CHAPTER V

CONCERNING THE INTEGRATION OF FORMULAS INVOLVING ANGLES OR THE SINES OF ANGLES

PROBLEM 23

234. *To investigate the integral of the proposed formula* $Xdx\text{Ang.sin.}x$.

[Note the use here of Ang.sin. for expressing the inverse sine function, and also the full stops used to abbreviate the words *sinus, cosinus, and tangentum*.]

SOLUTION

Since there arises

$$d.\text{Ang.sin.}x = \frac{dx}{\sqrt{(1-xx)}}$$

the proposed formula can thus be separated into factors $Ang.sin.x \cdot Xdx$. If now Xdx is allowed to be integrated, and there arises $\int Xdx = P$, then our integral becomes

$$\int Xdx\text{Ang.sin.}x = P\text{Ang.sin.}x - \int \frac{Pdx}{\sqrt{(1-xx)}}$$

and thus the work has been reduced to the integration of algebraic formulas, for which the instructions have been set out above.

Otherwise if there should be put $X = \frac{dx}{\sqrt{(1-xx)}}$, the integral evidently becomes

$$\int \frac{dx}{\sqrt{(1-xx)}}\text{Ang.sin.}x = \frac{1}{2}\left(\text{Ang.sin.}x\right)^2$$

in which case alone the square of the angle enters into the integral.

EXAMPLE 1

235. *To integrate this formula* $dy = x^n dx\text{Ang.sin.}x$.

Since there is present

$$P = \int x^n dx = \frac{x^{n+1}}{n+1},$$

we have

$$y = \frac{x^{n+1}}{n+1}\text{Ang.sin.}x - \frac{1}{n+1}\int \frac{x^{n+1}dx}{\sqrt{(1-xx)}}.$$

Hence for the various values of *n* the integrals have been elicited with the help of §120, as follow :

$$\int dx\text{Ang.sin.}x = x\text{Ang.sin.}x + \sqrt{(1-xx)} - 1,$$
$$\int xdx\text{Ang.sin.}x = \tfrac{1}{2}xx\text{Ang.sin.}x + \tfrac{1}{4}x\sqrt{(1-xx)} - \tfrac{1}{4}\text{Ang.sin.}x,$$
$$\int x^2dx\text{Ang.sin.}x = \tfrac{1}{3}x^3\text{Ang.sin.}x + \tfrac{1}{3}\left(\tfrac{1}{3}x^2 + \tfrac{2}{3}\right)\sqrt{(1-xx)} - \tfrac{1}{3}\cdot\tfrac{2}{3},$$
$$\int x^3dx\text{Ang.sin.}x = \tfrac{1}{4}x^4\text{Ang.sin.}x + \tfrac{1}{4}\left(\tfrac{1}{4}x^3 + \tfrac{1\cdot 3}{2\cdot 4}\right)\sqrt{(1-xx)} - \tfrac{1}{4}\cdot\tfrac{1\cdot 3}{2\cdot 4}\text{Ang.sin.}x,$$

which have been taken thus, so that they vanish on putting $x = 0$.

EXAMPLE 2

236. *To integrate this formula* $dy = \frac{xdx}{\sqrt{(1-xx)}}\text{Ang.sin.}x$.

Since there shall be

$$\int \frac{xdx}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)} = P,$$

the integral sought becomes

$$y = C - \sqrt{(1-xx)}\text{Ang.sin.}x + \int \frac{dx\sqrt{(1-xx)}}{\sqrt{(1-xx)}}$$

and thus there is had

$$y = \int \frac{xdx}{\sqrt{(1-xx)}}\text{Ang.sin.}x = C - \sqrt{(1-xx)}\text{Ang.sin.}x + x.$$

EXAMPLE 3

237. *To integrate this formula* $dy = \frac{dx}{(1-xx)^{\frac{3}{2}}}\text{Ang.sin.}x$.

Here there is

$$P = \int \frac{dx}{(1-xx)^{\frac{3}{2}}} = \frac{x}{\sqrt{(1-xx)}},$$

from which there becomes

$$y = \frac{x}{\sqrt{(1-xx)}}\text{Ang.sin.}x - \int \frac{xdx}{1-xx}$$

or

$$y = \int \frac{dx}{(1-xx)^{\frac{3}{2}}}\text{Ang.sin.}x = \frac{x}{\sqrt{(1-xx)}}\text{Ang.sin.}x + l\sqrt{(1-xx)},$$

which integral vanishes on putting $x = 0$.

SCHOLIUM

238. In a similar manner the formula $dy = Xdx\text{Ang.cos.}x.$ is integrated. For since there is found

$$d.\text{Ang.cos.}x = \frac{-dx}{\sqrt{(1-xx)}},$$

if we put $\int Xdx = P$, then

$$y = P\text{Ang.cos.}x + \int \frac{Pdx}{\sqrt{(1-xx)}}.$$

Indeed also if the formula $dy = Xdx\text{Ang.tang.}x$ is proposed, because there is

$$d.\text{Ang.tang.}x = \frac{dx}{1+xx}$$

on putting $\int Xdx = P$, here the integral is

$$y = \int Xdx\text{Ang.tang.}x = P\text{Ang.tang.}x - \int \frac{Pdx}{1+xx}.$$

Hence just as $\int Xdx$ is able to be expressed algebraically, so the whole integration can be reduced to an algebraic formula and thus the complete calculation can be obtained. Therefore since in these formulas the angle is present, of which the sine, the cosine, or the tangent is equal to *x*, we can consider also formulas of this kind, in which the square of this angle or of higher powers can be present.

PROBLEM 24

239. *Let φ denote the angle, of which the sine or tangent is a certain function of x, from which there is produced $d\varphi = udx$, and this formula is proposed $dy = Xdx \cdot \varphi^n$, that it is required to integrate.*

SOLUTION

Let $\int Xdx = P$, since we have $dy = \varphi^n dP$, and the integral shall be

$$y = \varphi^n P - n\int \varphi^{n-1} Pudx.$$

Now in a similar manner there shall be $\int Pudx = Q$; then

$$\int \varphi^{n-1} Pudx = \varphi^{n-1} Q - (n-1)\int \varphi^{n-2} Qudx;$$

then on putting $\int Qudx = R$ there shall be

$$\int \varphi^{n-2} Qudx = \varphi^{n-2} R - (n-2)\int \varphi^{n-3} Rudx.$$

And in this manner the power of the angle φ is continually given lower values, until finally it arrives at a formula free from the angle φ ; because that always comes about, as long as n is a positive whole number, and these integrals are allowed to be taken continually

$$\int Xdx = P,\ \int Pudx = Q,\ \int Qudx = R \quad \text{etc.},$$

which integrations if they do not succeed, then the integration is undertaken in vain.

EXAMPLE

240. *Let φ be the angle, the sine of which is equal to x, so that there becomes $d\varphi = \frac{dx}{\sqrt{(1-xx)}}$; to integrate the formula $dy = \varphi^n dx$.*

Hence there shall be :

$$X = 1,\quad P = x,\ Q = \int \frac{Pdx}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)},\ R = \int \frac{Qdx}{\sqrt{(1-xx)}} = -x,$$
$$S = \int \frac{Rdx}{\sqrt{(1-xx)}} = \sqrt{(1-xx)},\ T = x \quad \text{etc.},$$

from which with the values in place there is found :

$$y = \int \varphi^n dx = \varphi^n x + n\varphi^{n-1}\sqrt{(1-xx)} - n(n-1)\varphi^{n-2}x$$
$$-n(n-1)(n-2)\varphi^{n-3}\sqrt{(1-xx)} + \text{etc.}$$

Hence for the various values of the exponent n we have :

$$\int \varphi dx = \varphi x + \sqrt{(1-xx)} - 1,$$
$$\int \varphi^2 dx = \varphi^2 x + 2\varphi\sqrt{(1-xx)} - 2\cdot 1x,$$
$$\int \varphi^3 dx = \varphi^3 x + 3\varphi^2\sqrt{(1-xx)} - 3\cdot 2\varphi x - 3\cdot 2\cdot 1\sqrt{(1-xx)} + 6$$
$$\text{etc.}$$

with the integrals thus determined, in order that they vanish on putting $x = 0$.

SCHOLION

241. If there should be [the condition] $Xdx = udx = d\varphi$, then the integral of the formula $\varphi^n d\varphi$ becomes $\frac{1}{n+1}\varphi^{n+1}$; and in a like manner if Φ should be some function of the angle φ, then the integration of the formula $\Phi udx = \Phi d\varphi$ may be found without difficulty. Much more general formulas involving either the sines, cosines or tangents of angles are known, the integration of which finds a use for the most distinguished analysis, as the theory of astronomy especially can be reduced to formulas of this kind. But the first foundations must be sought from the differential calculus ; since from which there shall be

$$d.sin.n\varphi = nd\varphi cos.n\varphi,\ d.cos.n\varphi = -nd\varphi sin.n\varphi,\ d.\text{tang}.n\varphi = \frac{nd\varphi}{cos.^2 n\varphi},$$
$$d.\text{cot}.n\varphi = \frac{-nd\varphi}{sin.^2 n\varphi},\ d.\frac{1}{sin.n\varphi} = \frac{-nd\varphi cos.n\varphi}{sin.^2 n\varphi},\ d.\frac{1}{cos.n\varphi} = \frac{nd\varphi sin.n\varphi}{cos.^2 n\varphi},$$

and we obtain these elementary integrations

$$\int d\varphi cos.n\varphi = \tfrac{1}{n} sin.n\varphi,\ \int d\varphi sin.n\varphi = -\tfrac{1}{n} cos.n\varphi,$$
$$\int \frac{d\varphi}{cos.^2 n\varphi} = \frac{1}{n}\text{tang}.n\varphi,\ \int \frac{d\varphi}{sin.^2 n\varphi} = -\frac{1}{n}\text{cot}.n\varphi,$$
$$\int \frac{d\varphi cos.n\varphi}{sin.^2 n\varphi} = -\frac{1}{n\,sin.n\varphi},\ \int \frac{d\varphi sin.n\varphi}{cos.^2 n\varphi} = \frac{1}{n\,cos.n\varphi},$$

from which the integration of this kind of differential formulas follows,

$$d\varphi\left(A + Bcos.\varphi + \mathrm{C}\,cos.2\varphi + \mathrm{D}\,cos.3\varphi + \mathrm{E}\,cos.4\varphi + \text{etc.}\right),$$

since the integral is clearly :

$$A\varphi + B\,sin.\varphi + \tfrac{1}{2}\mathrm{C}\,sin.2\varphi + \tfrac{1}{3}\mathrm{D}\,sin.3\varphi + \tfrac{1}{4}\mathrm{E}\,sin.4\varphi + \text{etc.}$$

From there also it is convenient to call in support, these formulas treated in the elements in the composition of angles, clearly

$$sin.\alpha \cdot sin.\beta = \tfrac{1}{2}cos.(\alpha-\beta) - \tfrac{1}{2}cos.(\alpha+\beta),$$
$$cos.\alpha \cdot cos.\beta = \tfrac{1}{2}cos.(\alpha-\beta) + \tfrac{1}{2}cos.(\alpha+\beta),$$
$$sin.\alpha \cdot cos\beta \ = \tfrac{1}{2}sin.(\alpha+\beta) + \tfrac{1}{2}sin.(\alpha-\beta)$$
$$= \tfrac{1}{2}sin.(\alpha+\beta) - \tfrac{1}{2}sin.(\beta-\alpha),$$

from which products of many sines and cosines can be resolved into simple sines or cosines.

PROBLEM 25

242. *To investigate the integration of the formula* $d\varphi sin.^{n}\varphi$.

SOLUTION

[The formula] is shown resolved into these factors $sin.^{n-1}\varphi . d\varphi sin.\varphi$, and because

$$\int d\varphi sin.\varphi = -cos.\varphi,$$

here shall be

$$\int d\varphi sin.^{n}\varphi = -sin.^{n-1}\varphi cos.\varphi + (n-1)\int d\varphi sin.^{n-2}\varphi cos.^{2}\varphi.$$

Hence on account of $cos.^{2}\varphi = 1 - sin.^{2}\varphi$ there is put in place

$$\int d\varphi sin.^{n}\varphi = -sin.^{n-1}\varphi cos.\varphi + (n-1)\int d\varphi sin.^{n-2}\varphi - (n-1)\int d\varphi sin.^{n}\varphi,$$

where since the last formula is like that proposed, hence that reduction is deduced :

$$\int d\varphi sin.^{n}\varphi = -\tfrac{1}{n}sin.^{n-1}\varphi cos.\varphi + \tfrac{n-1}{n}\int d\varphi sin.^{n-2}\varphi,$$

in which the integration is returned in that simpler formula $d\varphi sin.^{n-2}\varphi$. Therefore since the simplest cases correspond to

$$\int d\varphi sin.^{0}\varphi = \varphi \quad \text{and} \quad \int d\varphi sin.\varphi = -cos.\varphi,$$

hence the way to continually greater exponents *n* is prepared :

$$\int d\varphi \sin.^0\varphi = \varphi,$$

$$\int d\varphi \sin.\varphi = -\cos.\varphi,$$

$$\int d\varphi \sin.^2\varphi = -\tfrac{1}{2}\sin.\varphi\cos.\varphi + \tfrac{1}{2}\varphi,$$

$$\int d\varphi \sin.^3\varphi = -\tfrac{1}{3}\sin.^2\varphi\cos.\varphi - \tfrac{2}{3}\varphi,$$

$$\int d\varphi \sin.^4\varphi = -\tfrac{1}{4}\sin.^3\varphi\cos.\varphi - \tfrac{1\cdot 3}{2\cdot 4}\sin.\varphi\cos.\varphi + \tfrac{1\cdot 3}{2\cdot 4}\varphi,$$

$$\int d\varphi \sin.^5\varphi = -\tfrac{1}{5}\sin.^4\varphi\cos.\varphi - \tfrac{1\cdot 4}{3\cdot 5}\sin.^2\varphi\cos.\varphi - \tfrac{2\cdot 4}{3\cdot 5}\cos\varphi,$$

$$\int d\varphi \sin.^6\varphi = -\tfrac{1}{6}\sin.^5\varphi\cos.\varphi - \tfrac{1\cdot 5}{4\cdot 6}\sin.^3\varphi\cos.\varphi - \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin.\varphi\cos.\varphi + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\varphi,$$

etc.

COROLLARY 1

243. As long as n is an odd number, the integral can be shown by sines and cosines alone, but if n is an even number, in addition the integral contains the angle itself and thus is a transcending function.

COROLLARY 2

244. Hence in the cases, in which n is an odd number, it is convenient to note at first, that even if the angle or the arc φ increases to infinity, the integral is unable still to grow beyond a certain limit, while yet, if n is an even number, it can also increase to infinity.

SCHOLIUM

245. The formula $d\varphi\cos.^n\varphi$, which can be resolved into these factors $\cos^{n-1}.\varphi.d\varphi\cos.\varphi$, is treated in a similar manner, and provides

$$\int d\varphi\cos.^n\varphi = \cos.^{n-1}\varphi\sin.\varphi + (n-1)\int d\varphi\cos.^{n-2}\varphi\sin.^2\varphi.$$
$$= \cos.^{n-1}\varphi\sin.\varphi + (n-1)\int d\varphi\cos.^{n-2}\varphi - (n-1)\int d\varphi\cos.^n\varphi,$$

from which, since the last formula is similar to that proposed, there is deduced

$$\int d\varphi\cos.^n\varphi = \tfrac{1}{n}\sin.\varphi\cos.^{n-1}\varphi + \tfrac{n-1}{n}\int d\varphi\cos.^{n-2}\varphi.$$

Whereby since in the cases $n = 0$ and $n = 1$ the integration shall be as set out, to the higher powers the progression is apparent :

$$\int d\varphi \cos.^0\varphi = \varphi,$$
$$\int d\varphi \cos.\varphi = \sin.\varphi,$$
$$\int d\varphi \cos.^2\varphi = \tfrac{1}{2}\sin.\varphi\cos.\varphi + \tfrac{1}{2}\varphi,$$
$$\int d\varphi \cos.^3\varphi = \tfrac{1}{3}\sin.\varphi\cos.^2\varphi + \tfrac{2}{3}\sin\varphi,$$
$$\int d\varphi \cos.^4\varphi = \tfrac{1}{4}\sin.\varphi\cos.^3\varphi + \tfrac{1\cdot 3}{2\cdot 4}\sin.\varphi\cos.\varphi + \tfrac{1\cdot 3}{2\cdot 4}\varphi,$$
$$\int d\varphi \cos.^5\varphi = \tfrac{1}{5}\sin.\varphi\cos.^4\varphi + \tfrac{1\cdot 4}{3\cdot 5}\sin.\varphi\cos.^2\varphi + \tfrac{2\cdot 4}{3\cdot 5}\cos\varphi,$$
$$\int d\varphi \cos.^6\varphi = \tfrac{1}{6}\sin.\varphi\cos.^5\varphi + \tfrac{1\cdot 5}{4\cdot 6}\sin.\varphi\cos.^3\varphi + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin.\varphi\cos.\varphi + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\varphi,$$

etc.

PROBLEMA 26

246. *To find the integral of the formula* $d\varphi \sin.^m\varphi\cos.^n\varphi$.

SOLUTION

So that this can be set out easier, we may consider the factor $\sin.^\mu\varphi\cos.^\nu\varphi$ which differentiated becomes $\mu d\varphi\sin.^{\mu-1}\varphi\cos.^{\nu+1}\varphi - \nu d\varphi\sin.^{\mu+1}\varphi\cos.^{\nu-1}\varphi$. Now since either in the first part $\cos.^2\varphi = 1-\sin.^2\varphi$ or in the second part $\sin.^2\varphi = 1-\cos.^2\varphi$ are put in place, there arises either

$$d.\sin.^\mu\varphi\cos.^\nu\varphi = +\mu d\varphi\sin.^{\mu-1}\varphi\cos.^{\nu-1}\varphi - (\mu+\nu)d\varphi\sin.^{\mu+1}\varphi.\cos.^{\nu-1}\varphi$$

or

$$d.\sin.^\mu\varphi\cos.^\nu\varphi = -\nu d\varphi\sin.^{\mu-1}\varphi\cos.^{\nu-1}\varphi + (\mu+\nu)d\varphi\sin.^{\mu-1}\varphi.\cos.^{\nu+1}\varphi.$$

Hence we obtain therefore the double reduction:

$$\text{I. } \int d\varphi\sin.^{\mu+1}\varphi\cos.^{\nu-1}\varphi = -\tfrac{1}{\mu+\nu}\sin.^\mu\varphi\cos.^\nu\varphi + \tfrac{\mu}{\mu+\nu}\int d\varphi\sin.^{\mu-1}\varphi.\cos.^{\nu-1}\varphi$$
$$\text{II. } \int d\varphi\sin.^{\mu-1}\varphi\cos.^{\nu+1}\varphi = \tfrac{1}{\mu+\nu}\sin.^\mu\varphi\cos.^\nu\varphi + \tfrac{\nu}{\mu+\nu}\int d\varphi\sin.^{\mu-1}\varphi.\cos.^{\nu-1}\varphi.$$

Whereby the proposed formula $\int d\varphi\sin.^m\varphi\cos.^n\varphi$ is reduced continually by successive simpler powers both of $\sin.\varphi$ as well as of $\cos.\varphi$, while either becomes completely simple or be present simpler, in which case the integration itself is apparent, since it shall be

$$\int d\varphi \sin.^m \varphi \cos.\varphi = \tfrac{1}{m+1}\sin.^{m+1}\varphi \text{ and } \int d\varphi \sin.\varphi \cos.^n \varphi = -\tfrac{1}{n+1}\cos.^{n+1}\varphi .$$

EXAMPLE

247. *To find the integral of the formula* $d\varphi \sin.^8 \varphi \cos.^7 \varphi$.

By the first reduction [I] on account of $\mu = 7$ and $\nu = 8$ we obtain

$$\int d\varphi \sin.^8 \varphi \cos.^7 \varphi = -\tfrac{1}{15}\sin.^7 \varphi \cos.^8 \varphi + \tfrac{7}{15}\int d\varphi \sin.^6 \varphi.\cos.^7 \varphi ;$$

by that latter reduction [II] we set out

$$\int d\varphi \sin.^6 \varphi \cos.^7 \varphi = \tfrac{1}{13}\sin.^7 \varphi \cos.^6 \varphi + \tfrac{6}{13}\int d\varphi \sin.^6 \varphi.\cos.^5 \varphi;$$

in this manner we can progress further

$$\int d\varphi \sin.^6 \varphi \cos.^5 \varphi = -\tfrac{1}{11}\sin.^5 \varphi \cos.^6 \varphi + \tfrac{5}{11}\int d\varphi \sin.^4 \varphi \cos.^5 \varphi,$$
$$\int d\varphi \sin.^4 \varphi \cos.^5 \varphi = \tfrac{1}{9}\sin.^5 \varphi \cos.^4 \varphi + \tfrac{4}{9}\int d\varphi \sin.^4 \varphi \cos.^3 \varphi,$$
$$\int d\varphi \sin.^4 \varphi \cos.^3 \varphi = -\tfrac{1}{7}\sin.^3 \varphi \cos.^4 \varphi + \tfrac{3}{7}\int d\varphi \sin.^2 \varphi \cos.^3 \varphi,$$
$$\int d\varphi \sin.^2 \varphi \cos.^3 \varphi = \tfrac{1}{5}\sin.^3 \varphi \cos.^2 \varphi + \tfrac{2}{5}\int d\varphi \sin.^2 \varphi \cos.\varphi,$$
$$\int d\varphi \sin.^2 \varphi \cos.\varphi = -\tfrac{1}{3}\sin.\varphi \cos.^2 \varphi + \tfrac{1}{3}\int d\varphi \cos.\varphi + \left(\tfrac{1}{3}\sin\varphi\right).$$

From these there is deduced the integral of the proposed formulas

$$\begin{aligned}\int d\varphi \sin.^8 \varphi \cos.^7 \varphi = &-\tfrac{1}{15}\sin.^7 \varphi \cos.^8 \varphi + \tfrac{1\cdot 7}{15\cdot 13}\sin.^7 \varphi \cos.^6 \varphi - \tfrac{1\cdot 7\cdot 6}{15\cdot 13\cdot 11}\sin.^5 \varphi \cos.^6 \varphi \\ &+ \tfrac{1\cdot 7\cdot 6\cdot 5}{15\cdot 13\cdot 11\cdot 9}\sin.^5 \varphi.\cos.^4 \varphi - \tfrac{1\cdot 7\cdot 6\cdot 5\cdot 4}{15\cdot 13\cdot 11\cdot 9\cdot 7}\sin.^3 \varphi.\cos.^4 \varphi \\ &+ \tfrac{1\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5}\sin.^3 \varphi.\cos.^2 \varphi - \tfrac{1\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3}\sin.\varphi.\cos.^2 \varphi \\ &+ \tfrac{1\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3}\sin.\varphi.\end{aligned}$$

SCHOLIUM

248. But when cases of this kind occur, always there is presented a product $\sin.^m \varphi \cos.^n \varphi$ to be resolved in the sin or cosine of multiples of the angles, with which done the individual parts are integrated easily. Besides here for the sake of brevity here I have indicated the angle simply by the letter φ and in no way does the problem become more general, if we express it by $\alpha\varphi+\beta$, just as also before this expression Ang.sin.x is equally extended more, if some function is written in place of x. Hence we shall consider formulas of this kind, in which the sine or cosine occupy the denominator, where indeed they are the simplest

$$\text{I.}\frac{d\varphi}{\sin.\varphi},\quad \text{II}\frac{d\varphi}{\cos.\varphi},\quad \text{III}\frac{d\varphi\cos.\varphi}{\sin.\varphi},\quad \text{IV.}\quad \frac{d\varphi\sin.\varphi}{\cos.\varphi},$$

the integrals of which in the first place it is required to know. For the first these transformations may be used:

$$\frac{d\varphi}{\sin.\varphi}=\frac{d\varphi\sin\varphi}{\sin^2.\varphi}=\frac{d\varphi\sin.\varphi}{1-\cos.^2\varphi}=\frac{-dx}{1-xx}\quad(\text{on putting } \cos.\varphi=x),$$

from which there comes about

$$\int\frac{d\varphi}{\sin.\varphi}=-\frac{1}{2}l\frac{1+x}{1-x}=-\frac{1}{2}l\frac{1+\cos.\varphi}{1-\cos.\varphi}.$$

For the second

$$\frac{d\varphi}{\cos.\varphi}=\frac{d\varphi\cos.\varphi}{\cos.^2\varphi}=\frac{d\varphi\cos.\varphi}{1-\sin.^2\varphi}=\frac{dx}{1-xx}\quad(\text{on putting } \sin.\varphi=x),$$

hence

$$\int\frac{d\varphi}{\cos.\varphi}=\frac{1}{2}l\frac{1+x}{1-x}=\frac{1}{2}l\frac{1+\sin.\varphi}{1-\sin.\varphi}.$$

The integration of the third and fourth clearly is performed with logarithms ; whereby it is helpful to know these integrals properly

$$\text{I.}\int\frac{d\varphi}{\sin.\varphi}=-\frac{1}{2}l\frac{1+\cos.\varphi}{1-\cos.\varphi}=l\frac{\sqrt{(1-\cos.\varphi)}}{\sqrt{1+\cos.\varphi}}=l\,\text{tang.}\tfrac{1}{2}\varphi,$$

$$\text{II.}\int\frac{d\varphi}{\cos.\varphi}=\frac{1}{2}l\frac{1+\sin.\varphi}{1-\sin.\varphi}=l\frac{\sqrt{(1+\sin.\varphi)}}{\sqrt{1-\sin.\varphi}}=l\,\text{tang.}\left(45^0+\tfrac{1}{2}\varphi\right),$$

$$\text{III.}\int\frac{d\varphi\cos.\varphi}{\sin.\varphi}=l\sin.\varphi=\int\frac{d\varphi}{\text{tang.}\varphi}=\int d\varphi\,\text{cot.}\varphi,$$

$$\text{IV.}\int\frac{d\varphi\sin.\varphi}{\cos.\varphi}=-l\cos.\varphi=\int d\varphi\,\text{tang.}\varphi\ ;$$

and hence it follows that III +IV

$$\int\frac{d\varphi}{\sin.\varphi\cos.\varphi}=l\frac{\sin.\varphi}{\cos.\varphi}=l\,\text{tang.}\varphi.$$

PROBLEMA 27

249. *To investigate the integration of the formulas* $\frac{d\varphi \sin.^m \varphi}{\cos.^n \varphi}$ *and* $\frac{d\varphi \cos.^m \varphi}{\sin.^n \varphi}$.

SOLUTION

In the first place it is seen immediately that the one formula can be transformed into the other on putting $\varphi = 90^0 - \psi$, since then there becomes $\sin.\varphi = \cos.\psi$ and $\cos.\varphi = \sin.\psi$, as long as it is noted that $d\varphi = -d\psi$. Whereby it suffices for only the first part to be discussed. Moreover the previous reduction §246 given provides, on taking $\mu + 1 = m$ and $v - 1 = -n$,

$$\int \frac{d\varphi \sin.^m \varphi}{\cos.^n \varphi} = -\frac{1}{m-n} \cdot \frac{\sin.^{m-1} \varphi}{\cos.^{n-1} \varphi} + \frac{m-1}{m-n} \int \frac{d\varphi \sin.^{m-2} \varphi}{\cos.^n \varphi},$$

from which it is agreed that the exponent of $\sin.\varphi$ is lowered by two continuously in the numerator, thus in order that finally it may come to either $\int \frac{d\varphi}{\cos.^n \varphi}$ or to $\int \frac{d\varphi \sin.\varphi}{\cos.^n \varphi} = \frac{1}{(n-1)} \cdot \frac{1}{\cos.^{n-1} \varphi}$ and thus only the formula $\int \frac{d\varphi}{\cos.^n \varphi}$ remains to be treated. But the other reduction treated there in (§ 246) on taking $\mu - 1 = m$ and $v - 1 = -n$ gives

$$\int \frac{d\varphi \sin.^m \varphi}{\cos.^{n-2} \varphi} = \frac{1}{m-n+2} \cdot \frac{\sin.^{m+1} \varphi}{\cos.^{n-1} \varphi} - \frac{n-1}{m-n+2} \int \frac{d\varphi \sin.^m \varphi}{\cos.^n \varphi},$$

from which it is deduced

$$\int \frac{d\varphi \sin.^m \varphi}{\cos.^n \varphi} = \frac{1}{n-1} \cdot \frac{\sin.^{m+1} \varphi}{\cos.^{n-1} \varphi} - \frac{m-n+2}{n-1} \int \frac{d\varphi \sin.^m \varphi}{\cos.^{n-2} \varphi},$$

with the aid of which reduction the exponent $\cos.\varphi$ is continually diminished by two in the denominator, thus in order that it may come to either $\int d\varphi \sin.^m \varphi$ or $\int \frac{d\varphi \sin^m.\varphi}{\cos.\varphi}$. The integration of this has been shown above, and indeed the form of this, if $m > 1$, by the first reduction is returned to either $\int \frac{d\varphi}{\cos.\varphi}$ or to $\int \frac{d\varphi \sin.\varphi}{\cos.\varphi}$; but the integral of that is $l\ \text{tang.}\left(45^0 + \frac{1}{2}\varphi\right)$, and of this truly $-l \cos.\varphi$.

COROLLARY 1

250. The first reduction cannot be used while $m = n$, clearly in this case $\int \frac{d\varphi \sin^n .\varphi}{\cos^n .\varphi}$ cannot be reduced to the formula $\int \frac{d\varphi \sin^{n-2} .\varphi}{\cos^n .\varphi}$. But the other reduction is always allowed to be used ; and if indeed the case $n = 1$ thereupon is excluded, the integration of this by the first method can be effected.

COROLLARY 2

251. But the account of this exclusion has been put into this, that the formula $\int \frac{d\varphi \sin^{n-2} .\varphi}{\cos^n .\varphi}$ is completely integrable having the integral equal to $\frac{1}{n-1} \cdot \frac{\sin^{n-1} .\varphi}{\cos^{n-1} .\varphi}$. It becomes in these cases

$$\int \frac{d\varphi}{\cos^2 .\varphi} = \frac{\sin .\varphi}{\cos .\varphi} = \text{tang}.\varphi, \qquad \int \frac{d\varphi \sin .\varphi}{\cos^3 .\varphi} = \frac{1}{2} \cdot \frac{d\varphi \sin^2 .\varphi}{\cos^2 .\varphi} = \frac{1}{2} \text{tang}^2 .\varphi,$$

$$\int \frac{d\varphi \sin^2 .\varphi}{\cos^4 .\varphi} = \frac{1}{3} \cdot \frac{d\varphi \sin^3 .\varphi}{\cos^3 .\varphi} = \frac{1}{3} \text{tang}^3 .\varphi, \qquad \int \frac{d\varphi \sin^3 .\varphi}{\cos^5 .\varphi} = \frac{1}{4} \cdot \frac{d\varphi \sin^4 .\varphi}{\cos^4 .\varphi} = \frac{1}{4} \text{tang}^4 .\varphi.$$

EXAMPLE 1

252. *To assign the integral to the formula* $\frac{d\varphi \sin .^m \varphi}{\cos .\varphi}$.

The first reduction gives

$$\int \frac{d\varphi \sin .^m \varphi}{\cos .\varphi} = \frac{-1}{m-1} \sin .^{m-1} \varphi + \int \frac{d\varphi \sin .^{m-2} \varphi}{\cos .\varphi}.$$

Hence from the cases noted from the beginning we have :

$$\int \frac{d\varphi}{cos.\varphi} = l\,\text{tang.}\left(45^0 + \tfrac{1}{2}\varphi\right),$$

$$\int \frac{d\varphi\, sin.\varphi}{cos.\varphi} = -l\, cos.\varphi = l\, sec.\varphi,$$

$$\int \frac{d\varphi\, sin.^2\varphi}{cos.\varphi} = -sin.\varphi + \int \frac{d\varphi}{cos.\varphi},$$

$$\int \frac{d\varphi\, sin.^3\varphi}{cos.\varphi} = -\tfrac{1}{2} sin.^2\varphi + l\, sec.\varphi,$$

$$\int \frac{d\varphi\, sin.^4\varphi}{cos.\varphi} = -\tfrac{1}{3} sin.^3\varphi - sin.\varphi + \int \frac{d\varphi}{cos.\varphi},$$

$$\int \frac{d\varphi\, sin.^5\varphi}{cos.\varphi} = -\tfrac{1}{4} sin.^4\varphi - \tfrac{1}{2} sin.^2\varphi + l\, sec.\varphi$$

$$\int \frac{d\varphi\, sin.^6\varphi}{cos.\varphi} = -\tfrac{1}{5} sin.^5\varphi - \tfrac{1}{3} sin.^3\varphi - sin.\varphi + \int \frac{d\varphi}{cos.\varphi},$$

$$\int \frac{d\varphi\, sin.^7\varphi}{cos.\varphi} = -\tfrac{1}{6} sin.^6\varphi - \tfrac{1}{4} sin.^4\varphi - \tfrac{1}{2} sin.^2\varphi + l\, sec.\varphi$$

etc.

SCHOLIUM

253. For the remaining cases of the denominators the whole calculation may be done from these reductions

$$\int \frac{d\varphi\, sin.^m\varphi}{cos.^2\varphi} = \frac{sin.^{m+1}\varphi}{cos.\varphi} - m\int d\varphi\, sin.^m\varphi,$$

$$\int \frac{d\varphi\, sin.^m\varphi}{cos.^3\varphi} = \frac{1}{2}\cdot\frac{sin.^{m+1}\varphi}{cos.^2\varphi} - \frac{m-1}{2}\int \frac{d\varphi\, sin.^m\varphi}{cos.\varphi},$$

$$\int \frac{d\varphi\, sin.^m\varphi}{cos.^4\varphi} = \frac{1}{3}\cdot\frac{sin.^{m+1}\varphi}{cos.^3\varphi} - \frac{m-2}{3}\int \frac{d\varphi\, sin.^m\varphi}{cos.^2\varphi},$$

$$\int \frac{d\varphi\, sin.^m\varphi}{cos.^5\varphi} = \frac{1}{4}\cdot\frac{sin.^{m+1}\varphi}{cos.^4\varphi} - \frac{m-3}{4}\int \frac{d\varphi\, sin.^m\varphi}{cos.^3\varphi}$$

etc.

EXAMPLE 2

254. *To assign the integral to the formula* $\frac{d\varphi}{cos.^n\varphi}$.

The other reduction on account of $m = 0$ becomes

$$\int \frac{d\varphi}{cos.^n\varphi} = \frac{1}{n-1}\cdot\frac{sin.\varphi}{cos.^{n-1}\varphi} + \frac{n-2}{n-1}\int \frac{d\varphi}{cos.^{n-2}\varphi};$$

because now the most simple cases are known

$$\int d\varphi = \varphi \quad \text{and} \quad \int \frac{d\varphi}{cos.\varphi} = l\,\text{tang.}\left(45^0 + \tfrac{1}{2}\varphi\right),$$

and all these following can all be recalled ,

$$\int\frac{d\varphi}{cos.^2\varphi}=\frac{sin.\varphi}{cos.\varphi},$$
$$\int\frac{d\varphi}{cos.^3\varphi}=\frac{1}{2}\cdot\frac{sin.\varphi}{cos.^2\varphi}+\frac{1}{2}\int\frac{d\varphi}{cos.\varphi},$$
$$\int\frac{d\varphi}{cos.^4\varphi}=\frac{1}{3}\cdot\frac{sin.\varphi}{cos.^3\varphi}+\frac{2}{3}\cdot\frac{sin.\varphi}{cos.\varphi},$$
$$\int\frac{d\varphi}{cos.^5\varphi}=\frac{1}{4}\cdot\frac{sin.\varphi}{cos.^4\varphi}+\frac{1\cdot 3}{2\cdot 4}\frac{sin.\varphi}{cos.^2\varphi}+\frac{1\cdot 3}{2\cdot 4}\int\frac{d\varphi}{cos.\varphi},$$
$$\int\frac{d\varphi}{cos.^6\varphi}=\frac{1}{5}\cdot\frac{sin.\varphi}{cos.^5\varphi}+\frac{1\cdot 4}{3\cdot 5}\frac{sin.\varphi}{cos.^3\varphi}+\frac{2\cdot 4}{3\cdot 5}\frac{sin.\varphi}{cos.\varphi}$$

etc.

COROLLARY 1

255. We have these integrations in a like manner :

$$\int\frac{d\varphi}{sin.\varphi}=l\,\text{tang.}\tfrac{1}{2}\varphi,\quad \int\frac{d\varphi}{sin.^2\varphi}=-\frac{cos.\varphi}{sin.\varphi},$$
$$\int\frac{d\varphi}{sin.^3\varphi}=-\frac{1}{2}\cdot\frac{cos.\varphi}{sin.^2\varphi}+\frac{1}{2}\int\frac{d\varphi}{sin.\varphi},$$
$$\int\frac{d\varphi}{sin.^4\varphi}=-\frac{1}{3}\cdot\frac{cos.\varphi}{sin.^3\varphi}-\frac{2}{3}\cdot\frac{cos.\varphi}{sin.\varphi},$$
$$\int\frac{d\varphi}{sin.^5\varphi}=-\frac{1}{4}\cdot\frac{cos.\varphi}{sin.^4\varphi}-\frac{1\cdot 3}{2\cdot 4}\frac{cos.\varphi}{sin.^2\varphi}+\frac{1\cdot 3}{2\cdot 4}\int\frac{d\varphi}{sin.\varphi}$$

etc.

COROLLARY 2

256. Following on there is

$$\int\frac{d\varphi\, sin.\varphi}{cos.^n\varphi}=\frac{1}{n-1}\cdot\frac{1}{cos.^{n-1}\varphi}\quad\text{and}\quad\int\frac{d\varphi\, cos.\varphi}{sin.^n\varphi}=\frac{-1}{n-1}\cdot\frac{1}{sin.^{n-1}\varphi}.$$

Again,

$$\int\frac{d\varphi\, sin.^2\varphi}{cos.^n\varphi}=\int\frac{d\varphi}{cos.^n\varphi}-\int\frac{d\varphi}{cos.^{n-2}\varphi},\quad \int\frac{d\varphi\, cos.^2\varphi}{sin.^n\varphi}=\int\frac{d\varphi}{sin.^n\varphi}-\int\frac{d\varphi}{sin.^{n-2}\varphi}$$

and

$$\int\frac{d\varphi\, sin.^3\varphi}{cos.^n\varphi}=\int\frac{d\varphi\, sin.\varphi}{cos.^n\varphi}-\int\frac{d\varphi\, sin.\varphi}{cos.^{n-2}\varphi},\quad \int\frac{d\varphi\, cos.^3\varphi}{sin.^n\varphi}=\int\frac{d\varphi\, cos.\varphi}{sin.^n\varphi}-\int\frac{d\varphi\, cos.\varphi}{sin.^{n-2}\varphi},$$

from which reductions it is permitted to progress continually.

PROBLEM 28

257. *To investigate the integral of the formula* $\frac{d\varphi}{sin.^m\varphi cos.^n\varphi}$.

SOLUTION

It is permitted here to adapt the above reductions used on taking *m* negative in the preceding problem; thus there becomes

$$\int\frac{d\varphi}{sin.^m\varphi cos.^n\varphi}=+\frac{1}{m+n}\cdot\frac{1}{sin.^{m+1}\varphi cos.^{n-1}\varphi}+\frac{m+1}{m+n}\int\frac{d\varphi}{sin.^{m+2}\varphi cos.^n\varphi},$$

from which on writing $m-2$ in place of *m*, by re-arrangement there becomes

$$\int\frac{d\varphi}{sin.^m\varphi cos.^n\varphi}=-\frac{1}{m-1}\cdot\frac{1}{sin.^{m-1}\varphi cos.^{n-1}\varphi}+\frac{m+n-2}{m-1}\int\frac{d\varphi}{sin.^{m-2}\varphi cos.^n\varphi};$$

from the other reduction similar to this there is

$$\int\frac{d\varphi}{sin.^m\varphi cos.^n\varphi}=\frac{1}{n-1}\cdot\frac{1}{sin.^{m-1}\varphi cos.^{n-1}\varphi}+\frac{m+n-2}{n-1}\int\frac{d\varphi}{sin.^m\varphi cos.^{n-2}\varphi}.$$

Now since in this generally the most simple forms shall be

$$\int\frac{d\varphi}{sin.\varphi}=l\,\text{tang.}\tfrac{1}{2}\varphi,\quad \int\frac{d\varphi}{cos.\varphi}=l\,\text{tang.}\left(45^0+\tfrac{1}{2}\varphi\right),\quad \int\frac{d\varphi}{sin.\varphi cos.\varphi}=l\,\text{tang.}\varphi,$$
$$\int\frac{d\varphi}{sin.^2\varphi}=-\text{cot.}\varphi,\quad \int\frac{d\varphi}{cos.^2\varphi}=\text{tang.}\varphi,$$

hence we can elicit the more composite forms :

$$\int\frac{d\varphi}{sin.\varphi cos.^2\varphi}=\frac{1}{cos.\varphi}+\int\frac{d\varphi}{sin.\varphi},$$

$$\int\frac{d\varphi}{sin.^2\varphi cos.\varphi}=-\frac{1}{sin.\varphi}+\int\frac{d\varphi}{cos.\varphi},$$

$$\int\frac{d\varphi}{sin.\varphi cos.^4\varphi}=\frac{1}{3}\frac{1}{cos.^3\varphi}+\int\frac{d\varphi}{sin.\varphi cos.^2\varphi},$$

$$\int\frac{d\varphi}{sin.^4\varphi cos.\varphi}=-\frac{1}{3}\frac{1}{sin.^3\varphi}+\int\frac{d\varphi}{sin.^2\varphi cos.\varphi},$$

$$\int\frac{d\varphi}{sin.\varphi cos.^6\varphi}=\frac{1}{5}\frac{1}{cos.^5\varphi}+\int\frac{d\varphi}{sin.\varphi cos.^4\varphi},$$

$$\int\frac{d\varphi}{sin.^6\varphi cos.\varphi}=-\frac{1}{5}\frac{1}{sin.^5\varphi}+\int\frac{d\varphi}{sin.^4\varphi cos.\varphi},$$

$$\int\frac{d\varphi}{sin.\varphi cos.^3\varphi}=\frac{1}{2}\frac{1}{cos.^2\varphi}+\int\frac{d\varphi}{sin.\varphi cos.\varphi},$$

$$\int\frac{d\varphi}{sin.^3\varphi cos.\varphi}=-\frac{1}{2}\frac{1}{sin.^2\varphi}+\int\frac{d\varphi}{sin.\varphi cos.\varphi},$$

$$\int\frac{d\varphi}{sin.\varphi cos.^5\varphi}=\frac{1}{cos.^4\varphi}+\int\frac{d\varphi}{sin.\varphi cos.^3\varphi},$$

$$\int\frac{d\varphi}{sin.^5\varphi cos.\varphi}=-\frac{1}{4}\frac{1}{sin.^4\varphi}+\int\frac{d\varphi}{sin.^3\varphi cos.\varphi},$$

$$\int\frac{d\varphi}{sin.\varphi cos.^7\varphi}=\frac{1}{6}\frac{1}{cos.^6\varphi}+\int\frac{d\varphi}{sin.\varphi cos.^5\varphi},$$

$$\int\frac{d\varphi}{sin.^7\varphi cos.\varphi}=-\frac{1}{6}\frac{1}{sin.^6\varphi}+\int\frac{d\varphi}{sin.^5\varphi cos.\varphi},$$

$$\int\frac{d\varphi}{sin.^2\varphi cos.^2\varphi}=\frac{1}{sin.\varphi cos.\varphi}+2\int\frac{d\varphi}{sin.^2\varphi}=-\frac{1}{sin.\varphi cos.\varphi}+2\int\frac{d\varphi}{cos.^2\varphi},$$

$$\int\frac{d\varphi}{sin.^2\varphi cos.^4\varphi}=\frac{1}{3}\cdot\frac{1}{sin.\varphi cos.^3\varphi}+\frac{4}{3}\int\frac{d\varphi}{sin.^2\varphi cos.^2\varphi},$$

$$\int\frac{d\varphi}{sin.^4\varphi cos.^2\varphi}=-\frac{1}{3}\cdot\frac{1}{sin.^3\varphi cos.\varphi}+\frac{4}{3}\int\frac{d\varphi}{sin.^2\varphi cos.^2\varphi}.$$

etc.

And thus however many composite formulas as can be brought to view are reduced to simpler integrals.

COROLLARY 1

258. Both the exponents of $sin.\varphi$ and $cos.\varphi$ can be reduced by two simultaneously ; for by the first reduction, there shall be

$$\int \frac{d\varphi}{sin.^{\mu}\varphi cos.^{\nu}\varphi} = -\frac{1}{\mu-1}\cdot\frac{1}{sin.^{\mu-1}\varphi cos.^{\nu-1}\varphi} + \frac{\mu+\nu-2}{\mu-1}\int \frac{d\varphi}{sin.^{\mu-2}\varphi cos.^{\nu}\varphi} ;$$

now this formula by the latter reduction on account of $m = \eta - 2$ and $n = \nu$ gives

$$\int \frac{d\varphi}{sin.^{\mu-2}\varphi cos.^{\nu}\varphi} = \frac{1}{\nu-1}\cdot\frac{1}{sin.^{\mu-3}\varphi cos.^{\nu-1}\varphi} + \frac{\mu+\nu-4}{\nu-1}\int \frac{d\varphi}{sin.^{\mu-2}\varphi cos.^{\nu-2}\varphi} ,$$

from which it is deduced,

$$\int \frac{d\varphi}{sin.^{\mu}\varphi cos.^{\nu}\varphi} = -\frac{1}{\mu-1}\cdot\frac{1}{sin.^{\mu-1}\varphi cos.^{\nu-1}\varphi} + \frac{\mu+\nu-2}{(\mu-1)(\nu-1)}\cdot\frac{1}{sin.^{\mu-3}\varphi cos.^{\nu-1}\varphi}$$
$$+ \frac{(\mu+\nu-2)(\mu+\nu-4)}{(\mu-1)(\nu-1)}\int \frac{d\varphi}{sin.^{\mu-2}\varphi cos.^{\nu-2}\varphi} .$$

COROLLARY 2

259. From the previous members reduced to a common denominator there is obtained :

$$\int \frac{d\varphi}{sin.^{\mu}\varphi cos.^{\nu}\varphi}$$
$$= \frac{(\mu-1)sin.^{2}\varphi-(\nu-1)cos.^{2}\varphi}{(\mu-1)(\nu-1)sin.^{\mu-1}\varphi cos.^{\nu-1}\varphi} + \frac{(\mu+\nu-2)(\mu+\nu-4)}{(\mu-1)(\nu-1)}\int \frac{d\varphi}{sin.^{\mu-2}\varphi cos.^{\nu-2}\varphi} ,$$

from which reduction as it is always permitted for a calculation to be assembled, unless either $\mu = 1$ or $\nu = 1$.

SCHOLIUM

260. Formulas of this kind $\frac{d\varphi}{sin.^{m}\varphi cos.^{n}\varphi}$ also are especially easy to be reduced in this way, while the numerator is multiplied by $sin.^{2}\varphi + cos.^{2}\varphi = 1$, from which there becomes

$$\int \frac{d\varphi}{sin.^{m}\varphi cos.^{n}\varphi} = \int \frac{d\varphi}{sin.^{m-2}\varphi cos.^{n}\varphi} + \int \frac{d\varphi}{sin.^{m}\varphi cos.^{n-2}\varphi} ,$$

which with that continued as far as it can be, until only a single power is left in the denominator. Thus there becomes

$$\int \frac{d\varphi}{sin.\varphi cos.\varphi} = \int \frac{d\varphi sin.\varphi}{cos.\varphi} + \int \frac{d\varphi cos.\varphi}{sin.\varphi} = l\,\frac{sin.\varphi}{cos.\varphi},$$
$$\int \frac{d\varphi}{sin.^{2}\varphi cos.^{2}\varphi} = \int \frac{d\varphi}{sin.^{2}\varphi} + \int \frac{d\varphi}{cos.^{2}\varphi} = \frac{sin.\varphi}{cos.\varphi} - \frac{cos.\varphi}{sin.\varphi}.$$

But if this formula should be proposed, $\frac{d\varphi}{sin.^m\varphi cos.^n\varphi}$, the formula is called on to help

$$sin.\varphi cos.\varphi = \tfrac{1}{2} sin.2\varphi ,$$

from which here is had

$$\int \frac{2^n d\varphi}{sin.^n 2\varphi} = 2^{n-1}\int \frac{d\omega}{sin.^n\omega}$$

on putting $\omega = 2\varphi$, which formula can be resolved by the above rules.

Therefore with these means observed concerning the formula $d\varphi sin.^m\varphi cos.^n\varphi$, if m and n should be certain whole numbers, either positive or negative, nothing further should be required ; but if they should be fractions, nothing very instructive comes to mind, since the cases, in which the integration succeeds, are produced as if individually.

But just as the integrals which cannot be displayed, they can be agreed to be expressed by a series, which we set out more carefully in the following chapter.

Now we consider fractional formulas, of which the denominator is $a + bcos.\varphi$ and powers of this ; for such formulas occur most frequently in theoretical astronomy.

PROBLEM 29

261. *To investigate the integral of the differential* $\frac{d\varphi}{a+bcos.\varphi}$.

SOLUTION

This investigation cannot be put in place more conveniently, than so that the proposed formula is reduced to a regular form by putting $cos.\varphi = \frac{1-xx}{1+xx}$ so that there becomes reasonably

$$sin.\varphi = \frac{2x}{1+xx} \quad \text{and hence} \quad d\varphi cos.\varphi = \frac{2dx(1-xx)}{(1+xx)^2}$$

and thus $d\varphi = \frac{2dx}{1+xx}$. Therefore because

$$a + bcos.\varphi = \frac{a+b+(a-b)xx}{1+xx},$$

our formula will become

$$\frac{d\varphi}{a+bcos.\varphi} = \frac{2dx}{a+b+(a-b)xx}$$

which, since there must be either $a > b$ or $a < b$, either an angle or a logarithm appears.

In the case $a > b$ there is found

$$\int \frac{d\varphi}{a+bcos.\varphi} = \frac{2}{\sqrt{(aa-bb)}} \text{Ang.tang.} \frac{(a-b)x}{\sqrt{(aa-bb)}},$$

now in the case $a < b$ there becomes

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \frac{2}{\sqrt{(bb-aa)}}\, l\, \frac{\sqrt{(bb-aa)}+x(b-a)}{\sqrt{(bb-aa)}-x(b-a)}.$$

Now indeed there is

$$x = \sqrt{\frac{1-\cos.\varphi}{1+\cos.\varphi}} = \text{tang.}\tfrac{1}{2}\varphi = \frac{\sin.\varphi}{1+\cos.\varphi}\ ;$$

with which factor restored when there arises

$$2\text{Ang.tang.}\frac{(a-b)x}{\sqrt{(aa-bb)}} = \text{Ang.tang.}\frac{2x\sqrt{(aa-bb)}}{a+b-(a-b)xx}$$

$$= \text{Ang.tang.}\frac{2\sin.\varphi\sqrt{(aa-bb)}}{(a+b)(1+\cos.\varphi)-(a-b)(1-\cos.\varphi)} = \text{Ang.tang.}\frac{\sin.\varphi\sqrt{(aa-bb)}}{a\cos.\varphi+b},$$

wherefore for the case $a > b$ we obtain

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \frac{2}{\sqrt{(aa-bb)}}\text{Ang.tang.}\frac{\sin.\varphi\sqrt{(aa-bb)}}{a\cos.\varphi+b}$$

or

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \frac{1}{\sqrt{(aa-bb)}}\text{Ang.sin.}\frac{\sin.\varphi\sqrt{(aa-bb)}}{a+b\cos.\varphi}$$

or in turn

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \frac{1}{\sqrt{(aa-bb)}}\text{Ang.cos.}\frac{a\cos.\varphi+b}{a+b\cos.\varphi}.$$

But for the case $a < b$,

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \frac{1}{\sqrt{(bb-aa)}}\, l\, \frac{\sqrt{(b+a)(1+\cos.\varphi)}+\sqrt{(b-a)(1-\cos.\varphi)}}{\sqrt{(b+a)(1+\cos.\varphi)}-\sqrt{(b-a)(1-\cos.\varphi)}},$$

or

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \frac{1}{\sqrt{(bb-aa)}}\, l\, \frac{a\cos.\varphi+b+\sin.\varphi\sqrt{(bb-aa)}}{a+b\cos.\varphi}.$$

But in the case $b = a$ the integral is equal to $\frac{x}{a} = \frac{1}{a}\text{tang.}\frac{1}{2}\varphi$, from which there becomes

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \text{tang.}\tfrac{1}{2}\varphi = \frac{\sin.\varphi}{1+\cos.\varphi},$$

which integrals vanish on making $\varphi = 0$.

COROLLARY 1

262. But the integral of the formula $\frac{d\varphi \sin.\varphi}{a+b\cos.\varphi}=\frac{-d.\cos.\varphi}{a+b\cos.\varphi}$ is $\frac{1}{b}l\frac{a+b}{a+b\cos.\varphi}$ thus taken, so that it vanishes on putting $\varphi=0$; and thus we have

$$\int\frac{d\varphi \sin.\varphi}{a+b\cos.\varphi}=\frac{1}{b}l\frac{a+b}{a+b\cos.\varphi}.$$

COROLLARY 2

263. Moreover the formula $\frac{d\varphi \cos.\varphi}{a+b\cos.\varphi}$ is transformed into $\frac{d\varphi}{b}-\frac{ad\varphi}{b(a+b\cos.\varphi)}$ from which the integral can be shown by the solution of the problem

$$\int\frac{d\varphi \cos.\varphi}{a+b\cos.\varphi}=\frac{\varphi}{b}-\frac{a}{b}\int\frac{d\varphi}{(a+b\cos.\varphi)}.$$

SCHOLIUM 1

264. With this integration found also the integral of this formula $\frac{d\varphi \cos.\varphi}{(a+b\cos.\varphi)^n}$ can be found, providing *n* is a whole number; for by making the form of the integral available most conveniently, it is seen:

$$\int\frac{d\varphi \cos.\varphi}{(a+b\cos.\varphi)^2}=\frac{A\sin.\varphi}{a+b\cos.\varphi}+m\int\frac{d\varphi}{a+b\cos.\varphi}$$

and there is found :

$$A=\frac{-b}{aa-bb} \text{ and } m=\frac{a}{aa-bb};$$

$$\int\frac{d\varphi}{(a+b\cos.\varphi)^3}=\frac{(A+B\cos.\varphi)\sin.\varphi}{(a+b\cos.\varphi)^2}+m\int\frac{d\varphi}{(a+b\cos.\varphi)^2}$$

and again there is found

$$A=\frac{-b}{aa-bb},\quad B=\frac{-bb}{2a(aa-bb)},\quad m=\frac{2aa+bb}{2a(aa-bb)}$$

and in the same way the investigation to greater powers can be continued, indeed not without a little tedious labour.
But in the following way the calculation is observed to be brought about most easily.
Clearly the more general formula is to be considered $\frac{d\varphi\cos.\varphi(f+g\cos.\varphi)}{(a+b\cos.\varphi)^{n+1}}$ and there is put

$$\int\frac{d\varphi\cos.\varphi(f+g\cos.\varphi)}{(a+b\cos.\varphi)^{n+1}}=\frac{A\sin.\varphi}{(a+b\cos.\varphi)^n}+\int\frac{d\varphi(B+C\cos.\varphi)}{(a+b\cos.\varphi)^n}$$

and on taking the differentials from that the equation is produced

$$f+g\cos.\varphi = A\cos.\varphi(a+b\cos.\varphi)+nAb\sin.^2\varphi+(B+C\cos.\varphi)(a+b\cos.\varphi),$$

which on account of $\sin.^2\varphi = 1-\cos.^2\varphi$, this form is put in place

$$\left.\begin{array}{llll} -f & -\ g\cos.\varphi+ & Ab\cos.^2\varphi \\ +nAb & +Aa\cos.\varphi- & nAb\cos.^2\varphi \\ +Ba & +Bb\cos.\varphi+ & Cb\cos.^2\varphi \\ & +Ca\cos.\varphi & \end{array}\right\}=0,$$

from which with the individual members equated to zero, there is elicited

$$A=\frac{ag-bf}{n(aa-bb)},\quad B=\frac{af-bg}{aa-bb}\quad\text{and}\quad C=\frac{(n-1)(ag-bf)}{n(aa-bb)},$$

thus in order that this reduction is obtained

$$\int\frac{d\varphi\cos.\varphi(f+g\cos.\varphi)}{(a+b\cos.\varphi)^{n+1}}$$
$$=\frac{(ag-bf)\sin.\varphi}{n(aa-bb)(a+b\cos.\varphi)^n}+\frac{1}{n(aa-bb)}\int\frac{d\varphi(n(af-bg)+(n-1)(ag-bf)\cos.\varphi)}{(a+b\cos.\varphi)^n},$$

with the help of which finally the formula is come upon $\int\frac{d\varphi(h+k\cos.\varphi)}{(a+b\cos.\varphi)}$, the integral of which equal to

$$\frac{k}{b}\varphi+\frac{bh-ak}{b}\int\frac{d\varphi}{a+b\cos.\varphi}$$

agrees with the above. But it is evident that $k=0$ always.

SCHOLIUM 2

265. Also formulas of this kind occur, in which the above angle quantity φ enters bearing the exponent of the exponential $e^{\alpha\varphi}$, which as is required to be treated, it is considered to be shown, as hence the above reduction method established may be illustrated very well. For here by that reduction a similar formula to that proposed is arrived at, from which the integral itself can be deduced. To this end, it is to be noted that $\int e^{\alpha\varphi}d\varphi=\frac{1}{\alpha}e^{\alpha\varphi}$.

PROBLEM 30

266. *To investigate the integral of the differential formula* $dy = e^{\alpha\varphi}d\varphi \sin.^{n}\varphi$.

SOLUTION

On taking $e^{\alpha\varphi}d\varphi$ for the factor of the differential there becomes

$$y = \tfrac{1}{\alpha}e^{\alpha\varphi}\sin.^{n}\varphi - \tfrac{n}{\alpha}\int e^{\alpha\varphi}d\varphi \sin.^{n-1}\varphi \cos.\varphi;$$

in a similar manner there is found

$$\int e^{\alpha\varphi}d\varphi \sin.^{n-1}\varphi \cos.\varphi$$
$$= \tfrac{1}{\alpha}e^{\alpha\varphi}\sin.^{n-1}\varphi\cos.\varphi - \tfrac{1}{\alpha}\int e^{\alpha\varphi}d\varphi\Big((n-1)\sin.^{n-2}\varphi\cos.^{2}\varphi - \sin.^{n}\varphi\Big),$$

which latter formula on account of $\cos.^{2}\varphi = 1-\sin.^{2}\varphi$ is reduced to this

$$(n-1)\int e^{\alpha\varphi}d\varphi\sin.^{n-2}\varphi - n\int e^{\alpha\varphi}d\varphi\sin.^{n}\varphi,$$

from which there is found

$$\int e^{\alpha\varphi}d\varphi\sin.^{n}\varphi = \tfrac{1}{\alpha}e^{\alpha\varphi}\sin.^{n}\varphi - \tfrac{n}{\alpha\alpha}e^{\alpha\varphi}\sin.^{n-1}\varphi\cos.\varphi + \tfrac{n(n-1)}{\alpha\alpha}\int e^{\alpha\varphi}d\varphi\sin.^{n-2}\varphi$$
$$-\tfrac{nn}{\alpha\alpha}\int e^{\alpha\varphi}d\varphi\sin.^{n}\varphi.$$

Whereby from this final formula on being taken with the first there is elicited

$$\int e^{\alpha\varphi}d\varphi\sin.^{n}\varphi = \frac{e^{\alpha\varphi}\sin.^{n-1}\varphi(\alpha\sin.\varphi - n\cos.\varphi)}{\alpha\alpha+nn} + \frac{n(n-1)}{\alpha\alpha+nn}\int e^{\alpha\varphi}d\varphi\sin.^{n-2}\varphi.$$

Hence in two cases the integral is given absolutely, clearly $n = 0$ and $n = 1$, and there becomes

$$\int e^{\alpha\varphi}d\varphi = \tfrac{1}{\alpha}e^{\alpha\varphi} - \tfrac{1}{\alpha} \quad\text{and}\quad \int e^{\alpha\varphi}d\varphi\sin.\varphi = \frac{e^{\alpha\varphi}(\alpha\sin.\varphi - \cos.\varphi)}{\alpha\alpha+1} + \frac{1}{\alpha\alpha+1}$$

and to these all the following are reduced, where n is a whole number greater than one.

COROLLARY 1

267. Thus if $n = 2$, we acquire this integration

$$\int e^{\alpha\varphi} d\varphi \sin.^2 \varphi = \frac{e^{\alpha\varphi} \sin.\varphi(\alpha \sin.\varphi - 2\cos.\varphi)}{\alpha\alpha+4} + \frac{1\cdot 2}{\alpha(\alpha\alpha+4)} e^{\alpha\varphi} - \frac{1\cdot 2}{\alpha(\alpha\alpha+4)};$$

but if there shall be $n = 3$, this integration:

$$\int e^{\alpha\varphi} d\varphi \sin.^3 \varphi = \frac{e^{\alpha\varphi} \sin.^2\varphi(\alpha \sin.\varphi - 3\cos.\varphi)}{\alpha\alpha+9} + \frac{2\cdot 3 e^{\alpha\varphi}(\alpha \sin.\varphi - \cos.\varphi)}{(\alpha\alpha+1)(\alpha\alpha+9)} + \frac{2\cdot 3}{(\alpha\alpha+1)(\alpha\alpha+9)}$$

with the integrals thus taken, so that the vanish on putting $\varphi = 0$.

COROLLARY 2

268. Therefore if with the integrations determined in this manner there is put in place $\alpha\varphi = -\infty$, in order that $e^{\alpha\varphi}$ vanished, then in general there arises

$$\int e^{\alpha\varphi} d\varphi \sin.^n \varphi = \frac{n(n-1)}{\alpha\alpha+nn} \int e^{\alpha\varphi} d\varphi \sin.^{n-2} \varphi$$

and hence the integrals for that case $\alpha\varphi = -\infty$, shall be

$$\int e^{\alpha\varphi} d\varphi = -\frac{1}{\alpha}, \qquad \int e^{\alpha\varphi} d\varphi \sin.\varphi = \frac{1}{\alpha\alpha+1},$$

$$\int e^{\alpha\varphi} d\varphi \sin.^2 \varphi = \frac{-1\cdot 2}{\alpha(\alpha\alpha+4)}, \qquad \int e^{\alpha\varphi} d\varphi \sin.^3 \varphi = \frac{1\cdot 2\cdot 3}{(\alpha\alpha+1)(\alpha\alpha+9)},$$

$$\int e^{\alpha\varphi} d\varphi \sin.^4 \varphi = \frac{-1\cdot 2\cdot 3\cdot 4}{\alpha(\alpha\alpha+4)(\alpha\alpha+16)}, \qquad \int e^{\alpha\varphi} d\varphi \sin.^5 \varphi = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{(\alpha\alpha+1)(\alpha\alpha+9)(\alpha\alpha+25)}.$$

COROLLARY 3

269. Whereby if this infinite series is put in place

$$s = 1 + \frac{1\cdot 2}{\alpha(\alpha\alpha+4)} + \frac{1\cdot 2\cdot 3\cdot 4}{\alpha(\alpha\alpha+4)(\alpha\alpha+16)} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{(\alpha\alpha+4)(\alpha\alpha+16)(\alpha\alpha+36)} + \text{ etc.}$$

then there becomes

$$s = -\alpha \int e^{\alpha\varphi} d\varphi \left(1 + \sin.^2\varphi + \sin.^4\varphi + \sin.^6\varphi + \text{ etc.}\right)$$

or

$$s = -\alpha \int \frac{e^{\alpha\varphi} d\varphi}{\cos.^2 \varphi}$$

on putting $\alpha\varphi = -\infty$ after the integration.

PROBLEM 31

270. *To investigate the integral of the differential formula* $e^{\alpha\varphi}d\varphi\cos.^n\varphi$.

SOLUTION

On proceeding in a similar manner as before there shall be

$$\int e^{\alpha\varphi}d\varphi\cos.^n\varphi=\tfrac{1}{\alpha}e^{\alpha\varphi}\cos.^n\varphi+\tfrac{n}{\alpha}\int e^{\alpha\varphi}d\varphi\sin.\varphi\cos.^{n-1}\varphi,$$

then

$$\int e^{\alpha\varphi}d\varphi\sin.\varphi\cos.^{n-1}\varphi$$
$$=\tfrac{1}{\alpha}e^{\alpha\varphi}\sin.\varphi\cos.^{n-1}\varphi-\tfrac{1}{\alpha}\int e^{\alpha\varphi}d\varphi\left(\cos.^n\varphi-(n-1)\cos.^{n-2}\varphi\sin.^2\varphi\right),$$

which final formula becomes

$$-(n-1)\int e^{\alpha\varphi}d\varphi\cos.^{n-2}\varphi+n\int e^{\alpha\varphi}d\varphi\cos.^n\varphi,$$

thus in order that there becomes

$$\int e^{\alpha\varphi}d\varphi\cos.^n\varphi=\tfrac{1}{\alpha}e^{\alpha\varphi}\cos.^n\varphi+\tfrac{n}{\alpha\alpha}e^{\alpha\varphi}\sin.\varphi\cos.^{n-1}\varphi+\tfrac{n(n-1)}{\alpha\alpha}\int e^{\alpha\varphi}d\varphi\cos.^{n-2}\varphi$$
$$-\tfrac{nn}{\alpha\alpha}\int e^{\alpha\varphi}d\varphi\cos.^n\varphi,$$

from which we deduce

$$\int e^{\alpha\varphi}d\varphi\cos.^n\varphi=\frac{e^{\alpha\varphi}\cos.^{n-1}\varphi(\alpha\cos.\varphi+n\sin.\varphi)}{\alpha\alpha+nn}+\frac{n(n-1)}{\alpha\alpha+nn}\int e^{\alpha\varphi}d\varphi\cos.^{n-2}\varphi\ ;$$

hence the simplest cases are therefore

$$\int e^{\alpha\varphi}d\varphi=\tfrac{1}{\alpha}e^{\alpha\varphi}+C,\qquad \int e^{\alpha\varphi}d\varphi\cos.\varphi=\frac{e^{\alpha\varphi}(\alpha\cos.\varphi+\sin.\varphi)}{\alpha\alpha+1}+C,$$

to which all the following cases are reduced, where *n* is a positive whole number.

SCHOLIUM

271. With the simplest cases noted another integral is given in the way of the proposed formula, why not also the further extension of this to $e^{\alpha\varphi}d\varphi\sin.^{m}\varphi\cos.^{n}\varphi$, to be elicited. For since the product $\sin.^{m}\varphi\cos.^{n}\varphi$ can be resolved in several sums of sines or cosines, each of which is of this form $M\sin.\lambda\varphi$ or $M\cos.\lambda\varphi$ and the integration is reduced to either of these formulas $e^{\alpha\varphi}d\varphi\sin.\lambda\varphi$ or $e^{\alpha\varphi}d\varphi\cos.\lambda\varphi$. Hence we put $\lambda\varphi=\omega$, so that we shall have

$$e^{\alpha\varphi}d\varphi\sin.\lambda\varphi=\tfrac{1}{\lambda}e^{\frac{\alpha}{\lambda}\omega}d\omega\sin.\omega \quad \text{and} \quad e^{\alpha\varphi}d\varphi\cos.\lambda\varphi=\tfrac{1}{\lambda}e^{\frac{\alpha}{\lambda}\omega}d\omega\cos.\omega,$$

the integrals of which are given by the above :

$$\int e^{\frac{\alpha}{\lambda}\omega}d\omega\sin.\omega=\frac{e^{\frac{\alpha}{\lambda}\omega}(\alpha\sin.\omega-\lambda\cos.\omega)}{\alpha\alpha+\lambda\lambda}=\frac{e^{\frac{\alpha}{\lambda}\omega}(\alpha\sin.\lambda\varphi-\lambda\cos.\lambda\varphi)}{\alpha\alpha+\lambda\lambda},$$

$$\int e^{\frac{\alpha}{\lambda}\omega}d\omega\cos.\omega=\frac{e^{\frac{\alpha}{\lambda}\omega}(\alpha\cos.\omega+\lambda\sin.\omega)}{\alpha\alpha+\lambda\lambda}=\frac{e^{\frac{\alpha}{\lambda}\omega}(\alpha\cos.\lambda\varphi+\lambda\sin.\lambda\varphi)}{\alpha\alpha+\lambda\lambda}.$$

From which finally we deduce

$$\int e^{\alpha\varphi}d\varphi\sin.\lambda\varphi=\frac{e^{\alpha\varphi}(\alpha\sin.\lambda\varphi-\lambda\cos.\lambda\varphi)}{\alpha\alpha+\lambda\lambda}$$

and

$$\int e^{\alpha\varphi}d\varphi\cos.\lambda\varphi=\frac{e^{\alpha\varphi}(\alpha\cos.\lambda\varphi+\lambda\sin.\lambda\varphi)}{\alpha\alpha+\lambda\lambda}.$$

If in general I should write at once $\sin.\varphi$ and $\cos.\varphi$ in place of $\sin.\lambda\varphi$ and $\cos.\lambda\varphi$, there would be no need for this reduction, but since here nothing is difficulty, I have valued the judgement of brevity.

CAPUT V

DE INTEGRATIONE FORMULARUM ANGULOS SINUSVE ANGULORUM IMPLICANTIUM

PROBLEMA 23

234. *Proposita formula differentiali* $X dx \text{Ang.sin.} x$ *eius integrale investigare.*

SOLUTIO

Cum sit

$$d.\text{Ang.sin.}x = \frac{dx}{\sqrt{(1-xx)}}$$

formula proposita ita in factores discerpatur $Ang.sin.x \cdot Xdx$. Si iam Xdx integrationem patiatur sitque $\int Xdx = P$, erit nostrum integrale

$$\int Xdx\text{Ang.sin.}x = P\text{Ang.sin.}x - \int \frac{Pdx}{\sqrt{(1-xx)}}$$

itaque opus reductum est ad integrationem formulae algebraicae, pro qua supra praecepta sunt tradita.

Caeterum si fuerit $X = \frac{dx}{\sqrt{(1-xx)}}$, manifestum est integrale fore

$$\int \frac{dx}{\sqrt{(1-xx)}} \text{Ang.sin.}x = \frac{1}{2}\left(\text{Ang.sin.}x\right)^2$$

quo solo casu quadratum anguli in integrale ingreditur.

EXEMPLUM 1

235. *Hanc formulam* $dy = x^n dx Ang.sin.x$ *integrare.*

Cum sit

$$P = \int x^n dx = \frac{x^{n+1}}{n+1},$$

habebimus

$$y = \frac{x^{n+1}}{n+1}\text{Ang.sin.}x - \frac{1}{n+1}\int \frac{x^{n+1}dx}{\sqrt{(1-xx)}}.$$

Hinc pro variis valoribus ipsius n erunt integralia ope § 120 eruta, ut sequentur

$$\int dx\,\text{Ang.sin.}x = x\,\text{Ang.sin.}x + \sqrt{(1-xx)} - 1,$$

$$\int x\,dx\,\text{Ang.sin.}x = \tfrac{1}{2}xx\,\text{Ang.sin.}x + \tfrac{1}{4}x\sqrt{(1-xx)} - \tfrac{1}{4}\text{Ang.sin.}x,$$

$$\int x^2dx\,\text{Ang.sin.}x = \tfrac{1}{3}x^3\text{Ang.sin.}x + \tfrac{1}{3}\left(\tfrac{1}{3}x^2 + \tfrac{2}{3}\right)\sqrt{(1-xx)} - \tfrac{1}{3}\cdot\tfrac{2}{3},$$

$$\int x^3dx\,\text{Ang.sin.}x = \tfrac{1}{4}x^4\text{Ang.sin.}x + \tfrac{1}{4}\left(\tfrac{1}{4}x^3 + \tfrac{1\cdot 3}{2\cdot 4}\right)\sqrt{(1-xx)} - \tfrac{1}{4}\cdot\tfrac{1\cdot 3}{2\cdot 4}\text{Ang.sin.}x,$$

quae ita sunt sumta, ut evanescant posito $x = 0$.

EXEMPLUM 2

236. *Hanc formulam* $dy = \frac{xdx}{\sqrt{(1-xx)}}\,Ang.sin.x$ *integrare.*

Cum sit

$$\int \frac{xdx}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)} = P,$$

erit integrale quaesitum

$$y = C - \sqrt{(1-xx)}\text{Ang.sin.}x + \int \frac{dx\sqrt{(1-xx)}}{\sqrt{(1-xx)}}$$

sicque habebitur

$$y = \int \frac{xdx}{\sqrt{(1-xx)}}\text{Ang.sin.}x = C - \sqrt{(1-xx)}\text{Ang.sin.}x + x.$$

EXEMPLUM 3

237. *Hanc formulam* $dy = \frac{dx}{(1-xx)^{\frac{3}{2}}}\text{Ang.sin.}x$ *integrare.*

Hic est

$$P = \int \frac{dx}{(1-xx)^{\frac{3}{2}}} = \frac{x}{\sqrt{(1-xx)}},$$

unde fit

$$y = \frac{x}{\sqrt{(1-xx)}}\text{Ang.sin.}x - \int \frac{xdx}{1-xx}$$

seu

$$y = \int \frac{dx}{(1-xx)^{\frac{3}{2}}}\text{Ang.sin.}x = \frac{x}{\sqrt{(1-xx)}}\text{Ang.sin.}x + l\sqrt{(1-xx)},$$

quod integrale evanescit posito $x = 0$.

SCHOLION

238. Simili modo integratur formula $dy = Xdx\text{Ang.cos.}x.$ Cum enim sit

$$d.\text{Ang.cos.}x = \frac{-dx}{\sqrt{(1-xx)}},$$

si ponamus $\int Xdx = P$, erit

$$y = P\text{Ang.cos.}x + \int \frac{Pdx}{\sqrt{(1-xx)}}.$$

Quin etiam si proponatur formula $dy = Xdx\text{Ang.tang.}x$, quia est

$$d.\text{Ang.tang.}x = \frac{dx}{1+xx}$$

posito $\int Xdx = P$ erit hoc integrale

$$y = \int Xdx\text{Ang.tang.}x = P\text{Ang.tang.}x - \int \frac{Pdx}{1+xx}.$$

Quoties ergo $\int Xdx$ algebraice dari potest, toties integratio reducitur ad formulam algebraicam sicque negotium confectum est habendum. Cum igitur in his formulis angulus, cuius sinus, cosinus vel tangens erat = x, inesset, consideremus etiam eiusmodi formulas, in quas quadratum huius anguli altiorve potestas ingreditur.

PROBLEMA 24

239. *Denotet φ angulum, cuius sinus tangensve est functio quaedam ipsius x, unde fiat $d\varphi = udx$, propositaque sit haec formula $dy = Xdx \cdot \varphi^n$, quam integrare oporteat.*

SOLUTIO

Sit $\int Xdx = P$, ut habeamus $dy = \varphi^n dP$, eritqua integrando

$$y = \varphi^n P - n\int \varphi^{n-1} Pudx.$$

Iam simili modo sit $\int Pudx = Q$; erit

$$\int \varphi^{n-1} Pudx = \varphi^{n-1} Q - (n-1)\int \varphi^{n-2} Qudx;$$

tum posito $\int Qudx = R$ erit

$$\int \varphi^{n-2} Qudx = \varphi^{n-2} R - (n-2)\int \varphi^{n-3} Rudx.$$

Hocque modo potestas anguli φ continuo deprimitur, donec tandem ad formulam ab angulo φ liberam perveniatur; id quod semper eveniet, dummodo n sit numerus integer positivus et haec integralia continuo sumere liceat

$$\int Xdx = P,\ \int Pudx = Q,\ \int Qudx = R \quad \text{etc.},$$

quae integrationes si non succedant, frustra integratio suscipitur.

EXEMPLUM

240. *Sit φ angulus, cuius sinus = x, ut sit $d\varphi = \frac{dx}{\sqrt{(1-xx)}}$; integrare formulam $dy = \varphi^n dx$.*

Erit ergo

$$X = 1,\quad P = x,\ Q = \int \frac{Pdx}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)},\ R = \int \frac{Qdx}{\sqrt{(1-xx)}} = -x,$$
$$S = \int \frac{Rdx}{\sqrt{(1-xx)}} = \sqrt{(1-xx)},\ T = x \quad \text{etc.},$$

quibus valoribus inventis reperietur

$$y = \int \varphi^n dx = \varphi^n x + n\varphi^{n-1}\sqrt{(1-xx)} - n(n-1)\varphi^{n-2}x$$
$$-n(n-1)(n-2)\varphi^{n-3}\sqrt{(1-xx)} + \text{etc.}$$

Pro variis ergo valoribus exponentis n habebimus

$$\int \varphi dx = \varphi x + \sqrt{(1-xx)} - 1,$$
$$\int \varphi^2 dx = \varphi^2 x + 2\varphi\sqrt{(1-xx)} - 2\cdot 1x,$$
$$\int \varphi^3 dx = \varphi^3 x + 3\varphi^2\sqrt{(1-xx)} - 3\cdot 2\varphi x - 3\cdot 2\cdot 1\sqrt{(1-xx)} + 6$$
$$\text{etc.}$$

integralibus ita determinatis, ut evanescant posito $x = 0$.

SCHOLION

241. Si sit $Xdx = udx = d\varphi$, formulae $\varphi^n d\varphi$ integrale est $\frac{1}{n+1}\varphi^{n+1}$; similique modo si fuerit Φ functio quaecunque anguli φ, formulae $\Phi udx = \Phi d\varphi$ integratio nihil habet difficultatis. Multo latius patent formulae sinus cosinusve angulorum et tangentes implicantes, quarum integratio per universam Analysin amplissimum habet usum, cum praecipue Theoria Astronomiae ad huiusmodi formulas sit reducta. Prima autem fundamenta peti debent ex calculo differentiali;

unde cum sit

$$d.sin.n\varphi = nd\varphi cos.n\varphi,\ d.cos.n\varphi = -nd\varphi sin.n\varphi,\ d.\text{tang}.n\varphi = \frac{nd\varphi}{cos.^2 n\varphi},$$
$$d.\text{cot}.n\varphi = \frac{-nd\varphi}{sin.^2 n\varphi},\ d.\frac{1}{sin.n\varphi} = \frac{-nd\varphi cos.n\varphi}{sin.^2 n\varphi},\ d.\frac{1}{cos.n\varphi} = \frac{nd\varphi sin.n\varphi}{cos.^2 n\varphi},$$

nanciscimur has integrationes elementares

$$\int d\varphi cos.n\varphi = \tfrac{1}{n} sin.n\varphi,\ \int d\varphi sin.n\varphi = -\tfrac{1}{n} cos.n\varphi,$$
$$\int \frac{d\varphi}{cos.^2 n\varphi} = \frac{1}{n}\text{tang}.n\varphi,\ \int \frac{d\varphi}{sin.^2 n\varphi} = -\frac{1}{n}\text{cot}.n\varphi,$$
$$\int \frac{d\varphi cos.n\varphi}{sin.^2 n\varphi} = -\frac{1}{n\,sin.n\varphi},\ \int \frac{d\varphi sin.n\varphi}{cos.^2 n\varphi} = \frac{1}{n\,cos.n\varphi},$$

unde statim huiusmodi formularum differentialium

$$d\varphi\left(A + Bcos.\varphi + \text{C}\,cos.2\varphi + \text{D}\,cos.3\varphi + \text{E}\,cos.4\varphi + \text{etc.}\right)$$

[integratio] consequitur, cum integrale manifesto sit

$$A\varphi + B\,sin.\varphi + \tfrac{1}{2}\text{C}\,sin.2\varphi + \tfrac{1}{3}\text{D}\,sin.3\varphi + \tfrac{1}{4}\text{E}\,sin.4\varphi + \text{etc.}$$

Deinde etiam in subsidium vocari convenit, quae in elementis de angulorum compositione traduntur, scilicet

$$sin.\alpha \cdot sin.\beta = \tfrac{1}{2}cos.(\alpha-\beta) - \tfrac{1}{2}cos.(\alpha+\beta),$$
$$cos.\alpha \cdot cos.\beta = \tfrac{1}{2}cos.(\alpha-\beta) + \tfrac{1}{2}cos.(\alpha+\beta),$$
$$sin.\alpha \cdot cos\,\beta = \tfrac{1}{2}sin.(\alpha+\beta) + \tfrac{1}{2}sin.(\alpha-\beta)$$
$$= \tfrac{1}{2}sin.(\alpha+\beta) - \tfrac{1}{2}sin.(\beta-\alpha),$$

unde producta plurium sinuum et cosinuum in simplices sinus cosinusve resolvuntur.

PROBLEMA 25

242. *Formulae* $d\varphi \sin.^n \varphi$ *integrale investigare.*

SOLUTIO

Repraesentetur in hos factores resoluta $\sin.^{n-1}\varphi . d\varphi \sin.\varphi$, et quia

$$\int d\varphi \sin.\varphi = -\cos.\varphi,$$

erit

$$\int d\varphi \sin.^n \varphi = -\sin.^{n-1}\varphi \cos.\varphi + (n-1)\int d\varphi \sin.^{n-2}\varphi \cos.^2\varphi.$$

Hinc ob $\cos.^2\varphi = 1 - \sin.^2\varphi$ habebitur

$$\int d\varphi \sin.^n \varphi = -\sin.^{n-1}\varphi \cos.\varphi + (n-1)\int d\varphi \sin.^{n-2}\varphi - (n-1)\int d\varphi \sin.^n \varphi,$$

ubi cum postrema formula ipsi propositae sit similis, hinc colligitur ista reductio

$$\int d\varphi \sin.^n \varphi = -\tfrac{1}{n}\sin.^{n-1}\varphi \cos.\varphi + \tfrac{n-1}{n}\int d\varphi \sin.^{n-2}\varphi,$$

qua integratio ad hanc formulam simpliciorem $d\varphi \sin.^{n-2}\varphi$ revocatur. Cum igitur casus simplicissimi constent

$$\int d\varphi \sin.^0 \varphi = \varphi \quad \text{et} \quad \int d\varphi \sin.\varphi = -\cos.\varphi,$$

hinc via ad continuo maiores exponentes n paratur:

$$\int d\varphi \sin.^0 \varphi = \varphi,$$

$$\int d\varphi \sin.\varphi = -\cos.\varphi,$$

$$\int d\varphi \sin.^2 \varphi = -\tfrac{1}{2}\sin.\varphi \cos.\varphi + \tfrac{1}{2}\varphi,$$

$$\int d\varphi \sin.^3 \varphi = -\tfrac{1}{3}\sin.^2\varphi \cos.\varphi - \tfrac{2}{3}\varphi,$$

$$\int d\varphi \sin.^4 \varphi = -\tfrac{1}{4}\sin.^3\varphi \cos.\varphi - \tfrac{1\cdot 3}{2\cdot 4}\sin.\varphi \cos.\varphi + \tfrac{1\cdot 3}{2\cdot 4}\varphi,$$

$$\int d\varphi \sin.^5 \varphi = -\tfrac{1}{5}\sin.^4\varphi \cos.\varphi - \tfrac{1\cdot 4}{3\cdot 5}\sin.^2\varphi \cos.\varphi - \tfrac{2\cdot 4}{3\cdot 5}\cos\varphi,$$

$$\int d\varphi \sin.^6 \varphi = -\tfrac{1}{6}\sin.^5\varphi \cos.\varphi - \tfrac{1\cdot 5}{4\cdot 6}\sin.^3\varphi \cos.\varphi - \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\sin.\varphi \cos.\varphi + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\varphi,$$

etc.

COROLLARIUM 1

243. Quoties n est numerus impar, integrale per solum sinum et cosinum exhibetur, at si n est numerus par, integrale insuper ipsum angulum involvit ideoque est functio transcendens.

COROLLARIUM 2

244. Casibus ergo, quibus n est numerus impar, id imprimis notari convenit, etiamsi angulus seu arcus φ in infinitum crescat, integrale tamen nunquam ultra certum limitem excrescere posse, cum tamen, si n sit numerus par, etiam in infinitum excrescat.

SCHOLION

245. Simili modo formula $d\varphi cos.^n\varphi$ tractatur, quae in hos factores resoluta $cos^{n-1}.\varphi.d\varphi cos.\varphi$ praebet

$$\int d\varphi cos.^n\varphi = cos.^{n-1}\varphi sin.\varphi + (n-1)\int d\varphi cos.^{n-2}\varphi sin.^2\varphi.$$

$$= cos.^{n-1}\varphi sin.\varphi + (n-1)\int d\varphi cos.^{n-2}\varphi - (n-1)\int d\varphi cos.^n\varphi,$$

unde, cum postrema formula propositae sit similis, colligitur

$$\int d\varphi cos.^n\varphi = \tfrac{1}{n} sin.\varphi cos.^{n-1}\varphi + \tfrac{n-1}{n}\int d\varphi cos.^{n-2}\varphi.$$

Quare cum casibus $n = 0$ et $n = 1$ integratio sit in promtu, ad altiores potestates patet progressio:

$$\int d\varphi cos.^0\varphi = \varphi,$$

$$\int d\varphi cos.\varphi = sin.\varphi,$$

$$\int d\varphi cos.^2\varphi = \tfrac{1}{2} sin.\varphi cos.\varphi + \tfrac{1}{2}\varphi,$$

$$\int d\varphi cos.^3\varphi = \tfrac{1}{3} sin.\varphi cos.^2\varphi + \tfrac{2}{3} sin\varphi,$$

$$\int d\varphi cos.^4\varphi = \tfrac{1}{4} sin.\varphi cos.^3\varphi + \tfrac{1\cdot3}{2\cdot4} sin.\varphi cos.\varphi + \tfrac{1\cdot3}{2\cdot4}\varphi,$$

$$\int d\varphi cos.^5\varphi = \tfrac{1}{5} sin.\varphi cos.^4\varphi + \tfrac{1\cdot4}{3\cdot5} sin.\varphi cos.^2\varphi + \tfrac{2\cdot4}{3\cdot5} cos\varphi,$$

$$\int d\varphi cos.^6\varphi = \tfrac{1}{6} sin.\varphi cos.^5\varphi + \tfrac{1\cdot5}{4\cdot6} sin.\varphi cos.^3\varphi + \tfrac{1\cdot3\cdot5}{2\cdot4\cdot6} sin.\varphi cos.\varphi + \tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\varphi,$$

etc.

PROBLEMA 26

246. *Formulae* $d\varphi \sin.^{m}\varphi \cos.^{n}\varphi$ *integrale invenire.*

SOLUTIO

Quo hoc facilius praestetur, consideremus factum $\sin.^{\mu}\varphi \cos.^{\nu}\varphi$ quod differentiatum fit $\mu d\varphi \sin.^{\mu-1}\varphi \cos.^{\nu+1}\varphi - \nu d\varphi \sin.^{\mu+1}\varphi \cos.^{\nu-1}\varphi$. Iam prout vel in parte priori $\cos.^{2}\varphi = 1-\sin.^{2}\varphi$ vel in posteriori $\sin.^{2}\varphi = 1-\cos.^{2}\varphi$ statuitur, oritur vel

$$d.\sin.^{\mu}\varphi \cos.^{\nu}\varphi = +\mu d\varphi \sin.^{\mu-1}\varphi \cos.^{\nu-1}\varphi - (\mu+\nu) d\varphi \sin.^{\mu+1}\varphi.\cos.^{\nu-1}\varphi$$

vel

$$d.\sin.^{\mu}\varphi \cos.^{\nu}\varphi = -\nu d\varphi \sin.^{\mu-1}\varphi \cos.^{\nu-1}\varphi + (\mu+\nu) d\varphi \sin.^{\mu-1}\varphi.\cos.^{\nu+1}\varphi.$$

Hinc igitur duplicem reductionem adipiscimur

$$\text{I. } \int d\varphi \sin.^{\mu+1}\varphi \cos.^{\nu-1}\varphi = -\tfrac{1}{\mu+\nu}\sin.^{\mu}\varphi \cos.^{\nu}\varphi + \tfrac{\mu}{\mu+\nu}\int d\varphi \sin.^{\mu-1}\varphi.\cos.^{\nu-1}\varphi$$

$$\text{II. } \int d\varphi \sin.^{\mu-1}\varphi \cos.^{\nu+1}\varphi = \tfrac{1}{\mu+\nu}\sin.^{\mu}\varphi \cos.^{\nu}\varphi + \tfrac{\nu}{\mu+\nu}\int d\varphi \sin.^{\mu-1}\varphi.\cos.^{\nu-1}\varphi.$$

Quare formula proposita $\int d\varphi \sin.^{m}\varphi \cos.^{n}\varphi$ successive continuo ad simpliciores potestates tam ipsius $\sin.\varphi$ quam ipsius $\cos.\varphi$ reducitur, donec alter vel penitus abeat vel simpliciter adsit, quo casu integratio per se patet, cum sit

$$\int d\varphi \sin.^{m}\varphi \cos.\varphi = \tfrac{1}{m+1}\sin.^{m+1}\varphi \quad \text{et} \quad \int d\varphi \sin.\varphi \cos.^{n}\varphi = -\tfrac{1}{n+1}\cos.^{n+1}\varphi .$$

EXEMPLUM

247. *Formulae* $d\varphi \sin.^{8}\varphi \cos.^{7}\varphi$ *integrale invenire.*

Per priorem reductiollem ob $\mu = 7$ et $\nu = 8$ impetramus

$$\int d\varphi \sin.^{8}\varphi \cos.^{7}\varphi = -\tfrac{1}{15}\sin.^{7}\varphi \cos.^{8}\varphi + \tfrac{7}{15}\int d\varphi \sin.^{6}\varphi.\cos.^{7}\varphi;$$

istam per posteriorem reductionem tractemus

$$\int d\varphi \sin.^{6}\varphi \cos.^{7}\varphi = \tfrac{1}{13}\sin.^{7}\varphi \cos.^{6}\varphi + \tfrac{6}{13}\int d\varphi \sin.^{6}\varphi.\cos.^{5}\varphi;$$

hoc modo ulterius progrediamur

$$\int d\varphi \sin.^6\varphi \cos.^5\varphi = -\tfrac{1}{11}\sin.^5\varphi\cos.^6\varphi + \tfrac{5}{11}\int d\varphi\sin.^4\varphi\cos.^5\varphi,$$
$$\int d\varphi \sin.^4\varphi \cos.^5\varphi = \ \tfrac{1}{9}\sin.^5\varphi\cos.^4\varphi + \tfrac{4}{9}\int d\varphi\sin.^4\varphi\cos.^3\varphi,$$
$$\int d\varphi \sin.^4\varphi \cos.^3\varphi = -\tfrac{1}{7}\sin.^3\varphi\cos.^4\varphi + \tfrac{3}{7}\int d\varphi\sin.^2\varphi\cos.^3\varphi,$$
$$\int d\varphi \sin.^2\varphi \cos.^3\varphi = \tfrac{1}{5}\sin.^3\varphi\cos.^2\varphi + \tfrac{2}{5}\int d\varphi\sin.^2\varphi\cos.\varphi,$$
$$\int d\varphi \sin.^2\varphi \cos.\varphi = -\tfrac{1}{3}\sin.\varphi\cos.^2\varphi + \tfrac{1}{3}\int d\varphi\cos.\varphi + \left(\tfrac{1}{3}\sin\varphi\right).$$

Ex his colligitur formulae propositae integrale

$$\begin{aligned}\int d\varphi \sin.^8\varphi\cos.^7\varphi = &-\tfrac{1}{15}\sin.^7\varphi\cos.^8\varphi + \tfrac{1\cdot 7}{15\cdot 13}\sin.^7\varphi\cos.^6\varphi - \tfrac{1\cdot 7\cdot 6}{15\cdot 13\cdot 11}\sin.^5\varphi\cos.^6\varphi\\ &+\tfrac{1\cdot 7\cdot 6\cdot 5}{15\cdot 13\cdot 11\cdot 9}\sin.^5\varphi.\cos.^4\varphi - \tfrac{1\cdot 7\cdot 6\cdot 5\cdot 4}{15\cdot 13\cdot 11\cdot 9\cdot 7}\sin.^3\varphi.\cos.^4\varphi\\ &+\tfrac{1\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3}{15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5}\sin.^3\varphi.\cos.^2\varphi - \tfrac{1\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3}\sin.\varphi.\cos.^2\varphi\\ &+\tfrac{1\cdot 7\cdot 6\cdot 5\cdot 4\cdot 3\cdot 2}{15\cdot 13\cdot 11\cdot 9\cdot 7\cdot 5\cdot 3}\sin.\varphi.\end{aligned}$$

SCHOLION

248. Quando autem huiusmodi casus occurrunt, semper praestat productum $\sin.^m\varphi\cos.^n\varphi$ in sinus vel cosinus angulorum multiplorum resolvere, quo facto singulae partes facillime integrantur. Caeterum hic brevitatis gratia angulum simpliciter littera φ indicavi nihiloque res foret generalior, si per $\alpha\varphi+\beta$ exprimeretur, quemadmodum etiam ante haec expressio Ang.*sin*.*x* aeque late patet, ac si loco *x* functio quaecunque scriberetur. Contemplemur ergo eiusmodi formulas, in quibus sinus cosinusve denominatorem occupant, ubi quidem simplicissimae sunt

$$\text{I.}\frac{d\varphi}{\sin.\varphi},\quad \text{II}\frac{d\varphi}{\cos.\varphi},\quad \text{III}\frac{d\varphi\cos.\varphi}{\sin.\varphi},\quad \text{IV.}\ \frac{d\varphi\sin.\varphi}{\cos.\varphi},$$

quarum integralia imprimis nosse oportet. Pro prima adhibeantur hae transformationes

$$\frac{d\varphi}{\sin.\varphi} = \frac{d\varphi\sin\varphi}{\sin^2.\varphi} = \frac{d\varphi\sin.\varphi}{1-\cos.^2\varphi} = \frac{-dx}{1-xx}\quad (\text{posito } \cos.\varphi = x),$$

unde fit

$$\int\frac{d\varphi}{\sin.\varphi} = -\frac{1}{2}l\frac{1+x}{1-x} = -\frac{1}{2}l\frac{1+\cos.\varphi}{1-\cos.\varphi}.$$

Pro secunda

$$\frac{d\varphi}{cos.\varphi} = \frac{d\varphi cos.\varphi}{cos.^2\varphi} = \frac{d\varphi cos.\varphi}{1-sin.^2\varphi} = \frac{dx}{1-xx} \quad (\text{posito } sin.\varphi = x),$$

ergo

$$\int \frac{d\varphi}{cos.\varphi} = \frac{1}{2} l \frac{1+x}{1-x} = \frac{1}{2} l \frac{1+sin.\varphi}{1-sin.\varphi}.$$

Tertiae et quartae integratio manifesto logarithmis conficitur; quare haec integralia probe notasse iuvabit

$$\text{I.} \int \frac{d\varphi}{sin.\varphi} = -\frac{1}{2} l \frac{1+cos.\varphi}{1-cos.\varphi} = l \frac{\sqrt{(1-cos.\varphi)}}{\sqrt{1+cos.\varphi}} = l\, \text{tang.} \tfrac{1}{2}\varphi,$$

$$\text{II.} \int \frac{d\varphi}{cos.\varphi} = \frac{1}{2} l \frac{1+sin.\varphi}{1-sin.\varphi} = l \frac{\sqrt{(1+sin.\varphi)}}{\sqrt{1-sin.\varphi}} = l\, \text{tang.}\left(45^0 + \tfrac{1}{2}\varphi\right),$$

$$\text{III.} \int \frac{d\varphi cos.\varphi}{sin.\varphi} = l\, sin.\varphi = \int \frac{d\varphi}{\text{tang.}\varphi} = \int d\varphi \,\text{cot.}\varphi,$$

$$\text{IV.} \int \frac{d\varphi sin.\varphi}{cos.\varphi} = -l\, cos.\varphi = \int d\varphi \,\text{tang.}\varphi \;;$$

hincque sequitur III +IV

$$\int \frac{d\varphi}{sin.\varphi cos.\varphi} = l \frac{sin.\varphi}{cos.\varphi} = l\, \text{tang.}\varphi.$$

PROBLEMA 27

249. *Formularum* $\frac{d\varphi sin.^m\varphi}{cos.^n\varphi}$ *et* $\frac{d\varphi cos.^m\varphi}{sin.^n\varphi}$ *integralia investigare.*

SOLUTIO

Primo statim perspicitur alteram formulam in alteram transmutari posito $\varphi = 90^0 - \psi$, quia tum fit $sin.\varphi = cos.\psi$ et $cos.\varphi = sin.\psi$, dummodo notetur fore $d\varphi = -d\psi$. Quare sufficit priorem tantum tractasse. Reductio autem prior § 246 data sumto $\mu + 1 = m$ et $\nu - 1 = -n$ praebet

$$\int \frac{d\varphi sin.^m\varphi}{cos.^n\varphi} = -\frac{1}{m-n} \cdot \frac{sin.^{m-1}\varphi}{cos.^{n-1}\varphi} + \frac{m-1}{m-n} \int \frac{d\varphi sin.^{m-2}\varphi}{cos.^n\varphi},$$

quo pacto in numeratore exponens ipsius $sin.\varphi$ continuo binario deprimitur, ita ut tandem perveniatur vel ad $\int \frac{d\varphi}{cos.^n\varphi}$ vel ad $\int \frac{d\varphi sin.\varphi}{cos.^n\varphi} = \frac{1}{(n-1)} \cdot \frac{1}{cos.^{n-1}\varphi}$ ideoque sola formula $\int \frac{d\varphi}{cos.^n\varphi}$ tractanda supersit. Altera autem reductio ibidem tradita (§ 246) sumto $\mu - 1 = m$ et $\nu - 1 = -n$ dat

$$\int \frac{d\varphi \sin.^m \varphi}{\cos.^{n-2} \varphi} = \frac{1}{m-n+2} \cdot \frac{\sin.^{m+1} \varphi}{\cos.^{n-1} \varphi} - \frac{n-1}{m-n+2} \int \frac{d\varphi \sin.^m \varphi}{\cos.^n \varphi},$$

unde colligitur

$$\int \frac{d\varphi \sin.^m \varphi}{\cos.^n \varphi} = \frac{1}{n-1} \cdot \frac{\sin.^{m+1} \varphi}{\cos.^{n-1} \varphi} - \frac{m-n+2}{n-1} \int \frac{d\varphi \sin.^m \varphi}{\cos.^{n-2} \varphi},$$

cuius reductionis ope exponens ipsius $\cos.\varphi$ in denominatore continuo binario deprimitur, ita ut tandem vel ad $\int d\varphi \sin.^m \varphi$ vel ad $\int \frac{d\varphi \sin^m.\varphi}{\cos.\varphi}$ perveniatur. Illius integratio iam supra est monstrata, huius vero forma, si $m > 1$, per priorem reductionem tandem vel ad $\int \frac{d\varphi}{\cos.\varphi}$ vel ad $\int \frac{d\varphi \sin.\varphi}{\cos.\varphi}$ revocatur; illius autem integrale est $l \operatorname{tang.}\left(45^0 + \frac{1}{2}\varphi\right)$, huius vero $-l \cos.\varphi$.

COROLLARIUM 1

250. Prior reductio non habet locum, quoties est $m = n$, hoc scilicet casu $\int \frac{d\varphi \sin^n.\varphi}{\cos^n.\varphi}$ non reduci potest ad formulam $\int \frac{d\varphi \sin^{n-2}.\varphi}{\cos^n.\varphi}$. Altera autem reductione semper uti licet; etsi enim casus $n = 1$ inde excluditur, eius tamen integratio per priorem effici potest.

COROLLARIUM 2

251. Ratio autem illius exclusionis in hoc est posita, quod formula $\int \frac{d\varphi \sin^{n-2}.\varphi}{\cos^n.\varphi}$ est absolute integrabilis habens integrale $= \frac{1}{n-1} \cdot \frac{\sin^{n-1}.\varphi}{\cos^{n-1}.\varphi}$ Erit ergo pro his casibus

$$\int \frac{d\varphi}{\cos^2.\varphi} = \frac{\sin.\varphi}{\cos.\varphi} = \operatorname{tang.}\varphi, \qquad \int \frac{d\varphi \sin.\varphi}{\cos^3.\varphi} = \frac{1}{2} \cdot \frac{d\varphi \sin^2.\varphi}{\cos^2.\varphi} = \frac{1}{2} \operatorname{tang}^2.\varphi,$$

$$\int \frac{d\varphi \sin^2.\varphi}{\cos^4.\varphi} = \frac{1}{3} \cdot \frac{d\varphi \sin^3.\varphi}{\cos^3.\varphi} = \frac{1}{3} \operatorname{tang}^3.\varphi, \qquad \int \frac{d\varphi \sin^3.\varphi}{\cos^5.\varphi} = \frac{1}{4} \cdot \frac{d\varphi \sin^4.\varphi}{\cos^4.\varphi} = \frac{1}{4} \operatorname{tang}^4.\varphi.$$

EXEMPLUM 1

252. *Formulae* $\frac{d\varphi \sin.^m \varphi}{\cos.\varphi}$ *integrale assignare.*

Prior reductio dat

$$\int \frac{d\varphi \sin.^m \varphi}{\cos.\varphi} = \frac{-1}{m-1} \sin.^{m-1} \varphi + \int \frac{d\varphi \sin.^{m-2} \varphi}{\cos.\varphi}.$$

Hinc a casibus per se notis incipiendo habebimus

$$\int\frac{d\varphi}{\cos.\varphi}=l\,\text{tang.}\left(45^0+\tfrac{1}{2}\varphi\right),$$
$$\int\frac{d\varphi\sin.\varphi}{\cos.\varphi}=-l\cos.\varphi=l\sec.\varphi,$$
$$\int\frac{d\varphi\sin.^2\varphi}{\cos.\varphi}=-\sin.\varphi+\int\frac{d\varphi}{\cos.\varphi},$$
$$\int\frac{d\varphi\sin.^3\varphi}{\cos.\varphi}=-\tfrac{1}{2}\sin.^2\varphi+l\sec.\varphi,$$
$$\int\frac{d\varphi\sin.^4\varphi}{\cos.\varphi}=-\tfrac{1}{3}\sin.^3\varphi-\sin.\varphi+\int\frac{d\varphi}{\cos.\varphi},$$
$$\int\frac{d\varphi\sin.^5\varphi}{\cos.\varphi}=-\tfrac{1}{4}\sin.^4\varphi-\tfrac{1}{2}\sin.^2\varphi+l\sec.\varphi$$
$$\int\frac{d\varphi\sin.^6\varphi}{\cos.\varphi}=-\tfrac{1}{5}\sin.^5\varphi-\tfrac{1}{3}\sin.^3\varphi-\sin.\varphi+\int\frac{d\varphi}{\cos.\varphi},$$
$$\int\frac{d\varphi\sin.^7\varphi}{\cos.\varphi}=-\tfrac{1}{6}\sin.^6\varphi-\tfrac{1}{4}\sin.^4\varphi-\tfrac{1}{2}\sin.^2\varphi+l\sec.\varphi$$

etc.

SCHOLION

253. Pro reliquis casibus denominatoris totum negotium conficietur his reductionibus

$$\int\frac{d\varphi\sin.^m\varphi}{\cos.^2\varphi}=\frac{\sin.^{m+1}\varphi}{\cos.\varphi}-m\int d\varphi\sin.^m\varphi,$$
$$\int\frac{d\varphi\sin.^m\varphi}{\cos.^3\varphi}=\frac{1}{2}\cdot\frac{\sin.^{m+1}\varphi}{\cos.^2\varphi}-\frac{m-1}{2}\int\frac{d\varphi\sin.^m\varphi}{\cos.\varphi},$$
$$\int\frac{d\varphi\sin.^m\varphi}{\cos.^4\varphi}=\frac{1}{3}\cdot\frac{\sin.^{m+1}\varphi}{\cos.^3\varphi}-\frac{m-2}{3}\int\frac{d\varphi\sin.^m\varphi}{\cos.^2\varphi},$$
$$\int\frac{d\varphi\sin.^m\varphi}{\cos.^5\varphi}=\frac{1}{4}\cdot\frac{\sin.^{m+1}\varphi}{\cos.^4\varphi}-\frac{m-3}{4}\int\frac{d\varphi\sin.^m\varphi}{\cos.^3\varphi}$$

etc.

EXEMPLUM 2

254. *Formulae* $\frac{d\varphi}{\cos.^n\varphi}$ *integrale assignare.*

Altera reductio ob $m = 0$ fit

$$\int\frac{d\varphi}{\cos.^n\varphi}=\frac{1}{n-1}\cdot\frac{\sin.\varphi}{\cos.^{n-1}\varphi}+\frac{n-2}{n-1}\int\frac{d\varphi}{\cos.^{n-2}\varphi},$$

quia iam casus simplicissimi

$$\int d\varphi=\varphi\quad\text{et}\quad\int\frac{d\varphi}{\cos.\varphi}=l\,\text{tang.}\left(45^0+\tfrac{1}{2}\varphi\right)$$

sunt cogniti, ad eos sequentes omnes revocabuntur

$$\int\frac{d\varphi}{cos.^2\varphi}=\frac{sin.\varphi}{cos.\varphi},$$

$$\int\frac{d\varphi}{cos.^3\varphi}=\frac{1}{2}\cdot\frac{sin.\varphi}{cos.^2\varphi}+\frac{1}{2}\int\frac{d\varphi}{cos.\varphi},$$

$$\int\frac{d\varphi}{cos.^4\varphi}=\frac{1}{3}\cdot\frac{sin.\varphi}{cos.^3\varphi}+\frac{2}{3}\cdot\frac{sin.\varphi}{cos.\varphi},$$

$$\int\frac{d\varphi}{cos.^5\varphi}=\frac{1}{4}\cdot\frac{sin.\varphi}{cos.^4\varphi}+\frac{1\cdot3}{2\cdot4}\frac{sin.\varphi}{cos.^2\varphi}+\frac{1\cdot3}{2\cdot4}\int\frac{d\varphi}{cos.\varphi},$$

$$\int\frac{d\varphi}{cos.^6\varphi}=\frac{1}{5}\cdot\frac{sin.\varphi}{cos.^5\varphi}+\frac{1\cdot4}{3\cdot5}\frac{sin.\varphi}{cos.^3\varphi}+\frac{2\cdot4}{3\cdot5}\frac{sin.\varphi}{cos.\varphi}$$

etc.

COROLLARIUM 1

255. Simili modo habebimus has integrationes

$$\int\frac{d\varphi}{sin.\varphi}=l\,\mathrm{tang.}\frac{1}{2}\varphi,\quad \int\frac{d\varphi}{sin.^2\varphi}=-\frac{cos.\varphi}{sin.\varphi},$$

$$\int\frac{d\varphi}{sin.^3\varphi}=-\frac{1}{2}\cdot\frac{cos.\varphi}{sin.^2\varphi}+\frac{1}{2}\int\frac{d\varphi}{sin.\varphi},$$

$$\int\frac{d\varphi}{sin.^4\varphi}=-\frac{1}{3}\cdot\frac{cos.\varphi}{sin.^3\varphi}-\frac{2}{3}\cdot\frac{cos.\varphi}{sin.\varphi},$$

$$\int\frac{d\varphi}{sin.^5\varphi}=-\frac{1}{4}\cdot\frac{cos.\varphi}{sin.^4\varphi}-\frac{1\cdot3}{2\cdot4}\frac{cos.\varphi}{sin.^2\varphi}+\frac{1\cdot3}{2\cdot4}\int\frac{d\varphi}{sin.\varphi}$$

etc.

COROLLARIUM 2

256. Deinde est

$$\int\frac{d\varphi\,sin.\varphi}{cos.^n\varphi}=\frac{1}{n-1}\cdot\frac{1}{cos.^{n-1}\varphi}\quad\text{et}\quad\int\frac{d\varphi\,cos.\varphi}{sin.^n\varphi}=\frac{-1}{n-1}\cdot\frac{1}{sin.^{n-1}\varphi}.$$

Porro

$$\int\frac{d\varphi\,sin.^2\varphi}{cos.^n\varphi}=\int\frac{d\varphi}{cos.^n\varphi}-\int\frac{d\varphi}{cos.^{n-2}\varphi},\quad \int\frac{d\varphi\,cos.^2\varphi}{sin.^n\varphi}=\int\frac{d\varphi}{sin.^n\varphi}-\int\frac{d\varphi}{sin.^{n-2}\varphi}$$

et

$$\int\frac{d\varphi\,sin.^3\varphi}{cos.^n\varphi}=\int\frac{d\varphi\,sin.\varphi}{cos.^n\varphi}-\int\frac{d\varphi\,sin.\varphi}{cos.^{n-2}\varphi},\quad \int\frac{d\varphi\,cos.^3\varphi}{sin.^n\varphi}=\int\frac{d\varphi\,cos.\varphi}{sin.^n\varphi}-\int\frac{d\varphi\,cos.\varphi}{sin.^{n-2}\varphi},$$

quibus reductionibus continuo ulterius progredi licet.

PROBLEMA 28

257. *Formulae* $\frac{d\varphi}{\sin.^m \varphi \cos.^n \varphi}$ *integrale investigare.*

SOLUTIO

Reductiones supra adhibitas hue accommodare licet sumendo in praecedente problemate *m* negative; ita erit

$$\int \frac{d\varphi}{\sin.^m \varphi \cos.^n \varphi} = +\frac{1}{m+n} \cdot \frac{1}{\sin.^{m+1} \varphi \cos.^{n-1} \varphi} + \frac{m+1}{m+n} \int \frac{d\varphi}{\sin.^{m+2} \varphi \cos.^n \varphi},$$

unde loco *m* scribendo $m-2$ per conversionem fit

$$\int \frac{d\varphi}{\sin.^m \varphi \cos.^n \varphi} = -\frac{1}{m-1} \cdot \frac{1}{\sin.^{m-1} \varphi \cos.^{n-1} \varphi} + \frac{m+n-2}{m-1} \int \frac{d\varphi}{\sin.^{m-2} \varphi \cos.^n \varphi};$$

altera huic similis est

$$\int \frac{d\varphi}{\sin.^m \varphi \cos.^n \varphi} = \frac{1}{n-1} \cdot \frac{1}{\sin.^{m-1} \varphi \cos.^{n-1} \varphi} + \frac{m+n-2}{n-1} \int \frac{d\varphi}{\sin.^m \varphi \cos.^{n-2} \varphi}.$$

Cum iam in hoc genere formae simplicissimae sint

$$\int \frac{d\varphi}{\sin.\varphi} = l\,\text{tang.}\tfrac{1}{2}\varphi, \quad \int \frac{d\varphi}{\cos.\varphi} = l\,\text{tang.}\left(45^0 + \tfrac{1}{2}\varphi\right), \quad \int \frac{d\varphi}{\sin.\varphi \cos.\varphi} = l\,\text{tang.}\varphi,$$

$$\int \frac{d\varphi}{\sin.^2 \varphi} = -\cot.\varphi, \quad \int \frac{d\varphi}{\cos.^2 \varphi} = \text{tang.}\varphi,$$

hinc magis compositas eliciemus

$$\int\frac{d\varphi}{sin.\varphi cos.^2\varphi}=\frac{1}{cos.\varphi}+\int\frac{d\varphi}{sin.\varphi},$$

$$\int\frac{d\varphi}{sin.^2\varphi cos.\varphi}=-\frac{1}{sin.\varphi}+\int\frac{d\varphi}{cos.\varphi},$$

$$\int\frac{d\varphi}{sin.\varphi cos.^4\varphi}=\frac{1}{3}\frac{1}{cos.^3\varphi}+\int\frac{d\varphi}{sin.\varphi cos.^2\varphi},$$

$$\int\frac{d\varphi}{sin.^4\varphi cos.\varphi}=-\frac{1}{3}\frac{1}{sin.^3\varphi}+\int\frac{d\varphi}{sin.^2\varphi cos.\varphi},$$

$$\int\frac{d\varphi}{sin.\varphi cos.^6\varphi}=\frac{1}{5}\frac{1}{cos.^5\varphi}+\int\frac{d\varphi}{sin.\varphi cos.^4\varphi},$$

$$\int\frac{d\varphi}{sin.^6\varphi cos.\varphi}=-\frac{1}{5}\frac{1}{sin.^5\varphi}+\int\frac{d\varphi}{sin.^4\varphi cos.\varphi},$$

$$\int\frac{d\varphi}{sin.\varphi cos.^3\varphi}=\frac{1}{2}\frac{1}{cos.^2\varphi}+\int\frac{d\varphi}{sin.\varphi cos.\varphi},$$

$$\int\frac{d\varphi}{sin.^3\varphi cos.\varphi}=-\frac{1}{2}\frac{1}{sin.^2\varphi}+\int\frac{d\varphi}{sin.\varphi cos.\varphi},$$

$$\int\frac{d\varphi}{sin.\varphi cos.^5\varphi}=\frac{1}{cos.^4\varphi}+\int\frac{d\varphi}{sin.\varphi cos.^3\varphi},$$

$$\int\frac{d\varphi}{sin.^5\varphi cos.\varphi}=-\frac{1}{4}\frac{1}{sin.^4\varphi}+\int\frac{d\varphi}{sin.^3\varphi cos.\varphi},$$

$$\int\frac{d\varphi}{sin.\varphi cos.^7\varphi}=\frac{1}{6}\frac{1}{cos.^6\varphi}+\int\frac{d\varphi}{sin.\varphi cos.^5\varphi},$$

$$\int\frac{d\varphi}{sin.^7\varphi cos.\varphi}=-\frac{1}{6}\frac{1}{sin.^6\varphi}+\int\frac{d\varphi}{sin.^5\varphi cos.\varphi},$$

$$\int\frac{d\varphi}{sin.^2\varphi cos.^2\varphi}=\frac{1}{sin.\varphi cos.\varphi}+2\int\frac{d\varphi}{sin.^2\varphi}=-\frac{1}{sin.\varphi cos.\varphi}+2\int\frac{d\varphi}{cos.^2\varphi},$$

$$\int\frac{d\varphi}{sin.^2\varphi cos.^4\varphi}=\frac{1}{3}\cdot\frac{1}{sin.\varphi cos.^3\varphi}+\frac{4}{3}\int\frac{d\varphi}{sin.^2\varphi cos.^2\varphi},$$

$$\int\frac{d\varphi}{sin.^4\varphi cos.^2\varphi}=-\frac{1}{3}\cdot\frac{1}{sin.^3\varphi cos.\varphi}+\frac{4}{3}\int\frac{d\varphi}{sin.^2\varphi cos.^2\varphi}.$$

etc.

Sicque formulae quantumvis compositae ad simpliciores, quarum integratio est in promtu, reducuntur.

COROLLARIUM 1

258. Ambo exponentes ipsius $sin.\varphi$ et $cos.\varphi$ simul binario minui possunt; erit enim per priorem reductionem

$$\int \frac{d\varphi}{sin.^{\mu}\varphi cos.^{\nu}\varphi} = -\frac{1}{\mu-1}\cdot\frac{1}{sin.^{\mu-1}\varphi cos.^{\nu-1}\varphi} + \frac{\mu+\nu-2}{\mu-1}\int \frac{d\varphi}{sin.^{\mu-2}\varphi cos.^{\nu}\varphi};$$

nunc haec formula per posteriorem ob $m=\eta-2$ et $n=\nu$ dat

$$\int \frac{d\varphi}{sin.^{\mu-2}\varphi cos.^{\nu}\varphi} = \frac{1}{\nu-1}\cdot\frac{1}{sin.^{\mu-3}\varphi cos.^{\nu-1}\varphi} + \frac{\mu+\nu-4}{\nu-1}\int \frac{d\varphi}{sin.^{\mu-2}\varphi cos.^{\nu-2}\varphi},$$

unde concluditur

$$\int \frac{d\varphi}{sin.^{\mu}\varphi cos.^{\nu}\varphi} = -\frac{1}{\mu-1}\cdot\frac{1}{sin.^{\mu-1}\varphi cos.^{\nu-1}\varphi} + \frac{\mu+\nu-2}{(\mu-1)(\nu-1)}\cdot\frac{1}{sin.^{\mu-3}\varphi cos.^{\nu-1}\varphi}$$
$$+\frac{(\mu+\nu-2)(\mu+\nu-4)}{(\mu-1)(\nu-1)}\int \frac{d\varphi}{sin.^{\mu-2}\varphi cos.^{\nu-2}\varphi}.$$

COROLLARIUM 2

259. Prioribus membris ad communem denominatorem reductis obtinebitur

$$\int \frac{d\varphi}{sin.^{\mu}\varphi cos.^{\nu}\varphi}$$
$$= \frac{(\mu-1)sin.^{2}\varphi-(\nu-1)cos.^{2}\varphi}{(\mu-1)(\nu-1)sin.^{\mu-1}\varphi cos.^{\nu-1}\varphi} + \frac{(\mu+\nu-2)(\mu+\nu-4)}{(\mu-1)(\nu-1)}\int \frac{d\varphi}{sin.^{\mu-2}\varphi cos.^{\nu-2}\varphi},$$

qua reductione semper ad calculum contrahendum uti licet, nisi vel $\mu=1$ vel $\nu=1$.

SCHOLION

260. Huiusmodi formulae $\frac{d\varphi}{sin.^{m}\varphi cos.^{n}\varphi}$ etiam hoc modo maxime obvio ad simpliciores reduci possunt, dum numerator per $sin.^{2}\varphi + cos.^{2}\varphi = 1$ multiplicatur, unde fit

$$\int \frac{d\varphi}{sin.^{m}\varphi cos.^{n}\varphi} = \int \frac{d\varphi}{sin.^{m-2}\varphi cos.^{n}\varphi} + \int \frac{d\varphi}{sin.^{m}\varphi cos.^{n-2}\varphi},$$

quae eousque continuari potest, donec in denominatore unica tantum potestas relinquatur. Ita erit

$$\int \frac{d\varphi}{sin.\varphi cos.\varphi} = \int \frac{d\varphi sin.\varphi}{cos.\varphi} + \int \frac{d\varphi cos.\varphi}{sin.\varphi} = l\frac{sin.\varphi}{cos.\varphi},$$
$$\int \frac{d\varphi}{sin.^{2}\varphi cos.^{2}\varphi} = \int \frac{d\varphi}{sin.^{2}\varphi} + \int \frac{d\varphi}{cos.^{2}\varphi} = \frac{sin.\varphi}{cos.\varphi} - \frac{cos.\varphi}{sin.\varphi}.$$

Quodsi proposita sit haec formula $\frac{d\varphi}{sin.^m\varphi cos.^n\varphi}$, in subsidium vocari potest esse

$$sin.\varphi cos.\varphi = \tfrac{1}{2} sin.2\varphi,$$

unde habetur

$$\int \frac{2^n d\varphi}{sin.^n 2\varphi} = 2^{n-1}\int \frac{d\omega}{sin.^n \omega}$$

posito $\omega = 2\varphi$, quae formula per superiora praecepta resolvitur.

His igitur adminiculis observatis circa formulam $d\varphi sin.^m\varphi cos.^n\varphi$, si quidem *m* et *n* fuerint numeri integri sive positivi sive negativi, nihil amplius desideratur; sin autem fuerint numeri fracti, nihil admodum praecipiendum occurrit, quandoquidem casus, quibus integratio succedit, quasi sponte se produnt.

Quemadmodum autem integralia, quae exhiberi nequeunt, per series exprimi conveniat, in capite sequente accuratius exponamus.

Nunc vero formulas fractas consideremus, quarum denominator est $a + b cos.\varphi$ eiusque potestas; tales enim formulae in Theoria Astronomiae frequentissime occurrunt.

PROBLEMA 29

261. *Formulae differentialis* $\frac{d\varphi}{a+b cos.\varphi}$ *integrale investigare.*

SOLUTIO

Haec investigatio commodius institui nequit, quam ut formula proposita ad formam ordinariam reducatur ponendo $cos.\varphi = \frac{1-xx}{1+xx}$ ut rationaliter fiat

$$sin.\varphi = \frac{2x}{1+xx} \quad \text{hincque} \quad d\varphi cos.\varphi = \frac{2dx(1-xx)}{(1+xx)^2}$$

sicque $d\varphi = \frac{2dx}{1+xx}$. Quia igitur

$$a + b cos.\varphi = \frac{a+b+(a-b)xx}{1+xx}$$

erit formula nostra

$$\frac{d\varphi}{a+b cos.\varphi} = \frac{2dx}{a+b+(a-b)xx}$$

quae, prout fuerit $a > b$ vel $a < b$, vel angulum vel logarithmum praebet.

Casu $a > b$ reperitur

$$\int \frac{d\varphi}{a+b cos.\varphi} = \frac{2}{\sqrt{(aa-bb)}} \text{Ang.tang.} \frac{(a-b)x}{\sqrt{(aa-bb)}},$$

casu $a < b$ vero est

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \frac{2}{\sqrt{(bb-aa)}} l \frac{\sqrt{(bb-aa)}+x(b-a)}{\sqrt{(bb-aa)}-x(b-a)}.$$

Nunc vero est

$$x = \sqrt{\frac{1-\cos.\varphi}{1+\cos.\varphi}} = \text{tang.}\tfrac{1}{2}\varphi = \frac{\sin.\varphi}{1+\cos.\varphi} \;;$$

qua restitutione facta cum sit

$$2\text{Ang.tang.}\frac{(a-b)x}{\sqrt{(aa-bb)}} = \text{Ang.tang.}\frac{2x\sqrt{(aa-bb)}}{a+b-(a-b)xx}$$

$$= \text{Ang.tang.}\frac{2\sin.\varphi\sqrt{(aa-bb)}}{(a+b)(1+\cos.\varphi)-(a-b)(1-\cos.\varphi)} = \text{Ang.tang.}\frac{\sin.\varphi\sqrt{(aa-bb)}}{a\cos.\varphi+b},$$

quocirca pro casu $a > b$ adipiscimur

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \frac{2}{\sqrt{(aa-bb)}} \text{Ang.tang.}\frac{\sin.\varphi\sqrt{(aa-bb)}}{a\cos.\varphi+b}$$

seu

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \frac{1}{\sqrt{(aa-bb)}} \text{Ang.sin.}\frac{\sin.\varphi\sqrt{(aa-bb)}}{a+b\cos.\varphi}$$

sive

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \frac{1}{\sqrt{(aa-bb)}} \text{Ang.cos.}\frac{a\cos.\varphi+b}{a+b\cos.\varphi}.$$

Pro casu autem $a < b$

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \frac{1}{\sqrt{(bb-aa)}} l \frac{\sqrt{(b+a)(1+\cos.\varphi)}+\sqrt{(b-a)(1-\cos.\varphi)}}{\sqrt{(b+a)(1+\cos.\varphi)}-\sqrt{(b-a)(1-\cos.\varphi)}}.$$

seu

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \frac{1}{\sqrt{(bb-aa)}} l \frac{a\cos.\varphi+b+\sin.\varphi\sqrt{(bb-aa)}}{a+b\cos.\varphi}.$$

At casu $b = a$ integrale est $= \frac{x}{a} = \frac{1}{a}\text{tang.}\frac{1}{2}\varphi,$ unde fit

$$\int \frac{d\varphi}{a+b\cos.\varphi} = \text{tang.}\tfrac{1}{2}\varphi = \frac{\sin.\varphi}{1+\cos.\varphi}$$

quae integralia evanescunt facto $\varphi = 0$.

COROLLARIUM 1

262. Formula autem $\frac{d\varphi \sin.\varphi}{a+b\cos.\varphi} = \frac{-d.\cos.\varphi}{a+b\cos.\varphi}$ integrale est $\frac{1}{b} l \frac{a+b}{a+b\cos.\varphi}$ ita sumtum, ut evanescat posito $\varphi = 0$; sicque habebimus

$$\int \frac{d\varphi \sin.\varphi}{a+b\cos.\varphi} = \frac{1}{b} l \frac{a+b}{a+b\cos.\varphi}.$$

COROLLARIUM 2

263. Formula autem $\frac{d\varphi \cos.\varphi}{a+b\cos.\varphi}$ transformatur in $\frac{d\varphi}{b} - \frac{ad\varphi}{b(a+b\cos.\varphi)}$ unde integrale per solutionem problematis exhiberi potest

$$\int \frac{d\varphi \cos.\varphi}{a+b\cos.\varphi} = \frac{\varphi}{b} - \frac{a}{b}\int \frac{d\varphi}{(a+b\cos.\varphi)}.$$

SCHOLION 1

264. Integratione hac inventa etiam huius formulae $\frac{d\varphi \cos.\varphi}{(a+b\cos.\varphi)^n}$ integrale inveniri potest existente *n* numero integro; quod fingendo integralis forma commodissime praestari videtur:

$$\int \frac{d\varphi \cos.\varphi}{(a+b\cos.\varphi)^2} = \frac{A\sin.\varphi}{a+b\cos.\varphi} + m\int \frac{d\varphi}{a+b\cos.\varphi}$$

ac reperitur

$$A = \frac{-b}{aa-bb} \quad \text{et} \quad m = \frac{a}{aa-bb} \;;$$

$$\int \frac{d\varphi}{(a+b\cos.\varphi)^3} = \frac{(A+B\cos.\varphi)\sin.\varphi}{(a+b\cos.\varphi)^2} + m\int \frac{d\varphi}{(a+b\cos.\varphi)^2}$$

reperiturque

$$A = \frac{-b}{aa-bb}, \quad B = \frac{-bb}{2a(aa-bb)}, \quad m = \frac{2aa+bb}{2a(aa-bb)}$$

similique modo investigatio ad maiores potestates continuari potest, labore quidem non parum taedioso.

Sequenti autem modo negotium facillime expediri videtur. Consideretur scilicet formula generalior $\frac{d\varphi \cos.\varphi(f+g\cos.\varphi)}{(a+b\cos.\varphi)^{n+1}}$ ac ponatur

$$\int \frac{d\varphi \cos.\varphi(f+g\cos.\varphi)}{(a+b\cos.\varphi)^{n+1}} = \frac{A\sin.\varphi}{(a+b\cos.\varphi)^n} + \int \frac{d\varphi(B+C\cos.\varphi)}{(a+b\cos.\varphi)^n}$$

sumtisque differentialibus ista prodibit aequatio

$$f+g\cos.\varphi = A\cos.\varphi(a+b\cos.\varphi)+nAb\sin.^2\varphi+(B+C\cos.\varphi)(a+b\cos.\varphi),$$

quae ob $\sin.^2\varphi = 1-\cos.^2\varphi$ hanc formam induit

$$\left.\begin{array}{llll} -f & -\ g\cos.\varphi + & Ab\cos.^2\varphi \\ +nAb & +Aa\cos.\varphi - & nAb\cos.^2\varphi \\ +Ba & +Bb\cos.\varphi + & Cb\cos.^2\varphi \\ & +Ca\cos.\varphi & \end{array}\right\} = 0,$$

unde singulis membris nihilo aequatis elicitur

$$A = \frac{ag-bf}{n(aa-bb)},\quad B = \frac{af-bg}{aa-bb} \quad \text{et} \quad C = \frac{(n-1)(ag-bf)}{n(aa-bb)},$$

ita ut haec obtineatur reductio

$$\int \frac{d\varphi\cos.\varphi(f+g\cos.\varphi)}{(a+b\cos.\varphi)^{n+1}}$$
$$= \frac{(ag-bf)\sin.\varphi}{n(aa-bb)(a+b\cos.\varphi)^n} + \frac{1}{n(aa-bb)}\int \frac{d\varphi(n(af-bg)+(n-1)(ag-bf)\cos.\varphi)}{(a+b\cos.\varphi)^n},$$

cuius ope tandem ad formulam $\int \frac{d\varphi(h+k\cos.\varphi)}{(a+b\cos.\varphi)}$ pervenitur, cuius integrale

$$\frac{k}{b}\varphi + \frac{bh-ak}{b}\int \frac{d\varphi}{a+b\cos.\varphi}$$

ex superioribus constat. Perspicuum autem est semper fore $k=0$.

SCHOLION 2

265. Occurrunt etiam eiusmodi formulae, in quas insuper quantitas exponentialis $e^{\alpha\varphi}$ angulum ipsum φ in exponente gerens ingreditur, quas quomodo tractari oporteat, ostendendum videtur, cum hinc methodus reductionum supra exposita maxime illustretur. Hic enim per illam reductionem ad formulam propositae similem pervenitur, unde ipsum integrale colligi poterit. In hunc finem notetur esse $\int e^{\alpha\varphi}d\varphi = \frac{1}{\alpha}e^{\alpha\varphi}$.

PROBLEMA 30

266. *Formulae differentialis* $dy = e^{\alpha\varphi} d\varphi \sin.^n \varphi$ *integrale investigare.*

SOLUTIO

Sumto $e^{\alpha\varphi} d\varphi$ pro factore differentiali erit

$$y = \frac{1}{\alpha} e^{\alpha\varphi} \sin.^n \varphi - \frac{n}{\alpha} \int e^{\alpha\varphi} d\varphi \sin.^{n-1} \varphi \cos.\varphi ;$$

simili modo reperitur

$$\int e^{\alpha\varphi} d\varphi \sin.^{n-1} \varphi \cos.\varphi$$
$$= \frac{1}{\alpha} e^{\alpha\varphi} \sin.^{n-1} \varphi \cos.\varphi - \frac{1}{\alpha} \int e^{\alpha\varphi} d\varphi \left((n-1) \sin.^{n-2} \varphi \cos.^2 \varphi - \sin.^n \varphi \right),$$

quae postrema formula ob $\cos.^2 \varphi = 1 - \sin.^2 \varphi$ reducitur ad has

$$(n-1) \int e^{\alpha\varphi} d\varphi \sin.^{n-2} \varphi - n \int e^{\alpha\varphi} d\varphi \sin.^n \varphi,$$

unde habebitur

$$\int e^{\alpha\varphi} d\varphi \sin.^n \varphi = \frac{1}{\alpha} e^{\alpha\varphi} \sin.^n \varphi - \frac{n}{\alpha\alpha} e^{\alpha\varphi} \sin.^{n-1} \varphi \cos.\varphi + \frac{n(n-1)}{\alpha\alpha} \int e^{\alpha\varphi} d\varphi \sin.^{n-2} \varphi$$
$$- \frac{nn}{\alpha\alpha} \int e^{\alpha\varphi} d\varphi \sin.^n \varphi.$$

Quare hanc postremam formulam cum prima coniungendo elicitur

$$\int e^{\alpha\varphi} d\varphi \sin.^n \varphi = \frac{e^{\alpha\varphi} \sin.^{n-1} \varphi (\alpha \sin.\varphi - n \cos.\varphi)}{\alpha\alpha + nn} + \frac{n(n-1)}{\alpha\alpha + nn} \int e^{\alpha\varphi} d\varphi \sin.^{n-2} \varphi.$$

Duobus ergo casibus integrale absolute datur, scilicet $n = 0$ et $n = 1$, eritque

$$\int e^{\alpha\varphi} d\varphi = \frac{1}{\alpha} e^{\alpha\varphi} - \frac{1}{\alpha} \quad \text{et} \quad \int e^{\alpha\varphi} d\varphi \sin.\varphi = \frac{e^{\alpha\varphi} (\alpha \sin.\varphi - \cos.\varphi)}{\alpha\alpha + 1} + \frac{1}{\alpha\alpha + 1}$$

atque ad hos sequentes omnes, ubi n est numerus integer unitate maior, reducuntur.

COROLLARIUM 1

267. Ita si $n = 2$, acquirimus hanc integrationem

$$\int e^{\alpha\varphi} d\varphi \sin.^2 \varphi = \frac{e^{\alpha\varphi} \sin.\varphi (\alpha \sin.\varphi - 2 \cos.\varphi)}{\alpha\alpha + 4} + \frac{1 \cdot 2}{\alpha(\alpha\alpha + 4)} e^{\alpha\varphi} - \frac{1 \cdot 2}{\alpha(\alpha\alpha + 4)};$$

at si sit $n = 3$, istam

$$\int e^{\alpha\varphi} d\varphi \sin.^3\varphi = \frac{e^{\alpha\varphi}\sin.^2\varphi(\alpha\sin.\varphi - 3\cos.\varphi)}{\alpha\alpha+9} + \frac{2\cdot 3e^{\alpha\varphi}(\alpha\sin.\varphi - \cos.\varphi)}{(\alpha\alpha+1)(\alpha\alpha+9)} + \frac{2\cdot 3}{(\alpha\alpha+1)(\alpha\alpha+9)}$$

integralibus ita sumtis, ut evanescant posito $\varphi = 0$.

COROLLARIUM 2

268. Si igitur determinatis hoc modo integralibus statuatur $\alpha\varphi = -\infty$, ut $e^{\alpha\varphi}$ evanescat, erit in genere

$$\int e^{\alpha\varphi} d\varphi \sin.^n\varphi = \frac{n(n-1)}{\alpha\alpha+nn}\int e^{\alpha\varphi} d\varphi \sin.^{n-2}\varphi$$

hincque integralia pro isto casu $\alpha\varphi = -\infty$, erunt

$$\int e^{\alpha\varphi} d\varphi = -\frac{1}{\alpha}, \qquad \int e^{\alpha\varphi} d\varphi \sin.\varphi = \frac{1}{\alpha\alpha+1},$$

$$\int e^{\alpha\varphi} d\varphi \sin.^2\varphi = \frac{-1\cdot 2}{\alpha(\alpha\alpha+4)}, \qquad \int e^{\alpha\varphi} d\varphi \sin.^3\varphi = \frac{1\cdot 2\cdot 3}{(\alpha\alpha+1)(\alpha\alpha+9)},$$

$$\int e^{\alpha\varphi} d\varphi \sin.^4\varphi = \frac{-1\cdot 2\cdot 3\cdot 4}{\alpha(\alpha\alpha+4)(\alpha\alpha+16)}, \quad \int e^{\alpha\varphi} d\varphi \sin.^5\varphi = \frac{1\cdot 2\cdot 3\cdot 4\cdot 5}{(\alpha\alpha+1)(\alpha\alpha+9)(\alpha\alpha+25)}.$$

COROLLARIUM 3

269. Quare si proponatur haec series infinita

$$s = 1 + \frac{1\cdot 2}{\alpha(\alpha\alpha+4)} + \frac{1\cdot 2\cdot 3\cdot 4}{\alpha(\alpha\alpha+4)(\alpha\alpha+16)} + \frac{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}{(\alpha\alpha+4)(\alpha\alpha+16)(\alpha\alpha+36)} + \text{ etc.}$$

erit

$$s = -\alpha\int e^{\alpha\varphi} d\varphi\left(1 + \sin.^2\varphi + \sin.^4\varphi + \sin.^6\varphi + \text{ etc.}\right)$$

seu

$$s = -\alpha\int \frac{e^{\alpha\varphi} d\varphi}{\cos.^2\varphi}$$

posito post integrationem $\alpha\varphi = -\infty$, .

PROBLEMA 31

270. *Formulae differentialis* $e^{\alpha\varphi}d\varphi\cos.^n\varphi$ *integrale investigare.*

SOLUTIO

Simili modo procedendo ut ante erit

$$\int e^{\alpha\varphi}d\varphi\cos.^n\varphi=\tfrac{1}{\alpha}e^{\alpha\varphi}\cos.^n\varphi+\tfrac{n}{\alpha}\int e^{\alpha\varphi}d\varphi\sin.\varphi\cos.^{n-1}\varphi,$$

tum vero

$$\int e^{\alpha\varphi}d\varphi\sin.\varphi\cos.^{n-1}\varphi$$
$$=\tfrac{1}{\alpha}e^{\alpha\varphi}\sin.\varphi\cos.^{n-1}\varphi-\tfrac{1}{\alpha}\int e^{\alpha\varphi}d\varphi\left(\cos.^n\varphi-(n-1)\cos.^{n-2}\varphi\sin.^2\varphi\right),$$

quae postrema formula abit in

$$-(n-1)\int e^{\alpha\varphi}d\varphi\cos.^{n-2}\varphi+n\int e^{\alpha\varphi}d\varphi\cos.^n\varphi,$$

ita ut sit

$$\int e^{\alpha\varphi}d\varphi\cos.^n\varphi=\tfrac{1}{\alpha}e^{\alpha\varphi}\cos.^n\varphi+\tfrac{n}{\alpha\alpha}e^{\alpha\varphi}\sin.\varphi\cos.^{n-1}\varphi+\tfrac{n(n-1)}{\alpha\alpha}\int e^{\alpha\varphi}d\varphi\cos.^{n-2}\varphi$$
$$-\tfrac{nn}{\alpha\alpha}\int e^{\alpha\varphi}d\varphi\cos.^n\varphi,$$

unde colligimus

$$\int e^{\alpha\varphi}d\varphi\cos.^n\varphi=\frac{e^{\alpha\varphi}\cos.^{n-1}\varphi(\alpha\cos.\varphi+n\sin.\varphi)}{\alpha\alpha+nn}+\frac{n(n-1)}{\alpha\alpha+nn}\int e^{\alpha\varphi}d\varphi\cos.^{n-2}\varphi\ ;$$

hinc ergo casus simplicissimi sunt

$$\int e^{\alpha\varphi}d\varphi=\tfrac{1}{\alpha}e^{\alpha\varphi}+C,\qquad\int e^{\alpha\varphi}d\varphi\cos.\varphi=\frac{e^{\alpha\varphi}(\alpha\cos.\varphi+\sin.\varphi)}{\alpha\alpha+1}+C,$$

ad quos sequentes omnes, ubi *n* est numerus integer positivus, reducuntur.

SCHOLION

271. Casibus simplicissimis notatis alia datur via integrale formularum propositarum, quin etiam huius magis patentis $e^{\alpha\varphi}d\varphi\, sin.^{m}\varphi\, cos.^{n}\varphi$ eruendi. Cum enim productum $sin.^{m}\varphi\, cos.^{n}\varphi$ resolvi possit in aggregatum plurium sinuum vel cosinuum, quorum quisque est huius formae $M\, sin.\lambda\varphi$ vel $M\, cos.\lambda\varphi$ integratio reducitur ad alterutram harum formularum $e^{\alpha\varphi}d\varphi\, sin.\lambda\varphi$ vel $e^{\alpha\varphi}d\varphi\, cos.\lambda\varphi$. Ponamus ergo $\lambda\varphi=\omega$, ut habeamus

$$e^{\alpha\varphi}d\varphi\, sin.\lambda\varphi=\tfrac{1}{\lambda}e^{\frac{\alpha}{\lambda}\omega}d\omega\, sin.\omega \qquad et \qquad e^{\alpha\varphi}d\varphi\, cos.\lambda\varphi=\tfrac{1}{\lambda}e^{\frac{\alpha}{\lambda}\omega}d\omega\, cos.\omega,$$

quarum integralia per superiora ita dantur

$$\int e^{\frac{\alpha}{\lambda}\omega}d\omega\, sin.\omega=\frac{e^{\frac{\alpha}{\lambda}\omega}(\alpha\, sin.\omega-\lambda\, cos.\omega)}{\alpha\alpha+\lambda\lambda}=\frac{e^{\frac{\alpha}{\lambda}\omega}(\alpha\, sin.\lambda\varphi-\lambda\, cos.\lambda\varphi)}{\alpha\alpha+\lambda\lambda},$$

$$\int e^{\frac{\alpha}{\lambda}\omega}d\omega\, cos.\omega=\frac{e^{\frac{\alpha}{\lambda}\omega}(\alpha\, cos.\omega+\lambda\, sin.\omega)}{\alpha\alpha+\lambda\lambda}=\frac{e^{\frac{\alpha}{\lambda}\omega}(\alpha\, cos.\lambda\varphi+\lambda\, sin.\lambda\varphi)}{\alpha\alpha+\lambda\lambda}.$$

Unde tandem colligimus

$$\int e^{\alpha\varphi}d\varphi\, sin.\lambda\varphi=\frac{e^{\alpha\varphi}(\alpha\, sin.\lambda\varphi-\lambda\, cos.\lambda\varphi)}{\alpha\alpha+\lambda\lambda}$$

et

$$\int e^{\alpha\varphi}d\varphi\, cos.\lambda\varphi=\frac{e^{\alpha\varphi}(\alpha\, cos.\lambda\varphi+\lambda\, sin.\lambda\varphi)}{\alpha\alpha+\lambda\lambda}.$$

Si in genere statim loco $sin.\varphi$ et $cos.\varphi$ scripsissem $sin.\lambda\varphi$ et $cos.\lambda\varphi$, hac reductione non fuisset opus, sed quia hic nihil est difficultatis, brevitati consulendum existimavi.