CHAPTER III

CONCERNING THE INTEGRATION OF DIFFERENTIAL FORMULAS BY INFINITE SERIES

PROBLEM 12

126. *If X is a fractional rational function of x, to show the integration of the differential formula $dy = Xdx$ by a series expansion.*

SOLUTION

Since X is a fractional rational function, the value of this can always be expanded out ; so that it becomes :

$$X = Ax^m + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n} + Ex^{m+4n} + \text{etc.},$$

where the coefficients *A*, *B*, *C* etc. establish a recurrent series to be determined from the denominator of the fraction. Hence the individual terms can be multiplied by *dx* and integrated, with which performed the integral *y* can be expressed by the following series :

$$y = \frac{Ax^{m+1}}{m+1} + \frac{Bx^{m+n+1}}{m+n+1} + \frac{Cx^{m+2n+1}}{m+2n+1} + \text{etc.} + \text{Const.};$$

where if in the series for X a term of the form $\frac{M}{x}$ occurs, then in the integral a term $M\,lx$ is introduced.

SCHOLIUM

127. Since the integral $\int Xdx$, unless it is algebraic, can be expressed by logarithms and angles, hence the values of the logarithms and of the angles are able to be shown by infinite series. Now since several series of this kind have been examined in the *Introductione* [L. Euler : *Introductio in analysin infinitorum,* Book I, Ch's. VI-VIII], not only the same series treated, but an infinite number of other series can be elicited by integration. This it will be helpful to indicate from examples, where chiefly we establish formulas of the kind, in which the denominator is a binomial [by this Euler simply means that the denominator has two terms, and should not be confused with the modern meaning of the term]; then indeed also we will consider some cases provided with trinomial denominator or a multinomial. But in the first place we elicit [functions] of this kind, in which the denominator of the fraction is binomial, and which is able to be transformed [into an infinite series].

EXAMPLE 1

128. *To integrate the differential formula* $\frac{ax}{a+x}$ *by a series.*

Let $y=\int\frac{dx}{a+x}$; then $y=l(a+x)+\text{Const.}$, from which integral thus to be determined, so that it vanishes on putting $x = 0$, will be $y=l(a+x)-la.$ Now since

$$\frac{1}{a+x}=\frac{1}{a}-\frac{x}{a^2}+\frac{xx}{a^3}-\frac{x^3}{4a^4}+\frac{x^4}{5a^5}-\text{etc.},$$

then by the same rule the integral can be defined

$$y=\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\frac{x^5}{5a^5}-\text{etc.},$$

from which we can deduce, as indeed it is now agreed,

$$l(a+x)=la+\frac{x}{a}-\frac{x^2}{2a^2}+\frac{x^3}{3a^3}-\frac{x^4}{4a^4}+\text{etc.},$$

COROLLARY 1

129. If we take x negative, so that $dy=\frac{-dx}{a-x}$, in the same manner it is clear, that

$$l(a-x)=la-\frac{x}{a}-\frac{x^2}{2a^2}-\frac{x^3}{3a^3}-\frac{x^4}{4a^4}-\text{etc.},$$

and with these combined :

$$l(aa-xx)=2la-\frac{xx}{aa}-\frac{x^4}{2a^4}-\frac{x^6}{3a^6}-\frac{x^8}{4a^8}-\text{etc.},$$

and

$$l\,\frac{a+x}{a-x}=\frac{2x}{a}+\frac{2x^3}{3a^3}+\frac{2x^5}{5a^5}+\frac{2x^7}{7a^7}+\text{etc.}$$

COROLLARY 2

130. These last series are elicited by the integration of the formulas

$$\frac{-2xdx}{aa-xx}=-2xdx\left(\frac{1}{aa}+\frac{xx}{a^4}+\frac{x^4}{a^6}+\text{etc.}\right),$$

and

$$\frac{2adx}{aa-xx}=2adx\left(\frac{1}{aa}+\frac{xx}{a^4}+\frac{x^4}{a^6}+\text{etc.}\right).$$

But there is

$$\int\frac{-2xdx}{aa-xx}=l(aa-xx)-laa \quad\text{and}\quad \int\frac{2adx}{aa-xx}=l\,\frac{a+x}{a-x},$$

thus so that now we can desist from these formulas for the integration through series.

EXAMPLE 2

131. *To integrate the differential formula* $\int\frac{adx}{aa+xx}$ *by a series.*

Let $dy=\frac{adx}{aa+xx}$, and since there arises $y=$ Arc. tang.$\frac{x}{a}$, the same angle can be expressed by an infinite series. For since we have

$$\frac{a}{aa+xx}=\frac{1}{a}-\frac{xx}{a^3}+\frac{x^4}{a^5}-\frac{x^6}{a^7}+\frac{x^8}{a^9}-\text{etc.},$$

on integration there arises

$$y=\text{Arc. tang.}\frac{x}{a}=\frac{x}{a}-\frac{x^3}{3a^3}+\frac{x^5}{5a^5}-\frac{x^7}{7a^7}+\text{etc.}$$

EXAMPLE 3

132. *To express the integrals of these formulas* $\int\frac{dx}{1+x^3}$ *and* $\int\frac{xdx}{1+x^3}$ *by series.*

Since there shall be

$$\frac{1}{1+x^3}=1-x^3+x^6-x^9+x^{12}-\text{etc.},$$

then there becomes

$$\int\frac{dx}{1+x^3}=x-\frac{1}{4}x^4+\frac{1}{7}x^7-\frac{1}{10}x^{10}+\frac{1}{13}x^{13}-\text{etc.}$$

and

$$\int\frac{xdx}{1+x^3}=\frac{1}{2}x^2-\frac{1}{5}x^5+\frac{1}{8}x^8-\frac{1}{11}x^{11}+\frac{1}{14}x^{14}-\text{etc.}$$

Now by § 77 we have by logarithms and angles :

$$\int\frac{dx}{1+x^3}=\frac{1}{3}l(1+x)-\frac{2}{3}cos.\frac{\pi}{3}l\sqrt{\left(1-2x\,cos.\frac{\pi}{3}+xx\right)}$$

$$+\frac{2}{3}sin.\frac{\pi}{3}\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{3}}{1-x\,cos.\frac{\pi}{3}},$$

and

$$\int\frac{xdx}{1+x^3}=-\frac{1}{3}l(1+x)-\frac{2}{3}cos.\frac{\pi}{3}l\sqrt{\left(1-2x\,cos.\frac{\pi}{3}+xx\right)}$$

$$+\frac{2}{3}sin.\frac{\pi}{3}\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{3}}{1-x\,cos.\frac{\pi}{3}},$$

But in this case, $cos.\frac{\pi}{3}=\frac{1}{2}$, $cos.\frac{2\pi}{3}=-\frac{1}{2}$, $sin.\frac{\pi}{3}=\frac{\sqrt{3}}{2}$, $sin.\frac{2\pi}{3}=\frac{\sqrt{3}}{2}$, from which there becomes :

$$\int\frac{dx}{1+x^3}=\frac{1}{3}l(1+x)-\frac{1}{3}l\sqrt{(1-x+xx)}+\frac{1}{\sqrt{3}}\text{Arc.tang.}\frac{x\sqrt{3}}{2-x},$$

$$\int\frac{xdx}{1+x^3}=-\frac{1}{3}l(1+x)+\frac{1}{3}l\sqrt{(1-x+xx)}+\frac{1}{\sqrt{3}}\text{Arc.tang.}\frac{x\sqrt{3}}{2-x}$$

as with both the integrals and the series taken, so that they vanish on putting $x=0$.

COROLLARY 1

133. Hence with the series added there is produced :

$$\frac{2}{\sqrt{3}}\text{Arc.tang.}\frac{x\sqrt{3}}{2-x}=x+\frac{1}{2}xx-\frac{1}{4}x^4-\frac{1}{5}x^5+\frac{1}{7}x^7+\frac{1}{8}x^8-\frac{1}{10}x^{10}-\frac{1}{11}x^{11}+\text{etc.},$$

moreover with the latter subtracted from the former there is made :

$$\frac{2}{3}l\frac{1+x}{\sqrt{(1-x+xx)}}=x-\frac{1}{2}x^2-\frac{1}{4}x^4+\frac{1}{5}x^5+\frac{1}{7}x^7-\frac{1}{8}x^8-\frac{1}{10}x^{10}+\frac{1}{11}x^{11}+\text{etc.},$$

the value of which is also :

$$=\frac{1}{3}l\frac{(1+x)^2}{1-x+xx}=\frac{1}{3}l\frac{(1+x)^3}{1+x^3}.$$

COROLLARY 2

134. Since there shall be :

$$\int\frac{xxdx}{1+x^3}=\frac{1}{3}l\left(1+x^3\right),$$

in the same manner there is :

$$\frac{1}{3}l\left(1+x^3\right)=\frac{1}{3}x^3-\frac{1}{6}x^6+\frac{1}{9}x^9-\frac{1}{12}x^{12}+\text{etc.},$$

in which series all the powers of *x* occurring are added.

EXAMPLE 4

135. *To express this integral* $y=\int\frac{(1+xx)dx}{1+x^4}$ *by a series.*

Since there arises :

$$\frac{1}{1+x^4}=1-x^4+x^8-x^{12}+x^{16}-\text{etc.},$$

then there shall be

$$y=x+\tfrac{1}{3}x^3-\tfrac{1}{5}x^5-\tfrac{1}{7}x^7+\tfrac{1}{11}x^{11}-\tfrac{1}{13}x^{13}-\tfrac{1}{15}x^{15}+\text{etc.}$$

Now by §82, when m = 1 and n = 4, on putting $\frac{\pi}{4}=\omega$ the integral becomes likewise :

$$y=\sin.\omega \text{ Arc.tang.}\frac{x\sin.\omega}{1-x\cos.\omega}+\sin.3\omega \text{ Arc.tang.}\frac{x\sin.3\omega}{1-x\cos.3\omega};$$

but on account of

$\frac{\pi}{4}=\omega=45°$ there is $\sin.\omega=\frac{1}{\sqrt{2}}$, $\cos.\omega=\frac{1}{\sqrt{2}}$, $\sin.3\omega=\frac{1}{\sqrt{2}}$, $\cos.3\omega=\frac{-1}{\sqrt{2}}$;

and we have

$$y=\frac{1}{\sqrt{2}}\text{Arc.tang.}\frac{x}{\sqrt{2}-x}+\frac{1}{\sqrt{2}}\text{Arc.tang.}\frac{x}{\sqrt{2}+x}=\frac{1}{\sqrt{2}}\text{Arc.tang.}\frac{x\sqrt{2}}{1-xx}.$$

EXAMPLE 5

136. *To express this integral* $y=\int\frac{(1+x^4)dx}{1+x^6}$ *by a series..*

Since there arises :

$$\frac{1}{1+x^6}=1-x^6+x^{12}-x^{18}+x^{24}-\text{etc.},$$

then there shall be :

$$y=x+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7-\tfrac{1}{11}x^{11}+\tfrac{1}{13}x^{13}+\tfrac{1}{17}x^{17}-\text{etc.}$$

But by §82, when $m=1, n=6$ and $\omega=\frac{\pi}{6}=30°$, there is :

$$y=\tfrac{2}{3}\sin.\omega \text{ Arc.tang.}\frac{x\sin.\omega}{1-x\cos.\omega}+\tfrac{2}{3}\sin.3\omega \text{ Arc.tang.}\frac{x\sin.3\omega}{1-x\cos.3\omega};$$

$$+\tfrac{2}{3}\sin.5\omega \text{ Arc.tang.}\frac{x\sin.5\omega}{1-x\cos.5\omega};$$

now there is :

$\sin.\omega=\frac{1}{2}$, $\cos.\omega=\frac{\sqrt{3}}{2}$, $\sin.3\omega=1$, $\cos.3\omega=0$, $\sin.5\omega=\frac{1}{2}$, $\cos.5\omega=-\frac{\sqrt{3}}{2}$, hence

$$y=\tfrac{1}{3}\text{Arc.tang.}\frac{x}{2-x\sqrt{3}}+\tfrac{2}{3}\text{Arc.tang.}x+\tfrac{1}{3}\text{Arc.tang.}\frac{x}{2+x\sqrt{3}}$$

or

$$y = \tfrac{1}{3}\text{Arc.tang.}\tfrac{x}{1-xx} + \tfrac{2}{3}\text{Arc.tang.}x = \tfrac{1}{3}\text{Arc.tang.}\tfrac{3x(1-xx)}{1-4xx+x^4}.$$

COROLLARY 1

137. Let

$$z = \int \tfrac{xxdx}{1+x^6} = \tfrac{1}{3}x^3 - \tfrac{1}{9}x^9 + \tfrac{1}{15}x^{15} - \tfrac{1}{21}x^{21} + \text{etc.},$$

but on making $x^3 = u$ there is produced :

$$z = \tfrac{1}{3}\int \tfrac{xxdx}{1+x^6} = \tfrac{1}{3}\text{Arc.tang.}u = \tfrac{1}{3}\text{Arc.tang.}x^3.$$

Hence a mixed series of this kind is produced :

$$x + \tfrac{n}{3}x^3 + \tfrac{1}{5}x^5 - \tfrac{1}{7}x^7 - \tfrac{n}{9}x^9 - \tfrac{1}{11}x^{11} + \tfrac{1}{13}x^{13} + \tfrac{n}{15}x^{15} + \tfrac{1}{17}x^{17} - \text{etc.},$$

the sum of which is

$$\tfrac{1}{3}\text{Arc.tang.}\tfrac{3x(1-xx)}{1-4xx+x^4} + \tfrac{n}{3}\text{Arc.tang.}x^3.$$

COROLLARY 2

138. Here if there is taken $n = -1$, the two angles can be gathered together into a single angle :

$$\tfrac{1}{3}\text{Arc.tang.}\tfrac{3x(1-xx)}{1-4xx+x^4} - \tfrac{1}{3}\text{Arc.tang.}x^3 = \tfrac{1}{3}\text{Arc.tang.}\tfrac{3x-4x^3+4x^5-x^7}{1-4xx+4x^4-3x^6}$$

which [final] fraction on being divided by $1 - xx + x^4$ is reduced to $\tfrac{3x-x^3}{1-3xx}$ which is the tangent of the triple of the angle having x for the tangent, thus in order that

$$\tfrac{1}{3}\text{Arc.tang.}\tfrac{3x-x^3}{1-3xx} = \text{Arc.tang.}x,$$

which likewise shows the same series is found.

EXAMPLE 6

139. *To integrate this series* $dy = \dfrac{\left(x^{m-1}+x^{n-m-1}\right)dx}{1+x^n}$ *by a series.*

On account of

$$\tfrac{1}{1+x^n} = 1 - x^n + x^{2n} - x^{3n} + x^{4n} - \text{etc.}$$

there is obtained :

$$y=\frac{x^m}{m}+\frac{x^{n-m}}{n-m}-\frac{x^{n+m}}{n+m}-\frac{x^{2n-m}}{2n-m}+\frac{x^{2n+m}}{2n+m}+\frac{x^{3n-m}}{3n-m}-\text{etc.}$$

Hence this series by § 82 can be expressed by some sum of the arcs of circles, which can be seen there.

COROLLARY

140. From the same proposed formula $dz=\frac{\left(x^{m-1}-x^{n-m-1}\right)dx}{1-x^n}$, on account of

$$\frac{1}{1-x^n}=1+x^n+x^{2n}+x^{3n}+x^{4n}+\text{etc.}$$

there is found :

$$z=\frac{x^m}{m}-\frac{x^{n-m}}{n-m}+\frac{x^{n+m}}{n+m}-\frac{x^{2n-m}}{2n-m}+\frac{x^{2n+m}}{2n+m}-\frac{x^{3n-m}}{3n-m}+\text{etc.}$$

and the value of this has been shown in §84.

EXAMPLE 7

141. *To integrate this formula* $dy=\frac{(1+2x)dx}{1+x+xx}$ *by a series..*

In the first place the integral is evidently $y=l\left(1+x+xx\right)$; but in order that it may be changed into a series, the numerator and the denominator are multiplied by $1-x$, so that there becomes :

$$dy=\frac{(1+x-2xx)dx}{1-x^3}.$$

Now since there is present :

$$\frac{1}{1-x^3}=1+x^3+x^6+x^9+x^{12}+\text{etc.},$$

on integrating,

$$y=x+\frac{x^2}{2}-\frac{2x^3}{3}+\frac{x^4}{4}+\frac{x^5}{5}-\frac{2x^6}{6}+\frac{x^7}{7}+\frac{x^8}{8}-\frac{2x^9}{9}+\text{etc.}$$

COROLLARY 1

142. In the same manner it can be found that $y = l\left(1 + x + xx + x^3\right)$ through a series expansion. For since there arises $y + l(1-x) = l\left(1-x^4\right)$, then

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{10} + \text{etc.}$$
$$\qquad\qquad - x^4 \qquad\qquad\qquad\qquad - \frac{x^8}{2}$$

or

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3x^8}{8} + \frac{x^9}{9} + \text{etc.}$$

COROLLARY 2

143. But the fraction $\frac{1+2x}{1+x+xx}$ gives on expansion in a series,

$$1 + x - 2xx + \mathrm{x}^3 + x^4 - 2x^5 + x^6 + x^7 - 2x^8 + \text{etc.},$$

from which by integration the same series as before is obtained.

EXEMPLE 8

144. *To integrate this formula* $dy = \frac{dx}{1-2x\cos.\zeta + xx}$ *by a series.*

By § 64, when $A = 1, B = 0, a = 1$ and $b = 1$, the integral of this formula is :

$$y = \frac{1}{\sin.\zeta}\,\text{Arc.tang.}\frac{x\sin.\zeta}{1 - x\cos.\zeta}.$$

But we find from the recurrent series :

$$\frac{1}{1-2x\cos.\zeta+xx} = 1 + 2x\cos.\zeta + \left(4\cos^2.\zeta - 1\right)xx + \left(8\cos^3.\zeta - 4\cos.\zeta\right)x^3$$
$$+\left(16\cos^4.\zeta - 12\cos^2.\zeta + 1\right)x^4 + \left(32\cos^5.\zeta - 32\cos^3.\zeta + 6\cos.\zeta\right)x^5 + \text{etc.},$$

by which the series sought is obtained on multiplication by dx and integrating. But with the powers of $\cos.\zeta$ changed into the cosines of multiples of angles there is found :

$$y = x + \tfrac{1}{2}xx\left(2\cos.\zeta\right) + \tfrac{1}{3}x^3\left(2\cos.2\zeta + 1\right) + \tfrac{1}{4}x^4\left(2\cos.3\zeta + 2\cos.\zeta\right)$$
$$+\tfrac{1}{5}x^5\left(2\cos.4\zeta + 2\cos.2\zeta + 1\right) + \tfrac{1}{6}x^6\left(2\cos.5\zeta + 2\cos.3\zeta + 2\cos.\zeta\right) + \text{etc.},$$

COROLLARY 1

145. If there is put

$$dz = \frac{(1-x\cos.\zeta)dx}{1-2x\cos.\zeta+xx},$$

there is by § 63 [Ch. I] $A=1$, $B=-\cos.\zeta$, $a=1$ and $b=1$, and thus

$$z = -\cos.\zeta\; l\sqrt{(1-2x\cos.\zeta+xx)} + \sin.\zeta \,\text{Arc. tang}\, \frac{x\sin.\zeta}{1-x\cos.\zeta} \;;$$

but with a series on account of

$$\frac{1-x\cos.\zeta}{1-2x\cos.\zeta+xx} = 1 + x\cos.\zeta + x^2\cos.2\zeta + x^3\cos.3\zeta + x^4\cos.4\zeta + \text{etc.}$$

there is produced

$$z = x + \tfrac{1}{2}xx\cos.\zeta + \tfrac{1}{3}x^3\cos.2\zeta + \tfrac{1}{4}x^4\cos.3\zeta + \tfrac{1}{5}x^5\cos.4\zeta + \text{etc.}$$

COROLLARY 2

146. But since

$$dz = \frac{dx(-x\cos.\zeta + \cos^2.\zeta + \sin^2.\zeta)}{1-2x\cos.\zeta+xx},$$

then

$$z = -\cos.\zeta\; l\sqrt{(1-2x\cos.\zeta+xx)} + \sin^2.\zeta\int\frac{dx}{1-2x\cos.\zeta+xx}.$$

Hence therefore for

$$y = \int\frac{dx}{1-2x\cos.\zeta+xx}$$

another infinite series is found involving logarithms, clearly

$$y = \frac{\cos.\zeta}{\sin^2.\zeta}\, l\sqrt{(1-2x\cos.\zeta+xx)}$$
$$+\frac{1}{\sin^2.\zeta}\left(x + \tfrac{1}{2}xx\cos.\zeta + \tfrac{1}{3}x^3\cos.2\zeta + \tfrac{1}{4}x^4\cos.3\zeta + \text{etc.}\right).$$

PROBLEM 12 [a]

(*We must assume that an incorrect number has been applied here, as 12 is repeated..)*

147. *To integrate the irrational differential formula* $dy = x^{m-1}dx\left(a+bx^n\right)$ *by an infinite series.*

SOLUTION

Let $a^{\frac{\mu}{\nu}} = c$; then there is produced

$$dy = cx^{m-1}dx\left(1+\tfrac{b}{a}x^n\right)^{\frac{\mu}{\nu}},$$

where indeed we assume that c is not an imaginary quantity. Therefore since there is

$$\left(1+\tfrac{b}{a}x^n\right)^{\frac{\mu}{\nu}} = 1+\frac{\mu b}{1\nu\cdot a}x^n+\frac{\mu(\mu-\nu)b^2}{1\nu\cdot 2\nu\cdot aa}x^{2n}+\frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu\cdot 2\nu\cdot 3\nu\cdot a^3}x^{3n}+\text{etc.},$$

on integrating :

$$y = c\left(\frac{x^m}{m}+\frac{\mu b}{1\nu\cdot a}\frac{x^{m+n}}{m+n}+\frac{\mu(\mu-\nu)bb}{1\nu\cdot 2\nu\cdot aa}\frac{x^{m+2n}}{m+2n}+\frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu\cdot 2\nu\cdot 3\nu\cdot a^3}\frac{x^{m+3n}}{m+3n}+\text{etc.}\right),$$

which series recurs indefinitely to infinity, unless $\frac{\mu}{\nu}$ is a positive whole number. But if in the case, in which ν is an even number, a should be a negative quantity, our expression can be represented thus

$$dy = x^{m-1}dx\left(bx^n-a\right)^{\frac{\mu}{\nu}} = b^{\frac{\mu}{\nu}}x^{m+\frac{\mu n}{\nu}-1}dx\left(1-\tfrac{a}{b}x^{-n}\right)^{\frac{\mu}{\nu}}.$$

Therefore since there shall be

$$\left(1-\tfrac{a}{b}x^{-n}\right)^{\frac{\mu}{\nu}} = 1-\frac{\mu b}{1\nu\cdot a}x^{-n}+\frac{\mu(\mu-\nu)a^2}{1\nu\cdot 2\nu\cdot b^2}x^{-2n}-\frac{\mu(\mu-\nu)(\mu-2\nu)a^3}{1\nu\cdot 2\nu\cdot 3\nu\cdot b^3}x^{-3n}+\text{etc.},$$

and it becomes on integrating,

$$y = b^{\frac{\mu}{\nu}}\left(\frac{\nu x^{m+\frac{\mu n}{\nu}}}{m\nu+\mu n}-\frac{\mu a}{1\nu\cdot b}\cdot\frac{\nu x^{m+\frac{(\mu-\nu)n}{\nu}}}{m\nu+(\mu-\nu)n}+\frac{\mu(\mu-\nu)a^2}{1\nu\cdot 2\nu\cdot b^2}\cdot\frac{\nu x^{m+\frac{(\mu-2\nu)n}{\nu}}}{m\nu+(\mu-2\nu)n}-\text{etc.}\right).$$

If a and b are positive numbers, each expansion is allowed to be used.

EXAMPLE 1

148. *To integrate the formula* $dy = \frac{dx}{\sqrt{(1-xx)}}$ *by series.*

First from above it is evident that y = Arc. sin. x, which angle hence can be expressed by an infinite series. For since there shall be

$$\frac{1}{\sqrt{(1-xx)}} = 1 + \frac{1}{2}x^2 + \frac{1\cdot 3}{2\cdot 4}x^4 + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8 + \text{etc.},$$

then

$$y = x + \frac{1}{2}\cdot\frac{x^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{x^9}{9} + \text{etc.},$$

and with each value thus defined, so that it vanishes on putting $x = 0$.

COROLLARY 1

149. Hence if there is put $x = 1$, on account of $Arc.sin.1 = \frac{\pi}{2}$ then

$$\frac{\pi}{2} = 1 + \frac{1}{2\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 9} + \text{etc.},$$

But if there is put $x = \frac{1}{2}$, on account of $Arc.sin.\frac{1}{2} = 30^\circ = \frac{\pi}{6}$ then

$$\frac{\pi}{6} = \frac{1}{2} + \frac{1}{2\cdot 2^3\cdot 3} + \frac{1\cdot 3}{2\cdot 4\cdot 2^5\cdot 5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 2^7\cdot 7} + \frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 2^9\cdot 9} + \text{etc.},$$

the ten terms of which series added together gives 0,52359877, of which the six fold gives 3,14159262 only disagreeing in the eighth figure from the truth.

COROLLARY 2

150. This proposed formula $dy = \frac{dx}{\sqrt{(x-xx)}}$ on putting $x = uu$ becomes

$$dy = \frac{2udu}{\sqrt{(uu-u^4)}} = \frac{2du}{\sqrt{(1-uu)}},$$

hence

$$y = 2Arc.sin.u = 2\,Arc.sin.\sqrt{x};$$

then indeed in terms of a series it becomes :

$$y = 2\left(u + \frac{1}{2}\cdot\frac{u^3}{3} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{u^5}{5} + \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{u^7}{7} + \text{etc.}\right)$$

or

$$y = 2\left(1 + \tfrac{1}{2}\cdot\tfrac{x}{3} + \tfrac{1\cdot 3}{2\cdot 4}\cdot\tfrac{xx}{5} + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\tfrac{x^3}{7} + \text{etc.}\right)\sqrt{x}.$$

EXAMPLE 2

151. *To integrate the formula* $dy = dx\sqrt{(2ax - xx)}$ *by a series.*

On putting $x=uu$ there becomes $dy = 2uudu\sqrt{(2a - uu)}$ but by the reduction I (§118) there is
$n = 2,\ m = 1,\ a = 2a,\ b = -1,\ \mu = 1,\ v = 2,$ from which

$$\int uudu\sqrt{(2a - uu)} = -\tfrac{1}{4}u(2a - uu)^{\frac{3}{2}} + \tfrac{1}{2}a\int du\sqrt{(2a - uu)},$$

and by III on taking $m = 1,\quad a = 2a,\quad b = -1,\quad n = 2,\quad \mu = -1,\quad v = 2$ there becomes

$$\int du\sqrt{(2a - uu)} = \tfrac{1}{2}u\sqrt{(2a - uu)} + a\int \tfrac{du}{\sqrt{(2a - uu)}};$$

but there is the relation

$$\int \tfrac{du}{\sqrt{(2a - uu)}} = Arc.sin\tfrac{u}{\sqrt{2a}} = Arc.sin\tfrac{\sqrt{x}}{\sqrt{2a}}$$

and thus

$$\int uudu\sqrt{(2a - uu)} = -\tfrac{1}{4}u(2a - uu)^{\frac{3}{2}} + \tfrac{1}{4}au\sqrt{(2a - uu)} + \tfrac{1}{2}aaArc.sin\tfrac{\sqrt{x}}{\sqrt{2a}}$$

$$= \tfrac{1}{4}u(uu - a)\sqrt{(2a - uu)} + \tfrac{1}{2}aaArc.sin\tfrac{\sqrt{x}}{\sqrt{2a}}.$$

Hence

$$y = \tfrac{1}{2}(x - a)\sqrt{(2ax - xx)} + aa\,Arc.sin\tfrac{\sqrt{x}}{\sqrt{2a}}.$$

But with the series to be found there is :

$$dy = dx\sqrt{2ax}\left(1 - \tfrac{x}{2a}\right)^{\frac{1}{2}}$$

$$= x^{\frac{1}{2}}dx\left(1 - \tfrac{1}{2}\cdot\tfrac{x}{2a} - \tfrac{1\cdot 1}{2\cdot 4}\cdot\tfrac{xx}{4aa} - \tfrac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\tfrac{x^3}{8a^3} - \text{etc.}\right)\sqrt{2a}$$

and hence on integration,

$$y = \left(\tfrac{2}{3}x^{\frac{3}{2}} - \tfrac{1}{2}\cdot\tfrac{2x^{\frac{5}{2}}}{5\cdot 2a} - \tfrac{1\cdot 1}{2\cdot 4}\cdot\tfrac{2x^{\frac{7}{2}}}{7\cdot 4aa} - \tfrac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\tfrac{2x^{\frac{9}{2}}}{9\cdot 8a^3} - \text{etc.}\right)\sqrt{2a}$$

or

$$y = \left(\frac{x}{3} - \frac{1}{2}\cdot\frac{x^2}{5\cdot 2a} - \frac{1\cdot 1}{2\cdot 4}\cdot\frac{x^3}{7\cdot 4aa} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\frac{x^4}{9\cdot 8a^3} - \text{etc.}\right)2\sqrt{2ax}.$$

COROLLARY

152. The integral can be found more easily on putting $x = a - v$, from which there becomes :

$$dy = -dv\sqrt{(aa - vv)}$$

and by the reduction III [§ 118]

$$\int dv\sqrt{(aa - vv)} = \tfrac{1}{2}v\sqrt{(aa - vv)} + \tfrac{1}{2}aa\int\frac{dv}{\sqrt{(aa - vv)}},$$

hence

$$y = C - \tfrac{1}{2}v\sqrt{(aa - vv)} - \tfrac{1}{2}aa\,Arc.sin.\tfrac{v}{a}$$

or

$$y = C - \tfrac{1}{2}(a - v)\sqrt{(2ax - xx)} - \tfrac{1}{2}aa\ Arc.sin.\tfrac{a-x}{a};$$

so that therefore on putting $x = 0$, also $y = 0$, it is required to put $C = \tfrac{1}{2}aa\ Arc.sin.1$, thus so that :

$$y = -\tfrac{1}{2}(a - x)\sqrt{(2ax - xx)} + \tfrac{1}{2}aa\,Arc.cos.\tfrac{a-x}{a}.$$

Indeed there is:

$$Arc.sin.\frac{\sqrt{x}}{\sqrt{2a}} = \tfrac{1}{2}Arc.cos.\tfrac{a-x}{a}.$$

COROLLARY 2

153. If we put $x = \frac{a}{2}$, then there becomes $y = \frac{-aa\sqrt{3}}{8} + \frac{\pi aa}{6}$; but the series gives

$$y = 2aa\left(\frac{1}{2\cdot 3} - \frac{1}{2\cdot 5\cdot 2^3} - \frac{1\cdot 1}{2\cdot 4\cdot 7\cdot 2^5} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6\cdot 9\cdot 2^7} - \text{etc.}\right),$$

from which it is deduced that

$$\pi = \frac{3\sqrt{3}}{4} + 6\left(\frac{1}{3} - \frac{1}{2\cdot 5\cdot 2^3} - \frac{1\cdot 1}{2\cdot 4\cdot 7\cdot 2^4} - \frac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6\cdot 9\cdot 2^6} - \text{etc.}\right);$$

but by the above [§ 149] there is

$$\pi = 3\left(1+\tfrac{1}{2\cdot3\cdot2^2}+\tfrac{1\cdot3}{2\cdot4\cdot5\cdot2^4}+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot7\cdot2^6}+\text{etc.}\right),$$

from the combination of which several others can be formed.

EXAMPLE 3

154. *To integrate the formula* $dy=\frac{dx}{\sqrt{(1+xx)}}$ *by means of a series.*

The integral is $y=l\left(x+\sqrt{(1+xx)}\right)$ taken thus, so that it vanishes on putting $x=0$. But on account of

$$\frac{1}{\sqrt{(1+xx)}} = 1-\tfrac{1}{2}x^2+\tfrac{1\cdot3}{2\cdot4}x^4-\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}x^6+\text{etc.}$$

the same integral expressed by a series shall be

$$y=x-\tfrac{1}{2}\cdot\tfrac{x^3}{3}+\tfrac{1\cdot3}{2\cdot4}\cdot\tfrac{x^5}{5}-\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\tfrac{x^7}{7}+\text{etc.}$$

EXAMPLE 4

155. *To integrate the formula* $dy=\frac{dx}{\sqrt{(xx-1)}}$ *by means of a series.* .

The integration gives $y=l\left(x+\sqrt{(xx-1)}\right)$ which vanishes on putting $x=1$. Now on account of

$$\frac{1}{\sqrt{(xx-1)}}=\tfrac{1}{x}+\tfrac{1}{2x^3}+\tfrac{1\cdot3}{2\cdot4x^5}+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6x^7}+\text{etc.}$$

the same integral shall be

$$y=C+lx-\tfrac{1}{2\cdot2x^2}-\tfrac{1\cdot3}{2\cdot4\cdot4x^4}-\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot6x^6}-\text{etc.}$$

which in order that it vanishes on putting $x = 1$, the constant is defined thus, so that the series becomes :

$$y=lx+\tfrac{1}{2\cdot2}\left(1-\tfrac{1}{x^2}\right)+\tfrac{1\cdot3}{2\cdot4\cdot4}\left(1-\tfrac{1}{x^4}\right)+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot6}\left(1-\tfrac{1}{x^6}\right)+\text{etc.}$$

COROLLARY

156. On putting $x = 1 + u$ there becomes

$$dy = \frac{du}{\left(\sqrt{2u+uu}\right)} = \frac{du}{\sqrt{2u}}\left(1+\frac{u}{2}\right)^{-\frac{1}{2}}$$
$$= \frac{du}{\sqrt{2u}}\left(1 - \frac{1}{2}\cdot\frac{u}{2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{uu}{4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{u^3}{8} - \text{etc.}\right),$$

from which there is had on integration

$$y = \frac{1}{\sqrt{2}}\left(2\sqrt{u} - \frac{1}{2}\cdot\frac{2u^{\frac{3}{2}}}{3\cdot 2} + \frac{1\cdot 3}{2\cdot 4}\cdot\frac{2u^{\frac{5}{2}}}{5\cdot 4} - \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{2u^{\frac{7}{2}}}{7\cdot 8} + \text{etc.}\right)$$

or

$$y = \left(1 - \frac{1u}{2\cdot 3\cdot 2} + \frac{1\cdot 3uu}{2\cdot 4\cdot 5\cdot 4} - \frac{1\cdot 3\cdot 5u^3}{2\cdot 4\cdot 6\cdot 7\cdot 8} + \text{etc.}\right)\sqrt{2u}.$$

EXAMPLE 5

157. *To integrate the formula* $dy = \frac{dx}{(1-x)^n}$ *by means of a series..*

By integration there arises

$$y = \frac{1}{(n-1)(1-x)^{n-1}} - \frac{1}{n-1}$$

on making $y = 0$, if $x = 0$, or

$$y = \frac{(1-x)^{-n+1}-1}{n-1}$$

Now indeed by the series there is :

$$dy = dx\left(1 + nx + \frac{n(n+1)}{1\cdot 2}x^2 + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3 + \text{etc.}\right),$$

from which the same integral is thus expressed :

$$y = x + \frac{nx^2}{2} + \frac{n(n+1)x^3}{1\cdot 2\cdot 3} + \frac{n(n+1)(n+2)x^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

Moreover it is hence evident also that :

$$(n-1)y + 1 = \frac{1}{(1-x)^{n-1}}.$$

SCHOLIUM

158. But since these can be exceedingly clumsy to use, and as there shall be a need to apply these more fully, I will set out another more abstract method to elicit the series, which often proves to be of outstanding use in analysis.

[This is an important result : the differential involves two powers of *x*, *m* and *n*, a fractional power of the binomial, $\frac{\mu}{\nu}$, and two coefficients, *a* and *b*, and hence is of general applicability]

PROBLEM 13

159. *For the proposed formula of the differential*

$$dy = x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}-1}$$

to convert the integral of this into a series by another method.

SOLUTION

Putting in place $y = \left(a+bx^n\right)^{\frac{\mu}{\nu}} z$ *;* then

$$dy = \left(a+bx^n\right)^{\frac{\mu}{\nu}-1}\left(dz\left(a+bx^n\right)+\frac{n\mu}{\nu}bx^{n-1}zdx\right),$$

from which there is prepared

$$x^{m-1}dx = dz\left(a+bx^n\right)+\frac{n\mu}{\nu}bx^{n-1}zdx$$

or

$$\nu x^{m-1}dx = \nu dz\left(a+bx^n\right)+n\mu bx^{n-1}zdx.$$

Now before we investigate the series in which the value of *z* is to be defined, that case should be noted in which *b* vanishes, that the differential becomes : [*x* is used rather than *b* in the first edition, and also in the *O.O*. edition, which is obviously an error]

$$dy = a^{\frac{\mu}{\nu}-1}x^{m-1}dx = a^{\frac{\mu}{\nu}}dz,$$

in order that $dz = \frac{1}{a}x^{m-1}dx$. Hence we may put in place

$$z = Ax^m + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n} + \text{etc.}$$

and there becomes :

$$\frac{dz}{dx} = mAx^{m-1} + \left(m+n\right)Bx^{m+n-1} + \left(m+2n\right)Cx^{m+2n-1} + \text{etc.}$$

These series are substituted in place of z and $\frac{dz}{dx}$ in the [above] equation

$$\frac{vdz}{dx}\left(a+bx^n\right)+n\mu bx^{n-1}z-vx^{m-1}=0$$

and with the individual terms set out following the powers of x this equation arises :

$$\left.\begin{array}{llll} mvaAx^{m-1} & +(m+n)vaBx^{m+n-1} & +\ (m+2n)vaCx^{m+2n-1} & +\text{etc.} \\ -v & +\quad mvbA & +\ (m+n)vbB & \\ & +\quad n\mu bA & +\quad n\mu bB & \end{array}\right\}=0,$$

from which the individual terms produced from the equations with positive terms with the equations equal to zero, are defined by the following formulas :

$$mvaA-v=0, \qquad \text{hence} \qquad A=\frac{1}{ma},$$

$$(m+n)vaB+(mv+n\mu)bA=0, \qquad B=-\frac{(mv+n\mu)b}{(m+n)va}A,$$

$$(m+2n)vaC+((m+n)v+n\mu)bB=0, \qquad C=-\frac{((m+n)v+n\mu)b}{(m+2n)va}B,$$

$$(m+3n)vaD+((m+2n)v+n\mu)bC=0, \qquad D=-\frac{((m+2n)v+n\mu)b}{(m+3n)va}C$$

and thus any coefficient is easily found from the preceding. Now indeed there shall be

$$y=\left(a+bx^n\right)^{\frac{\mu}{v}}\left(x^m+Bx^{m+n}+Cx^{m+2n}+Dx^{m+3n}+\text{etc.}\right)$$

SOLUTION 2

Just as this series we have assumed that this series follows ascending powers of x, thus it is also allowed to set up a series of descending powers :

$$z=Ax^{m-n}+Bx^{m-2n}+Cx^{m-3n}+Dx^{m-4n}+\text{etc.}$$

in order that

$$\frac{dz}{dx}=(m-n)Ax^{m-n-1}+(m-2n)Bx^{m-2n-1}+(m-3n)Cx^{m-3n-1}+\text{etc.}$$

from which series on substitution there is produced

$$\left.\begin{array}{l}(m-n)vbAx^{m-1}+(m-n)vaAx^{m+n-1}+(m-2n)vaCx^{m+2n-1}+(m-3n)vaCx^{m-3n-1}+\text{etc.}\\ \quad+n\mu bA \quad +(m-2n)vbB \quad +(m-3n)vbC \quad +(m-4n)vbD\\ \quad\quad -v \quad + \quad n\mu bB \quad + \quad n\mu bC \quad + \quad n\mu bD\end{array}\right\}=0.$$

Hence the letters *A*, *B*, *C* etc can be determined in the following manner :

$$(m-n)vbA+n\mu bA-v=0, \qquad \text{hence} \qquad A=\tfrac{v}{(m-n)v+n\mu}\cdot\tfrac{1}{b},$$

$$(m-n)vaA+(m-2n)vbB+n\mu bB=0, \qquad B=\tfrac{-(m-n)v}{(m-2n)v+n\mu}\cdot\tfrac{a}{b}A,$$

$$(m-2n)vaB+(m-3n)vbC+n\mu bC=0, \qquad C=\tfrac{-(m-2n)v}{(m-3n)v+n\mu}\cdot\tfrac{a}{b}B,$$

$$(m-3n)vaC+(m-4n)vbD+n\mu bD=0, \qquad D=\tfrac{-(m-3n)v}{(m-4n)v+n\mu}\cdot\tfrac{a}{b}C,$$

where again the law of the progression of these letters is evident.

COROLLARY 1

160. The first series therefore is memorable, since in the cases in which

$$(m+in)v+n\mu=0 \quad \text{or} \quad -\tfrac{m}{n}-\tfrac{\mu}{v}=i,$$

the series is truncated and the integral produced is algebraic. Now the latter series is truncated, as often as $m-in=0$ or $\frac{m}{n}=i$, with *i* denoting a whole positive number.

COROLLARY 2

161. Now each series also is troubled with a certain inconvenience, since it is not always possible to be called to use. For when either $m = 0$ or $m + in = 0$, the first cannot be used, and truly when $(m-in)v+n\mu=0$ or $\frac{m}{n}+\frac{\mu}{v}=i$, the use is removed from the latter, since the terms become infinite.

COROLLARY 3

162. Now this inconvenience in use comes about, so that, as often as one cannot be applied, the other certainly can be called into service, only with these cases excepted, in which both $-\frac{m}{n}$ and $\frac{\mu}{v}+\frac{m}{n}$ are positive whole numbers. But since then $v=1$, these cases are rational whole numbers and present no difficulty.

COROLLARY 4

163. Also both series for z can be joined together in this manner. Let the first series be equal to P, and the latter truly equal to Q, as it is possible to take both $z = P$ as well as $z = Q$. Moreover with the two joined together there arises $z = \alpha P + \beta Q$, provided $a + \beta = 1$.

SCHOLIUM

164. Moreover from this, since we produce two series for z, it follows that at least these two series are equal to each other; nor indeed is it necessary that the values of y to arise from that to become equal, as long as they differ from each other by a constant amount. Thus if the first series found is indicated by P, and the second by Q, since from that first one there arises $y = \left(a + bx^n\right)^{\frac{\mu}{\nu}} P$, and indeed from the other $y = \left(a + bx^n\right)^{\frac{\mu}{\nu}} Q$, certainly there shall be the constant quantity $\left(a + bx^n\right)^{\frac{\mu}{\nu}} \left(P - Q\right)$ and therefore $P - Q = C\left(a + bx^n\right)^{-\frac{\mu}{\nu}}$. Clearly each series only shows a particular integral, since it involves no constant, which now is not contained in the formula of the differential. Yet meanwhile by the same method the complete value for z can be elicited ; for in addition the series allows P or Q to be put in place and with a substitution made, the series P is defined as before ; now for the other a new series is required to be put in place, so that there becomes

$$\left.\begin{array}{lllll} n\nu a\beta x^{n-1} & +2n\nu a\gamma x^{2n-1} & +3n\nu a\delta x^{3n-1} & +4n\nu a\varepsilon x^{4n-1} & +\text{etc.} \\ +n\mu b\alpha & +n\nu b\beta & +2n\nu b\gamma & +3n\nu b\delta & \\ & +n\mu b\beta & +\ n\mu b\gamma & +\ n\mu b\delta & \end{array}\right\} = 0,$$

from which determinations are deduced :

$$\beta = \tfrac{-\mu b}{\nu a}\cdot\alpha,\quad \gamma = \tfrac{-(\mu+\nu)b}{2\nu a}\cdot\beta,\quad \delta = \tfrac{-(\mu+2\nu)b}{3\nu a}\cdot\gamma,\quad \delta = \tfrac{-(\mu+3\nu)b}{4\nu a}\cdot\delta,\text{etc.},$$

thus so that there is produced :

$$z = P + \alpha\left(1 - \tfrac{\mu}{\nu}\cdot\tfrac{b}{a}x^n + \tfrac{\mu(\mu+\nu)}{\nu\cdot 2\nu}\cdot\tfrac{b^2}{a^2}x^{2n} - \tfrac{\mu(\mu+\nu)(\mu+2\nu)}{\nu\cdot 2\nu\cdot 3\nu}\cdot\tfrac{b^3}{a^3}x^{3n} + \text{etc.}\right)$$

or

$$z = P + \alpha\left(a + bx^n\right)^{-\frac{\mu}{\nu}}$$

and hence

$$y = P\left(a+bx^n\right)^{\frac{\mu}{\nu}} + \alpha a^{\frac{\mu}{\nu}},$$

which is the complete integral, since the constant α remains arbitrary.

EXAMPLE 1

165. *To integrate the formula* $dy = \frac{dx}{\sqrt{(1-xx)}}$ *by a series in this manner.*

Since by comparison with the general form established :
$a = 1, b = -1, m = 1, n = 2, \mu = 1, \nu = 2$, from which on putting $y = z\sqrt{(1-xx)}$ the first solution

$$z = Ax + Bx^3 + Cx^5 + Dx^7 + \text{etc.}$$

produces

$$A = 1,\quad B = \tfrac{2}{3}A,\quad C = \tfrac{4}{5}B,\quad D = \tfrac{6}{7}C,\quad E = \tfrac{8}{9}D,\quad \text{etc.},$$

from which we deduce

$$y = \left(x + \tfrac{2}{3}x^3 + \tfrac{2\cdot4}{3\cdot5}x^5 + \tfrac{2\cdot4\cdot6}{3\cdot5\cdot7}x^7 + \text{etc.}\right)\sqrt{(1-xx)},$$

since the integral vanishes on putting $x = 0$; hence there arises $y = Arc.\ sin.x$. The other method here is attempted in vain on account of $\frac{m}{n} + \frac{\mu}{\nu} = 1$.

COROLLARY 1

166. On putting $x = 1$, hence it is seen that $y = 0$ on account of $\sqrt{(1-xx)} = 0$; but the product must be considered carefully in this case as the sum of the infinite series becomes infinite, thus so that there is no obstacle for $y = \frac{\pi}{2}$. If we put $x = \frac{1}{2}$, there becomes $y = 30^{\text{o}} = \frac{\pi}{6}$ and thus

$$\tfrac{\pi}{6} = \left(1 + \tfrac{2}{3\cdot4} + \tfrac{2\cdot4}{3\cdot5\cdot4^2} + \tfrac{2\cdot4\cdot6}{3\cdot5\cdot7\cdot4^3} + \text{etc.}\right)\tfrac{\sqrt{3}}{4}.$$

COROLLARY 2

167. In a similar manner the proposed formula $dy = \frac{dx}{\sqrt{(1+xx)}}$ is found :

$$y = \left(x - \tfrac{2}{3}x^3 + \tfrac{2\cdot4}{3\cdot5}x^5 - \tfrac{2\cdot4\cdot6}{3\cdot5\cdot7}x^7 + \text{etc.}\right)\sqrt{(1+xx)},$$

and this becomes

$$y = l\left(x + \sqrt{(1+xx)}\right)$$

EXAMPLE 2

168. *To integrate the formula* $dy = \frac{dx}{x\sqrt{(1-xx)}}$ *by a series in this manner.*

Hence [in the general term $dy = x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}-1}$] there is now put : $m=0, n=2,\ \mu=1,\ \nu=\ 2, a=1$ and $b=-1$; therefore the other series is taken to be summed [recall that $y=\left(a+bx^{n}\right)^{\frac{\mu}{\nu}} z$ in general]:

$$z=\frac{y}{\sqrt{(1-xx)}}=Ax^{-2}+Bx^{-4}+Cx^{-6}+Dx^{-8}+\text{etc.}$$

and there is produced :

$$A=1,\quad B=\tfrac{2}{3}A,\quad C=\tfrac{4}{5}B,\quad D=\tfrac{6}{7}C,\quad \text{etc.},$$

We may hence therefore deduce :

$$y=\left(\frac{1}{xx}+\frac{2}{3x^{4}}+\frac{2\cdot4}{3\cdot5x^{6}}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7x^{8}}+\text{etc.}\right)\sqrt{(1-xx)}.$$

But on integration there is produced :

$$y=l\,\frac{1-\sqrt{(1-xx)}}{x},$$

which values are in agreement, since each vanishes on putting $x = 1$.

COROLLARY 1

169. Moreover since this series does not converge unless $x>1$, as in this case the formula $\sqrt{(1-xx)}$ becomes imaginary, this series is of no use.

COROLLARIUM: 2

170. If $dy=\frac{dx}{x\sqrt{(xx-1)}}$ is proposed, the same series for *y* emerges multiplied by $\sqrt{-1}$ and there arises

$$y=\left(\frac{1}{xx}+\frac{2}{3x^{4}}+\frac{2\cdot4}{3\cdot5x^{6}}+\frac{2\cdot4\cdot6}{3\cdot5\cdot7x^{8}}+\text{etc.}\right)\sqrt{(xx-1)}.$$

But on putting $x=\frac{1}{u}$ then $dy=\frac{-du}{\sqrt{(1-uu)}}$ and $y=C-Arc.sin.u$ or

$$y=C-Arc.sin.\tfrac{1}{x}$$

where it is required to take $C = 0$, since that series vanishes on putting $x=\infty$, thus so that $y=-Arc.sin.\frac{1}{x}$, which agrees with the above [§ 165] on setting $\frac{1}{x}=u$.

EXAMPLE 3

171. *To integrate the formula* $dy=\frac{dx}{\sqrt{(a+bx^4)}}$ *in this manner by a series.*

Here $m=1, n=4, \mu=1, v=2$ and thus on putting $y=z\sqrt{(a+bx^4)}$ the first resolution gives

$$z=Ax+Bx^5+Cx^9+Dx^{13}+\text{etc.},$$

with

$$A=\frac{1}{a},\quad B=\frac{-3b}{5a}A,\quad C=\frac{-7b}{9a}B,\quad D=\frac{-11b}{13a}C,\quad \text{etc., arising,}$$

thus so that there becomes

$$y=\left(\frac{x}{a}-\frac{3bx^5}{5aa}+\frac{3\cdot7b^2x^9}{5\cdot9a^3}-\frac{3\cdot7\cdot11b^3x^{13}}{5\cdot9\cdot13a^4}+\text{etc.}\right)\sqrt{(a+bx^4)}.$$

But here the other resolution can also be put in place

$$z=Ax^{-3}+Bx^{-7}+Cx^{-11}+Dx^{-15}+\text{etc.}$$

with

$$A=\frac{-1}{b},\quad B=\frac{-3a}{5b}A,\quad C=\frac{-7a}{9b}B,\quad D=\frac{-11a}{13b}C,\quad \text{etc., arising,}$$

from which is deduced :

$$y=-\left(\frac{x}{bx^3}-\frac{3a}{5b^2x^7}+\frac{3\cdot7aa}{5\cdot9b^3x^{11}}-\frac{3\cdot7\cdot11a^3}{5\cdot9\cdot13b^4x^{15}}+\text{etc.}\right)\sqrt{(a+bx^4)},$$

of which series the first one vanishes on putting $x = 0$, now this one on putting $x=\infty$.

COROLLARY 1

172. Hence the difference of these two series is constant, clearly :

$$\left\{\begin{array}{l}+\frac{x}{a}-\frac{3bx^5}{5aa}+\frac{3\cdot7b^2x^9}{5\cdot9a^3}-\frac{3\cdot7\cdot11b^3x^{13}}{5\cdot9\cdot13a^4}+\text{etc.}\\ +\frac{x}{bx^3}-\frac{3a}{5b^2x^7}+\frac{3\cdot7aa}{5\cdot9b^3x^{11}}-\frac{3\cdot7\cdot11a^3}{5\cdot9\cdot13b^4x^{15}}+\text{etc}\end{array}\right\}\sqrt{(a+bx^4)}=\text{Const.}$$

[The editor of the *O.O.* volume points out that these two series do not both converge unless $x=\sqrt[4]{\frac{a}{b}}$, with $a+bx^4$ not equal to zero.]

COROLLARY 2

173. Hence we have on taking these two series together

$$\frac{a+bx^4}{abx^3}-\frac{3}{5}\cdot\frac{a^3+b^3x^{12}}{a^2b^2x^7}+\frac{3\cdot 7}{5\cdot 9}\cdot\frac{a^5+b^5x^{20}}{a^3b^3x^{11}}-\text{etc.}=\frac{C}{\sqrt{\left(a+bx^4\right)}},$$

where, whatever the value attributed to *x*, always the same value is obtained for *C*.

COROLLARY 3

174. Thus if *a* = 1 and *b* = 1, this series taken by $\sqrt{\left(1+x^4\right)}$ remains constant, clearly

$$\left(\frac{1+x^4}{x^3}-\frac{3}{5}\cdot\frac{1+x^{12}}{x^7}+\frac{3\cdot 7}{5\cdot 9}\cdot\frac{1+x^{20}}{x^{11}}-\text{etc.}\right)\sqrt{\left(a+bx^4\right)}=C.$$

Therefore on putting *x* = 1 there arises

$$C=\left(1-\tfrac{3}{5}+\tfrac{3\cdot 7}{5\cdot 9}-\tfrac{3\cdot 7\cdot 11}{5\cdot 9\cdot 13}+\text{etc.}\right)2\sqrt{2}$$

and also whatever the value given to *x*, the series is equal to that value.

COROLLARY 4

175. This last series proceeding with the alternating signs can be transformed easily into another series provided with the same signs, from which it is concluded that the constant

$$C=\left(1+\tfrac{1}{5}+\tfrac{1\cdot 3}{5\cdot 9}+\tfrac{1\cdot 3\cdot 5}{5\cdot 9\cdot 13}+\tfrac{1\cdot 3\cdot 5\cdot 7}{5\cdot 9\cdot 13\cdot 17}+\text{etc.}\right)\sqrt{2},$$

which series converges quickly enough, and it becomes approximately $C=\frac{13}{7}$.

SCHOLIUM

176. Such a method is established from this, that a certain undefined series is produced and the determination of this is derived from the nature of the problem. Moreover the use of this is chiefly concerned with resolving differential equations ; now also in the present arrangement it is often useful to summoned. Also with the help of the same method transcending reciprocal quantities, such as both the sines or cosines of angles, can be expressed by exponential series ; which even if now they are known from elsewhere, yet it is helpful to have the investigation of these explained by integration, since they are able to elicit other things clearly in a similar way.

PROBLEM 14

177. *To convert the magnitude of the exponent $y = a^x$ into a series.*

SOLUTION

With the logarithms taken we have $ly = x\,la$ and by differentiation

$$\frac{dy}{y} = dx\,la \text{ or } \frac{dy}{dx} = y\,la,$$

from which it is required to search for the value of y by means of a series. But since the complete integral appears more general, in our case it is to be noted that on putting $x = 0$ it must be that $y = 1$; whereby this series for y is put in place :

$$y = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{etc.}$$

from which there is mad,

$$\frac{dy}{dx} = A + 2Bx + 3Cx^2 + 4Dx^3 + \text{etc.},$$

with which substituted into the equation $\frac{dy}{dx} - y\,la = 0$ then

$$\left.\begin{aligned} &A + 2Bx + 3Cx^2 + 4Dx^3 + \text{etc.} \\ &-la - Ala - Bla - Cla - Dla \end{aligned}\right\} = 0$$

and hence the coefficients are determined thus :

$$A = la,\quad B = \tfrac{1}{2}Ala,\quad C = \tfrac{1}{3}Bla,\quad D = \tfrac{1}{4}Cla \quad \text{etc.}$$

and thus we follow upon

$$y = a^x = 1 + \frac{xla}{1} + \frac{x^2(la)^2}{1\cdot 2} + \frac{x^3(la)^3}{1\cdot 2\cdot 3} + \frac{x^4(la)^4}{1\cdot 2\cdot 3\cdot 4} + \text{ etc.},$$

which is that most noteworthy series given in the *Introductione* [*Introductio in analysin infinitorum*, Book I, Ch.Vll].

SCHOLIUM

178. For the sines and cosines of angles there must be a reduction to differentials of the second order, from which henceforth the series referring to the integral must be elicited. But since the integration requires a twin two fold determination, a series thus is to be put in place, so that it should be satisfactory to both conditions demanded by the nature of the equation. Now this method also can be extended to other investigations, which thus are changed into algebraic quantities, and here we shall begin with an example of this kind.

PROBLEM 15

179. *This expression* $y=\left(x+\sqrt{(1+xx)}\right)^n$ *is to be changed into a series following the progressive powers of x.*

SOLUTION

Since there is the equation $ly=nl\left(x+\sqrt{(1+xx)}\right)$, then

$$\frac{dy}{y}=\frac{ndx}{\sqrt{(1+xx)}};$$

now the squares are taken towards removing the root sign ; then

$$(1+xx)dy^2=nnyydx^2.$$

The equation is again differentiated with dx constant, as on division by $2dy$ it produces

$$ddy(1+xx)+xdxdy-nnydx^2=0,$$

from which y must be elicited by a series. But first it is evident, if there is put $x = 0$, that $y = 1$ and, if x is infinitely small, $y=(1+x)^n=1+nx$. Therefore such a series is formed :

$$y=1+nx+Ax^2+Bx^3+Dx^4+Dx^5+Ex^6+\text{ etc.},$$

from which it is deduced,

$$\frac{dy}{dx}=n+2Ax+3Bxx+4Cx^3+5Dx^4+6Ex^5+\text{etc.}$$

and

$$\frac{ddy}{dx^2}=2\text{A}+6Bx+12Cxx+20D^3+30Ex^4+\text{etc.}$$

Hence with the substitution made we arrive at :

$$\left.\begin{array}{l} 2A+6Bx+12Cxx+20Dx^3+30Ex^4+42Fx^5+\text{ etc.}\\ \qquad\qquad +\ 2A\quad +\ 6B\quad +12C\quad +20D\\ \qquad +n\ +\ 2A\quad +\ 3B\quad +\ 4C\quad +\ 5D\\ -nn-n^3\ -An^2\ -\ Bn^2\ -\ Cn^2\ -Dn^2 \end{array}\right\}=0$$

and hence the following determinations are derived :

$$A=\tfrac{nn}{2},\ B=\tfrac{n(nn-1)}{2\cdot 3},\ C=\tfrac{A(nn-4)}{3\cdot 4},\ D=\tfrac{B(nn-9)}{4\cdot 5}\ \text{ etc.},$$

thus so that there arises:

$$y=1+nx+\tfrac{nn}{1\cdot 2}x^2+\tfrac{n(nn-1)}{1\cdot 2\cdot 3}x^3+\tfrac{nn(nn-4)}{1\cdot 2\cdot 3\cdot 4}x^4+\tfrac{n(nn-1)(nn-9)x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$
$$+\tfrac{nn(nn-4)(nn-16)x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\tfrac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7+\text{etc.}$$

COROLLARY 1

180. As there is $y=\left(x+\sqrt{(1+xx)}\right)^n$, if we put in place $z=\left(-x+\sqrt{(1+xx)}\right)^n$, a similar series is produced for *z*, in which only *x* is taken negative; hence it is therefore concluded,

$$\tfrac{y+z}{2}=1+\tfrac{nn}{1\cdot 2}x^2+\tfrac{nn(nn-4)}{1\cdot 2\cdot 3\cdot 4}x^4+\tfrac{nn(nn-4)(nn-16)x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\text{etc.}$$

and

$$\tfrac{y-z}{2}=nx+\tfrac{n(nn-1)}{1\cdot 2\cdot 3}x^3+\tfrac{n(nn-1)(nn-9)x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}+\tfrac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7+\text{etc.}$$

COROLLARY 2

181. If there is put in place $x=\sqrt{-1}\cdot\sin.\varphi$, then $\sqrt{(1+xx)}=\cos.\varphi$ and hence

$$y=\left(\cos.\varphi+\sqrt{-1}\cdot\sin.\varphi\right)^n=\cos.n\varphi+\sqrt{-1}\cdot\sin.n\varphi$$

and

$$z=\left(\cos.\varphi-\sqrt{-1}\cdot\sin.\varphi\right)^n=\ \cos.n\varphi-\sqrt{-1}\cdot\sin.n\varphi,$$

from which we deduce :

$$\cos.n\varphi=1-\tfrac{nn}{1\cdot 2}\sin.^2\varphi+\tfrac{nn(nn-4)}{1\cdot 2\cdot 3\cdot 4}\sin.^4\varphi-\tfrac{4nn(nn-4)(nn-16)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\sin.^6\varphi\ +\ \text{etc.},$$

$$sin.n\varphi = n\,sin.\varphi - \frac{nn(nn-1)}{1\cdot2\cdot3}sin.^3\varphi + \frac{nn(nn-1)(nn-9)}{1\cdot2\cdot3\cdot4\cdot5}sin.^5\varphi$$

$$-\frac{4nn(nn-1)(nn-9)(nn-25)}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7}sin.^7\varphi + \text{etc.},$$

COROLLARY 3

182. These series relate to the multiplication of angles, and this series is terminated by having individual values which in the first cases occur only when *n* is even, and in the second cases, when *n* is odd.

PROBLEM 16

183. *For the proposed angle φ both the sine and the cosine are to be expressed by infinite series.*

SOLUTION

Let $y = sin.\varphi$ and $z = cos.\varphi$; then there arises

$$dy = d\varphi\sqrt{(1-yy)} \text{ and } dz = -d\varphi\sqrt{(1-zz)}.$$

The squares are taken:

$$dy^2 = d\varphi^2(1-yy) \text{ and } dz^2 = d\varphi^2(1-zz).$$

this is differentiated on taking $d\varphi$ constant and there comes about

$$ddy = -yd\varphi^2 \text{ et } ddz = -zd\varphi^2.$$

and thus it is required to define both y and z from the same equation. But for $y = sin.\varphi$ it is to be observed, if φ vanishes, to become $y = \varphi$, for $z = cos.\varphi$, if φ vanishes, to become $z = 1 - \frac{1}{2}\varphi\varphi$ or $z = 1 + 0\varphi$. Hence on making

$$y = \varphi + A\varphi^3 + B\varphi^5 + C\varphi^7 + \text{etc.},$$
$$z = 1 + \alpha\varphi^2 + \beta\varphi^4 + \gamma\varphi^6 + \text{etc.}$$

and with the substitution made :

$$\left.\begin{matrix} 2\cdot3A\varphi + 4\cdot5B\varphi^3 + 6\cdot7C\varphi^5 + \text{etc.} \\ +1 \quad + \quad A \quad + \quad B \end{matrix}\right\} = 0$$

and

$$\left.\begin{matrix} 1\cdot2\alpha + 3\cdot4\beta\varphi^2 + 5\cdot6\gamma\varphi^4 + \text{etc.} \\ +1 \quad + \quad \alpha \quad + \quad \beta \end{matrix}\right\} = 0,$$

from which we deduce

$$A=\frac{-1}{2\cdot 3},\quad B=\frac{-A}{4\cdot 5},\quad C=\frac{-B}{6\cdot 7},\quad D=\frac{-C}{8\cdot 9}\quad \text{etc.},$$

$$\alpha=\frac{-1}{1\cdot 2},\quad \beta=\frac{-\alpha}{3\cdot 4},\quad \gamma=\frac{-\beta}{5\cdot 6},\quad \delta=\frac{-\gamma}{7\cdot 8}\quad \text{etc.},$$

from which the most noteworthy series can now be obtained :

$$sin.\varphi=\frac{\varphi}{1}-\frac{\varphi^3}{1\cdot 2\cdot 3}+\frac{\varphi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{\varphi^7}{1\cdot 2\cdots 7}+\text{etc.},$$

$$cos.\varphi=1-\frac{\varphi^2}{1\cdot 2}+\frac{\varphi^4}{1\cdot 2\cdot 3\cdot 4}-\frac{\varphi^6}{1\cdot 2\cdots 6}+\text{etc.},$$

SCHOLION

184. There was no need to descend to differentials of the second order, but from the differentials of the formulas $y=sin.\varphi$ and $z=cos.\varphi$, which are $dy=zd\varphi$ and $dz=-yd\varphi$, the same series are found easily. For with the series produced as before

$$y=\varphi+A\varphi^3+B\varphi^5+C\varphi^7+\text{etc. and } z=1+\alpha\varphi^2+\beta\varphi^4+\gamma\varphi^6+\text{etc.}$$

with the substitution made there is obtained from the former :

$$\left.\begin{array}{l} 1\ +\ 3A\varphi^2+5B\varphi^4+7C\varphi^6+\text{etc.}\\ -1\ -\ \ \alpha\ \ -\ \beta\ \ -\ \gamma \end{array}\right\}=0,$$

and from the latter

$$\left.\begin{array}{l} 2\alpha\varphi+4\beta\varphi^3+6\gamma\varphi^5+\text{etc.}\\ +1\ +\ A\ \ +\ B \end{array}\right\}=0,$$

from which these values are deduced

$$\alpha=\frac{-1}{2},\quad A=\frac{\alpha}{3},\quad \beta=\frac{-A}{4},\quad B=\frac{\beta}{5},\quad \gamma=\frac{-B}{6},\quad C=\frac{\gamma}{7}\ \text{etc.},$$

and thus

$$\alpha=-\frac{1}{2},\quad \beta=+\frac{1}{2\cdot 3\cdot 4},\quad \gamma=-\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6}\quad \text{etc.},$$

$$A=-\frac{1}{2\cdot 3},\quad B=+\frac{1}{2\cdot 3\cdot 4\cdot 5},\quad C=-\frac{1}{2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\quad \text{etc.},$$

which values are in agreement with the preceding. Hence it is understood, how often the two equations are easily set out by series at the same time, as if we should wish to extract one separately.

PROBLEM 17

185. *To express the value of the quantity y, which satisfies this equation*

$$\frac{mdy}{\sqrt{(a+byy)}}=\frac{ndx}{\sqrt{(f+gxx)}}.$$

SOLUTION

The integration of this equation gives rise to

$$\frac{m}{\sqrt{b}}l\left(\sqrt{(a+byy)}+y\sqrt{b}\right)=\frac{n}{\sqrt{g}}l\left(\sqrt{(f+gxx)}+x\sqrt{g}\right)+C,$$

from which we deduce

$$y=\frac{1}{2\sqrt{b}}\left(\frac{\sqrt{(f+gxx)}+x\sqrt{g}}{h}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}-\frac{a}{2\sqrt{b}}\left(\frac{\sqrt{(f+gxx)}-x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}$$

thus on taking the constants *h* and *k*, in order that $hk=f$. Hence we learn, if *x* is taken as vanishing, to become :

$$y=\frac{1}{2\sqrt{b}}\left(\frac{\sqrt{f}+x\sqrt{g}}{h}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}-\frac{a}{2\sqrt{b}}\left(\frac{\sqrt{f}-x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}$$

or

$$y=\frac{1}{2\sqrt{b}}\left(\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}-a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right)+\frac{nx}{2m\sqrt{f}}\left(\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}+a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right)$$

or on putting $y=A+Bx$ then

$$B=\frac{n\sqrt{(AAb+a)}}{m\sqrt{f}},$$

thus in order that the constant *B* can be defined from the constant

$$A=\frac{1}{2\sqrt{b}}\left(\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}-a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right)$$

and in turn

$$\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}=A\sqrt{b}+\sqrt{(a+bAA)}\quad\text{and}\quad a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}=-A\sqrt{b}+\sqrt{(a+bAA)}.$$

Now towards finding the series, the proposed equation with the square taken

$$mm(f+gxx)dy^2 = nn(a+byy)dx^2$$

may be differentiated anew with *dx* taken as constant, so that on division by *2dy* there is produced :

$$mmddy(f+gxx)+mmgxdxdy-nnbydx^2=0.$$

Now for *y* the series is formed

$$y = A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\text{etc.},$$

with which substituted there is had :

$$\left.\begin{array}{rrrrr} & 2mmfC & +6mmfDx & +12mmfEx^2 & +20mmfFx^3+\text{ etc.}\\ & & & +2mmgC & +6mmgD\\ & & +mmgB & +2mmgC & +3mmgD\\ -nnbA & -nnbB & -nnbC & -nnbD & \end{array}\right\}=0.$$

Hence since *A* and *B* may be given, the remaining letters are determined thus :

$$C=\frac{nnb}{2mmf}A,$$

$$D=\frac{nnb-mmg}{2\cdot 3mmf}B,\quad E=\frac{nnb-4mmg}{3\cdot 4mmf}C,$$

$$F=\frac{nnb-9mmg}{4\cdot 5mmf}D,\quad G=\frac{nnb-16mmg}{5\cdot 6mmf}E,$$

$$H=\frac{nnb-25mmg}{6\cdot 7mmf}F,\quad I=\frac{nnb-36mmg}{7\cdot 8mmf}G$$

and thus the series for *y* will be known.

EXAMPLE 1

186. *To express the transcendent function $c^{Arc.\sin.x}$ by a series of successive progressing powers of x.*

Putting $y=c^{Arc.\sin.x}$; then there becomes $ly=lc\cdot Arc.\sin.x$ and $\frac{dy}{y}=\frac{dxlc}{\sqrt{(1-xx)}}$, hence

$$dy^2(1-xx)=yydx^2(lc)^2$$

and on differentiating,

$$dyy(1-xx)-xdxdy-ydx^2(lc)^2=0.$$

Now it may be noted on making x vanishing that the proposed function becomes $y=c^x=1+xlc$; hence the series is formed :

$$y=1+xlc+Ax^2+Bx^3+Cx^4+Dx^5+\text{etc.},$$

with which substituted there is had :

$$\left.\begin{array}{l} 1\cdot 2A+2\cdot 3Bx+3\cdot 4Cx^2+4\cdot 5Dx^3+5\cdot 6Ex^4+\text{etc.}\\ \qquad\qquad -1\cdot 2A\quad -2\cdot 3B\quad -3\cdot 4C\\ \quad -lc\quad -2A\quad -3B\quad -4C\\ -(le)^2-(lc)^3-A(lc)^2\quad -B(lc)^2\quad -C(lc)^2 \end{array}\right\}=0,$$

from which the remaining coefficient are determined thus :

$$A=\frac{(lc)^2}{1\cdot 2},\qquad C=\frac{4+(lc)^2}{3\cdot 4}A,\quad E=\frac{16+(lc)^2}{5\cdot 6}C\ \text{etc.},$$
$$B=\frac{\left(1+(lc)^2\right)lc}{2\cdot 3},\quad D=\frac{9+(lc)^2}{4\cdot 5}B,\quad F=\frac{25+(lc)^2}{6.7}D\ \text{etc.}$$

For the sake of brevity let $lc=\gamma$ and there becomes :

$$c^{Arc.sin.x}=1+\gamma x+\frac{\gamma\gamma}{1\cdot 2}x^2+\frac{\gamma(1+\gamma\gamma)}{1\cdot 2\cdot 3}x^3+\frac{\gamma(4+\gamma\gamma)}{1\cdot 2\cdot 3\cdot 4}x^4$$
$$+\frac{\gamma(1+\gamma\gamma)(9+\gamma\gamma)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5+\frac{\gamma(4+\gamma\gamma)(16+\gamma\gamma)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}x^6+\text{etc.}$$

EXAMPLE 2

187. *On putting $x=sin.\varphi$, to find the series of successive progressing powers of x that expresses the sine of the angle $n\varphi$.*

There is put in place $y=sin.n\varphi$ and it may be noted with vanishing φ to become $x=\varphi$ and $y=n\varphi=nx$, that is $y=0+nx$, which is the beginning for the series sought. But now there is :

$$d\varphi=\frac{dx}{\sqrt{(1-xx)}}\quad\text{and}\quad nd\varphi=\frac{dy}{\sqrt{(1-yy)}}.$$

Hence

$$\frac{dy}{\sqrt{(1-yy)}}=\frac{ndx}{\sqrt{(1-xx)}}$$

and with the squares taken

$$(1-xx)dy^2 = nndx^2(1-yy),$$

hence

$$ddy(1-xx) - xdxdy + nnydx^2 = 0.$$

Whereby this series is formed :

$$y = nx + Ax^3 + Bx^5 + Cx^7 + Dx^9 + \text{etc.};$$

with which put in place there is had :

$$\left.\begin{array}{llllllll} 2\cdot 3Ax & +4\cdot 5Bx^3 & + & 6\cdot 7Cx^5 & + & 8\cdot 9Dx^7 & + \text{etc.} \\ & -2\cdot 3A & - & 4\cdot 5B & - & 6\cdot 7C & \\ -n & - \ \ 3A & - & 5B & - & 7C & \\ +n^3 & + \ \ nnA & + & nnB & + & nnC & \end{array}\right\} = 0,$$

from which these values are deduced

$$A = \tfrac{-n(nn-1)}{2\cdot 3},\ B = \tfrac{-(nn-9)A}{4\cdot 5},\ C = \tfrac{-(nn-25)}{6\cdot 7}B \ \text{ etc. },$$

thus in order that

$$y = nx - \tfrac{n(nn-1)}{1\cdot 2\cdot 3}x^3 + \tfrac{n(nn-1)(nn-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5 - \tfrac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \text{ etc.}$$

or

$$sin.n\varphi = n\,sin\varphi - \tfrac{n(nn-1)}{1\cdot 2\cdot 3}sin^3\varphi + \tfrac{n(nn-1)(nn-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5}sin^5\varphi - \tfrac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}sin^7\varphi + \text{ etc.}$$

SCHOLION

188. Since this series is terminated only in cases in which *n* is an odd number, it is to be observed that for even numbers the series can be expressed conveniently by a product of $sin.\varphi$ with another series of the progressive successive powers of the cosine of φ. Towards finding that we put $cos.\varphi = u$ and there becomes:

$$sin.n\varphi = z\,sin.\varphi = z\sqrt{(1-uu)},$$

from which on account of

$$d\varphi = -\tfrac{du}{\sqrt{(1-uu)}}$$

then on differentiating :

$$-\frac{ndu\cos.n\varphi}{\sqrt{(1-uu)}}=dz\sqrt{(1-uu)}-\frac{zdu}{\sqrt{(1-uu)}}$$

or

$$-ndu\cos.n\varphi=dz(1-uu)-zdu,$$

which on taking *du* constant on differentiating again gives

$$-\frac{nndu^2\sin.n\varphi}{\sqrt{(1-uu)}}=ddz(1-uu)-3ududz-zdu^2=-nnzdu^2$$

on account of $\frac{\sin.n\varphi}{\sqrt{(1-uu)}}=z$.

On account of which the series sought for $z=\frac{\sin.n\varphi}{\sin.\varphi}$ must be elicited from the equation :

$$ddz(1-uu)-3ududz-zdu^2+nnzdu^2=0,$$

where it is to be noted, since $u=\cos.\varphi$, with vanishing *u*, in which case there is made $\varphi=90^\circ$, there becomes either $z=0$, if n is an even number, or $z=1$, if $n=4\alpha+1$, or $z=-1$, if $n=4\alpha-1$. Which individual cases are to be worked out separately; and for which the beginning of each series becomes apparent, if there is put $\varphi=90^\circ-\omega$ and on vanishing there becomes

$$u=\cos.\varphi=\omega,\ \sin.\varphi=1,\ \sin.n\varphi=\sin.(90^\circ\cdot n-n\omega)=z.$$

Now for the individual cases [*i.e.* in the four quadrants]:

I. if $n=4\alpha$, then $z=-\sin.n\omega=-nu$
II. if $n=4\alpha+1$, then $z=\cos.n\omega=1$
III. if $n=4\alpha+2$, then $z=\sin.n\omega=+nu$
IV. if $n=4\alpha+3$, then $z=-\cos.n\omega=-1$,

from which the known series are now deduced well enough.

CAPUT III

DE INTEGRATIONE FORMULARUM DIFFERENTIALIUM PER SERIES INFINITAS

PROBLEMA 12

126. *Si X fuerit functio rationalis fracta ipsius x, formulae differentialis* $dy = Xdx$ *integrale per seriem infinitam exhibere.*

SOLUTIO

Cum X sit functio rationalis fracta, eius valor semper ita evolvi potest;
ut fiat

$$X = Ax^m + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n} + Ex^{m+4n} + \text{etc.},$$

ubi coefficientes A, B, C etc. seriem recurrentem constituent ex denominatore fractionis determinandam. Multiplicentur ergo singuli termini per dx et integrentur, quo facto integrale y per sequentem seriem exprimetur

$$y = \frac{Ax^{m+1}}{m+1} + \frac{Bx^{m+n+1}}{m+n+1} + \frac{Cx^{m+2n+1}}{m+2n+1} + \text{etc.} + \text{Const.};$$

ubi si in serie pro X occurrat huiusmodi terminus $\frac{M}{x}$ inde in integrale ingredietur terminus $M\,lx$.

SCHOLION

127. Cum integrale $\int Xdx$, nisi sit algebraicum, per logarithmos et angulos exprimatur, hinc valores logarithmorum et angulorum per series infinitas exhiberi possunt. Cuiusmodi series cum iam in *Introductione* [L. Euleri *Introductio in analysin infinitorum,* t.I cap. VI-VIII] plures sint traditae, non solum eaedem, sed etiam infinitae aliae hic per integrationem erui possunt. Hoc exemplis declarasse iuvabit, ubi potissimnm eiusmodi formulas evolvemus, in quibus denominator est binomium; tum vero etiam casus aliquot denominatore trinomio vel multinomio praeditos contemplabimur. Imprimis autem eiusmodi eligemus, quibus fractio in aliam, cuius denominator est binomius, transmutari potest.

EXEMPLUM 1

128. *Formulam differentialem* $\frac{ax}{a+x}$; *per seriem integrare.*

Sit $y = \int \frac{dx}{a+x}$; erit $y = l(a+x) + \text{Const.}$,unde integrali ita determinato, ut evanescat posito $x = 0$, erit $y = l(a+x) - la.$ Iam cum sit

$$\frac{1}{a+x} = \frac{1}{a} - \frac{x}{a^2} + \frac{xx}{a^3} - \frac{x^3}{4a^4} + \frac{x^4}{5a^5} - \text{etc.},,$$

erit eadem lege integrale definiendo

$$y = \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \text{etc.},$$

unde colligemus, uti quidem iam constat,

$$l(a+x) = la + \frac{x}{a} - \frac{x^2}{2a^2} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \text{etc.},$$

COROLLARIUM 1

129. Si capiamus x negativum, ut sit $dy = \frac{-dx}{a-x}$, eodem modo patebit esse

$$l(a-x) = la - \frac{x}{a} - \frac{x^2}{2a^2} - \frac{x^3}{3a^3} - \frac{x^4}{4a^4} - \text{etc.},$$

hisque combinandis

$$l(aa-xx) = 2la - \frac{xx}{aa} - \frac{x^4}{2a^4} - \frac{x^6}{3a^6} - \frac{x^8}{4a^8} - \text{etc.},$$

et

$$l\,\frac{a+x}{a-x} = \frac{2x}{a} + \frac{2x^3}{3a^3} + \frac{2x^5}{5a^5} + \frac{2x^7}{7a^7} + \text{etc.}$$

COROLLARIUM 2

130. Hae posteriores series eruuntur per integrationem formularum

$$\frac{-2xdx}{aa-xx} = -2xdx\left(\frac{1}{aa} + \frac{xx}{a^4} + \frac{x^4}{a^6} + \text{etc.}\right),$$

et

$$\frac{2adx}{aa-xx} = 2adx\left(\frac{1}{aa} + \frac{xx}{a^4} + \frac{x^4}{a^6} + \text{etc.}\right).$$

Est autem

$$\int \frac{-2xdx}{aa-xx} = l(aa-xx) - laa \quad \text{et} \quad \int \frac{2adx}{aa-xx} = l\,\frac{a+x}{a-x},$$

ita ut iam his formulis per series integrandis supersedere possimus.

EXEMPLUM 2

131. *Formulam differentialem* $\int \frac{adx}{aa+xx}$ *per seriem integrare.*

Sit $dy = \frac{adx}{aa+xx}$, et cum sit $y = \text{Arc. tang.}\frac{x}{a}$, idem angulus series infinita exprimetur. Quia enim habemus

$$\frac{a}{aa+xx} = \frac{1}{a} - \frac{xx}{a^3} + \frac{x^4}{a^5} - \frac{x^6}{a^7} + \frac{x^8}{a^9} - \text{etc.},$$

erit integrando

$$y = \text{Arc. tang.}\frac{x}{a} = \frac{x}{a} - \frac{x^3}{3a^3} + \frac{x^5}{5a^5} - \frac{x^7}{7a^7} + \text{etc.}$$

EXEMPLUM 3

132. *Integralia harum formularum* $\int \frac{dx}{1+x^3}$ *et* $\int \frac{xdx}{1+x^3}$ *per series exprimere.*

Cum sit

$$\frac{1}{1+x^3} = 1 - x^3 + x^6 - x^9 + x^{12} - \text{etc.},$$

erit

$$\int \frac{dx}{1+x^3} = x - \frac{1}{4}x^4 + \frac{1}{7}x^7 - \frac{1}{10}x^{10} + \frac{1}{13}x^{13} - \text{etc.}$$

et

$$\int \frac{xdx}{1+x^3} = \frac{1}{2}x^2 - \frac{1}{5}x^5 + \frac{1}{8}x^8 - \frac{1}{11}x^{11} + \frac{1}{14}x^{14} - \text{etc.}$$

Verum per § 77 habemus per logarithmos et angulos

$$\int \frac{dx}{1+x^3} = \frac{1}{3}l(1+x) - \frac{2}{3}cos.\frac{\pi}{3}\,l\sqrt{\left(1-2x\,cos.\frac{\pi}{3}+xx\right)}$$

$$+\frac{2}{3}sin.\frac{\pi}{3}\,\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{3}}{1-x\,cos.\frac{\pi}{3}},$$

et

$$\int \frac{xdx}{1+x^3} = -\frac{1}{3}l(1+x) - \frac{2}{3}cos.\frac{\pi}{3}\,l\sqrt{\left(1-2x\,cos.\frac{\pi}{3}+xx\right)}$$

$$+\frac{2}{3}sin.\frac{\pi}{3}\,\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{3}}{1-x\,cos.\frac{\pi}{3}},$$

At est $cos.\frac{\pi}{3} = \frac{1}{2}$, $cos.\frac{2\pi}{3} = -\frac{1}{2}$, $sin.\frac{\pi}{3} = \frac{\sqrt{3}}{2}$, $sin.\frac{2\pi}{3} = \frac{\sqrt{3}}{2}$, unde fit

$$\int \frac{dx}{1+x^3} = \frac{1}{3}l(1+x) - \frac{1}{3}l\sqrt{(1-x+xx)} + \frac{1}{\sqrt{3}}\text{Arc.tang.}\frac{x\sqrt{3}}{2-x},$$

$$\int \frac{xdx}{1+x^3} = -\frac{1}{3}l(1+x) + \frac{1}{3}l\sqrt{(1-x+xx)} + \frac{1}{\sqrt{3}}\text{Arc.tang.}\frac{x\sqrt{3}}{2-x}$$

integralibus ut seriebus ita sumtis, ut evanescant posito $x = 0$.

COROLLARIUM 1

133. His igitur seriebus additis prodit

$$\frac{2}{\sqrt{3}}\text{Arc.tang.}\frac{x\sqrt{3}}{2-x}=x+\frac{1}{2}xx-\frac{1}{4}x^4-\frac{1}{5}x^5+\frac{1}{7}x^7+\frac{1}{8}x^8-\frac{1}{10}x^{10}-\frac{1}{11}x^{11}+\text{etc.},$$

subtracta autem posteriori a priori fit

$$\frac{2}{3}l\frac{1+x}{\sqrt{(1-x+xx)}}=x-\frac{1}{2}x^2-\frac{1}{4}x^4+\frac{1}{5}x^5+\frac{1}{7}x^7-\frac{1}{8}x^8-\frac{1}{10}x^{10}+\frac{1}{11}x^{11}+\text{etc.},$$

cuius valor etiam est

$$=\frac{1}{3}l\frac{(1+x)^2}{1-x+xx}=\frac{1}{3}l\frac{(1+x)^3}{1+x^3}.$$

COROLLARIUM 2

134. Cum sit

$$\int\frac{xxdx}{1+x^3}=\frac{1}{3}l\left(1+x^3\right),$$

erit eodem modo

$$\frac{1}{3}l\left(1+x^3\right)=\frac{1}{3}x^3-\frac{1}{6}x^6+\frac{1}{9}x^9-\frac{1}{12}x^{12}+\text{etc.},$$

qua serie illis adiecta omnes potestates ipsius x occurrent.

EXEMPLUM 4

135. *Integrale hoc* $y=\int\frac{(1+xx)dx}{1+x^4}$ *per seriem exprimere.*

Cum sit

$$\frac{1}{1+x^4}=1-x^4+x^8-x^{12}+x^{16}-\text{etc.},$$

erit

$$y=x+\frac{1}{3}x^3-\frac{1}{5}x^5-\frac{1}{7}x^7+\frac{1}{11}x^{11}-\frac{1}{13}x^{13}-\frac{1}{15}x^{15}+\text{etc.}$$

Verum per §82, ubi m = 1 et n = 4, posito $\frac{\pi}{4}=\omega$ fit integrale idem

$$y=\sin.\omega\ \text{Arc.tang.}\frac{x\sin.\omega}{1-x\cos.\omega}+\sin.3\omega\ \text{Arc.tang.}\frac{x\sin.3\omega}{1-x\cos.3\omega};$$

at ob $\frac{\pi}{4}=\omega=45°$ est $\sin.\omega=\frac{1}{\sqrt{2}}$, $\cos.\omega=\frac{1}{\sqrt{2}}$, $\sin.3\omega=\frac{1}{\sqrt{2}}$, $\cos.3\omega=\frac{-1}{\sqrt{2}}$;
habebimus

$$y=\frac{1}{\sqrt{2}}\text{Arc.tang.}\frac{x}{\sqrt{2}-x}+\frac{1}{\sqrt{2}}\text{Arc.tang.}\frac{x}{\sqrt{2}+x}=\frac{1}{\sqrt{2}}\text{Arc.tang.}\frac{x\sqrt{2}}{1-xx}.$$

EXEMPLUM 5

136. *Integrale hoc* $y=\int\frac{(1+x^4)dx}{1+x^6}$ *per seriem exprimere.*

Cum sit
$$\frac{1}{1+x^6}=1-x^6+x^{12}-x^{18}+x^{24}-\text{etc.},$$
erit
$$y=x+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7-\tfrac{1}{11}x^{11}+\tfrac{1}{13}x^{13}+\tfrac{1}{17}x^{17}-\text{etc.}$$

At per §82, ubi $m=1, n=6$ et $\omega=\frac{\pi}{6}=30°$, est

$$y=\tfrac{2}{3}\sin.\omega\ \text{Arc.tang.}\frac{x\sin.\omega}{1-x\cos.\omega}+\tfrac{2}{3}\sin.3\omega\ \text{Arc.tang.}\frac{x\sin.3\omega}{1-x\cos.3\omega};$$

$$+\tfrac{2}{3}\sin.5\omega\ \text{Arc.tang.}\frac{x\sin.5\omega}{1-x\cos.5\omega};$$

est vero

$\sin.\omega=\frac{1}{2},\ \cos.\omega=\frac{\sqrt{3}}{2}, \sin.3\omega=1,\ \cos.3\omega=0, \sin.5\omega=\frac{1}{2},\ \cos.5\omega=-\frac{\sqrt{3}}{2}$, ergo

$$y=\tfrac{1}{3}\text{Arc.tang.}\frac{x}{2-x\sqrt{3}}+\tfrac{2}{3}\text{Arc.tang.}x+\tfrac{1}{3}\text{Arc.tang.}\frac{x}{2+x\sqrt{3}}$$
seu
$$y=\tfrac{1}{3}\text{Arc.tang.}\frac{x}{1-xx}+\tfrac{2}{3}\text{Arc.tang.}x=\tfrac{1}{3}\text{Arc.tang.}\frac{3x(1-xx)}{1-4xx+x^4}.$$

COROLLARIUM 1

137. Sit
$$z=\int\frac{xxdx}{1+x^6}=\tfrac{1}{3}x^3-\tfrac{1}{9}x^9+\tfrac{1}{15}x^{15}-\tfrac{1}{21}x^{21}+\text{etc.},$$
at facto $x^3=u$ est
$$z=\tfrac{1}{3}\int\frac{xxdx}{1+x^6}=\tfrac{1}{3}\text{Arc.tang.}u=\tfrac{1}{3}\text{Arc.tang.}x^3.$$

Hinc series huiusmodi mixta formatur

$$x+\tfrac{n}{3}x^3+\tfrac{1}{5}x^5-\tfrac{1}{7}x^7-\tfrac{n}{9}x^9-\tfrac{1}{11}x^{11}+\tfrac{1}{13}x^{13}+\tfrac{n}{15}x^{15}+\tfrac{1}{17}x^{17}-\text{etc.},$$
cuius summa est
$$\tfrac{1}{3}\text{Arc.tang.}\frac{3x(1-xx)}{1-4xx+x^4}+\tfrac{n}{3}\text{Arc.tang.}x^3.$$

COROLLARIUM 2

138. Si hic capiatur $n=-1$, binos angulos in unum colligendo fit

$$\tfrac{1}{3}\text{Arc.tang.}\tfrac{3x(1-xx)}{1-4xx+x^4}-\tfrac{1}{3}\text{Arc.tang.}x^3=\tfrac{1}{3}\text{Arc.tang.}\tfrac{3x-4x^3+4x^5-x^7}{1-4xx+4x^4-3x^6}$$

quae fractio per $1-xx+x^4$ dividendo reducitur ad $\frac{3x-x^3}{1-3xx}$ quae est tangens tripli anguli x pro tangente habentis, ita ut sit

$$\tfrac{1}{3}\text{Arc.tang.}\tfrac{3x-x^3}{1-3xx}=\text{Arc.tang.}x\,,$$

quod idem series inventa manifesto indicat.

EXEMPLUM 6

139. *Hanc formulam* $dy=\frac{\left(x^{m-1}+x^{n-m-1}\right)dx}{1+x^n}$ *per seriem integrare.*

Ob

$$\frac{1}{1+x^n}=1-x^n+x^{2n}-x^{3n}+x^{4n}-\text{etc.}$$

habebitur

$$y=\frac{x^m}{m}+\frac{x^{n-m}}{n-m}-\frac{x^{n+m}}{n+m}-\frac{x^{2n-m}}{2n-m}+\frac{x^{2n+m}}{2n+m}+\frac{x^{3n-m}}{3n-m}-\text{etc.}$$

Haec ergo series per § 82 aggregatum aliquot arcuum circularium exprimit, quos ibi videre licet.

COROLLARIUM

140. Eodem proposita formula $dz=\frac{\left(x^{m-1}-x^{n-m-1}\right)dx}{1-x^n}$ ob

$$\frac{1}{1-x^n}=1+x^n+x^{2n}+x^{3n}+x^{4n}+\text{etc.}$$

invenitur

$$z=\frac{x^m}{m}-\frac{x^{n-m}}{n-m}+\frac{x^{n+m}}{n+m}-\frac{x^{2n-m}}{2n-m}+\frac{x^{2n+m}}{2n+m}-\frac{x^{3n-m}}{3n-m}+\text{etc.}$$

euius valor §84 est exhibitus.

EXEMPLUM 7

141. *Hanc formulam* $dy=\frac{(1+2x)dx}{1+x+xx}$ *per seriem integrare.*

Primo integrale est manifesto $y=l\left(1+x+xx\right)$; ut autem in seriem convertatur, multiplicetur numerator et denominator per $1-x$, ut fiat

$$dy = \frac{(1+x-2xx)dx}{1-x^3}.$$

Cum nunc sit

$$\frac{1}{1-x^3} = 1 + x^3 + x^6 + x^9 + x^{12} + \text{etc.}$$

erit integrando

$$y = x + \frac{x^2}{2} - \frac{2x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} - \frac{2x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} - \frac{2x^9}{9} + \text{etc.}$$

COROLLARIUM 1

142. Eodem modo inveniri potest $y = l\left(1 + x + xx + x^3\right)$ per seriem. Cum enim fiat y $+l(1-x) = l\left(1-x^4\right)$, erit

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} + \frac{x^8}{8} + \frac{x^9}{9} + \frac{x^{10}}{10} + \text{etc.}$$
$$-x^4 \qquad\qquad\qquad -\frac{x^8}{2}$$

sive

$$y = x + \frac{x^2}{2} + \frac{x^3}{3} - \frac{3x^4}{4} + \frac{x^5}{5} + \frac{x^6}{6} + \frac{x^7}{7} - \frac{3x^8}{8} + \frac{x^9}{9} + \text{etc.}$$

COROLLARIUM 2

143. At fractio $\frac{1+2x}{1+x+xx}$ per seriem recurrentem evoluta dat

$$1 + x - 2xx + \mathrm{x}^3 + x^4 - 2x^5 + x^6 + x^7 - 2x^8 + \text{etc.},$$

unde per integrationem eadem series obtinetur quae ante.

EXEMPLUM 8

144. *Hanc formulam* $dy = \frac{dx}{1-2x\cos.\zeta + xx}$ *per seriem integrare.*

Per § 64, ubi $A = 1, B = 0, a = 1$ et $b = 1,$ est huius formulae integrale

$$y = \frac{1}{\sin.\zeta}\,\text{Arc.tang.}\frac{x\sin.\zeta}{1-x\cos.\zeta}.$$

At per seriem recurrentem reperimus

$$\frac{1}{1-2x\cos.\zeta+xx} = 1 + 2x\cos.\zeta + \left(4\cos^2.\zeta - 1\right)xx + \left(8\cos^3.\zeta - 4\cos.\zeta\right)x^3$$
$$+\left(16\cos^4.\zeta - 12\cos^2.\zeta + 1\right)x^4 + \left(32\cos^5.\zeta - 32\cos^3.\zeta + 6\cos.\zeta\right)x^5 + \text{etc.},$$

qua serie per dx multiplicata et integrata obtinetur quaesitum. Potestatibus autem ipsius $\cos.\zeta$ in cosinus angulorum multiplorum conversis reperitur

$$y = x + \tfrac{1}{2}xx\left(2cos.\zeta\right) + \tfrac{1}{3}x^3\left(2cos.2\zeta + 1\right) + \tfrac{1}{4}x^4\left(2cos.3\zeta + 2cos.\zeta\right)$$
$$+ \tfrac{1}{5}x^5\left(2cos.4\zeta + 2cos.2\zeta + 1\right) + \tfrac{1}{6}x^6\left(2cos.5\zeta + 2cos.3\zeta + 2cos.\zeta\right) + \text{etc.},$$

COROLLARIUM 1

145. Si ponatur

$$dz = \frac{(1 - x\,cos.\zeta)dx}{1 - 2x\,cos.\zeta + xx},$$

erit per § 63 $A = 1, B = -cos.\zeta, a = 1$ et $b = 1$ ideoque

$$z = -cos.\zeta\; l\sqrt{(1 - 2x\,cos.\zeta + xx)} + sin.\zeta\,\text{Arc. tang}\ \frac{x\,sin.\zeta}{1 - x\,cos.\zeta}\ ;$$

at per seriem ob

$$\frac{1 - x\,cos.\zeta}{1 - 2x\,cos.\zeta + xx} = 1 + x\,cos.\zeta + x^2\,cos.2\zeta + x^3\,cos.3\zeta + x^4\,cos.4\zeta + \text{etc.}$$

fit

$$z = x + \tfrac{1}{2}xx\,cos.\zeta + \tfrac{1}{3}x^3\,cos.2\zeta + \tfrac{1}{4}x^4\,cos.3\zeta + \tfrac{1}{5}x^5\,cos.4\zeta + \text{etc.}$$

COROLLARIUM 2

146. At quia

$$dz = \frac{dx\left(-x\,cos.\zeta + cos^2.\zeta + sin^2.\zeta\right)}{1 - 2x\,cos.\zeta + xx},$$

erit

$$z = -cos.\zeta\; l\sqrt{(1 - 2x\,cos.\zeta + xx)} + sin^2.\zeta\int\frac{dx}{1 - 2x\,cos.\zeta + xx}.$$

Hinc ergo pro

$$y = \int\frac{dx}{1 - 2x\,cos.\zeta + xx}$$

alia reperitur series infinita cum logarithmo connexa, scilicet

$$y = \frac{cos.\zeta}{sin^2.\zeta}\, l\sqrt{(1 - 2x\,cos.\zeta + xx)}$$
$$+ \frac{1}{sin^2.\zeta}\left(x + \tfrac{1}{2}xx\,cos.\zeta + \tfrac{1}{3}x^3\,cos.2\zeta + \tfrac{1}{4}x^4\,cos.3\zeta + \text{etc.}\right).$$

PROBLEMA 12[a]

147. *Formulam differentialem irrationalem* $dy = x^{m-1}dx\left(a + bx^n\right)$ *per seriem infinitam integrare.*

SOLUTIO

Sit $a^{\frac{\mu}{\nu}} = c$; erit

$$dy = cx^{m-1}dx\left(1+\tfrac{b}{a}x^n\right)^{\frac{\mu}{\nu}},$$

ubi quidem assumimus c non esse quantitatem imaginariam. Cum igitur sit

$$\left(1+\tfrac{b}{a}x^n\right)^{\frac{\mu}{\nu}} = 1+\frac{\mu b}{1\nu\cdot a}x^n+\frac{\mu(\mu-\nu)b^2}{1\nu\cdot 2\nu\cdot aa}x^{2n}+\frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu\cdot 2\nu\cdot 3\nu\cdot a^3}x^{3n}+\text{etc.},$$

erit integrando

$$y = c\left(\frac{x^m}{m}+\frac{\mu b}{1\nu\cdot a}\frac{x^{m+n}}{m+n}+\frac{\mu(\mu-\nu)bb}{1\nu\cdot 2\nu\cdot aa}\frac{x^{m+2n}}{m+2n}+\frac{\mu(\mu-\nu)(\mu-2\nu)b^3}{1\nu\cdot 2\nu\cdot 3\nu\cdot a^3}\frac{x^{m+3n}}{m+3n}+\text{etc.}\right),$$

quae series in infinitum excurrit, nisi $\frac{\mu}{\nu}$ sit numerus integer positivus. Sin autem casu, quo ν numerus par, a fuerit quantitas negativa, expressio nostra ita est repraesentanda

$$dy = x^{m-1}dx\left(bx^n-a\right)^{\frac{\mu}{\nu}} = b^{\frac{\mu}{\nu}}x^{m+\frac{\mu n}{\nu}-1}dx\left(1-\tfrac{a}{b}x^{-n}\right)^{\frac{\mu}{\nu}}.$$

Cum igitur sit

$$\left(1-\tfrac{a}{b}x^{-n}\right)^{\frac{\mu}{\nu}} = 1-\frac{\mu b}{1\nu\cdot a}x^{-n}+\frac{\mu(\mu-\nu)a^2}{1\nu\cdot 2\nu\cdot b^2}x^{-2n}-\frac{\mu(\mu-\nu)(\mu-2\nu)a^3}{1\nu\cdot 2\nu\cdot 3\nu\cdot b^3}x^{-3n}+\text{etc.},$$

erit integrando

$$y = b^{\frac{\mu}{\nu}}\left(\frac{\nu x^{m+\frac{\mu n}{\nu}}}{m\nu+\mu n}-\frac{\mu a}{1\nu\cdot b}\cdot\frac{\nu x^{m+\frac{(\mu-\nu)n}{\nu}}}{m\nu+(\mu-\nu)n}+\frac{\mu(\mu-\nu)a^2}{1\nu\cdot 2\nu\cdot b^2}\cdot\frac{\nu x^{m+\frac{(\mu-2\nu)n}{\nu}}}{m\nu+(\mu-2\nu)n}-\text{etc.}\right).$$

Si a et b sint numeri positivi, utraque evolutione uti licet.

EXEMPLUM 1

148. *Formulam* $dy = \frac{dx}{\sqrt{(1-xx)}}$ *per seriem integrare.*

Primo ex superioribus patet esse y = Arc. sin. x, qui ergo angulus etiam per seriem infinitam exprimetur. Cum enim sit

$$\frac{1}{\sqrt{(1-xx)}} = 1+\frac{1}{2}x^2+\frac{1\cdot 3}{2\cdot 4}x^4+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x^8+\text{etc.},$$

erit

$$y = x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{x^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^7}{7}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{x^9}{9}+\text{etc.},$$

utroque valore ita definito, ut evanescat posito $x = 0$.

COROLLARIUM 1

149. Si ergo sit $x=1$, ob $Arc.sin.1=\frac{\pi}{2}$ erit

$$\frac{\pi}{2}=1+\frac{1}{2\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 7}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 9}+\text{ etc.},$$

At si ponatur $x=\frac{1}{2}$, ob $Arc.sin.\frac{1}{2}=30°=\frac{\pi}{6}$ erit

$$\frac{\pi}{6}=\frac{1}{2}+\frac{1}{2\cdot 2^3\cdot 3}+\frac{1\cdot 3}{2\cdot 4\cdot 2^5\cdot 5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 2^7\cdot 7}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8\cdot 2^9\cdot 9}+\text{ etc.},$$

cuius seriei decem termini additi dant 0,52359877, cuius sextuplum 3,14159262 tantum in octava figura a veritate discrepat.

COROLLARIUM 2

150. Proposita hac formula $dy=\frac{dx}{\sqrt{(x-xx)}}$ posito $x=uu$ fit

$$dy=\frac{2udu}{\sqrt{(uu-u^4)}}=\frac{2du}{\sqrt{(1-uu)}},$$

ergo

$$y=2Arc.sin.u=2\,Arc.sin.\sqrt{x}\;;$$

tum vero per seriem erit

$$y=2\left(u+\frac{1}{2}\cdot\frac{u^3}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{u^5}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{u^7}{7}+\frac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\cdot\frac{u^9}{9}+\text{ etc.}\right)$$

seu

$$y=2\left(1+\frac{1}{2}\cdot\frac{x}{3}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{xx}{5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{x^3}{7}+\text{etc.}\right)\sqrt{x}.$$

EXEMPLUM 2

151. *Formulam $dy=dx\sqrt{(2ax-xx)}$ per seriem integrare.*

Posito x=uu fit $dy=2uudu\sqrt{(2a-uu)}$ at per reductionem I (§ 118) est $n=2,\ m=1,\ a=2a,\ b=-1,\ \mu=1,\ \nu=2,$ unde

$$\int uudu\sqrt{(2a-uu)}=-\frac{1}{4}u(2a-uu)^{\frac{3}{2}}+\frac{1}{2}a\int du\sqrt{(2a-uu)},$$

et per III sumendo $m=1,\quad a=2a,\quad b=-1,\quad n=2,\quad \mu=-1,\quad \nu=2$ fit

$$\int du\sqrt{(2a-uu)} = \tfrac{1}{2}u\sqrt{(2a-uu)} + a\int\frac{du}{\sqrt{(2a-uu)}};$$

at est

$$\int\frac{du}{\sqrt{(2a-uu)}} = Arc.sin\frac{u}{\sqrt{2a}} = Arc.sin\frac{\sqrt{x}}{\sqrt{2a}}$$

ideoque

$$\int uudu\sqrt{(2a-uu)} = -\tfrac{1}{4}u(2a-uu)^{\frac{3}{2}} + \tfrac{1}{4}au\sqrt{(2a-uu)} + \tfrac{1}{2}aaArc.sin\frac{\sqrt{x}}{\sqrt{2a}}$$

$$= \tfrac{1}{4}u(uu-a)\sqrt{(2a-uu)} + \tfrac{1}{2}aaArc.sin\frac{\sqrt{x}}{\sqrt{2a}}.$$

Ergo

$$y = \tfrac{1}{2}(x-a)\sqrt{(2ax-xx)} + aaArc.sin\frac{\sqrt{x}}{\sqrt{2a}}.$$

Pro serie autem invenienda est

$$dy = dx\sqrt{2ax}\left(1-\tfrac{x}{2a}\right)^{\frac{1}{2}}$$
$$= x^{\frac{1}{2}}dx\left(1-\tfrac{1}{2}\cdot\tfrac{x}{2a}-\tfrac{1\cdot 1}{2\cdot 4}\cdot\tfrac{xx}{4aa}-\tfrac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\tfrac{x^3}{8a^3}-\text{etc.}\right)\sqrt{2a}$$

hincque integrando

$$y = \left(\tfrac{2}{3}x^{\frac{3}{2}}-\tfrac{1}{2}\cdot\tfrac{2x^{\frac{5}{2}}}{5\cdot 2a}-\tfrac{1\cdot 1}{2\cdot 4}\cdot\tfrac{2x^{\frac{7}{2}}}{7\cdot 4aa}-\tfrac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\tfrac{2x^{\frac{9}{2}}}{9\cdot 8a^3}-\text{etc.}\right)\sqrt{2a}$$

seu

$$y = \left(\tfrac{x}{3}-\tfrac{1}{2}\cdot\tfrac{x^2}{5\cdot 2a}-\tfrac{1\cdot 1}{2\cdot 4}\cdot\tfrac{x^3}{7\cdot 4aa}-\tfrac{1\cdot 1\cdot 3}{2\cdot 4\cdot 6}\cdot\tfrac{x^4}{9\cdot 8a^3}-\text{etc.}\right)2\sqrt{2ax}.$$

COROLLARIUM

152. Integrale facilius inveniri potest ponendo $x = a - v$, unde fit

$$dy = -dv\sqrt{(aa-vv)}$$

et per reductionem III [§ 118]

$$\int dv\sqrt{(aa-vv)} = \tfrac{1}{2}v\sqrt{(aa-vv)} + \tfrac{1}{2}aa\int\frac{dv}{\sqrt{(aa-vv)}}$$

hinc

$$y = C - \tfrac{1}{2}v\sqrt{(aa-vv)} - \tfrac{1}{2}aaArc.sin.\frac{v}{a}$$

seu

$$y = C - \tfrac{1}{2}(a-v)\sqrt{(2ax-xx)} - \tfrac{1}{2}aa\ Arc.sin.\frac{a-x}{a};$$

ut igitur fiat $y = 0$ posito $x = 0$, capi debet $C=\frac{1}{2}aa\ Arc.sin.1$, ita ut sit

$$y=-\tfrac{1}{2}(a-x)\sqrt{(2ax-xx)}+\tfrac{1}{2}aa\,Arc.cos.\tfrac{a-x}{a}.$$

Est vero

$$Arc.sin.\tfrac{\sqrt{x}}{\sqrt{2a}}=\tfrac{1}{2}Arc.cos.\tfrac{a-x}{a}.$$

COROLLARIUM 2

153. Si ponamus $x=\frac{a}{2}$, fit $y=\frac{-aa\sqrt{3}}{8}+\frac{\pi aa}{6};$ series autem dat

$$y\ =\ 2aa\left(\tfrac{1}{2\cdot3}-\tfrac{1}{2\cdot5\cdot2^3}-\tfrac{1\cdot1}{2\cdot4\cdot7\cdot2^5}-\tfrac{1\cdot1\cdot3}{2\cdot4\cdot6\cdot9\cdot2^7}-\text{etc.}\right),$$

unde colligitur

$$\pi\ =\ \tfrac{3\sqrt{3}}{4}+6\left(\tfrac{1}{3}-\tfrac{1}{2\cdot5\cdot2^3}-\tfrac{1\cdot1}{2\cdot4\cdot7\cdot2^4}-\tfrac{1\cdot1\cdot3}{2\cdot4\cdot6\cdot9\cdot2^6}-\text{etc.}\right);$$

at per superiorem [§ 149] est

$$\pi\ =\ 3\left(1+\tfrac{1}{2\cdot3\cdot2^2}+\tfrac{1\cdot3}{2\cdot4\cdot5\cdot2^4}+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6\cdot7\cdot2^6}+\text{etc.}\right),$$

ex quarum combinatione plures aliae formari possunt.

EXEMPLUM 3

154. *Formulam* $dy=\frac{dx}{\sqrt{(1+xx)}}$ *per seriem integrare.*

Integrale est $y=l\left(x+\sqrt{(1+xx)}\right)$ ita sumtum, ut evanescat posito $x=0$.
At ob

$$\tfrac{1}{\sqrt{(1+xx)}}=\ 1-\tfrac{1}{2}x^2+\tfrac{1\cdot3}{2\cdot4}x^4-\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}x^6+\text{etc.}$$

erit idem integrale per seriem expressum

$$y=x-\tfrac{1}{2}\cdot\tfrac{x^3}{3}+\tfrac{1\cdot3}{2\cdot4}\cdot\tfrac{x^5}{5}-\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\cdot\tfrac{x^7}{7}+\text{etc.}$$

EXEMPLUM 4

155. *Formulam* $dy=\frac{dx}{\sqrt{(xx-1)}}$ *per seriem integrare.*

Integratio dat $y=l\left(x+\sqrt{(xx-1)}\right)$ quod evanescit posito $x=1$. Iam ob

$$\frac{1}{\sqrt{(xx-1)}}=\frac{1}{x}+\frac{1}{2x^3}+\frac{1\cdot 3}{2\cdot 4x^5}+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6x^7}+\text{etc.}$$

erit idem integrale

$$y=C+lx-\frac{1}{2\cdot 2x^2}-\frac{1\cdot 3}{2\cdot 4\cdot 4x^4}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6x^6}-\text{etc.}$$

quod ut evanescat posito $x = 1$, constans ita definitur, ut fiat

$$y=lx+\frac{1}{2\cdot 2}\left(1-\frac{1}{x^2}\right)+\frac{1\cdot 3}{2\cdot 4\cdot 4}\left(1-\frac{1}{x^4}\right)+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\cdot 6}\left(1-\frac{1}{x^6}\right)+\text{etc.}$$

COROLLARIUM

156. Posito $x=1+u$ fit

$$dy=\frac{du}{\left(\sqrt{2u+uu}\right)}=\frac{du}{\sqrt{2u}}\left(1+\frac{u}{2}\right)^{-\frac{1}{2}}$$
$$=\frac{du}{\sqrt{2u}}\left(1-\frac{1}{2}\cdot\frac{u}{2}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{uu}{4}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{u^3}{8}-\text{etc.}\right),$$

unde integrando habebitur

$$y=\frac{1}{\sqrt{2}}\left(2\sqrt{u}-\frac{1}{2}\cdot\frac{2u^{\frac{3}{2}}}{3\cdot 2}+\frac{1\cdot 3}{2\cdot 4}\cdot\frac{2u^{\frac{5}{2}}}{5\cdot 4}-\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\frac{2u^{\frac{7}{2}}}{7\cdot 8}+\text{etc.}\right)$$

seu

$$y=\left(1-\frac{1u}{2\cdot 3\cdot 2}+\frac{1\cdot 3uu}{2\cdot 4\cdot 5\cdot 4}-\frac{1\cdot 3\cdot 5u^3}{2\cdot 4\cdot 6\cdot 7\cdot 8}+\text{etc.}\right)\sqrt{2u}.$$

EXEMPLUM 5

157. *Formulam* $dy=\frac{dx}{(1-x)^n}$ *per seriem integrare.*

Per integrationem fit

$$y=\frac{1}{(n-1)(1-x)^{n-1}}-\frac{1}{n-1}$$

facto $y = 0$, si $x = 0$, seu

$$y = \frac{(1-x)^{-n+1}-1}{n-1}$$

Iam vero per seriem est

$$dy = dx\left(1 + nx + \frac{n(n+1)}{1\cdot 2}x^2 + \frac{n(n+1)(n+2)}{1\cdot 2\cdot 3}x^3 + \text{etc.}\right),$$

unde idem integrale ita exprimetur

$$y = x + \frac{nx^2}{2} + \frac{n(n+1)x^3}{1\cdot 2\cdot 3} + \frac{n(n+1)(n+2)x^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

Hinc autem quoque manifesto fit

$$(n-1)y + 1 = \frac{1}{(1-x)^{n-1}}.$$

SCHOLION

158. Haec autem cum sint nimis obvia, quam ut iis fusius inhaerere sit opus, aliam methodum series eliciendi exponam magis absconditam, quae saepe in Analysi eximium usum afferre potest.

PROBLEMA 13

159. *Proposita formula differentiali*

$$dy = x^{m-1}dx\left(a + bx^n\right)^{\frac{\mu}{\nu}-1}$$

eius integrale altera methodo in seriem convertere.

SOLUTIO

Ponatur $y = \left(a + bx^n\right)^{\frac{\mu}{\nu}} z$; erit

$$dy = \left(a + bx^n\right)^{\frac{\mu}{\nu}-1}\left(dz\left(a + bx^n\right) + \frac{n\mu}{\nu}bx^{n-1}zdx\right),$$

unde fit

$$x^{m-1}dx = dz\left(a + bx^n\right) + \frac{n\mu}{\nu}bx^{n-1}zdx$$

seu

$$\nu x^{m-1}dx = \nu dz\left(a + bx^n\right) + n\mu bx^{n-1}zdx.$$

Iam antequam seriem, qua valor ipsius z definiatur, investigemus, notandum est casu, quo x evanescit, fieri

$$dy = a^{\frac{\mu}{\nu}-1} x^{m-1} dx = a^{\frac{\mu}{\nu}} dz,$$

ut sit $dz = \frac{1}{a} x^{m-1} dx$. Statuamus ergo

$$z = Ax^m + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n} + \text{etc.}$$

eritque

$$\frac{dz}{dx} = mAx^m + (m+n)Bx^{m+n-1} + (m+2n)Cx^{m+2n-1} + \text{etc.}$$

Substituantur hae series loco z et $\frac{dz}{dx}$ in aequatione

$$\frac{\nu dz}{dx}\left(a + bx^n\right) + n\mu bx^{n-1}z - \nu x^{m-1} = 0$$

singulisque terminis secundum potestates ipsius x dispositis orietur ista aequatio

$$\left.\begin{array}{llll} m\nu aAx^{m-1} & + (m+n)\nu aBx^{m+n-1} & + (m+2n)\nu aCx^{m+2n-1} & + \text{etc.} \\ -\nu & + m\nu bA & + (m+n)\nu bB & \\ & + n\mu bA & + n\mu bB & \end{array}\right\} = 0,$$

unde singulis terminis nihilo aequalibus positis coefficientes ficti per sequentes formulas definientur

$$m\nu aA - \nu = 0, \qquad \text{hinc} \quad A = \frac{1}{ma},$$

$$(m+n)\nu aB + (m\nu + n\mu)bA = 0, \qquad B = -\frac{(m\nu+n\mu)b}{(m+n)\nu a} A,$$

$$(m+2n)\nu aC + ((m+n)\nu + n\mu)bB = 0, \qquad C = -\frac{((m+n)\nu+n\mu)b}{(m+2n)\nu a} B,$$

$$(m+3n)\nu aD + ((m+2n)\nu + n\mu)bC = 0, \qquad D = -\frac{((m+2n)\nu+n\mu)b}{(m+3n)\nu a} C$$

sicque quilibet coefficiens facile ex praecedente reperitur. Tum vero erit

$$y = \left(a + bx^n\right)^{\frac{\mu}{\nu}} \left(x^m + Bx^{m+n} + Cx^{m+2n} + Dx^{m+3n} + \text{etc.}\right)$$

SOLUTIO 2

Quemadmodum hic seriem secundum potestates ipsius x ascendentem assumsimus, ita etiam descendentem constituere licet

$$z = Ax^{m-n} + Bx^{m-2n} + Cx^{m-3n} + Dx^{m-4n} + \text{etc.}$$

ut sit

$$\tfrac{dz}{dx} = (m-n)Ax^{m-n-1} + (m-2n)Bx^{m-2n-1} + (m-3n)Cx^{m-3n-1} + \text{etc.}$$

quibus seriebus substitutis prodit

$$\left.\begin{array}{llll}(m-n)vbAx^{m-1} & +(m-n)vaAx^{m+n-1} & +(m-2n)vaCx^{m+2n-1} & +(m-3n)vaCx^{m-3n-1} + \text{etc.}\\ +\ n\mu bA & +(m-2n)vbB & +(m-3n)vbC & +(m-4n)vbD\\ -v & +\ n\mu bB & +\qquad n\mu bC & +\qquad n\mu bD\end{array}\right\} = 0.$$

Hinc ergo sequenti modo litterae *A*, *B*, *C* etc. determinantur

$$(m-n)vbA + n\mu bA - v = 0, \qquad \text{ergo} \qquad A = \tfrac{v}{(m-n)v+n\mu}\cdot\tfrac{1}{b},$$

$$(m-n)vaA \ + (m-2n)vbB + n\mu bB = 0, \qquad B = \tfrac{-(m-n)v}{(m-2n)v+n\mu}\cdot\tfrac{a}{b}A,$$

$$(m-2n)vaB + (m-3n)vbC + n\mu bC = 0, \qquad C = \tfrac{-(m-2n)v}{(m-3n)v+n\mu}\cdot\tfrac{a}{b}B,$$

$$(m-3n)vaC + (m-4n)vbD + n\mu bD = 0, \qquad D = \tfrac{-(m-3n)v}{(m-4n)v+n\mu}\cdot\tfrac{a}{b}C,$$

ubi iterum lex progressionis harum litterarum est manifesta.

COROLLARIUM 1

160. Prior series ideo est memorabilis, quod casibus, quibus

$$(m+in)v + n\mu = 0 \quad \text{seu} \quad -\tfrac{m}{n} - \tfrac{\mu}{v} = i,$$

abrumpitur atque ipsum integrale algebraicum exhibet. Posterior vero abrumpitur, quoties $m - in = 0$ seu $\tfrac{m}{n} = i$ denotante *i* numerum integrum positivum.

COROLLARIUM 2

161. Utraque vero series etiam incommodo quodam laborat, quod non semper in usum vocari potest. Quando enim vel $m = 0$ vel $m + in = 0$, priori uti non licet, quando vero $(m-in)v + n\mu = 0$ seu $\tfrac{m}{n} + \tfrac{\mu}{v} = i$, usus posterioris tollitur, quia termini fierent infiniti.

COROLLARIUM 3

162. Hoc vero commode usu venit, ut, quoties altera applicari nequit, altera certo in usum vocari possit, iis tantum casibus exceptis, quibus et $-\frac{m}{n}$ et $\frac{\mu}{\nu}+\frac{m}{n}$ sunt numeri integri positivi. Quia autem tum est $\nu=1$, hi casus sunt rationales integri nihilque difficultatis habent.

COROLLARIUM 4

163. Possunt etiam ambae series simul pro z coniungi hoc modo. Sit prior series = P, posterior vero = Q, ut capi possit tam $z=P$ quam $z=Q$. Binis autem coniungendis erit $z=\alpha P+\beta Q$, dummodo sit $a+\beta=1$.

SCHOLION

164. Inde autem, quod duas series pro z exhibemus, minime sequitur has duas series inter se esse aequales; neque enim necesse est, ut valores ipsius y inde orti fiant aequales, dummodo quantitate constante a se invicem differant. Ita si prior series inventa per P, posterior per Q indicetur, quia ex illa fit $y=\left(a+bx^n\right)^{\frac{\mu}{\nu}}P$, ex hac vero $y=\left(a+bx^n\right)^{\frac{\mu}{\nu}}Q$, certo erit $\left(a+bx^n\right)^{\frac{\mu}{\nu}}(P-Q)$ quantitas constans ideoque $P-Q=C\left(a+bx^n\right)^{-\frac{\mu}{\nu}}$. Utraque scilicet series tantum integrale particulare praebet, quoniam nullum constantem involvit, quae non iam in formula differentiali contineatur. Interim tamen eadem methodo etiam valor completus pro z erui potest; praeter seriem enim assumtam P vel Q statui potest ac substitutione facta series P ut ante definitur; pro altera vero nova serie efficiendum est, ut sit

$$\left.\begin{array}{lllll} n\nu a\beta x^{n-1} & +2n\nu a\gamma x^{2n-1} & +3n\nu a\delta x^{3n-1} & +4n\nu a\varepsilon x^{4n-1} & +\text{etc.} \\ +n\mu b\alpha & +n\nu b\beta & +2n\nu b\gamma & +3n\nu b\delta & \\ & +n\mu b\beta & +\ n\mu b\gamma & +\ \ n\mu b\delta & \end{array}\right\}=0,$$

unde ducuntur hae determinationes

$$\beta=\frac{-\mu b}{\nu a}\cdot\alpha,\quad \gamma=\frac{-(\mu+\nu)b}{2\nu a}\cdot\beta,\quad \delta=\frac{-(\mu+2\nu)b}{3\nu a}\cdot\gamma,\quad \delta=\frac{-(\mu+3\nu)b}{4\nu a}\cdot\delta,\text{etc.},$$

ita ut prodeat

$$z=P+\alpha\left(1-\frac{\mu}{\nu}\cdot\frac{b}{a}x^n+\frac{\mu(\mu+\nu)}{\nu\cdot 2\nu}\cdot\frac{b^2}{a^2}x^{2n}-\frac{\mu(\mu+\nu)(\mu+2\nu)}{\nu\cdot 2\nu\cdot 3\nu}\cdot\frac{b^3}{a^3}x^{3n}+\text{etc.}\right)$$

seu

$$z=P+\alpha\left(a+bx^n\right)^{-\frac{\mu}{\nu}}$$

hincque

$$y = P\left(a + bx^n\right)^{\frac{\mu}{\nu}} + \alpha a^{\frac{\mu}{\nu}},$$

quod est integrale completum, quia constans *a* mansit arbitraria.

EXEMPLUM 1

165. *Formulam* $dy = \frac{dx}{\sqrt{(1-xx)}}$ *hoc modo per seriem integrare.*

Comparatione cum forma generali instituta $a = 1, b = -1, m = 1, n = 2, \mu = 1, \nu = 2$, unde posito $y = z\sqrt{(1-xx)}$ prima solutio

$$z = Ax + Bx^3 + Cx^5 + Dx^7 + \text{etc.}$$

praebet

$$A = 1,\quad B = \tfrac{2}{3}A,\quad C = \tfrac{4}{5}B,\quad D = \tfrac{6}{7}C,\quad E = \tfrac{8}{9}D,\quad \text{etc.},$$

unde colligimus

$$y = \left(x + \tfrac{2}{3}x^3 + \tfrac{2\cdot 4}{3\cdot 5}x^5 + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \text{etc.}\right)\sqrt{(1-xx)},$$

quod integrale evanescit posito $x = 0$; est ergo $y = Arc.\ sin.x$. Altera methodus hic frustra tentatur ob $\frac{m}{n} + \frac{\mu}{\nu} = 1$.

COROLLARIUM 1

166. Posito $x = 1$ videtur hinc fieri $y = 0$ ob $\sqrt{(1-xx)} = 0$; at perpendendum est fieri hoc casu seriei infinitae summam infinitam, ita ut nihil obstet, quominus sit $y = \frac{\pi}{2}$. Si ponamus $x = \frac{1}{2}$, fit $y = 30^{\text{o}} = \frac{\pi}{6}$ ideoque

$$\tfrac{\pi}{6} = \left(1 + \tfrac{2}{3\cdot 4} + \tfrac{2\cdot 4}{3\cdot 5\cdot 4^2} + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7\cdot 4^3} + \text{etc.}\right)\tfrac{\sqrt{3}}{4}.$$

COROLLARIUM 2

167. Simili modo proposita formula $dy = \frac{dx}{\sqrt{(1+xx)}}$ reperitur

$$y = \left(x - \tfrac{2}{3}x^3 + \tfrac{2\cdot 4}{3\cdot 5}x^5 - \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \text{etc.}\right)\sqrt{(1+xx)},$$

estque

$$y = l\left(x + \sqrt{(1 + xx)}\right)$$

EXEMPLUM 2

168. *Formulam* $dy = \frac{dx}{x\sqrt{(1-xx)}}$ *hoc modo per seriem integrare.*

Est ergo $m = 0, n = 2, \mu = 1, \nu = 2, a = 1$ et $b = -1$; utendum igitur est altera serie sumendo

$$z = \frac{y}{\sqrt{(1-xx)}} = Ax^{-2} + Bx^{-4} + Cx^{-6} + Dx^{-8} + \text{etc.}$$

fitque

$$A = 1, \quad B = \tfrac{2}{3}A, \quad C = \tfrac{4}{5}B, \quad D = \tfrac{6}{7}C, \quad \text{etc.},$$

Hinc ergo colligimus

$$y = \left(\frac{1}{xx} + \frac{2}{3x^4} + \frac{2\cdot4}{3\cdot5x^6} + \frac{2\cdot4\cdot6}{3\cdot5\cdot7x^8} + \text{etc.}\right)\sqrt{(1-xx)}.$$

At integratio praebet

$$y = l\frac{1-\sqrt{(1-xx)}}{x},$$

qui valores conveniunt, quia uterque evanescit posito $x = 1$.

COROLLARIUM 1

169. Cum autem haec series non convergat, nisi capiatur $x > 1$, hoc autem casu formula $\sqrt{(1-xx)}$ fiat imaginaria, haec series nullius est usus.

COROLLARIUM: 2

170. Si proponatur $dy = \frac{dx}{x\sqrt{(xx-1)}}$, eadem pro y series emergit per $\sqrt{-1}$ multiplicata eritque

$$y = \left(\frac{1}{xx} + \frac{2}{3x^4} + \frac{2\cdot4}{3\cdot5x^6} + \frac{2\cdot4\cdot6}{3\cdot5\cdot7x^8} + \text{etc.}\right)\sqrt{(xx-1)}.$$

Posito autem $x = \frac{1}{u}$ erit $dy = \frac{-du}{\sqrt{(1-uu)}}$ et $y = C - Arc.sin.u$ seu

$$y = C - Arc.sin.\tfrac{1}{x}$$

ubi sumi oportet $C = 0$, quia series illa evanescit posito $x = \infty$, ita ut sit $y = -Arc.sin.\frac{1}{x}$, quae cum superiori [§ 165] convenit statuendo $\frac{1}{x} = u$.

EXEMPLUM 3

171. *Formulam* $dy = \frac{dx}{\sqrt{(a+bx^4)}}$ *hoc modo per seriem integrare.*

Est hic $m=1, n=4,\ \mu=1, v=2$ ideoque posito $y = z\sqrt{\left(a+bx^4\right)}$ prior resolutio dat

$$z = Ax + Bx^5 + Cx^9 + Dx^{13} + \text{etc.},$$

existente

$$A = \frac{1}{a}, \quad B = \frac{-3b}{5a}A, \quad C = \frac{-7b}{9a}B, \quad D = \frac{-11b}{13a}C, \quad \text{etc.},$$

ita ut sit

$$y = \left(\frac{x}{a} - \frac{3bx^5}{5aa} + \frac{3\cdot7b^2x^9}{5\cdot9a^3} - \frac{3\cdot7\cdot11b^3x^{13}}{5\cdot9\cdot13a^4} + \text{etc.}\right)\sqrt{\left(a+bx^4\right)}.$$

Hic autem quoque altera resolutio locum habet ponendo

$$z = Ax^{-3} + Bx^{-7} + Cx^{-11} + Dx^{-15} + \text{etc.}$$

existente

$$A = \frac{-1}{b}, \quad B = \frac{-3a}{5b}A, \quad C = \frac{-7a}{9b}B, \quad D = \frac{-11a}{13b}C, \quad \text{etc.},$$

unde colligitur

$$y = -\left(\frac{x}{bx^3} - \frac{3a}{5b^2x^7} + \frac{3\cdot7aa}{5\cdot9b^3x^{11}} - \frac{3\cdot7\cdot11a^3}{5\cdot9\cdot13b^4x^{15}} + \text{etc.}\right)\sqrt{\left(a+bx^4\right)}.$$

quarum serierum illa evanescit posito $x = 0$, haec vero posito $x = \infty$.

COROLLARIUM 1

172. Differentia ergo harum duarum serierum est constans, scilicet

$$\left\{\begin{matrix} +\frac{x}{a} & -\frac{3bx^5}{5aa} & +\frac{3\cdot7b^2x^9}{5\cdot9a^3} & -\frac{3\cdot7\cdot11b^3x^{13}}{5\cdot9\cdot13a^4} & +\text{etc.} \\ +\frac{x}{bx^3} & -\frac{3a}{5b^2x^7} & +\frac{3\cdot7aa}{5\cdot9b^3x^{11}} & -\frac{3\cdot7\cdot11a^3}{5\cdot9\cdot13b^4x^{15}} & +\text{etc} \end{matrix}\right\}\sqrt{\left(a+bx^4\right)} = \text{Const.}$$

COROLLARIUM 2

173. Has ergo binas series colligendo habebimus

$$\frac{a+bx^4}{abx^3}-\frac{3}{5}\cdot\frac{a^3+b^3x^{12}}{a^2b^2x^7}+\frac{3\cdot7}{5\cdot9}\cdot\frac{a^5+b^5x^{20}}{a^3b^3x^{11}}-\text{etc.}=\frac{C}{\sqrt{\left(a+bx^4\right)}},$$

ubi, quicunque valor ipsi x tribuatur, pro C semper eadem quantitas obtinetur.

COROLLARIUM 3

174. Ita si a = 1 et b = 1, erit haec series in $\sqrt{\left(1+x^4\right)}$ ducta semper constans, scilicet

$$\left(\frac{1+x^4}{x^3}-\frac{3}{5}\cdot\frac{1+x^{12}}{x^7}+\frac{3\cdot7}{5\cdot9}\cdot\frac{1+x^{20}}{x^{11}}-\text{etc.}\right)\sqrt{\left(a+bx^4\right)}=C.$$

Cum igitur posito x = 1 fiat

$$C=\left(1-\frac{3}{5}+\frac{3\cdot7}{5\cdot9}-\frac{3\cdot7\cdot11}{5\cdot9\cdot13}+\text{etc.}\right)2\sqrt{2}$$

huicque valori etiam ina series, quicunque valor ipsi x tribuatur, est aequalis.

COROLLARIUM 4

175. Haec postrema series signis alternantibus procedens per differentias facile in aliam iisdem signis praeditam transformatur, unde eadem constans concluditur

$$C=\left(1+\frac{1}{5}+\frac{1\cdot3}{5\cdot9}+\frac{1\cdot3\cdot5}{5\cdot9\cdot13}+\frac{1\cdot3\cdot5\cdot7}{5\cdot9\cdot13\cdot17}+\text{etc.}\right)\sqrt{2},$$

quae series satis cito convergit, eritque proxime $C=\frac{13}{7}$.

SCHOLION

176. lsta methodus in hoc consistit, ut series quaedam indefinita fingatur eiusque determinatio ex natura rei derivetur. Eius usus autem potissimum cernitur in aequationibus differentialibus resolvendis; verum etiam in praesenti instituto saepe utiliter adhibetur. Eiusdem quoque methodi ope quantitates transcendentes reciprocae, veluti exponentiales et sinus cosinusve angulorum, per series exprimuntur; quae etsi iam aliunde sint cognitae, tamen earum investigationem per integrationem exposuisse iuvabit, cum simili modo alia praeclara erui queant.

PROBLEMA 14

177. *Quantitatem exponentialem* $y = a^x$ *in seriem convertere.*

SOLUTIO

Sumtis logarithmis habemus $ly = x\,la$ et differentiando

$$\frac{dy}{y} = dx\,la \quad \text{seu} \quad \frac{dy}{dx} = y\,la,$$

unde valorem ipsius *y* per seriem quaeri oportet. Cum autem integrale completum latius pateat, notetur nostro casu posito $x = 0$ fieri debere $y = 1$;
quare fingatur haec pro *y* series

$$y = 1 + Ax + Bx^2 + Cx^3 + Dx^4 + \text{etc.}$$

unde fit

$$\frac{dy}{dx} = A + 2Bx + 3Cx^2 + 4Dx^3 + \text{etc.},$$

quibus substitutis in aequatione $\frac{dy}{dx} - y\,la = 0$ erit

$$\left.\begin{array}{l} A + 2Bx + 3Cx^2 + 4Dx^3 + \text{etc.} \\ -la - Ala - Bla - Cla - Dla \end{array}\right\} = 0$$

hincque coefficientes ita determinantur

$$A = la, \quad B = \tfrac{1}{2}A\,la, \quad C = \tfrac{1}{3}B\,la, \quad D = \tfrac{1}{4}C\,la \quad \text{etc.}$$

sicque consequimur

$$y = a^x = 1 + \frac{xla}{1} + \frac{x^2(la)^2}{1\cdot 2} + \frac{x^3(la)^3}{1\cdot 2\cdot 3} + \frac{x^4(la)^4}{1\cdot 2\cdot 3\cdot 4} + \text{etc.},$$

quae est ipsa series notissima in *Introductione* data [*Introductio in analysin infinitorum,* t. I, cap.Vll].

SCHOLION

178. Pro sinibus et cosinibus angulorum ad differentialia secundi gradus est descendendum, ex quibus deinceps series integrale referens elici debet. Cum autem gemina integratio duplicem determinationem requirat, series ita est fingenda, ut duabus conditionibus ex natura rei petitis satisfaciat. Verum haec methodus etiam ad alias investigationes extenditur, quae adeo in quantitatibus algebraicis versantur, a cuiusmodi exemplo hic inchoemus.

PROBLEMA 15

179. *Hanc expressionem* $y=\left(x+\sqrt{(1+xx)}\right)^n$ *in seriem secundum potestates ipsius x progredientem convertere.*

SOLUTIO

Quia est $ly=nl\left(x+\sqrt{(1+xx)}\right)$, erit

$$\frac{dy}{y}=\frac{ndx}{\sqrt{(1+xx)}};$$

iam ad signum radicale tollendum sumantur quadrata; erit

$$(1+xx)dy^2=nnyydx^2.$$

Aequatio sumto *dx* constante denuo differentietur, ut per 2*dy* diviso prodeat

$$ddy(1+xx)+xdxdy-nnydx^2=0,$$

unde *y* per seriem elici debet. Primo autem patet, si sit *x* = 0, fore *y* = 1 ac, si *x* infinite parvum, $y=(1+x)^n=1+nx$. Fingatur ergo talis series

$$y=1+nx+Ax^2+Bx^3+Dx^4+Dx^5+Ex^6+\text{ etc.},$$

ex qua colligitur

$$\frac{dy}{dx}=n+2Ax+3Bxx+4Cx^3+5Dx^4+6Ex^5+\text{etc.}$$

et

$$\frac{ddy}{dx^2}=2\text{A}+6Bx+12Cxx+20D^3+30Ex^4+\text{etc.}$$

Facta ergo substitutione adipiscimur

$$\left.\begin{array}{l}2A+6Bx+12Cxx+20Dx^3+30Ex^4+42Fx^5+\text{ etc.}\\ \qquad\qquad +\ 2A\quad +\ 6B\quad +12C\quad +20D\\ \qquad +n\ +\ 2A\quad +\ 3B\quad +\ 4C\quad +\ 5D\\ -nn-n^3\ -An^2\ -\ Bn^2\ -\ Cn^2\ -Dn^2\end{array}\right\}=0$$

hincque derivantur sequentes determinationes

$$A=\frac{nn}{2},\ B=\frac{n(nn-1)}{2\cdot 3},\ C=\frac{A(nn-4)}{3\cdot 4},\ D=\frac{B(nn-9)}{4\cdot 5}\ \text{ etc.},$$

ita ut sit

$$y = 1 + nx + \frac{nn}{1\cdot 2}x^2 + \frac{n(nn-1)}{1\cdot 2\cdot 3}x^3 + \frac{nn(nn-4)}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{n(nn-1)(nn-9)x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$
$$+ \frac{nn(nn-4)(nn-16)x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \frac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \text{etc.}$$

COROLLARIUM 1

180. Uti est $y = \left(x + \sqrt{(1+xx)}\right)^n$, si statuamus $z = \left(-x + \sqrt{(1+xx)}\right)^n$, pro *z* similis series prodit, in qua *x* tantum negative capitur; hinc ergo concluditur

$$\frac{y+z}{2} = 1 + \frac{nn}{1\cdot 2}x^2 + \frac{nn(nn-4)}{1\cdot 2\cdot 3\cdot 4}x^4 + \frac{nn(nn-4)(nn-16)x^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \text{etc.}$$

et

$$\frac{y-z}{2} = nx + \frac{n(nn-1)}{1\cdot 2\cdot 3}x^3 + \frac{n(nn-1)(nn-9)x^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} + \frac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7 + \text{etc.}$$

COROLLARIUM 2

181. Si ponatur $x = \sqrt{-1}\cdot \sin.\varphi$, erit $\sqrt{(1+xx)} = \cos.\varphi$ hincque

$$y = \left(\cos.\varphi + \sqrt{-1}\cdot \sin.\varphi\right)^n = \cos.n\varphi + \sqrt{-1}\cdot \sin.n\varphi$$

et

$$z = \left(\cos.\varphi - \sqrt{-1}\cdot \sin.\varphi\right)^n = \cos.n\varphi - \sqrt{-1}\cdot \sin.n\varphi\,,$$

unde deducimus

$$\cos.n\varphi = 1 - \frac{nn}{1\cdot 2}\sin.^2\varphi + \frac{nn(nn-4)}{1\cdot 2\cdot 3\cdot 4}\sin.^4\varphi - \frac{4nn(nn-4)(nn-16)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}\sin.^6\varphi + \text{ etc.},$$

$$\sin.n\varphi = n\sin.\varphi - \frac{nn(nn-1)}{1\cdot 2\cdot 3}\sin.^3\varphi + \frac{nn(nn-1)(nn-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\sin.^5\varphi$$

$$- \frac{4nn(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\sin.^7\varphi + \text{ etc.},$$

COROLLARIUM 3

182. Hae series ad multiplicationem angulorum pertinent atque hoc habent singulare, quod prior tantum casibus, quibuss *n* est numerus par, posterior vero, quibus est numerus impar, abrumpatur.

PROBLEMA 16

183. *Proposito angulo φ tam eius sinum quam cosinum per seriem infinitam exprimere.*

SOLUTIO

Sit $y = sin.\varphi$ et $z = cos.\varphi$; erit

$$dy = d\varphi\sqrt{(1-yy)} \text{ et } dz = -d\varphi\sqrt{(1-zz)}.$$

Sumantur quadrata

$$dy^2 = d\varphi^2(1-yy) \text{ et } dz^2 = d\varphi^2(1-zz).$$

differentietur sumto $d\varphi$ constante fietque

$$ddy = -yd\varphi^2 \text{ et } ddz = -zd\varphi^2.$$

sicque *y* et *z* ex eadem aequatione definiri oportet. Sed pro $y = sin.\varphi$ observandum est, si φ evanescat, fieri $y = \varphi$, pro $z = cos.\varphi$, si φ evanescat, fieri $z = 1 - \frac{1}{2}\varphi\varphi$ seu $z = 1 + 0\varphi$. Fingatur ergo

$$y = \varphi + A\varphi^3 + B\varphi^5 + C\varphi^7 + \text{ etc.},$$
$$z = 1 + \alpha\varphi^2 + \beta\varphi^4 + \gamma\varphi^6 + \text{ etc.}$$

fietque substitutione facta

$$\left.\begin{array}{l} 2\cdot 3A\varphi + 4\cdot 5B\varphi^3 + 6\cdot 7C\varphi^5 + \text{etc.} \\ +1 \quad\quad + \quad A \quad + \quad B \end{array}\right\} = 0$$

et

$$\left.\begin{array}{l} 1\cdot 2\alpha + 3\cdot 4\beta\varphi^2 + 5\cdot 6\gamma\varphi^4 + \text{ etc.} \\ +1 \quad + \quad \alpha \quad + \quad \beta \end{array}\right\} = 0,$$

unde colligimus

$$A = \frac{-1}{2\cdot 3}, \quad B = \frac{-A}{4\cdot 5}, \quad C = \frac{-B}{6\cdot 7}, \quad D = \frac{-C}{8\cdot 9} \quad \text{etc.},$$
$$\alpha = \frac{-1}{1\cdot 2}, \quad \beta = \frac{-\alpha}{3\cdot 4}, \quad \gamma = \frac{-\beta}{5\cdot 6}, \quad \delta = \frac{-\gamma}{7\cdot 8} \quad \text{etc.},$$

unde series iam notissimae obtinentur

$$sin.\varphi = \frac{\varphi}{1} - \frac{\varphi^3}{1\cdot 2\cdot 3} + \frac{\varphi^5}{1\cdot 2\cdot 3\cdot 4\cdot 5} - \frac{\varphi^7}{1\cdot 2\cdots 7} + \text{etc.},$$

$$cos.\varphi = 1 - \frac{\varphi^2}{1\cdot 2} + \frac{\varphi^4}{1\cdot 2\cdot 3\cdot 4} - \frac{\varphi^6}{1\cdot 2\cdots 6} + \text{etc.},$$

SCHOLION

184. Non opus erat ad differentialia secundi gradus descendere, sed ex formularum $y = sin.\varphi$ et $z = cos.\varphi$ differentialibus, quae sunt $dy = zd\varphi$ et $dz = -yd\varphi$, eaedem series facile reperiuntur. Fictis enim seriebus ut ante

$$y = \varphi + A\varphi^3 + B\varphi^5 + C\varphi^7 + \text{etc. et } z = 1 + \alpha\varphi^2 + \beta\varphi^4 + \gamma\varphi^6 + \text{etc.}$$

substitutione facta obtinebitur ex priore

$$\left.\begin{array}{l} 1 \;+\; 3A\varphi^2 + 5B\varphi^4 + 7C\varphi^6 + \text{etc.} \\ -1 \;-\; \alpha \;-\; \beta \;-\; \gamma \end{array}\right\} = 0,$$

ex posteriore

$$\left.\begin{array}{l} 2\alpha\varphi + 4\beta\varphi^3 + 6\gamma\varphi^5 + \text{ etc.} \\ +1 \;+\; A \;+\; B \end{array}\right\} = 0,$$

unde colliguntur hae determinationes

$$\alpha = \tfrac{-1}{2}, \quad A = \tfrac{\alpha}{3}, \quad \beta = \tfrac{-A}{4}, \quad B = \tfrac{\beta}{5}, \quad \gamma = \tfrac{-B}{6}, \quad C = \tfrac{\gamma}{7} \text{ etc.},$$

ideoque

$$\alpha = -\tfrac{1}{2}, \quad \beta = +\tfrac{1}{2\cdot3\cdot4}, \quad \gamma = -\tfrac{1}{2\cdot3\cdot4\cdot5\cdot6} \quad \text{etc.},$$
$$A = -\tfrac{1}{2\cdot3}, \quad B = +\tfrac{1}{2\cdot3\cdot4\cdot5}, \quad C = -\tfrac{1}{2\cdot3\cdot4\cdot5\cdot6\cdot7} \quad \text{etc.},$$

qui valores cum praecedentibus conveniunt. Hinc intelligitur, quomodo saepe duae aequationes simul facilius per series evolvuntur, quam si alteram seorsim tractare velimus.

PROBLEMA 17

185. *Per seriem exprimere valorem quantitatis y, qui satisfaciat huic aequationi*

$$\frac{mdy}{\sqrt{(a+byy)}} = \frac{ndx}{\sqrt{(f+gxx)}}.$$

SOLUTIO

lntegratio huius aequationis suppeditat

$$\tfrac{m}{\sqrt{b}} l\left(\sqrt{(a+byy)} + y\sqrt{b}\right) = \frac{n}{\sqrt{g}} l\left(\sqrt{(f+gxx)} + x\sqrt{g}\right) + C,$$

unde deducimus

$$y=\frac{1}{2\sqrt{b}}\left(\frac{\sqrt{(f+gxx)}+x\sqrt{g}}{h}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}-\frac{a}{2\sqrt{b}}\left(\frac{\sqrt{(f+gxx)}-x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}$$

constantes h et k ita capiendo, ut sit $hk=f$. Hinc discimus, si x sumatur evanescens, fore

$$y=\frac{1}{2\sqrt{b}}\left(\frac{\sqrt{f}+x\sqrt{g}}{h}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}-\frac{a}{2\sqrt{b}}\left(\frac{\sqrt{f}-x\sqrt{g}}{k}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}$$

seu

$$y=\frac{1}{2\sqrt{b}}\left(\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}-a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right)+\frac{nx}{2m\sqrt{f}}\left(\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}+a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right)$$

vel posito $y=A+Bx$ erit

$$B=\frac{n\sqrt{(AAb+a)}}{m\sqrt{f}},$$

ita ut constans B definiatur ex constante

$$A=\frac{1}{2\sqrt{b}}\left(\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}-a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}\right)$$

et vicissim

$$\left(\frac{\sqrt{k}}{\sqrt{h}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}=A\sqrt{b}+\sqrt{(a+bAA)} \quad \text{atque} \quad a\left(\frac{\sqrt{h}}{\sqrt{k}}\right)^{\frac{n\sqrt{b}}{m\sqrt{g}}}=-A\sqrt{b}+\sqrt{(a+bAA)}.$$

Nunc ad seriem inveniendam aequatio proposita sumtis quadratis

$$mm(f+gxx)dy^2=nn(a+byy)dx^2$$

denuo differentietur capto dx constante, ut facta divisione per $2dy$ prodeat

$$mmddy(f+gxx)+mmgxdxdy-nnbydx^2=0.$$

Iam pro y fingatur series

$$y=A+Bx+Cx^2+Dx^3+Ex^4+Fx^5+\text{etc.},$$

qua substituta habebitur

$$\left.\begin{array}{l} 2mmfC + 6mmfDx + 12mmfEx^2 + 20mmfFx^3 + \text{ etc.} \\ \qquad\qquad\qquad + \ 2mmgC \quad + 6mmgD \\ \qquad\quad + \ mmgB \ + \ 2mmgC \quad + 3mmgD \\ - \ nnbA - \ nnbB \ - \ nnbC \ - \ nnbD \end{array}\right\} = 0.$$

Cum ergo A et B dentur, reliquae litterae ita determinantur

$$C = \frac{nnb}{2mmf} A,$$
$$D = \frac{nnb-mmg}{2\cdot3mmf} B, \ E = \frac{nnb-4mmg}{3\cdot4mmf} C,$$
$$F = \frac{nnb-9mmg}{4\cdot5mmf} D, \ G = \frac{nnb-16mmg}{5\cdot6mmf} E,$$
$$H = \frac{nnb-25mmg}{6\cdot7mmf} F, \ I = \frac{nnb-36mmg}{7\cdot8mmf} G$$

sicque series pro y erit cognita.

EXEMPLUM 1

186. *Functionem transcendentem $c^{Arc.\sin.x}$ per seriem secundum potestates ipsius x progredientem exprimere.*

Ponatur $y = c^{Arc.\sin.x}$; erit $ly = lc \cdot Arc.\sin.x$ et $\frac{dy}{y} = \frac{dxlc}{\sqrt{(1-xx)}}$, hinc

$$dy^2(1-xx) = yydx^2(lc)^2$$

et differentiando

$$ddy(1-xx) - xdxdy - ydx^2(lc)^2 = 0.$$

Observetur iam posito x evanescente fore $y = c^x = 1 + xlc$; hinc fingatur series

$$y = 1 + xlc + Ax^2 + Bx^3 + Cx^4 + Dx^5 + \text{etc.},$$

qua substituta habebitur

$$\left.\begin{array}{l} 1\cdot2A + 2\cdot3Bx \ + 3\cdot4Cx^2 + \ 4\cdot5Dx^3 + 5\cdot6Ex^4 + \text{etc.} \\ \qquad\qquad -1\cdot2A \ - \ 2\cdot3B \ - 3\cdot4C \\ \quad - \quad lc \ - \ 2A \ - \ 3B \ - \ 4C \\ -(le)^2 - (lc)^3 - A(lc)^2 \ - B(lc)^2 \ - C(lc)^2 \end{array}\right\} = 0,$$

unde reliqui coefficientes ita definiuntur

$$A = \frac{(lc)^2}{1\cdot 2}, \qquad C = \frac{4+(lc)^2}{3\cdot 4}A, \quad E = \frac{16+(lc)^2}{5\cdot 6}C \text{ etc.},$$
$$B = \frac{\left(1+(lc)^2\right)lc}{2\cdot 3}, \quad D = \frac{9+(lc)^2}{4\cdot 5}B, \quad F = \frac{25+(lc)^2}{6.7}D \text{ etc.}$$

Sit brevitatis gratia $lc = \gamma$ eritque

$$c^{Arc.sin.x} = 1+\gamma x+\frac{\gamma\gamma}{1\cdot 2}x^2+\frac{\gamma(1+\gamma\gamma)}{1\cdot 2\cdot 3}x^3+\frac{\gamma(4+\gamma\gamma)}{1\cdot 2\cdot 3\cdot 4}x^4$$
$$+\frac{\gamma(1+\gamma\gamma)(9+\gamma\gamma)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5+\frac{\gamma(4+\gamma\gamma)(16+\gamma\gamma)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}x^6+\text{etc.}$$

EXEMPLUM 2

187. *Posito $x = sin.\varphi$ invenire seriem secundum potestates ipsius x progredientem, quae sinum anguli $n\varphi$ exprimat.*

Ponatur $y = sin.n\varphi$ ac notetur evanescente φ fieri $x = \varphi$ et $y = n\varphi = nx$, hoc est $y = 0+nx$, quod est seriei quaesitae initium. Nunc autem est

$$d\varphi = \frac{dx}{\sqrt{(1-xx)}} \quad \text{et} \quad nd\varphi = \frac{dy}{\sqrt{(1-yy)}}.$$

Ergo

$$\frac{dy}{\sqrt{(1-yy)}} = \frac{ndx}{\sqrt{(1-xx)}}$$

et sumtis quadratis

$$(1-xx)dy^2 = nndx^2(1-yy),$$

hinc

$$ddy(1-xx)-xdxdy+nnydx^2 = 0.$$

Quare fingatur haec series

$$y = nx+Ax^3+Bx^5+Cx^7+Dx^9+\text{etc.};$$

qua substituta habebitur

$$\left.\begin{array}{lllll} 2\cdot 3Ax & +4\cdot 5Bx^3+ & 6\cdot 7Cx^5+ & 8\cdot 9Dx^7 & +\text{etc.} \\ & -2\cdot 3A & -\ 4\cdot 5B & -\ 6\cdot 7C & \\ -n & -\ 3A & -\ 5B & -\ 7C & \\ +n^3 & +\ nnA & +\ nnB & +\ nnC & \end{array}\right\} =0,$$

unde hae determinationes colliguntur

$$A=\tfrac{-n(nn-1)}{2\cdot 3},\ B=\tfrac{-(nn-9)A}{4\cdot 5},\ C=\tfrac{-(nn-25)}{6\cdot 7}B \text{ etc. },$$

ita ut sit

$$y=nx-\tfrac{n(nn-1)}{1\cdot 2\cdot 3}x^3+\tfrac{n(nn-1)(nn-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5}x^5-\tfrac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}x^7+\text{ etc.}$$

sive

$$\sin.n\varphi=n\sin\varphi-\tfrac{n(nn-1)}{1\cdot 2\cdot 3}\sin^3\varphi+\tfrac{n(nn-1)(nn-9)}{1\cdot 2\cdot 3\cdot 4\cdot 5}\sin^5\varphi-\tfrac{n(nn-1)(nn-9)(nn-25)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}\sin^7\varphi+\text{ etc.}$$

SCHOLION

188. Quia haec series tantum casibus, quibus n est numerus impar, abrumpitur, pro paribus notandum est seriem commode exprimi posse per productum ex $\sin.\varphi$ in aliam seriem secundum cosinus ipsius φ potestates progredientem. Ad quam inveniendam ponamus $\cos.\varphi=u$ sitque

$$\sin.n\varphi=z\sin.\varphi=z\sqrt{(1-uu)},$$

unde ob

$$d\varphi=-\tfrac{du}{\sqrt{(1-uu)}}$$

erit differentiando

$$-\tfrac{ndu\cos.n\varphi}{\sqrt{(1-uu)}}=dz\sqrt{(1-uu)}-\tfrac{zdu}{\sqrt{(1-uu)}}$$

seu

$$-\,ndu\cos.n\varphi=dz(1-uu)-zdu,$$

quae sumto du constante denuo differentiata dat

$$-\tfrac{nndu^2\sin.n\varphi}{\sqrt{(1-uu)}}=ddz(1-uu)-3ududz-zdu^2=-nnzdu^2$$

ob $\tfrac{\sin.n\varphi}{\sqrt{(1-uu)}}=z$.

Quocirca series quaesita pro $z=\tfrac{\sin.n\varphi}{\sin.\varphi}$ ex hac aequatione erui debet

$$ddz(1-uu)-3ududz-zdu^2+nnzdu^2=0,$$

ubi notandum est, quia $u=cos.\varphi$, evanescente *u*, quo casu fit $\varphi=90°$, fore vel $z=0$, si *n* numerus par, vel $z=1$, si $n=4\alpha+1$, vel $z=-1$, si $n=4\alpha-1$. Qui singuli casus seorsim sunt evolvendi; et quo principium cuiusque serlet pateat, sit $\varphi=90°-\omega$ et evanescente fit $u=cos.\varphi=\omega$, $sin.\varphi=1$, $sin.n\varphi=sin.(90°\cdot n-n\omega)=z$. Nunc pro casibus singulis

I. si $n=4\alpha$, fit $z=-sin.n\omega=-nu$
II. si $n=4\alpha+1$, fit $z=cos.n\omega=1$
III. si $n=4\alpha+2$, fit $z=sin.n\omega=+nu$
IV. si $n=4\alpha+3$, fit $z=-cos.n\omega=-1$,

unde series iam satis notae deducuntur.