3yxxdx - 2x^3dx + y^3dx - x^3dy}{(y-x)^2}.$$

There may be put x for dx and y for dy; and there becomes

$$\frac{2y^4 - 2y^3x + 2yx^3 - 2x^4}{(y-x)^2} = \frac{2y^3 + 2x^3}{y-x} = 2V,$$

etc.

PROBLEM 64

482. *With the proposed differential equation, which is allowed to be reduced according to the separation of the variables, to find a multiplier, by which that is rendered integrable by itself.*

SOLUTION

Let $Pdx + Qdy = 0$, which by a certain substitution, provided that in place of x and y another two variables t are u introduced, it may be adapted to separation; hence with this substitution done we may put in place $Pdx + Qdy = Rdt + Sdu$, but now this formula $Rdt + Sdu$, if it is divided by V, becomes separated, thus so that in this formula the quantity $\frac{Rdt+Sdu}{V}$ shall be a function of t alone and $\frac{S}{V}$ a function of u alone. Therefore since the formula $\frac{Rdt+Sdu}{V}$ shall be integrable by itself, also this will be integrable, $\frac{Pdx+Qdy}{V}$, clearly equal to that, if indeed the variables x and y are restored in V. Hence therefore from reduction to the separation of the equation $Pdx + Qdy = 0$ we have shown the multiplier, by which that is reduced to integrability, to be $\frac{1}{V}$ and thus, which equations according to the separation of the variables it is allowed to produce, for from these we are able to assign the multiplier, which renders these integrable.

COROLLARY 1

483. Hence a method of integrating differential equations through multipliers becomes apparent and equally the previous method, with the aid of the separation of variables, because that separation for some equation where it should succeed, supplies the multiplier.

COROLLARY 2

484. But against the method of integrating by multipliers another is more widely apparent which does not agree [to the separation of the variables], if it should be allowed to assign multipliers to equations of this kind, just as these above must lead to the separation of the variables.

SCHOLIUM

485. But if on being reduced to a separation a suitable multiplier can be elicited, yet it is not yet understood, how from a known multiplier, the separation of the variables ought to be performed [*i.e.* the process can be performed in reverse]; whereby also by this reason the method of integrating by multipliers is considered to be preferred by far to the other. For although this far that separation has led us to finding the multipliers, yet there is no doubt, why a way of finding the multipliers cannot be given without being with respect to the separation of the variables, and that this way is allowed even if it should be unknown to us. But that becomes clearer little by little as, if we know suitable multipliers for several equations, which thus far it has been possible to elicit from the separation of the variables, from which we might investigate [this property of multipliers independently] in the following examples.

EXAMPLE 1

486. *With the proposed differential equation of the first order*

$$dx(ax+\beta y+\gamma)+dy(\delta x+\varepsilon y+\zeta)=0$$

for that to assign a suitable multiplier.

This equation according to the separation may be prepared by putting in the first place [§ 417]

$$ax+\beta y+\gamma=r \quad \text{and} \quad \delta x+\varepsilon y+\zeta=s$$

and thus

$$adx+\beta dy=dr \quad \text{and} \quad \delta dx+\varepsilon dy=ds,$$

from which there arises

$$dx=\tfrac{\varepsilon dr-\beta ds}{\alpha\varepsilon-\beta\delta} \ \text{ and } \ dy=\tfrac{\alpha ds-\delta dr}{\alpha\varepsilon-\beta\delta},$$

and hence our equation with the denominator omitted as constant will be

$$\varepsilon rdr-\beta rds+\alpha sds-\delta sdr=0;$$

which since it shall be homogeneous, divided by $\varepsilon rr-(\beta+\delta)rs+\alpha ss$ becomes integrable. Which likewise is deduced from separation ; for on putting $r=su$ there is produced

$$\varepsilon ssudu+\varepsilon suuds-\beta suds+\alpha sds-\delta ssdu-\delta suds=0$$

or

$$ssdu(\varepsilon u-\delta)+sds(\varepsilon uu-\beta u-\delta u+\alpha)=0,$$

which is separated on dividing by $ss(\varepsilon uu-\beta u-\delta u+\alpha)$. Whereby our proposed multiplier of the proposed equation is

$$\frac{1}{ss(\varepsilon uu-\beta u-\delta u+\alpha)}=\frac{1}{\varepsilon rr-\beta rs-\delta rs+\alpha ss}=\frac{1}{r(\varepsilon r-\beta s)+s(\alpha s-\delta r)}$$

which with the values restored shall make

$$\frac{1}{(\alpha x+\beta y+\gamma)((\alpha\varepsilon-\beta\delta)x+\gamma\varepsilon-\beta\zeta)+(\delta x+\varepsilon y+\zeta)((\alpha\varepsilon-\beta\delta)y+\alpha\zeta-\gamma\delta)}$$

and on expanding out becomes

$$1:\left\{\begin{array}{c}(\alpha\varepsilon-\beta\delta)(\alpha xx+(\beta+\delta)xy+\varepsilon yy+\gamma x+\zeta y)+\alpha\zeta\zeta-(\beta+\delta)\gamma\zeta+\gamma\gamma\varepsilon\\ +(\alpha\gamma\varepsilon-(\beta-\delta)\alpha\zeta-\gamma\delta\delta)x+(\alpha\varepsilon\zeta+(\beta-\delta)\gamma\varepsilon-\beta\beta\zeta)y\end{array}\right\}$$

Whereby this equation integrable by itself shall be

$$\frac{dx(ax+\beta y+\gamma)+dy(\delta x+\varepsilon y+\zeta)}{(\alpha\varepsilon-\beta\delta)(\alpha xx+(\beta+\delta)xy+\varepsilon yy+\gamma x+\zeta y)+Ax+By+C}=0$$

with arising

$$A=\alpha\gamma\varepsilon-(\beta-\delta)\alpha\zeta-\gamma\delta\delta,$$
$$B=\alpha\varepsilon\zeta+(\beta-\delta)\gamma\varepsilon-\beta\beta\zeta,$$
$$C=\alpha\zeta\zeta-(\beta+\delta)\gamma\zeta+\gamma\gamma\varepsilon.$$

COROLLARY

487. Even if perhaps it happens that $\alpha\varepsilon-\beta\delta=0$, here the multiplier is not disturbing, since the separation indeed does not succeed from this operation. For let $\alpha=ma,\quad \beta=mb,\quad \delta=na,\quad \varepsilon=nb,$ in order that this equation may be had

$$dx(m(ax+by)+\gamma)+dy(n(ax+by)+\zeta)=0;$$

on account of

$$A=a(na-mb)(m\zeta-n\gamma),\quad B=b(na-mb)(m\zeta-n\gamma)$$

and

$$C=(m\zeta-n\gamma)(a\zeta-b\gamma)$$

with the common factor omitted the multiplier is

$$\frac{1}{(na-mb)(ax+by)+a\zeta-b\gamma}=0$$

thus so that this equation is integrable by itself :

$$\frac{(ax+by)(mdx+ndy)+\gamma dx+\zeta dy}{(na-mb)(ax+by)+a\zeta-b\gamma}=0.$$

EXEMPLUM 2

488. *To find a suitable multiplier for the proposed differential equation*

$$ydx(c+nx)-dy(y+a+bx+nxx)=0.$$

Let the substitution be made [§ 433]

$$\frac{y(c+nx)}{y+a+bx+nxx}=u \quad \text{or} \quad y=\frac{u(a+bx+nxx)}{c+nx-u},$$

in order that our equation can be assembled in this form :

$$ydx(c+nx)-\frac{ydy(c+nx)}{u}=0 \quad \text{or} \quad \frac{(c+nx)}{u}(udx-dy)=0$$

or

$$\frac{yy(c+nx)}{u}\left(\frac{dy}{y}-\frac{udx}{y}\right)=0;$$

indeed properly there must be a warning here, lest any factor should be omitted. But with the substitution made, there is found :

$$\frac{dy}{y}-\frac{udx}{y}=\frac{du}{u}+\frac{dx(b+2nx)}{a+bx+nxx}+\frac{du-ndx}{c+nx-u}-\frac{dx(c+nx-u)}{a+bx+nxx}$$
$$=\frac{du(c+nx)}{u(c+nx-u)}-\frac{dx(na+cc-bc+(b-2c)u+uu)}{(c+nx-u)(a+bx+nxx)}.$$

From which our equation can adapt this form

$$\frac{yy(c+nx)^2}{u(c+nx-u)}\left(\frac{du}{u}-\frac{dx(na+cc-bc+(b-2c)u+uu)}{(a+bx+nxx)(c+nx)}\right)=0,$$

which hence will be separated on multiplying by this multiplier

$$\frac{u(c+nx-u)}{yy(c+nx)^2(na+cc-bc+(b-2c)u+uu)};$$

for then there emerges

$$\frac{du}{u\left(na+cc-bc+(b-2c)u+uu\right)}-\frac{dx}{(a+bx+nxx)(c+nx)}=0.$$

Therefore from which we can follow up with the multiplier sought, where in place of u it is needed only to be replaced by its own value ; but then the multiplier is found :

$$\frac{a+bx+nxx}{n(a+bx+nxx)y^3+(a+bx+nxx)\left(2na-bc+n(b-2c)x\right)yy+(na+cc-bc)(a+bx+nxx)^2y},$$

which is reduced to this form

$$\frac{1}{ny^3+(2na-bc)yy+n(b-2c)xyy+(na+cc-bc)(a+bx+nxx)y}.$$

EXAMPLE 3

489. *With the proposed differential equation*

$$\frac{ndx(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}+(x-y)dy=0$$

to find a multiplier, which renders that integrable.

Above we have placed (§ 434) :

$$y=\frac{x-u}{1+xu} \quad \text{or} \quad u=\frac{x-y}{1+xy}$$

from which there becomes

$$x-y=\frac{u(1+xx)}{1+xu} \quad \text{and} \quad 1+yy=\frac{(1+xx)(1+uu)}{(1+xu)^2},$$

and hence our equation adopts this form

$$\frac{ndx(1+xx)(1+uu)^{\frac{3}{2}}}{(1+xu)^3}+\frac{udx(1+xx)(1+uu)-udu(1+xx)^2}{(1+xu)^3}=0,$$

which first multiplied by $(1+xu)^3$, then divided by

$$(1+xx)^2(1+uu)\left(u+n\sqrt{(1+uu)}\right)$$

is separated. Whereby the multiplier of our equation shall be

$$\frac{(1+xu)^3}{(1+xx)^2(1+uu)\left(u+n\sqrt{(1+uu)}\right)},$$

which first on account of $1+uu=\frac{(1+yy)(1+xu)^2}{1+xx}$ becomes $\frac{1+xu}{(1+xx)(1+yy)\left(u+n\sqrt{(1+uu)}\right)}$.

Now on account of $u=\frac{x-y}{1+xy}$ there is

$$\sqrt{(1+uu)}=\frac{\sqrt{(1+xx)(1+yy)}}{1+xy} \text{ and } 1+xu=\frac{1+xx}{1+xy}$$

and thus our multiplier is deduced

$$\frac{1}{(1+yy)\left(x-y+n\sqrt{(1+xx)(1+yy)}\right)},$$

thus so that this equation shall be integrable by itself :

$$\frac{ndx(1+yy)\sqrt{(1+yy)}+(x-y)dy\sqrt{(1+xx)}}{(1+yy)\left(x-y+n\sqrt{(1+xx)(1+yy)}\right)\sqrt{(1+xx)}}=0,$$

for the integration of which I shall not tarry, since now I may have presented this integral above.

EXAMPLE 4

490. *This equation supplies another example worthy of note*

$$ydx-xdy+ax^n ydy\left(x^n+b\right)^{\frac{1}{n}}=0.$$

Which if it is represented in this form :

$$xdy-ydx+\tfrac{1}{b}x^{n+1}dy=\tfrac{1}{b}x^{n+1}dy+ax^n ydy\left(x^n+b\right)^{\frac{1}{n}},$$

it comes about, so that each side becomes integrable, if it is taken by this multiplier

$$\frac{y^{n-1}}{x^{n+1}+abx^n y\left(x^n+b\right)^{\frac{1}{n}}};$$

according to which being found from the separation of the variables this handy [non adeo obvia : not exactly in the way] substitution may be applied

$$\frac{x}{\left(x^n+b\right)^{\frac{1}{n}}}=vy,$$

from which there becomes fit $x^n=\frac{bv^ny^n}{1-v^ny^n}$, and hence the equation

$$\frac{ydx-xdy}{\left(x^n+b\right)^{\frac{1}{n}}}+ax^nydy=0$$

changes into this :

$$\frac{yydv+v^{n+1}y^{n+1}dy+abv^ny^{n+1}dy}{1-v^ny^n}=0,$$

which multiplied by $\frac{1-v^ny^n}{yyv^n(ab+v)}$ is separated :

$$\frac{dv}{v^n(ab+v)}+y^{n-1}dy=0,$$

from which likewise that multiplier is deduced.

EXAMPLE 5

491. *With the proposed differential equation*

$$dy+yydx-\frac{adx}{x^4}=0$$

to find the multiplier, by which that is rendered integrable.

Following § 436 there is put $x=\frac{1}{t}$ and on account of $dx=-\frac{dt}{tt}$ our formula becomes $dy-\frac{yydt}{tt}+attdt$, in which again there is put in place $y=t-ttz$, and there emerges

$$-\,tt\left(dz+zzdt-adt\right),$$

which divided by $tt\left(zz-a\right)$ is separated ; hence our equation divided by

$$tt\left(zz-a\right)=\frac{(t-y)^2-at^4}{tt}=\left(1-xy\right)^2-\frac{a}{xx}$$

shall become integrable, from which the multiplier shall be :

$$\frac{xx}{xx(1-xy)^2-a}$$

and the equation integrable by itself

$$\frac{x^4dy+x^4yydx-adx}{x^4(1-xy)^2-axx}=0.$$

Now *x* may be considered as constant and from *dy* the integral is produced

$$\frac{1}{2\sqrt{a}}l\frac{x(1-xy)+\sqrt{a}}{\sqrt{a}-x(1-xy)}+X;$$

from which in order that the value of X may be found, it may be differentiated again and there is produced

$$\frac{2xydx-dx}{xx(1-xy)^2-a}+dX=\frac{x^4yydx-adx}{x^4(1-xy)^2-axx},$$

from which

$$dX=\frac{x^4yydx-adx-2x^3ydx+xxdx}{x^4(1-xy)^2-axx}=\frac{dx}{xx}\text{ and }X=-\frac{1}{x}+C;$$

whereby the complete integration shall be :

$$l\frac{\sqrt{a}+x(1-xy)}{\sqrt{a}-x(1-xy)}=\frac{2\sqrt{a}}{x}+C.$$

SCHOLIUM

492. Behold therefore several cases of differential equations, for which we know the multipliers, from the consideration of which this noteworthy investigation may be seen to be have been helped quite a lot. But nevertheless at this stage we are a long way from a sure method of finding suitable multipliers for whatever case, yet hence we are able to deduce the forms of the equations, so that they may be returned by given multipliers ; because these calculations are of the greatest use in this arduous task, in the following chapter we will investigate equations, in which given multipliers can be agreed on ; clearly here examples have set out suitable forms of multipliers supplied by us, on which our investigation will be able to build.

CAPUT II

DE INTEGRATIONE AEQUATIONUM DIFFERENTIALIUM OPE MULTIPLICATORUM

PROBLEMA 58

443. *Propositam aequationem differentialem examinare, utrum per se sit integrabilis necne.*

SOLUTIO

Dispositis omnibus aequationis terminis ad eandem partem signi aequalitatis, ut huiusmodi habeatur forma $Pdx+Qdy=0$, aequatio per se erit integrabilis, si formula $Pdx+Qdy$ fuerit verum differentiale functionis cuiuspiam binarum variabilium *x* et *y*. Hoc autem evenit, uti in *Calculo Differentiali* ostendimus, si differentiale ipsius *P* sumta sola *y* variabili ad *dy* eandem habeat rationem ac differentiale ipsius *Q* sumta sola *x* variabili ad *dx*, seu adhibito signandi modo, quo in *Calculo Differentiali* sumus usi, si fuerit

$$\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right).$$

Nam si *Z* sit ea functio, cuius differentiale est $Pdx+Qdy$, erit hoc signandi modo hinc ergo sequitur

$$P=\left(\frac{dZ}{dx}\right) \text{ et } Q=\left(\frac{dZ}{dy}\right);$$

hinc ergo sequitur

$$\left(\frac{dP}{dy}\right)=\left(\frac{ddZ}{dxdy}\right) \text{ et } \left(\frac{dQ}{dx}\right)=\left(\frac{ddZ}{dydx}\right).$$

At est

$$\left(\frac{ddZ}{dxdy}\right)=\left(\frac{ddZ}{dydx}\right),$$

unde colligitur

$$\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right).$$

Quare proposita aequatione differentiali $Pdx+Qdy=0$ utrum ea per se sit integrabilis necne, hoc modo dignoscetur. Quaerantur per differentiationem valores $\left(\frac{dP}{dy}\right)$ et $\left(\frac{dQ}{dx}\right)$, qui si fuerint inter se aequales, aequatio per se erit integrabilis; sin autem hi valores sint inaequales, aequatio non erit per se integrabilis.

COROLLARIUM 1

444. Omnes ergo aequationes differentiales, in quibus variabiles sunt a se invicem separatae, per se sunt integrabiles; habebunt enim huiusmodi formam $Xdx + Ydy = 0$, ut X sit functio solius x et Y solius y, eritque propterea $\left(\frac{dX}{dy}\right) = 0$ et $\left(\frac{dY}{dx}\right) = 0$.

COROLLARIUM 2

445. Vicissim igitur si proposita aequatione differentiali $Pdx + Qdy = 0$ fuerit $\left(\frac{dP}{dy}\right) = 0$ et $\left(\frac{dQ}{dx}\right) = 0$, variabiles in ea erunt separatae; littera enim P erit functio tantum ipsius x et Q tantum ipsius y. Unde aequationes separatae quasi primum genus aequationum per se integrabilium constituunt.

COROLLARIUM 3

446. Evidens autem est fieri posse, ut sit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, etiamsi neuter horum valorum sit nihilo aequalis. Dantur ergo aequationes per se integrabiles, licet variabiles in iis non sint separatae.

SCHOLION

447. Criterium hoc, quo aequationes per se integrabiles agnoscimus, maximi est momenti in hac, quam tradere suscipimus, methodo integrandi. Quodsi enim aequatio deprehendatur per se integrabilis, eius integrale per praecepta iam exposita inveniri potest; sin autem aequatio non fuerit per se integrabilis, semper dabitur quantitas, per quam, si ea multiplicetur, fiat per se integrabilis; unde totum negotium eo revocabitur, ut proposita aequatione quacunque per se non integrabili inveniatur multiplicator idoneus, qui eam reddat per se integrabilem; qui si semper inveniri posset, nihil amplius in hac methodo integrandi esset desiderandum. Verum haec investigatio rarissime succedit ac vix adhuc latius patet quam ad eas aequationes, quas ope separationis variabilium iam tractare docuimus; interim tamen non dubito hanc methodum praecedenti longe praeferre, cum ad naturam aequationum magis videatur accommodata atque etiam ad aequationes differentiales altiorum graduum pateat, in quibus separatio variabilium nullius est usus.

PROBLEMA 59

448. *Aequationis differentialis, quam per se integrabilem esse constat, integrale invenire.*

SOLUTIO

Sit aequatio differentialis $Pdx + Qdy = 0$; in qua cum sit $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$, erit $Pdx + Qdy$ differentiale cuiuspiam functionis binarum variabilium x et y, quae sit Z, ut sit $dZ = Pdx + Qdy$. Cum ergo habeamus hanc aequationem $dZ = 0$, erit integrale quaesitum $Z = C$. Totum negotium ergo huc redit, ut ista functio Z eruatur, quod, eum

sciamus esse $dZ = Pdx + Qdy$, haud difficulter praestabitur. Nam quia sumta tantum *x* variabili et altera *y* ut constante spectata est $dZ = Pdx$, habemus hic formulam differentialem simplicem unicam variabilem *x* involventem, quae per praecepta superioris sectionis integrata dabit $Z = \int Pdx + \text{Const.}$, ubi autem notandum est in hac constante quantitatem hic pro constanti habitam *y* utcunque inesse posse,
unde eius loco scribatur *Y* ut sit

$$Z = \int Pdx + Y.$$

Deinde simili modo *x* pro constante habeatur spectata sola *y* ut variabili, et cum sit $dZ = Qdy$, erit quoque $Z = \int Qdy + \text{Const.}$, quae constans autem quantitatem *x* involvet, ita ut sit functio ipsius *x*, qua posita X erit

$$Z = \int Qdy + X.$$

Quanquam autem neque hic functio *X* neque ibi functio *Y* determinatur, tamen, quia esse debet $\int Pdx + Y = \int Qdy + \text{X}$, hinc utraque determinabitur.
Cum enim sit

$$\int Pdx - \int Qdy = \text{X} - Y$$

haec quantitas $\int Pdx - \int Qdy$ semper in eiusmodi binas partes distinguetur, quarum altera est functio ipsius *x* tantum et altera ipsius *y* tantum, unde valores *X* et *Y* sponte cognoscuntur.

COROLLARIUM 1

449. Cum sit $Q = \left(\frac{dZ}{dy}\right)$, duplici integratione ne opus quidem est. Invento enim integrali $\int Pdx$ id iterum differentietur sumta sola *y* variabili prodeatque *Vdy*, unde necesse est fiat $Vdy + dY = Qdy$ ideoque

$$dY = Qdy - Vdy = (Q - V)dy.$$

COROLLARIUM 2

450. Aequationum ergo per se integrabilium $Pdx + Qdy = 0$ integratio ita perficietur. Quaeratur integrale $\int Pdx$ spectata *y* constante idque rursus differentietur spectata sola *y* variabili, unde prodeat *Vdy;* tum $Q - V$ erit functio ipsius *y* tantum, unde quaeratur $Y = \int (Q - V)dy$, eritque aequatio integralis $\int Pdx + Y = \text{Const}.$

COROLLARIUM 3

451. Vel quaeratur $\int Qdy$ spectata x constante, quod integrale rursus differentietur sumta x variabili, y autem constante, unde prodeat Udx; tum certe erit $P-U$ functio ipsius x tantum, unde quaeratur $X=\int(p-U)dx$, eritque aequatio integralis quaesita $\int Qdy+X=\text{Const}$.

COROLLARIUM 4

452. Ex rei natura patet perinde esse, utra via procedatur; necesse enim est ad eandem aequationem integralem perveniri, si quidem aequatio differentialis proposita per se fuerit integrabilis. Tum autem certe eveniet, ut priori casu $Q-V$ sit functio solius y, posteriori autem $P-U$ functio solius x.

SCHOLION

453. Haec methodus integrandi etiam tentari posset, antequam exploratum esset, num aequatio integrabilis existat; si enim vel in modo Corollarii 2 eveniret, ut $Q-V$ esset functio ipsius y tantum, vel in modo Corollarii 3, ut $P-U$ esset functio ipsius x tantum, hoc ipsum indicio foret aequationem esse per se integrabilem. Verum. tamen praestat ante omnia scrutari, an aequatio integrabilis sit per se necne, seu an sit $\left(\frac{dP}{dy}\right)=\left(\frac{dQ}{dx}\right)$, quoniam hoc examen sola differentiatione absolvitur. Exempla igitur aliquot aequationum per se integrabilium afferamus, quo non solum methodus integrandi, sed etiam insignes illae proprietates, quas commemoravimus, clarius intelligantur.

EXEMPLUM 1

454. *Aequationem per se integrabilem*

$$dx(\alpha x+\beta y+\gamma)+dy(\beta x+\delta y+\varepsilon)=0$$

integrare.

Cum hic sit $P=\alpha x+\beta y+\gamma$ et $Q=\beta x+\delta y+\varepsilon$, erit $\left(\frac{dP}{dy}\right)=\beta$ et $\left(\frac{dQ}{dx}\right)=\beta$, qua aequalitate integrabilitas per se confirmatur. Quaeratur ergo per Corollarium 2 spectata y ut constante

$$\int Pdx=\tfrac{1}{2}\alpha xx+\beta yx+\gamma x;$$

erit $Vdy=\beta xdy$ et

$$(Q-V)dy=dy(\delta y+\varepsilon)=dY$$

ideoque

$$Y=\tfrac{1}{2}\delta yy+\varepsilon y,$$

unde integrale erit

$$\tfrac{1}{2}axx+\beta yx+\gamma x+\tfrac{1}{2}\delta yy+\varepsilon y=C.$$

Modo autem Corollarii 3 spectata x constante erit

$$\int Qdy=\beta xy+\tfrac{1}{2}\delta yy+\varepsilon y,$$

quae spectata y constante praebet $Udx=\beta ydx$ hincque

$$(P-U)dx=(\alpha x+\gamma)dx \text{ et } X=\tfrac{1}{2}axx+\gamma x,$$

unde $\int Qdy+X=C$ integraIe dat ut ante. Hinc simul etiam intelligitur esse

$$\int Pdx-\int Qdy=\tfrac{1}{2}\alpha xx+\gamma x-\tfrac{1}{2}\delta yy-\varepsilon y,$$

quae in duas functiones $X-Y$ sponte dispescitur.

EXEMPLUM 2

455. *Aequationem per se integrabilem*

$$\frac{dy}{y}=\frac{xdy-ydx}{y\sqrt{(xx+yy)}} \quad seu \quad \frac{dx}{\sqrt{(xx+yy)}}+\frac{dy}{y}\left(1-\frac{x}{\sqrt{(xx+yy)}}\right)=0$$

integrare.

Cum hic sit $P=\frac{1}{\sqrt{(xx+yy)}}$ et $Q=\frac{1}{y}-\frac{x}{y\sqrt{(xx+yy)}}$,

pro charactere integrabilitatis per se cognoscendo est

$$\left(\frac{dP}{dy}\right)=\frac{-y}{(xx+yy)^{\frac{3}{2}}} \quad \text{et} \quad \left(\frac{dQ}{dx}\right)=\frac{-y}{(xx+yy)^{\frac{3}{2}}}$$

qui bini valores utique sunt aequales. Iam pro integrali inveniendo utamur regula Corollarii 2 et habebimus

$$\int Pdx=l\left(x+\sqrt{(xx+yy)}\right) \quad \text{et} \quad Vdy=\frac{ydy}{\left(x+\sqrt{(xx+yy)}\right)\sqrt{(xx+yy)}}$$

seu supra et infra per $\sqrt{(xx+yy)}-x$ multiplicando

$$V=\frac{\sqrt{(xx+yy)}-x}{y\sqrt{(xx+yy)}}=\frac{1}{y}-\frac{x}{y\sqrt{(xx+yy)}}$$

unde

$$Q-V=0 \text{ et } Y=\int(Q-V)dy=0,$$

sicque integrale quaesitum

$$l\left(x+\sqrt{(xx+yy)}\right)=\text{Const.}$$

Per regulam Corollarii 3 habemus

$$\int Qdy=ly-x\int\frac{dy}{y\sqrt{(xx+yy)}},$$

at posito $y=\frac{1}{z}$ est

$$\int\frac{dy}{y\sqrt{(xx+yy)}}=-\int\frac{dz}{y\sqrt{(xxzz+1)}}=-\frac{1}{x}l\left(xz+\sqrt{(xxzz+1)}\right)$$

ergo

$$\int Qdy=ly+l\frac{x+\sqrt{(xx+yy)}}{y}=l\left(x+\sqrt{(xx+yy)}\right),$$

unde

$$Udx=\frac{dx}{\sqrt{(xx+yy)}},$$

hinc $(P-U)dx=0$.

EXEMPLUM 3

456. *Aequationem per se integrabilem*

$$(xx+yy-aa)dy+(aa+2xy+xx)dx=0$$

integrare.

Hic ergo est $P=aa+2xy+xx$ et $Q=xx+yy-aa$, unde $\left(\frac{dP}{dy}\right)=2x$ et $\left(\frac{dQ}{dx}\right)=2x$, quae aequalitas integrabilitatem per se innuit. Tum vero est

$$\int Pdx=aax+xxy+\tfrac{1}{3}x^3 \text{ et } Vdy=xxdy,$$

unde

$$(Q-V)dy=(yy-aa)dy \text{ et } Y=\tfrac{1}{3}y^3-aay.$$

Ergo integrale

$$aax+xxy+\tfrac{1}{3}x^3+\tfrac{1}{3}y^3-aay=\text{Const.}$$

Altero modo est

$$\int Qdy=xxy+\tfrac{1}{3}y^3-aay \text{ hincque } Udx=2xydx,$$

ergo

$$(P-U)dx=(aa+xx)dx \text{ et } X=aax+\tfrac{1}{3}x^3,$$

unde integrale oritur ut ante.

SCHOLION

457. In his exemplis licuit integrale $\int Pdx$ actu exhibere indeque eius differentiale Vdy sumta sola y variabili assignare. Quodsi autem hoc integrale $\int Pdx$ evolvi nequeat, haud liquet, quomodo inde differentiale Vdy elici possit, quandoquidem formula $\int Pdx$ in se spectata constantem quamcunque, quae etiam y in se implicet, complectitur. Tum igitur quomodo procedendum sit, videamus.

Ponamus

$$Z=\int Pdx+Y,$$

et cum quaeratur $\left(\frac{d.\int Pdx}{dy}\right)=V$, ob $\int Pdx=Z-Y$ erit $V=\left(\frac{dZ}{dy}\right)-\frac{dY}{dy}$. At est $\left(\frac{dZ}{dx}\right)=P$, ergo $\left(\frac{ddZ}{dxdy}\right)=\left(\frac{dP}{dy}\right)=\left(\frac{dV}{dx}\right)$ ob $\left(\frac{dZ}{dy}\right)=V+\frac{dY}{dy}$. Hinc erit

$$V=\int dx\left(\frac{dP}{dy}\right);$$

quare quantitas V invenitur per integrationem huius formulae $\int dx\left(\frac{dP}{dy}\right)$, in qua y ut constans spectatur, postquam in valore $\frac{dP}{dy}$ inveniendo sola y variabilis esset assumta. Verum cum hic denuo constans cum y implicetur, hinc illa functio Y, quam quaerimus, non determinatur. Ratio huius incommodi manifesto in ambiguitate integralium $\int Pdx$ et $\int dx\left(\frac{dP}{dy}\right)$ est sita, dum utrumque functiones arbitrarias ipsius y recipit. Remedium ergo afferetur, si utrumque integrale certa quadam conditione determinetur. Ita quando integrale $\int Pdx$ ita accipi ponimus, ut evanescat posito $x=f$, ubi quidem constantem f pro lubitu accipere licet, tum eadem lege alterum integrale $\int dx\left(\frac{dP}{dy}\right)$ capiatur. Quo facto erit $Q-\int dx\left(\frac{dP}{dy}\right)$ functio ipsius y tantum et aequationis $Pdx+Qdy=0$ integrale erit

$$\int Pdx+\int dy\left(Q-\int dx\left(\frac{dP}{dy}\right)\right)=\text{Const.}$$

dummodo ambo integralia $\int Pdx$ et $\int dx\left(\frac{dP}{dy}\right)$, in quibus y ut constans tractatur, ita determinentur, ut evanescant, dum in utroque ipsi x idem valor f tribuitur. Quare hinc istam colligimus regulam:

REGULA PRO INTEGRATIONE AEQUATIONIS PER SE INTEGRABILIS $Pdx + Qdy = 0$ IN QUA $\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right)$

458. *Quaerantur integralia* $\int Pdx$ *et* $\int dx\left(\frac{dP}{dy}\right)$, *spectando y ut constantem ita, ut ambo evanescant, dum ipsi x certus quidam valor, puta x = f, tribuitur.*
Tum erit $Q - \int dx\left(\frac{dP}{dy}\right)$ *functio ipsius y tantum, quae sit = Y, et integrale quaesitum erit* $\int Pdx + \int Ydy = \text{Const.}$ *Vel, quod eodem redit, quaerantur integralia* $\int Qdy$ *et* $\int dy\left(\frac{dQ}{dx}\right)$ *spectando x ut constantem ita, ut ambo evanescant, dum ipsi y certus quidam valor, puta* $y = g$, *tribuitur; tum* $P - \int dy\left(\frac{dP}{dx}\right)$ *erit functio ipsius x tantum, qua posita = X erit integrale quaesitum* $\int Qdy + \int Xdx = \text{Const}$.

DEMONSTRATIO

Veritatem huius regulae ex praecedentibus perspicere licet, si cui forte precario assumsisse videamur ambas formulas $\int Pdx$ et $\int dx\left(\frac{dP}{dy}\right)$ eadem lege determinari debere, ut, dum ipsi *x* certus quidam valor, puta *x = f,* tribuitur, ambae evanescant. Sed ne forte quis putet alteram integrationem pari iure secundum aliam legem determinari posse, hanc demonstrationem addo. Prima quidem integratio ab arbitrio nostro pendet, quam ergo ita determinari assumamus, ut integrale $\int Pdx$ evanescat posito *x = f;* quo facto dico alterum integrale $\int dx\left(\frac{dP}{dy}\right)$ necessario per eandem conditionem determinari oportere. Sit enim $\int Pdx = Z$ eritque *Z* eiusmodi functio ipsarum *x* et *y,* quae evanescit posito *x = f;* habebit ergo factorem $f - x$ vel eius quampiam potestatem positivam $Z = (f - x)^{\lambda}$, ita ut sit $Z = (f - x)^{\lambda} T$. Nunc quia $\int dx\left(\frac{dP}{dy}\right)$ exprimit valorem ipsius $\left(\frac{dZ}{dy}\right)$, erit $\int dx\left(\frac{dP}{dy}\right) = (f - x)^{\lambda}\left(\frac{dT}{dy}\right)$, ex quo manifestum est hoc integrale etiam evanescere posito *x = f,* ita ut huius integralis determinatio non amplius arbitrio nostro relinquatur. Hoc posito erit utique aequationis per se integrabilis $Pdx + Qdy = 0$ integrale

$$\int Pdx + \int Ydy = \text{Const}.$$

existente

$$Y = Q - \int dx\left(\frac{dP}{dy}\right);$$

nam posito $\int Pdx = Z$, quatenus scilicet in hac integratione *y* pro constante habetur, habetur haec aequatio $Z + \int Ydy = \text{Const.}$, quam esse integrale quaesitum vel ex ipsa differentiatione patebit. Cum emm sit

$$dZ = Pdx + dy\left(\tfrac{dZ}{dy}\right) = Pdx + dy\int dx\left(\tfrac{dP}{dy}\right),$$

erit aequationis inventae differentiale

$$Pdx + dy\int dx\left(\tfrac{dP}{dy}\right) + Ydy = 0,$$

sed $Y = Q - \int dx\left(\tfrac{dP}{dy}\right)$, unde prodit $Pdx + Qdy = 0$, quae est ipsa aequatio differentialis proposita. Quod autem sit $Q - \int dx\left(\tfrac{dP}{dy}\right)$ functio ipsius *y* tantum, inde sequitur, quoniam aequatio differentialis per se est integrabilis.

THEOREMA

459. *Pro omni aequatione, quae per se non est integrabilis, semper datur quantitas, per quam ea multiplicata redditur integrabilis.*

DEMONSTRATIO

Sit $Pdx + Qdy = 0$ aequatio differentialis et concipiamus eius integrale completum, quod erit aequatio quaedam inter *x* et *y*, in quam constans quantitas arbitraria ingrediatur. Ex hac aequatione eruatur haec ipsa constans arbitraria, ut prodeat huiusmodi aequatio

Const. = functioni cuidam ipsarum *x* et y,

quae differentiata praebeat

$$0 = Mdx + Ndy;$$

quae aequatio iam a constante illa arbitraria per integrationem ingressa est libera ideoque necesse est, ut haec aequatio differentialis conveniat cum proposita, alioquin integrale suppositum non esset verum. Oportet ergo, ut relatio inter *dx* et *dy* utrinque prodeat eadem, unde erit

$$\frac{P}{Q} = \frac{M}{N}$$

ideoque

$$M = LP \quad \text{et} \quad N = LQ.$$

Sed quia $Mdx + Ndy$ est verum differentiale ex differentiatione cuiuspiam functionis ipsarum *x* et *y* ortum, est $\left(\tfrac{dM}{dy}\right) = \left(\tfrac{dN}{dx}\right)$. Quare pro aequatione $Pdx + Qdy = 0$ dabitur certo quidam multiplicator *L*, ut sit

$$\left(\frac{d.LP}{dy}\right)=\left(\frac{d.LQ}{dx}\right),$$

seu ut aequatio per L multiplicata fiat per se integrabilis,

COROLLARIUM 1

460. Pro omni ergo aequatione $Pdx+Qdy=0$ datur eiusmodi functio L, ut sit $\left(\frac{d.LP}{dy}\right)=\left(\frac{d.LQ}{dx}\right)$ ideoque evolvendo

$$L\left(\frac{dP}{dy}\right)+P\left(\frac{dL}{dy}\right)=L\left(\frac{dQ}{dx}\right)+Q\left(\frac{dL}{dx}\right)$$

seu

$$L\left(\left(\frac{dP}{dy}\right)-\left(\frac{dQ}{dx}\right)\right)=Q\left(\frac{dL}{dx}\right)-P\left(\frac{dL}{dy}\right);$$

quae functio L si fuerit inventa, aequatio differentialis $LPdx+LQdy=0$ per se erit integrabilis.

COROLLARIUM 2

461. In aequatione proposita loco Q tuto unitatem scribere licet, quia omnis aequatio hac forma $Pdx+dy=0$ repraesentari potest. Hinc inventio multiplicatoris L, qui eam reddat per se integrabilem, pendet a resolutione huius aequationis

$$L\left(\frac{dP}{dy}\right)=\left(\frac{dL}{dx}\right)-P\left(\frac{dL}{dy}\right)$$

ubi notandum est esse

$$\mathrm{d}L=\left(\frac{dL}{dx}\right)dx+dy\left(\frac{dL}{dy}\right).$$

SCHOLION

462. Quoniam hic quaeritur functio binarum variabilium x et y, quarum relatio mutua minime spectatur, quam involvit aequatio $Pdx+Qdy=0$, haec investigatio in nostrum librum secundum incurrit, ubi huiusmodi functio ex data quadam differentialium relatione indagari debet. In hac enim investigatione non attendimus ad aequationem propositam, qua formula $Pdx+Qdy$ nihilo aequalis reddi debet, sed absolute quaeritur multiplicator L, per quem formula $Pdx+Qdy$ multiplicata abeat in verum differentiale cuiuspiam functionis finitae, quae sit Z, ita ut habeatur $dZ=LPdx+LQdy$. Quo multiplicatore L invento tum demum aequalitas $Pdx+Qdy=0$ spectatur indeque concluditur functionem Z quantitati constanti aequari oportere. Cum igitur minime expectari queat, ut methodum tradamus huiusmodi multiplicatores pro quavis aequatione differentiali proposita inveniendi, eos casus percurramus, quibus talis multiplicator constat, undecunque sit repertus. Interim tamen ad pleniorem usum huius methodi notasse iuvabit, statim atque unum multiplicatorem pro quapiam aequatione differentiali cognoverimus, ex eo facile innumerabiles alios deduci posse, qui pariter aequationem propositam per se integrabilem reddant.

PROBLEMA 60

463. *Dato uno multiplicatore L, qui aequationem* $Pdx+Qdy=0$ *per se integrabilem reddat, invenire innumerabiles alios multiplicatores, qui idem officium praestent.*

SOLUTIO

Cum ergo $L(Pdx+Qdy)$ sit differentiale verum cuiuspiam functionis *Z*, quaeratur per superiora praecepta haec functio *Z*, ita ut sit

$$L(Pdx+Qdy)=dZ\ ,$$

et nunc manifestum est hanc formulam *dZ* integrationem etiam esse admissuram, si per functionem quamcunque ipsius *Z*, quam ita $\varphi:Z$ indicemus, multiplicetur. Cum igitur etiam integrabilis sit haec formula

$$(Pdx+Qdy)L\varphi:Z\ ,$$

erit quoque $L\varphi:Z$ multiplicator aequationis propositae $Pdx+Qdy=0$, qui eam reddat integrabilem. Quare invento uno multiplicatore *L* quaeratur per integrationem $Z=\int L(Pdx+Qdy)$ ac tum expressio $L\varphi:Z$, ubi pro $\varphi:Z$ functio quaecunque ipsius *Z* assumi potest, dabit infinitos alios multiplicatores idem officium praestantes.

SCHOLION

464. Tametsi sufficiat pro quavis aequatione differentiali unicum multiplicatorem cognovisse, tamen occurrunt casus, quibus perquam utile est plures, imo infinitos multiplicatores in promtu habere. Veluti si aequatio proposita in duas partes commode discerpatur huiusmodi

$$(Pdx+Qdy)+(Rdx+Sdy)=0$$

atque omnes multiplicatores constent, quibus utraque pars seorsim $Pdx+Qdy$ et $Rdx+Sdy$ reddatur integrabilis, inde interdum communis multiplicator utramque integrabilem reddens concludi potest. Sit enim $L\varphi:Z$ expressio generalis pro omnibus multiplicatoribus formulae $Pdx+Qdy$ et $M\varphi:V$ expressio generalis pro omnibus multiplicatoribus formulae $Rdx+Sdy$, et quoniam $\varphi:Z$ et $\varphi:V$ functiones quascunque quantitatum *Z* et *V* denotant, si eas ita capere liceat, ut fiat $L\varphi:Z=M\varphi:V$; habebitur multiplicator idoneus pro aequatione

$$Pdx+Qdy+Rdx+Sdy=0.$$

Intelligitur autem hoc iis tantum casibus praestari posse, quibus multiplicator pro tota aequatione etiam singulas eius partes seorsim sumtas integrabiles reddat. Quare cavendum est, ne huic methodo nimium tribuatur et, quando ea non succedit, aequatio pro irresolubili habeatur; evenire enim utique potest, ut tota aequatio habeat multiplicatorem, qui singulis eius partibus non conveniat. Ita proposita aequatione $Pdx+Qdy=0$ multiplicator partem *Pdx* seorsim integrabilem reddens manifesto est $\frac{X}{P}$; denotante

X functionem quamcunque ipsius x, et multiplicator partem alteram Qdy integrabilem reddens est $\frac{Y}{Q}$; etiamsi autem neutiquam fieri possit, ut sit $\frac{X}{P}=\frac{Y}{Q}$ seu $\frac{P}{Q}=\frac{X}{Y}$, nisi casibus per se obviis, tamen tota formula $Pdx+Qdy$ certo semper habet multiplicatorem, quo ea integrabilis reddatur.

EXEMPLUM 1

465. *Invenire omnes multiplicatores, quibus formula $\alpha ydx+\beta xdy$ integrabilis redditur.*

Primus multiplicator sponte se offert $\frac{1}{xy}$, qui praebet $\frac{adx}{x}+\frac{\beta dy}{y}$, cuius integrale est $\alpha lx+\beta ly=lx^{\alpha}y^{\beta}$. Huius ergo functio quaecunque $\varphi{:}x^{\alpha}y^{\beta}$ in $\frac{1}{xy}$ ducta dabit multiplicatorem idoneum, cuius itaque forma generalis est $\frac{1}{xy}\varphi{:}x^{\alpha}y^{\beta}$.

Functio enim quantitatis $x^{\alpha}y^{\beta}$ etiam est functio logarithmi eiusdem quantitatis. Nam si P fuerit functio ipsius p et Π functio ipsius P, etiam Π est functio ipsius p et vicissim.

COROLLARIUM

466. Si pro functione sumatur potestas quaecunque $x^{n\alpha}y^{n\beta}$, formula $\alpha ydx+\beta xdy$ integrabilis redditur, si multiplicetur per $x^{n\alpha-1}y^{n\beta-1}$, quo quidem casu integrale sponte patet; est enim $\frac{1}{n}x^{n\alpha}y^{n\beta}$.

EXEMPLUM 2

467. *Invenire omnes multiplicatores, qui hanc formulam $Xydx+dy$ integrabilem reddant.*

Primus multiplicator $\frac{1}{y}$ sponte se offert, unde, cum sit

$$\int\left(Xdx+\frac{dy}{y}\right)=\int Xdx+ly \quad \text{seu} \quad le^{\int Xdx}y,$$

omnes functiones huius quantitatis seu huius $e^{\int Xdx}y$ per y divisae dabunt multiplicatores idoneos. Unde expressio generalis pro omnibus multiplicatoribus erit $=\frac{1}{y}\varphi{:}e^{\int Xdx}y$.

COROLLARIUM

468. Pro formula ergo $Xydx+dy$ multiplicator quoque est $e^{\int Xdx}$, qui est functio ipsius x tantum; quo ergo cum etiam formula $\mathfrak{X}dx$ denotante $\mathfrak{X}$ functionem quameunque ipsius x integrabilis reddatur, ille multiplicator etiam huic formulae $dy+Xydx+\mathfrak{X}dx$ conveniet.

PROBLEMA 61

469. *Proposita aequatione* $dy + Xydx = \mathfrak{X}dx$, *in qua X et* $\mathfrak{X}$ *sint functiones quaecunque ipsius x, invenire multiplicatorem idoneum eamque integrare.*

SOLUTIO

Cum alterum membrum $\mathfrak{X}dx$ per functionem quamcunque ipsius *x* multiplicatum fiat integrabile, dispiciatur, num etiam prius membrum $dy + Xydx$ per huiusmodi multiplicatorem integrabile reddi possit. Quod cum praestet multiplicator $e^{\int Xdx}$, hoc adhibito habebitur aequatio integralis quaesita

$$e^{\int Xdx}y = \int e^{\int Xdx}\mathfrak{X}dx$$

sive

$$y = e^{-\int Xdx}\int e^{\int Xdx}\mathfrak{X}dx,$$

uti iam supra [§ 420] invenimus.

COROLLARIUM 1

470. Patet etiam, si loco *y* adsit functio quaecunque ipsius *y*, ut habeatur haec aequatio $dy + Xydx = \mathfrak{X}dx$, eam per multiplicatorem $e^{\int Xdx}$ reddi integrabilem et integrale fore

$$e^{\int Xdx}Y = \int e^{\int Xdx}\mathfrak{X}dx.$$

COROLLARIUM 2

471. Quare etiam haec aequatio $dy + yXdx = y^n\mathfrak{X}dx$ quia per y^n divisa abit in

$$\frac{dy}{y^n} + \frac{Xdx}{y^{n-1}} = \mathfrak{X}dx,$$

ubi posito $\frac{1}{y^{n-1}} = Y$ ob $-\frac{(n-1)dy}{y^n} = dY$ seu $\frac{dy}{y^n} = -\frac{dY}{(n-1)}$ prodit

$$-\frac{dY}{(n-1)} + YXdx = \mathfrak{X}dx \quad \text{seu} \quad dY - (n-1)YXdx = -(n-1)\mathfrak{X}dx,$$

quae per multiplicatorem $e^{-(n-1)\int Xdx}$ fit integrabilis: eiusque integrale erit

$$e^{-(n-1)\int Xdx}Y = -(n-1)\int e^{-(n-1)\int Xdx}\mathfrak{X}dx$$

sive [§ 429]

$$\frac{1}{y^{n-1}} = -(n-1)e^{(n-1)\int Xdx}\int e^{-(n-1)\int Xdx}\mathfrak{X}dx.$$

SCHOLION

472. Cum pro membro $dy+yXdx$ multiplicator generalis sit $\frac{1}{y}\varphi:e^{\int Xdx}y$, sumta loco functionis potestate multiplicator idoneus erit $e^{m\int Xdx}y^{m-1}$ integrale praebens $\frac{1}{m}e^{m\int Xdx}y^{m}$. Efficiendum ergo est, ut etiam idem multiplicator alterum membrum $y^{n}\mathfrak{X}dx$ reddat integrabile; quod evenit sumendo $m-1=-n$ seu $m=1-n$, ex quo huius membri integrale fit $\int e^{m\int Xdx}\mathfrak{X}dx$, ita ut aequatio integralis quaesita obtineatur

$$\tfrac{1}{1-n}e^{(1-n)\int Xdx}Y^{1-n}=\int e^{(1-n)\int Xdx}\mathfrak{X}dx,$$

quae cum modo inventa prorsus congruit.

PROBLEMA 62

473. *Proposita aequatione differentiali*

$$\alpha ydx+\beta xdy=x^{m}y^{n}\left(\gamma ydx+\delta xdy\right)$$

invenire multiplicatorem idoneum, qui eam integrabilem reddat, ipsumque integrale assignare

SOLUTIO

Consideretur utrumque membrum seorsim; ac pro priori vidimus $\alpha ydx+\beta xdy$ omnes multiplicatores idoneos contineri in hac forma

$$\tfrac{1}{xy}\varphi:x^{\alpha}y^{\beta}.$$

Pro altera parte $x^{m}y^{n}\left(\gamma ydx+\delta xdy\right)$ primus multiplicator est

$$\tfrac{1}{x^{m+1}y^{n+1}},$$

quo prodit $\frac{\gamma dx}{x}+\frac{\delta dy}{y}$, cuius integrale est $lx^{\gamma}y^{\delta}$; ergo forma generalis pro eius multiplicatoribus est,

$$\tfrac{1}{x^{m+1}y^{n+1}}\varphi:x^{\gamma}y^{\delta}$$

Quo nunc hi duo multiplicatores pares reddantur, loco functionum sumantur potestates fiatque

$$x^{\mu\alpha-1}y^{\mu\beta-1}=x^{\nu\gamma-m-1}y^{\nu\delta-n-1},$$

unde statui oportet

$$\mu\alpha=\nu\gamma-m \text{ et } \mu\beta=\nu\delta-n$$

hincque colligitur

$$\mu=\tfrac{\gamma n-\delta m}{\alpha\delta-\beta\gamma} \text{ et } \nu=\tfrac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}$$

Quocirca multiplicator erit

$$x^{\mu\alpha-1}y^{\mu\beta-1}=x^{\nu\gamma-m-1}y^{\nu\delta-n-1},$$

unde aequatio nostra induit hanc formam

$$x^{\mu\alpha-1}y^{\mu\beta-1}(\alpha ydx+\beta xdy)=x^{\nu\gamma-1}y^{\nu\delta-1}(\gamma ydx+\delta xdy)$$

ubi utrumque membrum per se est integrabile ideoque integrale quaesitum

$$\tfrac{1}{\mu}x^{\mu\alpha}y^{\mu\beta}=\tfrac{1}{\nu}x^{\nu\gamma}y^{\nu\delta}+\text{Const.},$$

quod convenit eum eo, quod capite praecedente [§ 431] est inventum.

COROLLARIUM 1

474. Posito ergo brevitatis gratia $\mu=\frac{\gamma n-\delta m}{\alpha\delta-\beta\gamma}$ et $\nu=\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}$; aequationis differentialis

$$aydx+\beta xdy=x^m y^n(\gamma ydx+\delta xdy)$$

integrale completum est

$$\tfrac{1}{\mu}x^{\mu\alpha}y^{\mu\beta}=\tfrac{1}{\nu}x^{\nu\gamma}y^{\nu\delta}+\text{Const.}$$

COROLLARIUM 2

475. Si eveniat, ut sit $\mu=0$ seu $\gamma n=\delta m$, integrale ad logarithmos reducetur eritque

$$lx^{\alpha}y^{\beta}=\tfrac{1}{\nu}x^{\nu\gamma}y^{\nu\delta}+\text{Const.};$$

sin autem $\nu=0$ seu $\alpha n=\beta m$, erit integrale

$$\tfrac{1}{\mu}x^{\mu\alpha}y^{\mu\beta}=lx^{\gamma}y^{\delta}+\text{Const.}$$

SCHOLION

476. Hinc autem casus excipi videtur, quo $\alpha\delta=\beta\gamma$, quia tum ambo numeri μ et ν fiunt infiniti. Verum si $\delta=\frac{\beta\gamma}{\alpha}$, aequatio nostra hanc induit formam

$$aydx+\beta xdy=\tfrac{\gamma}{\alpha}x^m y^n(aydx+\beta xdy)\quad\text{seu}\quad(aydx+\beta xdy)\left(1-\tfrac{\gamma}{\alpha}x^m y^n\right)=0,$$

quae cum habeat duos factores, duplex solutio ex utroque seorsim ad nihilum reducto derivatur. Prior scilicet nascitur ex $aydx+\beta xdy=0$, cuius integrale est $x^{\alpha}y^{\beta}=\text{Const.}$, alter vero factor per se dat aequationem finitam $1-\frac{\gamma}{\alpha}x^m y^n=0$, quarum solutionum utraque aeque satisfacit. Atque hoc in genere tenendum est de omnibus aequationibus differentialibus, quas in factores resolvere licet, ubi perinde atque in aequationibus finitis singuli factores praebent solutiones. Plerumque autem factores finiti statim, antequam integratio suscipitur, per divisionem tolli solent, quandoquidem non ex natura rei, sed per operationes institutas demum accessisse censentur, ita ut, perinde ac in Algebra saepe fieri solet, ad solutiones inutiles essent perducturi.

PROBLEMA 63

477. *Proposita aequatione differentiali homogenea multiplicatorem idoneum invenire, qui eam integrabilem reddat, indeque eius integrale eruere.*

SOLUTIO

Sit $Pdx + Qdy = 0$ aequatio proposita, in qua *P* et *Q* sint functiones homogeneae *n* dimensionum ipsarum *x* et *y*, ac quaeramus multiplicatorem *L*, qui sit etiam functio homogenea, cuius dimensionum numerus sit λ. Cum iam formula $L(Pdx + Qdy)$ sit integrabilis, erit integrale functio $\lambda + n + 1$ dimensionum ipsarum *x* et *y*, quae functio si ponatur *Z*, erit ex natura functionum homogenearum [§ 481]

$$LPx + LQy = (\lambda + n + 1)Z.$$

Quare si λ sumatur $= -n - 1$, quantitas $LPx + LQy$ erit vel = 0 vel constans, unde obtinemus $L = \frac{1}{Px+Qy}$, qui ergo est multiplicator idoneus pro nostra aequatione.

Idem quoque ex separatione variabilium colligitur; posito enim $y = ux$ fiet $P = x^n U$ et $Q = x^n V$ existentibus *U* et *V* functionibus *u* ipsius tantum et ob $dy = udx + xdu$ erit

$$Pdx + Qdy = x^n Udx + x^n Vudx + x^n Vxdu$$

seu

$$Pdx + Qdy = x^n (U + Vu)dx + x^{n+1} Vdu$$

At haec formula per $x^{n+1}(U + Vu)$ divisa fit integrabilis ideoque et formula nostra $Pdx + Qdy$ divisa per $x^{n+1}(U + Vu) = Px + Qy$, restitutis valoribus $U = \frac{P}{x^n}$, $V = \frac{Q}{x^n}$, et $u = \frac{y}{x}$, fiet integrabilis; seu multiplicator idoneus est $\frac{1}{Px+Qy}$, unde haec aequatio $\frac{Pdx+Qdy}{Px+Qy} = 0$ semper per se est integrabilis.

Iam ad integrale ipsius inveniendum integretur formula $\int \frac{Pdx}{Px+Qy}$ spectando *y* ut constantem ac determinetur certa ratione, ut evanescat posito $x = f$. Tum posito brevitatis causa $\frac{P}{Px+Qy} = R$ sumatur valor $\left(\frac{dR}{dy}\right)$ et eadem lege quaeratur integrale $\int dx\left(\frac{dR}{dy}\right)$ spectando iterum *y* ut constantem. Tum erit $\frac{Q}{Px+Qy} - \int dx\left(\frac{dR}{dy}\right)$ functio ipsius *y* tantum seu $\frac{Q}{Px+Qy} - \int dx\left(\frac{dR}{dy}\right) = Y$ atque hinc erit integrale quaesitum

$$\int \frac{Pdx}{Px+Qy} + \int Ydy = \text{Const.}$$

COROLLARIUM 1

478. Cum ergo formula $Pdx + Qdy$ sit per se integrabilis, si brevitatis gratia ponamus

$$\frac{P}{Px+Qy} = R \quad \frac{Q}{Px+Qy} = S,$$

necesse est sit $\left(\frac{dR}{dy}\right) = \left(\frac{dS}{dx}\right)$. At est

$$\left(\frac{dR}{dy}\right) = \left(Qy\left(\frac{dP}{dy}\right) - Py\left(\frac{dQ}{dy}\right) - PQ\right) : \left(Px + Qy\right)^2$$

et

$$\left(\frac{dS}{dx}\right) = \left(Px\left(\frac{dQ}{dx}\right) - Qx\left(\frac{dP}{dy}\right) - PQ\right) : \left(Px + Qy\right)^2.$$

Quamobrem habebitur

$$Qy\left(\frac{dP}{dy}\right) - Py\left(\frac{dQ}{dy}\right) = Px\left(\frac{dQ}{dx}\right) - Qx\left(\frac{dP}{dx}\right).$$

COROLLARIUM 2

479. Haec aequalitas etiam ex natura functionum homogenearum concluditur. Cum enim *P* et *Q* sint functiones *n* dimensionum ipsarum *x* et *y*, ob

$$dP = dx\left(\frac{dP}{dx}\right) + dy\left(\frac{dP}{dy}\right) \text{ et } dQ = dx\left(\frac{dQ}{dx}\right) + dy\left(\frac{dQ}{dy}\right)$$

erit

$$nP = x\left(\frac{dP}{dx}\right) + y\left(\frac{dP}{dy}\right) \text{ et } nQ = x\left(\frac{dQ}{dx}\right) + y\left(\frac{dQ}{dy}\right).$$

Aequalitas autem inventa est

$$Q\left(x\left(\frac{dP}{dx}\right) + y\left(\frac{dP}{dy}\right)\right) = P\left(x\left(\frac{dQ}{dx}\right) + y\left(\frac{dQ}{dy}\right)\right),$$

quae hinc abit in identicam $nPQ = nPQ$.

COROLLARIUM 3

480. Si aequatio homogenea $Pdx + Qdy = 0$ fuerit per se integrabilis et *P* et *Q* sint functiones –1 dimensionis, erit $Px + Qy$ numerus constans. Veluti cum $\frac{xdx+ydy}{xx+yy} = 0$ huiusmodi sit aequatio, si loco *dx* et *dy* scribantur *x* et *y* prodit $\frac{xx+yy}{xx+yy} = 1$.

SCHOLION

481. In *Calculo Differentiali* ostendimus, si V fuerit functio homogenea n dimensionum ipsarum x et y ponaturque $dV = Pdx + Qdy$, fore

$$Px + Qy = nV.$$

Quare si $Pdx + Qdy$ fuerit formula integrabilis et P et Q functiones homogeneae $n-1$ dimensionum, integrale statim habetur; erit enim $V = \frac{1}{n}(Px + Qy)$ neque ad hoc ulla integratione est opus. Interim tamen videmus hinc excipi oportere casum, quo $n = 0$, uti fit in nostra aequatione per multiplicatorem integrabili reddita $\frac{Pdx+Qdy}{Px+Qy} = 0$, ubi dx et dy multiplicantur per functiones -1 dimensionis; neque enim hie integrale sine integratione obtineri potest. Ratio autem huius exceptionis in hoc est sita, quod formulae integrabilis $Pdx + Qdy$, in qua P et Q sunt functiones homogeneae $n-1$ dimensionum, integrale tum tantum sit functio homogenea n dimensionum, quando n non est = 0; hoc enim solo casu fieri potest, ut integrale non sit functio nullius dimensionis, quemadmodum fit in hac formula differentiali $\frac{xdx+ydy}{xx+yy}$, quippe cuius integrale est $\frac{1}{2}l(xx + yy)$. Quocirca, quod formula $\frac{Pdx+Qdy}{Px+Qy}$ sit integrabilis, hoc peculiari modo demonstravimus ex ratione separabilitatis deducto. Interim tamen sine ullo respectu, unde hoc cognoverimus, id in praesenti negotio maxime est notatu dignum omnes aequationes homogeneas $Pdx + Qdy = 0$ per multiplicatorem $\frac{1}{Px+Qy}$ per se reddi integrabiles. Methodus igitur desideratur, cuius beneficio hunc multiplicatorem a priori invenire liceret; qua methodo sane maxima incrementa in Analysin importarentur. Quamdiu autem eousque pertingere non licet, plurimum intererit huiusmodi multiplicatores pro pluribus casibus probe notasse; quod cum iam in duobus aequationum generibus praestiterimus, pro reliquis aequationibus, quas supra integrare docuimus, multiplicatores investigemus; ipsa autem reductio ad separationem nobis hos multiplicatores patefaciet, uti in sequente problemate docebimus.

PROBLEMA 64

482. *Proposita aequatione differentiali, quam ad separationem variabilium reducere liceat, invenire multiplicatorem, per quem ea per se integrabilis reddatur.*

SOLUTIO

Sit $Pdx + Qdy = 0$, quae certa quadam substitutione, dum loco x et $y$aliae binae variabiles t et u introducuntur; ad separationem accommodetur; ponamus ergo facta hac substitutione fieri $Pdx + Qdy = Rdt + Sdu$, nunc autem hanc formulam $Rdt + Sdu$, si per V dividatur, separari, ita ut in hac formula $\frac{Rdt+Sdu}{V}$ quantitas : sit functio solius t et $\frac{S}{V}$ functio solius u. Cum igitur formula $\frac{Rdt+Sdu}{V}$ per se sit integrabilis, etiam integrabilis erit haec $\frac{Pdx+Qdy}{V}$, quippe illi aequalis, siquidem in V variabiles x et y restituantur. Hinc ergo ex reductione ad separabilitatem aequationis $Pdx + Qdy = 0$ discimus multiplicatorem, quo ea integrabilis reddatur, esse $\frac{1}{V}$ sicque, quas aequationes ad separationem variabilium perducere licet, pro iisdem multiplicatorem, qui illas integrabiles reddat, assignare possumus.

COROLLARIUM 1

483. Methodus ergo per multiplicatores integrandi aequationes differentiales aeque late patet ac prior methodus ope separationis variabilium, propterea quod ipsa separatio pro quavis aequatione, ubi succedit, multiplicatorem suppeditat.

COROLLARIUM 2

484. Contra autem methodus per multiplicatores integrandi latius patet altera, si pro eiusmodi aequationibus multiplicatores assignare liceat, quas quomodo ad separationem perduci debeant, non constet.

SCHOLION

485. Etsi autem ex reductione ad separationem idoneum multiplicatorem elicere licet, tamen nondum intelligitur, quomodo cognito multiplicatore separatio variabilium institui debeat; quare etiam ob hanc rationem methodus per multiplicatores integrandi alteri longe praeferenda videtur. Quamvis enim hactenus ipsa separatio nos ad inventionem multiplicatorum perduxerit, nullum tamen est dubium, quin detur via multiplicatores inveniendi nullo respectu ad separationem habito, licet haec via etiamnum nobis sit incognita. Ea autem paullatim planior reddetur, si pro quam plurimis aequationibus multiplicatores idoneos cognoverimus, ex quo, quos adhuc ex separatione eruere licet, indagemus in subiunctis exemplis.

EXEMPLUM 1

486. *Proposita aequatione differentiali primi ordinis*
$$dx(ax+\beta y+\gamma)+dy(\delta x+\varepsilon y+\zeta)=0$$
pro ea multiplicatorem idoneum assignare.

Haec aequatio ad separationem praeparatur ponendo primo [§ 417]
$$ax+\beta y+\gamma=r \quad \text{et} \quad \delta x+\varepsilon y+\zeta=s$$
ideoque
$$adx+\beta dy=dr \quad \text{et} \quad \delta dx+\varepsilon dy=ds,$$
unde oritur
$$dx=\tfrac{\varepsilon dr-\beta ds}{\alpha\varepsilon-\beta\delta} \;\; \text{et} \;\; dy=\tfrac{\alpha ds-\delta dr}{\alpha\varepsilon-\beta\delta},$$

hincque aequatio nostra omisso denominatore utpote constante erit
$$\varepsilon rdr-\beta rds+\alpha sds-\delta sdr=0;$$
quae cum sit homogenea, per $\varepsilon rr-(\beta+\delta)rs+\alpha ss$ divisa fit integrabilis. Quod idem ex separatione colligitur; posito enim $r=su$ prodit

$$\varepsilon ssudu + \varepsilon suuds - \beta suds + \alpha sds - \delta ssdu - \delta suds = 0$$

seu

$$ssdu(\varepsilon u - \delta) + sds(\varepsilon uu - \beta u - \delta u + \alpha) = 0,$$

quae divisa per $ss(\varepsilon uu - \beta u - \delta u + \alpha)$ separatur. Quare multiplicator nostrae aequationis propositae est

$$\frac{1}{ss(\varepsilon uu - \beta u - \delta u + \alpha)} = \frac{1}{\varepsilon rr - \beta rs - \delta rs + \alpha ss} = \frac{1}{r(\varepsilon r - \beta s) + s(\alpha s - \delta r)}$$

qui restitutis valoribus fit

$$\frac{1}{(\alpha x + \beta y + \gamma)((\alpha\varepsilon - \beta\delta)x + \gamma\varepsilon - \beta\zeta) + (\delta x + \varepsilon y + \zeta)((\alpha\varepsilon - \beta\delta)y + \alpha\zeta - \gamma\delta)}$$

atque evolutione facta

$$1 : \left\{ \begin{array}{c} (\alpha\varepsilon - \beta\delta)(\alpha xx + (\beta + \delta)xy + \varepsilon yy + \gamma x + \zeta y) + \alpha\zeta\zeta - (\beta + \delta)\gamma\zeta + \gamma\gamma\varepsilon \\ + (\alpha\gamma\varepsilon - (\beta - \delta)\alpha\zeta - \gamma\delta\delta)x + (\alpha\varepsilon\zeta + (\beta - \delta)\gamma\varepsilon - \beta\beta\zeta)y \end{array} \right\}$$

Quare per se integrabilis erit haec aequatio

$$\frac{dx(\alpha x + \beta y + \gamma) + dy(\delta x + \varepsilon y + \zeta)}{(\alpha\varepsilon - \beta\delta)(\alpha xx + (\beta + \delta)xy + \varepsilon yy + \gamma x + \zeta y) + Ax + By + C} = 0$$

existente

$$A = \alpha\gamma\varepsilon - (\beta - \delta)\alpha\zeta - \gamma\delta\delta,$$

$$B = \alpha\varepsilon\zeta + (\beta - \delta)\gamma\varepsilon - \beta\beta\zeta,$$

$$C = \alpha\zeta\zeta - (\beta + \delta)\gamma\zeta + \gamma\gamma\varepsilon.$$

COROLLARIUM

487. Etiamsi forte fiat $\alpha\varepsilon - \beta\delta = 0$, hic multiplicator non turbatur, cum tamen separatio non succedat hac quidem operatione. Sit enim $\alpha = ma, \quad \beta = mb, \quad \delta = na, \quad \varepsilon = nb,$ ut habeatur haec aequatio

$$dx(m(ax + by) + \gamma) + dy(n(ax + by) + \zeta) = 0;$$

ob

$$A = a(na - mb)(m\zeta - n\gamma), \quad B = b(na - mb)(m\zeta - n\gamma)$$

et

$$C = (m\zeta - n\gamma)(a\zeta - b\gamma)$$

omisso factore communi multiplicator est

$$\frac{1}{(na-mb)(ax+by)+a\zeta-b\gamma}=0$$

ita ut haec aequatio per se sit integrabilis

$$\frac{(ax+by)(mdx+ndy)+\gamma dx+\zeta dy}{(na-mb)(ax+by)+a\zeta-b\gamma}=0$$

EXEMPLUM 2

488. *Proposita aequatione differentiali*

$$ydx(c+nx)-dy(y+a+bx+nxx)=0$$

multiplicatorem idoneum invenire.

Fiat substitutio [§ 433]

$$\frac{y(c+nx)}{y+a+bx+nxx}=u \quad \text{seu} \quad y=\frac{u(a+bx+nxx)}{c+nx-u},$$

ut contrahatur aequatio nostra in hanc formam

$$ydx(c+nx)-\frac{ydy(c+nx)}{u}=0 \;\; \text{seu} \;\; \frac{(c+nx)}{u}(udx-dy)=0$$

vel

$$\frac{yy(c+nx)}{u}\left(\frac{dy}{y}-\frac{udx}{y}\right)=0;$$

probe enim cavendum est, ne hic ullus factor omittatur. At facta substitutione reperitur

$$\frac{dy}{y}-\frac{udx}{y}=\frac{du}{u}+\frac{dx(b+2nx)}{a+bx+nxx}+\frac{du-ndx}{c+nx-u}-\frac{dx(c+nx-u)}{a+bx+nxx}$$
$$=\frac{du(c+nx)}{u(c+nx-u)}-\frac{dx\big(na+cc-bc+(b-2c)u+uu\big)}{(c+nx-u)(a+bx+nxx)}.$$

Unde aequatio nostra induet hanc formam

$$\frac{yy(c+nx)^2}{u(c+nx-u)}\left(\frac{du}{u}-\frac{dx\big(na+cc-bc+(b-2c)u+uu\big)}{(a+bx+nxx)(c+nx)}\right)=0,$$

quae ergo separabitur ducta in hunc multiplicatorem

$$\frac{u(c+nx-u)}{yy(c+nx)^2\big(na+cc-bc+(b-2c)u+uu\big)};$$

tum enim prodit

$$\frac{du}{u\big(na+cc-bc+(b-2c)u+uu\big)}-\frac{dx}{(a+bx+nxx)(c+nx)}=0.$$

Quo igitur multiplicatorem quaesitum consequamur, ibi loco u tantum opus est suum valorem restituere; tum autem reperitur multiplicator

$$\frac{a+bx+nxx}{n(a+bx+nxx)y^3+(a+bx+nxx)(2na-bc+n(b-2c)x)yy+(na+cc-bc)(a+bx+nxx)^2 y},$$

qui reducitur ad hanc formam

$$\frac{1}{ny^3+(2na-bc)yy+n(b-2c)xyy+(na+cc-bc)(a+bx+nxx)y}.$$

EXEMPLUM 3

489. *Proposita aequatione differentiali*

$$\frac{ndx(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}+(x-y)dy=0$$

invenire multiplicatorem, qui eam integrabilem reddat.

Posuimus supra (§ 434)

$$y=\frac{x-u}{1+xu} \quad \text{seu} \quad u=\frac{x-y}{1+xy}$$

unde fit

$$x-y=\frac{u(1+xx)}{1+xu} \quad \text{et} \quad 1+yy=\frac{(1+xx)(1+uu)}{(1+xu)^2},$$

hincque nostra aequatio hanc induit formam

$$\frac{ndx(1+xx)(1+uu)^{\frac{3}{2}}}{(1+xu)^3}+\frac{udx(1+xx)(1+uu)-udu(1+xx)^2}{(1+xu)^3}=0,$$

quae primo multiplicata per $(1+xu)^3$, tum divisa per

$$(1+xx)^2(1+uu)\left(u+n\sqrt{(1+uu)}\right)$$

separatur. Quare aequationis nostrae multiplicator erit

$$\frac{(1+xu)^3}{(1+xx)^2(1+uu)\left(u+n\sqrt{(1+uu)}\right)},$$

qui primo ob $1+uu=\frac{(1+yy)(1+xu)^2}{1+xx}$ abit $\frac{1+xu}{(1+xx)(1+yy)\left(u+n\sqrt{(1+uu)}\right)}$.

Nunc ob $u=\frac{x-y}{1+xy}$ est

$$\sqrt{(1+uu)} = \frac{\sqrt{(1+xx)(1+yy)}}{1+xy} \text{ et } 1+xu = \frac{1+xx}{1+xy}$$

ideoque noster multiplicator colligitur

$$\frac{1}{(1+yy)\left(x-y+n\sqrt{(1+xx)(1+yy)}\right)},$$

ita ut per se sit integrabilis haec aequatio

$$\frac{ndx(1+yy)\sqrt{(1+yy)}+(x-y)dy\sqrt{(1+xx)}}{(1+yy)\left(x-y+n\sqrt{(1+xx)(1+yy)}\right)\sqrt{(1+xx)}} = 0,$$

cuius integrationi non immoror, cum iam supra integrale exhibuerim.

EXEMPLUM 4

490. *Aliud exemplum memoratu dignum suppeditat haec aequatio*

$$ydx - xdy + ax^n ydy\left(x^n + b\right)^{\frac{1}{n}} = 0.$$

Quae si hac forma repraesentetur

$$xdy - ydx + \tfrac{1}{b}x^{n+1}dy = \tfrac{1}{b}x^{n+1}dy + ax^n ydy\left(x^n + b\right)^{\frac{1}{n}},$$

evenit, ut utrumque integrabile existat, si ducatur in hunc multiplicatorem

$$\frac{y^{n-1}}{x^{n+1}+abx^n y\left(x^n+b\right)^{\frac{1}{n}}};$$

ad quem inveniendum ex separatione variabilium adhibeatur haec substitutio non adeo obvia

$$\frac{x}{\left(x^n+b\right)^{\frac{1}{n}}} = vy,$$

unde fit $x^n = \frac{bv^n y^n}{1-v^n y^n}$, et hinc aequatio

$$\frac{ydx-xdy}{\left(x^n+b\right)^{\frac{1}{n}}} + ax^n ydy = 0$$

abit in hanc

$$\frac{yydv+v^{n+1}y^{n+1}dy+abv^n y^{n+1}dy}{1-v^n y^n} = 0,$$

quae multiplicata per $\frac{1-v^n y^n}{yyv^n(ab+v)}$ separatur

$$\frac{dv}{v^n(ab+v)}+y^{n-1}dy=0,\ ,$$

unde idem ille multiplicator colligitur.

EXEMPLUM 5

491. *Proposita aequatione differentiali*

$$dy+yydx-\frac{adx}{x^4}=0$$

invenire multiplicatorem, quo ea integrabilis reddatur.

Secundum § 436 ponatur $x=\frac{1}{t}$ et ob $dx=-\frac{dt}{tt}$ nostra formula erit $dy-\frac{yydt}{tt}+attdt$, in qua porro statuatur $y=t-ttz$, et prodibit

$$-\,tt\left(dz+zzdt-adt\right),$$

quae per $tt\left(zz-a\right)$ divisa separatur; ergo et nostra aequatio divisa per

$$tt\left(zz-a\right)=\frac{(t-y)^2-at^4}{tt}=\left(1-xy\right)^2-\frac{a}{xx}$$

fiet integrabilis, ex quo multiplicator erit

$$\frac{xx}{xx(1-xy)^2-a}$$

et aequatio per se integrabilis

$$\frac{x^4dy+x^4yydx-adx}{x^4(1-xy)^2-axx}=0.$$

Spectetur iam x ut constans eritque ex dy natum integrale

$$\frac{1}{2\sqrt{a}}\,l\,\frac{x(1-xy)+\sqrt{a}}{\sqrt{a}-x(1-xy)}+X;$$

pro quo ut valor ipsius X obtineatur, differentietur denuo ac prodibit

$$\frac{2xydx-dx}{xx(1-xy)^2-a}\ +dX\ =\frac{x^4yydx-adx}{x^4(1-xy)^2-axx},$$

unde

$$dX = \frac{x^4yydx - adx - 2x^3ydx + xxdx}{x^4(1-xy)^2 - axx} = \frac{dx}{xx} \text{ et } X = -\frac{1}{x} + C;$$

quare aequatio integralis completa erit

$$l\frac{\sqrt{a}+x(1-xy)}{\sqrt{a}-x(1-xy)} = \frac{2\sqrt{a}}{x} + C.$$

SCHOLION

492. En ergo plures casus aequationum differentialium, pro quibus multiplicatores novimus, ex quorum contemplatione haec insignis investigatio non parum adiuvari videtur. Quanquam autem adhuc longe absumus a certa methodo pro quovis casu multiplicatores idoneos inveniendi, hinc tamen formas aequationum colligere poterimus, ut per datos multiplicatores integrabiles reddantur; quod negotium cum in hac ardua doctrina maximam utilitatem allaturum videatur, in sequente capite aequationes investigabimus, quibus dati multiplicatores conveniant; exempla scilicet hic evoluta idoneas multiplicatorum formas nobis suppeditant, quibus nostram investigationem superstruere licebit.