INTEGRAL CALCULUS

FIRST BOOK.

FIRST PART

OR
THE METHOD OF INVESTIGATING FUNCTIONS
OF *ONE* VARIABLE FROM SOME GIVEN RELATION OF THE DIFFERENTIAL OF THE FIRST DEGREE.

SECOND SECTION :
CONCERNED
WITH THE INTEGRATION OF DIFFERENTIAL EQUATIONS.

CHAPTER I
THE SEPARATION OF VARIABLES

DEFINITION

397. *A differential equation is said to be treated by the separation of the variables, when the equation thus is allowed be separated into two parts, so that in each only a single variable with its differential is present.*

COROLLARY 1

398. Therefore when the differential equation has been prepared thus, so that it can be reduced to this form $Xdx = Ydy$, in which X is a function only of x and Y only of y, then the equation is said to permit a separation of variables.

COROLLARY 2

399. But if P and X are functions on x only, while Q and Y denote functions of y only, this equation $PYdx = QXdy$ permits a separation of the variables ; for on dividing by XY it is changed into $\frac{Pdx}{X} = \frac{Qdy}{Y}$, in which the variables have been separated.

COROLLARY 3

400. Hence in the general form $\frac{dy}{dx} = V$ the separation of the variables can be treated, if V should be a function of this kind of x and y, so that it is possible to be resolved into two factors, of which one contains only the variable x, while the other contains only y. For if there is the equation $V = XY$, from this the separated equation $\frac{dy}{Y} = Xdx$ appears.

SCHOLIUM

401. On putting the ratio of the differentials $\frac{dy}{dx} = p$, we have put in place in this section for consideration a relation of this kind between the variables x, y and p, in which p is equal to some function of x and y. Hence we consider that first case here, in which the function is resolved into two parts, of which one is a function of x only, and the other of y only, thus in order that the equation can be reduced to this form $Xdx = Ydy$, in which the two variables are said to separated in turn from each other. And the simple formulas treated before are contained in this case, when $Y = 1$, so that $dy = Xdx$ and $y = \int Xdx$, where the whole calculation is reduced to the integration of the formula Xdx. But the separated equation $Xdx = Ydy$ has no more difficulty, as likewise it is allowed to treat simple formulas, as we show in the following problem.

PROBLEM 49

402. *To integrate a differential equation in which the variables are separated, or to find an equation between the variables.*

SOLUTION

An equation allowing separation of the variables can always be reduced to this form $Ydy = Xdx$, where *Xdx* can be regarded as the differential of a function of *x* and *Ydy* as the differential of a certain function of *y*. Therefore since the differentials are equal, the integrals of these also are equal, or it is necessary to differ by a constant quantity. Hence both formulas may be integrated separately by the rules of the above sections or the integrals $\int Ydy$ and $\int Xdx$ are sought, with which found certainly there will be $\int Ydy = \int Xdx + \text{Const.}$, by which equation a finite relation is expressed between the quantities *x* and *y*.

COROLLARY 1

403. Hence just as often as the differential equation is allowed to have the variables separated, so the whole integration can be completed by the same rules which have been treated above for simple formulas.

COROLLARY 2

404. In the equation for the integral $\int Ydy = \int Xdx + \text{Const.}$ either both functions $\int Ydy$ and $\int Xdx$ are algebraic, or one is algebraic and now the other transcendental, or both are transcendental, and thus the relation *x* and *y* is either algebraic or transcendental.

SCHOLIUM

405. It is customary to establish the basis of the resolution of differential equations from some with the separation of variables, so that, when a proposed equation does not allow the separation of variables, a suitable substitution may be investigated, the benefit of which may allow the separation of the new variables introduced. Hence the whole calculation can be reduced thus, so that for some differential equation a substitution of this kind or the introduction of new variables is shown, in order that the separation of variables hence can be treated. Certainly it is to be wished, that a method of this kind can be revealed by making a suitable substitution for whatever case; but nothing for sure is to be ascertained entirely in this treatment, while many substitutions, which should they be used at some point, cannot rely on any clear principles. Hence moreover the separation of variables cannot be considered as the true foundation of all integrations, therefore in differential equations of the second or higher grade no outstanding use is offered ; but below I have set out another principle that appears of wider use. In this chapter meanwhile it is seen to be worth the effort to set out particular integrations with the help of the separation of variables, since in this difficult business it is of great interest to know most methods.

PROBLEM 50

406. *To reduce the differential equation $Pdx = Qdy$ according to the separation of variables, in which P and Q shall be homogeneous functions of the same number of dimensions of x and y, and to find the integral of this. .*

SOLUTION

Since *P* and *Q* shall be homogeneous functions of *x* and *y* of the same number of dimensions, the $\frac{P}{Q}$ will be a homogeneous function of zero dimensions, which hence on putting $y = ux$ changes into a function of u. Therefore there is put $y = ux$ and $\frac{P}{Q}$ is changed into a function *U* of *u*, thus in order that there becomes $dy = Udx$. But on account of $y = ux$ there is made $dy = udx + xdu$ with which substitution our equation adopts this form $udx + xdu = Udx$ between the two variables *x* and *u*, which clearly are separable. For with the terms containing *dx* placed in one part there is had $xdu = (U-u)dx$ and thus

$$\frac{dx}{x} = \frac{du}{U-u}$$

which on integrating gives $lx = \int \frac{du}{U-u}$ thus as now *x* may be determined from the variable *u*, from which again it is recognised that $y = ux$.

COROLLARIUM 1

407. But if it is possible also to express the integral $\int \frac{du}{U-u}$ by logarithms, thus so that *lx* is equal to the logarithm of some algebraic function of *u*, then an algebraic equation is produced between *x* and *u* and thus on putting the value $\frac{y}{x}$ for *u*, an equation between x et *y*.

COROLLARY 2

408. Since there is $y = ux$, then $ly = lu + lx$ and thus, since $lx = \int \frac{du}{U-u}$ then

$$ly = lu + \int \frac{du}{U-u} = \int \frac{du}{u} + \int \frac{du}{U-u},$$

from which integrals reduced into one there is made $ly = \int \frac{Udu}{u(U-u)}$. Now this is to be noted that it is not allowed to add an arbitrary constant for lx and ly; for once a constant has been added to one integral, likewise a constant is defined to be added to the other, since there must be $ly = lx + lu$.

COROLLARY3

409. Since there shall be

$$\int\frac{du}{U-u}=\int\frac{du-dU+dU}{U-u}=\int\frac{dU}{U-u}-\int\frac{dU-du}{U-u}$$

on this account the latter member integrable by logarithms there becomes :

$$lx=\int\frac{dU}{U-u}-l(U-u)\text{ or }lx(U-u)=\int\frac{dU}{U-u}.$$

Hence likewise either this formula $\int\frac{du}{U-u}$ or $\int\frac{dU}{U-u}$ can be integrated.

SCHOLIUM

410. Because this method extends to all homogeneous equations and also is not hindered on account of irrationality, which perhaps is present in the functions *P* and *Q*, in the first place it is required to consider how many are to be preferred by other methods, which are applicable only to very special equations. And hence also we learn about all the equations, which with the aid of certain substitutions are able to be reduced to homogeneity, that can be treated by the same method. Just as if this equation were put in place :

$$dz+zzdx=\frac{adx}{xx},$$

at once it is apparent on putting $z=\frac{1}{y}$ that is reduced to this homogeneous equation [§ 414]:

$$-\frac{dy}{yy}+\frac{dx}{yy}=\frac{adx}{xx}\text{ or }xxdy=dx(xx-ayy).$$

The remainder is seen without difficulty, each equation of this kind proposed can be induced to become a homogeneous equation. Generally, as often indeed as this can be done, it is sufficient that these be tried to be put in place $x=u^m$ and $y=v^n$, where it is easily judged, if the exponents *m* and *n* thus are allowed to be assumed, in order that a number of the same dimension is produced everywhere ; for more complicated substitutions scarcely a place is conceded for this kind, unless perhaps they should emerge at once of their own accord. But here it will be of help to set out the method of integration by some examples.

EXAMPLE 1

411. *To find the integral for this proposed homogeneous differential equation*

$$xdx + ydy = mydx\,.$$

Hence since there shall be $\frac{dy}{dx} = \frac{my-x}{y}$, putting $y = ux$ then $\frac{my-x}{y} = \frac{mu-1}{u}$ and thus on account of $dy = udx + xdu$ then

$$udx + xdu = \frac{mu-1}{u}dx$$

and hence

$$\frac{dx}{x} = \frac{udu}{mu-1-uu} = \frac{-udu}{1-mu+uu}$$

or

$$\frac{dx}{x} = \frac{-udu+\frac{1}{2}mdu}{1-mu+uu} - \frac{\frac{1}{2}mdu}{1-mu+uu}\,,$$

from which on integrating

$$lx = -\tfrac{1}{2}l\left(1-mu+uu\right) - \tfrac{1}{2}m\int\frac{du}{1-mu+uu} + \text{Const.},$$

where three cases are to be considered, according as $m > 2$, $m < 2$ or $m = 2$.

1) Let $m > 2$ and the form of $1-mu+uu$ becomes

$$\left(u-a\right)\left(u-\tfrac{1}{a}\right),$$

in order that $m = a + \frac{1}{a} = \frac{aa+1}{a}$, and on account of

$$\frac{du}{\left(u-a\right)\left(u-\frac{1}{a}\right)} = \frac{a}{aa-1}\cdot\frac{du}{u-a} - \frac{a}{aa-1}\cdot\frac{du}{u-\frac{1}{a}}$$

there becomes

$$lx = -\tfrac{1}{2}l\left(1-mu+uu\right) - \frac{aa+1}{2\left(aa-1\right)}l\,\frac{u-a}{u-\frac{1}{a}} + C$$

or

$$lx\sqrt{\left(1-mu+uu\right)} + \frac{aa+1}{2\left(aa-1\right)}l\,\frac{au-aa}{au-1} = lc$$

and with the value $u = \frac{y}{x}$ restored the equation of the integral will be :

$$l\sqrt{\left(xx-mxy+yy\right)} + \frac{aa+1}{2\left(aa-1\right)}l\,\frac{ay-aax}{ay-x} = lc$$

or

$$\left(\frac{ay-aax}{ay-x}\right)^{\frac{aa+1}{2\left(aa-1\right)}}\sqrt{\left(xx-mxy+yy\right)} = c.$$

2) Let $m < 2$ or $m = 2\cos.\alpha$; then

$$\int \frac{du}{1-u\cos.\alpha+uu} = \frac{1}{\sin.\alpha}\text{ Ang.tang.}\frac{u\sin.\alpha}{1-u\cos.\alpha},$$

from which

$$lx\sqrt{(1-mu+uu)} = C - \frac{\cos.\alpha}{\sin.\alpha}\text{ Ang.tang.}\frac{u\sin.\alpha}{1-u\cos.\alpha}$$

or

$$l\sqrt{(xx-mxy+yy)} = C - \frac{\cos.\alpha}{\sin.\alpha}\text{ Ang.tang.}\frac{y\sin.\alpha}{x-y\cos.\alpha}.$$

3) Let $m = 2$; then there will be

$$\int \frac{du}{(1-u)^2} = \frac{1}{1-u}$$

and hence

$$lx(1-u) = C - \frac{1}{1-u} \quad\text{or}\quad l(x-y) = C - \frac{x}{x-y}.$$

EXAMPLE 2

412. *To find the integral for the proposed homogeneous differential equation*

$$dx(\alpha x+\beta y) = dy(\gamma x+\delta y).$$

On putting y = ux then $udx + xdu = dx\cdot\frac{\alpha+\beta u}{\gamma+\delta u}$ and thus

$$\frac{dx}{x} = \frac{du(\gamma+\delta u)}{\alpha+\beta u-\gamma u-\delta uu} = \frac{du\left(\delta u+\frac{1}{2}\gamma-\frac{1}{2}\beta\right)+du\left(\frac{1}{2}\gamma+\frac{1}{2}\beta\right)}{\left(\alpha+(\beta-\gamma)u-\delta uu\right)},$$

from which on integrating

$$lx = C - l\sqrt{\left(\alpha+(\beta-\gamma)u-\delta uu\right)} + \frac{1}{2}(\beta+\gamma)\int\frac{du}{\alpha+(\beta-\gamma)u-\delta uu},$$

where the same cases which before are to be considered, as clearly the denominator $\alpha+(\beta-\gamma)u-\delta uu$ has either two real unequal factors, or imaginary or equal factors.

EXAMPLE 3

413. *To find the integral for the proposed homogeneous differential equation*

$$xdx + ydy = xdy - ydx.$$

Since hence there shall be $\frac{dy}{dx} = \frac{x+y}{x-y}$, on putting $y = ux$ there becomes $udx + xdu = \frac{1+u}{1-u}dx$ or $xdu = \frac{1+uu}{1-u}dx$,from which it is deduced, that

$$\frac{dx}{x} = \frac{du-udu}{1+uu}$$

and on integrating,

$$lx = \text{Ang.tang.}u - l\sqrt{(1+uu)} + C$$

or

$$l\sqrt{(xx+yy)} = C + \text{Ang.tang.}\tfrac{y}{x}$$

EXAMPLE 4

414. *To find the integral for the proposed homogeneous differential equation*

$$xxdy = (xx - ayy)dx .$$

Here therefore there shall be $\frac{dy}{dx} = \frac{xx-ayy}{xx}$
and on putting $y = ux$ there is produced

$$udx + xdu = (1 - auu)dx$$

and thus

$$\tfrac{dx}{x} = \tfrac{du}{1-u-auu}dx \quad \text{and} \quad lx = \int \tfrac{du}{1-u-auu},$$

and there is no need to delay on the setting out of this integration.

EXAMPLE 5

415. *To find the integral of the proposed homogeneous differential equation*

$$xdy - ydx = dx\sqrt{(xx+yy)} .$$

Hence there shall be $\frac{dy}{dx} = \frac{y+\sqrt{(xx+yy)}}{x}$ from which on putting $y = ux$ there becomes

$$udx + xdu = \left(u + \sqrt{(1+uu)}\right)dx$$

or

$$xdu = dx\sqrt{(1+uu)},$$

thus in order that

$$\tfrac{dx}{x} = \tfrac{du}{\sqrt{(1+uu)}},$$

and the integral of this is

$$lx = la + l\left(u + \sqrt{(1+uu)}\right) = la + l\left(\frac{y+\sqrt{(xx+yy)}}{x}\right)$$

or

$$lx = la + l\,\frac{x}{\sqrt{(xx+yy)}-y}$$

from which is deduced $x = \frac{ax}{\sqrt{(xx+yy)}-y}$ or $\sqrt{(xx+yy)} = a + y$ and hence

$$xx = aa + 2ay.$$

SCHOLION

416. Here also transcending functions can be considered, provided they affect functions of zero dimensions of x and y, because on putting $y = ux$ likewise they change into functions of u. Thus if in the equation $Pdx = Qdy$, besides P and Q being homogeneous functions of the same number of dimensions, they may be present as formulas of the kind $l\frac{\sqrt{(xx+yy)}}{x}$, $e^{y:x}$, Ang. sin.$\frac{x}{\sqrt{(xx+yy)}}$, $cos.\frac{nx}{y}$ etc., and the method set out can be used with equal success, since on putting $y = ux$ the ratio $\frac{dy}{dx}$ is equal to a function of the new variable u only.

PROBLEM 51

417. *To reduce and integrate the differential equation of the first order by separating variables*

$$dx(\alpha + \beta x + \gamma y) = dy(\delta + \varepsilon x + \zeta y).$$

SOLUTION

There is put

$$\alpha + \beta x + \gamma y = t \quad \text{and} \quad \delta + \varepsilon x + \zeta y = u$$

so that there is made $tdx = udy$. But from this we deduce that

$$x = \frac{\zeta t - \gamma u - \alpha\zeta + \gamma\delta}{\beta\zeta - \gamma\varepsilon} \quad \text{et} \quad y = \frac{\beta u - \varepsilon t + \alpha\varepsilon - \beta\delta}{\beta\zeta - \gamma\varepsilon}$$

and thence

$$dx : dy = \zeta dt - \gamma du : \beta du - \varepsilon dt,$$

from which we find this equation

$$\zeta tdt - \gamma tdu = \beta udu - \varepsilon udt$$

or

$$dt(\zeta t + \varepsilon u) = du(\beta u + \gamma t);$$

which since it is homogeneous and since it agrees with example § 412, the integration is now arranged.

Yet indeed the case exists, in which this reduction to homogeneity cannot be treated, since there should be $\beta\zeta - \gamma\varepsilon = 0$, because then the introduction of the new variables t and u cannot be made. Hence this case requires a special solution, which thus may be put in place. Because then the proposed equation is to take a form of this kind

$$\alpha dx+(\beta x+\gamma y)dx=\delta dy+n(\beta x+\gamma y)dy,$$

we may put $\beta x+\gamma y=z$; then

$$\frac{dy}{dx}=\frac{\alpha+z}{\delta+nz}.$$

But $dy=\frac{dz-\beta dx}{\gamma}$, hence

$$\frac{dz-\beta dx}{\gamma}=\frac{\alpha+z}{\delta+nz}dx,$$

where the variables evidently are separable ; for it becomes

$$dx=\frac{dz(\delta+nz)}{\alpha\gamma+\beta\delta+(\gamma+n\beta)z},$$

and the integration of this involves logarithms, unless there should be $\gamma+n\beta=0$, in which case there is given algebraically : $x=\frac{2\delta z+nzz}{2(\alpha\gamma+\beta\delta)}+C$.

COROLLARY 1

418. Hence an equation of the first order differential, as it is called, in general cannot be reduced to homogeneity, but the cases in which $\beta\zeta=\gamma\varepsilon$, thence they must be excepted, which also lead generally to a different separated equation.

COROLLARY 2

419. If in these excepted cases there is $n=0$, or this shall be the proposed equation : $dy=dx(a+\beta x+\gamma y)$,on putting $\beta x+\gamma y=z$ on account of $\delta=1$, this equation arises : $dx=\frac{dz}{\alpha\gamma+\beta+\gamma z}$, the integral of which is

$$\gamma x=l\frac{\beta+\alpha\gamma+\gamma z}{C}=l\frac{\beta+\alpha\gamma+\beta\gamma x+\gamma\gamma y}{C}$$

or

$$\beta+\gamma(\alpha+\beta x+\gamma y)=Ce^{\gamma x}.$$

PROBLEM 52

420. *With the proposed equation of this kind of differential*

$$dy + Pydx = Qdx,$$

in which P and Q are any functions of x, but with one of the variables y nowhere having a dimension greater than one, to lead that equation to a separation of the variables and to integrate.

SOLUTION

A function of this kind of x is sought, which shall be X, so that with the substitution $y = Xu$ made, a separable equation may be produced. Moreover there arises

$$Xdu + udX + PXudx = Qdx,$$

as it is evident that a separable equation is allowed, if there should be $dX + PXdx = 0$ or

$$\frac{dX}{X} = -Pdx,$$

from which the integration gives

$$lX = -\int Pdx \quad \text{and} \quad X = e^{-\int Pdx};$$

hence with this assumed for the function X our transformed equation will be $Xdu = Qdx$ or

$$du = \frac{Qdx}{X} = e^{\int Pdx}Qdx,$$

from which, since P and Q shall be given functions of x, then there shall be

$$u = \int e^{\int Pdx}Qdx = \frac{y}{X}.$$

On account of which the integral of the proposed equation is

$$y = e^{-\int Pdx}\int e^{\int Pdx}Qdx.$$

COROLLARY 1

421. Hence the resolution of this equation $dy + Pydx = Qdx$ required a double integration, the one of the formula $\int Pdx$, the other of the formula $\int e^{\int Pdx}Qdx$.

But is suffices that an arbitrary constant be added to the final integration, since together y does not take a greater value. For even if initially there is written $\int Pdx + C$ in place of $\int Pdx$, the formula for y remains the same.

COROLLARY 2

422. Hence the formula Pdx then is integrated, and it is sufficient that a particular integral of this is taken and thus it is agreed that a value of this kind be attributed to the constant, so that the form of the integral becomes the simplest.

SCHOLIUM

423. Hence behold, here is another kind of equations extending wider than the preceding kind of homogeneous equations, that leads to the separation of the variables and which can be integrated in this way. But thence in analysis there is an overabundance of usefulness, since here the letters P and Q may denote whichever functions of x. Hence this equation

$$Rdy + Pydx = Qdx$$

is evidently able to be treated in this manner, even if R should denote any function of x. For with the division made by R the proposed form is produced, in place of P and Q there is written $\frac{P}{R}$ and $\frac{Q}{R}$ thus so that on integration there will become

$$y = e^{-\int \frac{Pdx}{R}} \int \frac{e^{\int \frac{Pdx}{R}} Qdx}{R}.$$

We may add certain examples for the illustration of this problem.

EXAMPLE 1

424. *For the proposed equation of the differential* $dy + ydx = x^n dx$, *to find the integral of this.*

Since here there shall be $P = 1$ and $Q = x^n$, then $\int Pdx = x$ and the equation of the integral becomes

$$y = e^{-x} \int e^x x^n dx,,$$

which, if n shall be a positive whole number, there prevails [§ 223]

$$y = e^{-x}\left(e^x\left(x^n - nx^{n-1} + n(n-1)x^{n-2} - \text{etc.}\right) + C\right),$$

from which on expansion there is produced

$$y = Ce^{-x} + x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + \text{etc.},$$

from which for simpler values of n,

if $n = 0$, then $y = Ce^{-x} + 1$,

if $n = 1$, then $y = Ce^{-x} + x - 1$,

if $n = 2$, then $y = Ce^{-x} + x^2 - 2x + 2\cdot 1$,

if $n = 3$, then $y = Ce^{-x} + x^3 - 3x^2 + 3\cdot 2x - 3\cdot 2\cdot 1$

etc.

COROLLARY 1

425. Hence if the constant C is taken = 0, with the particular integral there is had

$$y = x^n - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + \text{etc.},$$

which hence is algebraic, while n is some positive integer.

COROLLARY 2

426. If the integral must thus be determined, so that on putting $x = 0$ the value of y vanishes, then the constant C must be taken equal to the constant final term with the sign changed, from which the function is always to be transcending.

EXAMPLE 2

427. *For the proposed equation of the differential* $(1-xx)dy + xydx = adx$ *, to find the integral of this.*

This equation on division by $1-xx$ is reduced to that form

$$dy + \frac{xydx}{(1-xx)} = \frac{adx}{(1-xx)},$$

thus so that there shall be $P = \frac{x}{1-xx}$, $Q = \frac{a}{1-xx}$, hence

$$\int Pdx = -l\sqrt{(1-xx)} \quad \text{and} \quad e^{\int Pdx} = \frac{1}{\sqrt{(1-xx)}},$$

from which the integral is found

$$y = \sqrt{(1-xx)}\int \frac{adx}{(1-xx)^{\frac{3}{2}}} = \left(\frac{a}{\sqrt{(1-xx)}} + C\right)\sqrt{(1-xx)},$$

on which account the integral sought will be

$$y = ax + C\sqrt{(1-xx)};$$

because if thus it must be determined, so that on putting $x = 0$, it is required to take $C=0$ and then $y = ax$.

EXAMPLE 3

428. *For the proposed equation of the differential* $dy+\frac{nydx}{\sqrt{(1+xx)}}=adx$ *, to find the integral of this.*

Since here there shall be $P=\frac{n}{\sqrt{(1+xx)}}$ and $Q=a$, then

$$\int Pdx = nl\left(x+\sqrt{(1+xx)}\right) \quad \text{and} \quad e^{\int Pdx}=\left(x+\sqrt{(1+xx)}\right)^n$$

and again

$$e^{-\int Pdx}=\left(\sqrt{(1+xx)}-x\right)^n,$$

from which the integral sought will be

$$y=\left(\sqrt{(1+xx)}-x\right)^n\int adx\left(x+\sqrt{(1+xx)}\right)^n,$$

to the solution of which there is put $x+\sqrt{(1+xx)}=u$ and there is made $x=\frac{uu-1}{2u}$, hence

$$dx=\frac{du(1+uu)}{2uu},$$

hence

$$\int u^n dx=\frac{u^{n-1}}{2(n-1)}+\frac{u^{n+1}}{2(n+1)}+C.$$

Now because $\left(\sqrt{(1+xx)}-x\right)^n=u^{-n}$, then

$$y=Cu^{-n}+\frac{au^{-1}}{2(n-1)}+\frac{au}{2(n+1)}$$

or

$$y=C\left(\sqrt{(1+xx)}-x\right)^n+\frac{a}{2(n-1)}\left(\sqrt{(1+xx)}-x\right)+\frac{a}{2(n+1)}\left(\sqrt{(1+xx)}+x\right),$$

which expression is reduced to this form

$$y=C\left(\sqrt{(1+xx)}-x\right)^n+\frac{na}{nn-1}\sqrt{(1+xx)}-\frac{ax}{nn-1}.$$

If the integral must be determined thus, so that on putting $x = 0$ there becomes $y = 0$, it is required to take $C = -\frac{na}{nn-1}$.

PROBLEM 53

429. *For the proposed equation of the differential*

$$dy + Pydx = Qy^{n+1}dx,$$

where P and Q denote some functions of x, to reduce that to the separation of the variables and to integrate.

SOLUTION

This equation on putting $\frac{1}{y^n} = z$ is reduced at once to the form treated lately ; for on account of $\frac{dy}{y} = -\frac{dz}{nz}$ our equation divided by y, clearly

$$\frac{dy}{y} + Pdx = Qy^n dx,$$

is changed at once into

$$-\frac{dz}{nz} + Pdx = \frac{Qdx}{z} \quad \text{or} \quad dz - nPzdx = -nQdx,$$

and the integral of this is

$$z = -e^{n\int Pdx}\int e^{-n\int Pdx}nQdx$$

and thus

$$\frac{1}{y^n} = -ne^{n\int Pdx}\int e^{-n\int Pdx}Qdx.$$

Moreover it can be treated as before by seeking a function X of this kind, so that with the substitution made $y = Xu$ a separable equation is produced ; but there emerges

$$Xdu + udX + PXudx = X^{n+1}u^{n+1}Qdx.$$

Now on making

$$dX + PXdx = 0 \text{ or } X = e^{-\int Pdz}$$

and then

$$\frac{du}{u^{n+1}} = X^n Qdx = e^{-n\int Pdz}Qdx$$

and on integrating

$$-\frac{1}{nu^n} = \int e^{-n\int Pdz}Qdx.$$

Now since

$$u = \frac{y}{X} = e^{\int Pdz}y,$$

there will be had as before

$$\frac{1}{y^n} = -ne^{n\int Pdx}\int e^{-n\int Pdx}Qdx.$$

SCHOLIUM

430. Hence this case is to be considered as no different from the preceding, thus as nothing new has been made available. And these two kinds are nearly as one, which they may exhibit with a little latitude, in which the separation of the variables is obtained. The rest of the cases, which with the aid of certain substitutions are able to be prepared according to the separation of the variables, generally are exceedingly special, as hence it can be expected of the assigned use. Yet meanwhile we set out here a certain case more noteworthy than the rest.

PROBLEM 54

431. *With this proposed equation of the differential*

$$\alpha ydx + \beta xdy + x^m y^n\left(\gamma ydx + \delta xdy\right) = 0,$$

to reduce that according to the separation of the variables and to integrate.

SOLUTION

With the whole equation divided by xy we obtain this form

$$\frac{\alpha dx}{x} + \frac{\beta dy}{y} + x^m y^n\left(\frac{\gamma dx}{x} + \frac{\delta dy}{y}\right) = 0,$$

from which we deduce at once the substitutions $x^\alpha y^\beta = t$ and $x^\gamma y^\delta = u$ distinguished by constant use ; thence indeed there becomes

$$\frac{\alpha dx}{x} + \frac{\beta dy}{y} = \frac{dt}{t} \quad \text{et} \quad \frac{\gamma dx}{x} + \frac{\delta dy}{y} = \frac{du}{u}$$

and hence our equation

$$\frac{dt}{t} + x^m y^m \frac{du}{u} = 0.$$

But from the substitution there follows

$$x^{\alpha\delta-\beta\gamma} = t^\delta u^{-\beta} \quad \text{and} \quad y^{\alpha\delta-\beta\gamma} = u^\alpha t^{-\gamma}$$

and thus

$$x = t^{\frac{\delta}{\alpha\delta-\beta\gamma}} u^{\frac{-\beta}{\alpha\delta-\beta\gamma}} \quad \text{and} \quad y = t^{\frac{-\gamma}{\alpha\delta-\beta\gamma}} u^{\frac{\alpha}{\alpha\delta-\beta\gamma}},$$

with which substituted there becomes

$$\frac{dt}{t} + t^{\frac{\delta m-\gamma n}{\alpha\delta-\beta\gamma}} u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}} \frac{du}{u} = 0$$

and thus

$$t^{\frac{\gamma n-\delta m}{\alpha\delta-\beta\gamma}-1}dt+u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}-1}du=0,$$

of which equation the integral is

$$\frac{t^{\frac{\gamma n-\delta m}{\alpha\delta-\beta\gamma}}}{\gamma n-\delta m}+\frac{u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}}}{\alpha n-\beta m}=C,$$

where it only remains, that the values $t=x^{\alpha}y^{\beta}$ and $u=x^{\gamma}y^{\delta}$ are restored. Further it is to be noted, if there should be either $\gamma n-\delta m=0$ or $\alpha n-\beta m=0$, in place of these members either *lt* or *lu* must be written.

SCHOLIUM

432. The question leads to the proposed equation, from which a relation of this kind is sought between the variables *x* and *y*, so that there becomes

$$\int ydx=axy+bx^{m+1}y^{n+1};$$

for this to be resolved the differential are to be taken, from which there arises

$$ydx=axdy+aydx+bx^{m}y^{n}\left((m+1)ydx+(n+1)xdy\right),$$

from which equation when compared with our form

$$a=\alpha-1,\ \beta=a,\ \gamma=(m+1)b \quad\text{and}\quad \delta=(n+1)b,$$

hence

$$\alpha\delta-\beta\gamma=(n-m)ab-(n+1)b,$$
$$an-\beta m=(n-m)a-n \ \text{ et } \ \gamma n-\delta m=(n-m)b,$$

from which the equation of the integral becomes evident.

PROBLEM 55

433. *From this proposed equation of the differential*

$$ydy+dy\left(a+bx+nxx\right)=ydx\left(c+nx\right)$$

to reduce that to the separation of the variables and to integrate.

SOLUTION

Since hence there becomes

$$\frac{dy}{dx}=\frac{y(c+nx)}{y+a+bx+nxx},$$

this substitution is tested :

$$u=\frac{y(c+nx)}{y+a+bx+nxx} \quad\text{or}\quad y=\frac{u(a+bx+nxx)}{c+nx-u}$$

and there must become $dy = udx$ or

$$\frac{dy}{y} = \frac{udx}{y} = \frac{dx(c+nx-u)}{a+bx+nxx}.$$

But it is deduced from logarithms,

$$\frac{dy}{y} = \frac{du}{u} = \frac{dx(b+2nx)}{a+bx+nxx} - \frac{ndx-du}{c+nx-u} = \frac{dx(c+2nx-u)}{a+bx+nxx},$$

which is drawn together into

$$du\frac{(c+nx)-nudx}{u(c+nx-u)} = \frac{dx(c-b-nx-u)}{a+bx+nxx}$$

or

$$\frac{du(c+nx)}{u(c+nx-u)} = \frac{dx(na+cc-bc+(b-2c)u+uu)}{(c+nx-u)(a+bx+nxx)},$$

which multiplied by $c+nx-u$ clearly is separable, and there emerges

$$\frac{dx}{(a+bx+nxx)(c+nx)} = \frac{du}{u(na+cc-bc+(b-2c)u+uu)},$$

hence the integration of this can be resolved by logarithms and angles. But here in a case barely foreseen it eventuates, that this substitution avowed to succeed, will be of little help in solving this problem.

PROBLEM 56

434. *This proposed differential equation*

$$(y-x)dy = \frac{ndx(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}$$

to be reduced according to the separation of the variables and integrated. .

SOLUTION

On account of the double irrationality scarcely by any way is it apparent what kind of substitution is appropriate to be used. Certainly it is agreed that a substitution of this kind is sought, by which the same sign for the root does not implicate both variables at the same time. Towards this goal this convenient substitution is considered

$$y = \frac{x-u}{1+xu},$$

from which there is made

$$y-x = \frac{-u(1+xx)}{1+xu}, \quad 1+yy = \frac{(1+xx)(1+yy)}{(1+xu)^2}$$

and

$$dy = \frac{dx(1+uu)-du(1+xx)}{(1+xu)^2},$$

and with these values substituted into our equation there emerges

$$-udx(1+uu)+udu(1+xx)=ndx(1+uu)\sqrt{(1+uu)},$$

which evidently permits the separation of the variables ; clearly there is deduced

$$\frac{dx}{1+xx}=\frac{udu}{(1+uu)\left(n\sqrt{(1+uu)}+u\right)},$$

which equation on putting $1+uu=tt$ becomes even neater

$$\frac{dx}{1+xx}=\frac{dt}{t\left(nt+\sqrt{(\mathrm{tt}-1)}\right)}$$

and with the aid of putting $t=\frac{1+ss}{2s}$ with the irrationality removed,

$$\frac{dx}{1+xx}=-\frac{2ds(1-ss)}{(1+ss)\left(n+1+(n-1)ss\right)}=-\frac{2ds}{1+ss}+\frac{2nds}{n+1+(n-1)ss},$$

of which the integration can be produced without further difficulty.

SCHOLIUM

435. In this case the substitution $y=\frac{x-u}{1+xu}$ is especially worthy of note, by which twofold irrationalities are removed, from which the work is seen to be worth the effort, which by this more general substitution is able to excel [the reader may observe here a chance similarity in this transformation to one of the equations regarding relative motion between bodies in special relativity]

$$y=\frac{\alpha x+u}{1+\beta xu};$$

but from this there becomes

$$\alpha-\beta yy=\frac{(\alpha-\beta uu)(1-\alpha\beta xx)\alpha x}{(1+\beta xu)^2},\quad y-\alpha x=\frac{u(1-\alpha\beta xx)}{1+\beta xu}$$

and

$$dy=\frac{dx(\alpha-\beta uu)+du(1-\alpha\beta xx)}{(1+\beta xu)^2}$$

and now it is easily seen in equations of this kind that this substitution is able to bring a use; clearly by the benefit of this these twofold irrationalities $\frac{\sqrt{(\alpha-\beta yy)}}{\sqrt{(1-\alpha\beta xx)}}$ is reduced to that simple irrationality $\frac{\sqrt{(\alpha-\beta uu)}}{1+\beta xu}$, which again is easy to be reduced to a rational expression.

And these are nearly all the cases, in which the reduction to separability has found a place, from which careful investigations the approach to the remaining cases can be readily extended, which indeed even now have been treated; now at this stage I put in place a single investigation about these cases, in which this equation $dy+yydx=ax^m dx$ allows the separation of the variables, since frequently one comes upon equations of this kind and this equation at one time was studied with enthusiasm by the geometers [§ 441].

PROBLEM 57

436. *To define the values of the exponent m for the equation* $dy+yydx=ax^m dx$ *, for which that can be reduced to the separation of the variables.*

SOLUTION

In the first place this equation is separable at once in the case $m=0$; for then on account of $dy=dx(a-yy)$ there becomes $dx=\frac{dy}{a-yy}$. Hence all the investigation turns on this, so that with the aid of a substitution all the cases are reduced to this.
We may put $y=\frac{b}{x}$ and there becomes

$$-bdz+bbdx=ax^m zzdx;$$

in order that which form of the proposed comparison prevails, there is put in place $x^{m+1}=t$, so that there becomes

$$x^m dx=\frac{dt}{m+1} \quad \text{and} \quad dx=\frac{t^{\frac{-m}{m+1}}dt}{m+1},$$

and then

$$bdz+\frac{azzdt}{m+1}=\frac{bb}{m+1}t^{\frac{-m}{m+1}}dt,$$

which on taking $b=\frac{a}{m+1}$ agrees closer with the proposed similarity, so that there becomes

$$dz+zzdt=\frac{a}{(m+1)^2}t^{\frac{-m}{m+1}}dt.$$

Hence if this should be separable, the proposition by this substitution becomes separable in turn ; from which we conclude, if the proposed equation admits to separation in the case $m=n$, that also is to be admitted in the case $m=\frac{-n}{n+1}$. But from the case $m=0$ another hence is not found.

We may put $y=\frac{1}{x}-\frac{z}{xx}$, so that there becomes

$$dy=\frac{dx}{xx}-\frac{dz}{xx}+\frac{2zdx}{x^3} \quad \text{and} \quad yydx=\frac{dx}{xx}-\frac{2zdx}{x^3}+\frac{zzdx}{x^4},$$

from which there emerges

$$-\frac{dz}{xx}+\frac{zzdx}{x^4}=ax^m dx \quad \text{or} \quad dz-\frac{zzdx}{xx}=-ax^{m+2}dx\ ;$$

if now $x=\frac{1}{t}$ and there becomes

$$dz+zzdt=at^{-m-4}dt\ ;$$

which since it shall be similar to the proposed, we learn, that if the separation should succeed in the case $m=n$, also to succeed in the case $m=-n-4$.

Hence from the single case $m=n$ we follow with two, clearly

$$m=-\frac{n}{n+1} \quad \text{and} \quad m=-n-4.$$

Therefore since the case $m=0$ may be agreed upon, hence in turn the following formulas are presented for use :

$$m=-4,\quad m=-\frac{4}{3},\quad m=-\frac{8}{3},\quad m=-\frac{8}{5},\quad m=-\frac{12}{5},$$
$$m=-\frac{12}{7},\quad m=-\frac{16}{7}\quad \text{etc.},$$

all which cases are contained in this formula $m=\frac{-4i}{2i\pm 1}$.

COROLLARY 1

437. But if hence there should be either

$$m=\frac{-4i}{2i+1} \quad \text{or} \quad m=\frac{-4i}{2i-1},$$

the equation $dy+yydx=ax^m dx$ by some substitutions repeated finally can be reduced to the form again $du+uudv=cdv$, the separation and integration of which is agreed on.

COROLLARY 2

438. Clearly if $m=\frac{-4i}{2i+1}$, the equation $dy+yydx=ax^m dx$ by the substitutions

$$x=t^{\frac{1}{m+1}} \text{ and } y=\frac{a}{(m+1)z}$$

is reduced to this

$$dz+zzdt=\frac{a}{(m+1)^2}t^n dt,$$

so that there becomes $n=\frac{-4i}{2i-1}$, which case is to be agreed on for one degree less.

COROLLARY 3

439. But if there should be $m=\frac{-4i}{2i-1}$, the equation $dy+yydx=ax^m dx$ through these substitutions

$$x=\frac{1}{t} \text{ and } y=\frac{1}{x}-\frac{z}{xx} \text{ or } y=t-ttz$$

is reduced to this $dz+zzdt=at^n dt$, in which there is

$$n=\frac{-4(i-1)}{2i-1}=\frac{-4(i-1)}{2(i-1)+1},$$

which case anew is less by one degree.

COROLLARY 4

440. Hence all the separable cases found in this way for the exponent *m* give negative numbers contained between the limits 0 and – 4 , and if *i* should be an infinite number, the case $m=-2$ arises, but which agrees by itself, since the equation

$$dy+yydx=\frac{adx}{xx}$$

on putting $y=\frac{1}{x}$ becomes homogeneous [§ 410].

SCHOLIUM 1

441. This equation $dy+yydx=ax^m dx$ is usually called RICCATIAN after the author Count RICCATI, who proposed the separable case first. [One of his papers is present in these translations.] Here indeed I have shown that in the simplest form , since there this equation $dy+Ayyt^{\mu}dt=Bt^{\lambda}dt$ on putting

$$At^{\mu}dt=dx \quad \text{and} \quad At^{\mu+1}=(\mu+1)x$$

is reduced at once.

Moreover even if the two substitutions, with which I have made use here, are the most simple, yet with greater compositions put to use no other separable cases are uncovered ; from which this has been considered entirely remarkable, this most rare equation that that allows the separation of the variables, even if the number of cases in which this can be performed actually is infinite.

Besides this investigation from the exponent to the simplest coefficient can be treated; for on putting $y=x^{\frac{m}{2}}z$ there is produced

$$dz+\frac{mzdz}{2x}+x^{\frac{m}{2}}zzdx=ax^{\frac{m}{2}}dx,$$

where if there is made

$$x^{\frac{m}{2}}dx=dt \quad \text{et} \quad x^{\frac{m+2}{2}}=\frac{m+2}{2}t,$$

then $\frac{dx}{x}=\frac{2dt}{(m+2)t}$ and hence

$$dz + \frac{mzdz}{(m+2)t} + zzdt = adt,$$

hence which equation, as often as there shall be $\frac{m}{m+2} = \pm\, 2i$ or an even positive or negative number, is able to be reduced, thus so that this equation

$$dz \pm \frac{2izdt}{t} + zzdt = adt$$

always shall be integrable. If in addition there is put $z = u - \frac{m}{2(m+2)t}$, there arises

$$du + uudt = adt - \frac{m(m+4)dt}{4(m+2)^2 tt}$$

and for the separable cases $m = \frac{-4i}{2i\pm 1}$ there may be had

$$du + uudt = adt + \frac{i(i\pm 1)dt}{tt}$$

But the more fruitful development of this equation, since it is of the greatest interest, I will show in what follows, where I have developed the integration of differential equations by infinite series ; for hence we may elicit easier the separable cases and likewise we can assign the integrals.

SCHOLIUM 2

442. Fuller instructions about the separation of variables, which indeed shall soon have to be used, scarcely seem able to be treated, from which it is understood that this method is able to be used in hardly any differential equations. Hence I may advance to the explanation of another principle, from which it is allowed to be drawn up, which extends much wider, while also it is possible for differential equations of higher order to be accommodated, thus so that in this the true and natural origin of all integration may be considered to be contained.

Moreover this principle is established from this, that for any proposed differential equation between two variables there is always a certain function given, by which the equation on multiplication becomes integrable ; clearly it is necessary for all the parts of an equation to be set out in the same part, so that such a form $Pdx + Qdy = 0$ may be obtained ; and then I say that there is always given a certain function of the variables *x* and *y*, for example *V*, so that on multiplication the formula of the integral $VPdx + VQdy$ may arise, or truly it shall have been produced from the differentiation of some function of the two variables *x* and *y*. But if indeed this function may be put $= S$, so that there becomes $dS = VPdx + VQdy$, because there is $Pdx + Qdy = 0$, then also $dS = 0$

and thus $S = \text{Const.},$ which equation therefore will be integrable and that the complete integral of the differential equation $Pdx + Qdy = 0$. Hence the whole work reverts to finding the function V of this multiplier.

CALCULI INTEGRALIS
LIBER PRIOR.
PARS PRIMA
SEU
METHODUS INVESTIGANDI FUNCTIONES
UNIUS **VARIABILIS EX DATA RELATIONE QUACUN·**
QUE DIFFERENTIALIUM PRIMI GRADUS.

SECTIO SECUNDA
DE
INTEGRATIONE AEQUATIONUM
DIFFERENTIALIUM.

CAPUT I
DE SEPARATIONE VARIABILIUM

DEFINITIO

397. *In aequatione differentiali separatio variabilium locum habere dicitur, cum aequationem ita in duo membra dispescere licet, ut in utroque unica tantum variabilis cum suo differentiali insit.*

COROLLARIUM 1

398. Quando igitur aequatio differentialis ita est comparata, ut ad hanc formam $Xdx = Ydy$ reduci possit, in qua X functio sit solius x et Y solius y, tum ea aequatio separationem variabilium admittere dicitur.

COROLLARIUM 2

399. Quodsi P et X functiones ipsius x tantum, at Q et Y functiones ipsius y tantum denotent, haec aequatio $PYdx = QXdy$ separationem variabilium admittit; nam per XY divisa abit in $\frac{Pdx}{X} = \frac{Qdy}{Y}$, in qua variabiles sunt separatae.

COROLLARIUM 3

400. In forma ergo generali $\frac{dy}{dx} = V$ separatio variabilium locum habet, si V eiusmodi fuerit functio ipsarum x et y, ut in duos factores resolvi possit, quorum alter solam variabilem x, alter solam y contineat. Si enim sit $V = XY$, inde prodit aequatio separata $\frac{dy}{Y} = Xdx$.

SCHOLION

401. Posita differentialium ratione $\frac{dy}{dx} = p$ in hac sectione eiusmodi relationem inter x, y et p considerare instituimus, qua p aequetur functioni cuicunque ipsarum x et y. Hic igitur primum eum casum contemplamur, quo ista functio in duos factores resolvitur, quorum alter est functio tantum ipsius x et alter ipsius y, ita ut aequatio ad hanc formam reduci possit $Xdx = Ydy$, in qua binae variabiles a se invicem separatae esse dicuntur. Atque in hoc casu formulae simplices ante tractatae continentur, quando $Y = 1$, ut sit $dy = Xdx$ et $y = \int Xdx$, ubi totum negotium ad integrationem formulae Xdx revocatur. Haud maiorem autem habet difficultatem aequatio separata $Xdx = Ydy$, quam perinde ac formulas simplices tractare licet, id quod in sequente problemate ostendemus.

PROBLEMA 49

402. *Aequationem differentialem, in qua variabiles sunt separatae, integrare seu aequationem inter ipsas variabiles invenire.*

SOLUTIO

Aequatio separationem variabilium admittens semper ad hanc formam $Ydy = Xdx$ reducitur, ubi Xdx tanquam differentiale functionis cuiusdam ipsius x et Ydy tanquam differentiale functionis cuiusdam ipsius y spectari potest. Cum igitur differentialia sint aequalia, eorum integralia quoque aequalia esse vel quantitate constante differre necesse est. Integrentur ergo per praecepta superioris sectionis seorsim ambae formulae seu quaerantur integralia $\int Ydy$ et $\int Xdx$, quibus inventis erit utique $\int Ydy = \int Xdx + \text{Const.}$, qua aequatione relatio finita inter quantitates x et y exprimitur.

COROLLARIUM 1

403. Quoties ergo aequatio differentialis separationem variabilium admittit, toties integratio per eadem praecepta, quae supra de formulis simplicibus sunt tradita, absolvi poteat

COROLLARIUM 2

404. In aequatione integrali $\int Ydy = \int Xdx + \text{Const.}$ vel ambae functiones $\int Ydy$ et $\int Xdx$ sunt algebraicae, vel altera algebraica, altera vero transcendens, vel ambae transcendentes, sicque relatio inter x et y vel erit algebraica vel transcendens.

SCHOLION

405. In separatione variabilium a nonnullis totum fundamentum resolutionis aequationum differentialium constitui solet, ita ut, cum aequatio proposita separationem variabilium non admittit, idonea substitutio sit investiganda, cuius beneficia novae variabiles introductae separationem patiantur. Totum ergo negotium huc reducitur, ut proposita aequatione differentiali quacunque eiusmodi substitutio seu novarum variabilium introductio doceatur, ut deinceps separatio variabilium locum sit habitura. Optandum utique esset, ut huiusmodi methodus pro quovis casu idoneam substitutionem inveniendi aperiretur; sed nihil omnino certi in hoc negotio est compertum, dum pleraeque substitutiones, quae adhuc in usu fuerunt, nullis certis principiis innituntur. Deinde autem variabilium separatio non tanquam verum fundamentum omnis integrationis spectari potest, propterea quod in aequationibus differentialibus secundi altiorisve gradus nullum usum praestat; infra autem aliud principium latissime patens sum expositurus. In hoc capite interim praecipuas integrationes ope separationis variabilium administratas exponere operae pretium videtur, quandoquidem in hoc ardua negotio quam plurimas methodos cognoscere plurimum interest.

PROBLEMA 50

406. *Aequationem differentialem Pdx = Qdy, in qua P et Q sint functiones homogeneae eiusdem dimensionum numeri ipsarum x et y, ad separationem variabilium reducere eiusque integrale invenire.*

SOLUTIO

Cum P et Q sint functiones homogeneae ipsarum x et y eiusdem dimensionum numeri, erit $\frac{P}{Q}$ functio homogenea nullius dimensionis, quae ergo posito $y = ux$ abit in functionem ipsius u. Ponatur igitur $y = ux$ abeatque $\frac{P}{Q}$ in U functionem ipsius u, ita ut sit $dy = Udx$. Sed ob $y = ux$ fit $dy = udx + xdu$ qua substitutione nostra aequatio induet hanc formam $udx + xdu = Udx$ inter binas variabiles x et u, quae manifesto sunt separabiles. Nam dispositis terminis dx continentibus ad unam partem habetur $xdu = (U - u)dx$ ideoque

$$\frac{dx}{x} = \frac{du}{U-u}$$

quae integrata dat $lx = \int \frac{du}{U-u}$ ita ut iam ex variabili u determinetur x,unde porro cognoscitur $y = ux$.

COROLLARIUM 1

407. Quodsi ergo integrale $\int \frac{du}{U-u}$ etiam per logarithmos exprimi possit, ita ut lx aequetur logarithmo functionis cuiuspiam [algebraicae] ipsius u, habebitur aequatio algebraica inter x et u ideoque pro u posito valore $\frac{y}{x}$ aequatio algebraica inter x et y.

COROLLARIUM 2

408. Cum sit $y = ux$, erit $ly = lu + lx$ ideoque, cum sit $lx = \int \frac{du}{U-u}$ erit

$$ly = lu + \int \frac{du}{U-u} = \int \frac{du}{u} + \int \frac{du}{U-u},$$

quibus integralibus in unum reductis fit $ly = \int \frac{Udu}{u(U-u)}$. Verum hic notandum est non in utraque integratione pro lx et ly constantem arbitrariam adiicere licere; statim enim atque alteri integrali est adiecta, simul constans alteri adiicienda definitur, cum esse debeat $ly = lx + lu$.

COROLLARIUM 3

409. Cum sit

$$\int\frac{du}{U-u}=\int\frac{du-dU+dU}{U-u}=\int\frac{dU}{U-u}-\int\frac{dU-du}{U-u}$$

ob hoc posterius membrum per logarithmos integrabile erit

$$lx=\int\frac{dU}{U-u}-l(U-u)\text{ seu }lx(U-u)=\int\frac{dU}{U-u}.$$

Perinde ergo est, sive haec formula $\int\frac{du}{U-u}$ sive $\int\frac{dU}{U-u}$ integretur.

SCHOLION

410. Quoniam haec methodus ad omnes aequationes homogeneas patet neque etiam ob irrationalitatem, quae forte in functionibus P et Q inest, impeditur, imprimis est aestimanda plurimumque aliis methodis anteferenda, quae tantum ad aequationes nimis speciales sunt accommodatae. Atque hinc etiam discimus omnes aequationes, quae ope cuiusdam substitutionis ad homogeneitatem revocari possunt, per eandem methodum tractari posse. Veluti si proponatur haec aequatio

$$dz+zzdx=\frac{adx}{xx},$$

statim patet posito $z=\frac{1}{y}$ eam ad hanc homogeneam

$$-\frac{dy}{yy}+\frac{dx}{yy}=\frac{adx}{xx}\text{ seu }xxdy=dx(xx-ayy)$$

reduci [§ 414].

Caeterum non difficulter perspicitur, utrum aequatio proposita huiusmodi substitutione ad homogeneitatem perduci queat. Plerumque, quoties quidem fieri potest, sufficit has positiones $x=u^m$ et $y=v^n$ tentasse, ubi facile iudicabitur, num exponentes m et n ita assumere liceat, ut ubique idem dimensionum numerus prodeat; magis enim complicatis substitutionibus in hoc genere vix locus conceditur, nisi forte quasi sponte se prodant. Methodum autem integrandi hic expositam aliquot exemplis illustrasse iuvabit.

EXEMPLUM 1

411. *Proposita aequatione differentiali homogenea* $xdx+ydy=mydx$ *eius integrale invenire.*

Cum ergo hinc sit $\frac{dy}{dx}=\frac{my-x}{y}$, posito $y=ux$ fit $\frac{my-x}{y}=\frac{mu-1}{u}$ ideoque ob $dy=udx+xdu$ erit

$$udx+xdu=\frac{mu-1}{u}dx$$

hincque

$$\frac{dx}{x}=\frac{udu}{mu-1-uu}=\frac{-udu}{1-mu+uu}$$

seu

$$\frac{dx}{x}=\frac{-udu+\frac{1}{2}mdu}{1-mu+uu}-\frac{\frac{1}{2}mdu}{1-mu+uu},$$

unde integrando

$$lx=-\frac{1}{2}l\left(1-mu+uu\right)-\frac{1}{2}m\int\frac{du}{1-mu+uu}+\text{Const.},$$

ubi tres casus sunt considerandi, prout $m > 2$ vel $m < 2$ vel $m = 2$.

1) Sit $m > 2$ et $1-mu+uu$ huiusmodi formam habebit

$$\left(u-a\right)\left(u-\frac{1}{a}\right),$$

ut sit $m=a+\frac{1}{a}=\frac{aa+1}{a}$, et ob

$$\frac{du}{\left(u-a\right)\left(u-\frac{1}{a}\right)}=\frac{a}{aa-1}\cdot\frac{du}{u-a}-\frac{a}{aa-1}\cdot\frac{du}{u-\frac{1}{a}}$$

fiat

$$lx=-\frac{1}{2}l\left(1-mu+uu\right)-\frac{aa+1}{2\left(aa-1\right)}l\frac{u-a}{u-\frac{1}{a}}+C$$

seu

$$lx\sqrt{\left(1-mu+uu\right)}+\frac{aa+1}{2\left(aa-1\right)}l\frac{au-aa}{au-1}=lc$$

et restituto valore $u=\frac{y}{x}$ aequatio integralis erit

$$l\sqrt{\left(xx-mxy+yy\right)}+\frac{aa+1}{2\left(aa-1\right)}l\frac{ay-aax}{ay-x}=lc$$

seu

$$\left(\frac{ay-aax}{ay-x}\right)^{\frac{aa+1}{2\left(aa-1\right)}}\sqrt{\left(xx-mxy+yy\right)}=c.$$

2) Sit $m < 2$ seu $m=2\cos.\alpha$; erit

$$\int\frac{du}{1-u\cos.\alpha+uu}=\frac{1}{\sin.\alpha}\text{Ang.tang.}\frac{u\sin.\alpha}{1-u\cos.\alpha},$$

unde

$$lx\sqrt{\left(1-mu+uu\right)}=C-\frac{\cos.\alpha}{\sin.\alpha}\text{Ang.tang.}\frac{u\sin.\alpha}{1-u\cos.\alpha}$$

seu

$$l\sqrt{\left(xx-mxy+yy\right)}=C-\frac{\cos.\alpha}{\sin.\alpha}\text{Ang.tang.}\frac{y\sin.\alpha}{x-y\cos.\alpha}.$$

3) Sit $m = 2$; erit

$$\int\frac{du}{\left(1-u\right)^{2}}=\frac{1}{1-u}$$

hincque

$$lx(1-u) = C - \tfrac{1}{1-u} \quad \text{seu} \quad l(x-y) = C - \tfrac{x}{x-y}.$$

EXEMPLUM 2

412. *Proposita aequatione differentiali homogenea* $dx(\alpha x + \beta y) = dy(\gamma x + \delta y)$ *eius integrale invenire.*

Posito y = *ux* erit $udx + xdu = dx \cdot \frac{\alpha+\beta u}{\gamma+\delta u}$ ideoque

$$\frac{dx}{x} = \frac{du(\gamma+\delta u)}{\alpha+\beta u-\gamma u-\delta uu} = \frac{du\left(\delta u+\frac{1}{2}\gamma-\frac{1}{2}\beta\right)+du\left(\frac{1}{2}\gamma+\frac{1}{2}\beta\right)}{\left(\alpha+(\beta-\gamma)u-\delta uu\right)},$$

unde integrando

$$lx = C - l\sqrt{\left(\alpha+(\beta-\gamma)u-\delta uu\right)} + \frac{1}{2}(\beta+\gamma)\int\frac{du}{\alpha+(\beta-\gamma)u-\delta uu},$$

ubi iidem casus qui ante sunt considerandi, prout scilicet denominator $\alpha+(\beta-\gamma)u-\delta uu$ vel duos factores habet reales et inaequales vel aequales vel imaginarios.

EXEMPLUM 3

413. *Proposita aequatione differentiali homogenea* $xdx + ydy = xdy - ydx$ *eius integrale invenire.*

Cum hinc sit $\frac{dy}{dx} = \frac{x+y}{x-y}$, posito $y = ux$ fit $udx + xdu = \frac{1+u}{1-u}dx$ seu $xdu = \frac{1+uu}{1-u}dx$, unde colligitur

$$\frac{dx}{x} = \frac{du-udu}{1+uu}$$

et integrando

$$lx = \text{Ang.tang.}u - l\sqrt{(1+uu)} + C$$

seu

$$l\sqrt{(xx+yy)} = C + \text{Ang.tang.}\frac{y}{x}$$

EXEMPLUM 4

414. *Proposita aequatione differentiali homogenea* $xxdy = (xx-ayy)dx$ *eius integrale invenire.*

Hic ergo est $\frac{dy}{dx} = \frac{xx-ayy}{xx}$

et posito $y = ux$ prodit

$$udx + xdu = (1 - auu)dx$$

ideoque

$$\frac{dx}{x} = \frac{du}{1-u-auu}dx \quad \text{et} \quad lx = \int \frac{du}{1-u-auu},$$

cuius evolutioni non opus est immorari.

EXEMPLUM 5

415. *Proposita aequatione differentiali homogenea* $xdy - ydx = dx\sqrt{(xx + yy)}$ *eius integrale invenire.*

Erit ergo $\frac{dy}{dx} = \frac{y+\sqrt{(xx+yy)}}{x}$ unde posito $y = ux$ fit

$$udx + xdu = \left(u + \sqrt{(1+uu)}\right)dx$$

seu

$$xdu = dx\sqrt{(1+uu)},$$

ita ut sit

$$\frac{dx}{x} = \frac{du}{\sqrt{(1+uu)}},$$

cuins integrale est

$$lx = la + l\left(u + \sqrt{(1+uu)}\right) = la + l\left(\frac{y+\sqrt{(xx+yy)}}{x}\right)$$

seu

$$lx = la + l\frac{x}{\sqrt{(xx+yy)}-y}$$

unde colligitur $x = \frac{ax}{\sqrt{(xx+yy)}-y}$ seu $\sqrt{(xx + yy)} = a + y$ hincque $xx = aa + 2ay$.

SCHOLION

416. Huc etiam functiones transcendentes numerari possunt, modo afficiant functiones nullius dimensionis ipsarum x et y, quia posito $y = ux$ simul in functiones ipsius u abeunt. Ita si in aequatione $Pdx = Qdy$, praeterquam quod P et Q sunt functiones homogeneae eiusdem dimensionum numeri, insint huiusmodi formulae $l\frac{\sqrt{(xx+yy)}}{x}$, $e^{y:x}$, Ang. sin.$\frac{x}{\sqrt{(xx+yy)}}$, $cos.\frac{nx}{y}$ etc., methodus exposita pari successu adhiberi potest, quia posito $y = ux$ ratio $\frac{dy}{dx}$ aequatur functioni solius novae variabilis u.

PROBLEMA 51

417. *Aequationem differentialem primi ordinis*

$$dx(\alpha+\beta x+\gamma y)=dy(\delta+\varepsilon x+\zeta y)$$

ad separationem variabilium revocare et integrare.

SOLUTIO

Ponatur

$$\alpha+\beta x+\gamma y=t \quad \text{et} \quad \delta+\varepsilon x+\zeta y=u$$

ut fiat $tdx=udy$. At inde colligimus

$$x=\frac{\zeta t-\gamma u-\alpha\zeta+\gamma\delta}{\beta\zeta-\gamma\varepsilon} \quad \text{et} \quad y=\frac{\beta u-\varepsilon t+\alpha\varepsilon-\beta\delta}{\beta\zeta-\gamma\varepsilon}$$

hincque

$$dx:\mathrm{d}y=\zeta dt-\gamma du:\beta du-\varepsilon dt,$$

unde nanciscimur hanc aequationem

$$\zeta tdt-\gamma tdu=\beta udu-\varepsilon udt$$

seu

$$dt(\zeta t+\varepsilon u)=du(\beta u+\gamma t);$$

quae cum sit homogenea et cum exemplo § 412 conveniat, integratio iam est expedita.

Verum tamen casus existit, quo haec reductio ad homogeneitatem locum non habet, cum fuerit $\beta\zeta-\gamma\varepsilon=0$, quoniam tum introductio novarum variabilium *t* et *u* tollitur. Hic ergo casus peculiarem requirit solutionem, quae ita instituatur. Quoniam tum aequatio proposita eiusmodi formam est habitura

$$\alpha dx+(\beta x+\gamma y)dx=\delta dy+n(\beta x+\gamma y)dy,$$

ponamus $\beta x+\gamma y=z$; erit

$$\frac{dy}{dx}=\frac{\alpha+z}{\delta+nz}.$$

At $dy=\frac{dz-\beta dx}{\gamma}$,ergo

$$\frac{dz-\beta dx}{\gamma}=\frac{\alpha+z}{\delta+nz}dx,$$

ubi variabiles manifesto sunt separabiles; fit enim

$$dx=\frac{dz(\delta+nz)}{\alpha\gamma+\beta\delta+(\gamma+n\beta)z},$$

cuius integratio logarithmos involvit, nisi sit $\gamma+n\beta=0$, quo casu algebraice dat $x=\frac{2\delta z+nzz}{2(\alpha\gamma+\beta\delta)}+C.$

COROLLARIUM 1

418. Aequatio ergo differentialis primi ordinis, uti vocatur, in genere ad homogeneitatem reduci nequit, sed casus, quibus $\beta\zeta=\gamma\varepsilon$, inde excipi debent, qui etiam ad aequationem separatam omnino diversam deducunt.

COROLLARIUM 2

419. Si in his casibus exceptis sit $n=0$ seu haec proposita sit aequatio $dy=dx(a+\beta x+\gamma y)$, posito $\beta x+\gamma y=z$ ob $\delta=1$ haec oritur aequatio $dx=\frac{dz}{\alpha\gamma+\beta+\gamma z}$, cuius integrale est

$$\gamma x=l\frac{\beta+\alpha\gamma+\gamma z}{C}=l\frac{\beta+\alpha\gamma+\beta\gamma x+\gamma\gamma y}{C}$$

seu

$$\beta+\gamma(\alpha+\beta x+\gamma y)=Ce^{\gamma x}.$$

PROBLEMA 52

420. *Proposita aequatione differentiali huiusmodi*

$$dy+Pydx=Qdx,$$

in qua P et Q sint functiones quaecunque ipsius x, altera autem variabilis y cum suo differentiali nusquam plus una habeat dimensionem, eam ad separationem variabilium perducere et integrare.

SOLUTIO

Quaeratur eiusmodi functio ipsius x, quae sit X, ut facta substitutione $y = Xu$ aequatio prodeat separabilis. Tum autem oritur

$$Xdu+udX+PXudx=Qdx,$$

quam aequationem separationem admittere evidens est, si fuerit $dX+PXdx=0$ seu

$$\frac{dX}{X}=-Pdx,$$

unde integratio dat

$$lX=-\int Pdx \quad \text{et} \quad X=e^{-\int Pdx};$$

hac ergo pro X sumta functione aequatio nostra transformata erit $Xdu=Qdx$ seu

$$du=\frac{Qdx}{X}=e^{\int Pdx}Qdx,$$

unde, cum P et Q sint functiones datae ipsius x, erit

$$u = \int e^{\int Pdx} Qdx = \frac{y}{X}.$$

Quocirca aequationis propositae integrale est

$$y = e^{-\int Pdx} \int e^{\int Pdx} Qdx.$$

COROLLARIUM 1

421. Resolutio ergo huius aequationis $dy + Pydx = Qdx$ duplicem requirit integrationem, alteram formulae $\int Pdx$, alteram formulae $\int e^{\int Pdx} Qdx$. Sufficit autem in posteriori constantem arbitrariam adiecisse, cum valor ipsius y plus una non recipiat. Etiamsi enim in priori loco $\int Pdx$ scribatur $\int Pdx + C$, formula pro y manet eadem.

COROLLARIUM 2

422. Dum ergo formula Pdx integratur, sufficit eius integrale particulare sumi ideoque constanti ingredienti eiusmodi valorem tribui convenit, ut integralis forma fiat simplicissima.

SCHOLION

423. En ergo aliud aequationum genus non minus late patens quam praecedens homogenearum, quod ad separationem variabilium perduci hocque modo integrari potest. Inde autem in Analysin maxima utilitas redundat, cum hic litterae P et Q functiones quascunque ipsius x denotent. Hoc ergo modo manifestum est tractari posse hanc aequationem

$$Rdy + Pydx = Qdx,$$

si etiam R functionem quamcunque ipsius x denotet. Facta enim divisione per R forma proposita prodit, modo loco P et Q scribatur $\frac{P}{R}$ et $\frac{Q}{R}$ ita ut integrale futurum sit

$$y = e^{-\int \frac{Pdx}{R}} \int \frac{e^{\int \frac{Pdx}{R}} Qdx}{R}.$$

Ad huius problematis illustrationem quaedam exempla adiiciamus.

EXEMPLUM 1

424. *Proposita aequatione differentiali $dy + ydx = x^n dx$ eius integrale invenire.*

Cum hic sit $P = 1$ et $Q = x^n$, erit $\int Pdx = x$ et aequatio integralis fiet

$$y = e^{-x} \int e^x x^n dx,,$$

quae, si n sit numerus integer positivus, evadet [§ 223]

$y = e^{-x}\left(e^{x}\left(x^{n} - nx^{n-1} + n(n-1)x^{n-2} - \text{etc.}\right) + C\right),$

qua evoluta prodit

$$y = Ce^{-x} + x^{n} - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + \text{etc.},$$

unde pro simplicioribus valoribus ipsius *n*,

si $n = 0$, erit $y = Ce^{-x} + 1$,

si $n = 1$, erit $y = Ce^{-x} + x - 1$,

si $n = 2$, erit $y = Ce^{-x} + x^{2} - 2x + 2\cdot 1$,

si $n = 3$, erit $y = Ce^{-x} + x^{3} - 3x^{2} + 3\cdot 2x - 3\cdot 2\cdot 1$

etc.

COROLLARIUM 1

425. Si ergo constans *C* sumatur = 0, habebitur integrale particulare

$$y = x^{n} - nx^{n-1} + n(n-1)x^{n-2} - n(n-1)(n-2)x^{n-3} + \text{etc.},$$

quod ergo est algebraicum, dummodo *n* sit numerus integer positivus.

COROLLARlUM 2

426. Si integrale ita determinari debeat, ut posito *x* = 0 valor ipsius *y* evanescat, constans *C* aequalis sumi debet ultimo termino constanti signo mutato, unde id semper erit transcendens.

EXEMPLUM 2

427. *Proposita aequatione differentiali* $(1 - xx)dy + xydx = adx$ *eius integrale invenire.*

Aequatio ista per $1 - xx$ divisa ad hanc formam reducitur

$$dy + \frac{xydx}{(1-xx)} = \frac{adx}{(1-xx)},$$

ita ut sit $P = \frac{x}{1-xx}$, $Q = \frac{a}{1-xx}$, hinc

$$\int Pdx = -l\sqrt{(1-xx)} \quad \text{et} \quad e^{\int Pdx} = \frac{1}{\sqrt{(1-xx)}},$$

ex quo integrale reperitur

$$y = \sqrt{(1-xx)}\int \frac{adx}{(1-xx)^{\frac{3}{2}}} = \left(\frac{a}{\sqrt{(1-xx)}} + C\right)\sqrt{(1-xx)},$$

quocirca integrale quaesitum erit

$$y = ax + C\sqrt{(1-xx)};$$

quod si ita determinari debeat, ut posito $x = 0$, sumi oportet $C=0$ eritque $y = ax$.

EXEMPLUM 3

428. *Proposita aequatione differentiali* $dy + \frac{nydx}{\sqrt{(1+xx)}} = adx$ *eius integrale invenire.*

Cum hic sit $P = \frac{n}{\sqrt{(1+xx)}}$ et $Q = a$, erit

$$\int Pdx = nl\left(x + \sqrt{(1+xx)}\right) \quad \text{et} \quad e^{\int Pdx} = \left(x + \sqrt{(1+xx)}\right)^n$$

et

$$e^{-\int Pdx} = \left(\sqrt{(1+xx)} - x\right)^n,$$

unde integrale quaesitum erit

$$y = \left(\sqrt{(1+xx)} - x\right)^n \int adx\left(x + \sqrt{(1+xx)}\right)^n,$$

ad quod evolvendum ponatur $x + \sqrt{(1+xx)} = u$ et fiet $x = \frac{uu-1}{2u}$, hinc

$$dx = \frac{du(1+uu)}{2uu},$$

ergo

$$\int u^n dx = \frac{u^{n-1}}{2(n-1)} + \frac{u^{n+1}}{2(n+1)} + C.$$

Nunc quia $\left(\sqrt{(1+xx)} - x\right)^n = u^{-n}$, erit

$$y = Cu^{-n} + \frac{au^{-1}}{2(n-1)} + \frac{au}{2(n+1)}$$

sive

$$y = C\left(\sqrt{(1+xx)} - x\right)^n + \frac{a}{2(n-1)}\left(\sqrt{(1+xx)} - x\right) + \frac{a}{2(n+1)}\left(\sqrt{(1+xx)} + x\right)$$

quae expressio ad hanc formam reducitur

$$y = C\left(\sqrt{(1+xx)} - x\right)^n + \frac{na}{nn-1}\sqrt{(1+xx)} - \frac{ax}{nn-1}.$$

Si integrale ita determinari debeat, ut posito $x = 0$ fiat $y = 0$, sumi oportet $C = -\frac{na}{nn-1}$.

PROBLEMA 53

429. *Proposita aequatione differentiali*

$$dy + Pydx = Qy^{n+1}dx,$$

ubi P et Q denotent functiones quascunque ipsius x, eam ad separationem variabilium reducere et integrare.

SOLUTIO

Haec aequatio posito $\frac{1}{y^n} = z$ statim ad formam modo tractatam reducitur; nam ob $\frac{dy}{y} = -\frac{dz}{nz}$ aequatio nostra per y divisa, scilicet

$$\frac{dy}{y} + Pdx = Qy^n dx,$$

statim abit in

$$-\frac{dz}{nz} + Pdx = \frac{Qdx}{z} \quad \text{seu} \quad dz - nPzdx = -nQdx,$$

cuius integrale est

$$z = -e^{n\int Pdx}\int e^{-n\int Pdx}nQdx$$

ideoque

$$\frac{1}{y^n} = -ne^{n\int Pdx}\int e^{-n\int Pdx}Qdx.$$

Tractari autem potest ut praecedens quaerendo eiusmodi functionem X, ut facta substitutione $y = Xu$ prodeat aequatio separabilis; prodit autem

$$Xdu + udX + PXudx = X^{n+1}u^{n+1}Qdx.$$

Fiat ergo

$$dX + PXdx = 0 \text{ seu } X = e^{-\int Pdz}$$

eritque

$$\frac{du}{u^{n+1}} = X^n Qdx = e^{-n\int Pdz}Qdx$$

et integrando

$$-\frac{1}{nu^n} = \int e^{-n\int Pdz}Qdx.$$

Iam quia

$$u = \frac{y}{X} = e^{\int Pdz}y,$$

habebitur ut ante

$$\frac{1}{y^n} = -ne^{n\int Pdx}\int e^{-n\int Pdx}Qdx.$$

SCHOLION

430. Hic ergo casus a praecedente non differre est censendus, ita ut hic nihil novi sit praestitum. Atque haec duo genera sunt fere sola, quae quidem aliquanto latius pateant, in quibus separatio variabilium obtineri queat. Caeteri casus, qui ope cuiusdam substitutionis ad variabilium separationem praeparari possunt, plerumque sunt nimis speciales, quam ut insignis usus inde expectari possit. Interim tamen aliquot casus prae caeteris memorabiles hic exponamus.

PROBLEMA 54

431. *Proposita hac aequatione differentiali* $\alpha ydx + \beta xdy + x^m y^n \left(\gamma ydx + \delta xdy\right) = 0$, *eam ad separationem variabilium reducere et integrare.*

SOLUTIO

Tota aequatione per *xy* divisa nanciscimur hanc formam

$$\frac{\alpha dx}{x} + \frac{\beta dy}{y} + x^m y^n \left(\frac{\gamma dx}{x} + \frac{\delta dy}{y}\right) = 0,$$

unde statim has substitutiones $x^\alpha y^\beta = t$ et $x^\gamma y^\delta = u$ insigni usu non esse carituras colligimus; inde enim fit

$$\frac{\alpha dx}{x} + \frac{\beta dy}{y} = \frac{dt}{t} \quad \text{et} \quad \frac{\gamma dx}{x} + \frac{\delta dy}{y} = \frac{du}{u}$$

hincque aequatio nostra

$$\frac{dt}{t} + x^m y^m \frac{du}{u} = 0.$$

At ex substitutione sequitur $x^{\alpha\delta-\beta\gamma} = t^\delta u^{-\beta}$ et $y^{\alpha\delta-\beta\gamma} = u^\alpha t^{-\gamma}$ ideoque

$$x = t^{\frac{\delta}{\alpha\delta-\beta\gamma}} u^{\frac{-\beta}{\alpha\delta-\beta\gamma}} \quad \text{et} \quad y = t^{\frac{-\gamma}{\alpha\delta-\beta\gamma}} u^{\frac{\alpha}{\alpha\delta-\beta\gamma}},$$

quibus substitutis fit

$$\frac{dt}{t} + t^{\frac{\delta m-\gamma n}{\alpha\delta-\beta\gamma}} u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}} \frac{du}{u} = 0$$

ideoque

$$t^{\frac{\gamma n-\delta m}{\alpha\delta-\beta\gamma}-1}dt + u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}-1}du = 0,$$

cuius aequationis integrale est

$$\frac{t^{\frac{\gamma n-\delta m}{\alpha\delta-\beta\gamma}}}{\gamma n-\delta m} + \frac{u^{\frac{\alpha n-\beta m}{\alpha\delta-\beta\gamma}}}{\alpha n-\beta m} = C,$$

ubi tantum superest, ut restituantur valores $t = x^\alpha y^\beta$ et $u = x^\gamma y^\delta$. Caeterum notetur, si fuerit vel $\gamma n - \delta m = 0$ vel $\alpha n - \beta m = 0$, loco illorum membrorum vel *lt* vel *lu* scribi debere.

SCHOLION

432. Ad aequationem propositam ducit quaestio, qua eiusmodi relatio inter variabiles x et y quaeritur, ut fiat

$$\int ydx = axy + bx^{m+1}y^{n+1};$$

ad hanc enim resolvendam differentialia sumi debent, quo prodit

$$ydx = axdy + aydx + bx^m y^n \left((m+1)\,ydx + (n+1)\,xdy\right),$$

qua aequatione cum nostra forma comparata est

$$a = \alpha - 1,\ \beta = a,\ \gamma = (m+1)b \quad \text{et} \quad \delta = (n+1)b,$$

ergo

$$\alpha\delta - \beta\gamma = (n-m)ab - (n+1)b,$$
$$an - \beta m = (n-m)a - n \ \text{ et } \ \gamma n - \delta m = (n-m)b,$$

unde aequatio integralis fit manifesta.

PROBLEMA 55

433. *Proposita hac aequatione differentiali*

$$ydy + dy(a + bx + nxx) = ydx(c + nx)$$

eam ad separationem variabilium reducere et integrare.

SOLUTIO

Cum hinc sit

$$\frac{dy}{dx} = \frac{y(c+nx)}{y+a+bx+nxx},$$

tentetur haec substitutio

$$u = \frac{y(c+nx)}{y+a+bx+nxx} \quad \text{seu} \quad y = \frac{u(a+bx+nxx)}{c+nx-u}$$

fierique debet $dy = udx$ seu

$$\frac{dy}{y} = \frac{udx}{y} = \frac{dx(c+nx-u)}{a+bx+nxx}.$$

At ex logarithmis colligitur

$$\frac{dy}{y} = \frac{du}{u} = \frac{dx(b+2nx)}{a+bx+nxx} - \frac{ndx-du}{c+nx-u} = \frac{dx(c+2nx-u)}{a+bx+nxx},$$

quae contrahitur in

$$du\,\frac{(c+nx)-nudx}{u(c+nx-u)} = \frac{dx(c-b-nx-u)}{a+bx+nxx}$$

seu

$$\frac{du(c+nx)}{u(c+nx-u)}=\frac{dx\left(na+cc-bc+(b-2c)u+uu\right)}{(c+nx-u)(a+bx+nxx)},$$

quae per $c+nx-u$ multiplicata manifesto est separabilis, proditque

$$\frac{dx}{(a+bx+nxx)(c+nx)}=\frac{du}{u\left(na+cc-bc+(b-2c)u+uu\right)},$$

cuius ergo integratio per logarithmos et angulos absolvi potest. Casu autem hic vix praevidendo evenit, ut haec substitutio ad votum successerit, neque hoc problema magnopere iuvabit.

PROBLEMA 56

434. *Propositam hanc aequationem differentialem*

$$(y-x)dy=\frac{ndx(1+yy)\sqrt{(1+yy)}}{\sqrt{(1+xx)}}$$

ad separationem variabilium reducere et integrare.

SOLUTIO

Ob irrationalitatem duplicem vix ullo modo patet, cuiusmodi substitutione uti conveniat. Eiusmodi certe quaeri convenit, qua eidem signa radicali non ambae variabiles simul implicentur. Ad hunc scopum commoda videtur haec substitutio

$$y=\frac{x-u}{1+xu},$$

qua fit

$$y-x=\frac{-u(1+xx)}{1+xu},\quad 1+yy=\frac{(1+xx)(1+yy)}{(1+xu)^2}$$

et

$$dy=\frac{dx(1+uu)-du(1+xx)}{(1+xu)^2},$$

atque his valoribus in nostra aequatione substitutis prodit

$$-udx(1+uu)+udu(1+xx)=ndx(1+uu)\sqrt{(1+uu)},$$

quae manifesto separationem variabilium admittit; colligitur scilicet

$$\frac{dx}{1+xx}=\frac{udu}{(1+uu)\left(n\sqrt{(1+uu)}+u\right)},$$

quae aequatio posito $1+uu=tt$ concinnior redditur

$$\frac{dx}{1+xx}=\frac{dt}{t\left(nt+\sqrt{(tt-1)}\right)}$$

et ope positionis $t=\frac{1+ss}{2s}$ sublata irrationalitate

$$\frac{dx}{1+xx}=-\frac{2ds(1-ss)}{(1+ss)\left(n+1+(n-1)ss\right)}=-\frac{2ds}{1+ss}+\frac{2nds}{n+1+(n-1)ss},$$

cuius integratio nulla amplius laborat difficultate.

SCHOLION

435. In hoc casu praecipue substitutio $y=\frac{x-u}{1+xu}$ notari meretur, qua duplex irrationalitas tollitur, unde operae pretium erit videre, quid hac substitutione generaliori praestari possit

$$y=\frac{\alpha x+u}{1+\beta xu};$$

inde autem fit

$$\alpha-\beta yy=\frac{(\alpha-\beta uu)(1-\alpha\beta xx)\alpha x}{(1+\beta xu)^2},\quad y-\alpha x=\frac{u(1-\alpha\beta xx)}{1+\beta xu}$$

et

$$dy=\frac{dx(\alpha-\beta uu)+du(1-\alpha\beta xx)}{(1+\beta xu)^2}$$

ac iam facile perspicitur, in cuiusmodi aequationibus haec substitutio usum afferre possit; eius scilicet benificio haec duplex irrationalitas $\frac{\sqrt{(\alpha-\beta yy)}}{\sqrt{(1-\alpha\beta xx)}}$ reducitur ad hanc simplicem $\frac{\sqrt{(\alpha-\beta uu)}}{1+\beta xu}$, quam porro facile rationalem reddere licet.

Atque hi fere sunt casus, in quibus reductio ad separabilitatem locum invenit, quibus probe perpensis aditus facile patebit ad reliquos casus, qui quidem etiamnum sunt tractati; unicam vero adhuc investigationem apponam circa casus, quibus haec aequatio $dy+yydx=ax^m dx$ separationem variabilium admittit, quandoquidem ad huiusmodi aequationes frequenter pervenitur atque haec ipsa aequatio olim inter Geometras omni studio est agitata [§ 441].

PROBLEMA 57

436. *Pro aequatione* $dy + yydx = ax^m dx$ *valores exponentis m definire, quibus eam ad separationem variabilium reducere licet.*

SOLUTIO

Primo haec aequatio sponte est separabilis casu $m = 0$; tum enim ob $dy = dx(a - yy)$ fit $dx = \frac{dy}{a-yy}$. Omnis ergo investigatio in hoc versatur, ut ope substitutionum alii casus ad hunc reducantur.

Ponamus $y = \frac{b}{x}$ et fit

$$-bdz + bbdx = ax^m zzdx;$$

quae forma ut propositae similis evadat, statuatur $x^{m+1} = t$, ut sit

$$x^m dx = \frac{dt}{m+1} \quad \text{et} \quad dx = \frac{t^{\frac{-m}{m+1}}dt}{m+1},$$

eritque

$$bdz + \frac{azzdt}{m+1} = \frac{bb}{m+1}t^{\frac{-m}{m+1}}dt,$$

quae sumto $b = \frac{a}{m+1}$ ad similitudinem propositae propius accedit, ut sit

$$dz + zzdt = \frac{a}{(m+1)^2}t^{\frac{-m}{m+1}}dt.$$

Si ergo haec esset separabilis, ipsa proposita ista substitutione separabilis fieret et vicissim; unde concludimus, si aequatio proposita separationem admittat casu $m = n$, eam quoque esse admissuram casu $m = \frac{-n}{n+1}$. Hinc autem ex casu $m = 0$ alius non reperitur.

Ponamus $y = \frac{1}{x} - \frac{z}{xx}$, ut sit

$$dy = \frac{dx}{xx} - \frac{dz}{xx} + \frac{2zdx}{x^3} \quad \text{et} \quad yydx = \frac{dx}{xx} - \frac{2zdx}{x^3} + \frac{zzdx}{x^4},$$

unde prodit

$$-\frac{dz}{xx} + \frac{zzdx}{x^4} = ax^m dx \quad \text{seu} \quad dz - \frac{zzdx}{xx} = -ax^{m+2}dx;$$

sit nunc $x = \frac{1}{t}$ et fit

$$dz + zzdt = at^{-m-4}dt;$$

quae cum propositae sit similis, discimus, si separatio succedat casu $m = n$, etiam succedere casu $m = -n - 4$.

Ex uno ergo casu $m = n$ consequimur duos, scilicet

$$m = -\frac{n}{n+1} \quad \text{et} \quad m = -n - 4.$$

Cum igitur constet casus $m=0$, hinc formulae alternatim adhibitae praebent sequentes

$$m=-4,\quad m=-\tfrac{4}{3},\quad m=-\tfrac{8}{3},\quad m=-\tfrac{8}{5},\quad m=-\tfrac{12}{5},$$
$$m=-\tfrac{12}{7},\quad m=-\tfrac{16}{7}\quad \text{etc.},$$

qui casus omnes in hac formula $m=\frac{-4i}{2i\pm1}$ continentur.

COROLLARIUM 1

437. Quodsi ergo fuerit vel

$$m=\tfrac{-4i}{2i+1}\quad \text{vel}\quad m=\tfrac{-4i}{2i-1},$$

aequatio $dy+yydx=ax^{m}dx$ per aliquot substitutiones repetitas tandem ad formam $du+uudv=cdv$, cuius separatio et integratio constat, reduci potest.

COROLLARIUM 2

438. Scilicet si fuerit $m=\frac{-4i}{2i+1}$, aequatio $dy+yydx=ax^{m}dx$ per substitutiones

$$x=t^{\frac{1}{m+1}} \text{ and } y=\tfrac{a}{(m+1)z}$$

reducitur ad hanc

$$dz+zzdt=\tfrac{a}{(m+1)^2}t^{n}dt,$$

ut sit $n=\frac{-4i}{2i-1}$, qui casus uno gradu inferior est censendus.

COROLLARIUM 3

439. Sin autem fuerit $m=\frac{-4i}{2i-1}$, aequatio $dy+yydx=ax^{m}dx$ per has substitutiones

$$x=\tfrac{1}{t}\ \text{ et }\ y=\tfrac{1}{x}-\tfrac{z}{xx}\ \text{ seu }\ y=t-ttz$$

reducitur ad hanc $dz+zzdt=at^{n}dt$, in qua est

$$n=\tfrac{-4(i-1)}{2i-1}=\tfrac{-4(i-1)}{2(i-1)+1},$$

qui casus denuo uno gradu inferior est.

COROLLARIUM 4

440. Omnes ergo casus separabiles hoc modo inventi pro exponente m dant numeros negativos intra limites 0 et – 4 contentos, ac si i sit numerus infinitus, prodit casus $m=-2$, qui autem per se constat, cum aequatio

$$dy+yydx=\tfrac{adx}{xx}$$

posito $y=\tfrac{1}{x}$ fiat homogenea [§ 410].

SCHOLION 1

441. Aequatio haec $dy+yydx=ax^{m}dx$ vocari solet RICCATIANA ab Auctore Comite RICCATI, qui primus casus separabiles proposuit. Hic quidem eam in forma simplicissima exhibui, cum eo haec $dy+Ayyt^{\mu}dt=Bt^{\lambda}dt$ ponendo $At^{\mu}dt=dx$ et $At^{\mu+1}=(\mu+1)x$ statim reducatur.

Caeterum etsi binae substitutiones, quibus hic sum usus, sunt simplicissimae, tamen magis compositis adhibendis nulli alii casus separabiles deteguntur; ex quo hoc omnino memorabile est visum hanc aequationem rarissime separationem admittere, tametsi numerus casuum, quibus hoc praestari queat, revera sit infinitus.

Caeterum haec investigatio ab exponente ad simplicem coefficientem traduci potest; posito enim $y=x^{\frac{m}{2}}z$ prodit

$$dz+\tfrac{mzdz}{2x}+x^{\frac{m}{2}}zzdx=ax^{\frac{m}{2}}dx,$$

ubi si fiat

$$x^{\frac{m}{2}}dx=dt \quad \text{et} \quad x^{\frac{m+2}{2}}=\tfrac{m+2}{2}t,$$

erit $\frac{dx}{x}=\frac{2dt}{(m+2)t}$ hincque

$$dz+\tfrac{mzdz}{(m+2)t}+zzdt=adt,$$

quae ergo aequatio, quoties fuerit $\frac{m}{m+2}=\pm\, 2i$ seu numerus par tam positivus quam negativus, separabilis reddi potest, ita ut haec aequatio

$$dz\pm\tfrac{2izdt}{t}+zzdt=adt$$

semper sit integrabilis. Si praeterea ponatur $z=u-\frac{m}{2(m+2)t}$, oritur

$$du+uudt=adt-\tfrac{m(m+4)dt}{4(m+2)^{2}tt}$$

et pro casibus separabilitatis $m=\frac{-4i}{2i\pm1}$ habetur

$$du + uudt = adt + \frac{i(i \pm 1)dt}{tt}$$

Uberiorem autem huius aequationis evolutionem, quandoquidem est maximi momenti, in sequentibus docebo, ubi de integratione aequationum differentialium per series infinitas sum acturus; hinc enim facilius casus separabiles eruemus simulque integralia assignare poterimus.

SCHOLION 2

442. Ampliora praecepta circa separationem variabilium, quae quidem usum sint habitura, vix tradi posse videntur, unde intelligitur in paucissimis aequationibus differentialibus hanc methodum adhiberi posse. Progrediar igitur ad aliud principium explicandum, unde integrationes haurire liceat, quod multo latius patet, dum etiam ad aequationes differentiales altiorum graduum accommodari potest, ita ut in eo verus ac naturalis fons omnium integrationum contineri videatur.

Istud autem principium in hoc consistit, quod proposita quacunque aequatione differentiali inter duas variabiles semper detur functio quaedam, per quam aequatio multiplicata fiat integrabilis; aequationis scilicet omnia membra ad eandem partem disponi oportet, ut talem formam obtineat $Pdx + Qdy = 0$; ac tum dico semper dari functionem quandam variabilium *x* et *y*, puta *V*, ut facta multiplicatione formula $VPdx + VQdy$ integrabilis existat seu ut verum sit differentiale ex differentiatione cuiuspiam functionis binarum variabilium *x* et *y* natum. Quodsi enim haec functio ponatur = *S*, ut sit $dS = VPdx + VQdy$, quia est $Pdx + Qdy = 0$, erit etiam $dS = 0$ ideoque $S = \text{Const.}$, quae ergo aequatio erit integrale idque completum aequationis differentialis $Pdx + Qdy = 0$. Totum ergo negotium ad inventionem illius multiplicatoris *V* redit.