CHAPTER VII

A GENERAL METHOD BY WHICH ANY INTEGRALS MAY BE FOUND APPROXIMATELY

PROBLEM 36

297. *To investigate approximately the true value of the integral of any formula* $y=\int Xdx$.

SOLUTION

Since the formula of the whole integral by itself is indeterminate, that is usually resolved thus, so that, if a certain value of the variable x is granted, for example a, then the integral itself $y=\int Xdx$ obtains a given value, for example b. Hence the answer sought from the integration is forthcoming here in this manner, in order that, if some other value should be attributed to the variable x different from the value a, a value is defined that then must be given to the integral y. Hence we can attribute first to x a value disagreeing a little from a, for example $x=a+\alpha$, so that α is a very small quantity, and since the function X is changed a little, or if for x there is written $a+\alpha$ for a, as that is allowed to be considered constant. Hence therefore the integral of the differential formula Xdx becomes $Xx+\text{Const.}=y$; but since on putting $x=a$ it must become $y=b$ and the value of X remains as if unchanged, then $Xa+\text{Const.}=b$ and thus the $\text{Const.}=b-Xa$, from which we follow with $y=b+X(x-a)$. Whereby if we attribute the value $a+\alpha$ to x, we have the value of y agreed upon, which is equal to $b+\beta$; and now in a like manner from this case we are able to define y, if to x there is granted another value a little greater than $a+\alpha$; therefore on putting $a+\alpha$ in place of x the value of X thus becomes apparent again and thus for the constant to become $y=b+\beta+X(x-a-\alpha)$. Hence therefore the operation is permitted to be continued, as far as it is wished; in order that the reasoning of this can be shown better, we represent the procedure thus :

if $x=a$, there becomes $X=A$ and $y=b$,
if $x=a'$, there becomes $X=A'$ and $y=b'=b+A(a'-a)$,
if $x=a''$, there becomes $X=A''$ and $y=b''=b'+A'(a''-a')$,
if $x=a'''$, there becomes $X=A'''$ and $y=b'''=b''+A''(a'''-a'')$

etc.,

where the values a, a', a'', a''' etc. are put in place to proceed following very small differences. Hence there shall be $b' = b + A(a' - a)$, clearly into which the formula found $y = b + X(x - a)$ changes ; since it therefore becomes $X = A$, on putting $x = a$; then on now attributing to x the value $= a'$, to that there corresponds $y = b'$; in a similar manner there shall be $b'' = b' + A'(a'' - a')$, then $b''' = b'' + A''(a''' - a'')$ etc., as we have put in place above.
Hence on restoring the preceding values we have :

$$b' = b + A(a' - a),$$
$$b'' = b + A(a' - a) + A'(a'' - a'),$$
$$b''' = b + A(a' - a) + A'(a'' - a') + A''(a''' - a''),$$
$$b'''' = b + A(a' - a) + A'(a'' - a') + A''(a''' - a'') + A'''(a'''' - a''')$$
$$\text{etc. },$$

from which, if x exceeds a by some amount, the series a', a'', a''' etc. by increasing is continued to x, but the final sum gives the value of y.

COROLLARY 1

298. If the increments, by which x is increased, are put in place clearly equal to a, so that then $a' = a + a,\ a'' = a + 2a,\ a''' = a + 3a$ etc., from which with the values substituted for x the function X changes into A', A'', A''' etc., and the final value of these, for example $a + n\alpha$, is put $= x$, of these other values indeed X, then there becomes

$$y = b + a(A + A' + A'' + A''' + \ \ldots\ + X).$$

COROLLARY 2

299. Hence the value of the integral y is elucidated through the summation of the series $A, A', A'', \ldots$ X , the terms of this are formed from the formula X by putting successively $a,\ a + \alpha, a + 2\alpha, \ldots a + n\alpha$ in place of x. For the sum of this series multiplied by the difference α and added to b gives the value of y, which corresponds to $x = a + n\alpha$.

COROLLARY 3

300. Since smaller differences can be put in place, according to which the value of x increases, from that in this manner the value of y is defined more accurately, if indeed the terms of the series A, A', A'' etc. thus also are progressing according to small differences ; for unless this comes about, that determination will be exceedingly uncertain.

COROLLARY 4

301. Hence this approximation can thus be explained from the principles of series. From these numbers indicated

$$a,\ a',\ a'',\ a'',\cdots x$$

the series is formed

$$A,\ A',\ A'',\ A''',\ \cdots X,$$

hence the general term of this X is given from the formula of the differential $dy = Xdx$. Then in this series the term preceding the final shall be $'X$, corresponding to the number indicated $'x$ and hence the new series is formed

$$A(a'-a),\ A'\left(a''-a'\right),\ A''\left(a'''-a''\right),\cdots{}'X\left(x-{}'x\right);$$

the sum of which if it is put equal to S, will be the integral
$y=\int Xdx = b+S$ approximately.

SCHOLIUM 1

302. The integral is generally explained in this manner, as it is said to be the summation of all the values of the differential formula Xdx, if to the variable x successively all the values from some given a as far as to x are given, which proceed according to the differential dx, but is required to take this differential as infinitely small. Therefore an account of integration is to be represented similar to that which, in the geometry of the line is accustomed to be considered as the infinite sum of the points; which just as this is possible to be admitted, if explained properly, thus also that explanation of integration can be tolerated, provided it is recalled to the true principles as we done here, so that all the trickery is put in place. Therefore from this method set out it is certainly apparent that the integration truly can be obtained approximately by summation and indeed not to be set out exactly unless the differentials are put in place infinitely small. And from this source so the name of integration, which is also accustomed to be called the summation, as well as the sign $\int$ of the integral has arisen, which in general must be retained in setting out integrals in good style.

SCHOLIUM 2

303. If for the individual intervals, in which we have divided the jump from a to x, the quantities A, A', A'', A''' etc. actually are to be constants, we may obtain the integral $\int Xdx$ accurately. Hence so far an error is present, in as much as for the individual intervals these quantities are not constants. And indeed for the first interval, in which the variable x proceeds from the term a to a', A is the value of X agreeing with the term a, moreover for the term a' there corresponds A'; from which, as far as A' is not equal to A, so far an error creeps in. Therefore since at the beginning of this interval there shall be $X = A$, moreover at the end there shall be $X = A'$, it would be rather convenient to assume a certain mean between A and A', that which in the correction of this method soon will be observed to be treated. Meanwhile here it will be helpful to observe to be equal by right that both the final as well as the initial value can be taken, and where likewise this is evident, if by one way there should be an erring towards excess, the other way generally errs towards deficiency. From which hence the two expressions are allowed to be elicited, of which the one value of y shall be produced exceedingly large, and the other exceedingly small, thus in order that these constitute bounds on the true limit of y . [Assuming an monotonic increasing or decreasing function in the interval of integration.] Hence just as we have represented the procedure in § 301, the value of $y = \int Xdx$ is contained between these two limits.

$$b + A(a' - a) + A'(a'' - a') + A''(a''' - a'') + \cdots + 'X(x - 'x)$$

and

$$b + A'(a' - a) + A''(a'' - a') + A'''(a''' - a'') + \cdots + X(x - 'x),$$

with which known it is allowed to proceed closer to the true value.

SCHOLIUM 3

304. Now we have noted these intervals, in which we have assumed x to increase successively, thus must be put in place to be definitely small, so that the values of the corresponding A, A',A'' etc. disagree little between themselves in turn ; and hence it is required to be judged chiefly, whether it is convenient for these intervals $a' - a,\ a'' - a',\ a''' - a''$ etc. to be taken equal or unequal to each other. For where the value of X is changed little by changing x, there the intervals through which x proceeds, without risk can be put larger; but where it comes about, with a slight change of x the function X is induced to change greatly, there the smallest intervals must be taken. Just as if there shall be $X = \frac{1}{\sqrt{(1-xx)}}$, it is evident, where x has approached almost to one, however small an interval is taken, through which x may be increased, the function X is able to undergo the maximum change, since yet on taking $x = 1$ that thus increases to infinity. Therefore for these cases, with that approximation for that interval at any rate, is not permitted to be used, in the other term of which X shall be infinite ; but an easy remedy can be brought to this inconvenience, while the formula with the help of a suitable substitution is transformed into another or while for this interval at any rate an interval particular to the integration is put in place. Just as if for the formula proposed shall be $\frac{xdx}{\sqrt{(1-x^3)}}$, for the interval from $x = 1 - \omega$ to $x = 1$ the integral is not found by that method, but on putting $x = 1 - z$, since the bounds of z are 0 and ω, then z is a minimum quantity, from which the formula becomes $\frac{dz(1-z)}{\sqrt{(3z-3z^2+z^3)}} = \frac{dz}{\sqrt{3z}}$, the integral of which $\frac{2\sqrt{z}}{\sqrt{3}}$ presents a part of the integral $\frac{2\sqrt{\omega}}{\sqrt{3}}$ for that interval. Because it is possible to use artifices in all of these cases ; moreover there is a need to illustrate this method described with some examples.

EXAMPLE 1

305. *To exhibit approximately the integral* $y=\int x^n dx$, *thus taken so that it vanishes on putting* $x = 0$.

Here there is $a = 0$ and $b = 0$, then $X = x^n$; now the values of x increase from 0 by the common difference α, so that there becomes

$$\begin{array}{lccccccc} \text{the indices} & 0, & \alpha, & 2\alpha, & 3\alpha, & 4\alpha, & \cdots & x, \\ \text{the series} & 0, & \alpha^n, & 2^n\alpha^n, & 3^n\alpha^n, & 4^n\alpha^n, & \cdots & x^n, \end{array}$$

and the term preceding the final term is $(x-\alpha)^n$, whereby the bounds of the integral

$$y=\int x^n dx = \tfrac{1}{n+1}x^{n+1}$$

are

$$\alpha\left(0+\alpha^n+2^n\alpha^n+3^n\alpha^n+\quad\cdots\quad+(x-\alpha)^n,\right)$$

and

$$\alpha\left(\alpha^n+2^n\alpha^n+3^n\alpha^n+\quad\cdots\quad+x^n\right),$$

which consequently are closer when the interval α is taken less. Thus if $\alpha=1$, the bounds become

$$0+1+2^n+3^n+4^n+\quad\cdots\quad+(x-1)^n$$

and

$$1+2^n+3^n+4^n+\quad\cdots\quad+x^n;$$

if $\alpha=\tfrac{1}{2}$ is taken, the bounds become

$$\tfrac{1}{2^{n+1}}\left(0+1+2^n+3^n+4^n+\quad\cdots\quad+(2x-1)^n\right)$$

and

$$\tfrac{1}{2^{n+1}}\left(1+2^n+3^n+4^n+\quad\cdots\quad+(2x)^n\right);$$

and if in general there shall be $\alpha=\tfrac{1}{m}$, the bounds become

$$\frac{1}{m^{n+1}}\left(0+1+2^n+3^n+4^n+\quad\cdots\quad+(mx-1)^n\right)$$

and

$$\frac{1}{m^{n+1}}\left(1+2^n+3^n+4^n+\quad\cdots\quad+(mx)^n\right),$$

of which the latter exceeds the former by the excess $\frac{x^n}{m}$, from which it is apparent, if the number m is taken infinite, each bound then provides the true value of the integral, $y=\frac{1}{n+1}x^{n+1}$.

COROLLARY 1

306. Hence the sum of the series $1+2^n+3^n+4^n+\quad\cdots\quad+(mx)^n$ therefore approaches closer to $\frac{1}{n+1}(mx)^{n+1}$, when the number m is taken greater; whereby on putting $mx=z$ the sum of this progression

$$1+2^n+3^n+4^n+\quad\cdots\quad+z^n$$

therefore approaches closer to $\frac{1}{n+1}z^{n+1}$, when the number z becomes greater.

COROLLARY 2

307. But from the first bound on setting $mx=z$ the same quantity $\frac{1}{n+1}z^{n+1}$ is shown to be the sum of this series

$$0+1+2^n+3^n+4^n+\quad\cdots\quad+(z-1)^n,$$

from which with the mean taken consequently becomes more accurate,

$$1+2^n+3^n+4^n+\quad\cdots\quad+(z-1)^n+\tfrac{1}{2}z^n=\frac{1}{n+1}z^{n+1}$$

or by adding $\frac{1}{2}z^n$ to both sides we have :

$$1+2^n+3^n+4^n+\quad\cdots\quad+z^n=\frac{1}{n+1}z^{n+1}+\tfrac{1}{2}z^n \text{ approximately,}$$

which agrees with these values, which are known from the true sum of this progression.

EXAMPLE 2

308. *To exhibit approximately the integral* $y = \int \frac{dx}{x^n}$ *,thus taken so that it vanishes on putting* $x = 1$.

Hence there becomes $a = 1$ and $b = 0$, from which, if the interval of the progression from a to x is put equal to α , there shall be

the indices $a,\ a+\alpha,\quad a+2\alpha,\quad a+3\alpha,\cdots x$

and

the terms of the series $\frac{1}{a^n},\ \frac{1}{(a+\alpha)^n},\ \frac{1}{(a+2\alpha)^n},\ \frac{1}{(a+3\alpha)^n},\cdots \frac{1}{(x)^n} = X,$

where the term preceding the final term is $\frac{1}{(x-\alpha)^n} = {}'X$. Now since our integral shall be

$$y = \frac{1}{n-1} - \frac{1}{(n-1)x^n},$$

the value of this integral is contained between these bounds :

$$\alpha\left(1 + \frac{1}{(1+\alpha)^n} + \frac{1}{(1+2\alpha)^n} + \frac{1}{(1+3\alpha)^n} + \cdots + \frac{1}{(x-\alpha)^n}\right)$$

and

$$\alpha\left(\frac{1}{(1+\alpha)^n} + \frac{1}{(1+2\alpha)^n} + \frac{1}{(1+3\alpha)^n} + \cdots + \frac{1}{x^n}\right).$$

Whereby on putting $\alpha = \frac{1}{m}$ these bounds become

$$m^{n-1}\left(\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \cdots + \frac{1}{(mx-1)^n}\right)$$

and

$$m^{n-1}\left(\frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \frac{1}{(m+4)^n} + \cdots + \frac{1}{(mx)^n}\right),$$

which, when the number m is taken greater, consequently they approach closer to the value of the integral $y = \frac{1}{n-1} - \frac{1}{(n-1)x^{n-1}}$. Moreover it is to be noted in the case $n = 1$ that the integral becomes equal to lx.

COROLLARY 1

309. But if we put $mx = m + z$, in order that $x = \frac{m+z}{m}$, these progressions are produced :

$$m^{n-1}\left(\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \cdots + \frac{1}{(m+z-1)^n}\right)$$

and

$$m^{n-1}\left(\frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \frac{1}{(m+4)^n} + \cdots + \frac{1}{(m+z)^n}\right),$$

of which the sum of the first is greater, and of the second less, as

$$\frac{1}{n-1} - \frac{m^{n-1}}{(n-1)(m+z)^{n-1}} = \frac{(m+z)^{n-1} - m^{n-1}}{(n-1)(m+z)^{n-1}} \;;$$

but in the case $n = 1$, this expression changes into $l\left(1+\frac{z}{m}\right)$.

COROLLARY 2

310. Since the former progression is greater than the latter, then

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \cdots + \frac{1}{(m+z-1)^n} > \frac{(m+z)^{n-1} - m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}},$$

$$\frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \cdots + \frac{1}{(m+z)^n} < \frac{(m+z)^{n-1} - m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}} \;;$$

on both sides $\frac{1}{m^n}$ is added here to the latter, and now $\frac{1}{(m+z)^n}$ is added there to the former and there is taken the arithmetic mean; there becomes more precisely,

$$\frac{1}{m^n} + \frac{1}{(m+1)^n} + \frac{1}{(m+2)^n} + \frac{1}{(m+3)^n} + \cdots + \frac{1}{(m+z)^n} = \frac{(2m+n-1)(m+z)^n - (2z+2m-n+1)m^n}{2(n-1)m^n(m+z)^n},$$

which expression in the case $n = 1$ changes into $l\left(1+\frac{z}{m}\right) + \frac{1}{2m} + \frac{1}{2(m+z)}$.

COROLLARY 3

311. There is put $z = mv$ and we have the sum of the following series expressed approximately

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\dots+\frac{1}{m^n(1+v)^n}=\frac{(2m+n-1)(1+v)^n-2m(1+v)+n-1}{2(n-1)m^n(1+v)^n}$$

and in the case $n=1$

$$\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\dots+\frac{1}{m+mv}=l(1+v)+\frac{2+v}{2m(1+v)},$$

from which, if $v=1$, then approximately,

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\dots+\frac{1}{2^n m^n}=\frac{2^n(2m+n-1)-4m+n-1}{2^{n+1}(n-1)m^n}$$

and

$$\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\dots+\frac{1}{2m}=l2+\frac{3}{4m}.$$

COROLLARY 4

312. Hence it comes about that the logarithms of numbers of any magnitude can be assigned approximately by the rule, while the common series only prevail for numbers a little different from unity. Indeed we can write u for $1+v$ and we have

$$lu=\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\dots+\frac{1}{mu}-\frac{1+u}{2mu},$$

from which lu consequently is defined more accurately, when the number m is taken larger.

EXAMPLE 3

313. *To exhibit approximately the integral $y=\int\frac{cdx}{cc+xx}$,thus taken so that it vanishes on putting $x = 0$.*

This integral, as we know, is y = Ang. tang.$\frac{x}{c}$, according to which the approximate value is to be shown, there is $a = 0$ and $b = 0$; hence if the value of x is put in place to increase from 0 through constant differences to a, on account of $X=\frac{cdx}{cc+xx}$ the values of this will be :

for the indices $0, \quad \alpha, \quad 2\alpha, \quad \cdots \quad x$

for the series $\frac{1}{c}, \quad \frac{c}{cc+\alpha\alpha}, \frac{c}{cc+4\alpha\alpha}, \cdots \frac{c}{cc+xx}$,

the term of this preceding the final is $'X = \frac{c}{cc+(x-\alpha)^2}$. Whereby the value of our integral

$y =$ Ang. tang.$\frac{x}{c}$ is approximately

$$\alpha\left(\frac{1}{c}+\frac{c}{cc+\alpha\alpha}+\frac{c}{cc+4\alpha\alpha}+\cdots+\frac{c}{cc+(x-\alpha)^2}\right),$$

now the other is approximately less, since this is exceedingly large, which is

$$\alpha\left(\frac{c}{cc+\alpha\alpha}+\frac{c}{cc+4\alpha\alpha}+\frac{c}{cc+9\alpha\alpha}+\cdots+\frac{c}{cc+xx}\right).$$

Between these if a mean is taken, now on $\alpha\cdot\frac{1}{c}$ being added here and there $\alpha\cdot\frac{c}{cc+xx}$, on both sides, a closer approximation will be

$$\alpha\left(\frac{c}{cc}+\frac{c}{cc+\alpha\alpha}+\frac{c}{cc+4\alpha\alpha}+\cdots+\frac{c}{cc+xx}\right)$$

$$= \text{Ang. tang.}\frac{x}{c}+\frac{\alpha}{2}\left(\frac{1}{c}+\frac{c}{cc+xx}\right) = \text{Ang. tang.}\frac{x}{c}+\frac{\alpha(2cc+xx)}{2c(cc+xx)}.$$

Hence for this angle we now have the approximation

$$\text{Ang. tang.}\frac{x}{c} = \alpha c\left(\frac{1}{cc}+\frac{1}{cc+\alpha\alpha}+\frac{1}{cc+4\alpha\alpha}+\cdots+\frac{1}{cc+xx}\right)-\frac{\alpha(2cc+xx)}{2c(cc+xx)},$$

which consequently disagrees less with the true value, in which the ratio of α to c is a smaller number.
But if hence we take a very large number for c, it is permitted to take unity for α , from which on putting $x = cv$ there will be

$$\text{Ang. tang.}v = c\left(\frac{1}{cc}+\frac{1}{cc+1}+\frac{1}{cc+4}+\frac{1}{cc+9}+\cdots+\frac{1}{cc+ccvv}\right)-\frac{2+vv}{2c(1+vv)}$$

and that consequently is more exact, where the number c it taken greater.

COROLLARY 1

314. If we put $c = 1$, in which case error must be conspicuous, there becomes

$$\text{Ang. tang.}v = 1 + \frac{1}{1+1} + \frac{1}{1+4} + \frac{1}{1+9} + \cdots + \frac{1}{1+vv} - \frac{2+vv}{2(1+vv)}.$$

Let $v = 1$; then $\text{Ang. tang.}1 = \frac{\pi}{4} = 1 + \frac{1}{2} - \frac{3}{4} = \frac{3}{4}$ and hence $\pi = 3$, which is not too far from the truth.

If we put $c = 2$, there is produced :

$$\text{Ang. tang.}v = 2\left(\frac{1}{4} + \frac{1}{4+1} + \frac{1}{4+4} + \frac{1}{4+9} + \cdots + \frac{1}{4+4vv}\right) - \frac{2+vv}{4(1+vv)},$$

from which, if $v = 1$, there is deduced

$$\text{Ang. tang.}1 = \frac{\pi}{4} = 2\left(\frac{1}{4} + \frac{1}{4+1} + \frac{1}{4+4}\right) - \frac{3}{8} = \frac{23}{20} - \frac{3}{8} = \frac{31}{40}$$

and thus $\pi = \frac{31}{10} = 3{,}1$ agreeing closer.

COROLLARY 2

315. Let $c = 6$ and there shall be

$$\text{Ang. tang.}v = 6\left(\frac{1}{36} + \frac{1}{36+1} + \frac{1}{36+4} + \cdots + \frac{1}{36+36vv}\right) - \frac{2+vv}{12(1+vv)},$$

from which, if $v = \frac{1}{2}$ and $v = \frac{1}{3}$, there arises

$$\text{Ang. tang.}\tfrac{1}{2} = 6\left(\frac{1}{36} + \frac{1}{36+1} + \frac{1}{36+4} + \frac{1}{36+9}\right) - \frac{3}{20},$$
$$\text{Ang. tang.}\tfrac{1}{3} = 6\left(\frac{1}{36} + \frac{1}{36+1} + \frac{1}{36+4}\right) - \frac{19}{120}.$$

But there is $\text{Ang.tang.}\frac{1}{2} + \text{Ang.tang.}\frac{1}{3} = \text{Ang. tang.}1 = \frac{\pi}{4}$. Hence

$$\frac{\pi}{4} = 12\left(\frac{1}{36} + \frac{1}{37} + \frac{1}{40}\right) + \frac{2}{15} - \frac{37}{120} = \frac{1063}{1110} - \frac{7}{40} = \frac{695}{888}$$

or $\pi = \frac{695}{222} = 3{,}1306.$

COROLLARY 3

316. But if we put $v = 1$ there at once, then

$$\frac{\pi}{4} = 6\left(\frac{1}{36} + \frac{1}{37} + \frac{1}{40} + \frac{1}{45} + \frac{1}{52} + \frac{1}{72}\right) - \frac{1}{8},$$

from which there is made $\pi = 3,13696$ much closer to the truth ; clearly the addition of more terms leads to a result closer to the truth.

PROBLEM 37

317. *The method of approaching the values of integrals set out previously is returned , so that there is less disagreement from the true value.*

SOLUTION

Let $y = \int Xdx$ be the proposed formula of the integral, the value of which is agreed to be $y = b$, if there is put $x = a$, this shall be given either by the same condition of integration or thus derived now from some operations ; and we attribute now to x a value surpassing a little that value a, to which there corresponds the value $y = b$, then there now becomes $X = A$, if there is put $x = a$. But in the above method we have assumed, while x increased a little above a, for X to remain constant and equal to A and thus there becomes

$$\int Xdx = A(x-a).$$

But in as much as X is not constant, to this extent it is not the case that $\int Xdx = A(x-a)$, but there is had actually,

$$\int Xdx = X(x-a) - \int (x-a)dX.$$

Therefore we put $dX = Pdx$ and then

$$\int (x-a)dX = \int P(x-a)dx,$$

and if now $P = \frac{dX}{dx}$, as long as x is not much greater than a, we may consider it as a constant, and we have

$$\int P(x-a)dx = \tfrac{1}{2}P(x-a)^2$$

and thus there becomes

$$y=\int Xdx=b+X\left(x-a\right)-\tfrac{1}{2}P\left(x-a\right)^2,$$

which value now approaches closer to the true value, and if there values are taken for *X* and *P*, which are put in place either for the position $x=a$ or for the position $x=a+\alpha$, clearly with a greater value, to which by this operation we have put *x* to increase ; hence from which, as we have put either $x=a$ or $x=a+\alpha$, we will obtain two boundaries, between which the truth lies. Moreover in a similar manner we are able to progress further ; for since *P* is not constant, then

$$\int P\left(x-a\right)dx=\tfrac{1}{2}P\left(x-a\right)^2-\tfrac{1}{2}\int\left(x-a\right)^2 dP,$$

from which, if we put in place $dP=Qdx$, then

$$\int\left(x-a\right)^2 dP=\int Q\left(x-a\right)^2 dx=\tfrac{1}{3}Q\left(x-a\right)^3,$$

if we regard a certain *Q* as a constant quantity, thus so that

$$y=\int Xdx=b+X\left(x-a\right)-\tfrac{1}{2}P\left(x-a\right)^2+\tfrac{1}{2\cdot 3}Q\left(x-a\right)^3.$$

Hence by the same method if we proceed further, on putting

$$X=\tfrac{dy}{dx},\quad P=\tfrac{dX}{dx},\quad Q=\tfrac{dP}{dx},\quad R=\tfrac{dQ}{dx},\quad S=\tfrac{dR}{dx}\quad \text{etc.}$$

we come upon

$$y=b+X\left(x-a\right)-\tfrac{1}{2}P\left(x-a\right)^2+\tfrac{1}{2\cdot 3}Q\left(x-a\right)^3-\tfrac{1}{2\cdot 3\cdot 4}R\left(x-a\right)^4$$
$$+\tfrac{1}{2\cdot 3\cdot 4\cdot 5}S\left(x-a\right)^5-\quad \text{etc.},$$

which series converges strongly, only if *x* should not be much greater than *a*, and thus, if it is continued to infinity, the true value of *y* will be shown, if indeed the same extreme value $x=a+\alpha$ is substituted into the functions X, *P*, *Q*, *R* etc. But unless we wish to extend that series to infinity, it would be better to proceed by dividing the interval by successive values of *x* : a, a', a'', a''', a'''' etc. and then it is required for the particular appropriate values of the letters *X*, *P*, *Q*, *R*, *S* etc. to be sought, which shall be, so that they follow : If there should be

$$x=a,\quad a',\quad a'',\quad a''', a^{\text{IV}},\quad a^{\text{V}}\quad \text{etc.},$$

then there arises

$$X = A,\ A',\quad A'',\quad A''', A^{\mathrm{IV}},\quad A^{\mathrm{V}}\quad \text{etc.},$$
$$\tfrac{dX}{dx} = P = B,\ B',\quad B'',\quad B''', B^{\mathrm{IV}},\quad B^{\mathrm{V}}\quad \text{etc.},$$
$$\tfrac{dP}{dx} = Q = C,\ C',\ C'',\quad C''', C^{\mathrm{IV}},\ C^{\mathrm{V}}\quad \text{etc.},$$
$$\tfrac{dQ}{dx} = R = D,\ D',\ D'',\ D''', D^{\mathrm{IV}},\quad D^{\mathrm{V}}\quad \text{etc.}$$
etc. ;

then there becomes

$$y = b,\quad b',\quad b'',\quad b''',\quad b^{\mathrm{IV}},\quad b^{\mathrm{V}}\ \text{etc.},$$

with which put in place, as can be deduced from the preceding,

$$b' = b + A'(a'-a) - \tfrac{1}{2}B'(a'-a)^2 + \tfrac{1}{6}C'(a'-a)^3$$
$$-\tfrac{1}{24}D'(a'-a)^4 + \text{etc.},$$
$$b'' = b' + A''(a''-a') - \tfrac{1}{2}B''(a''-a')^2 + \tfrac{1}{6}C''(a''-a')^3$$
$$-\tfrac{1}{24}D''(a''-a')^4 + \text{etc.},$$

$$b''' = b'' + A'''(a'''-a'') - \tfrac{1}{2}B'''(a'''-a'')^2 + \tfrac{1}{6}C'''(a'''-a'')^3$$
$$-\tfrac{1}{24}D'''(a'''-a'')^4 + \text{etc.},$$

$$b^{\mathrm{IV}} = b''' + A^{\mathrm{IV}}(a^{\mathrm{IV}}-a''') - \tfrac{1}{2}B^{\mathrm{IV}}(a^{\mathrm{IV}}-a''')^2 + \tfrac{1}{6}C^{\mathrm{IV}}(a^{\mathrm{IV}}-a''')^3$$
$$-\tfrac{1}{24}D^{\mathrm{IV}}(a^{\mathrm{IV}}-a''')^4 + \text{etc.},$$
etc.,

from which expressions and consequently as far as they can be contained, accordingly the value of *y* may be obtained for the value of *x*, for some difference from the initial value of *a*.

COROLLARIUM 1

318. This method of approximation hence uses that theorem, the truth of which has been shown now in the *Differential Calculus*
[Part II, Ch. 3; *i. e*. a form of Taylor's Theorem for an integral, which is further justified below, but note that the derivative is found at the end of each interval, rather than the beginning, as is now customary]
: for if *y* were a function of *x* of this kind, which on putting *x* = *a* becomes equal to *b*, and there is put in place

$$\frac{dy}{dx}=X,\quad \frac{dX}{dx}=P,\quad \frac{dP}{dx}=Q,\quad \frac{dQ}{dx}=R,\quad \text{etc.}$$

in general to be

$$y=b+X(x-a)-\tfrac{1}{2}P(x-a)^2+\tfrac{1}{6}Q(x-a)^3-\tfrac{1}{24}R(x-a)^4$$
$$+\tfrac{1}{120}S(x-a)^5-\quad \text{etc.}$$

COROLLARY 2

319. If we wish this series to continue to infinity, there is no need to assume that the value of this *x* is only a little different from *a*. Now because that series is returned more converging, it expedites a jump from *a* to *x* separated by an interval and for the single operation described here to be put in place.

COROLLARY 3

320. If we make the values of *x* from *a* to increase by constant differences equal to *a* and letting the final value $a+n\alpha=x$, thus so that, if there arises

$$x=a,\quad a+\alpha,\quad a+2\alpha,\quad a+3\alpha,\quad \cdots\quad x,$$

there is produced

$$X=A,\ A',\quad A'',\quad A''',\quad \cdots\quad X,$$
$$\frac{dX}{dx}=P=B,\ B',\quad B'',\quad B''',\quad \cdots\quad P,$$
$$\frac{dP}{dx}=Q=C,\ C',\ C'',\quad C''',\quad \cdots\quad Q,$$
$$\frac{dQ}{dx}=R=D,\ D',\ D'',\quad D''',\quad \cdots\quad R$$
etc.

and hence

$$y=b,\quad b',\quad b'',\quad b''',\quad \cdots\quad y,$$

for the value *x* = *x*, on deducing the whole series

$$y=b+\alpha\left(A'+A''+A'''+\cdots\ +X\right)$$
$$-\tfrac{1}{2}\alpha^2\left(B'+B''+B'''+\cdots\ +P\right)$$
$$+\tfrac{1}{6}\alpha^3\left(C'+C''+C'''+\cdots\ +Q\right)$$
$$-\tfrac{1}{24}\alpha^4\left(D'+D''+D'''+\cdots+R\right)$$
etc.

SCHOLION 1

321. The demonstration of the theorem mentioned in Corollary I, upon which this method of approximation depends, thus is established from the nature of differentiation. Let y be a function of x, which on putting $x = a$ becomes $y = b$, and we inquire about the value of y, if x exceeds a in some manner. We begin from the maximum value of this, which is x, and by differentiation we descend, and from the differentiations it is apparent,

if there were x	to become y
$x-dx$	$y-dy+ddy-d^3y+d^4y-\text{etc.}$
$x-2dx$	$y-2dy+3ddy-4d^3y+5d^4y-\text{etc.}$
$x-3dx$	$y-3dy+6ddy-10d^3y+15d^4y-\text{etc.}$
.	.
.	.
$x-ndx$	$y-ndy+\frac{n(n+1)}{1\cdot 2}ddy-\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}d^3y+\frac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4}d^4y-\text{etc}$

Now we put $x-ndx=a$; then $n=\frac{x-a}{dx}$ and thus an infinite number; then the resulting value for y by hypothesis must be equal to b, on account of which we have

$$b=y-\frac{(x-a)dy}{dx}+\frac{(x-a)^2ddy}{1\cdot 2dx^2}-\frac{(x-a)^3d^3y}{1\cdot 2\cdot 3dx^3}-\frac{(x-a)^4d^3y}{1\cdot 2\cdot 3\cdot 4dx^4}-\text{etc.},$$

But if now we put in place

$$\frac{dy}{dx}=X,\quad \frac{dX}{dx}=P,\quad \frac{dP}{dx}=Q,\quad \frac{dQ}{dx}=R,\quad \text{etc.}$$

we find as before

$$y=b+X(x-a)-\tfrac{1}{2}P(x-a)^2+\tfrac{1}{6}Q(x-a)^3-\tfrac{1}{24}R(x-a)^4+\text{etc.}$$

From which it is apparent, if x only exceeds a a little, it suffices to put in place

$$y=b+X(x-a),$$

which is the first basis of the proposed approximation, clearly of that boundary, by which X is defined from the greater value of x.

SCHOLIUM 2

322. Just as this reasoning revealed for us only the other assigned upper boundary, thus this reasoning will guide us to the other boundary. Clearly, as before we descended from x to a, thus now we ascend from a to x;

if a should be changed into	then b will be changed into
$a+da$	$b+db$
$a+2da$	$b+2db+ddb$
$a+3da$	$b+3db+3ddb+d^3b$
.	.
.	.
$a+nda$	$b+ndb+\frac{n(n+1)}{1\cdot 2}ddb+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}d^3b+\text{etc.}$

Now let $a+nda=x$ or $n=\frac{x-a}{da}$ and the value of b becomes equal to y. But A, B, C, D etc. shall be the upper values of the functions X, P, Q, R etc., if in place of x we write a, and there will be in the present case

$$A=\frac{db}{da},\quad B=\frac{ddb}{da^2},\quad C=\frac{d^3b}{da^3}\ \text{etc.}$$

On account of which we have

$$y=b+A(x-a)+\tfrac{1}{2}B(x-a)^2+\tfrac{1}{6}C(x-a)^3+\tfrac{1}{24}D(x-a)^4+\ \text{etc.},$$

which series is entirely similar to the above apart from the sign; and if x should exceed a a little, as $b+A(x-a)$ indicates the value of y well enough, hence the above limit assigned is produced. But if we separate the progression from a to x as above in § 320 into equal intervals following the difference α, and the preceding [*i. e.* the derivatives are evaluated at the start of the final interval rather than at the end] final terms in the individual series are denoted by $'X, 'P, 'Q, 'R$ etc., then we have for y as if the other limit

$$\begin{aligned} y=b+\ &\alpha\left(A+A'+A''+\cdots+'X\right)\\ &+\tfrac{1}{2}\alpha^2\left(B+B'+B''+\cdots+'P\right)\\ &+\tfrac{1}{6}\alpha^3\left(C+C'+C''+\cdots+'Q\right)\\ &+\tfrac{1}{24}\alpha^4\left(D+D'+D''+\cdots+'R\right)\\ &\text{etc.,}\end{aligned}$$

thus in order that also by this amended method we have two limits, between which the true value of y is contained. Hence we may proceed with a closer value, if we take the arithmetic mean between these limits, from which there is produced

$$y=b+\alpha\left(A+A'+A''+\cdots+'X\right)-\tfrac{1}{2}\alpha\left(A+X\right)+\tfrac{1}{4}\alpha^2\left(B-P\right)$$
$$+\tfrac{1}{6}\alpha^3\left(C+C'+C''+\cdots+'Q\right)-\tfrac{1}{12}\alpha^3\left(C+Q\right)+\tfrac{1}{48}\alpha^4\left(D-R\right)$$
$$+\tfrac{1}{120}\alpha^5\left(E+E'+E''+\cdots+'S\right)-\tfrac{1}{240}\alpha^5\left(E+S\right)+\tfrac{1}{1440}\alpha^6\left(F-T\right)$$
etc.,

And hence the above approximations can be largely corrected only by adding the member $\tfrac{1}{4}\alpha^2\left(B-P\right)$.

EXAMPLE 1

323. *To express the logarithm of any number x approximately.*

Here therefore there is $y=\int\frac{dx}{x}$, as with the integral thus taken, so that it vanishes on putting *x* = 1; hence therefore *a* = 1 and *b* = 0, and $X=\frac{1}{x}$. Now we suppose to be ascending from unity to *x* through an interval equal to α, and since there shall be

$$P=\frac{dX}{dx}=-\frac{1}{xx},\quad Q=\frac{dP}{dx}=\frac{2}{x^3},\quad R=\frac{dQ}{dx}=-\frac{6}{x^4},$$

for the indices

$$x=1,\qquad 1+\alpha,\qquad 1+2\alpha,\qquad 1+3\alpha,\quad\cdots\quad x$$

there will be

$$X=1,\qquad \frac{1}{1+\alpha},\qquad \frac{1}{1+2\alpha},\qquad \frac{1}{1+3\alpha},\quad\cdots\quad \frac{1}{x},$$
$$P=-1,\qquad -\frac{1}{(1+\alpha)^2},\qquad -\frac{1}{(1+2\alpha)^2},\qquad -\frac{1}{(1+3\alpha)^2},\quad\cdots\quad -\frac{1}{xx},$$
$$Q=2,\qquad \frac{2}{(1+\alpha)^3},\qquad \frac{2}{(1+2\alpha)^3},\qquad \frac{2}{(1+3\alpha)^3},\quad\cdots\quad \frac{2}{x^3},$$
$$R=-6,\qquad -\frac{6}{(1+\alpha)^4},\qquad -\frac{6}{(1+2\alpha)^4},\qquad -\frac{6}{(1+3\alpha)^4},\quad\cdots\quad -\frac{6}{x^4},$$
etc.,

from which we obtain

$$\begin{aligned}lx = \ &\alpha\left(1+\tfrac{1}{1+\alpha}+\tfrac{1}{1+2\alpha}+\tfrac{1}{1+3\alpha}+\cdots+\tfrac{1}{x}\right)\\
&-\tfrac{1}{2}\alpha\left(1+\tfrac{1}{x}\right)-\tfrac{1}{4}\alpha\alpha\left(1-\tfrac{1}{xx}\right)\\
&+\tfrac{1}{3}\alpha^3\left(1+\tfrac{1}{(1+\alpha)^3}+\tfrac{1}{(1+2\alpha)^3}+\tfrac{1}{(1+3\alpha)^3}+\cdots+\tfrac{1}{x^3}\right)\\
&-\tfrac{1}{6}\alpha^3\left(1+\tfrac{1}{x^3}\right)-\tfrac{1}{8}\alpha^4\left(1-\tfrac{1}{x^4}\right)\\
&+\tfrac{1}{5}\alpha^5\left(1+\tfrac{1}{(1+\alpha)^5}+\tfrac{1}{(1+2\alpha)^5}+\tfrac{1}{(1+3\alpha)^5}+\cdots+\tfrac{1}{x^5}\right)\\
&-\tfrac{1}{10}\alpha^5\left(1+\tfrac{1}{x^5}\right)-\tfrac{1}{12}\alpha^6\left(1-\tfrac{1}{x^6}\right)\\
&\text{etc.}\end{aligned}$$

From which if we take $\alpha=\frac{1}{m}$ then

$$\begin{aligned}lx = \ &\left(\tfrac{1}{m}+\tfrac{1}{m+1}+\tfrac{1}{m+2}+\tfrac{1}{m+3}+\cdots+\tfrac{1}{mx}\right)-\tfrac{x+1}{2mx}-\tfrac{xx-1}{4mmxx}\\
&+\tfrac{1}{3}\left(\tfrac{1}{m^3}+\tfrac{1}{(m+1)^3}+\tfrac{1}{(m+2)^3}+\tfrac{1}{(m+3)^3}+\cdots+\tfrac{1}{(mx)^3}\right)-\tfrac{x^3+1}{6m^3x^3}-\tfrac{x^4-1}{8m^4x^4}\\
&+\tfrac{1}{5}\left(\tfrac{1}{m^5}+\tfrac{1}{(m+1)^5}+\tfrac{1}{(m+2)^5}+\tfrac{1}{(m+3)^5}+\cdots+\tfrac{1}{(mx)^5}\right)-\tfrac{x^5+1}{10m^5x^5}-\tfrac{x^6-1}{12m^6x^6}\\
&\text{etc.}\end{aligned}$$

COROLLARY

324. If these progressions are continued to infinity, then the sum of the hindmost parts $=-\frac{1}{2}l\frac{m}{m-1}-\frac{1}{2}l\frac{mx+1}{mx}=-\frac{1}{2}l\frac{mx+1}{(m-1)x}$, now the sum of the first parts [summed vertically] is equal to $\frac{1}{2}l\frac{m+1}{m-1}$;

[The series formed without the minus signs from the last terms are :

$$\tfrac{1}{2}\left(\tfrac{1}{m}+\tfrac{1}{mx}\right)+\tfrac{1}{4}\left(\frac{1}{(m)^2}-\frac{1}{(m^2x^2)}\right)+\tfrac{1}{2}\left(\frac{1}{3m^3}+\frac{1}{3(mx)^3}\right)+\tfrac{1}{4}\left(\frac{1}{2(m^2)^2}-\frac{1}{2(m^2x^2)^2}\right)$$

$$+\tfrac{1}{2}\left(\frac{1}{5m^5}+\frac{1}{5(mx)^5}\right)+\tfrac{1}{4}\left(\frac{1}{3(m^2)^3}-\frac{1}{3(m^2x^2)^3}\right)+\ldots.$$

$$=\tfrac{1}{2}\left(\tfrac{1}{m}+\tfrac{1}{3m^3}+\tfrac{1}{5m^5}+\ldots\right)+\tfrac{1}{2}\left(\tfrac{1}{mx}+\frac{1}{3(mx)^3}+\frac{1}{5(mx)^5}+\ldots.\right)+\tfrac{1}{4}\left(\frac{1}{(m^2)}+\frac{1}{2(m^2)^2}+\frac{1}{3(m^2)^3}+\ldots.\right)-\tfrac{1}{4}\left(\frac{1}{(m^2x^2)}+\frac{1}{2(m^2x^2)^2}+\frac{1}{3(\ldots}\right.$$

$$=\tfrac{1}{4}ln\frac{1+\frac{1}{m}}{1-\frac{1}{m}}+\tfrac{1}{4}ln\frac{1+\frac{1}{mx}}{1-\frac{1}{mx}}+\tfrac{1}{4}ln\frac{m^2}{m^2-1}-\tfrac{1}{4}ln\frac{m^2x^2-1}{m^2x^2}=\tfrac{1}{4}ln\frac{m+1}{m-1}\times\frac{mx+1}{mx-1}\times\frac{m^2}{m^2-1}\times\frac{m^2x^2-1}{m^2x^2}=\tfrac{1}{4}ln\frac{1}{(m-1)^2}\times\frac{m^2x^2-1}{x^2}=\tfrac{1}{2}ln\frac{(mx+1)}{(m-1)x}$$

as required.]

from which, since

$$lx+\tfrac{1}{2}l\frac{mx+1}{(m-1)x}+\tfrac{1}{2}l\frac{m-1}{m+1}=\tfrac{1}{2}l\frac{x(mx+1)}{m+1}$$

then

$$l\frac{x(mx+1)}{m+1}=2\left(\tfrac{1}{m+1}+\tfrac{1}{m+2}+\tfrac{1}{m+3}+\cdots+\tfrac{1}{mx}\right)$$
$$+\tfrac{2}{3}\left(\frac{1}{(m+1)^3}+\frac{1}{(m+2)^3}+\frac{1}{(m+3)^3}+\cdots+\frac{1}{m^3x^3}\right)$$
$$+\tfrac{2}{5}\left(\frac{1}{(m+1)^5}+\frac{1}{(m+2)^5}+\frac{1}{(m+3)^5}+\cdots+\frac{1}{m^5x^5}\right)$$

etc.

which expression thus, if it should be continued to infinity, produces the true value $\log.\frac{x(mx+1)}{m+1}$.

EXAMPLE 2

325. *To express approximately by this method the arc of the circle, of which the tangent is equal to* $\frac{x}{c}$.

The investigation therefore concerns the integral $y=\int\frac{cdx}{cc+xx}$ so that it vanishes on putting $x=0$, and then both $a=0$ and $b=0$, then for sure

$$X=\frac{c}{cc+xx},\quad P=\frac{dX}{dx}=\frac{-2cx}{(cc-3xx)^2},\quad Q=\frac{dP}{dx}=\frac{-2c(cc-3xx)}{(cc+xx)^3},$$
$$R=\frac{dQ}{dx}=\frac{6cx(3cc-4xx)}{(cc+xx)^4},\ S=\frac{dR}{dx}=\frac{6c(3c^4-33ccxx+20x^4)}{(cc+xx)^5}\quad\text{etc.,}$$

which expressions continued to infinity give :

$$y=\frac{cx}{cc+xx}+\frac{cx^3}{(cc+xx)^2}-\frac{cx^3(cc-3xx)}{3(cc+xx)^3}-\frac{cx^5(3cc-4xx)}{4(cc+xx)^4}+\frac{cx^5\left(3c^4-33ccxx+20x^4\right)}{20(cc+xx)^5}+\text{etc.}$$

Now if we may put to increase x through the interval = 1, so that $\alpha=1$, then

$$\begin{array}{llll}
A=\frac{c}{cc}, & B=0, & C=\frac{-2c^3}{c^6}, & D=0 \quad \text{etc.}\\
A'=\frac{c}{cc+1}, & B'=\frac{-2c}{(cc+1)^2}, & C'=\frac{-2c(cc-3)}{(cc+1)^3}, & D'=\frac{6c(3cc-4)}{(cc+1)^4},\\
A''=\frac{c}{cc+4}, & B''=\frac{-4c}{(cc+4)^2}, & C''=\frac{-2c(cc-12)}{(cc+4)^3}, & D''=\frac{12c(3cc-16)}{(cc+4)^4},\\
A'''=\frac{c}{cc+9}, & B'''=\frac{-6c}{(cc+9)^2}, & C'''=\frac{-2c(cc-27)}{(cc+9)^3}, & D'''=\frac{18c(3cc-36)}{(cc+9)^4},\\
\cdot & \cdot & \cdot & \cdot\\
\cdot & \cdot & \cdot & \cdot\\
X=\frac{c}{cc+xx}, & P=\frac{-2cx}{(cc+xx)^2}, & Q=\frac{-2c(cc-3xx)}{(cc+xx)^3}, & R=\frac{6cx(3cc-4xx)}{(cc+xx)^4}
\end{array}$$

and hence

$$\begin{aligned}
y= &\; c\left(\frac{1}{cc}+\frac{1}{cc+1}+\frac{1}{cc+2}+\frac{1}{cc+3}+\cdots+\frac{1}{cc+xx}\right)\\
&-\frac{1}{2c}-\frac{c}{2(cc+xx)}+\frac{cx}{2(cc+xx)^2}\\
&-\frac{c}{3}\left(\frac{1}{c^4}+\frac{cc-3}{(cc+1)^3}+\frac{cc-12}{(m+4)^3}+\frac{cc-27}{(m+9)^3}+\cdots+\frac{cc-3xx}{(cc+xx)^3}\right)\\
&+\frac{1}{6c^3}+\frac{c(cc-3xx)}{6(cc+xx)^3}-\frac{cx(3cc-4xx)}{8(cc+xx)^4}\\
&\text{etc.}
\end{aligned}$$

COROLLARY

326. Hence on putting $c=x=4$, in order that the arises $y=\text{Ang.tang.}1=\frac{\pi}{4}$ then

$$\begin{aligned}
\frac{\pi}{4}=&\frac{1}{4}+\frac{4}{17}+\frac{4}{20}+\frac{4}{25}+\frac{1}{8}-\frac{1}{8}-\frac{1}{16}+\frac{1}{128}\\
&-\frac{4}{3}\left(\frac{1}{256}+\frac{13}{17^3}+\frac{4}{20^3}-\frac{32}{32^3}\right)+\frac{1}{384}-\frac{1}{1536}+\frac{1}{128\cdot256},
\end{aligned}$$

the value of which does not depart much from the truth ; but I offer this example only in the cause of illustration, not as an easier approximation, than is made available by the other method, as expected.

EXAMPLE 3

327. *To assign an approximate value to the integral* $y=\int\frac{e^{-\frac{1}{x}}dx}{x}$ *thus taken, so that it vanishes on putting* $x = 0$.

From the reductions set out above there is :

$$y=\int\frac{e^{-\frac{1}{x}}dx}{x}=e^{-\frac{1}{x}}x-\int e^{-\frac{1}{x}}dx$$

and the part $e^{-\frac{1}{x}}dx$ vanishes on putting $x=0$. Hence we search for the integral

$$z=\int e^{-\frac{1}{x}}dx,$$

because with that found there is had $y=e^{-\frac{1}{x}}x-z$, and now the above method of approximating is used in vain. Therefore since on putting $x = 0$, z vanished, there shall be $a = 0$ and $b = 0$, then indeed as $X=e^{-\frac{1}{x}}$, and hence

$$P=\tfrac{dX}{dx}=e^{-\frac{1}{x}}\tfrac{1}{xx},\quad Q=\tfrac{dP}{dx}=e^{-\frac{1}{x}}\left(\tfrac{1}{x^4}-\tfrac{2}{x^3}\right),$$

$$R=\tfrac{dQ}{dx}=e^{-\frac{1}{x}}\left(\tfrac{1}{x^6}-\tfrac{6}{x^5}+\tfrac{6}{x^4}\right),\ S=\tfrac{dR}{dx}=e^{-\frac{1}{x}}\left(\tfrac{1}{x^8}-\tfrac{12}{x^7}+\tfrac{36}{x^6}-\tfrac{24}{x^5}\right)\quad \text{etc.},$$

with which values continued to infinity there shall be :

$$z=e^{-\frac{1}{x}}\left\{\begin{array}{l}x-\frac{1}{2}+\frac{1}{6}x^3\left(\frac{1}{x^4}-\frac{2}{x^3}\right)-\frac{1}{24}x^4\left(\frac{1}{x^6}-\frac{6}{x^5}+\frac{6}{x^4}\right)\\+\frac{1}{120}x^5\left(\frac{1}{x^8}-\frac{12}{x^7}+\frac{36}{x^6}-\frac{24}{x^5}\right)-\quad\text{etc.}\end{array}\right\}$$

or

$$z=e^{-\frac{1}{x}}\left\{\begin{array}{l}x-\frac{1}{2}+\frac{1}{6}\left(\frac{1}{x}-2\right)-\frac{1}{24}\left(\frac{1}{xx}-\frac{6}{x}+6\right)\\+\frac{1}{120}\left(\frac{1}{x^3}-\frac{12}{x^2}+\frac{36}{x}-24\right)-\frac{1}{720}\left(\frac{1}{x^4}-\frac{20}{x^3}+\frac{120}{x^2}-\frac{240}{x}+120\right)\quad\text{etc.}\end{array}\right\}$$

which series converges little, for whatever value of x is put in place. Therefore through the interval from 0 rising as far as x on putting for x successively $0,\alpha,\ 2\alpha,3\alpha$ etc., where it is to be noted that $A=0,B=0,C=0,D=0$ etc., and our rule presents :

$$\begin{aligned}z=\ &\alpha\left(e^{-\frac{1}{\alpha}}+e^{-\frac{1}{2\alpha}}+e^{-\frac{1}{3\alpha}}+\cdots+e^{-\frac{1}{x}}\right)-\tfrac{1}{2}\alpha e^{-\frac{1}{x}}-\tfrac{1}{4}\alpha^2e^{-\frac{1}{x}}\tfrac{1}{xx}\\&+\tfrac{1}{6}\alpha^3\left(e^{-\frac{1}{\alpha}}\left(\tfrac{1}{\alpha^4}-\tfrac{2}{\alpha^3}\right)+e^{-\frac{1}{2\alpha}}\left(\tfrac{1}{16\alpha^4}-\tfrac{2}{8\alpha^3}\right)+e^{-\frac{1}{3\alpha}}\left(\tfrac{1}{81\alpha^4}-\tfrac{2}{27\alpha^3}\right)+\cdots+e^{-\frac{1}{x}}\left(\tfrac{1}{x^4}-\tfrac{2}{x^3}\right)\right)\\&-\tfrac{1}{12}\alpha^3e^{-\frac{1}{x}}\left(\tfrac{1}{x^4}-\tfrac{2}{x^3}\right)-\tfrac{1}{48}\alpha^4e^{-\frac{1}{x}}\left(\tfrac{1}{x^6}-\tfrac{6}{x^5}+\tfrac{6}{x^4}\right).\end{aligned}$$

Hence if we wish to determine the value of *z* for the case $x = 1$ and for the small fraction α we assume $\frac{1}{n}$, we will have :

$$z=\frac{1}{n}\left(e^{-\frac{n}{1}}+e^{-\frac{n}{2}}+e^{-\frac{n}{3}}+e^{-\frac{n}{4}}+\cdots+e^{-\frac{n}{n}}\right)-\frac{1}{2ne}-\frac{1}{4nne}$$
$$+\frac{1}{6}\left(e^{-\frac{n}{1}}\frac{n-2}{1}+e^{-\frac{n}{2}}\frac{n-4}{16}+e^{-\frac{n}{3}}\frac{n-6}{81}+\cdots+e^{-\frac{n}{n}}\frac{n-2n}{n^4}\right)$$
$$+\frac{1}{12n^3e}-\frac{1}{48n^4e}.$$

Here if for *n* there is taken a number not very large, such as 10, the value of *z* to the millionth part of one is found exactly and in turn that will be produced more accurately if we should take 20 for *n*.

SCHOLIUM 1

328. This example suffices to show the excellent use of this method of approximation. Yet meanwhile cases arise, in which indeed this method cannot be used, even if we divide the whole interval through which the variable *x* increases into the smallest intervals. This eventuates when the function *X* for some interval, while a certain value is attributed to the variable *x*, increases to infinity, since yet the value of the integral $y=\int Xdx$ in this case does not become infinite ; just as if there should be

$$y=\int\frac{dx}{\sqrt{(a-x)}},$$

where $X=\frac{1}{\sqrt{(a-x)}}$, which on putting $x=a$ is made infinite, now the integral

$$y=C-2\sqrt{(a-x)}$$

in this case is finite. But this is used to eventuate, whenever a factor of this kind $a-x$ has the exponent in the denominator less than one ; while indeed the same factor in the integral is moved into the numerator ; but if the exponent of the same kind of factors in the denominator is unity or thus greater than unity, then also in the case $x = a$ the integral itself becomes infinite ; in which case because the approximation ceases, here for these there can only be talk, where the exponent is less than one, since then the approximation is actually disturbed. Now it is easy to offer a cure for this inconvenience ; for since the differential of a form of this kind shall be had $\frac{Xdx}{(a-x)^{\lambda:\mu}}$ with $\lambda<\mu$ being present, there is put $a-x=z^{\mu}$, so that there becomes $x=a-z^{\mu}$ and $dx=-\mu z^{\mu-1}dz$, and our differential will be equal to $-\mu Xz^{\mu-\lambda-1}dz,$ in which case $x = a$ or $z = 0$ shall no further be infinite. Or, which returns the same, for these intervals, in which the function *X* becomes infinite, the integration separately is put in place on putting $x=a\pm\omega$; for then the formula *Xdx* is made simple enough on account of ω being very small, so that the integration has no difficulty. Just as if we are able now to attend to the value of this $y=\int\frac{xxdx}{\sqrt{(a^4-x^4)}}$ for the interval from $x = 0$ as far as $x=a-\alpha$, for this final interval we put $x=a-\omega$ and it is required to integrate

$$\frac{(a-\omega)^2 d\omega}{\sqrt{(4a^3\omega-6aa\omega\omega+4a\omega^3-\omega^4)}},$$

which on account of ω definitely being small changes into

$$\frac{d\omega\sqrt{a}}{2\sqrt{\omega}}\left(1-\frac{5\omega}{4a}-\frac{5\omega\omega}{32aa}\right),$$

the integral of this on taking $\omega=\alpha$ is

$$\sqrt{a\alpha}-\frac{5\alpha\sqrt{\alpha}}{12\sqrt{a}}-\frac{5\alpha\alpha\sqrt{\alpha}}{32a\sqrt{a}},$$

[here we have followed the correction offered in the *O. O.* edition] because, if it is continued to more terms, not only for the final interval but also it can be obtained for two or more of the latter intervals on putting $\omega=2\alpha$ or $\omega=3\alpha$. For which intervals the denominator now becomes small enough, and this method is better to be used than that which was presented before.

SCHOLIUM 2

329. Meanwhile another inconvenience also happens, as the denominator may vanish in two cases, just as if there should be

$$y=\int\frac{Xdx}{\sqrt{(a-x)(x-b)}}$$

where the variable *x* must always be contained between the two bounds *b* and *a*, thus in order that, since *x* may have increased from *b* to *a*, then again it may decrease from *a* to *b*; but meanwhile the integral *y* continually goes on increasing, therefore the value of this cannot be determined conveniently. Hence in this case this substitution is called in aid

$$x=\tfrac{1}{2}(a+b)-\tfrac{1}{2}(a-b)\cos.\varphi,$$

from which there arises

$$dx=\tfrac{1}{2}(a-b)d\varphi\sin.\varphi$$

and

$$(a-x)(x-b)=\left(\tfrac{1}{2}(a-b)+\tfrac{1}{2}(a-b)\cos.\varphi\right)\left(\tfrac{1}{2}(a-b)-\tfrac{1}{2}(a-b)\cos.\varphi\right)$$

or

$$(a-x)(x-b)=\tfrac{1}{4}(a-b)^2\sin.^2\varphi,$$

from which there arises $y = \int Xd\varphi$, which no longer labours under an inconvenience, since the angle φ is allowed to increase continually further uniformly.

Also this extends to the cases, where the two factors in the denominator do not have the same exponent, just as if there should be

$$y = \int \frac{Xdx}{\sqrt[2\lambda]{(a-x)^{\mu}(x-b)^{\nu}}}$$

thus in order that μ and ν are less than 2λ, which I consider an even exponent. Now if μ and ν should not be equal, but $\nu < \mu$, then the integrand is reduced to equality in this manner

$$y = \int \frac{Xdx\sqrt[2\lambda]{(x-b)^{\mu-\nu}}}{\sqrt[2\lambda]{(a-x)^{\mu}(x-b)^{\nu}}}$$

Because if now as before there is put

$$x = \tfrac{1}{2}(a+b) - \tfrac{1}{2}(a-b)\cos.\varphi,$$

there will be obtained :

$$y = \left(\frac{a-b}{2}\right)^{\frac{2\lambda-\mu-\nu}{2\lambda}} \int Xd\varphi \sin^{\frac{\lambda-\mu}{\lambda}} (1-\cos.\varphi)^{\frac{\mu-\nu}{2\lambda}},$$

where the angle φ continues as far as it is wished, and the method for the preceding interval is allowed to be used. With which observations, this method of approximation need not be lingered over further.

CAPUT VII

METHODUS GENERALIS INTEGRALlA QUAECUNQUE PROXIME INVENIENDI

PROBLEMA 36

297. *Formulae integralis cuiuscunque* $y=\int Xdx$ *valorem vero proxime indagare.*

SOLUTIO

Cum omnis formula integralis per se sit indeterminata, ea semper ita determinari solet, ut, si variabili *x* certus quidam valor, puta *a*, tribuatur, ipsum integrale $y=\int Xdx$ datum valorem, puta *b*, obtineat. Integratione igitur hoc modo determinata quaestio huc redit, ut, si variabili *x* alius quicun que valor ab *a* diversus tribuatur, valor, quem tum integrale *y* sit habiturum, definiatur. Tribuamus ergo ipsi *x* primo valorem parum ab *a* discrepantem, puta $x=a+\alpha$, ut α sit quantitas valde parva, et quia functio X parum variatur, sive pro *x* scribatur *a* sive $a+\alpha$, eam tanquam constantem spectare licebit. Hinc ergo formulae differentialis *Xdx* integrale erit $Xx+\text{Const.}=y$; sed quia posito *x* = *a* fieri debet *y* = *b* et valor ipsius X quasi manet immutatus, erit *Xa* + Const. = *b* ideoque $\text{Const.}=b-Xa$, unde consequimur $y=b+X\left(x-a\right)$. Quare si ipsi *x* valorem $a+\alpha$ tribuamus, habebimus valorem convenientem ipsius *y*, qui sit = $b+\beta$; ac iam simili modo ex hoc casu definire
poterimus *y*, si ipsi *x* tribuatur alius valor parum superans $a+\alpha$; posito igitur $a+\alpha$ loco *x* valor ipsius X inde ortus denuo pro constante haberi poterit indeque fiet $y=b+\beta+X\left(x-a-\alpha\right)$. Hanc igitur operationem continuare licet, quousque lubuerit; cuius ratio quo melius perspiciatur, rem ita repraesentemus:

$$\begin{aligned}
&\text{si } x=a, && \text{fiat } X=A && \text{et } y=b,\\
&\text{si } x=a', && \text{fiat } X=A' && \text{et } y=b'=b+A(a'-a),\\
&\text{si } x=a'', && \text{fiat } X=A'' && \text{et } y=b''=b'+A'(a''-a'),\\
&\text{si } x=a''', && \text{fiat } X=A''' && \text{et } y=b'''=b''+A''(a'''-a'')
\end{aligned}$$

etc.,

ubi valores *a, a', a'', a'''* etc. secundum differentias valde parvas procedere ponuntur. Erit ergo $b'=b+A\left(a'-a\right)$, quippe in quam abit formula inventa $y=b+X\left(x-a\right)$; fit enim $X = A$, quia ponitur $x = a$; tum vero tribuitur ipsi x valor $= a'$, cui respondet $y = b'$; simili modo erit $b'' = b' + A'\left(a'' - a'\right)$, tum $b''' = b'' + A''\left(a''' - a''\right)$ etc., uti supra posuimus. Restituendo ergo valores praecedentes habebimus

$$b' = b + A\left(a' - a\right),$$
$$b'' = b + A\left(a' - a\right) + A'\left(a'' - a'\right),$$
$$b''' = b + A\left(a' - a\right) + A'\left(a'' - a'\right) + A''\left(a''' - a''\right),$$
$$b'''' = b + A\left(a' - a\right) + A'\left(a'' - a'\right) + A''\left(a''' - a''\right) + A'''\left(a'''' - a'''\right)$$
etc. ,

unde, si x quantumvis excedet a, series a', a'', a''' etc. crescendo continuetur ad x at ultimum aggregatum dabit valorem ipsius y.

COROLLARIUM 1

298. Si incrementa, quibus x augetur, aequalia statuantur, scilicet a, ut sit $a' = a + a,\ a'' = a + 2a,\ a''' = a + 3a$ etc., quibus valoribus pro x substitutis functio X abeat in A', A'', A''' etc., atque ultimus illorum valorum, puta $a + n\alpha$, sit $= x$, horum vero X, erit

$$y = b + a\left(A + A' + A'' + A''' + \ ... \ + X\right).$$

COROLLARIUM 2

299. Valor ergo integralis y per summationem seriei $A, A', A'', ...\ X$, cuius termini ex formula X formantur ponendo loco x successive $a,\ a + \alpha, a + 2\alpha, ... a + n\alpha$, eruitur. Summa enim illius seriei per differentiam α multiplicata et ad b adiecta dabit valorem ipsius y, qui ipsi $x = a + n\alpha$ respondet.

COROLLARIUM 3

300. Quo minores statuuntur differentiae, secundum quas valor ipsius x increscat, eo accuratius hoc modo valor ipsius y definitur, siquidem termini seriei A, A', A'' etc. inde etiam secundum parvas differentias progrediantur; nisi enim hoc eveniat, illa determinatio nimis erit incerta.

COROLLARIUM 4

301. Haec ergo approximatio ex doctrina serierum ita explicatur.
Ex indicibus

$$a,\ a',\ a'',\ a'', \cdots x$$

formetur series

$$A, A', A'', A''',\ \cdots X,$$

cuius ergo terminus generalis X ex formula differentiali $dy = Xdx$ datur. Tum in hac serie sit terminus ultimum praecedens $'X$, respondens indici $'x$ hincque nova formetur series

$$A(a'-a),\ A'(a''-a'),\ A''(a'''-a''),\cdots'X(x-'x);$$

cuius summa si ponatur = S, erit integrale $y=\int Xdx=b+S$ proxime.

SCHOLION 1

302. Hoc modo integratio vulgo explicari solet, ut dicatur esse summatio omnium valorum formulae differentialis Xdx, si variabili x successive omnes valores a dato quodam a usque ad x tribuantur, qui secundum differentiam dx procedunt, hanc differentiam autem infinite parvam accipi oportere. Similis igitur haec ratio integrationem repraesentandi est illi, qua in Geometria lineae ut aggregata infinitorum punctorum concipi solent; quae idea quemadmodum, si rite explicetur, admitti potest, ita etiam illa integrationis explicatio tolerari potest, dummodo ad vera principia, uti hic fecimus, revocetur, ut omni cavillationi occurratur. Ex methodo igitur exposita utique patet integrationem per summationem vero proxime obtineri posse neque vero exacte expediri, nisi differentiae infinite parvae, hoc est nullae, statuantur. Atque ex hoc fonte tam nomen integrationis, quae etiam summatio vocari solet, quam signum integralis $\int$ est natum, quae re bene explicata omnino retineri possunt.

SCHOLION 2

303. Si pro singulis intervallis, in quae saltum ab a ad x distinximus, quantitates A, A', A'', A''' etc. revera essent constantes, integrale $\int Xdx$ accurate impetraremus. Eatenus ergo error inest, quatenus pro singulis illis intervallis istae quantitates non sunt constantes. Ac pro primo quidem intervallo, quo variabilis x a termino a ad a' procedit, A est valor ipsius X termino a conveniens, alteri autem termino a' respondet A'; unde, quatenus non est $A'=A$, eatenus error irrepit. Cum igitur in istius intervalli initio sit $X=A$, in fine autem $X=A'$, conveniret potius medium quoddam inter A et A' assumi, id quod in correctione huius methodi mox tradenda observabitur. Interim hic notasse iuvabit pari iure pro quovis intervallo valorem tam finalem quam initialem capi posse, ubi simul hoc perspicitur, si altero modo in excessu peccetur, altero plerumque in defectu errari. Ex quo hinc binas expressiones eruere licet, quarum altera valorem ipsius y nimis magnum, altera nimis parvum sit praebitura, ita ut illae quasi limites veri valoris ipsius y constituant. Quemadmodum ergo rem repraesentavimus § 301, valor ipsius $y=\int Xdx$ intra hos duos limites continebitur

$$b+A(a'-a)+A'(a''-a')+A''(a'''-a'')+\cdots+'X(x-'x)$$

et

$$b+A'(a'-a)+A''(a''-a')+A'''(a'''-a'')+\cdots+X(x-'x),$$

quibus cognitis ad veritatem propius accedere licet.

SCHOLION 3

304. Iam notavimus intervalla illa, per quae x successive increscere assumimus, ideo valde parva statui debere, ut valores respondentes A, A',A'' etc. parum a se invicem discrepent; atque hinc potissimum iudicari oportet, utrum illa intervalla $a'-a,\ a''-a',\ a'''-a''$ etc. inter se aequalia an inaequalia capi conveniat. Ubi enim valor ipsius X mutando x parum mutatur, ibi intervalla, per quae x procedit, tuto maiora constitui possunt; ubi autem evenit, ut ipsi x levi mutatione inducta functio X vehementer varietur, ibi intervalla minima accipi debent. Veluti si sit $X=\frac{1}{\sqrt{(1-xx)}}$, perspicuum est, ubi x proxime ad unitatem accedit, quantumvis parvum intervalIum, per quod x augeatur, accipiatur, functionem X maximam mutationem pati posse, quia tandem sumto $x = 1$ ea adeo in infinitum excrescit. His igitur casibus ista approximatione pro eo saltem intervallo, in cuius altero termino X fit infinita, uti non licet; sed huic incommodo facile remedium affertur, dum formula ope idoneae substitutionis in aliam transformatur vel dum pro hoc saltem intervallo peculiaris integratio instituitur. Veluti si proposita sit formula $\frac{xdx}{\sqrt{\left(1-x^3\right)}}$, , pro intervallo ab $x=1-\omega$ ad $x = 1$ illa methodo integrale non reperitur, at posito $x=1-z$, quia termini ipsius z sunt 0 et ω, erit z quantitas minima, unde formula erit $\frac{dz(1-z)}{\sqrt{\left(3z-3z^2+z^3\right)}}=\frac{dz}{\sqrt{3z}}$, cuius integrale $\frac{2\sqrt{z}}{\sqrt{3}}$ pro intervallo illo praebet partem integralis $\frac{2\sqrt{\omega}}{\sqrt{3}}$. Quod artificium in omnibus huiusmodi casibus adhiberi potest; ipsam autem methodum descriptam aliquot exemplis illustrari opus est.

EXEMPLUM 1

305. *Integrale* $y=\int x^n dx$ *ita sumtum, ut evanescat posito* $x = 0$, *proxime exhibere.*

Hic est $a = 0$ et $b= 0$, tum $X=x^n$; iam valores ipsius x a 0 crescant per communem differentiam α, ut sint

$$\begin{array}{lllllll} \text{indices } 0, & \alpha, & 2\alpha, & 3\alpha, & 4\alpha, & \cdots & x, \\ \text{series } 0, & \alpha^n, & 2^n\alpha^n, & 3^n\alpha^n, & 4^n\alpha^n, & \cdots & x^n, \end{array}$$

et terminus ultimum praecedens est $(x-\alpha)^n$, quare integralis

$$y=\int x^n dx=\frac{1}{n+1}x^{n+1}$$

limites sunt

$$\alpha\left(0+\alpha^n+2^n\alpha^n+3^n\alpha^n+\quad\cdots\quad+(x-\alpha)^n,\right)$$

et

$$\alpha\left(\alpha^n+2^n\alpha^n+3^n\alpha^n+\quad\cdots\quad+x^n\right),$$

qui eo erunt arctiores, quo minus intervallum α accipiatur. Ita si $\alpha=1$, erunt limites

$$0+1+2^n+3^n+4^n+\quad\cdots\quad+(x-1)^n$$

et

$$1+2^n+3^n+4^n+\quad\cdots\quad+x^n;$$

si sumatur $\alpha=\frac{1}{2}$, erunt limites

$$\tfrac{1}{2^{n+1}}\left(0+1+2^n+3^n+4^n+\quad\cdots\quad+(2x-1)^n\right)$$

et

$$\tfrac{1}{2^{n+1}}\left(1+2^n+3^n+4^n+\quad\cdots\quad+(2x)^n\right);$$

ac si in genere sit $\alpha=\frac{1}{m}$, erunt limites

$$\tfrac{1}{m^{n+1}}\left(0+1+2^n+3^n+4^n+\quad\cdots\quad+(mx-1)^n\right)$$

et

$$\tfrac{1}{m^{n+1}}\left(1+2^n+3^n+4^n+\quad\cdots\quad+(mx)^n\right),$$

quorum hic illum superat excessu $\frac{x^n}{m}$ unde patet, si numerus *m* sumatur infinitus, utrumque limitem verum integralis $y=\frac{1}{n+1}x^{n+1}$ esse praebiturum valorem.

COROLLARIUM 1

306. Seriei ergo $1+2^n+3^n+4^n+\quad\cdots\quad+(mx)^n$ summa eo propius ad $\frac{1}{n+1}(mx)^{n+1}$ accedit, quo maior capiatur numerus *m;* quare posito $mx=z$ huius progressionis

$$1+2^n+3^n+4^n+\quad\cdots\quad+z^n$$

summa eo propius ad $\frac{1}{n+1}z^{n+1}$ accedit, quo maior fuerit numerus *z*.

COROLLARIUM 2

307. Ex priore autem limite posito $mx = z$ eadem quantitas $\frac{1}{n+1}z^{n+1}$ proxime exhibet summam huius seriei

$$0+1+2^n+3^n+4^n+\quad\cdots\quad+(z-1)^n,$$

unde medium sumendo erit accuratius

$$1+2^n+3^n+4^n+\quad\cdots\quad+(z-1)^n+\tfrac{1}{2}z^n=\tfrac{1}{n+1}z^{n+1}$$

seu addendo utrinque $\frac{1}{2}z^n$ habebimus

$$1+2^n+3^n+4^n+\quad\cdots\quad+z^n=\tfrac{1}{n+1}z^{n+1}+\tfrac{1}{2}z^n \text{ proxime,}$$

quod congruit cum iis, quae de vera huius progressionis summa sunt cognita.

EXEMPLUM 2

308. *Integrale* $y=\int\frac{dx}{x^n}$ *ita sumtum, ut evanescat posito* $x = 1$, *proxime exhibere.*

Erit ergo $a = 1$ et $b = 0$, unde, si ab a ad x intervallum progressionis statuatur $=\alpha$, erunt

$$\text{indices } a,\ a+\alpha,\quad a+2\alpha,\quad a+3\alpha,\cdots x$$

et

$$\text{termini seriei } \frac{1}{a^n},\ \frac{1}{(a+\alpha)^n},\ \frac{1}{(a+2\alpha)^n},\ \frac{1}{(a+3\alpha)^n},\cdots\frac{1}{(x)^n}=X,$$

ubi terminus ultimum praecedens est $\frac{1}{(x-\alpha)^n}={}'X$. Cum nunc nostrum integrale sit

$$y=\frac{1}{n-1}-\frac{1}{(n-1)x^n},$$

eius valor intra hos limites continebitur

$$\alpha\left(1+\frac{1}{(1+\alpha)^n}+\frac{1}{(1+2\alpha)^n}+\frac{1}{(1+3\alpha)^n}+\cdots+\frac{1}{(x-\alpha)^n}\right)$$

et

$$\alpha\left(\frac{1}{(1+\alpha)^n}+\frac{1}{(1+2\alpha)^n}+\frac{1}{(1+3\alpha)^n}+\cdots+\frac{1}{x^n}\right).$$

Quare posito $\alpha=\frac{1}{m}$ erunt hi limites

$$m^{n-1}\left(\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\cdots+\frac{1}{(mx-1)^n}\right)$$

et

$$m^{n-1}\left(\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\frac{1}{(m+4)^n}+\cdots+\frac{1}{(mx)^n}\right),$$

qui, quo maior accipiatur numerus m, eo propius ad valorem integralis $y=\frac{1}{n-1}-\frac{1}{(n-1)x^{n-1}}$ accedunt. Notandum autem est casu $n = 1$ integrale fore $= lx$.

COROLLARIUM 1

309. Quodsi ponamus $mx=m+z$, ut sit $x=\frac{m+z}{m}$, prodibunt hae progressiones

$$m^{n-1}\left(\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\cdots+\frac{1}{(m+z-1)^n}\right)$$

et

$$m^{n-1}\left(\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\frac{1}{(m+4)^n}+\cdots+\frac{1}{(m+z)^n}\right),$$

quarum summa alterius maior est, alterius minor quam

$$\frac{1}{n-1}-\frac{m^{n-1}}{(n-1)(m+z)^{n-1}}=\frac{(m+z)^{n-1}-m^{n-1}}{(n-1)(m+z)^{n-1}} \;;$$

casu autem $n = 1$ haec expressio abit in $l\left(1+\frac{z}{m}\right)$.

COROLLARIUM 2

310. Cum prior progressio maior sit quam posterior, erit

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\cdots+\frac{1}{(m+z-1)^n}>\frac{(m+z)^{n-1}-m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}},$$

$$\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\cdots+\frac{1}{(m+z)^n}<\frac{(m+z)^{n-1}-m^{n-1}}{(n-1)m^{n-1}(m+z)^{n-1}} \;;$$

addatur hic utrinque $\frac{1}{m^n}$ ibi vero $\frac{1}{(m+z)^n}$ et sumatur medium arithmeticum;

erit exactius

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\cdots+\frac{1}{(m+z)^n}=\frac{(2m+n-1)(m+z)^n-(2z+2m-n+1)m^n}{2(n-1)m^n(m+z)^n},$$

quae expressio casu $n = 1$ abit in $l\left(1+\frac{z}{m}\right)+\frac{1}{2m}+\frac{1}{2(m+z)}$.

COROLLARIUM 3

311. Ponatur $z = mv$ et habebimus sequentis seriei summam proxime expressam

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\cdots+\frac{1}{m^n(1+v)^n}=\frac{(2m+n-1)(1+v)^n-2m(1+v)+n-1}{2(n-1)m^n(1+v)^n}$$

et casu $n=1$

$$\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{m+mv}=l(1+v)+\frac{2+v}{2m(1+v)},$$

unde, si $v=1$, erit proxime

$$\frac{1}{m^n}+\frac{1}{(m+1)^n}+\frac{1}{(m+2)^n}+\frac{1}{(m+3)^n}+\cdots+\frac{1}{2^n m^n}=\frac{2^n(2m+n-1)-4m+n-1}{2^{n+1}(n-1)m^n}$$

et

$$\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{2m}=l2+\frac{3}{4m}.$$

COROLLARIUM 4

312. Hinc nascitur regula logarithmos quantumvis magnorum numerorum proxime assignandi, dum series vulgares tantum pro numeris parum ab unitate differentibus valent. Scribamus enim u pro $1+v$ et habebimus

$$lu=\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\cdots+\frac{1}{mu}-\frac{1+u}{2mu},$$

unde lu eo accuratius definitur, quo maior sumatur numerus m.

EXEMPLUM 3

313. *Integrale* $y=\int\frac{cdx}{cc+xx}$ *ita sumtum, ut evanescat posito* $x = 0$, *proxime exprimere.*

Hoc integrale, ut novimus, est $y = \text{Ang. tang.}\frac{x}{c}$, ad quem valorem proxime exhibendum est $a = 0$ et $b = 0$; si ergo valor ipsius x ab 0 per differentiam constantem a crescere statuatur, ob $X = \frac{cdx}{cc+xx}$ erunt eius valores

$$\begin{array}{lllll} \text{pro indicibus } 0, & a, & 2a, & \cdots & x \\ \text{series } \frac{1}{c}, & \frac{c}{cc+\alpha\alpha}, & \frac{c}{cc+4\alpha\alpha}, & \cdots & \frac{c}{cc+xx}, \end{array}$$

cuius terminus ultimum praecedens est $'X = \frac{c}{cc+(x-\alpha)^2}$. Quare integralis nostri $y = \text{Ang. tang.}\frac{x}{c}$ valor proxime est

$$\alpha\left(\frac{1}{c}+\frac{c}{cc+\alpha\alpha}+\frac{c}{cc+4\alpha\alpha}+\cdots+\frac{c}{cc+(x-\alpha)^2}\right),$$

alter vero proxime minor, quia hic est nimis magnus, est

$$\alpha\left(\frac{c}{cc+\alpha\alpha}+\frac{c}{cc+4\alpha\alpha}+\frac{c}{cc+9\alpha\alpha}+\cdots+\frac{c}{cc+xx}\right).$$

Inter quos si medium capiatur, ibi $\alpha\cdot\frac{1}{c}$ hic vero $\alpha\cdot\frac{c}{cc+xx}$ adiiciendo propius erit

$$\alpha\left(\frac{c}{cc}+\frac{c}{cc+\alpha\alpha}+\frac{c}{cc+4\alpha\alpha}+\cdots+\frac{c}{cc+xx}\right)$$

$$= \text{Ang. tang.}\frac{x}{c}+\frac{\alpha}{2}\left(\frac{1}{c}+\frac{c}{cc+xx}\right) = \text{Ang. tang.}\frac{x}{c}+\frac{\alpha(2cc+xx)}{2c(cc+xx)}.$$

Pro hoc ergo angulo valorem proxime verum habemus

$$\text{Ang. tang.}\frac{x}{c} = \alpha c\left(\frac{1}{cc}+\frac{1}{cc+\alpha\alpha}+\frac{1}{cc+4\alpha\alpha}+\cdots+\frac{1}{cc+xx}\right)-\frac{\alpha(2cc+xx)}{2c(cc+xx)},$$

qui eo minus a veritate discrepabit, quo minor fuerit α numerus ratione ipsius c. Quodsi ergo pro c numerum valde magnum sumamus, pro α unitatem accipere licet, unde posito $x = cv$ erit

$$\text{Ang. tang.}v = c\left(\frac{1}{cc}+\frac{1}{cc+1}+\frac{1}{cc+4}+\frac{1}{cc+9}+\cdots+\frac{1}{cc+ccvv}\right)-\frac{2+vv}{2c(1+vv)}$$

idque eo exactius, quo maior capiatur numerus c.

COROLLARIUM 1

314. Si ponamus $c = 1$, quo casu error insignis esse debet, fiet

$$\text{Ang. tang.}v = 1 + \tfrac{1}{1+1} + \tfrac{1}{1+4} + \tfrac{1}{1+9} + \cdots + \tfrac{1}{1+vv} - \tfrac{2+vv}{2(1+vv)}.$$

Sit $v = 1$; erit $\text{Ang. tang.}1 = \frac{\pi}{4} = 1 + \frac{1}{2} - \frac{3}{4} = \frac{3}{4}$ hincque $\pi = 3$, quod non multum abhorret a vero.

Si ponamus $c = 2$, prodit

$$\text{Ang. tang.}v = 2\left(\tfrac{1}{4} + \tfrac{1}{4+1} + \tfrac{1}{4+4} + \tfrac{1}{4+9} + \cdots + \tfrac{1}{4+4vv}\right) - \tfrac{2+vv}{4(1+vv)},$$

unde, si $v = 1$, colligitur $\text{Ang. tang.}1 = \frac{\pi}{4} = 2\left(\frac{1}{4} + \frac{1}{4+1} + \frac{1}{4+4}\right) - \frac{3}{8} = \frac{23}{20} - \frac{3}{8} = \frac{31}{40}$
sicque $\pi = \frac{31}{10} = 3{,}1$ propius accedens.

COROLLARIUM 2

315. Sit $c = 6$ eritque

$$\text{Ang. tang.}v = 6\left(\tfrac{1}{36} + \tfrac{1}{36+1} + \tfrac{1}{36+4} + \cdots + \tfrac{1}{36+36vv}\right) - \tfrac{2+vv}{12(1+vv)},$$

unde, si $v = \frac{1}{2}$ et $v = \frac{1}{3}$, oritur

$$\text{Ang. tang.}\tfrac{1}{2} = 6\left(\tfrac{1}{36} + \tfrac{1}{36+1} + \tfrac{1}{36+4} + \tfrac{1}{36+9}\right) - \tfrac{3}{20},$$
$$\text{Ang. tang.}\tfrac{1}{3} = 6\left(\tfrac{1}{36} + \tfrac{1}{36+1} + \tfrac{1}{36+4}\right) - \tfrac{19}{120}.$$

At est $\text{Ang.tang.}\frac{1}{2} + \text{Ang.tang.}\frac{1}{3} = \text{Ang. tang.}1 = \frac{\pi}{4}$. Ergo

$$\tfrac{\pi}{4} = 12\left(\tfrac{1}{36} + \tfrac{1}{37} + \tfrac{1}{40}\right) + \tfrac{2}{15} - \tfrac{37}{120} = \tfrac{1063}{1110} - \tfrac{7}{40} = \tfrac{695}{888}$$

seu $\pi = \frac{695}{222} = 3{,}1306.$

COROLLARIUM 3

316. Sin autem ibi statim ponamus $v = 1$, erit

$$\tfrac{\pi}{4} = 6\left(\tfrac{1}{36} + \tfrac{1}{37} + \tfrac{1}{40} + \tfrac{1}{45} + \tfrac{1}{52} + \tfrac{1}{72}\right) - \tfrac{1}{8},$$

unde fit $\pi = 3{,}13696$ multo propius veritati ; plurium scilicet terminorum additio propius ad veritatem perducit.

PROBLEMA 37

317. *Methodum ad integralium valores appropinquandi ante expositam perfectiorem reddere, ut minus a veritate aberretur.*

SOLUTIO

Sit $y=\int Xdx$ formula integralis proposita, cuius valorem iam constet esse $y = b$, si ponatur $x = a$, sive is sit datus per ipsam integrationis conditionem sive iam per aliquot operationes inde derivatus; ac tribuamus iam ipsi x valorem parum superantem illum a, cui respondet $y = b$, tum vero fiat $X = A$, si ponatur $x = a$. In superiori autem methodo assumsimus, dum x parum supra a excrescit, manere X constantem $=A$ ideoque fore

$$\int Xdx = A(x-a).$$

At quatenus X non est constans, eatenus non est $\int Xdx = A(x-a)$, sed revera habetur

$$\int Xdx = X(x-a)-\int (x-a)dX.$$

Ponamus igitur $dX = Pdx$ eritque

$$\int (x-a)dX = \int P(x-a)dx,$$

et si iam $P=\frac{dX}{dx}$, quamdiu x non multum a excedit, ut constantem spectemus, habebimus

$$\int P(x-a)dx = \tfrac{1}{2}P(x-a)^2$$

sicque fiet

$$y=\int Xdx = b + X(x-a)-\tfrac{1}{2}P(x-a)^2,$$

qui valor iam propius ad veritatem accedit, etsi pro X et P ii valores capiantur, quos induunt vel posito $x=a$ vel posito $x=a+\alpha$, maiore scilicet valore, ad quem hac operatione x crescere statuimus; ex quo hinc, prout vel $x = a$ vel $x=a+\alpha$ ponimus, geminos limites obtinebimus, inter quos veritas subsistit. Simili autem modo ulterius progredi poterimus; cum enim P non sit constans, erit

$$\int P(x-a)dx = \tfrac{1}{2}P(x-a)^2-\tfrac{1}{2}\int (x-a)^2 dP,$$

unde, si statuamus $dP = Qdx$, erit

$$\int (x-a)^2\, dP = \int Q(x-a)^2\, dx = \tfrac{1}{3}Q(x-a)^3,$$

siquidem Q ut quantitatem constantem spectemus, ita ut sit

$$y = \int Xdx = b + X(x-a) - \tfrac{1}{2}P(x-a)^2 + \tfrac{1}{2\cdot 3}Q(x-a)^3.$$

Eadem ergo methodo si ulterius procedamus, ponendo

$$X = \tfrac{dy}{dx}, \quad P = \tfrac{dX}{dx}, \quad Q = \tfrac{dP}{dx}, \quad R = \tfrac{dQ}{dx}, \quad S = \tfrac{dR}{dx} \quad \text{etc.}$$

invenimus

$$y = b + X(x-a) - \tfrac{1}{2}P(x-a)^2 + \tfrac{1}{2\cdot 3}Q(x-a)^3 - \tfrac{1}{2\cdot 3\cdot 4}R(x-a)^4$$
$$+ \tfrac{1}{2\cdot 3\cdot 4\cdot 5}S(x-a)^5 - \quad \text{etc.},$$

quae series vehementer convergit, si modo x non multum superet a, atque adeo, si in infinitum continuetur, verum valorem ipsius y exhibebit, siquidem in functionibus X, P, Q, R etc. valor extremus $x = a + \alpha$ substituatur. Nisi autem eam seriem in infinitum extendere velimus, praestabit per intervalla procedere tribuendo ipsi x successive valores a, a', a'', a''', a'''' etc. ac tum pro singulis valores litteris X, P, Q, R, S etc. convenientes quaeri oportet, qui sint, ut sequuntur: Si fuerit

$$x = a, \quad a', \quad a'', \quad a''', a^{\text{IV}}, \quad a^{\text{V}} \text{ etc.},$$

fiat

$$X = A, \ A', \quad A'', \quad A''', A^{\text{IV}}, \quad A^{\text{V}} \quad \text{etc.},$$
$$\tfrac{dX}{dx} = P = B, \ B', \quad B'', \quad B''', B^{\text{IV}}, \quad B^{\text{V}} \quad \text{etc.},$$
$$\tfrac{dP}{dx} = Q = C, \ C', \ C'', \quad C''', C^{\text{IV}}, \ C^{\text{V}} \quad \text{etc.},$$
$$\tfrac{dQ}{dx} = R = D, \ D', \ D'', \ D''', D^{\text{IV}}, \quad D^{\text{V}} \quad \text{etc.}$$
$$\text{etc.};$$

tum vero sit

$$y = b, \quad b', \quad b'', \quad b''', \quad b^{\text{IV}}, \quad b^{\text{V}} \text{ etc.},$$

quibus constitutis erit, ut ex antecedentibus colligere licet,

$$b' = b + A'(a' - a) - \tfrac{1}{2}B'(a' - a)^2 + \tfrac{1}{6}C'(a' - a)^3$$
$$- \tfrac{1}{24}D'(a' - a)^4 + \text{etc.},$$
$$b'' = b' + A''(a'' - a') - \tfrac{1}{2}B''(a'' - a')^2 + \tfrac{1}{6}C''(a'' - a')^3$$
$$- \tfrac{1}{24}D''(a'' - a')^4 + \text{etc.},$$

$$b''' = b'' + A'''(a''' - a'') - \tfrac{1}{2}B'''(a''' - a'')^2 + \tfrac{1}{6}C'''(a''' - a'')^3$$
$$- \tfrac{1}{24}D'''(a''' - a'')^4 + \text{etc.},$$

$$b^{\text{IV}} = b''' + A^{\text{IV}}\left(a^{\text{IV}} - a'''\right) - \tfrac{1}{2}B^{\text{IV}}\left(a^{\text{IV}} - a'''\right)^2 + \tfrac{1}{6}C^{\text{IV}}\left(a^{\text{IV}} - a'''\right)^3$$
$$- \tfrac{1}{24}D^{\text{IV}}\left(a^{\text{IV}} - a'''\right)^4 + \text{etc.},$$

etc.,

quae expressiones eousque continuentur, donec pro valore ipsius x quantumvis ab initiali a discrepante valor ipsius y obtineatur.

COROLLARIUM 1

318. Haec igitur approximandi methodus eo utitur theoremate, cuius veritas iam in *Calculo Differentiali* est demonstrata: quodsi y eiusmodi fuerit functio ipsius x, quae posito $x = a$ fiat $= b$, ac statuatur

$$\tfrac{dy}{dx} = X, \quad \tfrac{dX}{dx} = P, \quad \tfrac{dP}{dx} = Q, \quad \tfrac{dQ}{dx} = R, \quad \text{etc.}$$

fore generaliter

$$y = b + X(x - a) - \tfrac{1}{2}P(x - a)^2 + \tfrac{1}{6}Q(x - a)^3 - \tfrac{1}{24}R(x - a)^4$$
$$+ \tfrac{1}{120}S(x - a)^5 - \quad \text{etc.}$$

COROLLARIUM 2

319. Si hanc seriem in infinitum continuare vellemus, non opus esset valorem ipsius x parum tantum ab a diversum assumere. Verum quo ista series magis convergens reddatur, expedit saltum ab a ad x in intervalla dispesci et pro singulis operationem hic deseriptam institui.

COROLLARIUM 3

320. Si valores ipsius x ab a per differentias constantes = a crescere faciamus sitque ultimus $a + n\alpha = x$, ita ut, si fuerit

$$x = a, \quad a + \alpha, \quad a + 2\alpha, \quad a + 3\alpha, \quad \cdots \quad x,$$

fiat

$$\begin{aligned}
X &= A,\ A',\ \ A'',\ \ A''',\ \cdots\ X,\\
\tfrac{dX}{dx} = P &= B,\ B',\ \ B'',\ \ B''',\ \cdots\ P,\\
\tfrac{dP}{dx} = Q &= C,\ C',\ \ C'',\ \ C''',\ \cdots\ Q,\\
\tfrac{dQ}{dx} = R &= D,\ D',\ \ D'',\ \ D''',\ \cdots\ R
\end{aligned}$$

etc.

indeque

$$y = b,\quad b',\quad b'',\quad b''',\quad \cdots\quad y,$$

erit pro valore $x = x$ omnes series colligendo

$$\begin{aligned}
y = b &+ \alpha\left(A' + A'' + A''' + \cdots + X\right)\\
&- \tfrac{1}{2}\alpha^2\left(B' + B'' + B''' + \cdots + P\right)\\
&+ \tfrac{1}{6}\alpha^3\left(C' + C'' + C''' + \cdots + Q\right)\\
&- \tfrac{1}{24}\alpha^4\left(D' + D'' + D''' + \cdots + R\right)
\end{aligned}$$

etc.

SCHOLION 1

321. Demonstratio theorematis Corollario 1 memorati, cui haec methodus approximandi innititur, ex natura differentialium ita instruitur. Sit *y* functio ipsius *x*, quae posito $x = a$ fiat $y = b$, et quaeramus valorem ipsius *y*, si *x* utcunque excedat *a*. Incipiamus a valore ipsius maximo, qui est *x*, et per differentialia descendamus atque ex differentialibus patet,

si fuerit x	fore y
$x-dx$	$y - dy + ddy - d^3y + d^4y - \text{etc.}$
$x-2dx$	$y - 2dy + 3ddy - 4d^3y + 5d^4y - \text{etc.}$
$x-3dx$	$y - 3dy + 6ddy - 10d^3y + 15d^4y - \text{etc.}$
.	.
.	.
$x-ndx$	$y - ndy + \frac{n(n+1)}{1\cdot 2}ddy - \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}d^3y + \frac{n(n+1)(n+2)(n+3)}{1\cdot 2\cdot 3\cdot 4}d^4y - \text{etc}$

Ponamus nunc $x - ndx = a$; erit $n = \frac{x-a}{dx}$ ideoque numerus infinitus; tum vero valor pro *y* resultans per hypothesin esse debet $= b$, quamobrem habebimus

$$b = y - \frac{(x-a)dy}{dx} + \frac{(x-a)^2 ddy}{1\cdot 2dx^2} - \frac{(x-a)^3 d^3y}{1\cdot 2\cdot 3dx^3} - \frac{(x-a)^4 d^3y}{1\cdot 2\cdot 3\cdot 4dx^4} - \text{etc.},$$

Quodsi iam statuamus

$$\frac{dy}{dx}=X,\quad \frac{dX}{dx}=P,\quad \frac{dP}{dx}=Q,\quad \frac{dQ}{dx}=R,\quad \text{etc.}$$

reperimus ut ante

$$y=b+X\left(x-a\right)-\tfrac{1}{2}P\left(x-a\right)^2+\tfrac{1}{6}Q\left(x-a\right)^3-\tfrac{1}{24}R\left(x-a\right)^4+\text{etc.}$$

Unde patet, si x quam minime superet a, sufficere statui $y=b+X\left(x-a\right)$, quod est fundamentum approximationis primum propositae, illius scilicet limitis, quo X ex valore maiore ipsius x definitur.

SCHOLION 2

322. Quemadmodum hoc ratiocinium nobis alterum tantum limitem supra assignatum patefecit, ita ad alterum limitem hoc ratiocinium nos manuducet. Scilicet uti ante ab x ad a descendimus, ita nunc ab a ad x ascendamus;

si abeat a in	tum b abebit in
$a+da$	$b+db$
$a+2da$	$b+2db+ddb$
$a+3da$	$b+3db+3ddb+d^3b$
.	.
.	.
$a+nda$	$b+ndb+\frac{n(n+1)}{1\cdot 2}ddb+\frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}d^3b+\text{etc.}$

Sit iam $a+nda=x$ seu $n=\frac{x-a}{da}$ et valor ipsius b fiet = y. Sint autem A, B, C, D etc. valores superiorum functionum X, P, Q, R etc., si loco x scribatur a, eritque pro praesenti casu

$$A=\frac{db}{da},\quad B=\frac{ddb}{da^2},\quad C=\frac{d^3b}{da^3}\ \text{etc.}$$

Quocirca habebimus

$$y=b+A\left(x-a\right)+\tfrac{1}{2}B\left(x-a\right)^2+\tfrac{1}{6}C\left(x-a\right)^3+\tfrac{1}{24}D\left(x-a\right)^4+\ \text{etc.},$$

quae series superiori praeter signa omnino est similis; ac si x parum excedat a, ut $b+A\left(x-a\right)$ satis exacte valorem ipsius y indicet, hinc alter limes supra assignatus nascitur. Quodsi autem progressum ab a ad x ut supra § 320 in intervalla aequalia secundum differentiam α dispescamus et termini in singulis seriebus ultimos praecedentes notentur per $'X$, $'P$, $'Q$, $'R$ etc., habebimus pro y quasi alterum limitem

$$y = b + \alpha\left(A + A' + A'' + \cdots +' X\right)$$
$$+\tfrac{1}{2}\alpha^2\left(B + B' + B'' + \cdots +' P\right)$$
$$+\tfrac{1}{6}\alpha^3\left(C + C' + C'' + \cdots +' Q\right)$$
$$+\tfrac{1}{24}\alpha^4\left(D + D' + D'' + \cdots +' R\right)$$

etc.,

ita ut etiam in hac methodo emendata binos habebimus limites, inter quos verus valor ipsius y contineatur. Propius ergo valorem assequemur, si inter hos limites medium arithmeticum capiamus, unde prodibit

$$y = b + \alpha\left(A + A' + A'' + \cdots +' X\right) - \tfrac{1}{2}\alpha\left(A + X\right) + \tfrac{1}{4}\alpha^2\left(B - P\right)$$
$$+\tfrac{1}{6}\alpha^3\left(C + C' + C'' + \cdots +' Q\right) - \tfrac{1}{12}\alpha^3\left(C + Q\right) + \tfrac{1}{48}\alpha^4\left(D - R\right)$$
$$+\tfrac{1}{120}\alpha^5\left(E + E' + E'' + \cdots +' S\right) - \tfrac{1}{240}\alpha^5\left(E + S\right) + \tfrac{1}{1440}\alpha^6\left(F - T\right)$$

etc.,

Atque hinc superiores approximationes tantum addendo membrum : $\tfrac{1}{4}\alpha^2\left(B - P\right)$ non mediocriter corrigentur.

EXEMPLUM 1

323. *Logarithmum cuiusvis numeri x proxime exprimere.*

Hic igitur est $y = \int \frac{dx}{x}$, quod integrale ita capitur, ut evanescat posito $x = 1$; erit ergo $a = 1$ et $b = 0$ et $X = \frac{1}{x}$. Sumamus iam ab unitate ad x per intervalla $= \alpha$ ascendi, et cum sit

$$P = \frac{dX}{dx} = -\frac{1}{xx}, \quad Q = \frac{dP}{dx} = \frac{2}{x^3}, \quad R = \frac{dQ}{dx} = -\frac{6}{x^4},$$

pro indicibus

$$x = 1, \quad 1 + \alpha, \quad 1 + 2\alpha, \quad 1 + 3\alpha, \quad \cdots \quad x$$

erit

$$\begin{array}{llllll} X = 1, & \frac{1}{1+\alpha}, & \frac{1}{1+2\alpha}, & \frac{1}{1+3\alpha}, & \cdots & \frac{1}{x}, \\ P = -1, & -\frac{1}{(1+\alpha)^2}, & -\frac{1}{(1+2\alpha)^2}, & -\frac{1}{(1+3\alpha)^2}, & \cdots & -\frac{1}{xx}, \\ Q = 2, & \frac{2}{(1+\alpha)^3}, & \frac{2}{(1+2\alpha)^3}, & \frac{2}{(1+3\alpha)^3}, & \cdots & \frac{2}{x^3}, \\ R = -6, & -\frac{6}{(1+\alpha)^4}, & -\frac{6}{(1+2\alpha)^4}, & -\frac{6}{(1+3\alpha)^4}, & \cdots & -\frac{6}{x^4}, \end{array}$$

etc.,

unde adipiscimur

$$\begin{aligned} lx = {} & \alpha\left(1+\frac{1}{1+\alpha}+\frac{1}{1+2\alpha}+\frac{1}{1+3\alpha}+\cdots+\frac{1}{x}\right) \\ & -\frac{1}{2}\alpha\left(1+\frac{1}{x}\right)-\frac{1}{4}\alpha\alpha\left(1-\frac{1}{xx}\right) \\ & +\frac{1}{3}\alpha^3\left(1+\frac{1}{(1+\alpha)^3}+\frac{1}{(1+2\alpha)^3}+\frac{1}{(1+3\alpha)^3}+\cdots+\frac{1}{x^3}\right) \\ & -\frac{1}{6}\alpha^3\left(1+\frac{1}{x^3}\right)-\frac{1}{8}\alpha^4\left(1-\frac{1}{x^4}\right) \\ & +\frac{1}{5}\alpha^5\left(1+\frac{1}{(1+\alpha)^5}+\frac{1}{(1+2\alpha)^5}+\frac{1}{(1+3\alpha)^5}+\cdots+\frac{1}{x^5}\right) \\ & -\frac{1}{10}\alpha^5\left(1+\frac{1}{x^5}\right)-\frac{1}{12}\alpha^6\left(1-\frac{1}{x^6}\right) \end{aligned}$$

etc.

Quare si sumamus $\alpha = \frac{1}{m}$ erit

$$\begin{aligned} lx = {} & \left(\frac{1}{m}+\frac{1}{m+1}+\frac{1}{m+2}+\frac{1}{m+3}+\cdots+\frac{1}{mx}\right)-\frac{x+1}{2mx}-\frac{xx-1}{4mmxx} \\ & +\frac{1}{3}\left(\frac{1}{m^3}+\frac{1}{(m+1)^3}+\frac{1}{(m+2)^3}+\frac{1}{(m+3)^3}+\cdots+\frac{1}{(mx)^3}\right)-\frac{x^3+1}{6m^3x^3}-\frac{x^4-1}{8m^4x^4} \\ & +\frac{1}{5}\left(\frac{1}{m^5}+\frac{1}{(m+1)^5}+\frac{1}{(m+2)^5}+\frac{1}{(m+3)^5}+\cdots+\frac{1}{(mx)^5}\right)-\frac{x^5+1}{10m^5x^5}-\frac{x^6-1}{12m^6x^6} \end{aligned}$$

etc.

COROLLARIUM

324. Si hae progressiones in infinitum continuentur, erit postremarum partium $=-\frac{1}{2}l\frac{m}{m-1}-\frac{1}{2}l\frac{mx+1}{mx}=-\frac{1}{2}l\frac{mx+1}{(m-1)x}$ primarum vero $\frac{1}{2}l\frac{m+1}{m-1}$; unde, cum sit

$$lx+\frac{1}{2}l\frac{mx+1}{(m-1)x}+\frac{1}{2}l\frac{m-1}{m+1}=\frac{1}{2}l\frac{x(mx+1)}{m+1}$$

erit

$$l\frac{x(mx+1)}{m+1}=2\left(\frac{1}{m+1}+\frac{1}{m+2}+\frac{1}{m+3}+\cdots+\frac{1}{mx}\right)$$
$$+\frac{2}{3}\left(\frac{1}{(m+1)^3}+\frac{1}{(m+2)^3}+\frac{1}{(m+3)^3}+\cdots+\frac{1}{m^3x^3}\right)$$
$$+\frac{2}{5}\left(\frac{1}{(m+1)^5}+\frac{1}{(m+2)^5}+\frac{1}{(m+3)^5}+\cdots+\frac{1}{m^5x^5}\right)$$
etc.

quae expressio adeo, si in infinitum continuetur, verum valorem $\log.\frac{x(mx+1)}{m+1}$ praebet.

EXEMPLUM 2

325. *Arcum circuli, cuius tangens est $=\frac{x}{c}$ hac methodo proxime exprimere.*

Quaestio igitur est de integrali $y=\int\frac{cdx}{cc+xx}$ quod posito $x=0$ evanescit, eritque $a=0$ et $b=0$, tum vero

$$X=\frac{c}{cc+xx},\quad P=\frac{dX}{dx}=\frac{-2cx}{(cc-3xx)^2},\quad Q=\frac{dP}{dx}=\frac{-2c(cc-3xx)}{(cc+xx)^3},$$
$$R=\frac{dQ}{dx}=\frac{6cx(3cc-4xx)}{(cc+xx)^4},\ S=\frac{dR}{dx}=\frac{6c\left(3c^4-33ccxx+20x^4\right)}{(cc+xx)^5}\quad \text{etc.},$$

quae formae in infinitum continuatae dant

$$y=\frac{cx}{cc+xx}+\frac{cx^3}{(cc+xx)^2}-\frac{cx^3(cc-3xx)}{3(cc+xx)^3}-\frac{cx^5(3cc-4xx)}{4(cc+xx)^4}+\frac{cx^5\left(3c^4-33ccxx+20x^4\right)}{20(cc+xx)^5}+\text{etc.}$$

Verum si x per intervalla = 1, ut sit $\alpha=1$, crescere ponamus, erit

$$A=\frac{c}{cc},\qquad B=0,\qquad C=\frac{-2c^3}{c^6},\qquad D=0\quad \text{etc.}$$
$$A'=\frac{c}{cc+1},\qquad B'=\frac{-2c}{(cc+1)^2},\qquad C'=\frac{-2c(cc-3)}{(cc+1)^3},\qquad D'=\frac{6c(3cc-4)}{(cc+1)^4},$$
$$A''=\frac{c}{cc+4},\qquad B''=\frac{-4c}{(cc+4)^2},\qquad C''=\frac{-2c(cc-12)}{(cc+4)^3},\qquad D''=\frac{12c(3cc-16)}{(cc+4)^4},$$
$$A'''=\frac{c}{cc+9},\qquad B'''=\frac{-6c}{(cc+9)^2},\qquad C'''=\frac{-2c(cc-27)}{(cc+9)^3},\qquad D'''=\frac{18c(3cc-36)}{(cc+9)^4},$$
.
.
$$X=\frac{c}{cc+xx},\qquad P=\frac{-2cx}{(cc+xx)^2},\qquad Q=\frac{-2c(cc-3xx)}{(cc+xx)^3},\qquad R=\frac{6cx(3cc-4xx)}{(cc+xx)^4}$$

hincque

$$\begin{aligned} y = \ & c\left(\tfrac{1}{cc} + \tfrac{1}{cc+1} + \tfrac{1}{cc+2} + \tfrac{1}{cc+3} + \cdots + \tfrac{1}{cc+xx}\right) \\ & - \tfrac{1}{2c} - \tfrac{c}{2(cc+xx)} + \tfrac{cx}{2(cc+xx)^2} \\ & - \tfrac{c}{3}\left(\tfrac{1}{c^4} + \tfrac{cc-3}{(cc+1)^3} + \tfrac{cc-12}{(m+4)^3} + \tfrac{cc-27}{(m+9)^3} + \cdots + \tfrac{cc-3xx}{(cc+xx)^3}\right) \\ & + \tfrac{1}{6c^3} + \tfrac{c(cc-3xx)}{6(cc+xx)^3} - \tfrac{cx(3cc-4xx)}{8(cc+xx)^4} \\ & \text{etc.} \end{aligned}$$

COROLLARIUM

326. Posito ergo $c = x = 4$, ut fiat $y = \text{Ang.tang.}1 = \frac{\pi}{4}$ erit

$$\begin{aligned} \tfrac{\pi}{4} = & \tfrac{1}{4} + \tfrac{4}{17} + \tfrac{4}{20} + \tfrac{4}{25} + \tfrac{1}{8} - \tfrac{1}{8} - \tfrac{1}{16} + \tfrac{1}{128} \\ & - \tfrac{4}{3}\left(\tfrac{1}{256} + \tfrac{13}{17^3} + \tfrac{4}{20^3} - \tfrac{32}{32^3}\right) + \tfrac{1}{384} - \tfrac{1}{1536} + \tfrac{1}{128\cdot 256}, \end{aligned}$$

cuius valor non multum a veritate discedit; sed haec exempla tantum illustrationis causa affero, non ut approximatio facilior , quam aliae methodi suppeditant, inde expectetur.

EXEMPLUM 3

327. *Integrale* $y = \int \frac{e^{-\frac{1}{x}}dx}{x}$ *ita sumtum, ut evanescat posito* $x = 0$, *vero proxime assignare.*

Per reductiones supra expositas est

$$y = \int \frac{e^{-\frac{1}{x}}dx}{x} = e^{-\frac{1}{x}}x - \int e^{-\frac{1}{x}}dx$$

et pars $e^{-\frac{1}{x}}dx$ evanescit posito $x = 0$. Quaeramus ergo integrale $z = \int e^{-\frac{1}{x}}dx$,

quia eo invento habetur $y = e^{-\frac{1}{x}}x - z$, ac supra iam methodos approximandi in hoc exemplo frustra tentari. Cum igitur posito $x = 0$ evanescat z, erit $a = 0$ et $b = 0$, tum vero $X = e^{-\frac{1}{x}}$ hincque

$$P = \frac{dX}{dx} = e^{-\frac{1}{x}}\frac{1}{xx}, \quad Q = \frac{dP}{dx} = e^{-\frac{1}{x}}\left(\frac{1}{x^4} - \frac{2}{x^3}\right),$$

$$R = \frac{dQ}{dx} = e^{-\frac{1}{x}}\left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4}\right), S = \frac{dR}{dx} = e^{-\frac{1}{x}}\left(\frac{1}{x^8} - \frac{12}{x^7} + \frac{36}{x^6} - \frac{24}{x^5}\right) \quad \text{etc.},$$

quibus valoribus in infinitum continuatis erit

$$z = e^{-\frac{1}{x}} \left\{ \begin{array}{l} x - \frac{1}{2} + \frac{1}{6} x^3 \left(\frac{1}{x^4} - \frac{2}{x^3} \right) - \frac{1}{24} x^4 \left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4} \right) \\ + \frac{1}{120} x^5 \left(\frac{1}{x^8} - \frac{12}{x^7} + \frac{36}{x^6} - \frac{24}{x^5} \right) - \quad \text{etc.} \end{array} \right\}$$

seu

$$z = e^{-\frac{1}{x}} \left\{ \begin{array}{l} x - \frac{1}{2} + \frac{1}{6} \left(\frac{1}{x} - 2 \right) - \frac{1}{24} \left(\frac{1}{xx} - \frac{6}{x} + 6 \right) \\ + \frac{1}{120} \left(\frac{1}{x^3} - \frac{12}{x^2} + \frac{36}{x} - 24 \right) - \frac{1}{720} \left(\frac{1}{x^4} - \frac{20}{x^3} + \frac{120}{x^2} - \frac{240}{x} + 120 \right) \quad \text{etc.} \end{array} \right\}$$

quae series parum convergit, quicunque valor ipsi x tribuatur. Per intervalla igitur a 0 usque ad x ascendamus ponendo pro x successive $0, \alpha,\ 2\alpha, 3\alpha \quad \text{etc.}$, ubi notandum fore $A = 0, B = 0, C = 0, D = 0 \quad \text{etc.}$, ac regula nostra praebet

$$\begin{aligned} z = {} & \alpha \left(e^{-\frac{1}{\alpha}} + e^{-\frac{1}{2\alpha}} + e^{-\frac{1}{3\alpha}} + \cdots + e^{-\frac{1}{x}} \right) - \frac{1}{2} \alpha e^{-\frac{1}{x}} - \frac{1}{4} \alpha^2 e^{-\frac{1}{x}} \frac{1}{xx} \\ & + \frac{1}{6} \alpha^3 \left(e^{-\frac{1}{\alpha}} \left(\frac{1}{\alpha^4} - \frac{2}{\alpha^3} \right) + e^{-\frac{1}{2\alpha}} \left(\frac{1}{16\alpha^4} - \frac{2}{8\alpha^3} \right) + e^{-\frac{1}{3\alpha}} \left(\frac{1}{81\alpha^4} - \frac{2}{27\alpha^3} \right) + \cdots + e^{-\frac{1}{x}} \left(\frac{1}{x^4} - \frac{2}{x^3} \right) \right) \\ & - \frac{1}{12} \alpha^3 e^{-\frac{1}{x}} \left(\frac{1}{x^4} - \frac{2}{x^3} \right) - \frac{1}{48} \alpha^4 e^{-\frac{1}{x}} \left(\frac{1}{x^6} - \frac{6}{x^5} + \frac{6}{x^4} \right). \end{aligned}$$

Si hinc valorem ipsius z pro casu $x = 1$ determinare velimus et pro α fractionem parvam $\frac{1}{n}$ assumamus, habebimus

$$\begin{aligned} z = {} & \frac{1}{n} \left(e^{-\frac{n}{1}} + e^{-\frac{n}{2}} + e^{-\frac{n}{3}} + e^{-\frac{n}{4}} + \cdots + e^{-\frac{n}{n}} \right) - \frac{1}{2ne} - \frac{1}{4nne} \\ & + \frac{1}{6} \left(e^{-\frac{n}{1}} \frac{n-2}{1} + e^{-\frac{n}{2}} \frac{n-4}{16} + e^{-\frac{n}{3}} \frac{n-6}{81} + \cdots + e^{-\frac{n}{n}} \frac{n-2n}{n^4} \right) \\ & + \frac{1}{12n^3 e} - \frac{1}{48n^4 e}. \end{aligned}$$

Si hic pro n sumatur numerus mediocriter magnus, veluti 10, valor ipsius z ad partem millionesimam unitatis exactus reperitur ac vicies exactior prodiret, si pro n sumeremus 20.

SCHOLION 1

328. Hoc exemplum sufficiat eximium usum huius methodi approximandi ostendisse. Interim tamen occurrunt casus, quibus ne hac quidem methodo uti licet, etiamsi totum spatium, per quod variabilis x crescit, in minima intervalla dividamus. Evenit hoc, quando functio X pro quopiam intervallo, dum variabili x certus quidam valor tribuitur, in infinitum excrescit, cum tamen ipsa quantitas integralis $y=\int Xdx$ hoc casu non fiat infinita; veluti si fuerit

$$y=\int \frac{dx}{\sqrt{(a-x)}},$$

ubi $X=\frac{1}{\sqrt{(a-x)}}$, quae posito $x=a$ fit infinita, integrale vero $y=C-2\sqrt{(a-x)}$ hoc casu est finitum. Hoc autem semper usu venit, quoties huiusmodi factor $a-x$ in denominatore habet exponentem unitate minorem; tum enim idem factor in integrali in numeratorem transit; sin autem eiusdem factoris exponens in denominatore est unitas vel adeo unitate maior, tum etiam ipsum integrale casu $x=a$ fit infinitum; quo casu quia approximatio cessat, hic tantum de iis sermo est, ubi exponens unitate est minor, quoniam tum approximatio revera turbatur. Verum huic incommodo facile medela afferri potest; cum enim differentiale eiusmodi formam sit habiturum $\frac{Xdx}{(a-x)^{\lambda:\mu}}$ existente $\lambda<\mu$, ponatur $a-x=z^{\mu}$, ut sit $x=a-z^{\mu}$ et $dx=-\mu z^{\mu-1}dz$, et differentiale nostrum erit $=-\mu Xz^{\mu-\lambda-1}dz$, quod casu $x=a$ seu $z=0$ non amplius fit infinitum. Vel, quod eodem redit, pro iis intervallis, quibus functio X fit infinita, integratio seorsim revera instituatur ponendo $x=a\pm\omega$; tum enim formula Xdx satis fiet simplex ob ω valde parvum, ut integratio nihil habeat difficultatis. Veluti si valorem ipsius $y=\int \frac{xxdx}{\sqrt{(a^4-x^4)}}$ per intervalla ab $x=0$ usque ad $x=a-\alpha$ iam simus consecuti, pro hoc ultimo intervallo ponamus $x=a-\omega$ et integrari oportebit

$$\frac{(a-\omega)^2 d\omega}{\sqrt{(4a^3\omega-6aa\omega\omega+4a\omega^3-\omega^4)}},$$

quod ob ω valde parvum abit in

$$\frac{d\omega\sqrt{a}}{2\sqrt{\omega}}\left(1-\frac{5\omega}{4a}-\frac{5\omega\omega}{32aa}\right),$$

cuius integrale sumto $\omega=\alpha$ est

$$\sqrt{a\alpha}-\frac{5\alpha\sqrt{\alpha}}{12\sqrt{a}}-\frac{5\alpha\alpha\sqrt{\alpha}}{32a\sqrt{a}},$$

quod, si ad plures terminos continuetur, non solum pro ultimo intervallo, sed pro duobus pluribusve postremis ponendo $\omega=2\alpha$ vel $\omega=3\alpha$ adhiberi potest. Pro quibus enim intervallis denominator iam fit satis parvus, praestat hac methodo uti quam ea, quae ante est exposita.

SCHOLION 2

329. Interdum etiam aliud incommodum occurrit, ut denominator duobus casibus evanescat, veluti si fuerit

$$y=\int\frac{Xdx}{\sqrt{(a-x)(x-b)}}$$

ubi variabilis *x* semper inter limites *b* et *a* contineri debet, ita ut, cum a *b* ad *a* creverit, deinceps iterum ab *a* ad *b* decrescat; interea autem integrale *y* continuo crescere pergat, cuius igitur valor per intervalla commode determinari non potest. Hoc ergo casu in subsidium vocetur haec substitutio

$$x=\tfrac{1}{2}(a+b)-\tfrac{1}{2}(a-b)\cos.\varphi,$$

qua fit

$$dx=\tfrac{1}{2}(a-b)d\varphi\sin.\varphi$$

et

$$(a-x)(x-b)=\left(\tfrac{1}{2}(a-b)+\tfrac{1}{2}(a-b)\cos.\varphi\right)\left(\tfrac{1}{2}(a-b)-\tfrac{1}{2}(a-b)\cos.\varphi\right)$$

seu

$$(a-x)(x-b)=\tfrac{1}{4}(a-b)^2\sin.^2\varphi,$$

unde oritur $y=\int Xd\varphi$, quae nullo amplius incommodo laborat, cum angulum φ continuo ulterius aequabiliter augere licet.

Hoc etiam ad casus patet, ubi bini factores in denominatore non eundem habent exponentem, veluti si fuerit

$$y=\int\frac{Xdx}{\sqrt[2\lambda]{(a-x)^{\mu}(x-b)^{\nu}}}$$

ita ut μ et ν sint minores quam 2λ, quem exponentem parem suppono. Si iam μ et ν non sint aequales, sed $\nu<\mu$ ad aequalitatem reducantur hoc modo

$$y=\int\frac{Xdx\sqrt[2\lambda]{(x-b)^{\mu-\nu}}}{\sqrt[2\lambda]{(a-x)^{\mu}(x-b)^{\nu}}}$$

Quodsi iam ut ante ponatur

$$x=\tfrac{1}{2}(a+b)-\tfrac{1}{2}(a-b)\cos.\varphi,$$

obtinebitur

$$y=\left(\frac{a-b}{2}\right)^{\frac{2\lambda-\mu-\nu}{2\lambda}}\int Xd\varphi\sin^{\frac{\lambda-\mu}{\lambda}}(1-\cos.\varphi)^{\frac{\mu-\nu}{2\lambda}},$$

ubi angulum φ, quousque libuerit, continuare et methodo per intervalla procedente uti licet. Quibus observatis vix quicquam amplius hanc methodum approximandi remorabitur.