CHAPTER II

CONCERNING THE INTEGRATION OF IRRATIONAL DIFFERENTIAL FORMULAS

PROBLEM 6

88. *To find the integral of this proposed differential formula* $dy=\frac{dx}{\sqrt{(\alpha+\beta x+\gamma xx)}}$.

SOLUTION

The quantity $\alpha+\beta x+\gamma xx$ either has two real factors or otherwise.

[Note : Euler uses the word *formula* for the expression to be integrated. This usage has been preserved in the translations.]

I. In the first case the proposed formula is of this kind

$$dy=\frac{dx}{\sqrt{(a+bx)(f+gx)}};$$

and it is set up in order that the irrationality is removed

$$(a+bx)(f+gx)=(a+bx)^2zz;$$

then there arises

$$x=\frac{f-azz}{bzz-g} \quad \text{and thus} \quad dx=\frac{2(ag-bf)zdz}{(bzz-g)^2}$$

and

$$\sqrt{(a+bx)(f+gx)}=-\frac{(ag-bf)z}{bzz-g}$$

from which there becomes

$$dy=\frac{-2dz}{bzz-g}=\frac{2dz}{g-bzz} \quad \text{and} \quad z=\sqrt{\frac{f+gx}{a+bx}}.$$

Whereby if the letters are b et g are endowed with the same signs, then the integral can be expressed by logarithms, but if the signs are unequal, then it is expressed by angles.

II. In the latter case we have $dy=\frac{dx}{\sqrt{(aa-2abx\cos.\zeta+bbxx)}}$;

there is put in place

$$aa-2abx\cos.\zeta+bbxx=(bx-az)^2;$$

and there becomes

$$-2bx\cos.\zeta+a=-2bxz+azz$$

and

$$x=\frac{a(1-zz)}{2b(cos.\zeta-z)},\quad \text{hence}\quad dx=\frac{adz(1-2z\,cos.\zeta+zz)}{2b(cos.\zeta-z)^2}$$

and

$$\sqrt{(aa-2abx\,cos.\zeta+bbxx)}=\frac{a(1-2z\,cos.\zeta+zz)}{2(cos.\zeta-z)},$$

hence

$$dy=\frac{dz}{b(cos.\zeta-z)}\quad \text{and}\quad y=-\tfrac{1}{b}l\left(cos.\zeta-z\right).$$

But there is

$$z=\frac{bx-\sqrt{(aa-2abx\,cos.\zeta+bbxx)}}{a}$$

and thus

$$y=-\frac{1}{b}l\frac{a\,cos.\zeta-bx+\sqrt{(aa-2abx\,cos.\zeta+bbxx)}}{a}$$

or

$$y=\tfrac{1}{b}l\left(-a\,cos.\zeta+bx+\sqrt{(aa-2abx\,cos.\zeta+bbxx)}\right)+C.$$

COROLLARIUM 1

89. The second case can be extended further and it is possible for the formula

$$dy=\frac{dx}{\sqrt{(\alpha+\beta x+\gamma xx)}}$$

to be adapted, as long as γ is a positive quantity ; for on setting $b=\sqrt{\gamma}$ and $a\,cos.\gamma=\frac{-\beta}{2\sqrt{\gamma}}$ there arises

$$y=\frac{1}{\sqrt{\gamma}}l\left(\frac{\beta}{2\sqrt{\gamma}}+x\sqrt{\gamma}+\sqrt{(\alpha+\beta x+\gamma xx)}\right)+C.$$

or

$$y=\frac{1}{\sqrt{\gamma}}l\left(\frac{\beta}{2}+\gamma x+\sqrt{\gamma(\alpha+\beta x+\gamma xx)}\right)+C.$$

COROLLARY 2

90. For the first case since

$$\int\frac{2dz}{g-bzz}=\frac{1}{\sqrt{bg}}l\,\frac{\sqrt{g}+z\sqrt{b}}{\sqrt{g}-z\sqrt{b}}\quad \text{and}\quad \int\frac{2dz}{gzz+bzz}=\frac{2}{\sqrt{gb}}\text{Arc. tang.}\frac{z\sqrt{b}}{\sqrt{g}},$$

we have these cases :

$$\int\frac{dx}{\sqrt{(a+bx)(f+gx)}}=\frac{1}{\sqrt{bg}}l\frac{\sqrt{g(a+bx)}+\sqrt{b(f+gx)}}{\sqrt{g(a+bx)}-\sqrt{b(f+gx)}}+C$$

$$\int\frac{dx}{\sqrt{(bx-a)(f+gx)}}=\frac{1}{\sqrt{bg}}l\frac{\sqrt{g(bx-a)}+\sqrt{b(f+gx)}}{\sqrt{g(bx-a)}-\sqrt{b(f+gx)}}+C$$

$$\int\frac{dx}{\sqrt{(bx-a)(gx-f)}}=\frac{1}{\sqrt{bg}}l\frac{\sqrt{g(bx-a)}+\sqrt{b(gx-f)}}{\sqrt{g(bx-a)}-\sqrt{b(gx-f)}}+C$$

$$\int\frac{dx}{\sqrt{(a-bx)(f-gx)}}=\frac{-1}{\sqrt{bg}}l\frac{\sqrt{g(a-bx)}+\sqrt{b(f-gx)}}{\sqrt{g(a-bx)}-\sqrt{b(f-gx)}}+C$$

$$\int\frac{dx}{\sqrt{(a-bx)(f+gx)}}=\frac{2}{\sqrt{bg}}\text{Arc.tang.}\frac{\sqrt{b(f+gx)}}{\sqrt{g(a-bx)}}+C$$

$$\int\frac{dx}{\sqrt{(a-bx)(gx-f)}}=\frac{2}{\sqrt{bg}}\text{Arc.tang.}\frac{\sqrt{b(gx-f)}}{\sqrt{g(a-bx)}}+C.$$

COROLLARY 3

91. Of these six integration the first four are all contained in the case of Corollary 1, but the two final integrations are contained in this formula

$$dy=\frac{dx}{\sqrt{(a+\beta x-\gamma xx)}};$$

for if there is put for the penultimate

$$af=\alpha,\ ag-bf=\beta, bg=\gamma,$$

from which there is deduced

$$y=\frac{2}{\sqrt{\gamma}}\text{Arc.tang.}\frac{2\sqrt{\gamma(\alpha+\beta x-\gamma xx)}}{\beta-2\gamma x},$$

clearly if the arc is doubled. Moreover expressed by the cosine

$$y=\frac{2}{\sqrt{\gamma}}\text{Arc.cos.}\frac{\beta-2\gamma x}{\sqrt{(\beta\beta+4\alpha\gamma)}}+C,$$

the truth of which is apparent from differentiation.

SCHOLIUM 1

92. From the solution of this problem it is apparent also that this formula extends further :

$$\frac{Xdx}{\sqrt{(a+\beta x+\gamma xx)}},$$

if X were a rational function of x of some kind, possible to be integrated by the previous chapter. For with the variable z introduced in place of x, from which a rational function of the roots should be returned, also X will be changed into some rational function of z. Likewise at this point a more general situation is had, if in place of $\sqrt{(a+\beta x+\gamma xx)}=u$ there were some function X of the two quantities x and u; then indeed by the substitution used, since rational formulas of z are written both for x as well as for u, a formula is produced of rational differentials. Likewise it is possible thus to reveal this, so that we can say of the formula Xdx, if the function X involves no other irrationality in addition to $\sqrt{(a+\beta x+\gamma xx)}$, it is possible to assign the integral, therefore since with the aid of a substitution in the differential formula it can be transformed into a rational function.

SCHOLIUM 2

93. But with any proposed formula of the differentials before all else it is to be observed, whether with the help of some substitution it can be transformed into a rational function ; because if that should succeed, then the integration can be completed through the results of the previous chapter, from which likewise the integral is understood, unless it is algebraic, not to involve other transcending functions other than logarithms and angles. But if no such substitution of this kind can be found, a stop must be made to the labour of the integration, whenever it prevails that the integral cannot be expressed either algebraically or by logarithms of angles. Just as if Xdx were a differential formula of some kind, which could not be agreed upon to be reduced to a rational function, then the integral $\int Xdx$ of this has to be referred to a new kind of transcending function, in which nothing remains for us except that we try to find a value for this, that we can assign approximately. But with new kinds of transcending quantities admitted, innumerable other formulas can be reduced to that and can be integrated. Therefore in the first place care must be taken with this, so that the simplest formula for any kind you wish is to be noted, from which conceded it is allowed to define the integrals of the rest of the formulas. Hence we are led to the question of the greatest importance, just as the integrations of the more complicated formulas are required to be reduced to the more simple. Because before we progress, we must assess carefully other formulas of this kind, which by the aid of suitable substitutions are able to be freed from irrationality, just as we have now shown, as often as X should be a function of some rational quantities x and $u=\sqrt{(a+\beta x+\gamma xx)}$ **,** thus as no other irrationalities should be present besides the square root of a formula of this kind $a+\beta x+\gamma xx$, so also the differential formula Xdx can be transformed into a rational function.

PROBLEM 7

94. *For the proposed formula of the differential* $Xdx(a+bx)^{\frac{\mu}{\nu}}$ *in which* X *denotes some rational function of* x*, to free that from irrationality.*

SOLUTION

There is put in place $a+bx=z^{\nu}$, so that there becomes $(a+bx)^{\frac{\mu}{\nu}}=z^{\mu}$ then, since $x=\frac{z^{\nu}-a}{b}$, with this substitution made the function X becomes a rational function of z, which shall be called Z, and on account of $dx=\frac{\nu}{b}z^{\nu-1}dz$ our formula of the differential takes on this form $\frac{\nu}{b}Zz^{\mu+\nu-1}dz$; which as it is rational, it can be integrated from the above chapter and it can be expressed by logarithms or angles, unless it is algebraic. .

COROLLARY 1

95. From this more general substitution the calculation can be performed, if on putting $(a+bx)^{\frac{1}{\nu}}=u$ the letter V denotes some rational function of the two quantities x and u; for since on putting $x=\frac{u^{\nu}-a}{b}$, V becomes a rational function of u, and the formula $Vdx=\frac{\nu}{b}Vu^{\nu-1}du$ is rational.

COROLLARY 2

96. Also if two irrationalities of the same quantity $a + bx$, evidently $(a+bx)^{\frac{1}{\nu}}=u$ and $(a+bx)^{\frac{1}{n}}=\mathrm{v}$, should enter into the formula Xdx, on putting $a+bx=z^{n\nu}$ there becomes $x=\frac{z^{n\nu}-a}{b}, u=z^{n}$ and $\mathrm{v}=z^{\nu}$; from which since X becomes a rational function of z and $dx=\frac{n\nu}{b}z^{n\nu-1}dz$, with this substitution the formula Xdx becomes rational.

COROLLARY 3

97. It is understood in the same manner, if there is put

$$(a+bx)^{\frac{1}{\lambda}}=u,\ (a+bx)^{\frac{1}{\mu}}=\upsilon, (a+bx)^{\frac{1}{\nu}}=t \quad \text{etc.}$$

the letter X denotes some rational function of the quantities x, u, v, t etc., the rational differential formula Xdx being returned on making $a+bx=z^{\lambda\mu\nu}$;
for then there becomes :

$$x=\frac{z^{\lambda\mu\nu}-a}{b}, u=z^{\mu\nu}, \mathrm{v}=z^{\lambda\nu}, t=z^{\lambda\mu} \text{ etc. and } dx=\frac{\lambda\mu\nu}{b}z^{\lambda\mu\nu-1}.$$

EXAMPLE

98. From this formula

$$dy = \frac{xdx}{\sqrt[3]{(1+x)} - \sqrt{(1+x)}}$$

on making $1 + x = z^6$ there is found

$$dy = -\frac{6z^3dz(1-z^6)}{1-z} \;,$$

or

$$dy = -6dz\left(z^3 + z^4 + z^5 + z^6 + z^7 + z^8\right) \;,$$

and hence on integrating :

$$y = C - \tfrac{3}{2}z^4 - \tfrac{6}{5}z^5 - z^6 - \tfrac{6}{7}z^7 - \tfrac{3}{4}z^8 - \tfrac{2}{3}z^9\,;$$

and on the values being restored :

$$y = C - \tfrac{3}{2}\sqrt[3]{(1+x)^2} - \tfrac{6}{5}\sqrt[6]{(1+x)^5} - 1 - x - \tfrac{6}{7}(1+x)\sqrt[6]{(1+x)}$$
$$-\tfrac{3}{4}(1+x)\sqrt[3]{(1+x)} - \tfrac{2}{3}(1+x)\sqrt{(1+x)},$$

thus so that the integral is shown hence to be algebraic.

PROBLEM 8

99. *With the proposed formula of the differential* $Xdx\left(\frac{a+bx}{f+gx}\right)^{\frac{\mu}{\nu}}$ *, with X denoting some rational function of x, to free the same from irrationality.*

SOLUTION

On putting

$$\frac{a+bx}{f+gx} = z^\nu$$

there becomes

$$\left(\frac{a+bx}{f+gx}\right)^{\frac{\mu}{\nu}} = z^\mu$$

and

$$x = \frac{a - fz^\nu}{gz^\nu - b} \quad \text{and} \quad dx = \frac{\nu(bf - ag)z^{\nu-1}dz}{(gz^\nu - b)^2}$$

and thus in place of *X* some rational function of *z* is produced, on putting which equal to *Z* our formula of the differential becomes equal to

$$\frac{\nu(bf - ag)Zz^{\nu-1}dz}{(gz^\nu - b)^2};$$

which since it is rational, can be integrated as by the preceding Chapter I.

COROLLARY 1

100. On putting $\left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\nu}}=u$ if X were some rational function of the two quantities x and u, the formula of the differential Xdx through the substitution used is transformed into a rational function, and the integral of this is agreed upon.

COROLLARY 2

101. If X were a rational function both of x as well as quantities of the kind

$$\left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\lambda}}=u,\quad \left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\mu}}=\text{v},\quad \left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\nu}}=t$$

then the formula of the rational differential Xdx is returned from the substitution used

$$\frac{a+bx}{f+gx}=z^{\lambda\mu\nu},$$

from which there becomes :

$$x=\frac{a-fz^{\lambda\mu\nu}}{gz^{\lambda\mu\nu}-b}\quad\text{and}\quad u=z^{\mu\nu},\quad \text{v}=z^{\lambda\nu},\quad t=z^{\lambda\mu}.$$

SCHOLIUM 1

102. Therefore the reduction to rational cases has succeeded in these cases , even if more irrational formulas are present, because all these rational functions are brought about likewise by the same substitution and thus also the quantity x is expressed by a new variable z rationally. But if the proposed differential consists of two irrational formulas of this kind, in which both are unable to be reduced with the help of the same substitution likewise , even if this can be done in each in turn , the reduction cannot be put in place, unless perhaps the differential itself is allowed to be separated into two parts, each of which includes only a single irrational formula. Just as if this shall be the proposed formula of the differential :

$$dy=\frac{dx}{\sqrt{(1+xx)}-\sqrt{(1-xx)}},$$

on multiplying the numerator and the denominator by $\sqrt{(1+xx)}-\sqrt{(1-xx)}$ there becomes :

$$dy=\frac{dx\sqrt{(1+xx)}}{2xx}-\frac{dx\sqrt{(1-xx)}}{2xx},$$

and each part of this separately can be reduced to rationality and integrated. Moreover there is found :

$$y=C-\frac{\sqrt{(1-xx)}+\sqrt{(1+xx)}}{2x}+\frac{1}{2}l\left(x+\sqrt{(1+xx)}\right)-\frac{1}{2}\text{Arc.tang.}\frac{x}{\sqrt{(1-xx)}}.$$

[Here we have used the corrected form of this integration from the *O. O*. edition.]

But most conveniently when the irrationality is removed, if in the first part there is put in place
$\sqrt{(1+xx)} = px$, and in the second part $\sqrt{(1-xx)} = qx$. For although there shall be hence

$$x=\frac{1}{\sqrt{(pp-1)}} \quad \text{and} \quad x=\frac{1}{\sqrt{(1+qq)}},$$

yet there arises rationally :

$$dy=\frac{-ppdp}{2(pp-1)}-\frac{qqdq}{(1+qq)}.$$

SCHOLIUM 2

103. Concerning the general formulas, which are to be freed from irrationality, scarcely any further instruction is in order, as long as we add this case, in which the function X includes the two square root formulas of this kind $\sqrt{(a+bx)}$ and $\sqrt{(f+gx)}$.
For on putting

$$(a+bx) = (f+gx)tt$$

there is produced

$$x=\frac{a-ftt}{gtt-b}$$

and

$$\sqrt{(a+bx)}=\frac{t\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}}, \quad \sqrt{(f+gx)}=\frac{\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}}$$

and only a single formula of the irrational $\sqrt{(gtt-b)}$ is present in the formula of the differential, which on substitution anew is easily removed by that method, which we treated in Problem 6.

Since thus we move on to others, this formula of the differential deserves to be considered especially

$$x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$$

on account of the simplicity of this the use through the whole of analysis is greatest, where indeed we have taken the letters m, n,μ,ν to denote whole numbers ; for unless they should be such, they should be easily reduced to that form. Just as if we have $x^{-\frac{1}{3}}dx\left(a+b\sqrt{x}\right)^{\frac{\mu}{\nu}}$, it is necessary to put in place $x=u^6$,

hence $dx=6u^5du$, from which there is produced $6u^3du\left(a+bu^3\right)^{\frac{\mu}{\nu}}$. Then indeed it is allowed to assume a positive value for n; for if it should be negative, for example

$x^{m-1}dx\left(a+bx^{-n}\right)^{\frac{\mu}{\nu}}$, on putting $x=\frac{1}{u}$ the formula becomes $-u^{-m-1}du(a+bu^{n})^{\frac{\mu}{\nu}}$ by the same principal; which hence we investigate in cases that can be freed from irrationality. .

PROBLEM 9

104. *To define the cases in which the differential formula* $x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$ *can be reduced to rationality.*

SOLUTION

First it is apparent, if there is put ν = 1 or $\frac{\mu}{\nu}$ becomes a whole number, that the formula is rational itself without the need for any substitution. But if $\frac{\mu}{\nu}$ is a fraction, a two-fold substitution must be used .

I. There is put

$$a+bx^{n}=u^{\nu},$$

so that there becomes

$$\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}=u^{\mu};$$

then $x^{n}=\frac{u^{\nu}-a}{b}$, and hence

$$x^{m}=\left(\frac{u^{\nu}-a}{b}\right)^{\frac{m}{n}} \quad \text{and thus} \quad x^{m-1}dx=\frac{\nu}{nb}u^{\nu-1}du\left(\frac{u^{\nu}-a}{b}\right)^{\frac{m-n}{n}},$$

from which our formula becomes

$$\frac{\nu}{nb}u^{\mu+\nu-1}du\left(\frac{u^{\nu}-a}{b}\right)^{\frac{m-n}{n}}.$$

Hence it is therefore apparent, as often as the exponent $\frac{m-n}{n}$ or $\frac{m}{n}$ should be a whole number either positive or negative, then the formula is rational.

II. There is put

$$a+bx^{n}=x^{n}z^{\nu},$$

so that there becomes

$$x^{n}=\frac{a}{z^{\nu}-b} \quad \text{and} \quad \left(a+bx^{n}\right)^{\frac{\mu}{\nu}}=\frac{a^{\frac{\mu}{\nu}}z^{\mu}}{\left(z^{\nu}-b\right)^{\frac{\mu}{\nu}}};$$

then

$$x^{m}=\frac{a^{\frac{m}{n}}}{\left(z^{\nu}-b\right)^{\frac{m}{n}}}, \quad \text{hence} \quad x^{m-1}dx=\frac{-\nu a^{\frac{m}{n}}z^{\nu-1}dz}{n\left(z^{\nu}-b\right)^{\frac{m}{n}+1}}$$

and thus ours formula becomes

$$\frac{-\nu a^{\frac{m}{n}+\frac{\mu}{\nu}} z^{\mu+\nu-1} dz}{n\left(z^{\nu}-b\right)^{\frac{m}{n}+\frac{\mu}{\nu}+1}}.$$

From which it is apparent that this formula is rational, as often as $\frac{m}{n}+\frac{\mu}{\nu}$ should be a whole number. Moreover it is easily understood that other suitable substitutions to the same goal cannot be devised. Whereby we conclude that this irrational formula $x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$ can be freed from irrationality, if either $\frac{m}{n}$ or $\frac{m}{n}+\frac{\mu}{\nu}$ is a whole number.

COROLLARY 1

105. If the number $\frac{m}{n}$ is an integer, the case is itself is easy ; for there is put $m = in$ and then $x^{n}=v$*;* and then $x^{m}=v^{i}$ and thus our formula becomes

$$\tfrac{i}{m} v^{i-1}dv\left(a+bv\right)^{\frac{\mu}{\nu}},$$

which is carried out as in Problem 7.

COROLLARY 2

106. But if $\frac{m}{n}$ is not an integer, so that there is a place for the reduction to rationality, it is necessary that $\frac{m}{n}+\frac{\mu}{\nu}$ is a whole number, since this cannot happen unless $\nu = n$, and thus $m+\mu$ has to be a multiple of $n = \nu$.

COROLLARY 3

107. But if hence this formula $x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$ is able to be reduced to rationality, also this formula $x^{m\pm\alpha n-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}\pm\beta}$ likewise admits to the reduction, for whatever whole numbers are taken for α and β. From which for the reducible cases to be known it is sufficient to put $m < n$ and $\mu<\nu$.

COROLLARY 4

108. If $m = 0$, this formula $\frac{dx}{x}\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$ always is reduced by the first case to rationality on putting $x^{n}=\frac{a}{z^{\nu}-b}$; for it is transformed into this $\frac{\nu u^{\mu+\nu-1}du}{n\left(u^{\nu}-a\right)}$.

SCHOLIUM 1

109. Because the formula $x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$, as often as $m = in$ with i denoting either some positive or negative whole number, is able always to be reduced to rationality and these cases are by themselves evident, it is thought worth the effort to consider the reduction of these cases that require to be considered more carefully. Which in the end we put in place $v = n$ and $m < n$, likewise $\mu < n$, and it is necessary that $m + \mu = n$; from which the following forms which indeed are able to be reduced to rationality, are obtained in general in their simplest.

I. $dx(a+bx^{2})^{\frac{1}{2}}$,

II. $dx(a+bx^{3})^{\frac{2}{3}}, \quad xdx(a+bx^{3})^{\frac{1}{3}}$,

III. $dx(a+bx^{4})^{\frac{3}{4}}, \quad xxdx(a+bx^{4})^{\frac{1}{4}}$,

IV. $dx(a+bx^{5})^{\frac{4}{5}}, \quad xdx(a+bx^{5})^{\frac{3}{5}}, \quad x^{2}dx(a+bx^{5})^{\frac{2}{5}}, \quad x^{3}dx(a+bx^{5})^{\frac{1}{5}}$,

V. $dx(a+bx^{6})^{\frac{5}{6}}, \quad x^{4}dx(a+bx^{6})^{\frac{1}{6}}$,

from which also these forms admit to being reduced

$$x^{\pm 2\alpha}dx(a+bx^{2})^{\frac{1}{2}\pm\beta},$$
$$x^{\pm 3\alpha}dx(a+bx^{3})^{\frac{2}{3}\pm\beta}, \quad x^{1\pm 3\alpha}dx(a+bx^{3})^{\frac{1}{3}\pm\beta},$$
$$x^{\pm 4\alpha}dx(a+bx^{4})^{\frac{3}{4}\pm\beta}, \quad x^{2\pm 4\alpha}dx(a+bx^{4})^{\frac{1}{4}\pm\beta},$$
$$x^{\pm 5\alpha}dx(a+bx^{5})^{\frac{4}{5}\pm\beta}, \quad x^{1\pm 5\alpha}dx(a+bx^{5})^{\frac{2}{5}\pm\beta},$$
$$x^{2\pm 5\alpha}dx(a+bx^{5})^{\frac{2}{5}\pm\beta},$$
$$x^{3\pm 5\alpha}dx(a+bx^{5})^{\frac{1}{5}\pm\beta},$$
$$x^{\pm 6\alpha}dx(a+bx^{6})^{\frac{5}{6}\pm\beta}, \quad x^{4\pm 6\alpha}dx(a+bx^{6})^{\frac{1}{6}\pm\beta}$$

SCHOLIUM 2

110. Now even if the formula $x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$ cannot be freed from irrationality, yet it is allowed always to reduce all of these formula to that integration $x^{m\pm n\alpha-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}\pm\beta}$, thus so that since the integral of that is considered known also the integrals of these can be assigned. Which reduction this is necessary to set out here, since it brings the greatest utility to analysis. Here the remaining cases are affirmed without doubt besides these, which we have shown here to be allowed to be reduced to rationality, and no others exist, which by any substitution brought to bear, by which the formulas are able to be freed from irrationality. For with this proposed formula $\frac{dx}{\sqrt{(a+bx^3)}}$ no rational function of *z* can be put in place of *x*; in order that $a+bx^3$ admits the extraction of the square root; indeed that objection can be satisfied within the search, for if in place of *x* some irrational function of *z* may be substituted, provided a similar irrationality is contained in the denominator, which destroys that affecting the numerator *dx*, just as happens in this formula $\frac{dx}{\sqrt[3]{(a+bx^3)}}$, on making the substitution $x=\frac{\sqrt[3]{a}}{\sqrt[3]{(z^3-b)}}$; now since it conveniently comes about in use here, in no way is it to be considered how likewise that case should come about [in other situations]. Yet I wish to present this here as a small demonstration.

PROBLEM 10

111. *The integration of the formula* $\int x^{m+n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$ *leads to the integration of this formula* $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$.

SOLUTION

The function $x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}$ is considered ; since the differential of this is

$$\left(max^{m-1}\,dx+mb\,x^{m+n-1}\,dx+\tfrac{n(\mu+\nu)}{\nu}b\,x^{m+n-1}\,dx\right)\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

then

$$x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}=ma\int x^{m-1}\,dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}+\tfrac{(m\nu+n\mu+n\nu)b}{\nu}\int x^{m+n-1}\left(a+bx^n\right)^{\frac{\mu}{\nu}}dx,$$

from which it is elicited,

$$\int x^{m+n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}=\tfrac{\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{(m\nu+n\mu+n\nu)b}-\tfrac{m\nu a}{(m\nu+n\mu+n\nu)b}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}.$$

COROLLARY 1

112. Since thus also there shall be

$$\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}=\tfrac{x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{ma}-\tfrac{(m\nu+n\mu+n\nu)b}{m\nu a}\int x^{m+n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

in place of *m* we write $m-n$ and we have this reduction

$$\int x^{m-n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}=\tfrac{x^{m-n}\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{(m-n)a}-\tfrac{(m\nu+n\mu)b}{(m-n)\nu a}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}.$$

COROLLARY 2

113. Hence with the concession of the integral $\int x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$ also the integration of the formulas $\int x^{m\pm n-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$ are allowed; and in a like manner on progressing further all of these formulas shown $\int x^{m\pm\alpha n-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$ are able to be integrated.

PROBLEM 11

114. *The integration of the formula* $\int x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}+1}$ *leads to the integration of the this integral* $\int x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$.

SOLUTION

The differential of the function $x^{m}\left(a+bx^{n}\right)^{\frac{\mu}{\nu}+1}$ can be shown in this way :

$$\left(ma-\tfrac{(m\nu+n\mu+n\nu)a}{\nu}\right)x^{m-1}\,dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}+\tfrac{m\nu+n\mu+n\nu}{\nu}\,x^{m-1}\,dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}+1},$$

from which it is concluded :

$$x^{m}\left(a+bx^{n}\right)^{\frac{\mu}{\nu}+1}$$
$$=-\tfrac{(n\mu+n\nu)a}{\nu}\int x^{m-1}\,dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}+\tfrac{m\nu+n\mu+n\nu}{\nu}\int x^{m-1}\,dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}+1},$$

on account of which we have

$$\int x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}+1}=\frac{\nu x^{m}\left(a+bx^{n}\right)^{\frac{\mu}{\nu}+1}}{m\nu+n(\mu+\nu)}+\frac{n(\mu+\nu)a}{m\nu+n(\mu+\nu)}\int x^{m-1}\,dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}.$$

COROLLARY 1

115. Then from the same equation we elicit :

$$\int x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}=\frac{-\nu x^{m}\left(a+bx^{n}\right)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)a}+\frac{m\nu+n(\mu+\nu)}{n(\mu+\nu)a}\int x^{m-1}\,dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}};$$

now we write $\mu-\nu$ in place of μ so that we come upon this reduction.

COROLLARY 2

116. Hence with the concession of the integral $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$ also the integrals of the formulas $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}\pm 1}$ and on progressing further the integrals of these $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}\pm\beta}$ can be shown with β denoting some whole number.

COROLLARY 3

117. With these preceding adjoined to the integral

$$\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

all these integrals of the form

$$\int x^{m\pm\alpha n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}\pm\beta}$$

can be recalled, which hence all depend on the same transcending function.

SCHOLIUM 1

118. With the form of the differential of $x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}}$ thus prescribed

$$mx^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}+\tfrac{n\mu}{\nu}bx^{m+n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}-1},$$

we deduce this reduction

$$\int x^{m+n-1}\left(a+bx^n\right)^{\frac{\mu}{\nu}-1}=\frac{\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}}}{n\mu b}-\frac{m\nu}{n\mu b}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$$

and in addition rearranging this, and on writing $m-n$ and $\mu+\nu$ for m and μ :

$$\int x^{m-n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}=\frac{x^{m-n}\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{m-n}-\frac{n(\mu+\nu)b}{\nu(m-n)}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}.$$

Hence clearly in one operation the reduction is complete, since the above formulas finish in a two-fold reduction ; from which we have obtained six noteworthy reductions, which on that account we set out to be considered together.

$$\text{I. } \int x^{m+n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}} = \frac{\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{\left(m\nu+n(\mu+\nu)\right)b} - \frac{m\nu a}{\left(m\nu+n(\mu+\nu)\right)b}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

$$\text{II. } \int x^{m-n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}} = \frac{x^{m-n}\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{(m-n)a} - \frac{(m\nu+n\mu)b}{(m-n)\nu a}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

$$\text{III. } \int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}+1} = \frac{\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{m\nu+n(\mu+\nu)} + \frac{n(\mu+\nu)a}{m\nu+n(\mu+\nu)}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

$$\text{IV. } \int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}-1} = \frac{-\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}}}{n\mu a} + \frac{m\nu+n\mu}{n\mu a}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

$$\text{V. } \int x^{m+n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}-1} = \frac{\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}}}{n\mu b} - \frac{m\nu}{n\mu b}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

$$\text{VI. } \int x^{m-n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}+1} = \frac{x^{m-n}\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{m-n} - \frac{n(\mu+\nu)b}{\nu(m-n)}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}.$$

SCHOLION 2

119. Concerning these reductions at first it is to be noted that the first formula is to be integrated algebraically, if the coefficient of the last [integral] term vanishes. Thus there becomes,

$$\text{for I., if } m = 0, \int x^{n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}} = \frac{\nu\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)b},$$

$$\text{for II., if } \frac{\mu}{\nu} = \frac{-m}{n}, \int x^{m-n-1}dx\left(a+bx^n\right)^{\frac{-m}{n}} = \frac{x^{m-n}\left(a+bx^n\right)^{\frac{-m}{n}+1}}{(m-n)a},$$

$$\text{for IV., if } \frac{\mu}{\nu} = \frac{-m}{n}, \int x^{m-1}dx\left(a+bx^n\right)^{\frac{-m}{n}-1} = \frac{x^m\left(a+bx^n\right)^{\frac{-m}{n}}}{ma},$$

$$\text{for V., if } m = 0 \int x^{n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}-1} = \frac{\nu\left(a+bx^n\right)^{\frac{\mu}{\nu}}}{n\mu b}.$$

Then also the case is worthy of note, in which the coefficient of the final term of the formula is infinite; for then the reduction process ceases and the first formula has its own special integral to be evaluated.

In the first this comes about , if $\frac{\mu+\nu}{\nu} = \frac{-m}{n}$, and the formula $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{-m}{n}-1}$ on putting $a+bx^n = x^n z^n$ or $x^n = \frac{a}{z^n-b}$ changes into $\int \frac{-z^{-m-1}dz}{z^n-b}$, and the integral of this must be defined according to the first chapter.

In the second this comes about if $m = n$, and the formula $\int \frac{dx}{x}\left(a+bx^n\right)^{\frac{\mu}{\nu}}$ on putting $a+bx^n = z^n$ or $x^n = \frac{z^\nu - a}{b}$ changes into $\int \frac{\nu z^{\mu+\nu-1}dz}{n\left(z^\nu - a\right)}$

In the third it comes about, if $\frac{\mu}{\nu} = \frac{-m}{n} - 1$, and the formula $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{-m}{n}}$ on putting $a+bx^n = x^n z^n$ or $x^n = \frac{a}{z^n - b}$ changes into $\int \frac{-z^{-m+n-1}dz}{z^n - b}$ or on putting $z = \frac{1}{u}$ into $\int \frac{u^{m-1}du}{1-bu^n}$.

In the fourth it eventuates, if $\mu = 0$, and the formula $\int \frac{x^{m-1}dx}{a+bx^n}$ itself is rational.

In the fifth likewise it eventuates, if $\mu = 0$.

But in the sixth, if $m = n$, and the formula $\int \frac{dx}{x}\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}$ on putting $a+bx^n = z^\nu$ changes into $\frac{\nu}{n}\int \frac{z^{\mu+2\nu-1}dz}{z^\nu - a}$.

EXAMPLE 1

120. *To find the integral of this formula* $\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}}$, *for a given positive exponent m.*

Here on account of $a = 1, b = -1, n = 2, \mu = -1, \nu = 2$ in the first place the reduction gives

$$\int \frac{x^{m+1}dx}{\sqrt{(1-xx)}} = \frac{-x^m\sqrt{(1-xx)}}{m+1} + \frac{m}{m+1}\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}};$$

hence, as either even or odd numbers are taken for *m*, we obtain :

For odd numbers

$$\int \frac{xxdx}{\sqrt{(1-xx)}} = -\frac{1}{2}x\sqrt{(1-xx)} + \frac{1}{2}\int \frac{dx}{\sqrt{(1-xx)}},$$
$$\int \frac{x^4dx}{\sqrt{(1-xx)}} = -\frac{1}{4}x^3\sqrt{(1-xx)} + \frac{3}{4}\int \frac{x^2dx}{\sqrt{(1-xx)}},$$
$$\int \frac{x^6dx}{\sqrt{(1-xx)}} = -\frac{1}{6}x^5\sqrt{(1-xx)} + \frac{5}{6}\int \frac{x^4dx}{\sqrt{(1-xx)}},$$

etc.;

for the even numbers

$$\int \frac{x^3 dx}{\sqrt{(1-xx)}} = -\tfrac{1}{3}x^2\sqrt{(1-xx)} + \tfrac{2}{3}\int \frac{xdx}{\sqrt{(1-xx)}},$$
$$\int \frac{x^5 dx}{\sqrt{(1-xx)}} = -\tfrac{1}{5}x^4\sqrt{(1-xx)} + \tfrac{4}{5}\int \frac{x^3dx}{\sqrt{(1-xx)}},$$
$$\int \frac{x^7 dx}{\sqrt{(1-xx)}} = -\tfrac{1}{7}x^6\sqrt{(1-xx)} + \tfrac{6}{7}\int \frac{x^5dx}{\sqrt{(1-xx)}},$$
etc.

Now since there shall be

$$\int \frac{dx}{\sqrt{(1-xx)}} = \text{Arc.sin.}x \quad \text{and} \quad \int \frac{xdx}{\sqrt{(1-xx)}} = -\sqrt{(1-xx)},$$

we have the following integrals:
For the first order

$$\int \frac{dx}{\sqrt{(1-xx)}} = \text{Arc.sin.}x,$$
$$\int \frac{xxdx}{\sqrt{(1-xx)}} = -\tfrac{1}{2}x\sqrt{(1-xx)} + \tfrac{1}{2}\text{Arc.sin.}x,$$
$$\int \frac{x^4dx}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{4}x^3 + \tfrac{1\cdot 3}{2\cdot 4}x\right)\sqrt{(1-xx)} + \tfrac{1\cdot 3}{2\cdot 4}\text{Arc.sin.}x,$$
$$\int \frac{x^6dx}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{6}x^5 + \tfrac{1\cdot 5}{4\cdot 6}x^3 + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x\right)\sqrt{(1-xx)} + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\text{Arc.sin.}x,$$
$$\int \frac{x^8dx}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{8}x^7 + \tfrac{1\cdot 7}{6\cdot 8}x^5 + \tfrac{1\cdot 5\cdot 7}{4\cdot 6\cdot 8}x^3 + \tfrac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}x\right)\sqrt{(1-xx)} + \tfrac{1\cdot 3\cdot 5\cdot 7}{2\cdot 4\cdot 6\cdot 8}\text{Arc.sin.}x;$$

for the second order

$$\int \frac{xdx}{\sqrt{(1-xx)}} = -x\sqrt{(1-xx)},$$
$$\int \frac{x^3dx}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{3}x^2 + \tfrac{2}{3}\right)\sqrt{(1-xx)},$$
$$\int \frac{x^5dx}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{5}x^4 + \tfrac{1\cdot 4}{3\cdot 5}x^2 + \tfrac{2\cdot 4}{3\cdot 5}\right)\sqrt{(1-xx)},$$
$$\int \frac{x^7dx}{\sqrt{(1-xx)}} = -\left(\tfrac{1}{7}x^6 + \tfrac{1\cdot 6}{5\cdot 7}x^4 + \tfrac{1\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^2 + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}\right)\sqrt{(1-xx)}.$$

COROLLARIUM 1

121. Hence in general for the formula $\int \frac{x^{2i}dx}{\sqrt{(1-xx)}}$, if we put for the sake of brevity $\frac{1\cdot 3\cdot 5\cdot 7\ldots(2i-1)}{2\cdot 4\cdot 6\ldots\ldots\ldots 2i} = J$, we have this integral

$$\int \frac{x^{2i}dx}{\sqrt{(1-xx)}} =$$

$$= J\text{Arc.sin.}x - J\left(x + \tfrac{2}{3}x^3 + \tfrac{2\cdot 4}{3\cdot 5}x^5 + \tfrac{2\cdot 4\cdot 6}{3\cdot 5\cdot 7}x^7 + \ldots + \tfrac{2\cdot 4\cdot 6..(2i-2)}{3\cdot 5\cdot 7...(2i-1)}x^{2i-1}\right)\sqrt{(1-xx)}.$$

COROLLARY 2

122. In a similar manner for the formula $\int \frac{x^{2i+1}dx}{\sqrt{(1-xx)}}$, hence if we put for brevity $\frac{2\cdot 4\cdot 6..........2i}{1\cdot 3\cdot 5\cdot 7...(2i+1)} = K$, then we have this integral

$$\int \frac{x^{2i+1}dx}{\sqrt{(1-xx)}} =$$

$$= K - K\left(1 + \tfrac{1}{2}x^2 + \tfrac{1\cdot 3}{2\cdot 4}x^4 + \tfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}x^6 + \ldots + \tfrac{1\cdot 3\cdot 5\cdot...(2i-1)}{2\cdot 4\cdot 6\cdot..2i}x^{2i}\right)\sqrt{(1-xx)},$$

in order that the integral vanishes on putting $x = 0$.

EXAMPLE 2

123. *To find the integral of the formula* $\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}}$ *for the cases, in which for m negative numbers are taken.*

Here use is made of the second reduction, which gives

$$\int \frac{x^{m-3}dx}{\sqrt{(1-xx)}} = \frac{x^{m-2}\sqrt{(1-xx)}}{m-2} + \frac{m-1}{m-2}\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

from which it is apparent, if $m = 1$, that there becomes

$$\int \frac{dx}{xx\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{x};$$

then, if $m = 2$, by making the substitution $1 - xx = zz$, the formula $\int \frac{dx}{x\sqrt{(1-xx)}}$ is changed into $\int \frac{-dz}{1-zz}$, the integral of which is

$$-\tfrac{1}{2}l\,\tfrac{1+z}{1-z} = -\tfrac{1}{2}l\,\frac{1+\sqrt{(1-xx)}}{1-\sqrt{(1-xx)}} = -l\,\frac{1+\sqrt{(1-xx)}}{x},$$

from which we elicit the two-fold series of integrations :

$$\int\frac{dx}{x\sqrt{(1-xx)}}=-l\,\frac{1+\sqrt{(1-xx)}}{x}=l\,\frac{1-\sqrt{(1-xx)}}{x},$$

$$\int\frac{dx}{x^3\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{2xx}+\frac{1}{2}\int\frac{dx}{x\sqrt{(1-xx)}},$$

$$\int\frac{dx}{x^5\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{4x^4}+\frac{3}{4}\int\frac{dx}{x^3\sqrt{(1-xx)}},$$

$$\int\frac{dx}{x^7\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{6x^6}+\frac{5}{6}\int\frac{dx}{x^5\sqrt{(1-xx)}}$$

etc.

and

$$\int\frac{dx}{xx\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{x},$$

$$\int\frac{dx}{x^4\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{3x^3}+\frac{2}{3}\int\frac{dx}{xx\sqrt{(1-xx)}},$$

$$\int\frac{dx}{x^6\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{5x^5}+\frac{4}{5}\int\frac{dx}{x^4\sqrt{(1-xx)}}$$

etc.

Hence as in the two preceding corollaries, there is

$$\int\frac{dx}{x^{2i+1}\sqrt{(1-xx)}}=$$

$$=Jl\,\frac{1-\sqrt{(1-xx)}}{x}-J\left(\frac{1}{xx}+\frac{2}{3x^4}+\frac{2\cdot 4}{3\cdot 5x^6}+\ldots+\frac{2\cdot 4\cdot 6..(2i-2)}{3\cdot 5\cdot 7...(2i-1)x^{2i}}\right)\sqrt{(1-xx)},$$

$$\int\frac{dx}{x^{2i+2}\sqrt{(1-xx)}}=$$

$$=C-K\left(\frac{1}{x}+\frac{1}{2x^3}+\frac{1\cdot 3}{2\cdot 4x^5}+\ldots+\frac{1\cdot 3\cdot 5\cdot...(2i-1)}{2\cdot 4\cdot 6\cdot..2ix^{2i+1}}\right)\sqrt{(1-xx)}.$$

[Here a small correction has been made to the final integral in the first edition.]

SCHOLIUM 1

124. Hence now the integral of the formulas $\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}}$ are able to be assigned easily both for all the numbers *m* as for the odd values of μ. But our general reductions can be applied to this case :

I. $\int x^{m+1}dx(1-xx)^{\frac{\mu}{2}} = \frac{-x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{m}{m+\mu+2}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}}$,

II. $\int x^{m-3}dx(1-xx)^{\frac{\mu}{2}} = \frac{x^{m-2}(1-xx)^{\frac{\mu}{2}+1}}{m-2} + \frac{m+\mu}{m-2}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}}$,

III. $\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}+1} = \frac{x^m(1-xx)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{\mu+2}{m+\mu+2}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}}$,

IV. $\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}-1} = \frac{-x^m(1-xx)^{\frac{\mu}{2}}}{\mu} + \frac{m+\mu}{\mu}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}}$,

V. $\int x^{m+1}dx(1-xx)^{\frac{\mu}{2}-1} = \frac{-x^m(1-xx)^{\frac{\mu}{2}}}{\mu} + \frac{m}{\mu}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}}$,

VI. $\int x^{m-3}dx(1-xx)^{\frac{\mu}{2}+1} = \frac{x^{m-2}(1-xx)^{\frac{\mu}{2}+1}}{m-2} + \frac{\mu+2}{m-2}\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}}$.

For on putting $\mu = -1$ the four latter integrals give

$$\int x^{m-1}dx\sqrt{(1-xx)} = \frac{x^m\sqrt{(1-xx)}}{m+1} + \frac{1}{m+1}\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

$$\int\frac{x^{m-1}dx}{\sqrt{(1-xx)^3}}^{\frac{\mu}{2}} = \frac{x^m}{\sqrt{(1-xx)}} - (m-1)\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

$$\int\frac{x^{m+1}dx}{\sqrt{(1-xx)^3}} = \frac{x^m}{\sqrt{(1-xx)}} - m\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

$$\int x^{m-3}dx(1-xx) = \frac{x^{m-2}\sqrt{(1-xx)}}{m-2} + \frac{1}{m-2}\int\frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

from which the integrations for the cases $\mu = 1$ and $\mu = -3$ can be elicited, and thus again for the remaining cases.

SCHOLIUM 2

125. For other irrational formulas scarcely more complicated rules can be given, by which they can be reduced to the simpler form ; and as often as formulas of this kind occur, if they allow a reduction, generally this presents itself freely. Just as if the formula were of this kind $\int\frac{Pdx}{Q^{n+1}}$, n is either a whole number or a fraction, it can always be reduced to another formula of this kind $\int\frac{Sdx}{Q^n}$, which is certainly considered to be simpler. For as

$$d.\frac{R}{Q^n} = \frac{QdR-nRdQ}{Q^{n+1}},$$

on putting $\int\frac{Pdx}{Q^{n+1}} = y$, then

$$y + \frac{R}{Q^n} = \int\frac{Pdx+QdR-nRdQ}{Q^{n+1}}.$$

Now *R* can be defined thus, so that $Pdx+QdR-nRdQ$ is made divisible by *Q*, or because *QdB* now has a factor *Q*, so that it becomes $Pdx-nRdQ=QTdx$, and there is produced

$$y+\frac{R}{Q^n}=\int\frac{dR+Tdx}{Q^n}$$

or

$$\int\frac{Pdx}{Q^{n+1}}=-\frac{R}{Q^n}+\int\frac{dR+Tdx}{Q^n};$$

but it is always thus necessary to define the function *R*, so that $Pdx-nRdQ$ can give the factor *Q*; since if in general this cannot be fulfilled, yet by testing with examples it soon becomes apparent that the calculation always succeeds. But here with the assumption that *P* and *Q* are integral functions, and also for such functions *R* is always able to be elicited. If perhaps it comes about, that $dR+Tdx=0$, the proposed formula has an algebraic integral, which can be found in this way; but on the other hand this form can be reduced further into other forms, where the exponent of the denominator is diminished continually to unity ; and if *n* is a whole number, the calculation finally is reduced to a form of this kind $\frac{Vdx}{Q}$, which is the simplest without doubt. On account of which, since in this chapter scarcely anything more can be offered aiding in the solution of irrational formulas, we shall set out the method the same integrations can be performed by infinite series.

ADDENTUM

PROBLEM

To find the integral of this proposed formula $dy=(x+\sqrt{(1+xx)})^n dx$.

SOLUTION

Putting $x+\sqrt{(1+xx)}=u$, there is made $x=\frac{uu-1}{2u}$ and $dx=\frac{du(uu+1)}{2uu}$, by which our formula becomes :

$$dy=\tfrac{1}{2}u^{n-2}du\left(uu+1\right)$$

and thus the integral of this

$$y=\frac{u^{n+1}}{2(n+1)}+\frac{u^{n-1}}{2(n-1)}+\text{Const.},$$

which thus is always algebraic, unless either $n=1$ or $n=-1$.

COROLLARY 1

Also it is evident that this form can be extended further

$$dy = (x+\sqrt{(1+xx)})^n X dx$$

in this manner to be able to be integrated, provided X is some rational function of x. For on putting $x = \frac{uu-1}{2u}$, there is produced a rational function of u for X, which is equal to U, and hence there becomes

$$dy = \tfrac{1}{2} U u^{n-2} du \left(uu+1\right),$$

which formula if either rational, if n is a whole number, or which can easily be reduced to rationality, if n is a fraction.

COROLLARY 2

Since $x+\sqrt{(1+xx)} = \frac{uu+1}{2u}$, on putting $\sqrt{(1+xx)} = v$ this formula

$$dy = (x+\sqrt{(1+xx)})^n X dx$$

can also be integrated, if X is some rational function of the quantities x and v. For on making $x = \frac{uu-1}{2u}$ the function X is changed into a rational function of u, from which on putting equal to U as before there is obtained

$$dy = \tfrac{1}{2} U u^{n-2} du \left(uu+1\right).$$

EXAMPLE

The proposed formula shall be $dy = (ax+b\sqrt{(1+xx)})\,(x+\sqrt{(1+xx)})^n dx$.
On putting $x = \frac{uu-1}{2u}$ there becomes

$$dy = \frac{a(uu-1)+b(uu+1)}{2u} \cdot \tfrac{1}{2} u^{n-2} du \left(uu+1\right)$$

or

$$dy = \tfrac{1}{4} u^{n-3} du \left(a\left(u^4-1\right)+b\left(u^4+2uu+1\right)\right),$$

and the integral of this is

$$y = \frac{a+b}{4(n+2)} u^{n+2} + \frac{b}{2n} u^n + \frac{b-a}{4(n-2)} u^{n-2} + \text{Const.},$$

which is algebraic, unless either $n = 2$ or $n = -2$, or also $n = 0$.

CAPUT 2

DE INTEGRATIONE FORMULARUM DIFFERENTIALIUM IRRATIONALIUM

PROBLEMA 6

88. *Proposita formula differentiali* $dy = \frac{dx}{\sqrt{(\alpha+\beta x+\gamma xx)}}$ *eius integrale invenire.*

SOLUTIO

Quantitas $\alpha + \beta x + \gamma xx$ vel habet duos factores reales vel secus.

I. Priori casu formula proposita erit huiusmodi

$$dy = \frac{dx}{\sqrt{(a+bx)(f+gx)}};$$

statuatur ad irrationalitatem tollendam

$$(a+bx)(f+gx) = (a+bx)^2 zz;$$

erit

$$x = \frac{f-azz}{bzz-g} \quad \text{ideoque} \quad dx = \frac{2(ag-bf)zdz}{(bzz-g)^2}$$

et

$$\sqrt{(a+bx)(f+gx)} = -\frac{(ag-bf)z}{bzz-g}$$

unde fit

$$dy = \frac{-2dz}{bzz-g} = \frac{2dz}{g-bzz} \quad \text{and} \quad z = \sqrt{\frac{f+gx}{a+bx}}.$$

Quare si litterae b et g paribus signis sunt affectae, integrale per logarithmos, sin autem signis disparibus, per angulos exprimetur.

II. Posteriori casu habebimus

$$dy = \frac{dx}{\sqrt{(aa-2abx\cos.\zeta+bbxx)}}$$

statuatur

$$aa - 2abx\cos.\zeta + bbxx = (bx-az)^2;$$

erit

$$-2bx\cos.\zeta + a = -2bxz + azz$$

et

$$x = \frac{a(1-zz)}{2b(\cos.\zeta - z)}, \quad \text{hinc} \quad dx = \frac{adz(1-2z\cos.\zeta + zz)}{2b(\cos.\zeta - z)^2}$$

et

$$\sqrt{(aa - 2abx\cos.\zeta + bbxx)} = \frac{a(1-2z\cos.\zeta + zz)}{2(\cos.\zeta - z)},$$

ergo

$$dy = \frac{dz}{b(\cos.\zeta - z)} \quad \text{et} \quad y = -\frac{1}{b} l(\cos.\zeta - z).$$

At est

$$z = \frac{bx - \sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a}$$

ideoque

$$y = -\frac{1}{b} l \frac{a\cos.\zeta - bx + \sqrt{(aa - 2abx\cos.\zeta + bbxx)}}{a}$$

vel

$$y = \frac{1}{b} l\left(-a\cos.\zeta + bx + \sqrt{(aa - 2abx\cos.\zeta + bbxx)}\right) + C.$$

COROLLARIUM 1

89. Casus ultimus latius patet et ad formulam

$$dy = \frac{dx}{\sqrt{(\alpha + \beta x + \gamma xx)}}$$

accommodari potest, dummodo fuerit γ quantitas positiva ; namque ob $b = \sqrt{\gamma}$ et $a\cos.\gamma = \frac{-\beta}{2\sqrt{\gamma}}$ oritur

$$y = \frac{1}{\sqrt{\gamma}} l\left(\frac{\beta}{2\sqrt{\gamma}} + x\sqrt{\gamma} + \sqrt{(\alpha + \beta x + \gamma xx)}\right) + C.$$

seu

$$y = \frac{1}{\sqrt{\gamma}} l\left(\frac{\beta}{2} + \gamma x + \sqrt{\gamma(\alpha + \beta x + \gamma xx)}\right) + C.$$

COROLLARIUM 2

90. Pro casu priori cum sit

$$\int \frac{2dz}{g - bzz} = \frac{1}{\sqrt{bg}} l \frac{\sqrt{g} + z\sqrt{b}}{\sqrt{g} - z\sqrt{b}} \quad \text{et} \quad \int \frac{2dz}{gzz + bzz} = \frac{2}{\sqrt{gb}} \text{Arc. tang.} \frac{z\sqrt{b}}{\sqrt{g}},$$

habebimus hos casus

$$\int \frac{dx}{\sqrt{(a+bx)(f+gx)}} = \frac{1}{\sqrt{bg}} l \frac{\sqrt{g(a+bx)}+\sqrt{b(f+gx)}}{\sqrt{g(a+bx)}-\sqrt{b(f+gx)}} + C$$

$$\int \frac{dx}{\sqrt{(bx-a)(f+gx)}} = \frac{1}{\sqrt{bg}} l \frac{\sqrt{g(bx-a)}+\sqrt{b(f+gx)}}{\sqrt{g(bx-a)}-\sqrt{b(f+gx)}} + C$$

$$\int \frac{dx}{\sqrt{(bx-a)(gx-f)}} = \frac{1}{\sqrt{bg}} l \frac{\sqrt{g(bx-a)}+\sqrt{b(gx-f)}}{\sqrt{g(bx-a)}-\sqrt{b(gx-f)}} + C$$

$$\int \frac{dx}{\sqrt{(a-bx)(f-gx)}} = \frac{-1}{\sqrt{bg}} l \frac{\sqrt{g(a-bx)}+\sqrt{b(f-gx)}}{\sqrt{g(a-bx)}-\sqrt{b(f-gx)}} + C$$

$$\int \frac{dx}{\sqrt{(a-bx)(f+gx)}} = \frac{2}{\sqrt{bg}} \text{Arc.tang.} \frac{\sqrt{b(f+gx)}}{\sqrt{g(a-bx)}} + C$$

$$\int \frac{dx}{\sqrt{(a-bx)(gx-f)}} = \frac{2}{\sqrt{bg}} \text{Arc.tang.} \frac{\sqrt{b(gx-f)}}{\sqrt{g(a-bx)}} + C.$$

COROLLARIUM 3

91. Harum sex integrationum quatuor priores omnes in casu Corollarii 1continentur, binae autem postremae in hac formula

$$dy = \frac{dx}{\sqrt{(a+\beta x-\gamma xx)}}$$

continentur; sit enim pro penultima

$$af = \alpha,\ ag - bf = \beta, bg = \gamma,$$

unde colligitur

$$y = \frac{2}{\sqrt{\gamma}} \text{Arc.tang.} \frac{2\sqrt{\gamma(\alpha+\beta x-\gamma xx)}}{\beta-2\gamma x},$$

si scilicet ille arcus duplicetur. Per cosinum autem erit

$$y = \frac{2}{\sqrt{\gamma}} \text{Arc.cos.} \frac{\beta-2\gamma x}{\sqrt{(\beta\beta+4\alpha\gamma)}} + C,$$

cuius veritas ex differentiatione patet.

SCHOLION 1

92. Ex solutione huius problematis patet etiam hanc formulam latius patentem

$$\frac{Xdx}{\sqrt{(a+\beta x+\gamma xx)}},$$

si *X* fuerit functio rationalis quaecunque ipsius *x*, per praecepta capitis praecedentis integrari posse. Introducta enim loco *x* variabili *z*, qua formula radicalis rationalis redditur, etiam *X* abibit in functionem rationalem ipsius *z*.Idem adhuc generalius locum habet, si posito $\sqrt{(a+\beta x+\gamma xx)}=u$ fuerit X functio quaecunque rationalis binarum quantitatum x et u; tum enim per substitutionem adhibitam, quia tam pro x quam pro u formulae rationales ipsius z scribuntur, prodibit formula differentialis rationalis. Hoc idem etiam ita enunciari potest, ut dicamus formulae Xdx, si functio X nullam aliam irrationalem praeter $\sqrt{(a+\beta x+\gamma xx)}$ involvat, integrale assignari posse, propterea quod ea ope substitutionis in formulam differentialem rationalem transformari potest.

SCHOLION 2

93. Proposita autem formula differentiali quacunque irrationali ante omnia videndum est, num ea ope cuiuspiam substitutionis in rationalem transformari possit; quod si succedat, integratio per praecepta capitis praecedentis absolvi poterit, unde simul intelligitur integrale, nisi sit algebraicum, alias quantitates transcendentes non involvere praeter logarithmos et angulos. Quodsi autem nulla substitutio ad hoc idonea inveniri possit, ab integrationis labore est desistendum, quandoquidem integrale neque algebraice neque per logarithmos vel angulos exprimere valemus. Veluti si Xdx fuerit eiusmodi formula differentialis, quae nullo pacta ad rationalitatem reduci queat, eius integrale $\int Xdx$ ad novum genus functionum transcendentium erit referendum, in quo nihil aliud nobis relinquitur, nisi ut eius valorem. vero proxime assignare conemur. Admisso autem novo genere quantitatum transcendentium innumerabiles aliae formulae eo reduci atque integrari poterunt. Imprimis igitur in hoc erit elaborandum, ut pro quolibet genere formula simplicissima notetur, qua concessa reliquarum formularum integralia definire liceat. Hinc deducimur ad quaestionem maximi momenti, quomodo integrationem formularum magis complicatarum ad simpliciores reduci oporteat. Quod antequam aggrediamur, alias eiusmodi formulas perpendamus, quae ope idoneae substitutionis ab irrationalitate liberari queant, quemadmodum iam ostendimus, quoties X fuerit functio rationalis quantitatum x et $u=\sqrt{(a+\beta x+\gamma xx)}$ **,**ita ut alia irrationalitas non ingrediatur praeter radicem quadratam huiusmodi formulae $a+\beta x+\gamma xx$, toties formulam differentialem Xdx in rationalem transformari posse.

PROBLEMA 7

94. *Proposita formula differentiali* $Xdx(a+bx)^{\frac{\mu}{\nu}}$ *in qua X denotet functionent quamcunque rationalem ipsius x, eam ab irrationalitate liberare.*

SOLUTIO

Statuatur $a+bx=z^{\nu}$, ut fiat $(a+bx)^{\frac{\mu}{\nu}}=z^{\mu}$ tum, quia $x=\frac{z^{\nu}-a}{b}$, facta hac substitutione functio X abibit in functionem rationalem ipsius z, quae sit Z, et ob $dx=\frac{\nu}{b}z^{\nu-1}dz$ formula nostra differentialis induet hanc formam $\frac{\nu}{b}Zz^{\mu+\nu-1}dz$; quae cum sit rationalis, per caput superius integrari potest et integrale, nisi sit algebraicum, per logarithmos et angulos exprimetur.

COROLLARIUM 1

95. Hac substitutione generalius negotium confici poterit, si posito $(a+bx)^{\frac{1}{\nu}}=u$ littera *V* denotet functionem quamcunque rationalem binarum quantitatum *x* et *u;* cum enim posito $x=\frac{u^{\nu}-a}{b}$ fiat *V* functio rationalis ipsius *u*, formula $Vdx=\frac{\nu}{b}Vu^{\nu-1}du$ erit rationalis.

COROLLARIUM 2

96. Quin etiam si binae irrationalitates eiusdem quantitatis *a + bx,* scilicet $(a+bx)^{\frac{1}{\nu}}=u$ et $(a+bx)^{\frac{1}{n}}=\text{v}$, ingrediantur in formulam *Xdx*, posito $a+bx=z^{n\nu}$ fit $x=\frac{z^{n\nu}-a}{b}, u=z^{n}$ et $\text{v}=z^{\nu}$; unde cum *X* fiat functio rationalis ipsius *z* et $dx=\frac{n\nu}{b}z^{n\nu-1}dz$, hac substitutione formula *Xdx* evadet rationalis.

COROLLARIUM 3

97. Eodem modo intelligitur, si posito

$$(a+bx)^{\frac{1}{\lambda}}=u,\ (a+bx)^{\frac{1}{\mu}}=\upsilon, (a+bx)^{\frac{1}{\nu}}=t \quad \text{etc.}$$

littera *X* denotet functionem quamcunque rationalem quantitatum x, u, v, t etc., formulam differentialem *Xdx* rationalem reddi facto $a+bx=z^{\lambda\mu\nu}$;
erit enim

$$x=\frac{z^{\lambda\mu\nu}-a}{b}, u=z^{\mu\nu}, \text{v}=z^{\lambda\nu}, t=z^{\lambda\mu} \text{ etc. et } dx=\frac{\lambda\mu\nu}{b}z^{\lambda\mu\nu-1}.$$

EXEMPLUM

98. Proposita hac formula

$$dy=\frac{xdx}{\sqrt[3]{(1+x)}-\sqrt{(1+x)}}$$

facto $1+x=z^{6}$ reperitur

$$dy=-\frac{6z^{3}dz\left(1-z^{6}\right)}{1-z}$$

seu

$$dy=-6dz\left(z^{3}+z^{4}+z^{5}+z^{6}+z^{7}+z^{8}\right)$$

hincque integrando

$$y=C-\tfrac{3}{2}z^{4}-\tfrac{6}{5}z^{5}-z^{6}-\tfrac{6}{7}z^{7}-\tfrac{3}{4}z^{8}-\tfrac{2}{3}z^{9}$$

et restituendo

$$y = C - \tfrac{3}{2}\sqrt[3]{(1+x)^2} - \tfrac{6}{5}\sqrt[6]{(1+x)^5} - 1 - x - \tfrac{6}{7}(1+x)\sqrt[6]{(1+x)}$$
$$- \tfrac{3}{4}(1+x)\sqrt[3]{(1+x)} - \tfrac{2}{3}(1+x)\sqrt{(1+x)},$$

ita ut integrale adeo algebraice exhibeatur.

PROBLEMA 8

99. *Proposita formula differentiali* $Xdx\left(\frac{a+bx}{f+gx}\right)^{\frac{\mu}{\nu}}$*, denotante X functionem rationalem quamcunque ipsius x eam ab irrationalitate liberare.*

SOLUTIO

Posito

$$\frac{a+bx}{f+gx} = z^{\nu}$$

fit

$$\left(\frac{a+bx}{f+gx}\right)^{\frac{\mu}{\nu}} = z^{\mu}$$

et

$$x = \frac{a - fz^{\nu}}{gz^{\nu} - b} \quad \text{atque} \quad dx = \frac{\nu(bf - ag)z^{\nu-1}dz}{(gz^{\nu} - b)^2}$$

sicque loco *X* prodibit functio rationalis ipsius *z*, qua posita = *Z* erit formula nostra differentialis

$$= \frac{\nu(bf - ag)Zz^{\nu-1}dz}{(gz^{\nu} - b)^2}.$$

quae cum sit rationalis, per praecepta Capitis I integrari poterit.

COROLLARIUM 1

100. Posito $\left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\nu}} = u$ si *X* fuerit functio quaecunque rationalis binarum quantitatum *x* et *u*, formula differentialis *Xdx* per substitutionem usurpatam in rationalem transformabitur, cuius propterea integratio constat.

COROLLARIUM 2

101. Si *X* fuerit functio rationalis tam ipsius *x* quam quantitatum quotcunque huiusmodi

$$\left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\lambda}} = u, \quad \left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\mu}} = \mathrm{v}, \quad \left(\frac{a+bx}{f+gx}\right)^{\frac{1}{\nu}} = t$$

tum formula differentialis *Xdx* rationalis reddetur adhibita substitutione

$$\frac{a+bx}{f+gx}=z^{\lambda\mu\nu},$$

unde fit

$$x=\frac{a-fz^{\lambda\mu\nu}}{gz^{\lambda\mu\nu}-b} \quad \text{et} \quad u=z^{\mu\nu}, \quad \text{v}=z^{\lambda\nu}, \quad t=z^{\lambda\mu}.$$

SCHOLION 1

102. His casibus reductio ad rationalitatem ideo succedit, etiamsi plures formulae irrationales insint, quod eae omnes simul per eandem substitutionem rationales efficiantur indeque etiam ipsa quantitas x per novam variabilem z rationaliter exprimetur. Sin autem differentiale propositum duas eiusmodi formulas irrationales contineat, quae non ambae simul ope eiusdem substitutionis rationales reddi queant, etiamsi hoc in utraque seorsim fieri possit, reductio locum non habet, nisi forte ipsum differentiale in duas partes dispesci liceat, quarum utraque unam tantum formulam irrationalem complectatur. Veluti si proposita sit haec formula differentialis

$$dy=\frac{dx}{\sqrt{(1+xx)}-\sqrt{(1-xx)}},$$

eius numeratorem ac denominatorem per $\sqrt{(1+xx)}-\sqrt{(1-xx)}$ multiplicando fit

$$dy=\frac{dx\sqrt{(1+xx)}}{2xx}-\frac{dx\sqrt{(1-xx)}}{2xx},$$

cuius utraque pars seorsim rationalis reddi et integrari potest. Reperitur autem

$$y=C-\frac{\sqrt{(1-xx)}+\sqrt{(1+xx)}}{2x}+\frac{1}{2}l\left(x+\sqrt{(1+xx)}\right)-\frac{1}{2}\text{Arc.tang.}\frac{x}{\sqrt{(1-xx)}}.$$

Commodissime autem ibi irrationalitas tollitur, si in parte priori ponatur $\sqrt{(1+xx)}=px$, in posteriori $\sqrt{(1-xx)}=qx$. Etsi enim hinc sit

$$x=\frac{1}{\sqrt{(pp-1)}} \quad \text{et} \quad x=\frac{1}{\sqrt{(1+qq)}},$$

tamen oritur rationaliter

$$dy=\frac{-ppdp}{2(pp-1)}-\frac{qqdq}{(1+qq)}.$$

SCHOLION 2

103. Circa formulas generales, quae ab irrationalitate liberari queant, vix quicquam amplius praecipere licet, dummodo hunc casum addamus, quo functio X binas huiusmodi formulas radicales $\sqrt{(a+bx)}$ et $\sqrt{(f+gx)}$ complectitur.

Posito enim

$$(a+bx)=(f+gx)tt$$

fit

$$x = \frac{a - ftt}{gtt - b}$$

atque

$$\sqrt{(a+bx)} = \frac{t\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}}, \quad \sqrt{(f+gx)} = \frac{\sqrt{(ag-bf)}}{\sqrt{(gtt-b)}}$$

in formula differentiali unica tantum formula irrationalis $\sqrt{(gtt-b)}$, quae nova substitutione facile tolletur per ea, quae Problemate 6 tradidimus.

Ut igitur ad alia pergamus, imprimis considerari meretur haec formula differentialis

$$x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$$

cuius ob simplicitatem usus per universam Analysin est amplissimus, ubi quidem sumimus litteras $m,\ n, \mu, \nu$ numeros integros denotare; nisi enim tales essent, facile ad hanc formam reducerentur. Veluti si haberemus $x^{-\frac{1}{3}}dx\left(a+b\sqrt{x}\right)^{\frac{\mu}{\nu}}$, statui oportet $x = u^6$, hinc $dx = 6u^5du$, unde prodit $6u^3du\left(a+bu^3\right)^{\frac{\mu}{\nu}}$. Tum vero pro *n* valorem positivum assumere licet; si enim esset negativus, puta $x^{m-1}dx\left(a+bx^{-n}\right)^{\frac{\mu}{\nu}}$, ponatur $x = \frac{1}{u}$ fietque formula $-u^{-m-1}du(a+bu^n)^{\frac{\mu}{\nu}}$ similis principali; quae ergo quibus casibus ab irrationalitate liberari queat, investigemus.

PROBLEMA 9

104. *Definire casus, quibus formulam differentialem* $x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$ *ad rationalitatem perducere liceat.*

SOLUTIO

Primo patet, si fuerit $\nu = 1$ seu $\frac{\mu}{\nu}$ numerus integer, formulam per se fore rationalem neque substitutione opus esse. At si $\frac{\mu}{\nu}$ sit fractio, substitutione est utendum eaque duplici.

I. Ponatur

$$a + bx^n = u^\nu,$$

ut fiat

$$\left(a+bx^n\right)^{\frac{\mu}{\nu}} = u^\mu$$

erit $x^n = \frac{u^\nu - a}{b}$, hinc

$$x^m = \left(\frac{u^\nu - a}{b}\right)^{\frac{m}{n}} \quad \text{ideoque} \quad x^{m-1}dx = \frac{\nu}{nb}u^{\nu-1}du\left(\frac{u^\nu-a}{b}\right)^{\frac{m-n}{n}},$$

unde formula nostra fiet

$$\tfrac{\nu}{nb}u^{\mu+\nu-1}du\left(\tfrac{u^{\nu}-a}{b}\right)^{\frac{m-n}{n}}.$$

Hinc ergo patet, quoties exponens $\frac{m-n}{n}$ seu $\frac{m}{n}$ seu : fuerit numerus integer sive positivus sive negativus, hanc formulam esse rationalem.

II. Ponatur

$$a+bx^{n}=x^{n}z^{\nu},$$

ut fiat

$$x^{n}=\tfrac{a}{z^{\nu}-b} \quad \text{et} \quad \left(a+bx^{n}\right)^{\frac{\mu}{\nu}}=\frac{a^{\frac{\mu}{\nu}}z^{\mu}}{\left(z^{\nu}-b\right)^{\frac{\mu}{\nu}}};$$

tum

$$x^{m}=\frac{a^{\frac{m}{n}}}{\left(z^{\nu}-b\right)^{\frac{m}{n}}}, \quad \text{hinc} \quad x^{m-1}dx=\frac{-\nu a^{\frac{m}{n}}z^{\nu-1}dz}{n\left(z^{\nu}-b\right)^{\frac{m}{n}+1}}$$

ideoque formula nostra erit

$$\frac{-\nu a^{\frac{m}{n}+\frac{\mu}{\nu}}z^{\mu+\nu-1}dz}{n\left(z^{\nu}-b\right)^{\frac{m}{n}+\frac{\mu}{\nu}+1}}.$$

Ex quo patet hanc formam fore rationalem, quoties $\frac{m}{n}+\frac{\mu}{\nu}$ fuerit numerus integer. Facile autem intelligitur alias substitutiones huic scopo idoneas excogitari non posse. Quare concludimus formulam irrationalem hanc $x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$ ab irrationalitate liberari posse, si fuerit vel $\frac{m}{n}$ vel $\frac{m}{n}+\frac{\mu}{\nu}$ numerus integer.

COROLLARIUM 1

105. Si sit $\frac{m}{n}$ numerus integer, casus per se est facilis; ponatur enim $m = in$ et sit $x^{n}=v$; erit $x^{m}=v^{i}$ ideoque formula nostra

$$\tfrac{i}{m}v^{i-1}dv\left(a+bv\right)^{\frac{\mu}{\nu}},$$

quae per Problema 7 expeditur.

COROLLARIUM 2

106. At si $\frac{m}{n}$ non est numerus integer, ut reductio ad rationalitatem locum habeat, necesse est, ut $\frac{m}{n}+\frac{\mu}{\nu}$ sit numerus integer, quod fieri nequit, nisi sit ν = *n*, ideoque $m+\mu$ multiplum debet esse ipsius *n* = *ν*.

COROLLARY 3

107. Quodsi ergo haec formula $x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$ ad rationalitatem reduci queat, etiam haec formula $x^{m\pm\alpha n-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}\pm\beta}$ eandem reductionem admittet, quicunque numeri integri pro α et β assumantur. Unde ad casus reducibiles cognoscendos sufficit ponere $m<n$ et $\mu<\nu$.

COROLLARIUM 4

108. Si m = 0, haec formula $\frac{dx}{x}\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$; semper per casum primum ad rationalitatem reducitur ponendo $x^{n}=\frac{a}{z^{\nu}-b}$; transformatur enim in hanc $\frac{\nu u^{\mu+\nu-1}du}{n\left(u^{\nu}-a\right)}$.

SCHOLION 1

109. Quoniam formula $x^{m-1}dx\left(a+bx^{n}\right)^{\frac{\mu}{\nu}}$, quoties est $m = in$ denotante i numerum integrum sive positivum sive negativum quemcunque, semper ad rationalitatem reduci potest hique casus per se sunt perspicui, reliquos casus hanc reductionem admittentes accuratius contemplari operae pretium videtur. Quem in finem statuamus $\nu=n$ et $m<n$, item $\mu<n$, ac necesse est, ut sit $m+\mu=n$; unde sequentes formae in genere suo simplicissimae, quae quidem ad rationalitatem reduci queant, obtinentur.

I. $dx(a+bx^{2})^{\frac{1}{2}}$,

II. $dx(a+bx^{3})^{\frac{2}{3}}$, $\quad xdx(a+bx^{3})^{\frac{1}{3}}$,

III. $dx(a+bx^{4})^{\frac{3}{4}}$, $\quad xxdx(a+bx^{4})^{\frac{1}{4}}$,

IV. $dx(a+bx^{5})^{\frac{4}{5}}$, $\quad xdx(a+bx^{5})^{\frac{3}{5}}$, $\quad x^{2}dx(a+bx^{5})^{\frac{2}{5}}$, $\quad x^{3}dx(a+bx^{5})^{\frac{1}{5}}$,

V. $dx(a+bx^{6})^{\frac{5}{6}}$, $\quad x^{4}dx(a+bx^{6})^{\frac{1}{6}}$,

unde etiam hae reductionem admittent

$$x^{\pm 2\alpha}dx(a+bx^2)^{\frac{1}{2}\pm\beta},$$

$$x^{\pm 3\alpha}dx(a+bx^3)^{\frac{2}{3}\pm\beta},\quad x^{1\pm 3\alpha}dx(a+bx^3)^{\frac{1}{3}\pm\beta},$$

$$x^{\pm 4\alpha}dx(a+bx^4)^{\frac{3}{4}\pm\beta},\quad x^{2\pm 4\alpha}dx(a+bx^4)^{\frac{1}{4}\pm\beta},$$

$$x^{\pm 5\alpha}dx(a+bx^5)^{\frac{4}{5}\pm\beta},\quad x^{1\pm 5\alpha}dx(a+bx^5)^{\frac{2}{5}\pm\beta},$$

$$x^{2\pm 5\alpha}dx(a+bx^5)^{\frac{2}{5}\pm\beta},$$

$$x^{3\pm 5\alpha}dx(a+bx^5)^{\frac{1}{5}\pm\beta},$$

$$x^{\pm 6\alpha}dx(a+bx^6)^{\frac{5}{6}\pm\beta},\quad x^{4\pm 6\alpha}dx(a+bx^6)^{\frac{1}{6}\pm\beta}$$

SCHOLION 2

110. Verum etiamsi formula $x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$ ab irrationalitate liberari nequeat, tamen semper omnium harum formularum $x^{m\pm n\alpha-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}\pm\beta}$ integrationem ad eam reducere licet, ita ut illius integrali tanquam cognito spectato etiam harum integralia assignari queant. Quae reductio cum in Analysi summam afferat utilitatem, eam hic exponere necesse erit. Caeterum hic affirmare haud dubitamus praeter eos casus, quos reductionem ad rationalitatem admittere hic ostendimus, nullos alios existere, qui ulla substitutione adhibita ab irrationalitate liberari queant. Proposita enim hac formula $\frac{dx}{\sqrt{\left(a+bx^3\right)}}$ nulla functio rationalis ipsius *z* loco *x*; poni potest, ut $a+bx^3$ extractionem radicis quadratae admittat; obiici quidem potest scopo satisfieri posse, etiamsi loco *x* functio irrationalis ipsius *z* substituatur, dummodo similis irrationalitas in denominatore contineatur, qua illa numeratorem *dx* afficiens destruatur, quemadmodum fit in hac formula $\frac{dx}{\sqrt[3]{\left(a+bx^3\right)}}$ adhibendo substitutionem $x=\frac{\sqrt[3]{a}}{\sqrt[3]{\left(z^3-b\right)}}$; verum quod hic commode usu venit, nullo modo perspicitur, quomodo idem illo casu evenire possit. Hoc tamen minime pro demonstratione haberi volo.

PROBLEMA 10

111. *Integrationem formulae* $\int x^{m+n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$ *perducere ad integrationem huius formulae* $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$.

SOLUTIO

Consideretur functio $x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}$; cuius differentiale cum sit erit

$$\left(max^{m-1}\,dx+mb\,x^{m+n-1}\,dx+\frac{n(\mu+\nu)}{\nu}b\,x^{m+n-1}\,dx\right)\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

erit

$$x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}=ma\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}+\frac{(m\nu+n\mu+n\nu)b}{\nu}\int x^{m+n-1}\left(a+bx^n\right)^{\frac{\mu}{\nu}}dx,$$

unde elicitur

$$\int x^{m+n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}=\frac{\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{(m\nu+n\mu+n\nu)b}-\frac{m\nu a}{(m\nu+n\mu+n\nu)b}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}.$$

COROLLARIUM 1

112. Cum inde quoque sit

$$\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}=\frac{x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{ma}-\frac{(m\nu+n\mu+n\nu)b}{m\nu a}\int x^{m+n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

loco m scribamus $m-n$ et habebimus hanc reductionem

$$\int x^{m-n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}=\frac{x^{m-n}\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{(m-n)a}-\frac{(m\nu+n\mu)b}{(m-n)\nu a}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}.$$

COROLLARIUM 2

113. Concesso ergo integrali $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$ etiam harum formularum $\int x^{m\pm n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$; similique modo ulterius progrediendo omnium harum formularum $\int x^{m\pm\alpha n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$ integralia exhiberi possunt.

PROBLEMA 11

114. *Integrationem formulae* $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}$ *ad integrationem huius* $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$ *perducere.*

SOLUTIO

Functionis $x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}$ differentiale hoc modo exhiberi potest

$$\left(ma-\frac{(m\nu+n\mu+n\nu)a}{\nu}\right)x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}+\frac{m\nu+n\mu+n\nu}{\nu}x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}+1},$$

unde concluditur

$$x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}$$
$$=-\tfrac{(n\mu+n\nu)a}{\nu}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}+\tfrac{m\nu+n\mu+n\nu}{\nu}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}+1},$$

quocirca habebimus

$$\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}=\frac{\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{m\nu+n(\mu+\nu)}+\frac{n(\mu+\nu)a}{m\nu+n(\mu+\nu)}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}.$$

COROLLARIUM 1

115. Deinde ex eadem aequatione elicimus

$$\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}=\frac{-\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)a}+\frac{m\nu+n(\mu+\nu)}{n(\mu+\nu)a}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}};$$

scribamus iam $\mu-\nu$ loco μ ut nanciscamur hanc reductionem

COROLLARIUM 2

116. Concesso ergo integrali $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$ etiam harum formularum $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}\pm1}$ et ulterius progrediendo harum $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}\pm\beta}$ integralia exhiberi possunt denotante β numerum integrum quemcunque.

COROLLARIUM 3

117. His cum praecedentibus coniunctis ad integrationem

$$\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$$

omnia haec integralia

$$\int x^{m\pm\alpha n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}\pm\beta}$$

revocari possunt, quae ergo omnia ab eadem functione transcendente pendent.

SCHOLION 1

118. Ex formae $x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}}$ differentiali ita disposito

$$mx^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}+\tfrac{n\mu}{\nu}bx^{m+n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}-1}$$

deducimus hanc reductionem

$$\int x^{m+n-1}\left(a+bx^n\right)^{\frac{\mu}{\nu}-1}=\frac{\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}}}{n\mu b}-\frac{m\nu}{n\mu b}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}$$

ac praeterea hanc inversam pro m et μ scribendo $m - n$ et $\mu+\nu$

$$\int x^{m-n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}=\frac{x^{m-n}\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{m-n}-\frac{n(\mu+\nu)b}{\nu(m-n)}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}.$$

Hinc scilicet una operatione absolvitur reductio, cum superiores formulae duplicem reductionem exigant; ex quo sex reductiones sumus nacti omnino memorabiles, quas idcirco coniunctim conspectui exponamus.

I. $$\int x^{m+n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}=\frac{\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{\left(m\nu+n(\mu+\nu)\right)b}-\frac{m\nu a}{\left(m\nu+n(\mu+\nu)\right)b}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

II. $$\int x^{m-n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}=\frac{x^{m-n}\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{(m-n)a}-\frac{(m\nu+n\mu)b}{(m-n)\nu a}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

III. $$\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}=\frac{\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{m\nu+n(\mu+\nu)}+\frac{n(\mu+\nu)a}{m\nu+n(\mu+\nu)}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

IV. $$\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}-1}=\frac{-\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}}}{n\mu a}+\frac{m\nu+n\mu}{n\mu a}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

V. $$\int x^{m+n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}-1}=\frac{\nu x^m\left(a+bx^n\right)^{\frac{\mu}{\nu}}}{n\mu b}-\frac{m\nu}{n\mu b}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}},$$

VI. $$\int x^{m-n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}=\frac{x^{m-n}\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{m-n}-\frac{n(\mu+\nu)b}{\nu(m-n)}\int x^{m-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}}.$$

SCHOLION 2

119. Circa has reductiones primo observandum est formulam priorem algebraice esse integrabilem, si coefficiens posterioris evanescat. Ita fit pro I., si $m = 0, \int x^{n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}} = \frac{\nu\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}}{n(\mu+\nu)b}$,

pro II., si $\frac{\mu}{\nu} = \frac{-m}{n}, \int x^{m-n-1}dx\left(a+bx^n\right)^{\frac{-m}{n}} = \frac{x^{m-n}\left(a+bx^n\right)^{\frac{-m}{n}+1}}{(m-n)a}$,

pro IV., si $\frac{\mu}{\nu} = \frac{-m}{n}, \int x^{m-1}dx\left(a+bx^n\right)^{\frac{-m}{n}-1} = \frac{x^m\left(a+bx^n\right)^{\frac{-m}{n}}}{ma}$,

pro V., si $m = 0 \int x^{n-1}dx\left(a+bx^n\right)^{\frac{\mu}{\nu}-1} = \frac{\nu\left(a+bx^n\right)^{\frac{\mu}{\nu}}}{n\mu b}$.

Deinde etiam casus notari merentur, quibus coefficiens postremae formulae fit infinitus; tum enim reductio cessat et prior formula peculiare habet integrale seorsim evolvendum.
In prima hoc evenit, si $\frac{\mu+\nu}{\nu} = \frac{-m}{n}$, et formula $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{-m}{n}-1}$ posito $a+bx^n = x^n z^n$ seu $x^n = \frac{a}{z^n-b}$ abit in $\int \frac{-z^{-m-1}dz}{z^n-b}$, cuius integrale per caput primum definiri debet.
In secunda evenit, si $m = n$, et formula $\int \frac{dx}{x}\left(a+bx^n\right)^{\frac{\mu}{\nu}}$ posito $a+bx^n = z^n$ seu $x^n = \frac{z^\nu-a}{b}$ abit in $\int \frac{\nu z^{\mu+\nu-1}dz}{n\left(z^\nu-a\right)}$
In tertia evenit, si $\frac{\mu}{\nu} = \frac{-m}{n}-1$, et formula $\int x^{m-1}dx\left(a+bx^n\right)^{\frac{-m}{n}}$ posito $a+bx^n = x^n z^n$ seu $x^n = \frac{a}{z^n-b}$ abit in $\int \frac{-z^{-m+n-1}dz}{z^n-b}$ seu posito $z = \frac{1}{u}$ in $\int \frac{u^{m-1}du}{1-bu^n}$.
In quarta evenit, si $\mu = 0$, et formula $\int \frac{x^{m-1}dx}{a+bx^n}$ per se est rationalis.
In quinta idem evenit, $\mu = 0$.
In sexta autem si $m = n$, et formula $\int \frac{dx}{x}\left(a+bx^n\right)^{\frac{\mu}{\nu}+1}$ posito $a+bx^n = z^\nu$ abit in $\frac{\nu}{n}\int \frac{z^{\mu+2\nu-1}dz}{z^\nu-a}$.

EXEMPLUM 1

120. *Invenire integrale huius formulae* $\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}}$ *pro numeris positivis* exponenti *m datis.*

Hic ob $a = 1, b = -1, n = 2, \mu = -1, \nu = 2$ prima reductio dat

$$\int \frac{x^{m+1}dx}{\sqrt{(1-xx)}} = \frac{-x^m\sqrt{(1-xx)}}{m+1} + \frac{m}{m+1}\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}};$$

hinc, prout pro *m* sumantur numeri vel impares vel pares, obtinebimus:
Pro numeris imparibus

$$\int\frac{xxdx}{\sqrt{(1-xx)}}=-\tfrac{1}{2}x\sqrt{(1-xx)}+\tfrac{1}{2}\int\frac{dx}{\sqrt{(1-xx)}},$$
$$\int\frac{x^4dx}{\sqrt{(1-xx)}}=-\tfrac{1}{4}x^3\sqrt{(1-xx)}+\tfrac{3}{4}\int\frac{x^2dx}{\sqrt{(1-xx)}},$$
$$\int\frac{x^6dx}{\sqrt{(1-xx)}}=-\tfrac{1}{6}x^5\sqrt{(1-xx)}+\tfrac{5}{6}\int\frac{x^4dx}{\sqrt{(1-xx)}},$$

etc.;

pro numeris paribus

$$\int\frac{x^3dx}{\sqrt{(1-xx)}}=-\tfrac{1}{3}x^2\sqrt{(1-xx)}+\tfrac{2}{3}\int\frac{xdx}{\sqrt{(1-xx)}},$$
$$\int\frac{x^5dx}{\sqrt{(1-xx)}}=-\tfrac{1}{5}x^4\sqrt{(1-xx)}+\tfrac{4}{5}\int\frac{x^3dx}{\sqrt{(1-xx)}},$$
$$\int\frac{x^7dx}{\sqrt{(1-xx)}}=-\tfrac{1}{7}x^6\sqrt{(1-xx)}+\tfrac{6}{7}\int\frac{x^5dx}{\sqrt{(1-xx)}},$$

etc.;

Cum nunc sit

$$\int\frac{dx}{\sqrt{(1-xx)}}=\text{Arc.sin.}x \quad \text{et} \quad \int\frac{xdx}{\sqrt{(1-xx)}}=-\sqrt{(1-xx)},$$

habebimus sequentia integralia:

Pro ordine priore

$$\int\frac{dx}{\sqrt{(1-xx)}}=\text{Arc.sin.}x,$$
$$\int\frac{xxdx}{\sqrt{(1-xx)}}=-\tfrac{1}{2}x\sqrt{(1-xx)}+\tfrac{1}{2}\text{Arc.sin.}x,$$
$$\int\frac{x^4dx}{\sqrt{(1-xx)}}=-\left(\tfrac{1}{4}x^3+\tfrac{1\cdot3}{2\cdot4}x\right)\sqrt{(1-xx)}+\tfrac{1\cdot3}{2\cdot4}\text{Arc.sin.}x,$$
$$\int\frac{x^6dx}{\sqrt{(1-xx)}}=-\left(\tfrac{1}{6}x^5+\tfrac{1\cdot5}{4\cdot6}x^3+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}x\right)\sqrt{(1-xx)}+\tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}\text{Arc.sin.}x,$$
$$\int\frac{x^8dx}{\sqrt{(1-xx)}}=-\left(\tfrac{1}{8}x^7+\tfrac{1\cdot7}{6\cdot8}x^5+\tfrac{1\cdot5\cdot7}{4\cdot6\cdot8}x^3+\tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}x\right)\sqrt{(1-xx)}+\tfrac{1\cdot3\cdot5\cdot7}{2\cdot4\cdot6\cdot8}\text{Arc.sin.}x;$$

pro ordine posteriore

$$\int\frac{xdx}{\sqrt{(1-xx)}}=-x\sqrt{(1-xx)},$$
$$\int\frac{x^3dx}{\sqrt{(1-xx)}}=-\left(\tfrac{1}{3}x^2+\tfrac{2}{3}\right)\sqrt{(1-xx)},$$
$$\int\frac{x^5dx}{\sqrt{(1-xx)}}=-\left(\tfrac{1}{5}x^4+\tfrac{1\cdot4}{3\cdot5}x^2+\tfrac{2\cdot4}{3\cdot5}\right)\sqrt{(1-xx)},$$
$$\int\frac{x^7dx}{\sqrt{(1-xx)}}=-\left(\tfrac{1}{7}x^6+\tfrac{1\cdot6}{5\cdot7}x^4+\tfrac{1\cdot4\cdot6}{3\cdot5\cdot7}x^2+\tfrac{2\cdot4\cdot6}{3\cdot5\cdot7}\right)\sqrt{(1-xx)}.$$

COROLLARIUM 1

121. In genere ergo pro formula $\int \frac{x^{2i}dx}{\sqrt{(1-xx)}}$ si ponamus brevitatis gratia $\frac{1\cdot3\cdot5\cdot7\ldots(2i-1)}{2\cdot4\cdot6\ldots\ldots\ldots..2i} = J$, habebimus hoc integrale

$$\int \frac{x^{2i}dx}{\sqrt{(1-xx)}} =$$

$$= J\text{Arc.sin.}x - J\left(x + \tfrac{2}{3}x^3 + \tfrac{2\cdot4}{3\cdot5}x^5 + \tfrac{2\cdot4\cdot6}{3\cdot5\cdot7}x^7 + \ldots + \tfrac{2\cdot4\cdot6..(2i-2)}{3\cdot5\cdot7\ldots(2i-1)}x^{2i-1}\right)\sqrt{(1-xx)}.$$

COROLLARIUM 2

122.Simili modo pro formula $\int \frac{x^{2i+1}dx}{\sqrt{(1-xx)}}$, si ponamus brevitatis ergo $\frac{2\cdot4\cdot6\ldots\ldots\ldots..2i}{1\cdot3\cdot5\cdot7\ldots(2i+1)} = K$, habebimus hoc integrale

$$\int \frac{x^{2i+1}dx}{\sqrt{(1-xx)}} =$$

$$= K - K\left(1 + \tfrac{1}{2}x^2 + \tfrac{1\cdot3}{2\cdot4}x^4 + \tfrac{1\cdot3\cdot5}{2\cdot4\cdot6}x^6 + \ldots + \tfrac{1\cdot3\cdot5\cdot\ldots(2i-1)}{2\cdot4\cdot6\cdot..2i}x^{2i}\right)\sqrt{(1-xx)},$$

ut integrale evanescat posito $x = 0$.

EXEMPLUM 2

123. *Invenire integrale formulae* $\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}}$ *casibus, quibus pro m numeri negativi assumuntur.*

Hic utendum est secunda reductione, quae dat

$$\int \frac{x^{m-3}dx}{\sqrt{(1-xx)}} = \frac{x^{m-2}\sqrt{(1-xx)}}{m-2} + \frac{m-1}{m-2}\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

unde patet, si $m = 1$, fore

$$\int \frac{dx}{xx\sqrt{(1-xx)}} = -\frac{\sqrt{(1-xx)}}{x};$$

deinde, si $m = 2$, formula $\int \frac{dx}{x\sqrt{(1-xx)}}$ facta substitutione $1 - xx = zz$ abit in $\int \frac{-dz}{1-zz}$, cuius integrale est

$$-\tfrac{1}{2}l\,\frac{1+z}{1-z}=-\tfrac{1}{2}l\,\frac{1+\sqrt{(1-xx)}}{1-\sqrt{(1-xx)}}=-l\,\frac{1+\sqrt{(1-xx)}}{x}$$

unde duplicem seriem integrationum elicimus

$$\int\frac{dx}{x\sqrt{(1-xx)}}=-l\,\frac{1+\sqrt{(1-xx)}}{x}=l\,\frac{1-\sqrt{(1-xx)}}{x},$$

$$\int\frac{dx}{x^3\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{2xx}+\frac{1}{2}\int\frac{dx}{x\sqrt{(1-xx)}},$$

$$\int\frac{dx}{x^5\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{4x^4}+\frac{3}{4}\int\frac{dx}{x^3\sqrt{(1-xx)}},$$

$$\int\frac{dx}{x^7\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{6x^6}+\frac{5}{6}\int\frac{dx}{x^5\sqrt{(1-xx)}}$$

etc.

$$\int\frac{dx}{xx\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{x},$$

$$\int\frac{dx}{x^4\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{3x^3}+\frac{2}{3}\int\frac{dx}{xx\sqrt{(1-xx)}},$$

$$\int\frac{dx}{x^6\sqrt{(1-xx)}}=-\frac{\sqrt{(1-xx)}}{5x^5}+\frac{4}{5}\int\frac{dx}{x^4\sqrt{(1-xx)}}$$

etc.

Hinc erit ut in binis praecedentibus corollariis

$$\int\frac{dx}{x^{2i+1}\sqrt{(1-xx)}}=$$

$$=Jl\,\frac{1-\sqrt{(1-xx)}}{x}-J\left(\frac{1}{xx}+\frac{2}{3x^4}+\frac{2\cdot4}{3\cdot5x^6}+\ldots+\frac{2\cdot4\cdot6..(2i-2)}{3\cdot5\cdot7...(2i-1)x^{2i}}\right)\sqrt{(1-xx)},$$

$$\int\frac{dx}{x^{2i+2}\sqrt{(1-xx)}}=$$

$$=C-K\left(\frac{1}{x}+\frac{1}{2x^3}+\frac{1\cdot3}{2\cdot4x^5}+\ldots+\frac{1\cdot3\cdot5\cdot...(2i-1)}{2\cdot4\cdot6\cdot..2ix^{2i+1}}\right)\sqrt{(1-xx)}.$$

SCHOLION 1

124. Hinc iam facile integralia formularum $\int x^{m-1}dx(1-xx)^{\frac{\mu}{2}}$ tam pro omnibus numeris m quam pro imparibus μ assignari poterunt. Reductiones autem nostrae generales ad hunc casum accommodatae sunt:

$$\text{I. } \int x^{m+1}dx\left(1-xx\right)^{\frac{\mu}{2}} = \frac{-x^m\left(1-xx\right)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{m}{m+\mu+2}\int x^{m-1}dx\left(1-xx\right)^{\frac{\mu}{2}},$$

$$\text{II. } \int x^{m-3}dx\left(1-xx\right)^{\frac{\mu}{2}} = \frac{x^{m-2}\left(1-xx\right)^{\frac{\mu}{2}+1}}{m-2} + \frac{m+\mu}{m-2}\int x^{m-1}dx\left(1-xx\right)^{\frac{\mu}{2}},$$

$$\text{III. } \int x^{m-1}dx\left(1-xx\right)^{\frac{\mu}{2}+1} = \frac{x^m\left(1-xx\right)^{\frac{\mu}{2}+1}}{m+\mu+2} + \frac{\mu+2}{m+\mu+2}\int x^{m-1}dx\left(1-xx\right)^{\frac{\mu}{2}},$$

$$\text{IV. } \int x^{m-1}dx\left(1-xx\right)^{\frac{\mu}{2}-1} = \frac{-x^m\left(1-xx\right)^{\frac{\mu}{2}}}{\mu} + \frac{m+\mu}{\mu}\int x^{m-1}dx\left(1-xx\right)^{\frac{\mu}{2}},$$

$$\text{V. } \int x^{m+1}dx\left(1-xx\right)^{\frac{\mu}{2}-1} = \frac{-x^m\left(1-xx\right)^{\frac{\mu}{2}}}{\mu} + \frac{m}{\mu}\int x^{m-1}dx\left(1-xx\right)^{\frac{\mu}{2}},$$

$$\text{VI. } \int x^{m-3}dx\left(1-xx\right)^{\frac{\mu}{2}+1} = \frac{x^{m-2}\left(1-xx\right)^{\frac{\mu}{2}+1}}{m-2} + \frac{\mu+2}{m-2}\int x^{m-1}dx\left(1-xx\right)^{\frac{\mu}{2}}.$$

Posito enim $\mu = -1$quatuor posteriores dant

$$\int x^{m-1}dx\sqrt{\left(1-xx\right)} = \frac{x^m\sqrt{(1-xx)}}{m+1} + \frac{1}{m+1}\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

$$\int \frac{x^{m-1}dx}{\sqrt{(1-xx)^3}}^{\frac{\mu}{2}} = \frac{x^m}{\sqrt{(1-xx)}} - \left(m-1\right)\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

$$\int \frac{x^{m+1}dx}{\sqrt{(1-xx)^3}} = \frac{x^m}{\sqrt{(1-xx)}} - m\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

$$\int x^{m-3}dx\left(1-xx\right) = \frac{x^{m-2}\sqrt{(1-xx)}}{m-2} + \frac{1}{m-2}\int \frac{x^{m-1}dx}{\sqrt{(1-xx)}},$$

unde integrationes pro casibus $\mu = 1$ et $\mu = -\,3$ eliciuntur indeque porro reliqui.

SCHOLION 2

125. Pro aliis formulis irrationalibus magis complicatis vix regulas dare licet, quibus ad formam simpliciorem reduci queant; et quoties eiusmodi formulae occurrant, reductio, si quam admittunt, plerumque sponte se offert. Veluti si formula fuerit huiusmodi $\int \frac{Pdx}{Q^{n+1}}$, sive *n* sit numerus integer sive fractus, semper ad aliam huius formae $\int \frac{Sdx}{Q^n}$, quae utique simplicior aestimatur, reduci potest. Cum enim sit

$$d.\frac{R}{Q^n} = \frac{QdR-nRdQ}{Q^{n+1}},$$

positio $\int \frac{Pdx}{Q^{n+1}} = y$ erit

$$y + \frac{R}{Q^n} = \int \frac{Pdx+QdR-nRdQ}{Q^{n+1}}.$$

Iam definiatur R ita, ut $Pdx+QdR-nR\mathrm{d}Q$ per Q fiat divisibile, vel quia QdB iam factorem habet Q, ut fiat $Pdx-nRdQ=QTdx$, prodibitque

$$y+\frac{R}{Q^n}=\int\frac{dR+Tdx}{Q^n}$$

seu

$$\int\frac{Pdx}{Q^{n+1}}=-\frac{R}{Q^n}+\int\frac{dR+Tdx}{Q^n};$$

at semper functionem R ita definire licet, ut $Pdx-nRdQ$ factorem Q obtineat; quod etsi in genere praestari nequit, tamen rem in exemplis tentando mox perspicietur negotium semper succedere. Assumo autem hic P et Q esse functiones integras ac talis quoque semper pro R erui poterit. Si forte eveniat, ut $dR+Tdx=0$, formula proposita algebraicum habebit integrale, quod hoc modo reperietur; contra autem haec forma ulterius reduci poterit in alias, ubi denominatoris exponens continuo unitate diminuatur; ac si n sit numerus integer, negotium tandem reducitur ad huiusmodi formam $\frac{Vdx}{Q}$, quae sine dubio est simplicissima. Quamobrem cum in hoc capite vix quicquam amplius proferri possit ad integrationem formularum irrationalium iuvandam, methodum easdem integrationes per series infinitas perficiendi exponamus.

ADDITAMENTUM

PROBLEMA

Proposita formula $dy=(x+\sqrt{(1+xx)})^n dx$ *invenire eius integrale.*

SOLUTIO

Posito $x+\sqrt{(1+xx)}=u$ fit $x=\frac{uu-1}{2u}$ et $dx=\frac{du(uu+1)}{2uu}$ unde formula nostra

$$dy=\tfrac{1}{2}u^{n-2}du\left(uu+1\right)$$

ideoque eius integrale

$$y=\frac{u^{n+1}}{2(n+1)}+\frac{u^{n-1}}{2(n-1)}+\text{Const.},$$

quod ergo semper est algebraicum, nisi sit vel $n=1$ vel $n=-1$.

COROLLARIUM 1

Patet etiam hanc formam latius patentem

$$dy=(x+\sqrt{(1+xx)})^n Xdx$$

hoc modo integrari posse, dummodo X fuerit functio rationalis ipsius x.
Posito enim $x=\frac{uu-1}{2u}$ pro X prodit functio rationalis ipsius u, quae sit = U, hincque fit

$$dy = \tfrac{1}{2}Uu^{n-2}du\left(uu+1\right),$$

quae formula vel est rationalis, si n sit numerus integer, vel ad rationalitatem facile reducitur, si n sit numerus fractus.

COROLLARIUM 2

Cum sit $x+\sqrt{(1+xx)} = \frac{uu+1}{2u}$, posito $\sqrt{(1+xx)} = v$ etiam haec formula

$$dy = (x+\sqrt{(1+xx)})^n Xdx$$

integrabitur, si X fuerit functio rationalis quaecunque quantitatum x et v.
Facto enim $x = \frac{uu-1}{2u}$ functio X abit in functionem rationalem ipsius u, qua posita = U habebitur ut ante

$$dy = \tfrac{1}{2}Uu^{n-2}du\left(uu+1\right).$$

EXEMPLUM

Proposita sit formula $dy = (ax+b\sqrt{(1+xx)})\ (x+\sqrt{(1+xx)})^n dx$.
Posito $x = \frac{uu-1}{2u}$ fit

$$dy = \frac{a(uu-1)+b(uu+1)}{2u}\cdot\tfrac{1}{2}u^{n-2}du\left(uu+1\right)$$

seu

$$dy = \tfrac{1}{4}u^{n-3}du\left(a\left(u^4-1\right)+b\left(u^4+2uu+1\right)\right),$$

cuius integrale est

$$y = \frac{a+b}{4(n+2)}u^{n+2} + \frac{b}{2n}u^n + \frac{b-a}{4(n-2)}u^{n-2} + \text{Const.},$$

quae est algebraica, nisi sit vel n = 2 vel n = – 2 vel etiam n = 0.