CHAPER IX

CONCERNING THE DEVELOPMENT OF INTEGRALS THROUGH INFINITE PRODUCTS

PROBLEM 43

356. *The value of this integral* $\int\frac{dx}{\sqrt{(1-xx)}}$*, which it accepts in the case* $x=1$*, is to be developed into an infinite product.*

SOLUTION

Just as we have reduced the above formulas of higher powers to the simple, thus here we continually induce the formula $\int\frac{dx}{\sqrt{(1-xx)}}$ to higher powers. Thus, since on putting $x=1$ there becomes

$$\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}}=\frac{m+1}{m}\int\frac{x^{m+1}dx}{\sqrt{(1-xx)}},$$

[Recall from § 118 and § 120, Ch. II, that $\int\frac{x^{m+1}dx}{\sqrt{(1-xx)}}=\frac{-x^m\sqrt{(1-xx)}}{m+1}+\frac{m}{m+1}\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}}$]

there becomes

$$\int\frac{dx}{\sqrt{(1-xx)}}=\frac{2}{1}\int\frac{xxdx}{\sqrt{(1-xx)}}=\frac{2\cdot4}{1\cdot3}\int\frac{x^4dx}{\sqrt{(1-xx)}}=\frac{2\cdot4\cdot6}{1\cdot3\cdot5}\int\frac{x^6dx}{\sqrt{(1-xx)}}\quad\text{etc.},$$

from which thus also we conclude that the product becomes indefinitely, if an infinite number is taken for i

$$\int\frac{dx}{\sqrt{(1-xx)}}=\frac{2\cdot4\cdot6\cdot8\cdots\cdots2i}{1\cdot3\cdot5\cdot7\cdots\cdots(2i-1)}\int\frac{x^{2i}dx}{\sqrt{(1-xx)}}.$$

Now in a similar manner from the formula $\int\frac{xdx}{\sqrt{(1-xx)}}$ we may find on ascending [to higher powers]

$$\int\frac{xdx}{\sqrt{(1-xx)}}=\frac{3\cdot5\cdot7\cdot9\cdots(2i+1)}{2\cdot4\cdot6\cdot8\cdots\cdots2i}\int\frac{x^{2i+1}dx}{\sqrt{(1-xx)}},$$

and I note, that if i is an infinite number, those formulas

$$\int\frac{x^{2i}dx}{\sqrt{(1-xx)}}\quad\text{and}\quad\int\frac{x^{2i+1}dx}{\sqrt{(1-xx)}}$$

form an equal ratio. For it is evident from the reduction principle, that if *m* is an infinite number, there arises

$$\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}} = \int \frac{x^{m+1}dx}{\sqrt{(1-xx)}} = \int \frac{x^{m+3}dx}{\sqrt{(1-xx)}}$$

and thus generally,

$$\int \frac{x^{m+\mu}dx}{\sqrt{(1-xx)}} = \int \frac{x^{m+\nu}dx}{\sqrt{(1-xx)}}$$

whatever the finite difference in size should be between μ and ν. Since therefore there becomes

$$\int \frac{x^{2i}dx}{\sqrt{(1-xx)}} = \int \frac{x^{2i+1}dx}{\sqrt{(1-xx)}}$$

if we put

$$\frac{2\cdot 4\cdot 6\cdots 2i}{1\cdot 3\cdot 5\cdots(2i-1)} = M \quad \text{and} \quad \frac{3\cdot 5\cdot 7\cdot 9\cdots(2i+1)}{2\cdot 4\cdot 6\cdot 8\cdots 2i} = N,$$

then

$$\int \frac{dx}{\sqrt{(1-xx)}} : \int \frac{xdx}{\sqrt{(1-xx)}} = M:N = \tfrac{M}{N}:1$$

on putting $x=1$. But

$$\int \frac{xdx}{\sqrt{(1-xx)}} = 1 \quad \text{and} \quad \int \frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2},$$

from which it is deduced that

$$\int \frac{dx}{\sqrt{(1-xx)}} = \frac{M}{N}.$$

Because the products *M* and *N* are constructed from an equal number of factors, if the first factor $\frac{2}{1}$ of the product *M* by the first factor $\frac{3}{2}$ of the product *N*, the second of that $\frac{4}{3}$ by the second of this $\frac{5}{4}$ and thus again we may divide, there is made

$$\frac{M}{N} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdot \text{ etc.},$$

from which we obtain for the case $x=1$ by an infinite product

$$\int \frac{dx}{\sqrt{(1-xx)}} = \frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdot \text{ etc.} = \frac{\pi}{2}.$$

COROLLARY 1

357. Hence for the value of π we have elicited the same infinite product, which now some time ago Wallis found and the truth of which we have confirmed in the *Introductione* [Book I, Ch. XI, § 185] preceding in several ways ; and thus there shall be

$$\pi = 2\cdot\frac{2\cdot 2}{1\cdot 3}\cdot\frac{4\cdot 4}{3\cdot 5}\cdot\frac{6\cdot 6}{5\cdot 7}\cdot\frac{8\cdot 8}{7\cdot 9}\cdot \text{ etc.}$$

COROLLARY 2

358. There is no difference, in whatever order the individual factors in this product are set out, provided none are left out. Thus by taking some number from the beginning, the remainder can be set out in order, just as

$$\frac{\pi}{2} = \frac{2}{1}\cdot\frac{2\cdot 4}{3\cdot 3}\cdot\frac{4\cdot 6}{5\cdot 5}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{8\cdot 10}{9\cdot 9}\cdot\text{etc.}$$

or

$$\frac{\pi}{2} = \frac{2\cdot 4}{1\cdot 3}\cdot\frac{2\cdot 6}{3\cdot 5}\cdot\frac{4\cdot 8}{5\cdot 7}\cdot\frac{6\cdot 10}{7\cdot 9}\cdot\frac{8\cdot 12}{9\cdot 11}\cdot\text{etc.}$$

or

$$\frac{\pi}{2} = \frac{2}{3}\cdot\frac{2\cdot 4}{1\cdot 5}\cdot\frac{4\cdot 6}{3\cdot 7}\cdot\frac{6\cdot 8}{5\cdot 9}\cdot\frac{8\cdot 10}{7\cdot 11}\cdot\text{etc.}$$

or

$$\frac{\pi}{2} = \frac{2\cdot 4}{3\cdot 5}\cdot\frac{2\cdot 6}{1\cdot 7}\cdot\frac{4\cdot 8}{3\cdot 9}\cdot\frac{6\cdot 10}{5\cdot 11}\cdot\frac{8\cdot 12}{7\cdot 13}\cdot\text{etc.}$$

SCHOLIUM

359. Hence the fundamental of this development consists in this, that the value of the integral $\int\frac{x^{i+\alpha}dx}{\sqrt{(1-xx)}}$ shall be the same with *i* denoting an infinite number, in whatever manner the finite number α may be varied. And this indeed has been shown from the reduction

$$\int\frac{x^{i-1}dx}{\sqrt{(1-xx)}} = \frac{i+1}{i}\int\frac{x^{i+1}dx}{\sqrt{(1-xx)}},$$

if two different values α are taken. But then there is no doubt, why this integral $\int\frac{x^{i+1}dx}{\sqrt{(1-xx)}}$ should not be contained between these integrals $\int\frac{x^{i}dx}{\sqrt{(1-xx)}}$ and $\int\frac{x^{i+2}dx}{\sqrt{(1-xx)}}$ as if limits, which since the are equal to each other, by necessity all the intermediate formulas also from the same must be equal. And this can be extended wider to more complicated formulas, thus so that by denoting the infinite number *i* then there shall be

$$\int\frac{x^{i+\alpha}dx}{\sqrt{\left(1-x^n\right)^k}} = \int\frac{x^{i}dx}{\sqrt{\left(1-x^n\right)^k}}.$$

But since there is [the equality of these integrals as established in Ch. 8]

$$\int \frac{x^{m+n-1}dx}{(1-x^n)^{\frac{n-k}{n}}} = \frac{m}{m+k}\int \frac{x^{m-1}dx}{\sqrt{(1-x^n)^{\frac{n-k}{n}}}},$$

these formulas are equal on putting $m = \infty$; from which also the equality in other cases, in which $\alpha = n$ or $\alpha = 2n$ or $\alpha = 3n$ etc.,is considered; if moreover α holds some mean value, also the mean value of this formula must hold a certain value between equal values and thus will be equal to these themselves. Therefore we are able to resolve the following problems from this established principle.

PROBLEM 44

360. *To express the ratio of these two integrals*

$$\int x^{m-1}dx\left(1-x^n\right)^{\frac{k-n}{n}} \quad \textit{and} \quad \int x^{\mu-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}$$

by an infinite product of factors in the case x = 1.

SOLUTION

Since there shall be

$$\int x^{m-1}dx\left(1-x^n\right)^{\frac{k-n}{n}} = \frac{m+k}{n}\int x^{m+n-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}$$

in the case $x = 1$, the value of the integrals of this can be reduced to an integral infinitely removed in this manner

$$\int x^{m-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}$$
$$= \frac{(m+k)(m+k+n)(m+k+2n)\cdots(m+k+in)}{m\quad(m+n)\quad(m+2n)\cdots(m+in)}\int x^{m+in+n-1}dx\left(1-x^n\right)^{\frac{k-n}{n}},$$

where we assume i to denote an infinite number. Moreover in a similar manner for the other formula proposed there shall be

$$\int x^{\mu-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}$$
$$= \frac{(\mu+k)(\mu+k+n)(\mu+k+2n)\cdots(\mu+k+in)}{\mu\quad(\mu+n)\quad(\mu+2n)\cdots(\mu+in)}\int x^{\mu+in+n-1}dx\left(1-x^n\right)^{\frac{k-n}{n}},$$

and these integrals of these final formulas are equal on account of the infinite exponents not withstanding the inequality of the numbers m and μ; then these two equal infinite products are constructed from a number of factors. Whereby if the first are divided by the first, the second by the second and so forth, the ratio of the two proposed integrals can be expressed thus :

$$\frac{\int x^{m-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}}{\int x^{\mu-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}}=\frac{\mu(m+k)}{m(\mu+k)}\cdot\frac{(\mu+n)(m+k+n)}{(m+n)(\mu+k+n)}\cdot\frac{(\mu+2n)(m+k+2n)}{(m+2n)(\mu+k+2n)}\cdot\text{etc.},$$

if indeed both integrals are thus determined, so that they vanish on putting $x = 0$, then there is now put in place $x = 1$; moreover it is necessary for the letters $m,\ \mu,\ n,\ k$ to denote positive numbers.

COROLLARY 1

361. If the difference of the numbers m and μ is equal to a multiple of n, then in the product found, the infinite factors cancel each other and there remains a finite number of factors, as if $\mu = m+n$, there is had

$$\frac{(m+n)(m+k)}{m(m+k+n)}\cdot\frac{(m+2n)(m+k+n)}{(m+n)(m+k+2n)}\cdot\frac{(m+3n)(m+k+2n)}{(m+2n)(m+k+3n)}\cdot\ \text{etc.},$$

which is reduced to $\frac{m+k}{m}$.

COROLLARY 2

362. Moreover the value of this product by necessity is finite, since it is expressed from the ratio of the integral formulas, from which it is apparent, as the numerators and denominators in the individual factors are alternately greater or smaller.

COROLLARY 3

363. If we put $m=1, \mu=3, n=4$ and $k=2$, then

$$\frac{\int\frac{dx}{\sqrt{\left(1-x^4\right)}}}{\int\frac{xxdx}{\sqrt{\left(1-x^4\right)}}}=\frac{3\cdot3}{1\cdot5}\cdot\frac{7\cdot7}{5\cdot9}\cdot\frac{11\cdot11}{9\cdot13}\cdot\frac{15\cdot15}{13\cdot17}\cdot\text{etc.};$$

moreover we have found above that the product of these two formulas to be equal to $\frac{\pi}{4}$.

PROBLEM 45

364. *To express the value of this integral* $\int x^{m-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}$ *taken on putting* $x = 1$, *by an infinite product.*

SOLUTION

Since in the preceding problem the ratio of this integral to that other infinite product shall be assigned,

$$\int x^{\mu-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}$$

in this the exponent μ thus is taken, so that it is possible for the integral to be shown. Hence $\mu=n$ is taken and the integral is made equal to

$$C-\tfrac{1}{k}\left(1-x^n\right)^{\frac{k}{n}}=\frac{1-\left(1-x^n\right)^{\frac{k}{n}}}{k}$$

thus determined so that it vanishes on putting $x = 0$; now there is put $x = 1$, as the postulated condition, and because this integral becomes equal to $\frac{1}{k}$, and thus we have in the case $x = 1$ the expression for the proposed form of the formula

$$\int x^{m-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}=\frac{1}{k}\cdot\frac{n(m+k)}{m(k+n)}\cdot\frac{2n(m+k+n)}{(m+n)(k+2n)}\cdot\frac{3n(m+k+2n)}{(m+2n)(k+3n)}\cdot\text{etc.},$$

since the individual factors can be distributed to be represented thus :

$$\int x^{m-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}=\frac{n}{mk}\cdot\frac{2n(m+k)}{(m+n)(k+n)}\cdot\frac{3n(m+k+n)}{(m+2n)(k+2n)}\cdot\frac{4n(m+k+2n)}{(m+3n)(k+3n)}\cdot\text{etc.},$$

COROLLARY 1

365. Since in this expression the letters m and k are permutable, it follows also that these integrals are equal to each other on putting $x = 1$:

$$\int x^{m-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}=\int x^{k-1}dx\left(1-x^n\right)^{\frac{m-n}{n}},$$

as we have now elicited the equality above in § 349 .

COROLLARY 2

366. Since the value of our formula, if $m=n-k$, is equal to the value of this integral $\int\frac{z^{k-1}dz}{1+z^n}$ on putting $z=\infty$, if on account of $m+k=n$ we put $m=\frac{n+\alpha}{2}$ and $k=\frac{n-\alpha}{2}$, then we have :

$$\int\frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{n+\alpha}{2n}}}=\int\frac{x^{k-1}dx}{\left(1-x^n\right)^{\frac{n-\alpha}{2n}}}=\int\frac{z^{k-1}dz}{1+z^n}=\int\frac{z^{m-1}dz}{1+z^n}$$
$$=\frac{4n}{nn-\alpha\alpha}\cdot\frac{2\cdot 4nn}{9nn-\alpha\alpha}\cdot\frac{4\cdot 6nn}{25nn-\alpha\alpha}\cdot\frac{6\cdot 8nn}{49nn-\alpha\alpha}\cdot\text{etc.}$$

Which product can also be expressed in this manner

$$\frac{2}{n-\alpha}\cdot\frac{2n\cdot 2n}{(n+\alpha)(3n-\alpha)}\cdot\frac{4n\cdot 4n}{(3n+\alpha)(5n-\alpha)}\cdot\frac{6n\cdot 6n}{(5n+\alpha)(7n-\alpha)}\cdot\text{etc.,}$$

because hence also by § 351 it expresses the value of $\frac{\pi}{n\,sin.\frac{m\pi}{n}}=\frac{\pi}{n\,cos.\frac{\alpha\pi}{2n}}$.

COROLLARY 3

367. Or if we put simply $k=n-m$, there is made

$$\int\frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{m}{n}}}=\int\frac{x^{n-m-1}dx}{\left(1-x^n\right)^{\frac{n-m}{n}}}=\int\frac{z^{m-1}dz}{1+z^n}=\int\frac{z^{n-m-1}dz}{1+z^n}$$
$$=\frac{1}{n-m}\cdot\frac{nn}{m(2n-m)}\cdot\frac{4nn}{(n+m)(3n-m)}\cdot\frac{9nn}{(2n+m)(4n-m)}\cdot\text{etc.,}$$

which arises from the first form found. Hence therefore this form stands, if there is put $x=1$ and $z=\infty$.

SCHOLIUM 1

368. Moreover in the *Introductione* from the multiplication of angles we arrive at

$$sin.\frac{m\pi}{n}=\frac{m\pi}{n}\left(1-\frac{mm}{nn}\right)\left(1-\frac{mm}{4nn}\right)\left(1-\frac{mm}{9nn}\right)\left(1-\frac{mm}{16nn}\right)\cdot\text{etc.,}$$

and since

$$sin.\frac{(n-m)\pi}{n}=sin.\frac{m\pi}{n},$$

on account of $n-m=k$ there is also

$$sin.\frac{m\pi}{n}=\frac{k\pi}{n}\left(1-\frac{kk}{nn}\right)\left(1-\frac{kk}{4nn}\right)\left(1-\frac{kk}{9nn}\right)\left(1-\frac{kk}{16nn}\right)\cdot\text{etc.,}$$

which is reduced to this form

$$sin.\frac{m\pi}{n}=\frac{k\pi}{n}\cdot\frac{(n-k)(n+k)}{nn}\cdot\frac{(2n-k)(2n+k)}{4nn}\cdot\frac{(3n-k)(3n+k)}{9nn}.\ \text{etc.}$$

and for k with its own value restored

$$sin.\frac{m\pi}{n}=\frac{\pi}{n}(n-m)\cdot\frac{m(2n-m)}{nn}\cdot\frac{(n+m)(3n-m)}{4nn}\cdot\frac{(2n+m)(4n-m)}{9nn}\cdot\text{etc.},$$

from which clearly for $\frac{\pi}{n\,sin.\frac{m\pi}{n}}$ the same product is found, that expresses the value of our integration, and thus we have a new demonstration for that excellent theorem above [§ 351], by prevailing on many devious routes to be

$$\int\frac{x^{m-1}dx}{(1-x^n)^{\frac{m}{n}}}=\int\frac{x^{n-m-1}dx}{(1-x^n)^{\frac{n-m}{n}}}=\int\frac{z^{m-1}dz}{1+z^n}=\int\frac{z^{n-m-1}dz}{1+z^n}=\frac{\pi}{n\,sin.\frac{m\pi}{n}}.$$

SCHOLIUM 2

369. In order that our formula extends further, we put $\frac{k}{n}=\frac{\mu}{\nu}$ or $k=\frac{\mu n}{\nu}$ and we obtain

$$\int x^{m-1}dx(1-x^n)^{\frac{\mu}{\nu}-1}=\frac{\nu}{m\mu}\cdot\frac{2(m\nu+n\mu)}{(m+n)(\mu+\nu)}\cdot\frac{3(m\nu+n(\mu+\nu))}{(m+2n)(\mu+2\nu)}\cdot\frac{4(m\nu+n(\mu+2\nu))}{(m+3n)(\mu+3\nu)}\cdot\text{etc.}$$
$$=\frac{\nu}{m\mu}\cdot\frac{2(m\nu+n\mu)}{(m+n)(\mu+\nu)}\cdot\frac{3(m\nu+n\mu+n\nu)}{(m+2n)(\mu+2\nu)}\cdot\frac{4(m\nu+n\mu+2n\nu)}{(m+3n)(\mu+3\nu)}\cdot\frac{5(m\nu+n\mu+3n\nu)}{(m+4n)(\mu+4\nu)}\cdot\text{etc.}$$

in which expression the letters m, n and μ, ν are permutable except for the first factor, which is not connected with the rest by the law of continuation ; and if we multiply by *n*, the interchange will be perfect, from which we conclude

$$n\int x^{m-1}dx(1-x^n)^{\frac{\mu}{\nu}-1}=\nu\int x^{\mu-1}dx(1-x^\nu)^{\frac{m}{n}-1}$$

which equality is reduced to that observed above in the case $\nu=n$. It will be helpful to consider carefully particular cases, which we have chosen from the values of μ and ν.

EXAMPLE 1

370. Let $\mu=1$ and $\nu=2$ and there becomes

$$\int\frac{x^{m-1}dx}{\sqrt{(1-x^n)}}=\frac{2}{m}\cdot\frac{2(2m+n)}{3(m+n)}\cdot\frac{3(2m+3n)}{5(m+2n)}\cdot\frac{4(2m+5n)}{7(m+3n)}\cdot\text{etc.}=\frac{2}{n}\int\frac{dx}{\sqrt[n]{(1-x^2)^{n-m}}},$$

which expression can be more conveniently represented by

$$\int\frac{x^{m-1}dx}{\sqrt{(1-x^n)}}=\frac{2}{m}\cdot\frac{4(2m+n)}{3(2m+2n)}\cdot\frac{6(2m+3n)}{5(2m+4n)}\cdot\frac{8(2m+5n)}{7(2m+6n)}\cdot\text{etc.},$$

from which the most specific cases are deduced :

$$\int\frac{dx}{\sqrt{(1-xx)}}=2\cdot\frac{2\cdot4}{3\cdot3}\cdot\frac{4\cdot6}{5\cdot5}\cdot\frac{6\cdot8}{7\cdot7}\cdot\text{etc.}\qquad=\int\frac{dx}{\sqrt{(1-xx)}},$$

$$\int\frac{dx}{\sqrt{(1-x^3)}}=2\cdot\frac{4\cdot5}{3\cdot8}\cdot\frac{6\cdot11}{5\cdot14}\cdot\frac{8\cdot17}{7\cdot20}\cdot\frac{10\cdot23}{9\cdot26}\cdot\text{etc.}\qquad=\frac{2}{3}\int\frac{dx}{\sqrt[3]{(1-x^2)^2}},$$

$$\int\frac{xdx}{\sqrt{(1-x^3)}}=1\cdot\frac{4\cdot7}{3\cdot10}\cdot\frac{6\cdot13}{5\cdot16}\cdot\frac{8\cdot19}{7\cdot22}\cdot\frac{10\cdot25}{9\cdot28}\cdot\text{etc.}\qquad=\frac{2}{3}\int\frac{dx}{\sqrt[3]{(1-x^2)}},$$

$$\int\frac{dx}{\sqrt{(1-x^4)}}=2\cdot\frac{4\cdot3}{3\cdot5}\cdot\frac{6\cdot7}{5\cdot9}\cdot\frac{8\cdot11}{7\cdot13}\cdot\frac{10\cdot15}{9\cdot17}\cdot\text{etc.}\qquad=\frac{1}{2}\int\frac{dx}{\sqrt[4]{(1-x^2)^3}},$$

$$\int\frac{xdx}{\sqrt{(1-x^4)}}=1\cdot\frac{4\cdot4}{3\cdot6}\cdot\frac{6\cdot8}{5\cdot10}\cdot\frac{8\cdot12}{7\cdot14}\cdot\frac{10\cdot16}{9\cdot18}\cdot\text{etc.}\qquad=\frac{1}{2}\int\frac{dx}{\sqrt{(1-x^2)}},$$

or

$$=1\cdot\frac{2\cdot4}{3\cdot3}\cdot\frac{4\cdot6}{5\cdot5}\cdot\frac{6\cdot8}{7\cdot7}\cdot\frac{8\cdot10}{9\cdot9}\cdot\text{etc.},$$

$$\int\frac{xxdx}{\sqrt{(1-x^4)}}=\frac{2}{3}\cdot\frac{4\cdot5}{3\cdot7}\cdot\frac{6\cdot9}{5\cdot11}\cdot\frac{8\cdot13}{7\cdot15}\cdot\frac{10\cdot17}{9\cdot19}\cdot\text{etc.}\qquad=\frac{1}{2}\int\frac{dx}{\sqrt[4]{(1-xx)}},$$

$$\int\frac{x^3dx}{\sqrt{(1-x^4)}}=\frac{2}{4}\cdot\frac{4\cdot6}{3\cdot8}\cdot\frac{6\cdot10}{5\cdot12}\cdot\frac{8\cdot14}{7\cdot16}\cdot\frac{10\cdot18}{9\cdot20}\cdot\text{etc.}\qquad=\frac{1}{2}.$$

EXAMPLE 2

371. Let $\mu=1$ and $\nu=3$ and there becomes

$$\int\frac{x^{m-1}dx}{\sqrt[3]{(1-x^n)^2}}=\frac{3}{m}\cdot\frac{2(3m+n)}{4(m+n)}\cdot\frac{3(3m+4n)}{7(m+2n)}\cdot\frac{4(3m+7n)}{10(m+3n)}\cdot\text{etc.}=\frac{3}{n}\int\frac{dx}{\sqrt[n]{(1-x^3)^{n-m}}},$$

from which the following most specific cases are deduced

$$\int \frac{dx}{\sqrt[3]{(1-x^2)^2}} = \frac{3}{1}\cdot\frac{2\cdot5}{4\cdot3}\cdot\frac{3\cdot11}{7\cdot5}\cdot\frac{4\cdot17}{10\cdot7}\cdot\frac{5\cdot23}{13\cdot7}\cdot\text{etc.} \qquad = \frac{3}{2}\int \frac{dx}{\sqrt{(1-x^3)}},$$

$$\int \frac{dx}{\sqrt[3]{(1-x^3)^2}} = \frac{3}{1}\cdot\frac{2\cdot6}{4\cdot4}\cdot\frac{3\cdot15}{7\cdot7}\cdot\frac{4\cdot24}{10\cdot10}\cdot\frac{5\cdot33}{13\cdot13}\cdot\text{etc.} \qquad = \int \frac{dx}{\sqrt[3]{(1-x^3)^2}},$$

or
$$= \frac{3}{1}\cdot\frac{2\cdot6}{4\cdot4}\cdot\frac{5\cdot9}{7\cdot7}\cdot\frac{8\cdot12}{10\cdot10}\cdot\frac{11\cdot15}{13\cdot13}\cdot\text{etc.}$$

$$\int \frac{xdx}{\sqrt[3]{(1-x^3)^2}} = \frac{3}{2}\cdot\frac{2\cdot9}{4\cdot5}\cdot\frac{3\cdot18}{7\cdot8}\cdot\frac{4\cdot27}{10\cdot11}\cdot\frac{5\cdot36}{13\cdot14}\cdot\text{etc.} \qquad = \int \frac{dx}{\sqrt[3]{(1-x^3)}},$$

or
$$= \frac{3}{2}\cdot\frac{3\cdot6}{4\cdot5}\cdot\frac{6\cdot9}{7\cdot8}\cdot\frac{9\cdot12}{10\cdot11}\cdot\frac{12\cdot15}{13\cdot14}\cdot\text{etc.}$$

$$\int \frac{dx}{\sqrt[3]{(1-x^4)^2}} = \frac{3}{1}\cdot\frac{2\cdot7}{4\cdot5}\cdot\frac{3\cdot19}{7\cdot9}\cdot\frac{4\cdot31}{10\cdot13}\cdot\frac{5\cdot43}{13\cdot17}\cdot\text{etc.} \qquad = \frac{3}{4}\int \frac{dx}{\sqrt[4]{(1-x^3)^3}},$$

$$\int \frac{xxdx}{\sqrt[3]{(1-x^4)^2}} = 1\cdot\frac{2\cdot13}{4\cdot7}\cdot\frac{3\cdot25}{7\cdot11}\cdot\frac{4\cdot37}{10\cdot15}\cdot\frac{5\cdot49}{13\cdot19}\cdot\text{etc.} \qquad = \frac{3}{4}\int \frac{dx}{\sqrt[4]{(1-x^3)}}.$$

EXAMPLE 3

372. Let $\mu = 2$ and $\nu = 3$ and there becomes

$$\int \frac{x^{m-1}dx}{\sqrt[3]{(1-x^n)}} = \frac{3}{2m}\cdot\frac{2(3m+2n)}{5(m+n)}\cdot\frac{3(3m+5n)}{8(m+2n)}\cdot\frac{4(3m+8n)}{11(m+3n)}\cdot\text{etc.} = \frac{3}{n}\int \frac{xdx}{\sqrt[n]{(1-x^3)^{n-m}}},$$

from which the following specific cases are deduced

$$\int \frac{dx}{\sqrt[3]{(1-x^2)}} = \frac{3}{2}\cdot\frac{2\cdot7}{5\cdot3}\cdot\frac{3\cdot13}{8\cdot5}\cdot\frac{4\cdot19}{11\cdot7}\cdot\frac{5\cdot25}{14\cdot9}\cdot\text{etc.} \qquad = \frac{3}{2}\int \frac{xdx}{\sqrt{(1-x^3)}},$$

$$\int \frac{dx}{\sqrt[3]{(1-x^3)}} = \frac{3}{2}\cdot\frac{2\cdot9}{5\cdot4}\cdot\frac{3\cdot18}{8\cdot7}\cdot\frac{4\cdot27}{11\cdot10}\cdot\frac{5\cdot36}{14\cdot13}\cdot\text{etc.} \qquad = \int \frac{xdx}{\sqrt[3]{(1-x^3)^2}}$$

or
$$= \frac{3\cdot3}{2\cdot4}\cdot\frac{6\cdot6}{5\cdot7}\cdot\frac{9\cdot9}{8\cdot10}\cdot\frac{12\cdot12}{11\cdot13}\cdot\text{etc.},$$

$$\int \frac{xdx}{\sqrt[3]{(1-x^3)}} = \frac{3}{4}\cdot\frac{2\cdot12}{5\cdot5}\cdot\frac{3\cdot21}{8\cdot8}\cdot\frac{4\cdot30}{11\cdot11}\cdot\frac{5\cdot39}{14\cdot14}\cdot\text{etc.} \qquad = \int \frac{xdx}{\sqrt[3]{(1-x^3)}}$$

or
$$= \frac{3}{4}\cdot\frac{4\cdot6}{5\cdot5}\cdot\frac{7\cdot9}{8\cdot8}\cdot\frac{10\cdot12}{11\cdot11}\cdot\frac{13\cdot15}{14\cdot14}\cdot\text{etc.},$$

$$\int \frac{dx}{\sqrt[3]{(1-x^4)}} = \frac{3}{2}\cdot\frac{2\cdot11}{5\cdot5}\cdot\frac{3\cdot23}{8\cdot9}\cdot\frac{4\cdot35}{11\cdot13}\cdot\frac{5\cdot47}{14\cdot17}\cdot\text{etc.} \qquad = \frac{3}{4}\int \frac{xdx}{\sqrt[4]{(1-x^3)^3}},$$

$$\int \frac{x^2dx}{\sqrt[3]{(1-x^4)}} = \frac{1}{2}\cdot\frac{2\cdot17}{5\cdot7}\cdot\frac{3\cdot29}{8\cdot11}\cdot\frac{4\cdot41}{11\cdot15}\cdot\frac{5\cdot53}{14\cdot19}\cdot\text{etc.} \qquad = \frac{3}{4}\int \frac{xdx}{\sqrt[4]{(1-x^3)}}.$$

EXAMPLE 4

373. Let $\mu = 1$ and $\nu = 4$ and there is made

$$\int \frac{x^{m-1}dx}{\sqrt[4]{\left(1-x^{n}\right)^{3}}} = \frac{4}{m}\cdot\frac{2(4m+n)}{5(m+n)}\cdot\frac{3(4m+5n)}{9(m+2n)}\cdot\frac{4(4m+9n)}{13(m+3n)}\cdot\text{etc.} = \frac{4}{n}\int\frac{dx}{\sqrt[n]{\left(1-x^{4}\right)^{n-m}}},$$

from which the following specific cases are produced :

$$\int\frac{dx}{\sqrt[4]{\left(1-x^{2}\right)^{3}}} = \frac{4}{1}\cdot\frac{2\cdot 6}{5\cdot 3}\cdot\frac{3\cdot 14}{9\cdot 5}\cdot\frac{4\cdot 22}{13\cdot 7}\cdot\frac{5\cdot 30}{17\cdot 9}\cdot\text{etc.} \qquad = 2\int\frac{dx}{\sqrt{\left(1-x^{4}\right)}}$$

or

$$= \frac{4}{1}\cdot\frac{4\cdot 3}{3\cdot 5}\cdot\frac{6\cdot 7}{5\cdot 9}\cdot\frac{8\cdot 11}{7\cdot 13}\cdot\frac{10\cdot 15}{9\cdot 17}\cdot\text{etc.},$$

$$\int\frac{dx}{\sqrt[4]{\left(1-x^{3}\right)^{3}}} = \frac{4}{1}\cdot\frac{2\cdot 7}{5\cdot 4}\cdot\frac{3\cdot 19}{9\cdot 7}\cdot\frac{4\cdot 31}{13\cdot 10}\cdot\frac{5\cdot 43}{17\cdot 13}\cdot\text{etc.} \qquad = \frac{4}{3}\int\frac{xdx}{\sqrt[3]{\left(1-x^{3}\right)^{2}}}$$

$$\int\frac{xdx}{\sqrt[4]{\left(1-x^{3}\right)^{3}}} = \cdot\frac{2\cdot 11}{5\cdot 5}\cdot\frac{3\cdot 23}{9\cdot 8}\cdot\frac{4\cdot 35}{13\cdot 11}\cdot\frac{5\cdot 47}{17\cdot 14}\cdot\text{etc.} \qquad = \frac{4}{3}\int\frac{xdx}{\sqrt[3]{\left(1-x^{3}\right)}}$$

$$\int\frac{dx}{\sqrt[4]{\left(1-x^{4}\right)^{3}}} = \frac{4}{1}\cdot\frac{2\cdot 8}{5\cdot 5}\cdot\frac{3\cdot 24}{9\cdot 9}\cdot\frac{4\cdot 40}{13\cdot 13}\cdot\frac{5\cdot 56}{17\cdot 17}\cdot\text{etc.} \qquad = \int\frac{dx}{\sqrt[4]{\left(1-x^{4}\right)^{3}}}$$

or

$$= \frac{4}{1}\cdot\frac{4\cdot 4}{5\cdot 5}\cdot\frac{6\cdot 12}{9\cdot 9}\cdot\frac{8\cdot 20}{13\cdot 13}\cdot\frac{10\cdot 28}{17\cdot 17}\cdot\text{etc.}$$

or

$$= \frac{4}{1}\cdot\frac{2\cdot 8}{5\cdot 5}\cdot\frac{6\cdot 12}{9\cdot 9}\cdot\frac{10\cdot 16}{13\cdot 13}\cdot\frac{14\cdot 20}{17\cdot 17}\cdot\text{etc.},$$

$$\int\frac{xxdx}{\sqrt[3]{\left(1-x^{4}\right)^{3}}} = \frac{4}{3}\cdot\frac{2\cdot 16}{5\cdot 7}\cdot\frac{3\cdot 32}{9\cdot 11}\cdot\frac{4\cdot 48}{13\cdot 15}\cdot\frac{5\cdot 64}{17\cdot 19}\cdot\text{etc.} \qquad = \int\frac{dx}{\sqrt[4]{\left(1-x^{4}\right)}}$$

or

$$= \frac{4}{3}\cdot\frac{4\cdot 8}{5\cdot 7}\cdot\frac{6\cdot 16}{9\cdot 11}\cdot\frac{8\cdot 24}{13\cdot 15}\cdot\frac{10\cdot 32}{17\cdot 19}\cdot\text{etc.}$$

or

$$= \frac{4}{3}\cdot\frac{4\cdot 8}{5\cdot 7}\cdot\frac{8\cdot 12}{9\cdot 11}\cdot\frac{12\cdot 16}{13\cdot 15}\cdot\frac{16\cdot 20}{17\cdot 19}\cdot\text{etc.}$$

And now the case $\mu = 3$ and $\nu = 4$ is contained in these and in the preceding.

SCHOLIUM

374. These other formulas, in which I have introduced the letters μ and ν, do not extend beyond that first considered ; for the series depend on the fractions $\frac{m}{n}$ and $\frac{\mu}{\nu}$, which since they are always to be recalled with a common denominator, it is sufficient to consider the formulas

$$\int \frac{x^{m-1}dx}{\sqrt[n]{\left(1-x^n\right)^{n-k}}} = \int \frac{x^{k-1}dx}{\sqrt[n]{\left(1-x^n\right)^{n-m}}}.$$

Therefore since the value of these in the case $x=1$ is equal to this product

$$\frac{1}{k}\cdot\frac{n(m+k)}{m(k+n)}\cdot\frac{2n(m+k+n)}{(m+n)(k+2n)}\cdot\frac{3n(m+k+2n)}{(m+2n)(k+3n)}.\text{etc},$$

if we permute the factors in the individual members of the numerator and we partition the members otherwise, the same product adopts this form :

$$\frac{(m+k)}{mk}\cdot\frac{n(m+k+n)}{(m+n)(k+n)}\cdot\frac{2n(m+k+2n)}{(m+2n)(k+2n)}\cdot\frac{3n(m+k+3n)}{(m+3n)(k+3n)}.\text{ etc.},$$

which is considered more convenient to be remembered. In a similar manner since there shall be

$$\int \frac{x^{p-1}dx}{\sqrt[n]{\left(1-x^n\right)^{n-q}}} = \int \frac{x^{q-1}dx}{\sqrt[n]{\left(1-x^n\right)^{n-p}}}$$
$$=\frac{(p+q)}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\frac{3n(p+q+3n)}{(p+3n)(q+3n)}.\text{ etc.},$$

that form on division by this will be

$$\frac{\int x^{m-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}}{\int x^{p-1}dx\left(1-x^n\right)^{\frac{q-n}{n}}}$$
$$=\frac{pq(m+k)}{mk(p+q)}\cdot\frac{(p+n)(q+n)(m+k+n)}{(m+n)(k+n)(p+q+n)}\cdot\frac{(p+2n)(q+2n)(m+k+2n)}{(m+2n)(k+2n)(p+q+2n)}\cdot\text{ etc.},$$

all the members are retained by the same rule. But hence extraordinary combinations of formulas of this kind can be deduced, which in order that they are easier to be kept in mind, for the sake of brevity I will use in the following shorthand form.

DEFINITION

375. *The value of the integral formula* $\int x^{p-1}dx\left(1-x^{n}\right)^{\frac{q-n}{n}}$ *, that it takes on putting* $x = 1$*, we may indicate for the sake of brevity by this sign* $\left(\frac{p}{q}\right)$*, where it has to be realised that I assume a certain exponent n to remain the same, in the comparison of several formulas of this kind.*

COROLLARY 1

376. Therefore in the first place it is apparent that $\left(\frac{p}{q}\right)=\left(\frac{q}{p}\right)$ and each formula to be

$$\frac{(p+q)}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\text{etc.},$$

of which the progression of the members is clear, while the individual factors both of the numerator and the denominator are increased continually by the same number *n*, thus in order that from the known first member the sequences are easily formed.

COROLLARY 2

377. Then if there shall be $p=n$, on account of the integration formula it is clear that

$$\left(\frac{n}{q}\right)=\left(\frac{q}{n}\right)=\frac{1}{q},\quad \text{likewise}\quad \left(\frac{p}{n}\right)=\left(\frac{n}{p}\right)=\frac{1}{p}$$

Again since

$$\int x^{p-1}dx\left(1-x^{n}\right)^{\frac{-p}{n}}=\frac{\pi}{n\sin.\frac{p\pi}{n}},$$

on account of $q-n=-p$ or $p+q=n$ then

$$\left(\frac{p}{n-p}\right)=\left(\frac{n-p}{p}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}.$$

Whereby the value of the formula $\left(\frac{p}{q}\right)$ can be determined completely, as long as either $p=n$, $q=n$, or $p+q=n$.

COROLLARY 3

378. Because we have found this reduction also [§ 345]

$$\int x^{p+n-1}dx\left(1-x^{n}\right)^{\frac{q-n}{n}}=\frac{p}{p+q}\int x^{p-1}dx\left(1-x^{n}\right)^{\frac{q-n}{n}},$$

it follows that

$$\left(\frac{p+n}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right)$$

and hence

$$\left(\frac{p}{q}\right)=\left(\frac{q}{p}\right)=\frac{p-n}{p+q-n}\left(\frac{p-n}{q}\right)=\frac{q-n}{p+q-n}\left(\frac{p}{q-n}\right),$$

then indeed also

$$\left(\frac{p}{q}\right)=\frac{(p-n)(q-n)}{(p+q-n)(p+q-2n)}\cdot\left(\frac{p-n}{q-n}\right),$$

from which the numbers p et q can always be taken less than n.

PROBLEM 46

379. *To find different products from the two formulas of this kind, which are equal to each other.*

SOLUTION

Hence the numbers are sought a, b, c, d and p, q, r, s, so that it happens that

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right)=\left(\frac{p}{q}\right)\left(\frac{r}{s}\right),$$

which, since

$$\left(\frac{a}{b}\right)=\frac{a+b}{ab}\cdot\frac{n(a+b+n)}{(a+n)(b+n)}\cdot\text{etc.},\qquad \left(\frac{c}{d}\right)=\frac{c+d}{cd}\cdot\frac{n(c+d+n)}{(c+n)(d+n)}\cdot\text{etc.},$$

$$\left(\frac{p}{q}\right)=\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\text{etc.},\qquad \left(\frac{r}{s}\right)=\frac{r+s}{rs}\cdot\frac{n(r+s+n)}{(r+n)(s+n)}\cdot\text{etc.},$$

comes about, if

$$\frac{(a+b)(c+d)}{abcd}=\frac{(p+q)(r+s)}{pqrs}$$

or

$$abcd(p+q)(r+s)=pqrs(a+b)(c+d),$$

thus in order that, since there are six factors on both sides, individual factors shall be equal to individual factors. Hence from the four *abcd* and *pqrs* it is necessary that at least two are equal; and thus let $s=d$ and being effected, it is required that

$$abc(p+q)(r+d)=pqr(a+b)(c+d).$$

I. The other factor r is taken; which since it is unable to be equal to c, because otherwise it should make $\left(\frac{c}{d}\right)=\left(\frac{r}{s}\right)$, there is put in place $r=b$, so that there becomes

$$ac(p+q)(b+d) = pq(a+b)(c+d) \;;$$

here neither p nor q can be equal to $p+q$, hence there must be put :

1) Either $p+q=a+b$, so that there shall be $ac(b+d)=pq(c+d)$, because neither c nor $b+d$ can be equal to $c+d$; for there becomes either $d=0$ or $b=c$ and $\left(\frac{r}{s}\right)=\left(\frac{c}{d}\right)$; there is left $a=c+d$ and $pq=c(b+d)$ and thus $p=b+d$ and $q=c$, from which there is made

$$\left(\tfrac{c+d}{b}\right)\left(\tfrac{c}{d}\right)=\left(\tfrac{b+d}{c}\right)\left(\tfrac{b}{d}\right).$$

2) Or $p+q=c+d$, hence $ac(b+d)=pq(a+b)$; here c can neither be equal to p nor q; since there then arises $\left(\frac{p}{q}\right)=\left(\frac{c}{d}\right)$; from which there is made $c=a+b$, so that $pq=a(b+d)$, hence $p=a$, $q=b+d$, $r=b, s=d$, consequently

$$\left(\tfrac{a}{b}\right)\left(\tfrac{a+b}{d}\right)=\left(\tfrac{b+d}{a}\right)\left(\tfrac{b}{d}\right).$$

II. Because $r=a$ does not differ from the preceding on account of the interchangeability of a and b, there is put in place $r=p+q$ and there is produced $abc(d+p+q)=pq(a+b)(c+d)$. Because r is unable to be equal to c, then the factor $d+p+q$ cannot be put equal either to p or q or to $c+d$; hence there is left $d+p+q=a+b$ and $abc=pq(c+d)$; where, because c cannot be equal to $c+d$ and p and q enjoy being in the pair arrangement, there becomes $p=c$; then $q=a+b-c-d$ and $ab=(c+d)(a+b-c-d)$, from which $a=c+d$, $q=b, p=c$, $r=b+c$, $s=d$, and thus there is prepared

$$\left(\tfrac{c+d}{b}\right)\left(\tfrac{c}{d}\right)=\left(\tfrac{c}{b}\right)\left(\tfrac{b+c}{d}\right).$$

COROLLARY 1

380. These solutions are returned almost the same and hence the three equal products of the two formulas are elicited :

$$\left(\tfrac{c}{d}\right)\left(\tfrac{c+d}{b}\right)=\left(\tfrac{c}{b}\right)\left(\tfrac{b+c}{d}\right)=\left(\tfrac{b}{d}\right)\left(\tfrac{b+d}{c}\right)$$

or in terms of the letters p, q, r

$$\left(\tfrac{p}{q}\right)\left(\tfrac{p+q}{r}\right)=\left(\tfrac{q}{r}\right)\left(\tfrac{q+r}{p}\right)=\left(\tfrac{p}{r}\right)\left(\tfrac{p+r}{q}\right).$$

COROLLARUM 2

381. If these formulas are set out in infinite products, there is found :

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\frac{p+q+r}{pqr}\cdot\frac{nn(p+q+r+n)}{(p+n)(q+n)(r+n)}\cdot\frac{4nn(p+q+r+2n)}{(p+2n)(q+2n)(r+2n)}\cdot\text{etc.},$$

from which it appears that the three letters *p*, *q*, *r* can be permuted among themselves in any way, and hence it is possible to include these three formulas.

COROLLARY 3

382. We can restore these integral formulas and the three following products are equal to each other :

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}\cdot\int\frac{x^{p+q-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}=\int\frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{q+r-1}dx}{\sqrt[n]{(1-x^n)^{n-p}}}$$
$$=\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{p+r-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}.$$

COROLLARY 4

383. This case is noteworthy, in which $p+q=n$; then indeed on account of

$$\left(\frac{p+q}{r}\right)=\left(\frac{n}{r}\right)=\left(\frac{1}{r}\right)\text{ and }\left(\frac{p}{q}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}$$

these three products become equal to $\frac{\pi}{nr\sin.\frac{p\pi}{n}}$. Clearly there shall be

$$\int\frac{x^{n-p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{n-p+r-1}dx}{\sqrt[n]{(1-x^n)^{n-p}}}=\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{p+r-1}dx}{\sqrt[n]{(1-x^n)^{p}}}=\frac{\pi}{nr\sin.\frac{p\pi}{n}}.$$

SCHOLIUM

384. That threefold property of the products from the two formulas is to be noted especially, and for the various numbers to be put in place of *p*, *q*, *r* the following specific equalities are obtained :

p	*q*	*r*	
1	1	2	$\left(\frac{1}{1}\right)\left(\frac{2}{2}\right)=\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)$
1	2	2	$\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)=\left(\frac{2}{2}\right)\left(\frac{4}{1}\right)$
1	2	3	$\left(\frac{2}{1}\right)\left(\frac{3}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{1}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{2}\right)$
1	1	3	$\left(\frac{1}{1}\right)\left(\frac{3}{2}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)$

2	2	3	$\left(\frac{2}{2}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)$
1	3	3	$\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{1}\right)$
2	3	3	$\left(\frac{3}{2}\right)\left(\frac{5}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{2}\right)$
1	1	4	$\left(\frac{1}{1}\right)\left(\frac{4}{2}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)$
1	2	4	$\left(\frac{2}{1}\right)\left(\frac{4}{3}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{1}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{2}\right)$
1	3	4	$\left(\frac{3}{1}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{3}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{1}\right)$
1	4	4	$\left(\frac{4}{1}\right)\left(\frac{5}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{1}\right)$
2	2	4	$\left(\frac{2}{2}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)$
2	3	4	$\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{2}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{3}\right)$
2	4	4	$\left(\frac{4}{2}\right)\left(\frac{6}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{2}\right)$
3	3	4	$\left(\frac{3}{3}\right)\left(\frac{6}{4}\right)=\left(\frac{4}{8}\right)\left(\frac{7}{3}\right)$
3	4	4	$\left(\frac{4}{3}\right)\left(\frac{7}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{3}\right).$

Which formulas prevail for all the numbers *n*, and if numbers greater than *n* occur, we have seen above that these can be reduced to smaller numbers.

PROBLEM 47

385. *To find diverse products from three formulas of this kind, which are equal to each other*.

SOLUTION

The product $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)$ may be considered, which set out gives

$$\frac{p+q+r+s}{pqrs}\cdot\frac{n^3(p+q+r+s+n)}{(p+n)(q+n)(r+n)(s+n)}\cdot\text{etc},$$

which it is evident holds the same value, in whatever way the four letters are interchanged among themselves. Then indeed the same product is set out from this product $\left(\frac{p}{q}\right)\left(\frac{r}{s}\right)\left(\frac{p+q}{r+s}\right)$, where the same permutation is in place. Hence all these products are equal to each other :

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right),\quad \left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)\left(\frac{p+q+r}{s}\right),\quad \left(\frac{p}{s}\right)\left(\frac{p+s}{q}\right)\left(\frac{p+q+s}{r}\right),$$
$$\left(\frac{p}{q}\right)\left(\frac{p+q}{s}\right)\left(\frac{p+q+s}{r}\right),\quad \left(\frac{p}{r}\right)\left(\frac{p+r}{s}\right)\left(\frac{p+q+s}{q}\right),\quad \left(\frac{p}{s}\right)\left(\frac{p+s}{r}\right)\left(\frac{p+r+s}{q}\right),$$
$$\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)\left(\frac{p+q+r}{s}\right),\quad \left(\frac{q}{s}\right)\left(\frac{q+s}{p}\right)\left(\frac{p+q+s}{r}\right),\quad \left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right),$$
$$\left(\frac{q}{r}\right)\left(\frac{q+r}{s}\right)\left(\frac{p+q+s}{p}\right),\quad \left(\frac{q}{s}\right)\left(\frac{q+s}{r}\right)\left(\frac{q+r+s}{p}\right),\quad \left(\frac{r}{s}\right)\left(\frac{r+s}{q}\right)\left(\frac{q+r+s}{p}\right).$$

The products of other forms with the help of the preceding properties hence arise at once; indeed there is

$$\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right).$$

Then also this product $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)$ set out for the first member gives $\frac{(p+q+r)(p+r+s)}{pqrs(p+r)}$ in which both *p* and *r* as well as *q* and *s* can be permuted between themselves, thus so that there shall be

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right).$$

SCHOLIUM

386. However wide this is considered to be allowed, yet no new comparisons are supplied, which now are not present in the preceding. Finally indeed the equality

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right).$$

arises from the multiplication of these :

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right),\quad \left(\frac{p}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right).$$

Now the first form is apparent from this example : the equality

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right)$$

arises from the multiplication of these :

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r+s}\right)=\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right),\quad \left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right).$$

But these comparisons are especially useful and to be reduced in turn according to the values of diverse formulas of the same order or to a given number *n*, so that integration need hardy ever be recalled, from which given the rest can be defined by these.

PROBLEM 48

387. *To show the simplest formulas, to which the integration of all the cases contained in the form*

$$\left(\frac{p}{q}\right)=\int\frac{x^{p-1}dx}{\sqrt[n]{\left(1-x^{n}\right)^{n-q}}}$$

can be reduced.

SOLUTION

First there is $\left(\frac{n}{p}\right)=\frac{1}{p}$, from which these cases may be had :

$$\left(\frac{n}{1}\right)=1,\quad \left(\frac{n}{2}\right)=\frac{1}{2},\quad \left(\frac{n}{3}\right)=\frac{1}{3},\quad \left(\frac{n}{4}\right)=\frac{1}{4},\quad \left(\frac{n}{5}\right)=\frac{1}{5}\quad \text{etc.}$$

Then there is $\left(\frac{p}{n-p}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}$, from which the values of all the formulas are known, which we may indicate :

$$\left(\frac{n-1}{1}\right)=\alpha,\quad \left(\frac{n-2}{2}\right)=\beta,\quad \left(\frac{n-3}{3}\right)=\gamma,\quad \left(\frac{n-4}{4}\right)=\delta\quad \text{etc.}$$

Now these are not sufficient for setting out all the remaining cases, since in addition it is required to consider these :

$$\left(\frac{n-2}{1}\right)=A,\quad \left(\frac{n-3}{2}\right)=B,\quad \left(\frac{n-4}{3}\right)=C,\quad \left(\frac{n-5}{4}\right)=D\quad \text{etc.}$$

and from these all the rest can be determined with the aid of the equations shown above ; from which it helps to know these chiefly :

$$\left(\frac{n-a}{a}\right)\left(\frac{n}{b}\right)=\left(\frac{n-a}{b}\right)\left(\frac{n-a+b}{a}\right),$$

$$\left(\frac{n-a}{a}\right)\left(\frac{n-a-b}{b}\right)=\left(\frac{n-b}{b}\right)\left(\frac{n-a-b}{a}\right),$$

$$\left(\frac{n-a}{a}\right)\left(\frac{n-b-1}{b}\right)\left(\frac{n-a-b}{a-1}\right)=\left(\frac{n-b}{b}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-b}{a}\right).$$

From which first on putting $a=b+1$ there is found :

$$\left(\frac{n-1}{a}\right)=\left(\frac{n-a}{a}\right)\left(\frac{n}{a-1}\right):\left(\frac{n-a}{a-1}\right),$$

where $\left(\frac{n}{a-1}\right)=\left(\frac{1}{a-1}\right)$, and thus by the formulas assumed $\left(\frac{n-1}{a}\right)$ is defined .
From the second on putting $b = 1$ there is found :

$$\left(\frac{n-a-1}{a}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right).$$

From the third on putting $b = 1$ there is deduced :

$$\left(\frac{n-a-1}{a-1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-2}{1}\right)$$

and thus all the formulas $\left(\frac{n-a-2}{a-1}\right)$ are found and from these again on putting $b=2$ into the third :

$$\left(\frac{n-a-2}{a-1}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-3}{2}\right),$$

from which the forms $\left(\frac{n-a-3}{a-1}\right)$ are found and thus again all $\left(\frac{n-a-b}{a}\right)$, clearly for which form everything is completed. But the labour for the first equations is greatly reduced. For on finding $\left(\frac{n-a-2}{a-1}\right)$ there is deduced from the first ;

$$\left(\frac{n-2}{a+2}\right)=\left(\frac{n-a-2}{a+2}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-2}{a}\right),$$

and now for the second :

$$\left(\frac{n-a-2}{2}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right).$$

and in a similar manner from the formulas $\left(\frac{n-a-3}{a-1}\right)$ found these are derived :

$$\left(\frac{n-3}{a+3}\right)=\left(\frac{n-a-3}{a+3}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-3}{a}\right),$$
$$\left(\frac{n-a-3}{a-1}\right)=\left(\frac{n-3}{3}\right)\left(\frac{n-a-3}{a}\right):\left(\frac{n-a}{a}\right).$$

COROLLARY 1

388. From the equation $\left(\frac{n-1}{a}\right)=\frac{1}{a-1}\left(\frac{n-a}{a}\right):\left(\frac{n-a}{a-1}\right)$ there are defined :

$$\left(\frac{n-1}{2}\right)=\frac{\beta}{1A},\quad \left(\frac{n-1}{3}\right)=\frac{\gamma}{2B},\quad \left(\frac{n-1}{4}\right)=\frac{\delta}{3C},\quad \left(\frac{n-1}{5}\right)=\frac{\varepsilon}{4D}\quad \text{etc.,}$$

now from the equation $\left(\frac{n-a-1}{a}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)$ these formulas

$$\left(\tfrac{n-2}{1}\right)=\tfrac{\alpha A}{\alpha},\quad \left(\tfrac{n-3}{1}\right)=\tfrac{\alpha B}{\beta},\quad \left(\tfrac{n-4}{1}\right)=\tfrac{\alpha C}{\gamma},\quad \left(\tfrac{n-5}{1}\right)=\tfrac{\alpha D}{\delta}\quad \text{etc.}$$

COROLLARY 2

389. The equation $\left(\tfrac{n-a-1}{a-1}\right)=\left(\tfrac{n-1}{1}\right)\left(\tfrac{n-a}{a-1}\right)\left(\tfrac{n-a-1}{a}\right):\left(\tfrac{n-a}{a}\right)\left(\tfrac{n-2}{1}\right)$ gives

$$\left(\tfrac{n-3}{1}\right)=\tfrac{\alpha AB}{\beta A},\quad \left(\tfrac{n-4}{2}\right)=\tfrac{\alpha BC}{\gamma A},\quad \left(\tfrac{n-5}{3}\right)=\tfrac{\alpha CD}{\delta A},\quad \left(\tfrac{n-6}{4}\right)=\tfrac{\alpha DE}{\varepsilon A}\quad \text{etc.},$$

from which these formulas are found $\left(\tfrac{n-2}{a+2}\right)=\left(\tfrac{n-a-2}{a+2}\right)\left(\tfrac{n}{a}\right):\left(\tfrac{n-a-2}{a}\right)$, that is,

$$\left(\tfrac{n-2}{3}\right)=\tfrac{\gamma\beta A}{1\alpha AB},\quad \left(\tfrac{n-2}{4}\right)=\tfrac{\delta\gamma A}{2\alpha BC},\quad \left(\tfrac{n-2}{5}\right)=\tfrac{\varepsilon\delta A}{3\alpha CD},\quad \left(\tfrac{n-2}{6}\right)=\tfrac{\zeta\varepsilon A}{4\alpha DE}\quad \text{etc.}$$

and also these $\left(\tfrac{n-a-2}{2}\right)=\left(\tfrac{n-2}{2}\right)\left(\tfrac{n-a-2}{a}\right):\left(\tfrac{n-a}{a}\right)$, which are

$$\left(\tfrac{n-3}{2}\right)=\tfrac{\beta\alpha AB}{\alpha\beta A},\quad \left(\tfrac{n-4}{2}\right)=\tfrac{\beta\alpha BC}{\beta\gamma A},\quad \left(\tfrac{n-5}{2}\right)=\tfrac{\beta\alpha CD}{\gamma\delta A},\quad \left(\tfrac{n-6}{2}\right)=\tfrac{\beta\alpha DE}{\delta\varepsilon A}\quad \text{etc.}$$

COROLLARY 3

390. Then the equation $\left(\tfrac{n-a-2}{a-1}\right)=\left(\tfrac{n-2}{2}\right)\left(\tfrac{n-a}{a-1}\right)\left(\tfrac{n-a-2}{a}\right):\left(\tfrac{n-a}{a}\right)\left(\tfrac{n-3}{2}\right)$
gives

$$\left(\tfrac{n-4}{1}\right)=\tfrac{\alpha\beta ABC}{\beta\gamma AB},\quad \left(\tfrac{n-5}{2}\right)=\tfrac{\alpha\beta BCD}{\gamma\delta AB},\quad \left(\tfrac{n-6}{3}\right)=\tfrac{\alpha\beta CDE}{\delta\varepsilon AB},\quad \left(\tfrac{n-7}{4}\right)=\tfrac{\alpha\beta DEF}{\varepsilon\zeta AB}\quad \text{etc.},$$

hence $\left(\tfrac{n-3}{a+3}\right)=\left(\tfrac{n-a-3}{a+3}\right)\left(\tfrac{n}{a}\right):\left(\tfrac{n-a-3}{a}\right)$ gives

$$\left(\tfrac{n-3}{4}\right)=\tfrac{\beta\gamma\delta AB}{1\alpha\beta ABC},\quad \left(\tfrac{n-3}{5}\right)=\tfrac{\gamma\delta\varepsilon AB}{2\alpha\beta BCD},\quad \left(\tfrac{n-3}{6}\right)=\tfrac{\delta\varepsilon\zeta AB}{3\alpha\beta CDE},\quad \text{etc.}$$

and from $\left(\tfrac{n-a-3}{3}\right)=\left(\tfrac{n-3}{3}\right)\left(\tfrac{n-a-3}{a}\right):\left(\tfrac{n-a}{a}\right)$ there are deduced

$$\left(\tfrac{n-5}{3}\right)=\tfrac{\alpha\beta\gamma BCD}{\beta\gamma\delta AB},\quad \left(\tfrac{n-6}{3}\right)=\tfrac{\alpha\beta\gamma CDE}{\gamma\delta\varepsilon AB},\quad \left(\tfrac{n-7}{3}\right)=\tfrac{\alpha\beta\gamma DEF}{\delta\varepsilon\xi AB},\quad \text{etc.}$$

EXAMPLE 1

391. *The cases contained in this form* $\int \frac{x^{p-1}dx}{\sqrt[2]{(1-x^2)^{2-q}}} = \left(\frac{p}{q}\right)$ *are set out, where n=2, and where*

$$\left(\frac{p+2}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right).$$

It is evident that all these formulas can be extricated either algebraically or by angles ; yet by using these rules, because the numbers p et q must not be in excess of the two, we have the one formula depending on the circle :

$$\left(\frac{1}{1}\right) = \frac{\pi}{2\,sin.\frac{\pi}{2}} = \frac{\pi}{2} = \alpha,$$

from which our cases become :

$$\left(\frac{2}{1}\right) = 1, \quad \left(\frac{2}{2}\right) = \frac{1}{2},$$
$$\left(\frac{1}{1}\right) = \alpha.$$

EXAMPLE 2

392. *The cases contained in this form* $\int \frac{x^{p-1}dx}{\sqrt[3]{(1-x^3)^{3-q}}} = \left(\frac{p}{q}\right)$ *are set out, where* $n = 3$, *and where*

$$\left(\frac{p+3}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right).$$

Here the main cases, to which the rest can be reduced, are

$$\left(\frac{2}{1}\right) = \frac{\pi}{3\,sin.\frac{\pi}{3}} = \frac{2\pi}{3\sqrt{3}} = \alpha \quad \text{and} \quad \left(\frac{1}{1}\right) = \int \frac{dx}{\sqrt[3]{(1-x^3)^2}},$$

with which conceded the remainder are :

$$\left(\frac{3}{1}\right) = 1, \quad \left(\frac{3}{2}\right) = \frac{1}{2}, \quad \left(\frac{3}{3}\right) = \frac{1}{3},$$
$$\left(\frac{2}{1}\right) = \alpha, \quad \left(\frac{2}{2}\right) = \frac{\alpha}{A},$$
$$\left(\frac{1}{1}\right) = A.$$

EXAMPLE 3

393. *The cases contained in this formula are set out* $\int\frac{x^{p-1}dx}{\sqrt[4]{\left(1-x^{4}\right)^{4-q}}}=\left(\frac{p}{q}\right)$, *where* $n=4$, *and where there is*

$$\left(\frac{p+4}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right).$$

These two cases depend on the circle :

$$\left(\frac{3}{1}\right)=\frac{\pi}{4sin.\frac{\pi}{4}}=\frac{\pi}{2\sqrt{2}}=\alpha \quad \text{and} \quad \left(\frac{2}{2}\right)=\frac{\pi}{4sin.\frac{2\pi}{4}}=\frac{\pi}{4}=\beta,$$

now in addition there is the need for the single transcendent $\left(\frac{2}{1}\right)=A$, from which the others can be determined thus :

$$\left(\frac{4}{1}\right)=1,\quad \left(\frac{4}{2}\right)=\frac{1}{2},\quad \left(\frac{4}{3}\right)=\frac{1}{3},\quad \left(\frac{4}{4}\right)=\frac{1}{4},$$
$$\left(\frac{3}{1}\right)=\alpha,\quad \left(\frac{3}{2}\right)=\frac{\beta}{A},\quad \left(\frac{3}{3}\right)=\frac{\alpha}{2A},$$
$$\left(\frac{2}{1}\right)=A,\quad \left(\frac{2}{2}\right)=\beta,$$
$$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$$

EXAMPLE 4

394. *The cases contained in this form* $\int\frac{x^{p-1}dx}{\sqrt[5]{\left(1-x^{5}\right)^{5-q}}}=\left(\frac{p}{q}\right)$ *are to be set out, where* $n=5$, *and where*

$$\left(\frac{p+5}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right).$$

These two formulas depend on the circle :

$$\left(\frac{4}{1}\right)=\frac{\pi}{5sin.\frac{\pi}{5}}=\alpha \quad \text{and} \quad \left(\frac{3}{2}\right)=\frac{\pi}{5sin.\frac{2\pi}{5}}=\beta,$$

in addition it is required to assume two new transcendents :

$$\left(\frac{3}{1}\right)=A \ \text{ and } \left(\frac{2}{2}\right)=B,$$

by which all are determined in the following manner :

$$\left(\frac{5}{1}\right)=1,\quad \left(\frac{5}{2}\right)=\frac{1}{2},\quad \left(\frac{5}{3}\right)=\frac{1}{3},\quad \left(\frac{5}{4}\right)=\frac{1}{4},\quad \left(\frac{5}{5}\right)=\frac{1}{5},$$
$$\left(\frac{4}{1}\right)=\alpha,\quad \left(\frac{4}{2}\right)=\frac{\beta}{A},\quad \left(\frac{4}{3}\right)=\frac{\beta}{2B},\quad \left(\frac{4}{4}\right)=\frac{\alpha}{3A},$$
$$\left(\frac{3}{1}\right)=A,\quad \left(\frac{3}{2}\right)=\beta,\quad \left(\frac{3}{3}\right)=\frac{\beta\beta}{\alpha B},$$
$$\left(\frac{2}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{2}{2}\right)=B,$$
$$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$$

EXAMPLE 5

395. *The cases contained in this form* $\int\frac{x^{p-1}dx}{\sqrt[6]{\left(1-x^6\right)^{6-q}}}=\left(\frac{p}{q}\right)$ *are set out, where n* = 6.

These three formulas depend on the circle :

$$\left(\frac{5}{1}\right)=\frac{\pi}{6\,sin.\frac{\pi}{6}}=\frac{\pi}{3}=\alpha,\quad \left(\frac{4}{2}\right)=\frac{\pi}{6\,sin.\frac{2\pi}{6}}=\frac{\pi}{3\sqrt{3}}=\beta,\quad \left(\frac{3}{3}\right)=\frac{\pi}{6\,sin.\frac{3\pi}{6}}=\frac{\pi}{6}=\gamma\ ;$$

then these two transcendents are assumed now : $\left(\frac{4}{1}\right)=A$ and $\left(\frac{3}{2}\right)=B$

and through these all are determined in the following manner :

$$\left(\frac{6}{1}\right)=1,\quad \left(\frac{6}{2}\right)=\frac{1}{2},\quad \left(\frac{6}{3}\right)=\frac{1}{3},\quad \left(\frac{6}{4}\right)=\frac{1}{4},\quad \left(\frac{6}{5}\right)=\frac{1}{5},\quad \left(\frac{6}{6}\right)=\frac{1}{6},$$
$$\left(\frac{5}{1}\right)=\alpha,\quad \left(\frac{5}{2}\right)=\frac{\beta}{A},\quad \left(\frac{5}{3}\right)=\frac{\gamma}{2B},\quad \left(\frac{5}{4}\right)=\frac{\beta}{3B},\quad \left(\frac{5}{5}\right)=\frac{\alpha}{4A},$$
$$\left(\frac{4}{1}\right)=A,\quad \left(\frac{4}{2}\right)=\beta,\quad \left(\frac{4}{3}\right)=\frac{\beta\gamma}{\alpha B},\quad \left(\frac{4}{4}\right)=\frac{\beta\gamma A}{2\alpha BB},$$
$$\left(\frac{3}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{3}{2}\right)=B,\quad \left(\frac{3}{3}\right)=\gamma,$$
$$\left(\frac{2}{1}\right)=\frac{\alpha B}{\gamma},\quad \left(\frac{2}{2}\right)=\frac{\alpha BB}{\gamma A},$$
$$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$$

SCHOLIUM

396. These determinations are allowed to be continued as far as one would wish, in which the new kinds of transcending [formulas or functions] introduced must be especially noted ; the first of which occurs if *n=3*, and it is

$$\left(\frac{1}{1}\right)=\int\frac{dx}{\sqrt[3]{\left(1-x^3\right)^2}}$$

the value of which we have seen to be given above by the infinite product [§ 371]

$$=\frac{3}{1}\cdot\frac{2}{4}\cdot\frac{6}{4}\cdot\frac{5}{7}\cdot\frac{9}{7}\cdot\frac{8}{10}\cdot\frac{12}{10}\cdot\text{etc.},$$

which from the formula $\left(\frac{1}{1}\right)$ on account of $n=3$ is also equal to

$$\frac{2}{1\cdot1}\cdot\frac{3\cdot5}{4\cdot4}\cdot\frac{6\cdot8}{7\cdot7}\cdot\frac{9\cdot11}{10\cdot10}\cdot\frac{12\cdot14}{13\cdot13}\cdot\text{etc.}$$

Then from the class $n=4$ this new transcending form arises :

$$\left(\frac{2}{1}\right)=\int\frac{xdx}{\sqrt[4]{\left(1-x^4\right)^3}}=\int\frac{dx}{\sqrt[4]{\left(1-x^4\right)^2}}=\int\frac{dx}{\sqrt{\left(1-x^4\right)}},$$

which is equal to this infinite product

$$\frac{3}{1\cdot2}\cdot\frac{4\cdot7}{5\cdot6}\cdot\frac{8\cdot11}{9\cdot10}\cdot\frac{12\cdot15}{13\cdot14}\cdot\frac{16\cdot19}{17\cdot18}\cdot\text{etc.}=\frac{3}{2}\cdot\frac{2\cdot7}{5\cdot3}\cdot\frac{4\cdot11}{9\cdot5}\cdot\frac{6\cdot15}{13\cdot7}\cdot\frac{8\cdot19}{17\cdot9}\cdot\text{etc.}$$

From the class $n = 5$ we obtain these new transcending formulas

$$\left(\frac{3}{1}\right)=\int\frac{x^2dx}{\sqrt[5]{\left(1-x^5\right)^4}}=\int\frac{dx}{\sqrt[5]{\left(1-x^5\right)^2}}=\frac{4}{1\cdot3}\cdot\frac{5\cdot9}{6\cdot8}\cdot\frac{10\cdot14}{11\cdot13}\cdot\frac{15\cdot19}{16\cdot18}\cdot\text{etc.}$$

and

$$\left(\frac{2}{2}\right)=\int\frac{xdx}{\sqrt[5]{\left(1-x^5\right)^3}}=\frac{4}{2\cdot2}\cdot\frac{5\cdot9}{7\cdot7}\cdot\frac{10\cdot14}{12\cdot12}\cdot\frac{15\cdot19}{17\cdot17}\cdot\text{etc.},$$

thus so that there becomes :

$$\left(\frac{3}{1}\right):\left(\frac{2}{2}\right)=\frac{2\cdot2}{1\cdot3}\cdot\frac{7\cdot7}{6\cdot8}\cdot\frac{12\cdot12}{11\cdot13}\cdot\frac{17\cdot17}{16\cdot18}\cdot\text{etc.}$$

The class $n=6$ supplies these two transcending formulas :

$$\left(\frac{4}{1}\right)=\int\frac{x^3dx}{\sqrt[6]{\left(1-x^6\right)^5}}=\int\frac{dx}{\sqrt[3]{\left(1-x^6\right)}}=\frac{1}{2}\int\frac{ydy}{\sqrt[6]{\left(1-y^3\right)^5}}$$

on putting $xx=y$ and

$$\left(\frac{3}{2}\right)=\int\frac{x^2dx}{\sqrt[3]{\left(1-x^6\right)^2}}=\int\frac{xdx}{\sqrt{\left(1-x^6\right)}}=\frac{1}{2}\int\frac{dy}{\sqrt{\left(1-y^3\right)}}=\frac{1}{3}\int\frac{dz}{\sqrt[3]{\left(1-zz\right)^2}}$$

on taking $y=xx$ and $z=x^3$. Moreover it is to be observed that between these and the first relation given,

$$\int\frac{dx}{\sqrt[3]{\left(1-x^3\right)^2}}=2\int\frac{ydy}{\sqrt[6]{\left(1-y^6\right)^4}}=2\left(\frac{2}{2}\right),$$

which is [see § 384] $\gamma\left(\frac{4}{1}\right)\left(\frac{2}{2}\right)=\alpha\left(\frac{3}{2}\right)\left(\frac{3}{2}\right)$, thus so that the first suffices here for the other to be granted.

CAPUT IX

DE EVOLUTIONE INTEGRALIUM PER PRODUCTA INFINITA

PROBLEMA 43

356. *Valorem huius integralis* $\int\frac{dx}{\sqrt{(1-xx)}}$ *, quem casu* $x=1$ *recipit, in productum infinitum evolvere.*

SOLUTIO

Quemadmodum supra formulas altiores ad simplicem reduximus, ita hic formulam $\int\frac{dx}{\sqrt{(1-xx)}}$ continuo ad altiores perducamus. Ita, cum posito $x=1$ sit

$$\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}}=\frac{m+1}{m}\int\frac{x^{m+1}dx}{\sqrt{(1-xx)}}$$

erit

$$\int\frac{dx}{\sqrt{(1-xx)}}=\frac{2}{1}\int\frac{xxdx}{\sqrt{(1-xx)}}=\frac{2\cdot 4}{1\cdot 3}\int\frac{x^4dx}{\sqrt{(1-xx)}}=\frac{2\cdot 4\cdot 6}{1\cdot 3\cdot 5}\int\frac{x^6dx}{\sqrt{(1-xx)}}\quad \text{etc.},$$

unde concludimus fore indefinite

$$\int\frac{dx}{\sqrt{(1-xx)}}=\frac{2\cdot 4\cdot 6\cdot 8\cdots\cdots 2i}{1\cdot 3\cdot 5\cdot 7\cdots\cdots(2i-1)}\int\frac{x^{2i}dx}{\sqrt{(1-xx)}}$$

atque adeo etiam, si pro *i* sumatur numerus infinitus. Nunc simili modo a formula $\int\frac{xdx}{\sqrt{(1-xx)}}$ ascendamus reperiemusque

$$\int\frac{xdx}{\sqrt{(1-xx)}}=\frac{3\cdot 5\cdot 7\cdot 9\cdots\cdot(2i+1)}{2\cdot 4\cdot 6\cdot 8\cdots\cdots 2i}\int\frac{x^{2i+1}dx}{\sqrt{(1-xx)}}$$

atque observo, si *i* sit numerus infinitus, formulas istas

$$\int\frac{x^{2i}dx}{\sqrt{(1-xx)}}\quad \text{et}\quad \int\frac{x^{2i+1}dx}{\sqrt{(1-xx)}}$$

rationem aequalitatis esse habituras. Ex reductione enim principali perspicuum est, si *m* sit numerus infinitus, fore

$$\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}}=\int\frac{x^{m+1}dx}{\sqrt{(1-xx)}}=\int\frac{x^{m+3}dx}{\sqrt{(1-xx)}}$$

atque adeo in genere

$$\int \frac{x^{m+\mu}dx}{\sqrt{(1-xx)}} = \int \frac{x^{m+\nu}dx}{\sqrt{(1-xx)}}$$

quantumvis magna fuerit differentia inter μ et ν, modo finita. Cum igitur sit

$$\int \frac{x^{2i}dx}{\sqrt{(1-xx)}} = \int \frac{x^{2i+1}dx}{\sqrt{(1-xx)}}$$

si ponamus

$$\frac{2\cdot4\cdot6\cdots2i}{1\cdot3\cdot5\cdots(2i-1)} = M \quad \text{et} \quad \frac{3\cdot5\cdot7\cdot9\cdots(2i+1)}{2\cdot4\cdot6\cdot8\cdots2i} = N,$$

erit

$$\int \frac{dx}{\sqrt{(1-xx)}} : \int \frac{xdx}{\sqrt{(1-xx)}} = M:N = \frac{M}{N}:1$$

posito $x=1$. At est

$$\int \frac{xdx}{\sqrt{(1-xx)}} = 1 \quad \text{et} \quad \int \frac{dx}{\sqrt{(1-xx)}} = \frac{\pi}{2},$$

unde colligitur

$$\int \frac{dx}{\sqrt{(1-xx)}} = \frac{M}{N}.$$

Quia producta *M* et *N* ex aequali factorum numero constant, si primum factorem $\frac{2}{1}$ producti *M* per primum factorem $\frac{3}{2}$ producti *N*, secundum $\frac{4}{3}$ illius per secundum $\frac{5}{4}$ huius et ita porro dividamus, fiet

$$\frac{M}{N} = \frac{2\cdot2}{1\cdot3}\cdot\frac{4\cdot4}{3\cdot5}\cdot\frac{6\cdot6}{5\cdot7}\cdot\frac{8\cdot8}{7\cdot9}\cdot \text{ etc.},$$

unde obtinemus pro casu $x=1$ per productum infinitum

$$\int \frac{dx}{\sqrt{(1-xx)}} = \frac{2\cdot2}{1\cdot3}\cdot\frac{4\cdot4}{3\cdot5}\cdot\frac{6\cdot6}{5\cdot7}\cdot\frac{8\cdot8}{7\cdot9}\cdot \text{ etc.} = \frac{\pi}{2}.$$

COROLLARIUM 1

357. Pro valore ergo ipsius π idem productum infinitum elicuimus, quod olim iam WALLISIUS invenerat et cuius veritatem in *Introductione* confirmavimus diversissimis viis incedentes; erit itaque

$$\pi = 2\cdot\frac{2\cdot2}{1\cdot3}\cdot\frac{4\cdot4}{3\cdot5}\cdot\frac{6\cdot6}{5\cdot7}\cdot\frac{8\cdot8}{7\cdot9}\cdot \text{ etc.}$$

COROLLARIUM 2

358. Nihil interest, quonam ordine singuli factores in hoc producto disponantur, dummodo nulli relinquantur. Ita aliquot ab initio seorsim sumendo reliqui ordine debito disponi possunt, veluti

$$\frac{\pi}{2}=\frac{2}{1}\cdot\frac{2\cdot4}{3\cdot3}\cdot\frac{4\cdot6}{5\cdot5}\cdot\frac{6\cdot8}{7\cdot7}\cdot\frac{8\cdot10}{9\cdot9}\cdot\text{etc.}$$

vel

$$\frac{\pi}{2}=\frac{2\cdot4}{1\cdot3}\cdot\frac{2\cdot6}{3\cdot5}\cdot\frac{4\cdot8}{5\cdot7}\cdot\frac{6\cdot10}{7\cdot9}\cdot\frac{8\cdot12}{9\cdot11}\cdot\text{etc.}$$

vel

$$\frac{\pi}{2}=\frac{2}{3}\cdot\frac{2\cdot4}{1\cdot5}\cdot\frac{4\cdot6}{3\cdot7}\cdot\frac{6\cdot8}{5\cdot9}\cdot\frac{8\cdot10}{7\cdot11}\cdot\text{etc.}$$

vel

$$\frac{\pi}{2}=\frac{2\cdot4}{3\cdot5}\cdot\frac{2\cdot6}{1\cdot7}\cdot\frac{4\cdot8}{3\cdot9}\cdot\frac{6\cdot10}{5\cdot11}\cdot\frac{8\cdot12}{7\cdot13}\cdot\text{etc.}$$

SCHOLION

359. Fundamentum ergo huius evolutionis in hoc consistit, quod valor integralis $\int\frac{x^{i+\alpha}dx}{\sqrt{(1-xx)}}$ denotante i numerum infinitum idem sit, utcunque numerus finitus α varietur. Atque hoc quidem ex reductione

$$\int\frac{x^{i-1}dx}{\sqrt{(1-xx)}}=\frac{i+1}{i}\int\frac{x^{i+1}dx}{\sqrt{(1-xx)}}$$

manifestum est, si pro α valores binario differentes assumantur. Deinde autem nullum est dubium, quin hoc integrale $\int\frac{x^{i+1}dx}{\sqrt{(1-xx)}}$ inter haec $\int\frac{x^{i}dx}{\sqrt{(1-xx)}}$ et $\int\frac{x^{i+2}dx}{\sqrt{(1-xx)}}$ quasi limites contineatur, qui cum sint inter se aequales, necesse est omnes formulas intermedias iisdem quoque esse aequales. Atque hoc latius patet ad formulas magis complicatas, ita ut denotante i numerum infinitum sit

$$\int\frac{x^{i+\alpha}dx}{\sqrt{(1-x^n)^k}}=\int\frac{x^{i}dx}{\sqrt{(1-x^n)^k}}.$$

Cum enim sit

$$\int\frac{x^{m+n-1}dx}{(1-x^n)^{\frac{n-k}{n}}}=\frac{m}{m+k}\int\frac{x^{m-1}dx}{\sqrt{(1-x^n)^{\frac{n-k}{n}}}},$$

hae formulae posito $m=\infty$ sunt aequales; unde illarum quoque aequalitas casibus, quibus $\alpha=n$ vel $\alpha=2n$ vel $\alpha=3n$ etc., perspicitur; sin autem α medium quempiam valorem teneat, formulae ipsius quoque valor medium quoddam tenere debet inter valores aequales ideoque ipsis erit aequalis. Hoc igitur principio stabilito sequens problema resolvere poterimus.

PROBLEMA 44

360. *Rationem horum duorum integralium* $\int x^{m-1}dx\left(1-x^{n}\right)^{\frac{k-n}{n}}$ *et* $\int x^{\mu-1}dx\left(1-x^{n}\right)^{\frac{k-n}{n}}$ *in casu x* = 1 *per productum infinitorum factorum exprimere.*

SOLUTIO

Cum sit

$$\int x^{m-1}dx\left(1-x^{n}\right)^{\frac{k-n}{n}}=\frac{m+k}{n}\int x^{m+n-1}dx\left(1-x^{n}\right)^{\frac{k-n}{n}}$$

casu $x = 1$, valor istius integralis ad integrale infinite remotum reducetur hoc modo

$$\int x^{m-1}dx\left(1-x^{n}\right)^{\frac{k-n}{n}}$$
$$=\frac{(m+k)(m+k+n)(m+k+2n)\cdots(\mathrm{m}+k+in)}{m\quad(m+n)\quad(m+2n)\cdots(m+in)}\int x^{m+in+n-1}dx\left(1-x^{n}\right)^{\frac{k-n}{n}},$$

ubi *i* numerum infinitum denotare assumimus. Simili autem modo pro altera formula proposita erit

$$\int x^{\mu-1}dx\left(1-x^{n}\right)^{\frac{k-n}{n}}$$
$$=\frac{(\mu+k)(\mu+k+n)(\mu+k+2n)\cdots(\mu+k+in)}{\mu\quad(\mu+n)\quad(\mu+2n)\cdots(\mu+in)}\int x^{\mu+in+n-1}dx\left(1-x^{n}\right)^{\frac{k-n}{n}},$$

atque hae postremae formulae integrales ob exponentes infinitos aequales erunt non obstante inaequalitate numerorum m et μ; tum vero bina haec producta infinita pari factorum numero constant. Quare si singuli per singulos, hoc est primus per primum, secundus per secundum [et ita porro] dividantur, ratio binorum integralium propositorum ita exprimetur

$$\frac{\int x^{m-1}dx\left(1-x^{n}\right)^{\frac{k-n}{n}}}{\int x^{\mu-1}dx\left(1-x^{n}\right)^{\frac{k-n}{n}}}=\frac{\mu(m+k)}{m(\mu+k)}\cdot\frac{(\mu+n)(m+k+n)}{(m+n)(\mu+k+n)}\cdot\frac{(\mu+2n)(m+k+2n)}{(m+2n)(\mu+k+2n)}\cdot\text{etc.},$$

si quidem ambo integralia ita determinentur, ut posito $x = 0$ evanescant, tum vero statuatur $x = 1$; litteris autem $m,\ \mu,\ n,\ k$ numeros positivos denotari necesse est.

COROLLARIUM 1

361. Si differentia numerorum m et μ aequetur multiplo ipsius n, in producto invento infiniti factores se destruunt relinqueturque factorum numerus finitus, uti, si $\mu=m+n$, habebitur

$$\frac{(m+n)(m+k)}{m(m+k+n)}\cdot\frac{(m+2n)(m+k+n)}{(m+n)(m+k+2n)}\cdot\frac{(m+3n)(m+k+2n)}{(m+2n)(m+k+3n)}\cdot\text{ etc.,}$$

quod reducitur ad $\frac{m+k}{m}$.

COROLLARIUM 2

362. Valor autem illius producti necessario est finitus, id quod tam ex formulis integralibus, quarum rationem exprimit, patet quam inde, quod in singulis factoribus numeratores et denominatores sunt alternatim maiores et minores.

COROLLARIUM 3

363. Si ponamus $m=1, \mu=3, n=4$ et $k=2$, erit

$$\frac{\int\frac{dx}{\sqrt{(1-x^4)}}}{\int\frac{xxdx}{\sqrt{(1-x^4)}}}=\frac{3\cdot3}{1\cdot5}\cdot\frac{7\cdot7}{5\cdot9}\cdot\frac{11\cdot11}{9\cdot13}\cdot\frac{15\cdot15}{13\cdot17}\cdot\text{etc.;}$$

supra autem invenimus productum harum binarum formularum esse = $\frac{\pi}{4}$

PROBLEMA 45

364. *Valorem huius integralis* $\int x^{m-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}$*, quem posito* $x = 1$ *recipit, per productum infinitum exprimere.*

SOLUTIO

Cum in problemate praecedente ratio huius integralis ad hoc alterum productum infinitum sit assignata,

$$\int x^{\mu-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}$$

in hoc exponens μ ita accipiatur, ut integrale exhiberi possit. Capiatur ergo $\mu=n$ et integrale fit

$$C-\frac{1}{k}\left(1-x^n\right)^{\frac{k}{n}}=\frac{1-\left(1-x^n\right)^{\frac{k}{n}}}{k}$$

ita determinatum, ut posito $x = 0$ evanescat; ponatur nunc, ut conditio postulat, $x = 1$, et quia hoc integrale erit = $\frac{1}{k}$, habebimus formulae propositae integrale casu $x = 1$ ita expressum

$$\int x^{m-1}dx\left(1-x^n\right)^{\frac{k-n}{n}}=\frac{1}{k}\cdot\frac{n(m+k)}{m(k+n)}\cdot\frac{2n(m+k+n)}{(m+n)(k+2n)}\cdot\frac{3n(m+k+2n)}{(m+2n)(k+3n)}\cdot\text{etc.,}$$

quod singulos factores partiendo ita repraesentari potest

$$\int x^{m-1}dx\left(1-x^n\right)^{\frac{k-n}{n}} = \frac{n}{mk}\cdot\frac{2n(m+k)}{(m+n)(k+n)}\cdot\frac{3n(m+k+n)}{(m+2n)(k+2n)}\cdot\frac{4n(m+k+2n)}{(m+3n)(k+3n)}\cdot\text{etc.},$$

COROLLARIUM 1

365. Cum in hac expressione litterae *m* et *k* sint permutabiles, sequitur etiam haec integralia posito $x = 1$ inter se esse aequalia

$$\int x^{m-1}dx\left(1-x^n\right)^{\frac{k-n}{n}} = \int x^{k-1}dx\left(1-x^n\right)^{\frac{m-n}{n}},$$

quam aequalitatem iam supra § 349 elicuimus.

COROLLARIUM 2

366. Cum formulae nostrae valor, si $m = n-k$, aequalis sit valori huius $\int\frac{z^{k-1}dz}{1+z^n}$ posito $z=\infty$, si ob $m+k=n$ statuamus $m=\frac{n+\alpha}{2}$ et $k=\frac{n-\alpha}{2}$, habebimus

$$\int\frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{n+\alpha}{2n}}} = \int\frac{x^{k-1}dx}{\left(1-x^n\right)^{\frac{n-\alpha}{2n}}} = \int\frac{z^{k-1}dz}{1+z^n} = \int\frac{z^{m-1}dz}{1+z^n}$$
$$= \frac{4n}{nn-\alpha\alpha}\cdot\frac{2\cdot 4nn}{9nn-\alpha\alpha}\cdot\frac{4\cdot 6nn}{25nn-\alpha\alpha}\cdot\frac{6\cdot 8nn}{49nn-\alpha\alpha}\cdot\text{etc.}$$

Quod productum etiam hoc modo exponi potest

$$\frac{2}{n-\alpha}\cdot\frac{2n\cdot 2n}{(n+\alpha)(3n-\alpha)}\cdot\frac{4n\cdot 4n}{(3n+\alpha)(5n-\alpha)}\cdot\frac{6n\cdot 6n}{(5n+\alpha)(7n-\alpha)}\cdot\text{etc.},$$

quod ergo etiam exprimit valorem ipsius $\frac{\pi}{n\sin.\frac{m\pi}{n}} = \frac{\pi}{n\cos.\frac{\alpha\pi}{2n}}$ per § 351.

COROLLARIUM 3

367. Vel si simpliciter ponamus $k = n-m$, fiet

$$\int\frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{m}{n}}} = \int\frac{x^{n-m-1}dx}{\left(1-x^n\right)^{\frac{n-m}{n}}} = \int\frac{z^{m-1}dz}{1+z^n} = \int\frac{z^{n-m-1}dz}{1+z^n}$$
$$= \frac{1}{n-m}\cdot\frac{nn}{m(2n-m)}\cdot\frac{4nn}{(n+m)(3n-m)}\cdot\frac{9nn}{(2n+m)(4n-m)}\cdot\text{etc.},$$

quae ex forma primum inventa oritur. Haec ergo aequalitas subsistit, si ponatur $x=1$ et $z=\infty$.

SCHOLION 1

368. In *Introductione* autem pro multiplicatione angulorum inveneram

$$sin.\frac{m\pi}{n}=\frac{m\pi}{n}\left(1-\frac{mm}{nn}\right)\left(1-\frac{mm}{4nn}\right)\left(1-\frac{mm}{9nn}\right)\left(1-\frac{mm}{16nn}\right)\cdot\text{etc.},$$

et cum

$$sin.\frac{(n-m)\pi}{n}=sin.\frac{m\pi}{n},$$

ob $n-m=$ k erit etiam

$$sin.\frac{m\pi}{n}=\frac{k\pi}{n}\left(1-\frac{kk}{nn}\right)\left(1-\frac{kk}{4nn}\right)\left(1-\frac{kk}{9nn}\right)\left(1-\frac{kk}{16nn}\right)\cdot\text{etc.},$$

quae reducitur ad hanc formam

$$sin.\frac{m\pi}{n}=\frac{k\pi}{n}\cdot\frac{(n-k)(n+k)}{nn}\cdot\frac{(2n-k)(2n+k)}{4nn}\cdot\frac{(3n-k)(3n+k)}{9nn}.\ \text{etc.}$$

et pro k suo valore restituto

$$sin.\frac{m\pi}{n}=\frac{\pi}{n}(n-m)\cdot\frac{m(2n-m)}{nn}\cdot\frac{(n+m)(3n-m)}{4nn}\cdot\frac{(2n+m)(4n-m)}{9nn}\cdot\text{etc.},$$

unde manifesto pro $\frac{\pi}{n\,sin.\frac{m\pi}{n}}$ idem reperitur productum, quod valorem nostrorum integralium exprimit, sicque novam habemus demonstrationem pro theoremate illo eximio supra [§ 351] per multas ambages evicto esse

$$\int\frac{x^{m-1}dx}{\left(1-x^n\right)^{\frac{m}{n}}}=\int\frac{x^{n-m-1}dx}{\left(1-x^n\right)^{\frac{n-m}{n}}}=\int\frac{z^{m-1}dz}{1+z^n}=\int\frac{z^{n-m-1}dz}{1+z^n}=\frac{\pi}{n\,sin.\frac{m\pi}{n}}.$$

SCHOLION 2

369. Quo nostra formula latius pateat, ponamus $\frac{k}{n}=\frac{\mu}{\nu}$ seu $k=\frac{\mu n}{\nu}$ et nanciscemur

$$\int x^{m-1}dx\left(1-x^n\right)^{\frac{\mu}{\nu}-1}=\frac{\nu}{m\mu}\cdot\frac{2(m\nu+n\mu)}{(m+n)(\mu+\nu)}\cdot\frac{3\left(m\nu+n(\mu+\nu)\right)}{(m+2n)(\mu+2\nu)}\cdot\frac{4\left(m\nu+n(\mu+2\nu)\right)}{(m+3n)(\mu+3\nu)}\cdot\text{etc.}$$
$$=\frac{\nu}{m\mu}\cdot\frac{2(m\nu+n\mu)}{(m+n)(\mu+\nu)}\cdot\frac{3(m\nu+n\mu+n\nu)}{(m+2n)(\mu+2\nu)}\cdot\frac{4(m\nu+n\mu+2n\nu)}{(m+3n)(\mu+3\nu)}\cdot\frac{5(m\nu+n\mu+3n\nu)}{(m+4n)(\mu+4\nu)}\cdot\text{etc.}$$

in qua expressione litterae m, n et μ, v sunt permutabiles praeterquam in primo factore, qui cum reliquis lege continuitatis non connectitur; ac si per n multiplicemus, permutabilitas erit perfecta, unde concludimus fore

$$n\int x^{m-1}dx\left(1-x^{n}\right)^{\frac{\mu}{v}-1}=v\int x^{\mu-1}dx\left(1-x^{v}\right)^{\frac{m}{n}-1}$$

quae aequalitas casu $v=n$ ad supra observatam reducitur. Caeterum iuvabit casus praecipuos perpendisse, quos ex valoribus μ et v desumamus.

EXEMPLUM 1

370. Sit $\mu=1$ et $v=2$ fietque

$$\int\frac{x^{m-1}dx}{\sqrt{\left(1-x^{n}\right)}}=\frac{2}{m}\cdot\frac{2(2m+n)}{3(m+n)}\cdot\frac{3(2m+3n)}{5(m+2n)}\cdot\frac{4(2m+5n)}{7(m+3n)}\cdot\text{etc.}=\frac{2}{n}\int\frac{dx}{\sqrt[n]{\left(1-x^{2}\right)^{n-m}}},$$

quae expressio ita commodius repraesentatur

$$\int\frac{x^{m-1}dx}{\sqrt{\left(1-x^{n}\right)}}=\frac{2}{m}\cdot\frac{4(2m+n)}{3(2m+2n)}\cdot\frac{6(2m+3n)}{5(2m+4n)}\cdot\frac{8(2m+5n)}{7(2m+6n)}\cdot\text{etc.},$$

unde sequentes casus specialissimi deducuntur

$$\int\frac{dx}{\sqrt{(1-xx)}}=2\cdot\frac{2\cdot4}{3\cdot3}\cdot\frac{4\cdot6}{5\cdot5}\cdot\frac{6\cdot8}{7\cdot7}\cdot\text{etc.}\qquad=\int\frac{dx}{\sqrt{(1-xx)}},$$

$$\int\frac{dx}{\sqrt{\left(1-x^{3}\right)}}=2\cdot\frac{4\cdot5}{3\cdot8}\cdot\frac{6\cdot11}{5\cdot14}\cdot\frac{8\cdot17}{7\cdot20}\cdot\frac{10\cdot23}{9\cdot26}\cdot\text{etc.}\qquad=\frac{2}{3}\int\frac{dx}{\sqrt[3]{\left(1-x^{2}\right)^{2}}},$$

$$\int\frac{xdx}{\sqrt{\left(1-x^{3}\right)}}=1\cdot\frac{4\cdot7}{3\cdot10}\cdot\frac{6\cdot13}{5\cdot16}\cdot\frac{8\cdot19}{7\cdot22}\cdot\frac{10\cdot25}{9\cdot28}\cdot\text{etc.}\qquad=\frac{2}{3}\int\frac{dx}{\sqrt[3]{\left(1-x^{2}\right)}},$$

$$\int\frac{dx}{\sqrt{\left(1-x^{4}\right)}}=2\cdot\frac{4\cdot3}{3\cdot5}\cdot\frac{6\cdot7}{5\cdot9}\cdot\frac{8\cdot11}{7\cdot13}\cdot\frac{10\cdot15}{9\cdot17}\cdot\text{etc.}\qquad=\frac{1}{2}\int\frac{dx}{\sqrt[4]{\left(1-x^{2}\right)^{3}}},$$

$$\int\frac{xdx}{\sqrt{\left(1-x^{4}\right)}}=1\cdot\frac{4\cdot4}{3\cdot6}\cdot\frac{6\cdot8}{5\cdot10}\cdot\frac{8\cdot12}{7\cdot14}\cdot\frac{10\cdot16}{9\cdot18}\cdot\text{etc.}\qquad=\frac{1}{2}\int\frac{dx}{\sqrt{\left(1-x^{2}\right)}},$$

sive $\qquad =1\cdot\frac{2\cdot4}{3\cdot3}\cdot\frac{4\cdot6}{5\cdot5}\cdot\frac{6\cdot8}{7\cdot7}\cdot\frac{8\cdot10}{9\cdot9}\cdot\text{etc.},$

$$\int\frac{xxdx}{\sqrt{\left(1-x^{4}\right)}}=\frac{2}{3}\cdot\frac{4\cdot5}{3\cdot7}\cdot\frac{6\cdot9}{5\cdot11}\cdot\frac{8\cdot13}{7\cdot15}\cdot\frac{10\cdot17}{9\cdot19}\cdot\text{etc.}\qquad=\frac{1}{2}\int\frac{dx}{\sqrt[4]{(1-xx)}},$$

$$\int\frac{x^{3}dx}{\sqrt{\left(1-x^{4}\right)}}=\frac{2}{4}\cdot\frac{4\cdot6}{3\cdot8}\cdot\frac{6\cdot10}{5\cdot12}\cdot\frac{8\cdot14}{7\cdot16}\cdot\frac{10\cdot18}{9\cdot20}\cdot\text{etc.}\qquad=\frac{1}{2}.$$

EXEMPLUM 2

371. Sit $\mu=1$ et $\nu=3$ fietque

$$\int\frac{x^{m-1}dx}{\sqrt[3]{\left(1-x^{n}\right)^{2}}}=\frac{3}{m}\cdot\frac{2(3m+n)}{4(m+n)}\cdot\frac{3(3m+4n)}{7(m+2n)}\cdot\frac{4(3m+7n)}{10(m+3n)}\cdot\text{etc.}=\frac{3}{n}\int\frac{dx}{\sqrt[n]{\left(1-x^{3}\right)^{n-m}}},$$

unde sequentes casus specialissimi deducuntur

$$\int\frac{dx}{\sqrt[3]{\left(1-x^{2}\right)^{2}}}=\frac{3}{1}\cdot\frac{2\cdot5}{4\cdot3}\cdot\frac{3\cdot11}{7\cdot5}\cdot\frac{4\cdot17}{10\cdot7}\cdot\frac{5\cdot23}{13\cdot7}\cdot\text{etc.}\qquad=\frac{3}{2}\int\frac{dx}{\sqrt{\left(1-x^{3}\right)}},$$

$$\int\frac{dx}{\sqrt[3]{\left(1-x^{3}\right)^{2}}}=\frac{3}{1}\cdot\frac{2\cdot6}{4\cdot4}\cdot\frac{3\cdot15}{7\cdot7}\cdot\frac{4\cdot24}{10\cdot10}\cdot\frac{5\cdot33}{13\cdot13}\cdot\text{etc.}\quad=\int\frac{dx}{\sqrt[3]{\left(1-x^{3}\right)^{2}}},$$

sive
$$=\frac{3}{1}\cdot\frac{2\cdot6}{4\cdot4}\cdot\frac{5\cdot9}{7\cdot7}\cdot\frac{8\cdot12}{10\cdot10}\cdot\frac{11\cdot15}{13\cdot13}\cdot\text{etc.}$$

$$\int\frac{xdx}{\sqrt[3]{\left(1-x^{3}\right)^{2}}}=\frac{3}{2}\cdot\frac{2\cdot9}{4\cdot5}\cdot\frac{3\cdot18}{7\cdot8}\cdot\frac{4\cdot27}{10\cdot11}\cdot\frac{5\cdot36}{13\cdot14}\cdot\text{etc.}\quad=\int\frac{dx}{\sqrt[3]{\left(1-x^{3}\right)}},$$

sive
$$=\frac{3}{2}\cdot\frac{3\cdot6}{4\cdot5}\cdot\frac{6\cdot9}{7\cdot8}\cdot\frac{9\cdot12}{10\cdot11}\cdot\frac{12\cdot15}{13\cdot14}\cdot\text{etc.}$$

$$\int\frac{dx}{\sqrt[3]{\left(1-x^{4}\right)^{2}}}=\frac{3}{1}\cdot\frac{2\cdot7}{4\cdot5}\cdot\frac{3\cdot19}{7\cdot9}\cdot\frac{4\cdot31}{10\cdot13}\cdot\frac{5\cdot43}{13\cdot17}\cdot\text{etc.}\quad=\frac{3}{4}\int\frac{dx}{\sqrt[4]{\left(1-x^{3}\right)^{3}}},$$

$$\int\frac{xxdx}{\sqrt[3]{\left(1-x^{4}\right)^{2}}}=1\cdot\frac{2\cdot13}{4\cdot7}\cdot\frac{3\cdot25}{7\cdot11}\cdot\frac{4\cdot37}{10\cdot15}\cdot\frac{5\cdot49}{13\cdot19}\cdot\text{etc.}\quad=\frac{3}{4}\int\frac{dx}{\sqrt[4]{\left(1-x^{3}\right)}}.$$

EXEMPLUM 3

372. Sit $\mu=2$ et $\nu=3$ fietque

$$\int\frac{x^{m-1}dx}{\sqrt[3]{\left(1-x^{n}\right)}}=\frac{3}{2m}\cdot\frac{2(3m+2n)}{5(m+n)}\cdot\frac{3(3m+5n)}{8(m+2n)}\cdot\frac{4(3m+8n)}{11(m+3n)}\cdot\text{etc.}=\frac{3}{n}\int\frac{xdx}{\sqrt[n]{\left(1-x^{3}\right)^{n-m}}},$$

unde sequentes casus speciales deducuntur

$$\int\frac{dx}{\sqrt[3]{(1-x^2)}}=\frac{3}{2}\cdot\frac{2\cdot7}{5\cdot3}\cdot\frac{3\cdot13}{8\cdot5}\cdot\frac{4\cdot19}{11\cdot7}\cdot\frac{5\cdot25}{14\cdot9}\cdot\text{etc.}\qquad=\frac{3}{2}\int\frac{xdx}{\sqrt{(1-x^3)}},$$

$$\int\frac{dx}{\sqrt[3]{(1-x^3)}}=\frac{3}{2}\cdot\frac{2\cdot9}{5\cdot4}\cdot\frac{3\cdot18}{8\cdot7}\cdot\frac{4\cdot27}{11\cdot10}\cdot\frac{5\cdot36}{14\cdot13}\cdot\text{etc.}\qquad=\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}$$

sive $$=\frac{3\cdot3}{2\cdot4}\cdot\frac{6\cdot6}{5\cdot7}\cdot\frac{9\cdot9}{8\cdot10}\cdot\frac{12\cdot12}{11\cdot13}\cdot\text{etc.},$$

$$\int\frac{xdx}{\sqrt[3]{(1-x^3)}}=\frac{3}{4}\cdot\frac{2\cdot12}{5\cdot5}\cdot\frac{3\cdot21}{8\cdot8}\cdot\frac{4\cdot30}{11\cdot11}\cdot\frac{5\cdot39}{14\cdot14}\cdot\text{etc.}\qquad=\int\frac{xdx}{\sqrt[3]{(1-x^3)}}$$

sive $$=\frac{3}{4}\cdot\frac{4\cdot6}{5\cdot5}\cdot\frac{7\cdot9}{8\cdot8}\cdot\frac{10\cdot12}{11\cdot11}\cdot\frac{13\cdot15}{14\cdot14}\cdot\text{etc.},$$

$$\int\frac{dx}{\sqrt[3]{(1-x^4)}}=\frac{3}{2}\cdot\frac{2\cdot11}{5\cdot5}\cdot\frac{3\cdot23}{8\cdot9}\cdot\frac{4\cdot35}{11\cdot13}\cdot\frac{5\cdot47}{14\cdot17}\cdot\text{etc.}\qquad=\frac{3}{4}\int\frac{xdx}{\sqrt[4]{(1-x^3)^3}},$$

$$\int\frac{x^2dx}{\sqrt[3]{(1-x^4)}}=\frac{1}{2}\cdot\frac{2\cdot17}{5\cdot7}\cdot\frac{3\cdot29}{8\cdot11}\cdot\frac{4\cdot41}{11\cdot15}\cdot\frac{5\cdot53}{14\cdot19}\cdot\text{etc.}\qquad=\frac{3}{4}\int\frac{xdx}{\sqrt[4]{(1-x^3)}}.$$

EXEMPLUM 4

373. Sit $\mu=1$ et $\nu=4$ fietque

$$\int\frac{x^{m-1}dx}{\sqrt[4]{(1-x^n)^3}}=\frac{4}{m}\cdot\frac{2(4m+n)}{5(m+n)}\cdot\frac{3(4m+5n)}{9(m+2n)}\cdot\frac{4(4m+9n)}{13(m+3n)}\cdot\text{etc.}=\frac{4}{n}\int\frac{dx}{\sqrt[n]{(1-x^4)^{n-m}}},$$

unde sequentes casus speciales prodeunt

$$\int \frac{dx}{\sqrt[4]{(1-x^2)^3}} = \frac{4}{1}\cdot\frac{2\cdot 6}{5\cdot 3}\cdot\frac{3\cdot 14}{9\cdot 5}\cdot\frac{4\cdot 22}{13\cdot 7}\cdot\frac{5\cdot 30}{17\cdot 9}\cdot\text{etc.} \qquad = 2\int\frac{dx}{\sqrt{(1-x^4)}}$$

seu

$$= \frac{4}{1}\cdot\frac{4\cdot 3}{3\cdot 5}\cdot\frac{6\cdot 7}{5\cdot 9}\cdot\frac{8\cdot 11}{7\cdot 13}\cdot\frac{10\cdot 15}{9\cdot 17}\cdot\text{etc.},$$

$$\int \frac{dx}{\sqrt[4]{(1-x^3)^3}} = \frac{4}{1}\cdot\frac{2\cdot 7}{5\cdot 4}\cdot\frac{3\cdot 19}{9\cdot 7}\cdot\frac{4\cdot 31}{13\cdot 10}\cdot\frac{5\cdot 43}{17\cdot 13}\cdot\text{etc.} \qquad = \frac{4}{3}\int\frac{xdx}{\sqrt[3]{(1-x^3)^2}}$$

$$\int \frac{xdx}{\sqrt[4]{(1-x^3)^3}} = \cdot\frac{2\cdot 11}{5\cdot 5}\cdot\frac{3\cdot 23}{9\cdot 8}\cdot\frac{4\cdot 35}{13\cdot 11}\cdot\frac{5\cdot 47}{17\cdot 14}\cdot\text{etc.} \qquad = \frac{4}{3}\int\frac{xdx}{\sqrt[3]{(1-x^3)}}$$

$$\int \frac{dx}{\sqrt[4]{(1-x^4)^3}} = \frac{4}{1}\cdot\frac{2\cdot 8}{5\cdot 5}\cdot\frac{3\cdot 24}{9\cdot 9}\cdot\frac{4\cdot 40}{13\cdot 13}\cdot\frac{5\cdot 56}{17\cdot 17}\cdot\text{etc.} \qquad = \int\frac{dx}{\sqrt[4]{(1-x^4)^3}}$$

seu

$$= \frac{4}{1}\cdot\frac{4\cdot 4}{5\cdot 5}\cdot\frac{6\cdot 12}{9\cdot 9}\cdot\frac{8\cdot 20}{13\cdot 13}\cdot\frac{10\cdot 28}{17\cdot 17}\cdot\text{etc.}$$

seu

$$= \frac{4}{1}\cdot\frac{2\cdot 8}{5\cdot 5}\cdot\frac{6\cdot 12}{9\cdot 9}\cdot\frac{10\cdot 16}{13\cdot 13}\cdot\frac{14\cdot 20}{17\cdot 17}\cdot\text{etc.},$$

$$\int \frac{xxdx}{\sqrt[3]{(1-x^4)^3}} = \frac{4}{3}\cdot\frac{2\cdot 16}{5\cdot 7}\cdot\frac{3\cdot 32}{9\cdot 11}\cdot\frac{4\cdot 48}{13\cdot 15}\cdot\frac{5\cdot 64}{17\cdot 19}\cdot\text{etc.} \qquad = \int\frac{dx}{\sqrt[4]{(1-x^4)}}$$

seu

$$= \frac{4}{3}\cdot\frac{4\cdot 8}{5\cdot 7}\cdot\frac{6\cdot 16}{9\cdot 11}\cdot\frac{8\cdot 24}{13\cdot 15}\cdot\frac{10\cdot 32}{17\cdot 19}\cdot\text{etc.}$$

seu

$$= \frac{4}{3}\cdot\frac{4\cdot 8}{5\cdot 7}\cdot\frac{8\cdot 12}{9\cdot 11}\cdot\frac{12\cdot 16}{13\cdot 15}\cdot\frac{16\cdot 20}{17\cdot 19}\cdot\text{etc.}$$

Atque in his et praecedentibus iam casus $\mu = 3$ et $\nu = 4$ est contentus.

SCHOLION

374. Caeterum hae formulae, in quas litteras μ et ν introduxi, latius non patent quam primum consideratae; series enim pendent a binis fractionibus $\frac{m}{n}$ et $\frac{\mu}{\nu}$, quae cum semper ad communem denominatorem revocari queant, formulas

$$\int \frac{x^{m-1}dx}{\sqrt[n]{(1-x^n)^{n-k}}} = \int \frac{x^{k-1}dx}{\sqrt[n]{(1-x^n)^{n-m}}}$$

perpendisse sufficiet. Cum igitur earum valor casu $x = 1$ aequetur huic producto

$$\frac{1}{k}\cdot\frac{n(m+k)}{m(k+n)}\cdot\frac{2n(m+k+n)}{(m+n)(k+2n)}\cdot\frac{3n(m+k+2n)}{(m+2n)(k+3n)}.\text{etc},$$

si in singulis membris factores numeratorum permutemus et membra aliter partiamur, idem productum hanc induet formam

$$\frac{(m+k)}{mk}\cdot\frac{n(m+k+n)}{(m+n)(k+n)}\cdot\frac{2n(m+k+2n)}{(m+2n)(k+2n)}\cdot\frac{3n(m+k+3n)}{(m+3n)(k+3n)}.\ \text{etc.},$$

quae ad memoriam magis accommodata videtur. Simili modo cum sit

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}=\int\frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^{n-p}}}$$
$$=\frac{(p+q)}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\frac{3n(p+q+3n)}{(p+3n)(q+3n)}.\ \text{etc.},$$

illam formam per hanc dividendo erit

$$\frac{\int x^{m-1}dx(1-x^n)^{\frac{k-n}{n}}}{\int x^{p-1}dx(1-x^n)^{\frac{q-n}{n}}}$$
$$=\frac{pq(m+k)}{mk(p+q)}\cdot\frac{(p+n)(q+n)(m+k+n)}{(m+n)(k+n)(p+q+n)}\cdot\frac{(p+2n)(q+2n)(m+k+2n)}{(m+2n)(k+2n)(p+q+2n)}\cdot\ \text{etc.},$$

cuius omnia membra eadem lege continentur. Hinc autem eximiae comparationes buiusmodi formularum deduci possunt, quae quo facilius commemoran queant, brevitatis causa sequenti scriptionis compendio utar.

DEFINITIO

375. *Formulae integralis* $\int x^{p-1}dx(1-x^n)^{\frac{q-n}{n}}$ *valorem, quem posito* $x = 1$ *recipit, brevitatis gratia hoc signo* $\left(\frac{p}{q}\right)$ *indicemus, ubi quidem exponentem n, quem in comparatione plurium huiusmodi formularum eundem esse assumo, subintelligi oportet.*

COROLLARIUM 1

376. Primum igitur patet esse $\left(\frac{p}{q}\right)=\left(\frac{q}{p}\right)$ et utramque formulam esse

$$\frac{(p+q)}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\frac{2n(p+q+2n)}{(p+2n)(q+2n)}\cdot\text{etc.},$$

quorum membrorum progressio est manifesta, dum singuli factores tam numeratoris quam denominatoris continuo eodem numero *n* augentur, ita ut ex cognito primo membro sequentia facile formentur.

COROLLARIUM 2

377. Deinde si sit $p = n$, ob formulam integrabilem liquet esse

$$\left(\frac{n}{q}\right) = \left(\frac{q}{n}\right) = \frac{1}{q}, \quad \text{item} \quad \left(\frac{p}{n}\right) = \left(\frac{n}{p}\right) = \frac{1}{p}$$

Porro cum

$$\int x^{p-1}dx\left(1-x^n\right)^{\frac{-p}{n}} = \frac{\pi}{n\sin.\frac{p\pi}{n}},$$

ob $q-n=-p$ seu $p+q=n$ erit

$$\left(\frac{p}{n-p}\right) = \left(\frac{n-p}{p}\right) = \frac{\pi}{n\sin.\frac{p\pi}{n}}.$$

Quare valor formulae $\left(\frac{p}{q}\right)$ absolute assignari potest, quoties fuerit vel $p = n$ vel $q = n$ vel $p+q = n$.

COROLLARIUM 3

378. Quia etiam [§ 345] invenimus hanc reductionem

$$\int x^{p+n-1}dx\left(1-x^n\right)^{\frac{q-n}{n}} = \frac{p}{p+q}\int x^{p-1}dx\left(1-x^n\right)^{\frac{q-n}{n}},$$

sequitur fore

$$\left(\frac{p+n}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right)$$

hincque

$$\left(\frac{p}{q}\right) = \left(\frac{q}{p}\right) = \frac{p-n}{p+q-n}\left(\frac{p-n}{q}\right) = \frac{q-n}{p+q-n}\left(\frac{p}{q-n}\right),$$

tum vero etiam

$$\left(\frac{p}{q}\right) = \frac{(p-n)(q-n)}{(p+q-n)(p+q-2n)}\cdot\left(\frac{p-n}{q-n}\right),$$

unde semper numeri p et q infra n deprimi possunt.

PROBLEMA 46

379. *Invenire diversa producta ex binis huiusmodi formulis, quae inter se sint aequalia.*

SOLUTIO

Quaerantur ergo numeri a, b, c, d et $p, q, r,$ s, ut fiat

$$\left(\frac{a}{b}\right)\left(\frac{c}{d}\right) = \left(\frac{p}{q}\right)\left(\frac{r}{s}\right),$$

quod, cum sit

$$\left(\frac{a}{b}\right)=\frac{a+b}{ab}\cdot\frac{n(a+b+n)}{(a+n)(b+n)}\cdot\text{etc.},\qquad \left(\frac{c}{d}\right)=\frac{c+d}{cd}\cdot\frac{n(c+d+n)}{(c+n)(d+n)}\cdot\text{etc.},$$
$$\left(\frac{p}{q}\right)=\frac{p+q}{pq}\cdot\frac{n(p+q+n)}{(p+n)(q+n)}\cdot\text{etc.},\qquad \left(\frac{r}{s}\right)=\frac{r+s}{rs}\cdot\frac{n(r+s+n)}{(r+n)(s+n)}\cdot\text{etc.},$$

eveniet, si fuerit

$$\frac{(a+b)(c+d)}{abcd}=\frac{(p+q)(r+s)}{pqrs}$$

seu

$$abcd(p+q)(r+s)=pqrs(a+b)(c+d),$$

ita ut, cum utrinque sex sint factores, singuli singulis sint aequales. Ex quaternis ergo *abcd* et *pqrs* binos ad minimum aequales esse oportet; sit itaque $s=d$ efficique oportet

$$abc(p+q)(r+d)=pqr(a+b)(c+d).$$

I. Sumatur alter factor *r;* qui cum ipsi *c* aequari nequeat, quia alioquin fieret $\left(\frac{c}{d}\right)=\left(\frac{r}{s}\right)$, statuatur $r = b$, ut fiat

$$ac(p+q)(b+d)=pq(a+b)(c+d)\ ;$$

hic neque *p* neque *q* ipsi $p+q$ aequari potest, poni ergo debet:

1) Vel $p+q=a+b$, ut sit $ac(b+d)=pq(c+d)$, quia neque *c* neque $b+d$ ipsi $c+d$ aequari potest; fieret enim vel $d=0$ vel $b=c$ et $\left(\frac{r}{s}\right)=\left(\frac{c}{d}\right)$; relinquitur $a=c+d$ et $pq=c(b+d)$ ideoque $p=b+d$ et $q=c$, unde conficitur

$$\left(\frac{c+d}{b}\right)\left(\frac{c}{d}\right)=\left(\frac{b+d}{c}\right)\left(\frac{b}{d}\right).$$

2) Vel $p+q=c+d$, ergo $ac(b+d)=pq(a+b)$; hic *c* neque ipsi *p* neque *q* aequari potest; fieret enim $\left(\frac{p}{q}\right)=\left(\frac{c}{d}\right)$; unde fiat $c=a+b$, ut sit $pq=a(b+d)$, ergo $p=a$, $q=b+d$, $r=b$, $s=d$, consequenter

$$\left(\frac{a}{b}\right)\left(\frac{a+b}{d}\right)=\left(\frac{b+d}{a}\right)\left(\frac{b}{d}\right).$$

II. Quia $r=a$ non differt a praecedenti ob *a* et *b* permutabiles, statuatur $r=p+q$ fietque $abc(d+p+q)=pq(a+b)(c+d)$. Quoniam *r* ipsi *c* aequari nequit, factor $d+p+q$ neque ipsi *p* neque *q* neque $c+d$ aequalis poni potest; relinquitur ergo

$d+p+q=a+b$ et $abc=pq(c+d)$; ubi, quia c ipsi $c+d$ aequari nequit ac p et q pari conditione gaudent, fiat $p=c$; erit $q=a+b-c-d$ et $ab=(c+d)(a+b-c-d)$, unde $a=c+d$, $q=b, p=c,$ $r=b+c,$ $s=d$, sicque conficitur

$$\left(\frac{c+d}{b}\right)\left(\frac{c}{d}\right)=\left(\frac{c}{b}\right)\left(\frac{b+c}{d}\right).$$

COROLLARIUM 1

380. Hae solutiones eodem fere redeunt indeque tria producta binarum formularum aequalia eruuntur

$$\left(\frac{c}{d}\right)\left(\frac{c+d}{b}\right)=\left(\frac{c}{b}\right)\left(\frac{b+c}{d}\right)=\left(\frac{b}{d}\right)\left(\frac{b+d}{c}\right)$$

vel in litteris *p, q, r*

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right).$$

COROLLARUM 2

381. Si hae formulae in producta infinita evolvantur, reperietur

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\frac{p+q+r}{pqr}\cdot\frac{nn(p+q+r+n)}{(p+n)(q+n)(r+n)}\cdot\frac{4nn(p+q+r+2n)}{(p+2n)(q+2n)(r+2n)}\cdot\text{etc.},$$

unde patet tres litteras *p, q, r* utcunque inter se permutari posse, atque hinc ternas illas formulas concludere licet.

COROLLARIUM 3

382. Restituamus ipsas formulas integrales et sequentia tria producta erunt inter se aequalia

$$\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}\cdot\int\frac{x^{p+q-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}=\int\frac{x^{q-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{q+r-1}dx}{\sqrt[n]{(1-x^n)^{n-p}}}$$
$$=\int\frac{x^{p-1}dx}{\sqrt[n]{(1-x^n)^{n-r}}}\cdot\int\frac{x^{p+r-1}dx}{\sqrt[n]{(1-x^n)^{n-q}}}.$$

COROLLARIUM 4

383. Hic casus notatu dignus, quo $p+q=n$; tum enim ob

$$\left(\frac{p+q}{r}\right)=\left(\frac{n}{r}\right)=\left(\frac{1}{r}\right) \text{ et } \left(\frac{p}{q}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}$$

haec tria producta fient $=\frac{\pi}{nr\sin.\frac{p\pi}{n}}$. Erit scilicet

$$\int\frac{x^{n-p-1}dx}{\sqrt[n]{\left(1-x^{n}\right)^{n-r}}}\cdot\int\frac{x^{n-p+r-1}dx}{\sqrt[n]{\left(1-x^{n}\right)^{n-p}}}=\int\frac{x^{p-1}dx}{\sqrt[n]{\left(1-x^{n}\right)^{n-r}}}\cdot\int\frac{x^{p+r-1}dx}{\sqrt[n]{\left(1-x^{n}\right)^{p}}}=\frac{\pi}{nr\sin.\frac{p\pi}{n}}.$$

SCHOLION

384. Triplex ista proprietas productorum ex binis formulis maxime est notatu digna ac pro variis numeris loco p, q, r substituendis obtinebuntur sequentes aequalitates speciales

p	q	r	
1	1	2	$\left(\frac{1}{1}\right)\left(\frac{2}{2}\right)=\left(\frac{2}{1}\right)\left(\frac{3}{1}\right)$
1	2	2	$\left(\frac{2}{1}\right)\left(\frac{3}{2}\right)=\left(\frac{2}{2}\right)\left(\frac{4}{1}\right)$
1	2	3	$\left(\frac{2}{1}\right)\left(\frac{3}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{1}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{2}\right)$
1	1	3	$\left(\frac{1}{1}\right)\left(\frac{3}{2}\right)=\left(\frac{3}{1}\right)\left(\frac{4}{1}\right)$
2	2	3	$\left(\frac{2}{2}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{2}\right)\left(\frac{5}{2}\right)$
1	3	3	$\left(\frac{3}{1}\right)\left(\frac{4}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{1}\right)$
2	3	3	$\left(\frac{3}{2}\right)\left(\frac{5}{3}\right)=\left(\frac{3}{3}\right)\left(\frac{6}{2}\right)$
1	1	4	$\left(\frac{1}{1}\right)\left(\frac{4}{2}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{1}\right)$
1	2	4	$\left(\frac{2}{1}\right)\left(\frac{4}{3}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{1}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{2}\right)$
1	3	4	$\left(\frac{3}{1}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{1}\right)\left(\frac{5}{3}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{1}\right)$
1	4	4	$\left(\frac{4}{1}\right)\left(\frac{5}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{1}\right)$
2	2	4	$\left(\frac{2}{2}\right)\left(\frac{4}{4}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{2}\right)$
2	3	4	$\left(\frac{3}{2}\right)\left(\frac{5}{4}\right)=\left(\frac{4}{3}\right)\left(\frac{7}{2}\right)=\left(\frac{4}{2}\right)\left(\frac{6}{3}\right)$
2	4	4	$\left(\frac{4}{2}\right)\left(\frac{6}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{2}\right)$
3	3	4	$\left(\frac{3}{3}\right)\left(\frac{6}{4}\right)=\left(\frac{4}{8}\right)\left(\frac{7}{3}\right)$
3	4	4	$\left(\frac{4}{3}\right)\left(\frac{7}{4}\right)=\left(\frac{4}{4}\right)\left(\frac{8}{3}\right).$

Quae formulae pro omnibus numeris n valent, ac si numeri maiores quam n occurrant, eos ad minores reduci posse supra vidimus.

PROBLEMA 47

385. *Invenire producta diversa ex ternis huiusmodi formulis, quae inter se sint aequalia.*

SOLUTIO

Consideretur productum $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)$, quod evolutum praebet

$$\frac{p+q+r+s}{pqrs}\cdot\frac{n^3(p+q+r+s+n)}{(p+n)(q+n)(r+n)(s+n)}\cdot\text{etc.},$$

quod eundem valorem retinere evidens est, quomodocunque quatuor litterae inter se commutentur. Tum vero eadem evolutio prodit ex hoc producto $\left(\frac{p}{q}\right)\left(\frac{r}{s}\right)\left(\frac{p+q}{r+s}\right)$, ubi eadem permutatio locum habet. Aequalia ergo sunt inter se omnia haec producta

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right),\quad \left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right)\left(\frac{p+q+r}{s}\right),\quad \left(\frac{p}{s}\right)\left(\frac{p+s}{q}\right)\left(\frac{p+q+s}{r}\right),$$
$$\left(\frac{p}{q}\right)\left(\frac{p+q}{s}\right)\left(\frac{p+q+s}{r}\right),\quad \left(\frac{p}{r}\right)\left(\frac{p+r}{s}\right)\left(\frac{p+q+s}{q}\right),\quad \left(\frac{p}{s}\right)\left(\frac{p+s}{r}\right)\left(\frac{p+r+s}{q}\right),$$
$$\left(\frac{q}{r}\right)\left(\frac{q+r}{p}\right)\left(\frac{p+q+r}{s}\right),\quad \left(\frac{q}{s}\right)\left(\frac{q+s}{p}\right)\left(\frac{p+q+s}{r}\right),\quad \left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right),$$
$$\left(\frac{q}{r}\right)\left(\frac{q+r}{s}\right)\left(\frac{p+q+s}{p}\right),\quad \left(\frac{q}{s}\right)\left(\frac{q+s}{r}\right)\left(\frac{q+r+s}{p}\right),\quad \left(\frac{r}{s}\right)\left(\frac{r+s}{q}\right)\left(\frac{q+r+s}{p}\right).$$

Producta alterius formae ope praecedentis proprietatis hinc sponte fluunt;
est enim

$$\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right).$$

Deinde vero etiam hoc productum $\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)$ evolutum pro primo membro dat $\frac{(p+q+r)(p+r+s)}{pqrs(p+r)}$ in quo tam p et r quam q et s inter se permutare licet, ita ut sit

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right).$$

SCHOLION

386. Quantumvis late haec patere videantur, tamen nullas novas comparationes suppeditant, quae non iam in praecedenti contineantur. Postrema enim aequalitas

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r}{q}\right).$$

oritur ex multiplicatione harum

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)=\left(\frac{p}{r}\right)\left(\frac{p+r}{q}\right),\quad \left(\frac{p}{r}\right)\left(\frac{p+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right).$$

Priorum vero formatio ex hoc exemplo patebit : aequalitas

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right)$$

oritur ex multiplicatione harum

$$\left(\frac{p}{q}\right)\left(\frac{p+q}{r+s}\right)=\left(\frac{r+s}{p}\right)\left(\frac{p+r+s}{q}\right),\quad \left(\frac{p+q}{r}\right)\left(\frac{p+q+r}{s}\right)=\left(\frac{r}{s}\right)\left(\frac{r+s}{p+q}\right).$$

Istae autem comparationes praecipue utiles sunt ad valores diversarum formularum eiusdem ordinis seu pro dato numero n invicem reducendos, ut integratio ad paucissimas revocetur, quibus datis reliquae per eas definiri queant.

PROBLEMA 48

387. *Formulas simplicissimas exhibere, ad quas integratio omnium casuum in forma*

$$\left(\frac{p}{q}\right)=\int\frac{x^{p-1}dx}{\sqrt[n]{\left(1-x^n\right)^{n-q}}}$$

contentorum reduci queat.

SOLUTIO

Primo est $\left(\frac{n}{p}\right)=\frac{1}{p}$, unde habentur hi casus

$$\left(\frac{n}{1}\right)=1,\quad \left(\frac{n}{2}\right)=\frac{1}{2},\quad \left(\frac{n}{3}\right)=\frac{1}{3},\quad \left(\frac{n}{4}\right)=\frac{1}{4},\quad \left(\frac{n}{5}\right)=\frac{1}{5}\quad \text{etc.}$$

Deinde est $\left(\frac{p}{n-p}\right)=\frac{\pi}{n\sin.\frac{p\pi}{n}}$, unde omnium harum formularum valores sunt cogniti, quas indicemus

$$\left(\frac{n-1}{1}\right)=\alpha,\quad \left(\frac{n-2}{2}\right)=\beta,\quad \left(\frac{n-3}{3}\right)=\gamma,\quad \left(\frac{n-4}{4}\right)=\delta\quad \text{etc.}$$

Verum hi non sufficiunt ad reliquos omnes expediendos, praeterea tanquam cognitos spectari oportet hos

$$\left(\frac{n-2}{1}\right)=A,\quad \left(\frac{n-3}{2}\right)=B,\quad \left(\frac{n-4}{3}\right)=C,\quad \left(\frac{n-5}{4}\right)=D\quad \text{etc.}$$

atque ex his reliqui omnes determinari poterunt ope aequationum supra demonstratarum; unde potissimum has notasse iuvabit

$$\left(\frac{n-a}{a}\right)\left(\frac{n}{b}\right)=\left(\frac{n-a}{b}\right)\left(\frac{n-a+b}{a}\right),$$
$$\left(\frac{n-a}{a}\right)\left(\frac{n-a-b}{b}\right)=\left(\frac{n-b}{b}\right)\left(\frac{n-a-b}{a}\right),$$
$$\left(\frac{n-a}{a}\right)\left(\frac{n-b-1}{b}\right)\left(\frac{n-a-b}{a-1}\right)=\left(\frac{n-b}{b}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-b}{a}\right).$$

Ex harum prima posito $a=b+1$ invenitur
$$\left(\frac{n-1}{a}\right)=\left(\frac{n-a}{a}\right)\left(\frac{n}{a-1}\right):\left(\frac{n-a}{a-1}\right),$$
ubi $\left(\frac{n}{a-1}\right)=\left(\frac{1}{a-1}\right)$, ideoque per formulas assumtas definitur $\left(\frac{n-1}{a}\right)$.
Ex secunda posito b = 1 invenitur

$$\left(\frac{n-a-1}{a}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right).$$

Ex tertia posito b = 1 deducitur

$$\left(\frac{n-a-1}{a-1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-2}{1}\right)$$

sicque reperiuntur omnes formulae $\left(\frac{n-a-2}{a-1}\right)$ et ex his porro ponendo $b=2$ in tertia

$$\left(\frac{n-a-2}{a-1}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-3}{2}\right),$$

unde reperiuntur formae $\left(\frac{n-a-3}{a-1}\right)$ et ita porro omnes $\left(\frac{n-a-b}{a}\right)$, quippe quae forma omnes complectitur. Labor autem per priores aequationes non mediocriter contrahitur. Inventa enim $\left(\frac{n-a-2}{a-1}\right)$ ex prima colligitur
$$\left(\frac{n-2}{a+2}\right)=\left(\frac{n-a-2}{a+2}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-2}{a}\right),$$
ex secunda vero
$$\left(\frac{n-a-2}{2}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right).$$

similique modo ex inventis formulis $\left(\frac{n-a-3}{a-1}\right)$ derivantur hae

$$\left(\frac{n-3}{a+3}\right)=\left(\frac{n-a-3}{a+3}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-3}{a}\right),$$
$$\left(\frac{n-a-3}{a-1}\right)=\left(\frac{n-3}{3}\right)\left(\frac{n-a-3}{a}\right):\left(\frac{n-a}{a}\right).$$

COROLLARIUM 1

388. Ex aequatione $\left(\frac{n-1}{a}\right)=\frac{1}{a-1}\left(\frac{n-a}{a}\right):\left(\frac{n-a}{a-1}\right)$ definiuntur

$$\left(\frac{n-1}{2}\right)=\frac{\beta}{1A},\quad \left(\frac{n-1}{3}\right)=\frac{\gamma}{2B},\quad \left(\frac{n-1}{4}\right)=\frac{\delta}{3C},\quad \left(\frac{n-1}{5}\right)=\frac{\varepsilon}{4D}\quad \text{etc.},$$

ex aequatione vero $\left(\frac{n-a-1}{a}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)$ hae formulae

$$\left(\frac{n-2}{1}\right)=\frac{\alpha A}{\alpha},\quad \left(\frac{n-3}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{n-4}{1}\right)=\frac{\alpha C}{\gamma},\quad \left(\frac{n-5}{1}\right)=\frac{\alpha D}{\delta}\quad \text{etc.}$$

COROLLARIUM 2

389. Aequatio $\left(\frac{n-a-1}{a-1}\right)=\left(\frac{n-1}{1}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-1}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-2}{1}\right)$ praebet

$$\left(\frac{n-3}{1}\right)=\frac{\alpha AB}{\beta A},\quad \left(\frac{n-4}{2}\right)=\frac{\alpha BC}{\gamma A},\quad \left(\frac{n-5}{3}\right)=\frac{\alpha CD}{\delta A},\quad \left(\frac{n-6}{4}\right)=\frac{\alpha DE}{\varepsilon A}\quad \text{etc.},$$

unde reperiuntur istae formulae $\left(\frac{n-2}{a+2}\right)=\left(\frac{n-a-2}{a+2}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-2}{a}\right)$,

$$\left(\frac{n-2}{3}\right)=\frac{\gamma\beta A}{1\alpha AB},\quad \left(\frac{n-2}{4}\right)=\frac{\delta\gamma A}{2\alpha BC},\quad \left(\frac{n-2}{5}\right)=\frac{\varepsilon\delta A}{3\alpha CD},\quad \left(\frac{n-2}{6}\right)=\frac{\zeta\varepsilon A}{4\alpha DE}\quad \text{etc.}$$

atque etiam istae $\left(\frac{n-a-2}{2}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)$, quae sunt

$$\left(\frac{n-3}{2}\right)=\frac{\beta\alpha AB}{\alpha\beta A},\quad \left(\frac{n-4}{2}\right)=\frac{\beta\alpha BC}{\beta\gamma A},\quad \left(\frac{n-5}{2}\right)=\frac{\beta\alpha CD}{\gamma\delta A},\quad \left(\frac{n-6}{2}\right)=\frac{\beta\alpha DE}{\delta\varepsilon A}\quad \text{etc.}$$

COROLLARIUM 3

390. Tum aequatio $\left(\frac{n-a-2}{a-1}\right)=\left(\frac{n-2}{2}\right)\left(\frac{n-a}{a-1}\right)\left(\frac{n-a-2}{a}\right):\left(\frac{n-a}{a}\right)\left(\frac{n-3}{2}\right)$
dat

$$\left(\frac{n-4}{1}\right)=\frac{\alpha\beta ABC}{\beta\gamma AB},\quad \left(\frac{n-5}{2}\right)=\frac{\alpha\beta BCD}{\gamma\delta AB},\quad \left(\frac{n-6}{3}\right)=\frac{\alpha\beta CDE}{\delta\varepsilon AB},\quad \left(\frac{n-7}{4}\right)=\frac{\alpha\beta DEF}{\varepsilon\zeta AB}\quad \text{etc.},$$

hinc $\left(\frac{n-3}{a+3}\right)=\left(\frac{n-a-3}{a+3}\right)\left(\frac{n}{a}\right):\left(\frac{n-a-3}{a}\right)$ praebet

$$\left(\frac{n-3}{4}\right)=\frac{\beta\gamma\delta AB}{1\alpha\beta ABC},\quad \left(\frac{n-3}{5}\right)=\frac{\gamma\delta\varepsilon AB}{2\alpha\beta BCD},\quad \left(\frac{n-3}{6}\right)=\frac{\delta\varepsilon\zeta AB}{3\alpha\beta CDE},\quad \text{etc.}$$

atque ex $\left(\frac{n-a-3}{3}\right)=\left(\frac{n-3}{3}\right)\left(\frac{n-a-3}{a}\right):\left(\frac{n-a}{a}\right)$ deducuntur

$$\left(\frac{n-5}{3}\right)=\frac{\alpha\beta\gamma BCD}{\beta\gamma\delta AB},\quad \left(\frac{n-6}{3}\right)=\frac{\alpha\beta\gamma CDE}{\gamma\delta\varepsilon AB},\quad \left(\frac{n-7}{3}\right)=\frac{\alpha\beta\gamma DEF}{\delta\varepsilon\xi AB},\quad \text{etc.}$$

EXEMPLUM 1

391. *Casus in hac forma* $\int\frac{x^{p-1}dx}{\sqrt[2]{\left(1-x^2\right)^{2-q}}}=\left(\frac{p}{q}\right)$ *contentos, ubi n=2, evolvere, ubi est*

$$\left(\frac{p+2}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right).$$

Manifestum est has formulas omnes vel algebraice vel per angulos expediri; his tamen regulis utentes, quia numeri p et q binarium superare non debent, unam formulam a circulo pendentem habemus

$$\left(\frac{1}{1}\right)=\frac{\pi}{2\sin.\frac{\pi}{2}}=\frac{\pi}{2}=\alpha,$$

unde nostri casus erunt

$$\left(\frac{2}{1}\right)=1,\quad \left(\frac{2}{2}\right)=\frac{1}{2},$$
$$\left(\frac{1}{1}\right)=\alpha.$$

EXEMPLUM 2

392. *Casus in hac forma* $\int\frac{x^{p-1}dx}{\sqrt[3]{\left(1-x^3\right)^{3-q}}}=\left(\frac{p}{q}\right)$ *contentos, ubi* $n=3$, *evolvere, ubi est*

$$\left(\frac{p+3}{q}\right)=\frac{p}{p+q}\left(\frac{p}{q}\right).$$

Hic casus principales, ad quos caeteri reducuntur, sunt

$$\left(\frac{2}{1}\right)=\frac{\pi}{3\sin.\frac{\pi}{3}}=\frac{2\pi}{3\sqrt{3}}=\alpha \quad \text{et} \quad \left(\frac{1}{1}\right)=\int\frac{dx}{\sqrt[3]{\left(1-x^3\right)^2}},$$

qua concessa erunt reliqui

$$\left(\frac{3}{1}\right)=1,\quad \left(\frac{3}{2}\right)=\frac{1}{2},\quad \left(\frac{3}{3}\right)=\frac{1}{3},$$
$$\left(\frac{2}{1}\right)=\alpha,\quad \left(\frac{2}{2}\right)=\frac{\alpha}{A},$$
$$\left(\frac{1}{1}\right)=A.$$

EXEMPLUM 3

393. *Casus in hac forma* $\int \frac{x^{p-1}dx}{\sqrt[4]{(1-x^4)^{4-q}}} = \left(\frac{p}{q}\right)$ *contentos, ubi* $n = 4$ *, evolvere, ubi est*

$$\left(\frac{p+4}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right).$$

A circulo pendent hae duae

$$\left(\frac{3}{1}\right) = \frac{\pi}{4 sin.\frac{\pi}{4}} = \frac{\pi}{2\sqrt{2}} = \alpha \quad \text{et} \quad \left(\frac{2}{2}\right) = \frac{\pi}{4 sin.\frac{2\pi}{4}} = \frac{\pi}{4} = \beta,$$

praeterea vero una transcendente singulari opus est $\left(\frac{2}{1}\right) = A$, unde reliquae ita determinantur

$$\left(\frac{4}{1}\right) = 1, \quad \left(\frac{4}{2}\right) = \frac{1}{2}, \quad \left(\frac{4}{3}\right) = \frac{1}{3}, \quad \left(\frac{4}{4}\right) = \frac{1}{4},$$

$$\left(\frac{3}{1}\right) = \alpha, \quad \left(\frac{3}{2}\right) = \frac{\beta}{A}, \quad \left(\frac{3}{3}\right) = \frac{\alpha}{2A},$$

$$\left(\frac{2}{1}\right) = A, \quad \left(\frac{2}{2}\right) = \beta,$$

$$\left(\frac{1}{1}\right) = \frac{\alpha A}{\beta}.$$

EXEMPLUM 4

394. *Casus in hac forma* $\int \frac{x^{p-1}dx}{\sqrt[5]{(1-x^5)^{5-q}}} = \left(\frac{p}{q}\right)$ *contentos, ubi n* = 5, *evolvere, ubi est*

$$\left(\frac{p+5}{q}\right) = \frac{p}{p+q}\left(\frac{p}{q}\right).$$

A circulo pendent hae duae formulae

$$\left(\frac{4}{1}\right) = \frac{\pi}{5 sin.\frac{\pi}{5}} = \alpha \quad \text{and} \quad \left(\frac{3}{2}\right) = \frac{\pi}{5 sin.\frac{2\pi}{5}} = \beta,$$

praeter quas duas novas transcendentes assumi oportet

$$\left(\frac{3}{1}\right) = A \text{ et } \left(\frac{2}{2}\right) = B,$$

per quas omnes sequenti modo determinantur

$$\left(\frac{5}{1}\right)=1,\quad \left(\frac{5}{2}\right)=\frac{1}{2},\quad \left(\frac{5}{3}\right)=\frac{1}{3},\quad \left(\frac{5}{4}\right)=\frac{1}{4},\quad \left(\frac{5}{5}\right)=\frac{1}{5},$$
$$\left(\frac{4}{1}\right)=\alpha,\quad \left(\frac{4}{2}\right)=\frac{\beta}{A},\quad \left(\frac{4}{3}\right)=\frac{\beta}{2B},\quad \left(\frac{4}{4}\right)=\frac{\alpha}{3A},$$
$$\left(\frac{3}{1}\right)=A,\quad \left(\frac{3}{2}\right)=\beta,\quad \left(\frac{3}{3}\right)=\frac{\beta\beta}{\alpha B},$$
$$\left(\frac{2}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{2}{2}\right)=B,$$
$$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$$

EXEMPLUM 5

395. *Casus in hac forma* $\int\frac{x^{p-1}dx}{\sqrt[6]{\left(1-x^6\right)^{6-q}}}=\left(\frac{p}{q}\right)$ *contentos, ubi n* = 6, *evolvere.*

A circulo pendent hae tres formulae

$$\left(\frac{5}{1}\right)=\frac{\pi}{6\sin.\frac{\pi}{6}}=\frac{\pi}{3}=\alpha,\quad \left(\frac{4}{2}\right)=\frac{\pi}{6\sin.\frac{2\pi}{6}}=\frac{\pi}{3\sqrt{3}}=\beta,\quad \left(\frac{3}{3}\right)=\frac{\pi}{6\sin.\frac{3\pi}{6}}=\frac{\pi}{6}=\gamma\ ;$$

tum vero assumantur hae duae transcendentes

$$\left(\frac{4}{1}\right)=A\ \text{ et }\left(\frac{3}{2}\right)=B$$

atque per has omnes sequenti modo determinantur

$$\left(\frac{6}{1}\right)=1,\quad \left(\frac{6}{2}\right)=\frac{1}{2},\quad \left(\frac{6}{3}\right)=\frac{1}{3},\quad \left(\frac{6}{4}\right)=\frac{1}{4},\quad \left(\frac{6}{5}\right)=\frac{1}{5},\quad \left(\frac{6}{6}\right)=\frac{1}{6},$$
$$\left(\frac{5}{1}\right)=\alpha,\quad \left(\frac{5}{2}\right)=\frac{\beta}{A},\quad \left(\frac{5}{3}\right)=\frac{\gamma}{2B},\quad \left(\frac{5}{4}\right)=\frac{\beta}{3B},\quad \left(\frac{5}{5}\right)=\frac{\alpha}{4A},$$
$$\left(\frac{4}{1}\right)=A,\quad \left(\frac{4}{2}\right)=\beta,\quad \left(\frac{4}{3}\right)=\frac{\beta\gamma}{\alpha B},\quad \left(\frac{4}{4}\right)=\frac{\beta\gamma A}{2\alpha BB},$$
$$\left(\frac{3}{1}\right)=\frac{\alpha B}{\beta},\quad \left(\frac{3}{2}\right)=B,\quad \left(\frac{3}{3}\right)=\gamma,$$
$$\left(\frac{2}{1}\right)=\frac{\alpha B}{\gamma},\quad \left(\frac{2}{2}\right)=\frac{\alpha BB}{\gamma A},$$
$$\left(\frac{1}{1}\right)=\frac{\alpha A}{\beta}.$$

SCHOLION

396. Has determinationes, quousque libuerit, continuare licet, in quibus praecipue notari debent casus novas transcendentium species introducentes; quorum primus occurrit, si *n=3*, estque

$$\left(\frac{1}{1}\right)=\int\frac{dx}{\sqrt[3]{\left(1-x^3\right)^2}}$$

cuius valorem per productum infinitum supra [§ 371] vidimus esse

$$=\frac{3}{1}\cdot\frac{2}{4}\cdot\frac{6}{4}\cdot\frac{5}{7}\cdot\frac{9}{7}\cdot\frac{8}{10}\cdot\frac{12}{10}\cdot\text{etc.},$$

quod ex formula $\left(\frac{1}{1}\right)$ ob $n=3$ etiam est

$$\frac{2}{1\cdot 1}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{6\cdot 8}{7\cdot 7}\cdot\frac{9\cdot 11}{10\cdot 10}\cdot\frac{12\cdot 14}{13\cdot 13}\cdot\text{etc.}$$

Deinde ex classe $n=4$ nascitur haec nova forma transcendens

$$\left(\frac{2}{1}\right)=\int\frac{xdx}{\sqrt[4]{\left(1-x^4\right)^3}}=\int\frac{dx}{\sqrt[4]{\left(1-x^4\right)^2}}=\int\frac{dx}{\sqrt{\left(1-x^4\right)}},$$

quae aequatur huic producto infinito

$$\frac{3}{1\cdot 2}\cdot\frac{4\cdot 7}{5\cdot 6}\cdot\frac{8\cdot 11}{9\cdot 10}\cdot\frac{12\cdot 15}{13\cdot 14}\cdot\frac{16\cdot 19}{17\cdot 18}\cdot\text{etc.}=\frac{3}{2}\cdot\frac{2\cdot 7}{5\cdot 3}\cdot\frac{4\cdot 11}{9\cdot 5}\cdot\frac{6\cdot 15}{13\cdot 7}\cdot\frac{8\cdot 19}{17\cdot 9}\cdot\text{etc.}$$

Ex classe $n = 5$ impetramus duas novas formulas transcendentes

$$\left(\frac{3}{1}\right)=\int\frac{x^2dx}{\sqrt[5]{\left(1-x^5\right)^4}}=\int\frac{dx}{\sqrt[5]{\left(1-x^5\right)^2}}=\frac{4}{1\cdot 3}\cdot\frac{5\cdot 9}{6\cdot 8}\cdot\frac{10\cdot 14}{11\cdot 13}\cdot\frac{15\cdot 19}{16\cdot 18}\cdot\text{etc.}$$

et

$$\left(\frac{2}{2}\right)=\int\frac{xdx}{\sqrt[5]{\left(1-x^5\right)^3}}=\frac{4}{2\cdot 2}\cdot\frac{5\cdot 9}{7\cdot 7}\cdot\frac{10\cdot 14}{12\cdot 12}\cdot\frac{15\cdot 19}{17\cdot 17}\cdot\text{etc.},$$

ita ut sit

$$\left(\frac{3}{1}\right):\left(\frac{2}{2}\right)=\frac{2\cdot 2}{1\cdot 3}\cdot\frac{7\cdot 7}{6\cdot 8}\cdot\frac{12\cdot 12}{11\cdot 13}\cdot\frac{17\cdot 17}{16\cdot 18}\cdot\text{etc.}$$

Classis $n=6$ has duas formulas transcendentes suppeditat

$$\left(\frac{4}{1}\right)=\int\frac{x^3dx}{\sqrt[6]{\left(1-x^6\right)^5}}=\int\frac{dx}{\sqrt[3]{\left(1-x^6\right)}}=\frac{1}{2}\int\frac{ydy}{\sqrt[6]{\left(1-y^3\right)^5}}$$

posito $xx=y$ et

$$\left(\frac{3}{2}\right)=\int\frac{x^2dx}{\sqrt[3]{\left(1-x^6\right)^2}}=\int\frac{xdx}{\sqrt{\left(1-x^6\right)}}=\frac{1}{2}\int\frac{dy}{\sqrt{\left(1-y^3\right)}}=\frac{1}{3}\int\frac{dz}{\sqrt[3]{\left(1-zz\right)^2}}$$

sumto $y=xx$ et $z=x^3$. Notandum autem est inter has et primam

$$\int \frac{dx}{\sqrt[3]{\left(1-x^3\right)^2}} = 2\int \frac{ydy}{\sqrt[6]{\left(1-y^6\right)^4}} = 2\left(\frac{2}{2}\right)$$

relationem dari, quae est [§ 384]

$$\gamma\left(\frac{4}{1}\right)\left(\frac{2}{2}\right) = \alpha\left(\frac{3}{2}\right)\left(\frac{3}{2}\right),$$

ita ut prima admissa hic altera sufficiat.