INTEGRAL CALCULUS

BOOK ONE

PART ONE

OR

A METHOD FOR FINDING FUNCTIONS OF ONE VARIABLE FROM SOME GIVEN RELATION OF THE DIFFERENTIALS OF THE FIRST ORDER

FIRST SECTION

CONCERNING THE

INTEGRATION OF DIFFERENTIAL FORMULAS.

CHAPTER I

CONCERNING THE INTEGRATION OF RATIONAL DIFFERENTIAL FORMULAS

DEFINITION

40. *The formula of a differential is rational, when the differential dx of the variable x of which the function is sought is multiplied by a rational function of x; or if X designates a rational function of x, this formula Xdx of the differential is said to be rational.*

COROLLARY 1

41. Hence in this chapter a function of x is sought of this kind, which if it should be put as y, in order that $\frac{dy}{dx}$ is equal to a rational function of x, or with such a function put equal to X, so that $\frac{dy}{dx} = X$.

COROLLARY 2

42. Hence a function of x of this kind is sought, of which the differential is equal to Xdx; hence the integral of this, which now is accustomed to be indicated by $\int Xdx$, provides the function sought.

COROLLARY 3

43. But if P should be a function of x of this kind, so that the differential of this is $dP = Xdx$, since the quantity $P + C$ likewise is differential, the complete integral of the proposed formula Xdx is $P + C$ [, for some arbitrary constant C.]

SCHOLION 1

44. Questions of this kind are referred to in the first part of the first book, in which functions of the single variable x are sought from a given relation of the differentials of the first degree. It is evident that if the function sought is equal to y and $\frac{dy}{dx} = p$, it is required to prevail, in order that for some proposed equation between the three quantities x, y and p, from this the natural form of the function y is found, or an equation between x and y, with the letter p elicited. But the question thus proposed in general is considered to outdo the analytical powers so far, so that at no time can a solution be expected.

Therefore our strengths are to be exercised in the simpler cases, among which the case occurs, in which p is equal to some function of x, for example X, so that $\frac{dy}{dx} = X$ or $dy = Xdx$ and thus the integral $y = \int Xdx$ is required, as we arrange in the first section. Now this case extends widely to various natural forms of the function X and also it is involved in many more difficult cases, from which we put in place only questions to be resolved in this chapter, in which that function X is rational, and then to be progressing to these irrational functions. Hence this part is subdivided conveniently into two parts, in the first of which the integration of the simple formulas, in which $p = \frac{dy}{dx}$ is equal to a function of x only, is to be treated, but in the other it is appropriate to give an account of the integration, when the equation proposed is some function of x, y and p. And since in these two sections and chiefly in the first, most is an elaboration of the geometers, that occupy the greater part of the whole work.

SCHOLION 2

45. But first the principles of integration are to be desired from the differential calculus, and in the same way as the principles of division from multiplication, and the principles of the extraction of roots from an account of raising to powers, are accustomed to be taken. Since therefore, if a magnitude to be differentiated depends on several parts, such as $P + Q - R$, the differential of this is $dP + dQ - dR$, thus in turn, if the formula of the differential depends on several parts as $Pdx + Qdx - Rdx$, the integral is

$$\int Pdx + \int Qdx - \int Rdx$$

obviously from the integrals of the individual parts separately. Then, since the differential of the magnitude aP is adP, the integral of the differential formula $aPdx$ is $a\int Pdx$, clearly as the differential formula is multiplied by that constant magnitude, the integral must be multiplied by the same. Thus if the formula of the differentials is $aPdx + bQdx + cRdx$, whatever functions of x are designated by P, Q, R, the integral is

$$a\int Pdx + b\int Qdx + c\int Rdx,$$

thus in order that the integration is to be put in place of the forms Pdx, Qdx and Rdx, and thus the above must be increased by the addition of an arbitrary constant C, in order that the complete integral is obtained.

PROBLEM 1

46. *To find the function of x, in order that the differential of this is equal to $ax^n dx$, or to integrate the differential formula $ax^n dx$.*

SOLUTION

Since the differential of the power x^m is $mx^{m-1}dx$, there is in turn

$$\int mx^{m-1}dx = m\int x^{m-1}dx = x^m$$

and thus

$$\int x^{m-1}dx = \tfrac{1}{m}x^m;$$

on making $m - 1 = n$ or $m = n + 1$; there becomes

$$\int x^n dx = \tfrac{1}{n+1}x^{n+1} \quad \text{and} \quad a\int x^n dx = \tfrac{a}{n+1}x^{n+1}.$$

From which the completed integral of the proposed differential formula is

$$\tfrac{a}{n+1}x^{n+1} + C,$$

the reason or from which thus is apparent, because the differential actually is equal to $ax^n dx$. And this integration always has a place, whatever number is attributed to the exponent n, either positive or negative, either an integer or a fraction, or even irrational. Hence a single case is excepted, in which the exponent $n = -1$ or it is proposed to integrate this formula $\frac{adx}{x}$. Now in the differential calculus I have shown, if lx denotes the hyperbolic logarithm of x, the differential of this is equal to $\frac{dx}{x}$, from which in turn we can conclude that

$$\int \tfrac{dx}{x} = lx \quad \text{and} \quad \int \tfrac{adx}{x} = alx.$$

Whereby with an arbitrary constant added the complete integral of the formula $\frac{adx}{x}$ is equal to

$$alx + C = lx^a + C,$$

that also on putting lc for C can thus be expressed: lcx^a.

COROLLARY 1

47. Hence the integral of the differential formulas $ax^n dx$ is always algebraic with the only excepted case, in which $n = -1$ and the integral is expressed by logarithms, which are to be referred to as transcendent functions. Obviously, it is given by

$$\int \tfrac{adx}{x} = alx + C = lcx^n.$$

COROLLARY 2

48. If the exponent n denotes a positive number, then the following integrations are to be expressed properly, in as much as they are especially easy:

$$\int adx = ax + C, \quad \int axdx = \frac{a}{2}xx + C, \quad \int ax^2dx = \frac{a}{3}x^3 + C,$$
$$\int ax^3dx = \frac{a}{4}x^4 + C, \quad \int ax^4dx = \frac{a}{5}x^5 + C, \quad \int ax^5dx = \frac{a}{6}x^6 + C.$$

COROLLARY 3

49. If n is a negative number, on putting $n = -m$ there becomes

$$\int \frac{adx}{x^m} = \frac{a}{1-m}x^{1-m} + C = \frac{-a}{(m-l)x^{m-1}} + C,$$

from which these simpler cases can be noted :

$$\int \frac{adx}{x^2} = \frac{-a}{x} + C, \quad \int \frac{adx}{x^3} = \frac{-a}{2xx} + C, \quad \int \frac{adx}{x^4} = \frac{-a}{3x^3} + C,$$
$$\int \frac{adx}{x^5} = \frac{-a}{4x^4} + C, \quad \int \frac{adx}{x^6} = \frac{-a}{5x^5} + C, \quad \text{etc.}$$

COROLLARY 4

50. But also if n denotes a fractional number, hence the integrals can be obtained.
First let $n = \frac{m}{3}$; then

$$\int adx\sqrt{x^m} = \frac{2a}{m+2}x\sqrt{x^m} + C,$$

from which the cases are to be noted :

$$\int adx\sqrt{x} = \frac{2a}{3}x\sqrt{x} + C, \quad \int axdx\sqrt{x} = \frac{2a}{5}x^2\sqrt{x} + C,$$
$$\int axxdx\sqrt{x} = \frac{2a}{7}x^3\sqrt{x} + C, \quad \int ax^3dx\sqrt{x} = \frac{2a}{9}x^4\sqrt{x} + C.$$

COROLLARY 5

51. Also there can be put $n = \frac{-m}{2}$ and there is had

$$\int \frac{adx}{\sqrt{x^m}} = \frac{2a}{2-m\sqrt{x^m}} + C = \frac{-2a}{(m-2)\sqrt{x^{m-2}}} + C,$$

from which these cases are to be noted :

$$\int \frac{adx}{\sqrt{x}} = 2a\sqrt{x} + C, \quad \int \frac{adx}{x\sqrt{x}} = \frac{-2a}{\sqrt{x}} + C, \quad \int \frac{adx}{xx\sqrt{x}} = \frac{-2a}{3x\sqrt{x}} + C,$$

$$\int \frac{adx}{x^3\sqrt{x}} = \frac{-2a}{5x^2\sqrt{x}} + C.$$

COROLLARIUM 6

52. If in general we put $n = \frac{\mu}{\nu}$, then there arises

$$\int ax^{\frac{\mu}{\nu}} dx = \frac{\nu a}{\mu+\nu} x^{\frac{\mu+\nu}{\nu}} + C$$

or through roots,

$$\int adx\sqrt[\nu]{x^{\mu}} = \frac{\nu a}{\mu+\nu}\sqrt[\nu]{x^{\mu+\nu}} + C\,;$$

but if there is put $n = \frac{-\mu}{\nu}$ then there is had

$$\int \frac{adx}{x^{\frac{\mu}{\nu}}} = \frac{\nu a}{\nu-\mu} x^{\frac{\nu-\mu}{\nu}} + C$$

or through roots,

$$\int \frac{adx}{\sqrt[\nu]{x^{\mu}}} = \frac{\nu a}{\nu-\mu}\sqrt[\nu]{x^{\nu-\mu}} + C\,.$$

SCHOLIUM1

53. Nevertheless I have decided in this chapter to treat only rational functions, yet these irrationals can still be presented at once, as likewise they can be treated as rationals. Hence the remaining more complicated formulas are able to be integrated also, if functions of a certain variable *z* are taken for *x*. Just as if we put $x = f + gz$, then $dx = gdz$; whereby if for *a* we write $\frac{a}{g}$, there is had

$$\int adz\left(f + gz\right)^{n} = \frac{a}{(n+1)g}\left(f + gz\right)^{n+1} + C\,,$$

but in the particular case, in which $n = -1$,

$$\int \frac{adz}{\left(f+gz\right)} = \frac{a}{g}\, l\left(f + gz\right) + C\,.$$

Then if there shall be $n = -m$, the integral becomes

$$\int \frac{adz}{\left(f+gz\right)^{m}} = \frac{-a}{(m-1)g\left(f+gz\right)^{m-1}} + C\,.$$

But on putting $n = \frac{\mu}{\nu}$ there is produced

$$\int adz\left(f+gz\right)^{\frac{\mu}{\nu}}=\frac{\nu a}{(\nu+\mu)g}\left(f+gz\right)^{\frac{\mu}{\nu}+1}+C\,;$$

but on putting $n=-\frac{\mu}{\nu}$ there is obtained

$$\int\frac{adz}{(f+gz)^{\frac{\mu}{\nu}}}=\frac{\nu a(f+gz)}{(\nu-\mu)g(f+gz)^{\frac{\mu}{\nu}}}+C\,.$$

SCHOLIUM 2

54. Here another significant property deserves to be noted. Since here a function y is sought, in order that $dy=ax^n dx$, if we put $\frac{dy}{dx}=p$, this relation is obtained $p=ax^n$, from which the function y must be investigated. Therefore since

$$y=\frac{a}{n+1}x^{n+1}+C\,,$$

on account of $ax^n=p$ there is also

$$y=\frac{px}{n+1}+C$$

and thus we have the case, where the relation of the differentials through a certain equation is proposed between x, y and p and now we know for each to be satisfied by the equation $y=\frac{a}{n+1}x^{n+1}+C$. Now this no further is the complete integral for the relation held in the equation $y=\frac{px}{n+1}+C$, but only a particular one, because the new integral does not involve a constant, which is not in the relation of the differentials. Moreover the complete integral is

$$y=\frac{aD}{n+1}x^{n+1}+C$$

involving a new constant D; hence there becomes

$$\frac{dy}{dx}=aDx^n=p \quad \text{and thus} \quad y=\frac{px}{n+1}+C.$$

If this is not relevant to the present situation, yet it is helpful to be noted.

PROBLEM 2

55. To find a function of x, of which the differential is equal to Xdx, with X denoting some rational integer function of x, or to definite the integral $\int Xdx$.

SOLUTION

Since X is a function of x of the rational integers [*e. g.* a polynomial], by necessity it contained in this form :

$$X=\alpha+\beta x+\gamma x^2+\delta x^3+\varepsilon x^4+\text{etc.},$$

from which by the previous problem the integral sought is

$$\int Xdx = C + \alpha x + \tfrac{1}{2}\beta x^2 + \tfrac{1}{3}\gamma x^3 + \tfrac{1}{4}\delta x^4 + \tfrac{1}{5}\varepsilon x^5 + \tfrac{1}{5}\zeta x^6 + \text{etc.},$$

And in general if there should be

$$X = \alpha x^{\lambda} + \beta x^{\mu} + \gamma x^{\nu} + \text{etc.},$$

then there becomes

$$\int Xdx = C + \tfrac{\alpha}{\lambda+1}x^{\lambda+1} + \tfrac{\beta}{\beta+1}x^{\mu+1} + \tfrac{\gamma}{\gamma+1}x^{\nu+1} + \text{etc.},$$

where the exponents α, β, ν etc. are able to signify both positive and negative numbers as well as fractions, provided it is noted, if there should be $\lambda = -1$, then there becomes $\int \frac{\alpha dx}{x} = \alpha lx$, which is the single case to be referred to the order of transcendent functions.

PROBLEM 3

56. *If* X *denotes some rational fractional function of x, to describe the method, with the help of which it is convenient to investigate the integral of the formula Xdx.*

SOLUTION

Therefore let $X = \frac{M}{N}$, thus in order that *M* and *N* are to become integral functions of *x*, and in the first place it is to be considered, whether the sum of the powers of *x* in the numerator *M* is either greater or less than in the denominator *N*, in which case from the fraction $\frac{M}{N}$, integral [*i. e.* whole] parts can be elicited by division; the integration of which can be obtained without difficulty, and the whole calculation is reduced to a fraction of this kind $\frac{M}{N}$, in which the sum of the powers of *x* in the numerator *M* is less than in the denominator *N*.

Then all the factors of the denominator *N* itself are sought, both simple if they are real, as well as real squares, clearly in turn of the simple squares of [complex conjugate] imaginary factors arising ; likewise it is to be considered whether these factors are all unequal or not ; for indeed with the equality of the factors another method for the resolution of the fraction $\frac{M}{N}$ into simple fractions has to be put in place, since they arise from the factors of individual fractions, of which the sum of the fractions proposed is equal to $\frac{M}{N}$. Certainly from the simple factor $a + bx$ the fraction arises :

$$\frac{A}{a+bx};$$

if two are equal or the denominator *N* should have the factor $(a+bx)^2$, hence there arises the fractions :

$$\frac{A}{(a+bx)^2} + \frac{B}{a+bx};$$

moreover from the factor $(a+bx)^3$ of this kind these three fractions arise:

$$\frac{\mathrm{A}}{(a+bx)^3}+\frac{\mathrm{B}}{(a+bx)^2}+\frac{C}{a+bx}:$$

and thus henceforth.
But a double factor, the form of which is $aa-2abx\cos.\zeta+bbxx$, unless the two should be equal, gives a fraction of the form

$$\frac{A+Bx}{aa-2abx\cos.\zeta+bbxx};$$

but if the *N* involves two equal factors of this kind, from this the two partial fractions of this kind arise :

$$\frac{A+Bx}{\left(aa-2abx\cos.\zeta+bbxx\right)^2}+\frac{C+Dx}{aa-2abx\cos.\zeta+bbxx}$$

but if the cube, thus $\left(aa-2abx\cos.\zeta+bbxx\right)^3$ should be a factor of the denominator *N*, from that there arises the three partial fractions :

$$\frac{A+Bx}{\left(aa-2abx\cos.\zeta+bbxx\right)^3}+\frac{C+Dx}{\left(aa-2abx\cos.\zeta+bbxx\right)^2}+\frac{E+Fx}{aa-2abx\cos.\zeta+bbxx};$$

and thus henceforth.
Therefore since in this way the proposed fraction $\frac{M}{N}$ can be resolved into all its simple fractions, all are contained in one or other of these forms : either

$$\frac{A}{\left(a+bx\right)^n} \text{ or } \frac{A+Bx}{\left(aa-2abx\cos.\zeta+bbxx\right)^n}$$

and now it is required to integrate the individual terms multiplied by *dx*; the value of the function sought is the sum of all these integrals $\int Xdx=\int\frac{Mdx}{N}$.

COROLLARY 1

57. Hence for an integration of all the forms of this kind $\frac{Mdx}{N}$ the whole calculation is reduced to the integration of the two forms of this kind :

$$\int\frac{Adx}{\left(a+bx\right)^n} \text{ and } \int\frac{\left(A+Bx\right)dx}{\left(aa-2abx\cos.\zeta+bbxx\right)^n},$$

while for *n* there are written successively the numbers 1, 2, 3, 4, etc.

COROLLARIUM 2

58. And indeed the integral of the first form above (§ 53) is now obtained, from which it is apparent to be :

$$\int\frac{Adx}{a+bx}=\frac{A}{b}l\left(a+bx\right)+\text{Const.}$$

$$\int\frac{Adx}{\left(a+bx\right)^2}=\frac{-A}{b\left(a+bx\right)}+\text{Const.}$$

$$\int\frac{Adx}{\left(a+bx\right)^3}=\frac{-A}{2b\left(a+bx\right)^2}+\text{Const.}$$

et generally

$$\int\frac{Adx}{\left(a+bx\right)^n}=\frac{-A}{(n-1)b\left(a+bx\right)^{n-1}}+\text{Const.}$$

COROLLARY 3

59. Hence in order that the proposition can be completed, nothing else is required, except that the integration of this form

$$\int\frac{\left(A+Bx\right)dx}{\left(aa-2abx\cos.\zeta+bbxx\right)^n}$$

must be shown, indeed for the case $n = 1$, then for the cases $n = 2$, $n = 3$, $n = 4$, etc.

SCHOLIUM 1

60. Unless we want to avoid imaginary quantities, the whole calculation now can be completed from what has been treated; for with the denominator N resolved into all its simple factors, they are either real or imaginary, the proposed fraction always can be resolved into partial fractions of this form $\frac{Adx}{a+bx}$ or of this $\frac{Adx}{\left(a+bx\right)^n}$; since the integrals of which can be shown, the integral of the whole form $\frac{M}{N}dx$ can be obtained. But then as here there might be some inconvenience the pairs of imaginary parts [*i. e.* complex conjugates] thus are joined together, in order that a real expression results, which nevertheless completes the nature of the integration absolutely.

SCHOLION 2

61. Certainly here it is to be conceded that we postulate the resolution of an integral function into factors, even if the algebra so far has by no means led to that [conclusion], in order that this resolution actually can be put in place. But this is accustomed to be postulated in analysis everywhere, in order that, as the longer we progress, those things which are left behind, even if they have not been explored well enough, we assume as known ; evidently it suffices here that all the factors are able to be assigned by the method of approximation, however close they can be assigned. In a like manner in the

integral calculus the longer we proceed, the integrals of all the formulas of the form *Xdx*, whatever function of *x* may be signified by the letter *X*, we will consider as known and by us to be the most outstanding, if we can prevail to reduce more obscure integrals to these forms ; this also disturbs nothing in practical use, since the values of such formulas $\int Xdx$ are allowed to be assigned almost to any extend you wish, as we shall show in what follows. Otherwise about these integrals, the resolution of the denominator *N* into its factors is absolutely necessary, therefore in order that the individual factors are present in the expression of the integral ; there are just a few cases and these are especially easy, in which we are able to do without that resolution ; just as if this formula is proposed $\frac{x^{n-1}dx}{1+x^n}$ it is at once apparent on putting $x^n = v$ that it goes into $\frac{dv}{n(1+v)}$, the integral of which is $\frac{1}{n}l(1+v) = \frac{1}{n}l\left(1+x^n\right)$, where there was no need for the resolution of the factors. Now cases of this kind are so evident in themselves, so that no particular explanation of these may be needed.

PROBLEM 4

62. *To find the integral of this formula*

$$y = \int \frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$

SOLUTION

Since the numerator depends on the two parts *Adx*+*Bxdx*, this latter term *Bxdx* can be removed in the following manner. Since there arises

$$l\left(aa-2abx\cos.\zeta+bbxx\right) = \int \frac{-2abdx\cos.\zeta+2bbxdx}{aa-2abx\cos.\zeta+bbxx}$$

this equation is to be multiplied by $\frac{B}{2bb}$ and taken away from the proposed equation ; for thus there is produced

$$y-\frac{B}{2bb}l\left(aa-2abx\cos.\zeta+bbxx\right) = \int \frac{\left(A+\frac{Ba\cos.\zeta}{b}\right)dx}{aa-2abx\cos.\zeta+bbxx}$$

thus in order that only the formula to be integrated remains. For brevity the formula is put in place

$$A+\frac{Ba\cos.\zeta}{b} = C$$

so that this becomes

$$\int \frac{Cdx}{aa-2abx\cos.\zeta+bbxx}$$

which thus can be expressed thus

$$\int \frac{Cdx}{aa\sin.^2\zeta+(bx-a\cos.\zeta)^2}.$$

There is put in place $bx-a\cos.\zeta=av\sin.\zeta$ and hence $dx=\frac{adv\sin.\zeta}{b}$ from which our formula becomes

$$\int \frac{Cadv\sin.\zeta:b}{aa\sin.^2\zeta(1+vv)}=\frac{C}{ab\sin.\zeta}\int \frac{dv}{(1+vv)}.$$

But from the differential calculus we know that this becomes

$$\int \frac{dv}{(1+vv)}=\text{Arc.tang.}v=\text{Arc.tang.}\frac{bx-a\cos.\zeta}{a\sin.\zeta},$$

from which on account of

$$C=\frac{Ab+Ba\cos.\zeta}{b}$$

our integral is now

$$\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc.tang.}\frac{bx-a\cos.\zeta}{a\sin.\zeta}$$

On account of which the integral of the proposed formula

$$\int \frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$

is equal to

$$\frac{B}{2bb}l(aa-2abx\cos.\zeta+bbxx)+\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc.tang.}\frac{bx-a\cos.\zeta}{a\sin.\zeta},$$

which so that is complete, an arbitrary constant C is added above.

COROLLARY 1

63. If to $\text{Arc.tang.}\frac{bx-a\cos.\zeta}{a\sin.\zeta}$ we add $\text{Arc.tang.}\frac{\cos.\zeta}{\sin.\zeta}$, which obviously is considered agreeable to be the constant to be added, there is produced $\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}$ and thus we have

$$\int \frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$
$$=\frac{B}{2bb}l\left(aa-2abx\cos.\zeta+bbxx\right)+\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}$$

with the constant *C* added.

COROLLARY 2

64. If we wish, so that the integral thus vanishes on putting $x = 0$, the constant *C* may be taken as equal to $\frac{-B}{2bb}laa$ and thus there arises

$$\int \frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$
$$=\frac{B}{bb}l\frac{\sqrt{(aa-2abx\cos.\zeta+bbxx)}}{a}+\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}$$

Hence this integral depends on logarithms in part, and in part on the arc of circles or angles.

COROLLARY 3

65. If the letter *B* vanishes, the part depending on logarithms vanishes and there is produced

$$\int \frac{Adx}{aa-2abx\cos.\zeta+bbxx}=\frac{A}{ab\sin.\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}+C$$

and thus is defined by an angle alone.

COROLLARY 4

66. If the angle ζ is right and thus $\cos.\zeta = 0$ and $\sin.\zeta = 1$, there is obtained

$$\int \frac{(A+Bx)dx}{aa+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa+bbxx)}}{a}+\frac{A}{ab}\text{Arc.tang.}\frac{bx}{a}+C\,;$$

if the angle ζ is 60° and thus $\cos.\zeta=\frac{1}{2}$ and $\sin.\zeta=\frac{\sqrt{3}}{2}$, then there arises

$$\int \frac{(A+Bx)dx}{aa-abx+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa-abx+bbxx)}}{a}+\frac{2Ab+Ba}{abb\sqrt{3}}\text{Arc.tang.}\frac{bx\sqrt{3}}{2a-bx}$$

But if $\zeta = 120^{\circ}$ and thus $\cos.\zeta = -\frac{1}{2}$ and $\sin.\zeta=\frac{\sqrt{3}}{2}$, then there arises

$$\int\frac{(A+Bx)dx}{aa+abx+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa+abx+bbxx)}}{a}+\frac{2Ab-Ba}{abb\sqrt{3}}\text{Arc.tang.}\frac{bx\sqrt{3}}{2a+bx}$$

SCHOLIUM 1

67. Here it comes about entirely worthy to be noted, that in the case $\zeta = 0$, in which the denominator $aa-2abx+bbxx$ becomes a square, the account of the angle disappears from the integral. Indeed on putting the angle ζ to be infinitely small then there becomes $\cos.\zeta = 1$ and $\sin.\zeta = \zeta$, from which the logarithmic becomes $\frac{B}{bb}l\frac{a-bx}{a}$ and the other part $\frac{Ab+Ba}{abb\zeta}\text{Arc.tang.}\frac{bx\zeta}{a-bx}=\frac{(Ab+Ba)x}{ab(a-bx)}$, because the arc of the infinitely small tangent itself is equal to $\frac{bx\zeta}{a-bx}$, and thus this part becomes algebraic. On account of which there becomes

$$\int\frac{(A+Bx)dx}{(a-bx)^2}=\frac{B}{bb}l\frac{a-bx}{a}+\frac{(Ab+Ba)x}{ab(a-bx)}+\text{Const.},$$

the truth of which is obvious from the preceding; for there is

$$\frac{(A+Bx)}{(a-bx)^2}=\frac{-B}{b(a-bx)}+\frac{Ab+Ba}{b(a-bx)^2}$$

Now there becomes

$$\int\frac{-Bdx}{b(a-bx)}=\frac{B}{bb}l(a-bx)-\frac{B}{bb}la=\frac{B}{bb}l\frac{a-bx}{a},$$

$$\int\frac{(Ab+Ba)dx}{b(a-bx)^2}=\frac{Ab+Ba}{bb(a-bx)}-\frac{Ab+Ba}{abb}=\frac{(Ab+Ba)x}{ab(a-bx)},$$

if indeed each integration is thus determined, as in the case $x = 0$ the integrals vanish.

SCHOLION 2

68. In a similar manner, which we have used here, if in the formula with the differential fraction $\frac{Mdx}{N}$ the sum of the powers of x in the numerator M shall be one degree less than in the denominator N, also that boundary can be lifted. For let the equations be

$$M = Ax^{n-1}+Bx^{n-2}+Cx^{n-3}+\text{etc.}$$

and

$$N=\alpha x^n+\beta x^{n-1}+\gamma x^{n-2}+\text{etc.}$$

and there is put $\frac{Mdx}{N}=dy$. Now since there shall be

$$dN=n\alpha x^{n-1}dx+(n-1)dx\beta x^{n-2}+(n-2)dx\gamma x^{n-3}+\text{etc.}$$

then there will be

$$\frac{AdN}{n\alpha N}=\frac{dx}{N}\left(Ax^{n-1}+\frac{(n-1)A\beta}{n\alpha}x^{n-2}+\frac{(n-2)A\gamma}{n\alpha}x^{n-3}+\text{etc.}\right),$$

with which value then subtracted there remains

$$dy-\frac{AdN}{n\alpha N}=\frac{dx}{N}\left(\left(B-\frac{(n-1)A\beta}{n\alpha}\right)x^{n-2}+\left(C-\frac{(n-2)A\gamma}{n\alpha}\right)x^{n-3}+\text{etc.}\right)$$

Whereby if for the sake of brevity there is put

$$B-\frac{(n-1)A\beta}{n\alpha}=\mathfrak{B},\quad C-\frac{(n-2)A\gamma}{n\alpha}=\mathfrak{C},\quad D-\frac{(n-3)A\delta}{n\alpha}=\mathfrak{D},\quad \text{etc.},$$

there is obtained

$$y=\frac{A}{n\alpha}lN+\int\frac{dx\left(\mathfrak{B}x^{n-2}+\mathfrak{C}x^{n-3}+\mathfrak{D}x^{n-4}+\text{etc.}\right)}{\alpha x^{n}+\beta x^{n-1}+\gamma x^{n-2}+\delta x^{n-3}\text{etc.}}=\int\frac{Mdx}{N}.$$

Therefore in this manner all the differential formulas of the fractions can be reduced by that, so that the sum of the powers of x in the numerator shall be less by two or more degrees than in the denominator.

PROBLEM 5

69. *The integral formula*

$$y=\int\frac{(A+Bx)dx}{(aa-2abx\cos.\zeta+bbxx)^{n+1}}$$

is to be reduced to another like formula, where the power of the denominator is less by one degree.

SOLUTION

For the sake of brevity let the formula be $aa-2abx\cos.\zeta+bbxx=X$ and there is put

$$\int\frac{(A+Bx)dx}{X^{n+1}}=y\,.$$

Since on account of

$$dX=-2abdx\cos.\zeta+2bbdxx$$

let

$$d.\frac{C+Dx}{X^{n}}=\frac{-n(C+Dx)dX}{X^{n+1}}+\frac{Ddx}{X^{n}}\frac{1}{2}$$

and thus [for some constants C and D],

$$\frac{C+Dx}{X^{n}}=\int\frac{2nb(C+Dx)(a\cos.\zeta-bx)dx}{X^{n+1}}+\int\frac{Ddx}{X^{n}},$$

we have

$$y+\frac{C+Dx}{X^{n}}=\int\frac{dx\left(A+2nCab\cos.\zeta+x\left(B+2nDab\cos.\zeta-2nCbb\right)-2nDbbxx\right)}{X^{n+1}}+\int\frac{Ddx}{X^{n}}.$$

Now in the first formula the letters C and D thus are to be defined, in order that the numerator becomes divisible by X ; hence [see * below] it is necessary for the numerator to be equal to $-2nDXdx$,
[*i.e.* the numerator becomes $-2nDdx\left(aa-2abx\cos.\zeta+bbxx\right)=-2nDXdx$]
from which we arrive at

$$A+2nCab\cos.\zeta=-2nDaa$$

and

$$B+2nDab\cos.\zeta-2nCbb=4nDab\cos.\zeta$$

or $B-2nCbb=2nDab\cos.\zeta$ and hence

$$2nDa=\frac{B-2nCbb}{b\cos.\zeta};$$

but from the former condition there is

$$2nDa=\frac{-A-2nCab\cos.\zeta}{a}$$

from which equated there arises

$$Ba+Ab\cos.\zeta-2nCabb\sin^{2}.\zeta=0$$

or

$$C=\frac{Ba+Ab\cos.\zeta}{2nabb\sin^{2}.\zeta},$$

from which

$$B-2nCbb=\frac{Ba\sin.^{2}\zeta-Ba-Ab\cos.\zeta}{a\sin.^{2}\zeta}=\frac{-Ab\cos.\zeta-Ba\cos.^{2}\zeta}{a\sin.^{2}\zeta},$$

thus so that there is found

$$D=\frac{-Ab-Ba\cos.\zeta}{2naab\sin.^{2}\zeta}.$$

Hence with the letters taken :

$$C=\frac{Ba+Ab\cos.\zeta}{2nabb\sin^{2}.\zeta}\quad\text{and}\quad D=\frac{-Ab-Ba\cos.\zeta}{2naab\sin^{2}.\zeta}$$

then there becomes [*]

$$y+\frac{C+Dx}{X^{n}}=\int\frac{-2nDdx}{X^{n}}+\int\frac{Ddx}{X^{n}}=-\left(2n-1\right)D\int\frac{dx}{X^{n}}$$

and thus

$$\int\frac{\left(A+Bx\right)dx}{X^{n+1}}=\frac{-C-Dx}{X^{n}}-\left(2n-1\right)D\int\frac{dx}{X^{n}},$$

or

$$\int\frac{(A+Bx)dx}{X^{n+1}}=\frac{-Baa-Aab\cos.\zeta+(Abb+Bab\cos.\zeta)x}{2naabb\sin.^2\zeta X^n}$$
$$+\frac{(2n-1)(Ab+Ba\cos.\zeta)}{2naab\sin.^2\zeta}\int\frac{dx}{X^n}.$$

Whereby, if the formula $\int\frac{dx}{X^n}$ should be agreed upon, then it is also possible to assign the integral $\int\frac{(A+Bx)dx}{X^{n+1}}$.

COROLLARY 1

70. Therefore since on keeping $\mathrm{X}=aa-2abx\cos.\zeta+bbxx$ there shall be

$$\int\frac{dx}{\mathrm{X}}=\frac{1}{ab\sin.\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}+\text{Const.},$$

then there becomes

$$\int\frac{(A+Bx)dx}{X^2}=\frac{-Baa-Aab\cos.\zeta+(Abb+Bab\cos.\zeta)x}{2aabb\sin.^2\zeta\ X}+\frac{Ab+Ba\cos.\zeta}{2na^3bb\sin.^3\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}+\text{Const.}$$

And thus on putting $B=0$ and $A=1$ there becomes

$$\int\frac{dx}{X^2}=\frac{a\cos.\zeta+bx}{2aab\sin.^2\zeta X}+\frac{1}{2a^3b\sin.^3\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}+\text{Const.}$$

Hence the integral $\int\frac{(A+Bx)dx}{X^2}$ does not involve logarithms.

COROLLARY 2

71. Hence therefore since there becomes

$$\int\frac{dx}{X^3}=\frac{a\cos.\zeta+bx}{4aab\sin.^2\zeta X^2}+\frac{3}{4aa\sin.^2\zeta}\int\frac{dx}{X^2}+\text{Const.},$$

on substituting that value, it becomes

$$\int\frac{dx}{X^3}=\frac{-a\cos.\zeta+bx}{4aab\sin.^2\zeta\ X^2}+\frac{3(-a\cos.\zeta+bx)}{2\cdot4\ a^4b\sin.^4\zeta\ X}+\frac{1\cdot3}{2\cdot4\ a^5b\sin.^5\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}$$

and hence in turn it is concluded

$$\int\frac{dx}{X^4}=\frac{-a\cos.\zeta+bx}{6aab\sin.^2\zeta\; X^3}+\frac{5(-a\cos.\zeta+bx)}{4\cdot 6\, a^4 b\sin.^4\zeta\; X^2}+\frac{3(-a\cos.\zeta+bx)}{2\cdot 4\cdot 6\, a^6 b\sin.^6\zeta\; X}$$
$$+\frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6\, a^7 b\sin.^7\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}$$

COROLLARIUM 3

72. Thus by progressing further the integrals of all the formulas of this kind are obtained :

$$\int\frac{dx}{X},\quad \int\frac{dx}{X^2},\quad \int\frac{dx}{X^3},\quad \int\frac{dx}{X^4}\quad \text{etc.,}$$

the first of which is expressed by the arc of a circle only, while the remaining contain algebraic parts as well.

SCHOLIUM

73. Moreover it is sufficient to know the integral $\int\frac{dx}{X^{n+1}}$, because the formula $\int\frac{(A+Bx)dx}{X^{n+1}}$ is easily reduced to that ; for thus it is possible to represent this by

$$\frac{1}{2bb}\int\frac{2Abbdx+2Bbbxdx-2Babdx\cos.\zeta+2Babdx\cos.\zeta}{X^{n+1}},$$

which on account of $2bbxdx-2abdx\cos.\zeta=dX$, goes into this form

$$\frac{1}{2bb}\int\frac{BdX}{X^{n+1}}+\frac{1}{2b}\int\frac{(Ab+Ba)\cos.\zeta\, dx}{X^{n+1}}.$$

But

$$\int\frac{dX}{X^{n+1}}=-\frac{1}{nX^n},$$

from which there is obtained

$$\int\frac{(A+Bx)dx}{X^{n+1}}=\frac{-B}{2nbbX^n}+\frac{Ab+Ba\cos.\zeta}{b}\int\frac{dx}{X^{n+1}},$$

from which there is only the need to know the integral $\int\frac{dX}{X^{n+1}}$, but which we have shown. And these are all the aids we require by which all the fractional formulas $\frac{M}{N}dx$ are to be integrated. Provided *M* and *N* are integral [i. e. polynomial] functions of *x*. On account of which in general the integration of all the formulas of this kind $\int Vdx$, where *V* is some rational function of *x*, is in the rule; from which it is to be noted, unless the integration should be algebraic, it can always be shown in terms of either logarithms or angles.

Therefore nothing different remains, unless that we should illustrate the method with a few examples.

EXAMPLE 1

74. *With the proposed differential formula* $\frac{(A+Bx)dx}{\alpha+\beta x+\gamma xx}$, *to define the integral of this.*

Since in the numerator of the variable x there is fewer dimensions than in the denominator, this fraction includes no integral parts. Hence the nature of the denominator must be assessed carefully, whether it should have two real simple factors or not, and in the first case, if the factors are equal ; from which we have three cases to be explained.

1. The denominator has both factors equal and let it be equal to $(a+bx)^2$ and the fraction $\frac{(A+Bx)dx}{(a+bx)^2}$ is resolved into these two :

$$\frac{Ab-Ba}{b(a+bx)^2}+\frac{B}{b(a+bx)},$$

from which there is produced

$$\int\frac{(A+Bx)dx}{(a+bx)^2}=\frac{Ba-Ab}{bb(a+bx)}+\frac{B}{bb}l(a+bx)+\text{Const.};$$

if the integral is thus to be determined, so that it vanishes on putting $x=0$, then there is found :

$$\int\frac{(A+Bx)dx}{(a+bx)^2}=\frac{(Ab-Ba)x}{ab(a+bx)}+\frac{B}{bb}l\frac{a+bx}{a}.$$

II. If the denominator has two unequal factors and let this be the proposed formula : $\frac{A+Bx}{(a+bx)(f+gx)}dx$; and this fraction is resolved into these partial fractions :

$$\frac{Ab-Ba}{bf-ag}\cdot\frac{dx}{a+bx}+\frac{Ag-Bf}{ag-bf}\cdot\frac{dx}{f+gx},$$

from which there is obtained the integral sought

$$\int\frac{(A+Bx)dx}{(a+bx)(f+gx)}=\frac{Ab-Ba}{b(bf-ag)}l\frac{a+bx}{a}+\frac{Ag-Bf}{g(ag-bf)}l\frac{f+gx}{f}+\text{Const.}$$

Putting

$$\frac{Ab-Ba}{b(bf-ag)}=m+n\quad\text{and}\quad\frac{Ag-Bf}{g(ag-bf)}=m-n,$$

in order that the integral becomes

$$ml\frac{(a+bx)(f+gx)}{af}+nl\frac{f(a+bx)}{a(f+gx)};$$

there becomes :

$$2m=\frac{B(bf-ag)}{bg(bf-ag)}=\frac{B}{bg}\quad\text{and}\quad 2n=\frac{2Abg-Bag-Bbf}{bg(bf-ag)};$$

hence there becomes :

$$\int\frac{(A+Bx)dx}{(a+bx)(f+gx)}=\frac{B}{2bg}l\frac{(a+bx)(f+gx)}{af}+\frac{2Abg-B(ag+bf)}{2bg(bf\text{ -}ag)}l\frac{f(a+bx)}{a(f+gx)}.$$

III. Let the simple factors of the denominator both be imaginary, in which case there is had $aa-2abx\cos.\zeta+bbxx$; now as we have treated this case above [§ 64], it becomes

$$\int\frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$

$$=\frac{B}{bb}l\frac{\sqrt{(aa-2abx\cos.\zeta+bbxx)}}{a}+\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}$$

COROLLARIUM 1

75. In the second case, in which $f=a$ and $g=-b$, there will be

$$\int\frac{(A+Bx)dx}{aa-bbxx}=\frac{-B}{2bb}l\frac{aa-bbxx}{aa}+\frac{A}{2ab}l\frac{a+bx}{a-bx}$$

hence separately there follows :

$$\int\frac{Adx}{aa-bbxx}=\frac{A}{2ab}l\frac{a+bx}{a-bx}+C\,,$$

and

$$\int\frac{Bxdx}{aa-bbxx}=\frac{-B}{2bb}l\frac{aa-bbxx}{aa}=\frac{B}{bb}l\frac{a}{\sqrt{(aa-bbxx)}}+C.$$

COROLLARY 2

76. In the third case, if we put $\cos.\zeta=0$, then we have

$$\int\frac{(A+Bx)dx}{aa+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa+bbxx)}}{a}+\frac{A}{ab}\text{Arc.tang.}\frac{bx}{a}+c$$

and hence singly:

$$\int\frac{Adx}{aa+bbxx}=\frac{A}{ab}\text{Arc.tang.}\frac{bx}{a}+c\,,$$

and

$$\int\frac{Bxdx}{aa+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa+bbxx)}}{a}+c$$

EXAMPLE 2

77. *With the proposed differential formula* $\frac{x^{m-1}dx}{1+x^n}$, *if indeed the exponent m* – 1 *is less than n, to define the integral.*

In the final chapter of the *Institutionum Calculi Differentialis* we have found the simple fractions, into which this fraction $\frac{x^{m-1}dx}{1+x^n}$ is to be resolved, on taking π for the measure of two right angles, to be contained in this general form :

$$\frac{2\sin.\frac{(2k-1)\pi}{n}\sin.\frac{(2k-1)\pi}{n}-2\cos.\frac{m(2k-1)\pi}{n}\left(x-\cos.\frac{(2k-1)\pi}{n}\right)}{n\left(1-2x\cos.\frac{(2k-1)\pi}{n}+xx\right)},$$

where for *k* it is agreed to substitute all the numbers 1, 2, 3 etc., until *2k* - 1 begins to surpass the number *n* . [See the notes at the end of this translation.]
Hence this form is multiplied by *dx* and since it is to be compared with our general form

$$\frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$

then

$$a=1,\quad b=1,\quad \zeta=\frac{(2k-1)\pi}{n}$$

and

$$A=\tfrac{2}{n}\sin.\frac{(2k-1)\pi}{n}\sin.\frac{m(2k-1)\pi}{n}+\tfrac{2}{n}\cos.\frac{(2k-1)\pi}{n}\cos.\frac{m(2k-1)\pi}{n}$$

or

$$A=\tfrac{2}{n}\cos.\frac{(m-1)(2k-1)\pi}{n}\quad\text{and}$$

$$B=-\tfrac{2}{n}\cos.\frac{m(2k-1)\pi}{n},$$

from which there arises

$$Ab+Ba\cos.\zeta=\tfrac{2}{n}\sin.\frac{(2k-1)\pi}{n}\sin.\frac{m(2k-1)\pi}{n}$$

and therefore the integral of this part is :

$$-\tfrac{2}{n}\cos.\frac{m(2k-1)\pi}{n}\,l\sqrt{\left(1-2x\cos.\frac{(2k-1)\pi}{n}+xx\right)}$$

$$+\tfrac{2}{n}\sin.\frac{m(2k-1)\pi}{n}\,\text{Arc.tang.}\frac{x\sin.\frac{(2k-1)\pi}{n}}{1-x\cos.\frac{(2k-1)\pi}{n}}.$$

And if n is an odd number, in addition to the agreed fraction $\frac{\pm dx}{n(1+x)}$, the integral of which is $\pm\frac{1}{n}l(1+x)$, where the upper sign prevails, if m is odd, now the lower, if m is even. On account of which the integral sought $\int\frac{x^{m-1}dx}{1+x^n}$ can be expressed in the following manner

$$-\frac{2}{n}cos.\frac{m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{\pi}{n}+xx\right)}+\frac{2}{n}sin.\frac{m\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{n}}{1-x\,cos.\frac{\pi}{n}}$$

$$-\frac{2}{n}cos.\frac{3m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{3\pi}{n}+xx\right)}+\frac{2}{n}sin.\frac{3m\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{3\pi}{n}}{1-x\,cos.\frac{3\pi}{n}}$$

$$-\frac{2}{n}cos.\frac{5m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{5\pi}{n}+xx\right)}+\frac{2}{n}sin.\frac{5m\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{5\pi}{n}}{1-x\,cos.\frac{5\pi}{n}}$$

$$-\frac{2}{n}cos.\frac{7m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{7\pi}{n}+xx\right)}+\frac{2}{n}sin.\frac{7m\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{7\pi}{n}}{1-x\,cos.\frac{7\pi}{n}}$$

etc.

according to the odd numbers being less than to n itself ; and thus the whole integral is obtained, if n should be an even number, but if n should be an odd number, the part $\pm\frac{1}{n}l(1+x)$ is agreed upon above according to whether m should be either an odd or even number; from which if $m = 1$, $+\frac{1}{n}l(1+x)$ is agreed above.

COROLLARY 1

78. We take $m = 1$, in order that the form $\int\frac{dx}{1+x^n}$ is obtained, and for the various cases of n we arrive at :

I. $\int\frac{dx}{1+x}=l\left(1+x\right)$

II. $\int\frac{dx}{1+x^2}=\text{Arc.tang.}x$

III. $\int\frac{dx}{1+x^3}=-\frac{2}{3}cos.\frac{\pi}{3}l\sqrt{\left(1-2x\,cos.\frac{\pi}{3}+xx\right)}$

$$+\frac{2}{3}sin.\frac{\pi}{3}\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{3}}{1-x\,cos.\frac{\pi}{3}}+\frac{1}{3}l\left(1+x\right)$$

IV. $$\int\frac{dx}{1+x^4}=-\tfrac{2}{4}cos.\tfrac{\pi}{4}\,l\sqrt{\left(1-2x\,cos.\tfrac{\pi}{4}+xx\right)}\quad+\tfrac{2}{4}sin.\tfrac{\pi}{4}\,\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{4}}{1-x\,cos.\frac{\pi}{4}}$$
$$-\tfrac{2}{4}cos.\tfrac{3\pi}{4}\,l\sqrt{\left(1-2x\,cos.\tfrac{3\pi}{4}+xx\right)}\;+\tfrac{2}{4}sin.\tfrac{3\pi}{4}\,\text{Arc.tang.}\frac{x\,sin.\frac{3\pi}{4}}{1-x\,cos.\frac{3\pi}{4}}$$

V. $$\int\frac{dx}{1+x^5}=-\tfrac{2}{5}cos.\tfrac{\pi}{5}\,l\sqrt{\left(1-2x\,cos.\tfrac{\pi}{5}+xx\right)}\quad+\tfrac{2}{5}sin.\tfrac{\pi}{5}\,\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{5}}{1-x\,cos.\frac{\pi}{5}}$$
$$-\tfrac{2}{5}cos.\tfrac{3\pi}{5}\,l\sqrt{\left(1-2x\,cos.\tfrac{3\pi}{5}+xx\right)}\;+\tfrac{2}{5}sin.\tfrac{3\pi}{5}\,\text{Arc.tang.}\frac{x\,sin.\frac{3\pi}{5}}{1-x\,cos.\frac{3\pi}{5}}+\tfrac{1}{5}l(1+x)$$

VI. $$\int\frac{dx}{1+x^6}=-\tfrac{2}{6}cos.\tfrac{\pi}{6}\,l\sqrt{\left(1-2x\,cos.\tfrac{\pi}{6}+xx\right)}\quad+\tfrac{2}{6}sin.\tfrac{\pi}{6}\,\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{6}}{1-x\,cos.\frac{\pi}{6}}$$
$$-\tfrac{2}{6}cos.\tfrac{3\pi}{6}\,l\sqrt{\left(1-2x\,cos.\tfrac{3\pi}{5}+xx\right)}\;+\tfrac{2}{6}sin.\tfrac{3\pi}{6}\,\text{Arc.tang.}\frac{x\,sin.\frac{3\pi}{6}}{1-x\,cos.\frac{3\pi}{6}}$$
$$-\tfrac{2}{6}cos.\tfrac{5\pi}{6}\,l\sqrt{\left(1-2x\,cos.\tfrac{5\pi}{5}+xx\right)}\;+\tfrac{2}{6}sin.\tfrac{5\pi}{6}\,\text{Arc.tang.}\frac{x\,sin.\frac{5\pi}{6}}{1-x\,cos.\frac{5\pi}{6}}$$

COROLLARY 2

79. In place of the sines and cosines the values are to be substituted, where this can be conveniently done, and we obtain :

$$\int\frac{dx}{1+x^3}=-\tfrac{1}{3}l\sqrt{\left(1-x+xx\right)}\;+\frac{1}{\sqrt{3}}\text{Arc.tang.}\frac{x\sqrt{3}}{2-x}+\tfrac{1}{3}l(1+x)$$

or

$$\int\frac{dx}{1+x^3}=-\tfrac{1}{3}l\frac{1+x}{\sqrt{(1-x+xx)}}\;+\frac{1}{\sqrt{3}}\text{Arc.tang.}\frac{x\sqrt{3}}{2-x}$$

Then on account of $sin.\frac{\pi}{4}=cos.\frac{\pi}{4}=\frac{1}{\sqrt{2}}=sin.\frac{3\pi}{4}=-cos.\frac{3\pi}{4}$ there becomes

$$\int\frac{dx}{1+x^4}=+\frac{1}{2\sqrt{2}}l\frac{\sqrt{\left(1+x\sqrt{2}+xx\right)}}{\sqrt{\left(1-x\sqrt{2}+xx\right)}}\;+\frac{1}{2\sqrt{2}}\text{Arc.tang.}\frac{x\sqrt{2}}{1-xx}\,,$$

then

$$\int\frac{dx}{1+x^6}=+\frac{1}{2\sqrt{3}}l\frac{\sqrt{\left(1+x\sqrt{3}+xx\right)}}{\sqrt{\left(1-x\sqrt{3}+xx\right)}}\;+\frac{1}{6}\text{Arc.tang.}\frac{3x(1-xx)}{1-4xx+x^4}$$

EXAMPLE 3

80. *With the proposed differential formula* $\frac{x^{m-1}dx}{1+x^n}$, *if indeed the exponent* $m-1$ *is less than n, to define the integral itself.*

The part of the fraction of the function $\frac{x^{m-1}}{1+x^n}$ is required to come from some factor contained in this form :

$$\frac{2\,sin.\frac{2k\pi}{n}\,sin.\frac{2mk\pi}{n}-2\,cos.\frac{2mk\pi}{n}\left(x-cos.\frac{2k\pi}{n}\right)}{n\left(1-2x\,cos.\frac{2k\pi}{n}+xx\right)}$$

which on comparison with our form $\frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$ gives

$$a=1,\quad b=1,\quad \zeta=\frac{2k\pi}{n},$$

$$A=\frac{2}{n}sin.\frac{2k\pi}{n}sin.\frac{2mk\pi}{n}+\frac{2}{n}cos.\frac{2k\pi}{n}cos.\frac{2mk\pi}{n},\quad B=-\frac{2}{n}cos.\frac{2mk\pi}{n}$$

and hence

$$Ab+Ba\,cos.\zeta=\frac{2}{n}sin.\frac{2k\pi}{n}sin.\frac{2mk\pi}{n}.$$

From which the integral hence arising is equal to

$$-\frac{2}{n}cos.\frac{2mk\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{2k\pi}{n}+xx\right)}+\frac{2}{n}sin.\frac{2mk\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{2k\pi}{n}}{1-x\,cos.\frac{2k\pi}{n}},$$

where all the successive values for *k* ,1, 2, 3 etc. must be substituted, as long as 2*k* is less than *n*. These parts of the integral arising, $-\frac{1}{n}l(1-x)$ and $\mp\frac{1}{n}l(1+x)$, are to be added above from the fraction $\frac{1}{n(1-x)}$ and, if *n* is an even number, from the fraction $\frac{\mp 1}{n(1+x)}$, where the upper sign prevails if *m* is even, and truly the lower, if *m* is odd. [A correction to the first edition has been added here, by the editor of the *O. O*. edition]

On account of which the integral $\int\frac{x^{m-1}dx}{1+x^n}$ can be expressed in this way :

$$-\tfrac{1}{n}l(1-x)$$

$$-\tfrac{2}{n}cos.\frac{2m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{\pi}{n}+xx\right)}+\tfrac{2}{n}sin.\frac{2m\pi}{n}\,\text{Arc.tang.}\frac{x\,sin.\frac{2\pi}{n}}{1-x\,cos.\frac{2\pi}{n}}$$

$$-\tfrac{2}{n}cos.\frac{4m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{4\pi}{n}+xx\right)}+\tfrac{2}{n}sin.\frac{4m\pi}{n}\,\text{Arc.tang.}\frac{x\,sin.\frac{4\pi}{n}}{1-x\,cos.\frac{4\pi}{n}}$$

$$-\tfrac{2}{n}cos.\frac{6m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{6\pi}{n}+xx\right)}+\tfrac{2}{n}sin.\frac{6m\pi}{n}\,\text{Arc.tang.}\frac{x\,sin.\frac{6\pi}{n}}{1-x\,cos.\frac{6\pi}{n}}$$

etc.

COROLLARY

81. If $m = 1$ and for n successively the numbers 1, 2, 3 etc. are substituted, in order that we arrive at the following integrations :

I. $\int\frac{dx}{1-x}=-l(1-x)$

II. $\int\frac{dx}{1-xx}=-\frac{1}{2}l(1-x)+\frac{1}{2}l(1+x)=\frac{1}{2}l\frac{(1+x)}{(1-x)}$

III. $\int\frac{dx}{1-x^3}=-\frac{1}{3}l(1-x)-\frac{2}{3}cos.\frac{2\pi}{3}l\sqrt{\left(1-2x\,cos.\frac{2\pi}{3}+xx\right)}$

$$+\frac{2}{3}sin.\frac{2\pi}{3}\,\text{Arc.tang.}\frac{x\,sin.\frac{2\pi}{3}}{1-x\,cos.\frac{2\pi}{3}}$$

IV. $\int\frac{dx}{1-x^4}=-\frac{1}{4}l(1-x)-\frac{2}{4}cos.\frac{2\pi}{4}l\sqrt{\left(1-2x\,cos.\frac{2\pi}{4}+xx\right)}\quad+\frac{2}{4}sin.\frac{2\pi}{4}\,\text{Arc.tang.}\frac{x\,sin.\frac{2\pi}{4}}{1-x\,cos.\frac{2\pi}{4}}$

$$+\frac{2}{4}l(1+x)$$

V. $\int\frac{dx}{1-x^5}=-\frac{1}{5}l(1-x)-\frac{2}{5}cos.\frac{2\pi}{5}l\sqrt{\left(1-2x\,cos.\frac{2\pi}{5}+xx\right)}\quad+\frac{2}{5}sin.\frac{2\pi}{5}\,\text{Arc.tang.}\frac{x\,sin.\frac{2\pi}{5}}{1-x\,cos.\frac{2\pi}{5}}$

$$-\frac{2}{5}cos.\frac{4\pi}{5}l\sqrt{\left(1-2x\,cos.\frac{4\pi}{5}+xx\right)}\;+\frac{2}{5}sin.\frac{4\pi}{5}\,\text{Arc.tang.}\frac{x\,sin.\frac{4\pi}{5}}{1-x\,cos.\frac{4\pi}{5}}$$

VI. $\int\frac{dx}{1-x^6}=-\frac{1}{6}l(1-x)-\frac{2}{6}cos.\frac{2\pi}{6}l\sqrt{\left(1-2x\,cos.\frac{2\pi}{6}+xx\right)}\quad+\frac{2}{6}sin.\frac{2\pi}{6}\,\text{Arc.tang.}\frac{x\,sin.\frac{2\pi}{6}}{1-x\,cos.\frac{2\pi}{6}}$

$$-\frac{2}{6}cos.\frac{4\pi}{6}l\sqrt{\left(1-2x\,cos.\frac{4\pi}{5}+xx\right)}\;+\frac{2}{6}sin.\frac{4\pi}{6}\,\text{Arc.tang.}\frac{x\,sin.\frac{4\pi}{6}}{1-x\,cos.\frac{4\pi}{6}}$$

$$+\frac{1}{6}l(1+x)$$

EXAMPLE 4

82. *With the proposed differential formula* $\frac{\left(x^{m-1}+x^{n-m-1}\right)dx}{1+x^{n}}$ *on* $n>m-1$ *arising, to define the integral of this.*

From Example 2 it is apparent that any part of the integral in general, on taking i for some odd number not greater than n,

$$-\tfrac{2}{n}cos.\frac{im\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{i\pi}{n}+xx\right)}+\tfrac{2}{n}sin.\frac{im\pi}{n}\,\text{Arc.tang.}\frac{x\,sin.\frac{i\pi}{n}}{1-x\,cos.\frac{i\pi}{n}},$$

$$-\tfrac{2}{n}cos.\frac{i(n-m)\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{i\pi}{n}+xx\right)}+\tfrac{2}{n}sin.\frac{i(n-m)\pi}{n}\,\text{Arc.tang.}\frac{x\,sin.\frac{i\pi}{n}}{1-x\,cos.\frac{i\pi}{n}}.$$

Now there is

$$cos.\frac{i(n-m)\pi}{n}=cos.\left(i\pi-\tfrac{im\pi}{n}\right)=-cos.\tfrac{im\pi}{n}$$

and

$$sin.\frac{i(n-m)\pi}{n}=sin.\left(i\pi-\tfrac{im\pi}{n}\right)=+sin.\tfrac{im\pi}{n},$$

from which the logarithmic parts cancel each other, and the part of the integral in general is equal to :

$$\tfrac{4}{n}sin.\frac{im\pi}{n}\,\text{Arc.tang.}\frac{x\,sin.\frac{i\pi}{n}}{1-x\,cos.\frac{i\pi}{n}}.$$

Hence there is put in place the angle of convenience $\frac{\pi}{n}=\omega$ and there becomes :

$$\begin{aligned}\int\frac{\left(x^{m-1}+x^{n-m-1}\right)dx}{1+x^{n}}=&+\tfrac{4}{n}sin.m\omega\,\text{Arc.tang.}\tfrac{x\,sin.\omega}{1-x\,cos\,\omega}\\&+\tfrac{4}{n}sin.3m\omega\,\text{Arc.tang.}\tfrac{x\,sin.3\omega}{1-x\,cos\,3\omega}\\&+\tfrac{4}{n}sin.5m\omega\,\text{Arc.tang.}\tfrac{x\,sin.5\omega}{1-x\,cos\,5\omega}\\&\quad\vdots\\&+\tfrac{4}{n}sin.im\omega\,\text{Arc.tang.}\tfrac{x\,sin.i\omega}{1-x\,cos\,i\omega}\end{aligned}$$

on taking for i the greatest odd number not greater than n. If the number n is itself odd, the part arising from the position $i = n$ vanishes on account of sin. $mn = 0$. Hence it is to be noted here that the whole integral is to expressed by pure angles.

COROLLARY

83. In a like manner the following integral can be elicited, where only in this manner the following integral is elicited, where logarithms alone are left, on retaining $\frac{\pi}{n} = \omega$:

$$\int \frac{\left(x^{m-1} - x^{n-m-1}\right)dx}{1+x^n} = -\frac{4}{n} cos.m\omega \, l\sqrt{\left(1 - 2x\,cos.\omega + xx\right)}$$
$$-\frac{4}{n} cos.3m\omega \, l\sqrt{\left(1 - 2x\,cos.3\omega + xx\right)}$$
$$-\frac{4}{n} cos.5m\omega \, l\sqrt{\left(1 - 2x\,cos.5\omega + xx\right)}$$
$$\vdots$$
$$-\frac{4}{n} cos.im\omega \, l\sqrt{\left(1 - 2x\,cos.i\omega + xx\right)},$$

clearly as long as the odd number i does not exceed the exponent n.

EXAMPLE 5

84. *With the proposed differential formula* $\frac{\left(x^{m-1} - x^{n-m-1}\right)dx}{1+x^n}$ *with* $n > m-1$ *arising, to define the integral of this.*

From example 3 it can be deduced that any part of the integral, if indeed for the sake of gravity we put $\frac{\pi}{n} = \omega$, can be written as

$$-\frac{2}{n} cos.2mk\omega \, l\sqrt{\left(1 - 2x\,cos.2k\omega + xx\right)} + \frac{2}{n} sin.2mk\omega \text{ Arc.tang.}\frac{x\,sin.2mk\omega}{1 - x\,cos.2mk\omega}$$
$$+\frac{2}{n} cos.2k\left(n-m\right)\omega \, l\sqrt{\left(1 - 2x\,cos.2k\omega + xx\right)} + \frac{2}{n} sin.2k\left(n-m\right)\omega \text{ Arc.tang.}\frac{x\,sin.2mk\omega}{1 - x\,cos.2mk\omega}$$

But there is

$$cos.2k\left(n-m\right)\omega = cos.\left(2k\pi - 2km\omega\right) = cos.2km\omega$$

and

$$sin.2k\left(n-m\right)\omega = sin.\left(2k\pi - 2km\omega\right) = -sin.2km\omega,$$

from which the general part goes into

$$\frac{4}{n} sin.2km\omega \text{ Arc.tang.}\frac{x\,sin.2k\omega}{1 - x\,cos.2m\omega},$$

from which hence this integration is deduced :

$$\int\frac{\left(x^{m-1}-x^{n-m-1}\right)dx}{1+x^n}=+\frac{4}{n}\sin.2m\omega\text{Arc.tang.}\frac{x\sin.2\omega}{1-x\cos 2\omega}$$
$$+\frac{4}{n}\sin.4m\omega\text{Arc.tang.}\frac{x\sin.4\omega}{1-x\cos 4\omega}$$
$$+\frac{4}{n}\sin.6m\omega\text{Arc.tang.}\frac{x\sin.6\omega}{1-x\cos 6\omega},$$

with the even numbers increasing all the while, as long as they do not exceed the exponent *n*.

COROLLARY

85. Also from the same place this integral is completed with $\frac{\pi}{n}=\omega$ in place :

$$\int\frac{\left(x^{m-1}+x^{n-m-1}\right)dx}{1-x^n}=-\frac{2}{n}l\left(1-x\right)$$
$$-\frac{4}{n}\cos.2m\omega\, l\sqrt{\left(1-2x\cos.2\omega+xx\right)}$$
$$-\frac{4}{n}\cos.4m\omega\, l\sqrt{\left(1-2x\cos.4\omega+xx\right)}$$
$$-\frac{4}{n}\cos.6m\omega\, l\sqrt{\left(1-2x\cos.6\omega+xx\right)},$$

where also the even numbers are not to be continued beyond the term *n*.

EXAMPLE 6

86. *With the proposed differential* $dy=\frac{dx}{x^3(1+x)\left(1-x^4\right)}$*, to find the integral of this.*

The fractional function attached to *dx* following the factors of the denominator is

$$\frac{1}{x^3(1+x)^2(1-x)(1+xx)}$$

which is resolved into these simple fractions

$$\frac{1}{x^3}-\frac{1}{x^2}+\frac{1}{x}-\frac{1}{4(1+x)^2}-\frac{9}{8(1+x)}+\frac{1}{8(1-x)}+\frac{1+x}{4(1+xx)}=\frac{dy}{dx},$$

from which there is elicited by integration:

$$y=-\frac{1}{2x^2}+\frac{1}{x}+lx+\frac{1}{4(1+x)}-\frac{9}{8}l\left(1+x\right)-\frac{1}{8}l\left(1-x\right)$$
$$+\frac{1}{8}l\left(1+xx\right)+\frac{1}{4}\text{Arc.tang.}x,$$

which expression can be changed into this form :

$$y=C+\frac{-2+2x+5xx}{4xx(1+x)}-l\,\frac{1+x}{x}+\frac{1}{8}l\left(\frac{1+xx}{1-xx}\right)+\frac{1}{4}\text{Arc.tang.}x,$$

SCHOLIUM

87. Therefore by this the chapter has been made clear, so that nothing further from this kind can be desired. Hence as often as a function y of x of this kind is sought, so that $\frac{dy}{dx}$ is equal to some rational function of x, so the integration presents no difficulty, except perhaps in the elucidation of the particular factor of the denominator where the teaching of algebra has not been sufficient; then indeed the difficulty must be attributed to the defects of the algebra, not the method of integration which we have treated here. Then also chiefly it is agreed to be noted always, since $\frac{dy}{dx}$ is put equal to a rational function of x, the function of y, unless it is algebraic, does not involve other transcending functions in addition to logarithms and angles ; where indeed here it is to be observed always that hyperbolic logarithms are to be understood, since the differential of lx in not equal to $\frac{1}{x}$, unless the hyperbolic logarithm is taken ; but the reduction of these to common logarithms is very simple, thus so that hence the application of the calculus to the practice is not bothered by any impediment. Whereby we may progress to these other cases, in which the formula $\frac{dy}{dx}$ is equal to an irrational function of x, where indeed at first it must be noted, whenever this function by a suitable substitution can be reduced to rationals, the case to be returned to this chapter. Just as if there should be

$$dy = \frac{1+\sqrt{x}-\sqrt[3]{xx}}{1+\sqrt[3]{x}}dx,$$

it is clear on putting $x = z^6$, from which there arises $dx = 6z^5dz$, to become

$$dy = \frac{\left(1+z^3-z^4\right)}{1+zz}\cdot 6z^5dz$$

and thus

$$\frac{dy}{dz} = -6z^7 + 6z^6 + 6z^5 - 6z^4 + 6zz - 6 + \frac{6}{1+zz},$$

from which on integrating

$$y = -\frac{3}{4}z^8 + \frac{6}{7}z^7 + z^6 - \frac{6}{5}z^5 + 2z^3 - 6z + 6\text{Arc.tang.}z$$

and on restoring the value :

$$y = -\frac{3}{4}x\sqrt[3]{x} + \frac{6}{7}x\sqrt[6]{x} + x - \frac{6}{5}\sqrt[6]{x^5} + 2\sqrt{x} - 6\sqrt[6]{x} + 6\text{Arc.tang.}\sqrt[6]{x} + C.$$

APPENDICES FROM

INTRODUCTIO IN ANALYSIN INFINITORUM CH.2 & INSTITUTIONES CALCULI DIFFERENTIALIS CH.18,Part 2.

In Ch. 9 § 145 of the *Introductio...*, Euler proceeds as follows : The quadratic $p-qz+rzz$ in the variable *z* has imaginary roots if $4pr>qq$ or $\frac{q}{2\sqrt{pr}}<1.$ Since the sines and cosines of angles are less than 1, then the formula $p-qz+rzz$ has simple imaginary factors, if $\frac{q}{2\sqrt{pr}}$ should be the sine or cosine of some angle. Hence, let

$$\frac{q}{2\sqrt{pr}}=cos.\varphi,\quad \text{or}\quad q=2\sqrt{pr}\cdot cos.\varphi$$

and the trinomial $p-qz+zz$ contains the imaginary simple factors. But lest the irrationality should produce any trouble, I assume this form $pp-2pqz\,cos.\varphi+qqzz$, and the simple imaginary factors of this are these :

$$qz-p\left(cos.\varphi+\sqrt{-1}\cdot sin.\varphi\right)\quad\text{and}\quad qz-p\left(cos.\varphi-\sqrt{-1}\cdot sin.\varphi\right).$$

Where indeed it is apparent, if $cos.\varphi=\pm1$, then both the factors are equal and real since $sin.\varphi=0$. [In modern terms, we would say that $qz-pe^{i\varphi}$ and $qz-pe^{-i\varphi}$ are the factors, and powers of z can be expressed in the form $z^n=r^n\left(cos.\varphi\pm\sqrt{-1}\cdot sin.\varphi\right)^n$]. In § 147, Euler assumes DeMoivre's Theorem :

$$\left(cos.\varphi\pm\sqrt{-1}\cdot sin.\varphi\right)^n=cos.n\varphi\pm\sqrt{-1}\cdot sin.n\varphi\ ,$$

and hence

$$z^n=r^n\left(cos.n\varphi\pm\sqrt{-1}\cdot sin.n\varphi\right),\quad\text{where}\quad r=\tfrac{p}{q}$$

In § 150 Euler examines the function a^n+z^n, and the form of the trinomial factors of this factor $pp-2pqz\,cos.\varphi+qqzz$ is to be determined. On putting $r=\frac{p}{q}$ into the equation, there are produced these two equations : $a^n=r^n\,cos.n\varphi,$ and $0=r^n\,sin.n\varphi$. From the latter equation we must have, for some integer *k*, $n\varphi=2k\pi,$ or $n\varphi=(2k+1)\pi$; but in the former equation, we must choose $n\varphi=(2k+1)\pi$, in order that we have $a^n-r^n=0$, and hence $r=a=\frac{p}{q}$. Hence in the above equation, $p=a$, $q=1$, and $\varphi=(2k+1)\frac{\pi}{n}$.

Hence the quadratic factor with conjugate roots of the function $a^n + z^n$ can be shown to be $aa - 2az\cos.\frac{2k+1}{n}\pi + zz$, where *k* is less than *n*. Euler goes on to look at numerous cases involving this factorization.

The partial fraction reduction quoted above in Examples 2 and 3 comes from the *Institutionum....*, Part 2, Ch.18, which we now address at length, we take up the story at the end of § 406 to Example 1 of §417 in translation :

If the rational fraction is expressed by $\frac{P}{Q}$, then the values of the factors of the denominator can be found from the resolution of the equation $Q = 0$, and here we make use of a method involving the differential calculus in order that any simple factors of the denominator can thus be defined.

407. Therefore we arrange the denominator Q of the fraction $\frac{P}{Q}$ to have a factor $f + gx$, thus so that $Q = (f + gx)S$, and indeed here, neither does *S* contain in addition the same factor $f + gx$. Let the simple fraction arising from this factor be equal to $\frac{\mathfrak{A}}{f+gx}$ and the complement has a form of this kind $\frac{V}{S}$, thus so that $\frac{\mathfrak{A}}{f+gx} + \frac{V}{S} = \frac{P}{Q}$.

Hence there arises : $\frac{V}{S} = \frac{P}{Q} - \frac{\mathfrak{A}}{f+gx} = \frac{P}{(f+gx)S} - \frac{\mathfrak{A}}{f+gx} = \frac{P-\mathfrak{A}S}{(f+gx)S}$, and thus $V = \frac{P-\mathfrak{A}S}{(f+gx)}$.

Therefore, as *V* is an integral function of *x*, it is necessary that $P - \mathfrak{A}S$ should be divisible by $f + gx$; and therefore, if there is put $f + gx = 0$ or $x = \frac{-f}{g}$, then the expression $P - \mathfrak{A}S$ vanishes, [as $f + gx$ is a factor of $P - \mathfrak{A}S$ or $x = \frac{-f}{g}$ is a root of this polynomial].

Hence on making $x = \frac{-f}{g}$, since $P - \mathfrak{A}S = 0$, then $\mathfrak{A} = \frac{P}{S}$, [for this value of *x*] as now we have found above. But since $S = \frac{Q}{f+gx}$, then there arises $\mathfrak{A} = \frac{(f+gx)P}{Q}$, if there is put $f + gx = 0$ or $x = \frac{-f}{g}$ everywhere. Now since in this case both the numerator $(f + gx)P$ as well as the denominator Q vanish here, by means of that, which we are about to set out in the investigation of fractions of this kind, then if indeed there is put $x = \frac{-f}{g}$,

$$\mathfrak{A} = \frac{(f + gx)dP + Pgdx}{dQ}.$$

But in this case on account of $(f + gx)dP = 0$ then $\mathfrak{A} = \frac{Pgdx}{dQ}$.

[One can see that Euler is thinking of the ratio becoming zero on zero when $x = \frac{-f}{g}$, and so the ratio is to the differential of the numerator to the differential of the denominator. One can see also that the problem is essentially an application of L'Hôpital's Rule : as is well known, in the neighbourhood of a common simple zero $x = a$ of well-behaved functions r and s of x, or a common root of polynomials, the values of the functions at a are represented by their differentials, and it is an easy matter to show that $\lim_{x \to a} \frac{r(a)}{s(a)} = \frac{r'(a)}{s'(a)}$, where the rule can be extended to higher derivatives if necessary .]

408. Therefore if the denominator Q of the proposed fraction $\frac{P}{Q}$ has a simple factor $f + gx$, from that there arises the simple fraction $\frac{\mathfrak{A}}{f+gx}$ with $\mathfrak{A} = \frac{Pgdx}{dQ}$ being present, after everywhere here the value $\frac{-f}{g}$ is put in place of x arising from the equation $f + gx = 0$ must be substituted. Hence, if Q is not expressed in factors, this division can often be omitted without any bother, particularly if the denominator Q has indefinite exponents, since the value of $\mathfrak{A}$ can be obtained from the formula $\frac{Pgdx}{dQ}$. But if the denominator now was expressed in factors, thus in order that hence the value of S should be at once apparent, then the other expression is to be preferred, in which we found $\mathfrak{A} = \frac{P}{S}$ on putting everywhere equally $x = \frac{-f}{g}$. And thus for finding the value of $\mathfrak{A}$ itself in whatever case that formula can be used, which is seen to be the more convenient and expedient. But we illustrate the use of the new formula with some examples.

EXAMPLE 1

Let this fraction be proposed $\frac{x^9}{1+x^{17}}$, a simple fraction of which is required to be defined from the factor of the denominator $1+x$.

Since here $Q = 1 + x^{17}$, even if $1+x$ is agreed to be a factor of this, yet, if as the first method demands, we wish to divide by that, there is produced

$$S = 1 - x + xx - x^3 + \ldots + x^{16}$$

Therefore it is more convenient to use the new formula $\mathfrak{A} = \frac{Pgdx}{dQ}$; and thus since $f= 1$, $g = 1$ and $P= x^9$, on account of $dQ = 17x^{16}dx$ there becomes $\mathfrak{A} = \frac{x^9}{17x^{16}} = \frac{1}{17x^7}$ and on putting $x = -1$, from which there arises $\mathfrak{A} = -\frac{1}{17}$, and the simple fraction arising from the factor of the denominator $1 + x$ is $\frac{-1}{17(1+x)}$.

EXAMPLE 2

For the proposed fraction $\frac{x^m}{1-x^{2n}}$*, to find the simple fraction arising from the factor* $1-x$ *of the denominator.*

On account of the proposed factor $1-x$ there is $f = 1$ and $g = -1$. Then the denominator $1-x^{2n}$ gives $dQ = -2nx^{2n-1}dx$, from which because of $P = x^m$ there is obtained $\mathfrak{A} = \frac{Pgdx}{dQ} = \frac{-x^m}{-2nx^{2n-1}}$. And on putting from the equation $1-x$, $x = 1$ there becomes $\mathfrak{A} = \frac{1}{2n}$, thus so that the simple fraction is $\frac{1}{2n(1-x)}$

EXAMPLE 3

For the proposed fraction $\frac{x^m}{1-4x^k+3x^n}$*, to find the simple fraction arising from the factor* $1-x$ *of the denominator.*

Hence here there becomes $f = 1, g = -1, P = x^m$, and $Q = 1-4xk+3x^n$ and $\frac{dQ}{dx} = -4kx^{k-1} + 3nx^{n-1}$; from which there arises $\mathfrak{A} = \frac{-x^m}{-4kx^{k-1}+3nx^{n-1}}$ and on putting $x = 1$ then $\mathfrak{A} = \frac{1}{4k-3n}$. The simple fraction arising from this simple factor of the denominator $1-x$ is now $\frac{1}{(4k-3n)(1-x)}$.

409. Now we put the denominator Q of the fraction $\frac{P}{Q}$ to have the square factor $(f+gx)^2$ and the simple fractions hence arising are $\frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}$.

Let $Q = (f+gx)^2 S$ and the complement $= \frac{V}{S}$, thus in order that

$$\frac{V}{S} = \frac{P}{Q} - \frac{\mathfrak{A}}{(f+gx)^2} - \frac{\mathfrak{B}}{f+gx} \text{ and } V = \frac{P-\mathfrak{A}S-\mathfrak{B}(f+gx)S}{(f+gx)^2}$$

Now because V is an integral function, it is necessary that $P-\mathfrak{A}S-\mathfrak{B}(f+gx)S$ is divisible by $(f+gx)^2$; and since S does not contain a further factor $f+gx$, also this expression $\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx)$ is divisible by $(f+gx)^2$, and thus with the factor $f+gx = 0$ or $x = -\frac{f}{g}$ not only this expression, but also the differential of this vanishes, $d.\frac{P}{S} - \mathfrak{B}gdx$. Therefore on putting $x = -\frac{f}{g}$, there arises from the first equation $\mathfrak{A} = \frac{P}{S}$, and from the second equation there now arises $\mathfrak{B}\frac{1}{gdx} = d.\frac{P}{S}$; from which the values of the fractions sought $\frac{\mathfrak{A}}{(f+gx)^2} + \frac{\mathfrak{B}}{f+gx}$ can be found .

EXAMPLE

From the proposed fraction, $\frac{x^m}{1-4x^3+3x^4}$ *the denominator of which has the factor* $(1-x)^2$, *to find all the simple fractions thus arising.*

Since here there is $f = 1, g = -1, P = x^m$ and $Q = 1-4x^3+3x^4$
then [on factorising the denominator], $S = 1+2x+3xx$, then

$$\frac{P}{S} = \frac{x^m}{1+2x+3xx} \text{ and } d.\frac{P}{S} = \frac{mx^{m-1}dx+2(m-1)x^m dx+3(m-2)x^{m+1}dx}{(1+2x+3xx)^2}.$$

Hence on putting $x = 1$ there arises

$$\mathfrak{A} = \frac{1}{6} \text{ and } \mathfrak{B} = -1\cdot\frac{6m-8}{36} = \frac{4-3m}{18};$$

from which the fractions sought are

$$\frac{1}{6(1-x)^2} + \frac{4-3m}{18(1-x)}.$$

410. Let the denominator Q of the fraction $\frac{P}{Q}$ have three equal simple fractions, or $Q = (f+gx)^3 S$ and let the simple fractions arising from the cube of the factor $(f+gx)^3$ be these

$$\frac{\mathfrak{A}}{(f+gx)^3} + \frac{\mathfrak{B}}{(f+gx)^2} + \frac{\mathfrak{C}}{(f+gx)};$$

now the complement of these fractions to be established for the proposed fraction $\frac{P}{Q}$ is equal to $\frac{V}{S}$ and it follows that

$$V = \frac{P-\mathfrak{A}S-\mathfrak{B}S(f+gx)-\mathfrak{C}S(f+gx)^2}{(f+gx)^3}$$

Whereby this expression $\frac{P}{S} - \mathfrak{A} - \mathfrak{B}(f+gx) - \mathfrak{C}(f+gx)^2 = 0$ is divisible by $(f+gx)^3$; from which on putting $f+gx = 0$ or $x = \frac{-f}{g}$ not only this expression, but also the first and second differentials are able to become equal to 0. Clearly on putting $x = \frac{-f}{g}$

$$\frac{P}{S}-\mathfrak{A}-\mathfrak{B}(f+gx)-\mathfrak{C}(f+gx)^2=0$$
$$d.\frac{P}{S}-\mathfrak{B}gdx-2\mathfrak{C}gdx(f+gx)=0$$
$$dd.\frac{P}{S}-2\mathfrak{C}g^2dx^2=0.$$

Hence from the first equation there becomes

$$\mathfrak{A}=\frac{P}{S}.$$

Now from the second

$$\mathfrak{B}=\frac{1}{gdx}d.\frac{P}{S}.$$

And from the third there is defined

$$\mathfrak{C}=\frac{1}{2g^2dx^2}dd.\frac{P}{S}.$$

411. Hence generally, if the denominator Q of the fraction $\frac{P}{Q}$ has the factor $(f+gx)^n$, thus in order that $Q=(f+gx)^n S$,with these simple fractions arising from the factor $(f+gx)^n$:

$$\frac{\mathfrak{A}}{(f+gx)^n}+\frac{\mathfrak{B}}{(f+gx)^{n-1}}+\frac{\mathfrak{C}}{(f+gx)^{n-2}}+\frac{\mathfrak{D}}{(f+gx)^{n-3}}+\frac{\mathfrak{E}}{(f+gx)^{n-4}}+\text{etc.},$$

until finally the fraction is arrived at, of which the denominator is $f+gx$, if the method is put in place as before, then this expression is found:

$$\frac{P}{S}-\mathfrak{A}-\mathfrak{B}(f+gx)-\mathfrak{C}(f+gx)^2-\mathfrak{D}(f+gx)^3-\mathfrak{E}(f+gx)^4-\text{etc.}$$

which must be divisible by $(f+gx)^n$ *;* hence this expression as well as the individual differentials of this as far as the degree $n-1$ must vanish when we set $x=\frac{-f}{g}$

From which equations it can be concluded on putting $x=\frac{-f}{g}$ everywhere :

$$\mathfrak{A} = \frac{P}{S}$$
$$\mathfrak{B} = \frac{1}{1gdx} d.\frac{P}{S}$$
$$\mathfrak{C} = \frac{1}{1\cdot 2g^2dx^2} dd.\frac{P}{S}$$
$$\mathfrak{D} = \frac{1}{1\cdot 2\cdot 3g^3dx^3} d^3.\frac{P}{S}$$
$$\mathfrak{E} = \frac{1}{1\cdot 2\cdot 3\cdot 4g^4dx^4} d^4.\frac{P}{S}$$
etc.

Where indeed it is to be observed it is required to take the differentials themselves of $\frac{P}{S}$ beforehand, as in place of x there is to be put $\frac{-f}{g}$;

412. Hence in this way these numbers $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc can be expressed more easily than in the manner treated in the *Introductio*, and the values of these can be found more readily on many occasions too by this new method. Which comparison by which, in the previous manner, we can define the values of the letters $\mathfrak{A}, \mathfrak{B}, \mathfrak{C}, \mathfrak{D}$ etc , can be put in place more easily : Put $x = \frac{-f}{g}$

For the variable x , the remainder is put in place

$$\mathfrak{A} = \frac{P}{S} \qquad \frac{P - \mathfrak{A}S}{f+gx} = \mathfrak{P},$$
$$\mathfrak{B} = \frac{\mathfrak{P}}{S} \qquad \frac{\mathfrak{P} - \mathfrak{B}S}{f+gx} = \mathfrak{Q},$$
$$\mathfrak{C} = \frac{\mathfrak{Q}}{S} \qquad \frac{\mathfrak{Q} - \mathfrak{C}S}{f+gx} = \mathfrak{R},$$
$$\mathfrak{D} = \frac{\mathfrak{R}}{S} \qquad \frac{\mathfrak{R} - \mathfrak{D}S}{f+gx} = \mathfrak{S},$$
$$\mathfrak{E} = \frac{\mathfrak{S}}{S}$$ and thus henceforth.

413. But if the denominator Q of the fraction $\frac{P}{Q}$ should not have all real simple factors, then two of the imaginary factors are to be taken together, the product of which is real. Hence let a factor of the denominator Q be $ff - 2fgx\cos\varphi + ggxx$, which on being put equal to zero gives this two-fold imaginary value :

$$x = \frac{f}{g}\cos\varphi \pm \frac{f}{g\sqrt{-1}}\sin\varphi;$$

from which there arises

$$x^n = \frac{f^n}{g^n}\cos n\varphi \pm \frac{f^n}{g^n\sqrt{-1}}\sin n\varphi.$$

We put the denominator $Q = \left(ff - 2fgx\cos\varphi + ggxx \right) S$ and in addition S is not to be divisible by $ff - 2fgx\cos\varphi + ggxx$. Let the fraction originating from this denominator be

$$\frac{\mathfrak{A} + \mathfrak{a}x}{ff - 2fgx\cos\varphi + ggxx}$$

and the complement for the proposed fraction $\frac{P}{Q}$ is equal to $\frac{V}{S}$; then

$$V = \frac{P - (\mathfrak{A} + \mathfrak{a}x)S}{ff - 2fgx\cos\varphi + ggxx},$$

from which $P - (\mathfrak{A} + \mathfrak{a}x)S$ and in addition also $\frac{P}{S} - \mathfrak{A} - \mathfrak{a}x$ is divisible by $ff - 2fgx\cos\varphi + ggxx$. Hence $\frac{P}{S} - \mathfrak{A} - \mathfrak{a}x$ vanishes, if there is put $ff - 2fgx\cos\varphi + ggxx = 0$, that is, if there is put either

$$x = \frac{f}{g}\cos\varphi + \frac{f}{g\sqrt{-1}}\sin\varphi$$

or

$$x = \frac{f}{g}\cos\varphi - \frac{f}{g\sqrt{-1}}\sin\varphi$$

414. Because P and S are integral [*i. e.* polynomial] functions of x, the substitution can be made separately into each in turn ; and because for any power of x, for example x^n, this binomial must be substituted

$$x^n = \frac{f^n}{g^n}\cos n\varphi \pm \frac{f^n}{g^n\sqrt{-1}}\sin n\varphi,$$

in the first place we can put $\frac{f^n}{g^n}\cos n\varphi$ for x^n everywhere, and with this done P is changed into $\mathfrak{P}$ and S into $\mathfrak{S}$. Then in place of x^n everywhere there is put $\frac{f^n}{g^n}\sin n\varphi$ and with this done P becomes $\mathfrak{p}$ and S becomes $\mathfrak{s}$; where it is to be noted that before these substitutions each function P and S must be expanded out completely, thus in order that, if perhaps factors are to be involved, then these are to be removed. From these values $\mathfrak{P}, \mathfrak{p}, \mathfrak{S}, \mathfrak{s}$ found it is evident, if there is put $x = \frac{f}{g}\cos\varphi \pm \frac{f}{g\sqrt{-1}}\sin\varphi$, the function P is changed into $\mathfrak{P} \pm \frac{\mathfrak{p}}{\sqrt{-1}}$ and the function S changed into $\mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}}$

Hence since $\frac{P}{S} - \mathfrak{A} - \mathfrak{a}x$ or $P - (\mathfrak{A} + \mathfrak{a}x)S$ must vanish in each case, then

$$\mathfrak{P} \pm \frac{\mathfrak{p}}{\sqrt{-1}} = \left(\mathfrak{A} + \frac{\mathfrak{a}f}{g}\cos\varphi \pm \frac{\mathfrak{a}f}{g\sqrt{-1}}\sin\varphi \right)\left(\mathfrak{S} \pm \frac{\mathfrak{s}}{\sqrt{-1}} \right)$$

from which on account of the sign condition these two equations arise :

$$\mathfrak{P}=\mathfrak{A}\mathfrak{S}+\frac{\mathfrak{a}f\mathfrak{S}}{g}\cos\varphi-\frac{\mathfrak{a}f\mathfrak{s}}{g}\sin\varphi$$

$$\mathfrak{p}=\mathfrak{A}\mathfrak{s}+\frac{\mathfrak{a}f\mathfrak{s}}{g}\cos\varphi+\frac{\mathfrak{a}f\mathfrak{S}}{g}\sin\varphi,$$

from which on eliminating $\mathfrak{A}$ there arises

$$\mathfrak{S}\mathfrak{p}-\mathfrak{s}\mathfrak{P}=\frac{\mathfrak{a}f\left(\mathfrak{S}^2+\mathfrak{s}^2\right)}{g}\sin\varphi\ ;$$

and thus there becomes

$$\mathfrak{a}=\frac{g\left(\mathfrak{S}\mathfrak{p}-\mathfrak{s}\mathfrak{P}\right)}{f\left(\mathfrak{S}^2+\mathfrak{s}^2\right)\sin\varphi}.$$

Then on eliminating $\sin\varphi$ there becomes

$$\mathfrak{S}\mathfrak{P}+\mathfrak{s}\mathfrak{p}=\left(\mathfrak{S}^2+\mathfrak{s}^2\right)\left(\mathfrak{A}+\frac{\mathfrak{a}f}{g}\cos\varphi\right).$$

Hence

$$\mathfrak{A}=\frac{\mathfrak{S}\mathfrak{P}+\mathfrak{s}\mathfrak{p}}{\mathfrak{S}^2+\mathfrak{s}^2}-\frac{\left(\mathfrak{S}\mathfrak{p}-\mathfrak{s}\mathfrak{P}\right)\cos\varphi}{\left(\mathfrak{S}^2+\mathfrak{s}^2\right)\sin\varphi}.$$

415. Since now there shall be $S=\frac{Q}{ff-2fgx\cos\varphi+ggxx}$,

because on putting $ff-2fgx\cos\varphi+ggxx=0$
both the numerator as well as the denominator vanish, in this case there arises

$$S=\frac{dQ:dx}{2ggx-2fg\cos\varphi}$$

Now we may put, if $\frac{f^n}{g^n}\cos n\varphi$ is substituted everywhere, the function $\frac{dQ}{dx}$ to become $\mathfrak{Q}$,

but if there is put in place everywhere $\frac{f^n}{g^n}\sin n\varphi$, that goes into $\mathfrak{q}$; and it is evident, if

there is put $x=\frac{f}{g}\cos\varphi\pm\frac{f}{g\sqrt{-1}}\sin\varphi$, the function $\frac{dQ}{dx}$ becomes $\mathfrak{Q}\pm\frac{\mathfrak{q}}{\sqrt{-1}}$
From which the function S is changed into

$$\frac{\mathfrak{Q}\pm\mathfrak{q}:\sqrt{-1}}{\pm 2fg\sin\varphi:\sqrt{-1}}.$$

Therefore since $S=\mathfrak{S}\pm\frac{\mathfrak{s}}{\sqrt{-1}}$ with the same value for x put in place, there is obtained :

$$\mathfrak{Q}\pm\frac{\mathfrak{q}}{\sqrt{-1}}=\pm\frac{2fg\mathfrak{S}}{\sqrt{-1}}\sin\varphi-2fg\mathfrak{s}\sin\varphi.$$

Hence there becomes

$$\mathfrak{s}=\frac{-\mathfrak{Q}}{2fg\sin\varphi}\ \text{and}\ \mathfrak{S}=\frac{\mathfrak{q}}{2fg\sin\varphi}$$

With these values in place there arises

$$\mathfrak{a} = \frac{2gg(\mathfrak{p}\mathfrak{q}+\mathfrak{P}\mathfrak{Q})}{\mathfrak{Q}^2+\mathfrak{q}^2}$$

and

$$\mathfrak{A} = \frac{2fg(\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q})\sin\varphi}{\mathfrak{Q}^2+\mathfrak{q}^2} - \frac{2fg(\mathfrak{p}\mathfrak{q}+\mathfrak{P}\mathfrak{Q})\cos\varphi}{\mathfrak{Q}^2+\mathfrak{q}^2}.$$

416. Hence therefore a suitable ratio is obtained and here a simple fraction can be formed from whatever the factor of the second power, since the denominator of the proposed denominator is retained in the computation, since we avoid a division, in which the value of the letter *S* must be defined and which often is more than a little troublesome. Therefore if the denominator *Q* of the fraction $\frac{P}{Q}$ has such a factor $ff-2fgx\cos\varphi+ggxx=0$, a simple fraction arising from this factor can be defined in the following manner, so that we may produce :

$$= \frac{\mathfrak{A}+\mathfrak{a}x}{ff-2fgx\cos\varphi+ggxx}.$$

There is put $x=\frac{f}{g}\cos\varphi$ and for whatever the power of x^n there is written $\frac{f^n}{g^n}\cos n\varphi$; with which done *P* is changed into $\mathfrak{P}$ and the function $\frac{dQ}{dx}$ into $\mathfrak{Q}$. Then likewise there is put $x=\frac{f}{g}\sin\varphi$ and any power of this $x^n=\frac{f^n}{g^n}\sin n\varphi$ and *P* is changed into $\mathfrak{p}$ and $\frac{dQ}{dx}$ into $\mathfrak{q}$. And in this manner with the values of the letters $\mathfrak{P},\mathfrak{Q},\mathfrak{p}$ and $\mathfrak{q}$ found, $\mathfrak{A}$ and $\mathfrak{a}$ can thus be found, in order that there becomes

$$\mathfrak{A} = \frac{2fg(\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q})\sin\varphi}{\mathfrak{Q}^2+\mathfrak{q}^2} - \frac{2fg(\mathfrak{p}\mathfrak{q}+\mathfrak{P}\mathfrak{Q})\cos\varphi}{\mathfrak{Q}^2+\mathfrak{q}^2},$$

$$\mathfrak{a} = \frac{2gg(\mathfrak{p}\mathfrak{q}+\mathfrak{P}\mathfrak{Q})}{\mathfrak{Q}^2+\mathfrak{q}^2}.$$

Hence the fraction arising from the factor $ff-2fgx\cos\varphi+ggxx$ of the denominator *Q* is given by

$$\frac{2fg(\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q})\sin\varphi+2g(\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q})(gx-f\cos\varphi)}{(\mathfrak{Q}^2+\mathfrak{q}^2)(ff-2fgx\cos\varphi+ggxx)}.$$

EXAMPLE 1

If this fraction is proposed $\frac{x^m}{a+bx^n}$, the denominator of which $a+bx^n$ has this factor $ff-2fgx\cos\varphi+ggxx$, to find the simple fraction agreeing with this factor.

Because here $P=x^m$ and $Q=a+bx^n$, then $\frac{dQ}{dx}=nbx^n$ and thus there arises

$$\mathfrak{P}=\frac{f^m}{g^m}\cos m\varphi,\qquad \mathfrak{p}=\frac{f^n}{g^n}\sin m\varphi,$$

$$\mathfrak{Q}=\frac{nbf^{n-1}}{g^{n-1}}\cos(n-1)\varphi,\qquad \mathfrak{q}=\frac{nbf^{n-1}}{g^{n-1}}\sin(n-1)\varphi.$$

From these there becomes

$$\mathfrak{Q}^2+\mathfrak{q}^2=\frac{n^2b^2f^{2(n-1)}}{g^{2(n-1)}},$$

$$\mathfrak{P}\mathfrak{Q}-\mathfrak{p}\mathfrak{q}=\frac{nbf^{m+n-1}}{g^{m+n-1}}\sin(n-m-1)\varphi,$$

and

$$\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}=\frac{nbf^{m+n-1}}{g^{m+n-1}}\cos(n-m-1)\varphi.$$

On account of which the simple fraction sought is

$$\frac{2g^{n-m}\left(f\sin\varphi\cdot\sin(n-m-1)\varphi+gx\cos(n-m-1)\varphi-f\cos\varphi\cdot\cos(n-m-1)\varphi\right)}{nbf^{n-m-1}\left(ff-2fgx\cos\varphi+ggxx\right)}$$

or

$$\frac{2g^{n-m}\left(gx\cos(n-m-1)\varphi-f\cos(n-m)\varphi\right)}{nbf^{n-m-1}\left(ff-2fgx\cos\varphi+ggxx\right)}.$$

EXAMPLE 2

If this fraction should be proposed, $\frac{1}{x^m\left(a+bx^n\right)}$, the denominator of which has the factor $ff-2fgx\cos\varphi+ggxx$; to find the simple fraction arising from this.

As there is $p=1$ and $Q=ax^m+bx^{m+n}$, then $\frac{dQ}{dx}=max^{m-1}+(m+n)bx^{m+n-1}$

and thus on putting $x^n=\frac{f^n}{g^n}\cos n\varphi$, on account of $P=x^0$, $\mathfrak{P}=1$, and

$$\mathfrak{Q}=\frac{maf^{m-1}}{g^{m-1}}\cos(m-1)\varphi+\frac{(m+n)bf^{m+n-1}}{g^{m+n-1}}\cos(m+n-1)\varphi$$

and [on putting $x^n=\frac{f^n}{g^n}\sin n\varphi$] $\mathfrak{p}=0$ and

$$\mathfrak{q} = \frac{maf^{m-1}}{g^{m-1}} sin(m-1)\varphi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}} sin(m+n-1)\varphi .$$

Hence

$$\mathfrak{A}^2 + \mathfrak{q}^2 = \frac{m^2a^2f^{2(m-1)}}{g^{m-1}} + \frac{2m(m+n)abf^{2m+n-2}}{g^{2m+n-2}} cos\, n\varphi + \frac{(m+n)^2 b^2 f^{2(m+n-1)}}{g^{2(m+n-1)}} .$$

For if now $ff - 2fgx\,cos\,\varphi + ggxx$ is a divisor of $a + bx^n$, then $a + \frac{bf^n}{g^n} cos\, n\varphi = 0$ and $\frac{bf^n}{g^n} sin\, n\varphi = 0$, on account of which $aa = \frac{bbf^{2n}}{g^{2n}}$.
Hence there becomes

$$\mathfrak{A}^2 + \mathfrak{q}^2 = \frac{(m+n)^2 bbf^{2(m+n-1)}}{g^{2(m+n-1)}} - \frac{m(2n+m)aaf^{2(m-1)}}{g^{2(m-1)}} = \frac{nnaaf^{2(m-1)}}{g^{2(m-1)}} = \frac{nnbbf^{2(m+n-1)}}{g^{2(m+n-1)}} .$$

Then there now becomes

$$\begin{aligned}\mathfrak{P}\mathfrak{q} - \mathfrak{p}\mathfrak{Q} &= \frac{maf^{m-1}}{g^{m-1}} sin(m-1)\varphi + \frac{(m+n)bf^{m+n-1}}{g^{m+n-1}} sin(m+n-1)\varphi \\ &= \frac{bf^{m+n-1}}{g^{m+n-1}}\left((m+n)sin(m+n-1)\varphi - m\,cos\, n\varphi \cdot sin(m-1)\varphi\right) \\ &= \frac{bf^{m+n-1}}{g^{m+n-1}}\left(n\,cos\, n\varphi \cdot sin(m-1)\varphi + (m+n)sin\, n\varphi \cdot cos(m-1)\varphi\right)\end{aligned}$$

and

$$\mathfrak{P}\mathfrak{q} + \mathfrak{p}\mathfrak{Q} = \frac{bf^{m+n-1}}{g^{m+n-1}}\left((m+n)cos(m+n-1)\varphi - m\,cos\, n\varphi \cdot cos(m-1)\varphi\right).$$

Either since $ff - 2fg\,cos\,\varphi + ggxx$ is also a divisor of $ax^{m-1} + bx^{m+n-1}$, then

$$\frac{af^{m-1}}{g^{m-1}} cos(m-1)\varphi + \frac{bf^{m+n-1}}{g^{m+n-1}} cos(m+n-1)\varphi = 0$$

and

$$\frac{af^{m-1}}{g^{m-1}} sin(m-1)\varphi + \frac{bf^{m+n-1}}{g^{m+n-1}} sin(m+n-1)\varphi = 0,$$

from which there becomes

$$\mathfrak{Q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} cos(m+n-1)\varphi \text{ and } \mathfrak{q} = \frac{nbf^{m+n-1}}{g^{m+n-1}} sin(m+n-1)\varphi$$

or

$$\mathfrak{Q} = \frac{-naf^{m-1}}{g^{m-1}} cos(m-1)\varphi \text{ and } \mathfrak{q} = \frac{-naf^{m-1}}{g^{m-1}} sin(m-1)\varphi$$

From which there results the fraction sought

$$\frac{2g^m\left(f\cos m\varphi - gx\cos(m-1)\varphi\right)}{naf^{m-1}\left(ff-2fgx\cos\varphi+ggxx\right)}.$$

Which formula follows from the first example, if *m* is put negative, from which there was no need to have put in place this particular example.

EXAMPLE 3

If the denominator of this fraction $\frac{x^m}{a+bx^n+cx^{2n}}$ *should have a factor of the form* $ff-2fgx\cos\varphi+ggxx$, *to find the simple fraction arising from this factor.*

If $ff-2fgx\cos\varphi+ggxx$ is a factor of the denominator $a+bx^n+cx^{2n}$, then, as we have shown above,

$$a+\frac{bf^n}{g^n}\cos n\varphi+\frac{cf^{2n}}{g^{2n}}\cos 2n\varphi=0 \text{ and } \frac{bf^n}{g^n}\sin n\varphi+\frac{cf^{2n}}{g^{2n}}\sin 2n\varphi=0$$

Therefore since there shall be $P=x^m$ and $Q=a+bx^n+cx^{2n}$, then

$$\frac{dQ}{dx}=nbx^{n-1}+2ncx^{2n-1},$$

from which there is produced

$$\mathfrak{P}=\frac{f^m}{g^m}\cos m\varphi \text{ and } \mathfrak{p}=\frac{f^m}{g^m}\sin m\varphi,$$

$$\mathfrak{Q}=\frac{nbf^{n-1}}{g^{n-1}}\cos(n-1)\varphi+\frac{2ncf^{2n-1}}{g^{2n-1}}\cos(2n-1)\varphi,$$

$$\mathfrak{q}=\frac{nbf^{n-1}}{g^{n-1}}\sin(n-1)\varphi+\frac{2ncf^{2n-1}}{g^{2n-1}}\sin(2n-1)\varphi.$$

On account of which we have

$$\mathfrak{A}^2+\mathfrak{q}^2=\frac{n^2f^{2(n-1)}}{g^{2(n-1)}}\left(bb+\frac{4bcf^n}{g^n}\cos n\varphi+\frac{4ccf^{2n}}{g^{2n}}\right)$$

But from the two former equations there is

$$\frac{f^{2n}}{g^{2n}}\left(bb+\frac{2bcf^n}{g^n}\cos n\varphi+\frac{ccf^{2n}}{g^{2n}}\right)=aa$$

and thus

$$\frac{4bcf^n}{g^n}\cos n\varphi=\frac{2g^{2n}aa}{f^{2n}}-2bb-\frac{2ccf^{2n}}{g^{2n}};$$

from which value substituted there, there becomes

$$\mathfrak{A}^2+\mathfrak{q}^2=\frac{n^2 f^{2n-2}}{g^{2n-2}}\left(\frac{2aag^{2n}}{f^{2n}}-bb+\frac{2ccf^{2n}}{g^{2n}}\right)$$

or

$$\mathfrak{A}^2+\mathfrak{q}^2=\frac{n^2\left(2aag^{4n}-bbf^{2n}g^{2n}+2ccf^{4n}\right)}{ffg^{4n-2}}.$$

Then there becomes

$$\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q}=\frac{nbf^{m+n-1}}{g^{m+n-1}}sin\left(n-m-1\right)\varphi+\frac{2ncf^{m+2n-1}}{g^{m+2n-1}}sin\left(2n-m-1\right)\varphi,$$

$$\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}=\frac{nbf^{m+n-1}}{g^{m+n-1}}cos\left(n-m-1\right)\varphi+\frac{2ncf^{m+2n-1}}{g^{m+2n-1}}cos\left(2n-m-1\right)\varphi.$$

From which values found the simple fraction sought becomes

$$\frac{2\,fg\left(\mathfrak{P}\mathfrak{q}-\mathfrak{p}\mathfrak{Q}\right)sin\,\varphi+2g\left(\mathfrak{P}\mathfrak{Q}+\mathfrak{p}\mathfrak{q}\right)\left(gx-f\,cos\,\varphi\right)}{\left(\mathfrak{Q}^2+\mathfrak{q}^2\right)\left(ff-2\,fgx\,cos\,\varphi+ggxx\right)}.$$

417. But these values are more easily expressed if we determine these factors themselves from the denominator. Therefore let the denominator of the proposed fraction be

$$a+bx^n;$$

of which if the trinomial factor is put in place $ff-2\,fgx\,cos\,\varphi+ggxx$, there will be, as we have shown in the *Introductio*,

$$a+\frac{bf^n}{g^n}cosn\varphi=0 \text{ and } \frac{bf^n}{g^n}sin\,n\varphi=0$$

therefore since $sin\,n\varphi=0$, either $n\varphi=(2k\text{-}1)\pi$ or $n\varphi=2k\pi$ *;* in the first case then $cos\,n\varphi=-1$, in the latter $cos\,n\varphi=+1$. Hence if *a* and *b* are positive quantities, in the first case only the condition is had, in which there arises $a=\frac{bf^n}{g^n}$ and hence

$$f=a^{\frac{1}{n}} \text{ and } g=b^{\frac{1}{n}}.$$

But we may retain in place of these irrational quantities the letters *f* and *g* or we may put rather $a=f^n$ and $b=g^n$, thus in order that the factors of this function are to be found :

$$f^n+g^nx^n.$$

Therefore since $\varphi=\frac{(2k\text{-}1)\pi}{n}$, where *k* can be designated to some positive number, but indeed numbers for *k* greater than $\frac{2k\text{-}1}{n}$ are not to be taken, than which return $\frac{2k\text{-}1}{n}$ less than unity ; hence the factors of the proposed fraction $f^n+g^nx^n$ are the following :

$$ff-2fgx\cos\frac{\pi}{n}+ggxx$$
$$ff-2fgx\cos\frac{3\pi}{n}+ggxx$$
$$ff-2fgx\cos\frac{5\pi}{n}+ggxx$$
etc,

where it is to be noted that if n is an odd number, then this one binomial factor is present,
$$f+gx$$
but if n is an even number then no binomial factor is present.

EXAMPLE 1

To resolve this fraction $\frac{x^m}{f^n+g^nx^n}$ *into its simple fractions.*

Since a factor of each trinomial denominator is contained in this form
$$ff-2fgx\cos\frac{(2k-1)\pi}{n}+ggxx,$$
in example 1 in the preceding paragraph, $a=f^n, b=g^n$, and $\varphi=\frac{(2k-1)\pi}{n}$,
from which there shall be
$$\sin(n-m-1)\varphi=\sin(m+1)\varphi=\sin\frac{(m+1)(2k-1)\pi}{n},$$
and
$$\cos(n-m-1)\varphi=-\cos(m+1)\varphi=-\cos\frac{(m+1)(2k-1)\pi}{n}.$$

Hence from this factor this simple fraction arises
$$\frac{2f\sin.\frac{(2k-1)\pi}{n}\sin.\frac{(m+1)(2k-1)\pi}{n}-2\cos.\frac{(m+1)(2k-1)\pi}{n}\left(gx-f\cos.\frac{(2k-1)\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff-2fgx\cos.\frac{(2k-1)\pi}{n}+ggxx\right)}$$

On account of which the proposed fraction can be resolved into these simple parts :
$$\frac{2f\sin.\frac{\pi}{n}\sin.\frac{(m+1)\pi}{n}-2\cos.\frac{(m+1)\pi}{n}\left(gx-f\cos.\frac{\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff-2fgx\cos.\frac{\pi}{n}+ggxx\right)}$$
$$+\frac{2f\sin.\frac{3\pi}{n}\sin.\frac{3(m+1)\pi}{n}-2\cos.\frac{3(m+1)\pi}{n}\left(gx-f\cos.\frac{3\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff-2fgx\cos.\frac{3\pi}{n}+ggxx\right)}$$
$$+\frac{2f\sin.\frac{5\pi}{n}\sin.\frac{5(m+1)\pi}{n}-2\cos.\frac{5(m+1)\pi}{n}\left(gx-f\cos.\frac{5\pi}{n}\right)}{nf^{n-m-1}g^m\left(ff-2fgx\cos.\frac{5\pi}{n}+ggxx\right)}$$
etc.

Therefore if n were an even number, in this way all the simple fractions arise; but if n should be an odd number, on account of the binomial factor $f+gx$ this must be added to the resulting fractions above

$$\frac{\pm 1}{nf^{n-m-1}g^{m}(f+gx)}$$

where the sign + prevails, if m is an even number, and conversely for the $-$ sign. If m should be a number greater than n, then to these fractions the above integral parts of this kind must be added

$$Ax^{m-n}+Bx^{m-2n}+Cx^{m-3n}+Dx^{m-4n}+\text{etc.},$$

as long as the exponents remain positive, and there becomes

$$\begin{aligned}
Ag^{n}&=1 \quad \text{hence} & A&=\frac{1}{g^{n}}\\
Af^{n}+Bg^{n}&=0 & B&=-\frac{f^{n}}{g^{2n}}\\
Bf^{n}+Cg^{n}&=0 & C&=+\frac{f^{2n}}{g^{3n}}\\
Cf^{n}+Dg^{n}&=0 & D&=-\frac{f^{3n}}{g^{4n}}\\
&\text{etc.} & &\text{etc.}
\end{aligned}$$

LIBER PRIOR

PARS PRIMA
SEU
METHODUS INVESTIGANDI FUNCTIONES UNIUS VARIABILIS
EX DATA RELATIONE QUACUNQUE DIFFERENTIALIUM PRIMI GRADUS

SECTIO PRIMA

DE

INTEGRATIONE FORMULARUM

DIFFERENTIALIUM.

CAPUT I

DE INTEGRATIONE FORMULARUM DIFFERENTIALIUM RATIONALIUM

DEFINITIO

40. *Formula differentialis rationalis est, quando variabilis x, cuius functio quaeritur, differentiale dx multiplicatur in functionem rationalem ipsius x, seu si* X *designet functionem rationalem ipsius x, haec formula differentialis Xdx dicitur rationalis.*

COROLLARIUM 1

41. In hoc ergo capite eiusmodi functio ipsius x quaeritur, quae si ponatur y, ut $\frac{dy}{dx}$ aequetur functioni rationali ipsius x, seu posita tali functione = X ut sit $\frac{dy}{dx} = X$.

COROLLARIUM 2

42. Hinc quaeritur eiusmodi functio ipsius x, cuius differentiale sit $= Xdx$; huius ergo integrale, quod ita indicari solet $\int Xdx$, praebebit functionem quaesitam.

COROLLARIUM 3

43. Quodsi P fuerit eiusmodi functio ipsius x, ut eius differentiale dP sit $= Xdx$, quoniam. quantitatis $P + C$ idem est differentiale, formulae propositae Xdx integrale completum est $P + C$.

SCHOLION 1

44. Ad libri promi partem priorem huiusmodi referuntur quaestiones, quibus functiones solius variabilis x ex data differentialium primi gradus relatione quaeruntur. Scilicet si functio quaesita $= y$ et $\frac{dy}{dx} = p$, id praestari oportet, ut proposita aequatione quacunque inter ternas quantitates x, y et p inde indoles functionis y seu aequatio inter x et y, elisa littera p, inveniatur. Quaestio autem sic in genere proposita vires Analyseos adeo superare videtur, ut eius solutio nunquam expectari queat. In casibus igitur simplicioribus vires nostrae sunt exercendae, inter quos primum occurrit casus, quo p functioni cuipiam ipsius x, puta X, aequatur, ut sit $\frac{dy}{dx} = X$ seu $dy = Xdx$ ideoque integrale $y = \int Xdx$ requiratur, in quo primam sectionem collocamus. Verum et hic casus pro varia indole functionis X latissime patet ac plurimis difficultatibus implicatur, unde in hoc capite eiusmodi tantum quaestiones evolvere instituimus, in quibus ista functio X est rationalis, deinceps ad functiones irrationales

atque adeo transcendentes progressuri. Hinc ista pars commode in duas sectiones subdividitur, in quarum altera integratio formularum simplicium, quibus $p = \frac{dy}{dx}$ functioni tantum ipsius x aequatur, est tradenda, in altera autem rationem integrandi doceri conveniet, cum proposita fuerit aequatio quaecunque ipsarum x, y et p. Et cum in his duabus sectionibus ac potissimum priore a Geometris plurimum sit elaboratum, eae fere maximam partem totius operis complebunt.

SCHOLION 2

45. Prima autem integrationis principia ex ipso calculo differentiali sunt petenda, perinde ac principia divisionis ex multiplicatione et principia extractionis radicum ex ratione evectionis ad potestates sumi solent. Cum igitur, si quantitas differentianda ex pluribus partibus constet ut $P + Q - R$, eius differentiale sit $dP + dQ - dR$, ita vicissim, si formula differentialis ex pluribus partibus constet ut $Pdx + Qdx - Rdx$, integrale erit

$$\int Pdx + \int Qdx - \int Rdx$$

singulis scilicet partibus seorsim integrandis. Deinde cum quantitatis aP differentiale sit adP, formulae differentialis $aPdx$ integrale erit $a\int Pdx$ scilicet per quam quantitatem constantem formula differentialis multiplicatur, per eandem integrale multiplicari debet. Ita si formula differentialis sit $aPdx + bQdx + cRdx$, quaecunque functiones ipsius x litteris P, Q, R designentur, integrale erit

$$a\int Pdx + b\int Qdx + c\int Rdx,$$

ita ut integratio tantum in singulis formulis Pdx, Qdx et Rdx sit instituenda, hocque facto insuper adiici debet constans arbitraria C, ut integrale completum obtineatur.

PROBLEMA 1

46. *Invenire functionem ipsius x, ut eius differentiale sit* $= ax^n dx$, *seu integrare formulam differentialem* $ax^n dx$.

SOLUTIO

Cum potestatis x^m differentiale sit $mx^{m-1}dx$, erit vicissim

$$\int mx^{m-1}dx = m\int x^{m-1}dx = x^m$$

ideoque

$$\int x^{m-1}dx = \frac{1}{m}x^m;$$

fiat $m - 1 = n$ seu $m = n + 1$; erit

$$\int x^n dx = \frac{1}{n+1} x^{n+1} \quad \text{et} \quad a\int x^n dx = \frac{a}{n+1} x^{n+1}.$$

Unde formulae differentialis propositae $ax^n dx$ integrale completum erit

$$\frac{a}{n+1} x^{n+1} + C,$$

cuius ratio vel inde patet, quod eius differentiaIe revera fit $= ax^n dx$. Atque haec integratio semper locum habet, quicunque numerus exponenti n tribuatur, sive positivus sive negativus, sive integer sive fractus, sive etiam irrationalis. Unicus casus hinc excipitur, quo est exponens $n = -1$ seu haec formula integranda $\frac{adx}{x}$ proponitur. Verum in calculo differentiali iam ostendimus, si lx denotet logarithmum hyperbolicum ipsius x, fore eius differentiale $= \frac{dx}{x}$, unde vicissim concludimus esse

$$\int \frac{dx}{x} = lx \quad \text{et} \quad \int \frac{adx}{x} = alx.$$

Quare adiecta constante arbitraria erit formulae $\frac{adx}{x}$ integrale completum

$$= alx + C = lx^a + C,$$

quod etiam pro C ponendo $1c$ ita exprimitur: lcx^a.

COROLLARIUM 1

47. Formulae ergo differentialis $ax^n dx$ integrale semper est algebraicum solo excepto casu, quo $n = -1$ et integrale per logarithmos exprimitur, qui ad functiones transcendentes sunt referendi. Est scilicet

$$\int \frac{adx}{x} = alx + C = lcx^n.$$

COROLLARIUM 2

48. Si exponens n numeros positivos denotet, sequentes integrationes utpote maxime obviae probe sunt tenendae

$$\int adx = ax + C, \quad \int axdx = \frac{a}{2} xx + C, \quad \int ax^2 dx = \frac{a}{3} x^3 + C,$$
$$\int ax^3 dx = \frac{a}{4} x^4 + C, \quad \int ax^4 dx = \frac{a}{5} x^5 + C, \quad \int ax^5 dx = \frac{a}{6} x^6 + C.$$

COROLLARIUM 3

49. Si n sit numerus negativus, posito $n = -m$ fit

$$\int \frac{adx}{x^m} = \frac{a}{1-m} x^{1-m} + C = \frac{-a}{(m-l)x^{m-1}} + C,$$

unde hi casus simpliciores notentur

$$\int \frac{adx}{x^2} = \frac{-a}{x} + C, \quad \int \frac{adx}{x^3} = \frac{-a}{2xx} + C, \quad \int \frac{adx}{x^4} = \frac{-a}{3x^3} + C,$$
$$\int \frac{adx}{x^5} = \frac{-a}{4x^4} + C, \quad \int \frac{adx}{x^6} = \frac{-a}{5x^5} + C, \quad \text{etc.}$$

COROLLARIUM 4

50. Quin etiam si *n* denotet numeros fractos, integralia hinc obtinentur.
Sit primo $n = \frac{m}{3}$; erit

$$\int adx\sqrt{x^m} = \frac{2a}{m+2} x\sqrt{x^m} + C,$$

unde casus notentur

$$\int adx\sqrt{x} = \frac{2a}{3} x\sqrt{x} + C, \quad \int axdx\sqrt{x} = \frac{2a}{5} x^2\sqrt{x} + C,$$
$$\int axxdx\sqrt{x} = \frac{2a}{7} x^3\sqrt{x} + C, \quad \int ax^3dx\sqrt{x} = \frac{2a}{9} x^4\sqrt{x} + C.$$

COROLLARIUM 5

51. Ponatur etiam $n = \frac{-m}{2}$ et habebitur

$$\int \frac{adx}{\sqrt{x^m}} = \frac{2a}{2-m\sqrt{x^m}} + C = \frac{-2a}{(m-2)\sqrt{x^{m-2}}} + C,$$

unde hi casus notentur

$$\int \frac{adx}{\sqrt{x}} = 2a\sqrt{x} + C, \quad \int \frac{adx}{x\sqrt{x}} = \frac{-2a}{\sqrt{x}} + C, \quad \int \frac{adx}{xx\sqrt{x}} = \frac{-2a}{3x\sqrt{x}} + C,$$
$$\int \frac{adx}{x^3\sqrt{x}} = \frac{-2a}{5x^2\sqrt{x}} + C.$$

COROLLARIUM 6

52. Si in genere ponamus $n = \frac{\mu}{\nu}$, fiet

$$\int ax^{\frac{\mu}{\nu}} dx = \frac{\nu a}{\mu+\nu} x^{\frac{\mu+\nu}{\nu}} + C$$

seu per radicalia

$$\int adx\sqrt[\nu]{x^\mu} = \frac{\nu a}{\mu+\nu} \sqrt[\nu]{x^{\mu+\nu}} + C;$$

sin autem ponatur $n = \frac{-\mu}{\nu}$ habebitur

$$\int \frac{adx}{x^{\frac{\mu}{\nu}}} = \frac{\nu a}{\nu-\mu} x^{\frac{\nu-\mu}{\nu}} + C$$

seu per radicalia

$$\int \frac{adx}{\sqrt[\nu]{x^{\mu}}} = \frac{\nu a}{\nu-\mu} \sqrt[\nu]{x^{\nu-\mu}} + C\ .$$

SCHOLION 1

53. Quanquam in hoc capite functiones tantum rationales tractare institueram, tamen istae irrationalitates tam sponte se obtulerunt, ut perinde ac rationales tractari possint. Caeterum hinc quoque formulae magis complicatae integrari possunt, si pro x functiones alius cuiuspiam variabilis z statuantur. Veluti si ponamus $x = f + gz$, erit $dx = gdz;$ quare si pro a scribamus $\frac{a}{g}$, habebitur

$$\int adz\left(f+gz\right)^{n} = \frac{a}{(n+1)g}\left(f+gz\right)^{n+1} + C\ ,$$

casu autem singulari, quo $n = -1$,

$$\int \frac{adz}{(f+gz)} = \frac{a}{g} l\left(f+gz\right) + C\ .$$

Tum si sit $n = -m$, fiet

$$\int \frac{adz}{(f+gz)^{m}} = \frac{-a}{(m-1)g(f+gz)^{m-1}} + C\ .$$

At posito $n = \frac{\mu}{\nu}$ prodit

$$\int adz\left(f+gz\right)^{\frac{\mu}{\nu}} = \frac{\nu a}{(\nu+\mu)g}\left(f+gz\right)^{\frac{\mu}{\nu}+1} + C\ ;$$

posito autem $n = -\frac{\mu}{\nu}$ obtinetur

$$\int \frac{adz}{(f+gz)^{\frac{\mu}{\nu}}} = \frac{\nu a(f+gz)}{(\nu-\mu)g(f+gz)^{\frac{\mu}{\nu}}} + C\ .$$

SCHOLION 2

54. Caeterum hic insignis proprietas annotari meretur. Cum hic quaeratur functio y, ut sit $dy = ax^{n}dx$, si ponamus $\frac{dy}{dx} = p$, haec habebitur relatio $p = ax^{n}$, ex qua functio y investigari debet. Quoniam igitur est

$$y = \frac{a}{n+1} x^{n+1} + C\ ,$$

ob $ax^n = p$ erit quoque

$$y = \frac{px}{n+1} + C$$

sicque casum habemus, ubi relatio differentialium per aequationem quandam inter *x*, *y* et *p* proponitur cuique iam novimus satisfieri per aequationem $y = \frac{a}{n+1}x^{n+1} + C$. Verum haec non ampius erit integrale completum pro relatione in aequatione $y = \frac{px}{n+1} + C$ contenta, sed tantum particulare, quoniam integrale non involvit novam constantem, quae in relatione differentiali non insit. Integrale autem completum est

$$y = \frac{aD}{n+1}x^{n+1} + C$$

novam constantem *D* involvens; hinc fit

$$\frac{dy}{dx} = aDx^n = p \quad \text{ideoque} \quad y = \frac{px}{n+1} + C.$$

Etsi hoc non ad praesens institutum pertinet, tamen notasse iuvabit.

PROBLEMA 2

55. Invenire functionem ipsius x, cuius differentiale sit = Xdx denotante X functionem quamcunque rationalem integram ipsius x, seu definire integrale $\int Xdx$.

SOLUTIO

Cum *X* sit functio rationalis integra ipsius *x*, in hac forma contineatur necesse est

$$X = \alpha + \beta x + \gamma x^2 + \delta x^3 + \varepsilon x^4 + \text{etc.},$$

unde per problema praecedens integrale quaesitum est

$$\int Xdx = C + \alpha x + \tfrac{1}{2}\beta x^2 + \tfrac{1}{3}\gamma x^3 + \tfrac{1}{4}\delta x^4 + \tfrac{1}{5}\varepsilon x^5 + \tfrac{1}{5}\zeta x^6 + \text{etc.},$$

Atque in genere si sit

$$X = \alpha x^{\lambda} + \beta x^{\mu} + \gamma x^{\nu} + \text{etc.},$$

erit

$$\int Xdx = C + \frac{\alpha}{\lambda+1}x^{\lambda+1} + \frac{\beta}{\beta+1}x^{\mu+1} + \frac{\gamma}{\gamma+1}x^{\nu+1} + \text{etc.},$$

ubi exponentes α, β, ν etc. etiam numeros tam negativos quam fractos significare possunt, dummodo notetur, si fuerit $\lambda = -1$, fore $\int \frac{\alpha dx}{x} = \alpha lx$, qui est unicus casus ad ordinem transcendentium referendus.

PROBLEMA 3

56. *Si* X *denotet functionem quamcunque rationalem fractam ipsius x, methodum describere, cuius ope formulae Xdx integrale investigari conveniat.*

SOLUTIO

Sit igitur $X=\frac{M}{N}$, ita ut *M* et *N* futurae sint functiones integrae ipsius *x*, ac primo dispiciatur, num summa potestas ipsius *x* in numeratore *M* tanta sit vel etiam maior quam in denominatore *N*, quo casu ex fractione $\frac{M}{N}$ partes integrae per divisionem eliciantur; quarum integratio cum nihil habeat difficultatis, totum negotium reducitur ad eiusmodi fractionem $\frac{M}{N}$, in cuius numeratore *M* summa potestas ipsius *x* minor sit quam in denominatore *N*.

Tum quaerantur omnes factores ipsius denominatoris *N*, tam simplices, si fuerint reales, quam duplices reales, vicem scilicet binorum simplicium imaginariorum gerentes; simulque videndum est, utrum hi factores omnes sint inaequales necne; pro factorum enim aequalitate alio modo resolutio fractionis $\frac{M}{N}$ in fractiones simplices est instituenda, quandoquidem ex singulis factoribus fractiones partiales nascuntur, quarum aggregatum fractioni propositae $\frac{M}{N}$ aequatur. Scilicet ex factore simplici $a+bx$ nascitur fractio

$$\frac{A}{a+bx};$$

si bini sint aequales seu denominator *N* factorem habeat $(a+bx)^2$, hinc nascuntur fractiones

$$\frac{\mathrm{A}}{(a+bx)^2}+\frac{B}{a+bx};$$

ex huiusmodi autem factore $(a+bx)^3$ hae tres fractiones

$$\frac{\mathrm{A}}{(a+bx)^3}+\frac{\mathrm{B}}{(a+bx)^2}+\frac{C}{a+bx}:$$

et ita porro.

Factor autem duplex, cuius forma est $aa-2abx\cos.\zeta+bbxx$, nisi alius ipsi fuerit aequalis, dabit fractionem partialem

$$\frac{A+Bx}{aa-2abx\cos.\zeta+bbxx};$$

si autem denominator *N* duos huiusmodi factores aequales involvat, inde nascuntur binae huiusmodi fractiones partiales

$$\frac{A+Bx}{\left(aa-2abx\cos.\zeta+bbxx\right)^2}+\frac{C+Dx}{aa-2abx\cos.\zeta+bbxx}$$

at si cubus adeo $\left(aa-2abx\cos.\zeta+bbxx\right)^3$ fuerit factor denominatoris *N*, ex eo oriuntur huiusmodi tres fractiones partiales

$$\frac{A+Bx}{\left(aa-2abx\cos.\zeta+bbxx\right)^3}+\frac{C+Dx}{\left(aa-2abx\cos.\zeta+bbxx\right)^2}+\frac{E+Fx}{aa-2abx\cos.\zeta+bbxx};$$

et ita porro.
Cum igitur hoc modo fractio proposita $\frac{M}{N}$ in omnes suas fractiones simplices fuerit resoluta, omnes continebuntur in alterutra harum formarum vel

$$\frac{A}{(a+bx)^n}\ \text{vel}\ \frac{A+Bx}{\left(aa-2abx\cos.\zeta+bbxx\right)^n}$$

ac singulos iam per *dx* multiplicatos integrari oportet; erit omnium horum integralium aggregatum valor functionis quaesitae $\int Xdx=\int\frac{Mdx}{N}$.

COROLLARIUM 1

57. Pro integratione ergo omnium huiusmodi formularum $\frac{Mdx}{N}$ totum negotium reducitur ad integrationem huiusmodi binarum formularum

$$\int\frac{Adx}{(a+bx)^n}\ \text{et}\ \int\frac{(A+Bx)dx}{\left(aa-2abx\cos.\zeta+bbxx\right)^n},$$

dum pro *n* successive scribuntur numeri 1, 2, 3, 4 etc.

COROLLARIUM 2

58. Ac prioris quidem formae integrale iam supra (§ 53) est expeditum, unde patet fore

$$\int\frac{Adx}{a+bx}=\frac{A}{b}l(a+bx)+\text{Const.}$$
$$\int\frac{Adx}{(a+bx)^2}=\frac{-A}{b(a+bx)}+\text{Const.}$$
$$\int\frac{Adx}{(a+bx)^3}=\frac{-A}{2b(a+bx)^2}+\text{Const.}$$

et generatim

$$\int\frac{Adx}{(a+bx)^n}=\frac{-A}{(n-1)b(a+bx)^{n-1}}+\text{Const.}$$

COROLLARIUM 3

59. Ad propositum ergo absolvendum nihil aliud superest, nisi ut integratio huius formulae

$$\int \frac{(A+Bx)dx}{(aa-2abx\cos.\zeta+bbxx)^n}$$

doceatur, primo quidem casu $n = 1$, tum vero casibus $n = 2$, $n = 3$, $n = 4$ etc.

SCHOLION 1

60. Nisi vellemus imaginaria evitare, totum negotium ex iam traditis confici posset; denominatore enim N in omnes suos factores simplices resoluto, sive sint reales sive imaginarii, fractio proposita semper resolvi poterit in fractiones partiales huius formae $\frac{Adx}{a+bx}$ vel huius $\frac{Adx}{(a+bx)^n}$; quarum integralia cum sint in promtu, totius formae $\frac{M}{N}dx$ integrale habetur. Tum autem non parum molestum foret binas partes imaginarias ita coniungere, ut expressio realis resultaret, quod tamen rei natura absolute exigit.

SCHOLION 2

61. Hic utique postulamus resolutionem cuiusque functionis integrae in factores nobis concedi, etiamsi algebra neutiquam adhuc eo sit perducta, ut haec resolutio actu institui possit. Hoc autem in Analysi ubique postulari solet, ut, quo longius progrediamur, ea, quae retro sunt relicta, etiamsi non satis fuerint explorata, tanquam cognita assumamus; sufficere scilicet hic potest omnes factores per methodum approximationum quantumvis prope assignari posse. Simili modo cum in calculo integrali longius processerimus, integralia omnium huiusmodi formularum Xdx, quaecunque functio ipsius x littera X significetur, tanquam cognita spectabimus plurimumque nobis praestinisse videbimur, si integralia magis abscondita ad eas formas reducere valuerimus; atque hoc etiam in usu practico nihil turbat, cum valores talium formularum $\int Xdx$ quantumvis prope assignare liceat, uti in sequentibus ostendemus. Caeterum ad has integrationes resolutio denominatoris N in suos factores absolute est necessaria, propterea quod singuli hi factores in expressionem integralis ingrediuntur; paucissimi sunt casus iique maxime obvii, quibus ista resolutione carere possumus; veluti si proponatur haec formula $\frac{x^{n-1}dx}{1+x^n}$ statim patet posito $x^n = v$ eam abire in $\frac{dv}{n(1+v)}$, cuius integrale est $\frac{1}{n}l(1+v) = \frac{1}{n}l\left(1+x^n\right)$, ubi resolutione in factores non fuerat opus. Verum huiusmodi casus per se tam sunt perspicui, ut eorum tractatio nulla peculiari explicatione indigeat.

PROBLEMA 4

62. *Invenire integrale huius formulae*

$$y = \int \frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$

SOLUTIO

Cum numerator duabus constet partibus $Adx+Bxdx$, haec posterior $B\,xdx$ sequenti modi tolli poterit. Cum sit

$$l\left(aa-2abx\cos.\zeta+bbxx\right)=\int\frac{-2abdx\cos.\zeta+2bbxdx}{aa-2abx\cos.\zeta+bbxx}$$

multiplicetur haec aequatio per $\frac{B}{2bb}$ et a proposita auferatur; sic enim prodibit

$$y-\frac{B}{2bb}l\left(aa-2abx\cos.\zeta+bbxx\right)=\int\frac{\left(A+\frac{Ba\cos.\zeta}{b}\right)dx}{aa-2abx\cos.\zeta+bbxx}$$

ita ut haec tantum formula integranda supersit. Ponatur brevitatis gratia formula

$$A+\frac{Ba\cos.\zeta}{b}=C$$

ut habeatur haec

$$\int\frac{Cdx}{aa-2abx\cos.\zeta+bbxx}$$

quae ita exhiberi potest

$$\int\frac{Cdx}{aa\sin.^2\zeta+\left(bx-a\cos.\zeta\right)^2}.$$

Statuatur $bx-a\cos.\zeta=av\sin.\zeta$ hincque $dx=\frac{adv\sin.\zeta}{b}$ unde formula nostra erit

$$\int\frac{Cadv\sin.\zeta:b}{aa\sin.^2\zeta\left(1+vv\right)}=\frac{C}{ab\sin.\zeta}\int\frac{dv}{\left(1+vv\right)}.$$

Ex calculo autem differentiali novimus esse

$$\int\frac{dv}{\left(1+vv\right)}=\text{Arc.tang.}v=\text{Arc.tang.}\frac{bx-a\cos.\zeta}{a\sin.\zeta},$$

unde ob

$$C=\frac{Ab+Ba\cos.\zeta}{b}$$

erit nostrum integrale

$$\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc.tang.}\frac{bx-a\cos.\zeta}{a\sin.\zeta}$$

Quocirca formulae propositae

$$\int \frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$

integrale est

$$\frac{B}{2bb}l\left(aa-2abx\cos.\zeta+bbxx\right)+\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc.tang.}\frac{bx-a\cos.\zeta}{a\sin.\zeta},$$

quod ut fiat completum, constans arbitraria *C* insuper addatur.

COROLLARIUM 1

63. Si ad $\text{Arc.tang.}\frac{bx-a\cos.\zeta}{a\sin.\zeta}$ addamus $\text{Arc.tang.}\frac{\cos.\zeta}{\sin.\zeta}$, quippe qui in constante addenda contentus concipiatur, prodibit $\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}$ sicque habebimus

$$\int \frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$
$$=\frac{B}{2bb}l\left(aa-2abx\cos.\zeta+bbxx\right)+\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}$$

adiecta constante *C.*

COROLLARIUM 2

64. Si velimus, ut integrale hoc evanescat posito $x=0$, constans *C* sumi debet $=\frac{-B}{2bb}laa$ sicque fiet

$$\int \frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$
$$=\frac{B}{bb}l\frac{\sqrt{(aa-2abx\cos.\zeta+bbxx)}}{a}+\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}$$

Pendet ergo hoc integrale partim a logarithmis, partim ab arcubus circularibus seu angulis.

COROLLARIUM 3

65. Si littera *B* evanescat, pars a logarithmis pendens evanescit fitque

$$\int \frac{Adx}{aa-2abx\cos.\zeta+bbxx}=\frac{A}{ab\sin.\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}+C$$

sicque per solum angulum definitur.

COROLLARIUM 4

66. Si angulus ζ sit rectus ideoque $cos.\zeta = 0$ et $sin.\zeta = 1$, habebitur

$$\int\frac{(A+Bx)dx}{aa+bbxx} = \frac{B}{bb}l\frac{\sqrt{(aa+bbxx)}}{a} + \frac{A}{ab}\text{Arc.tang.}\frac{bx}{a} + C\,;$$

si angulus ζ sit 60° ideoque $\cos.\zeta = \frac{1}{2}$ et $\sin.\zeta = \frac{\sqrt{3}}{2}$, erit

$$\int\frac{(A+Bx)dx}{aa-abx+bbxx} = \frac{B}{bb}l\frac{\sqrt{(aa-abx+bbxx)}}{a} + \frac{2Ab+Ba}{abb\sqrt{3}}\text{Arc.tang.}\frac{bx\sqrt{3}}{2a-bx}$$

At si $\zeta = 120^{o}$ ideoque $\cos.\zeta = -\frac{1}{2}$ et $\sin.\zeta = \frac{\sqrt{3}}{2}$, erit

$$\int\frac{(A+Bx)dx}{aa+abx+bbxx} = \frac{B}{bb}l\frac{\sqrt{(aa+abx+bbxx)}}{a} + \frac{2Ab-Ba}{abb\sqrt{3}}\text{Arc.tang.}\frac{bx\sqrt{3}}{2a+bx}$$

SCHOLION 1

67. Omnino hic notatu dignum evenit, quod casu $\zeta = 0$, quo denominator $aa-2abx+bbxx$ fit quadratum, ratio anguli ex integrali discedat. Posito enim angulo ζ infinite parvo erit $\cos.\zeta = 1$ et $\sin.\zeta = \zeta$, unde pars logarithmica fit $\frac{B}{bb}l\frac{a-bx}{a}$ et altera pars $\frac{Ab+Ba}{abb\zeta}\text{Arc.tang.}\frac{bx\zeta}{a-bx} = \frac{(Ab+Ba)x}{ab(a-bx)}$, quia arcus infinite parvi $\frac{bx\zeta}{a-bx}$ tangens ipsi est aequalis, sicque haec pars fit algebraica. Quocirca erit

$$\int\frac{(A+Bx)dx}{(a-bx)^2} = \frac{B}{bb}l\frac{a-bx}{a} + \frac{(Ab+Ba)x}{ab(a-bx)} + \text{Const.},$$

cuius veritas ex praecedentibus est manifesta; est enim

$$\frac{(A+Bx)}{(a-bx)^2} = \frac{-B}{b(a-bx)} + \frac{Ab+Ba}{b(a-bx)^2}$$

Iam vero est

$$\int\frac{-Bdx}{b(a-bx)} = \frac{B}{bb}l(a-bx) - \frac{B}{bb}la = \frac{B}{bb}l\frac{a-bx}{a},$$

$$\int\frac{(Ab+Ba)dx}{b(a-bx)^2} = \frac{Ab+Ba}{bb(a-bx)} - \frac{Ab+Ba}{abb} = \frac{(Ab+Ba)x}{ab(a-bx)},$$

siquidem utraque integratio ita determinetur, ut casu $x = 0$ integralia evanescant.

SCHOLION 2

68. Simili modo, quo hic usi sumus, si in formula differentiali fracta $\frac{Mdx}{N}$ summa potestas ipsius x in numeratore M uno gradu minor sit quam in denominatore N, etiam is terminus tolli poterit. Sit enim

$$M = Ax^{n-1} + Bx^{n-2} + Cx^{n-3} + \text{etc.}$$

et

$$N = \alpha x^{n} + \beta x^{n-1} + \gamma x^{n-2} + \text{etc.}$$

ac ponatur $\frac{Mdx}{N} = dy$ Cum iam sit

$$dN = n\alpha x^{n-1}dx + (n-1)dx\beta x^{n-2} + (n-2)dx\gamma x^{n-3} + \text{etc.}$$

erit

$$\frac{AdN}{n\alpha N} = \frac{dx}{N}\left(Ax^{n-1} + \frac{(n-1)A\beta}{n\alpha}x^{n-2} + \frac{(n-2)A\gamma}{n\alpha}x^{n-3} + \text{etc.}\right),$$

quo valore inde subtracto remanebit

$$dy - \frac{AdN}{n\alpha N} = \frac{dx}{N}\left(\left(B - \frac{(n-1)A\beta}{n\alpha}\right)x^{n-2} + \left(C - \frac{(n-2)A\gamma}{n\alpha}\right)x^{n-3} + \text{etc.}\right)$$

Quare si brevitatis gratia ponatur

$$B - \frac{(n-1)A\beta}{n\alpha} = \mathfrak{B}, \quad C - \frac{(n-2)A\gamma}{n\alpha} = \mathfrak{C}, \quad D - \frac{(n-3)A\delta}{n\alpha} = \mathfrak{D}, \quad \text{etc.},$$

obtinebitur

$$y = \frac{A}{n\alpha}lN + \int \frac{dx\left(\mathfrak{B}x^{n-2} + \mathfrak{C}x^{n-3} + \mathfrak{D}x^{n-4} + \text{etc.}\right)}{\alpha x^{n} + \beta x^{n-1} + \gamma x^{n-2} + \delta x^{n-3}\text{etc.}} = \int \frac{Mdx}{N}.$$

Hoc igitur modo omnes formulae differentiales fractae eo reduci possunt, ut summa potestas ipsius x in numeratore duobus pluribusve gradibus minor sit quam in denominatore.

PROBLEMA 5

69. Formulam integralem

$$y = \int \frac{(A+Bx)dx}{(aa - 2abx\cos.\zeta + bbxx)^{n+1}}$$

ad aliam similem reducere, ubi potestas denominatoris sit unogradu inferior.

SOLUTIO

Sit brevitatis gratia $aa-2abx\cos.\zeta+bbxx=X$ ac ponatur

$$\int\frac{(A+Bx)dx}{X^{n+1}}=y\,.$$

Cum ob

$$dX=-2abdx\cos.\zeta+2bbdxx$$

sit

$$d.\frac{C+Dx}{X^{n}}=\frac{-n(C+Dx)dX}{X^{n+1}}+\frac{Ddx}{X^{n}}$$

ideoque

$$\frac{C+Dx}{X^{n}}=\int\frac{2nb(C+Dx)(a\cos.\zeta-bx)dx}{X^{n+1}}+\int\frac{Ddx}{X^{n}},$$

habebimus

$$y+\frac{C+Dx}{X^{n}}=\int\frac{dx\big(A+2nCab\cos.\zeta+x(B+2nDab\cos.\zeta-2nCbb)-2nDbbxx\big)}{X^{n+1}}+\int\frac{Ddx}{X^{n}}.$$

Iam in formula priori litterae C et D ita definiantur, ut numerator per X fiat divisibilis; oportet ergo sit $=-2nDXdx$, unde nanciscimur

$$A+2nCab\cos.\zeta=-2nDaa$$

et

$$B+2nDab\cos.\zeta-2nCbb=4nDab\cos.\zeta$$

seu $B-2nCbb=2nDab\cos.\zeta$ hincque

$$2nDa=\frac{B-2nCbb}{b\cos.\zeta};$$

at ex priori conditione est

$$2nDa=\frac{-A-2nCab\cos.\zeta}{a}$$

quibus aequatis fit

$$Ba+Ab\cos.\zeta-2nCabb\sin^{2}.\zeta=0$$

seu

$$C=\frac{Ba+Ab\cos.\zeta}{2nabb\sin^{2}.\zeta},$$

unde

$$B-2nCbb=\frac{Ba\sin.^{2}\zeta-Ba-Ab\cos.\zeta}{a\sin.^{2}\zeta}=\frac{-Ab\cos.\zeta-Ba\cos.^{2}\zeta}{a\sin.^{2}\zeta},$$

ita ut reperiatur

$$D=\frac{-Ab-Ba\cos.\zeta}{2naab\sin.^{2}\zeta}.$$

Sumtis ergo litteris

$$C=\frac{Ba+Ab\cos.\zeta}{2nabb\sin^{2}.\zeta}\quad\text{et}\quad D=\frac{-Ab-Ba\cos.\zeta}{2naab\sin^{2}.\zeta}$$

erit

$$y+\frac{C+Dx}{X^{n}}=\int\frac{-2nDdx}{X^{n}}+\int\frac{Ddx}{X^{n}}=-(2n-1)D\int\frac{dx}{X^{n}}$$

ideoque

$$\int\frac{(A+Bx)dx}{X^{n+1}}=\frac{-C-Dx}{X^{n}}-(2n-1)D\int\frac{dx}{X^{n}},$$

sive

$$\int\frac{(A+Bx)dx}{X^{n+1}}=\frac{-Baa-Aab\cos.\zeta+(Abb+Bab\cos.\zeta)x}{2naabb\sin.^{2}\zeta X^{n}}$$
$$+\frac{(2n-1)(Ab+Ba\cos.\zeta)}{2naab\sin.^{2}\zeta}\int\frac{dx}{X^{n}}.$$

Quare, si formula $\int\frac{dx}{X^{n}}$ constet, etiam integrale hoc $\int\frac{(A+Bx)dx}{X^{n+1}}$ assignari poterit.

COROLLARIUM 1

70. Cum igitur manente $X=aa-2abx\cos.\zeta+bbxx$ sit

$$\int\frac{dx}{X}=\frac{1}{ab\sin.\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}+\text{Const.},$$

erit

$$\int\frac{(A+Bx)dx}{X^{2}}=\frac{-Baa-Aab\cos.\zeta+(Abb+Bab\cos.\zeta)x}{2aabb\sin.^{2}\zeta X}$$
$$+\frac{Ab+Ba\cos.\zeta}{2na^{3}bb\sin.^{3}\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}+\text{Const.}$$

Ideoque posito $B = 0$ et $A = 1$ fiet

$$\int\frac{dx}{X^{2}}=\frac{a\cos.\zeta+bx}{2aab\sin.^{2}\zeta X}+\frac{1}{2a^{3}b\sin.^{3}\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}+\text{Const.}$$

Integrale ergo $\int\frac{(A+Bx)dx}{X^{2}}$ logarithmos non involvit.

COROLLARIUM 2

71. Hinc ergo cum sit

$$\int\frac{dx}{X^{3}}=\frac{a\cos.\zeta+bx}{4aab\sin.^{2}\zeta X^{2}}+\frac{3}{4aa\sin.^{2}\zeta}\int\frac{dx}{X^{2}}+\text{Const.},$$

erit ilIum valorem substituendo

$$\int \frac{dx}{X^3} = \frac{-a\cos.\zeta + bx}{4aab\sin.^2\zeta X^2} + \frac{3(-a\cos.\zeta + bx)}{2\cdot 4a^4 b\sin.^4\zeta X} + \frac{1\cdot 3}{2\cdot 4a^5 b\sin.^5\zeta} \text{Arc.tang.} \frac{bx\sin.\zeta}{a - bx\cos.\zeta}$$

hincque porro concluditur

$$\int \frac{dx}{X^4} = \frac{-a\cos.\zeta + bx}{6aab\sin.^2\zeta X^3} + \frac{5(-a\cos.\zeta + bx)}{4\cdot 6a^4 b\sin.^4\zeta X^2} + \frac{3(-a\cos.\zeta + bx)}{2\cdot 4\cdot 6a^6 b\sin.^6\zeta X}$$
$$+ \frac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6a^7 b\sin.^7\zeta} \text{Arc.tang.} \frac{bx\sin.\zeta}{a - bx\cos.\zeta}$$

COROLLARIUM 3

72. Sic ulterius progrediendo omnium huiusmodi formularum integralia obtinebuntur

$$\int \frac{dx}{X}, \quad \int \frac{dx}{X^2}, \quad \int \frac{dx}{X^3}, \quad \int \frac{dx}{X^4} \quad \text{etc.},$$

quorum primum arcu circulari solo exprimitur, reliqua vero praeterea partes algebraicas continent.

SCHOLION

73. Sufficit autem integralia $\int \frac{dx}{X^{n+1}}$ nosse, quia formula $\int \frac{(A+Bx)dx}{X^{n+1}}$ facile eo reducitur; ita enim repraesentari potest

$$\frac{1}{2bb} \int \frac{2Abbdx + 2Bbbxdx - 2Babdx\cos.\zeta + 2Babdx\cos.\zeta}{X^{n+1}}$$

quae ob $2bbxdx - 2abdx\cos.\zeta = dX$ abit in hanc

$$\frac{1}{2bb} \int \frac{BdX}{X^{n+1}} + \frac{1}{2b} \int \frac{(Ab+Ba)\cos.\zeta\, dx}{X^{n+1}}.$$

At

$$\int \frac{dX}{X^{n+1}} = -\frac{1}{nX^n},$$

unde habebitur

$$\int \frac{(A+Bx)dx}{X^{n+1}} = \frac{-B}{2nbbX^n} + \frac{Ab+Ba\cos.\zeta}{b} \int \frac{dx}{X^{n+1}},$$

unde tantum opus est nosse integralia $\int \frac{dX}{X^{n+1}}$, quae modo exhibuimus. Atque haec sunt omnia subsidia, quibus indigemus ad omnes formulas fractas $\frac{M}{N} dx$ integrandas, Dummodo *M* et *N* sint functiones integrae ipsius *x*. Quocirca in genere integratio omnium huiusmodi formularum $\int Vdx$, ubi *V* est functio rationalis ipsius *x* quaecunque, est in potestate; de quibus notandum est, nisi integralia fuerint algebraica, semper vel per logarithmos vel angulos exhibeti posse. Nihil aliud igitur superest, nisi ut hanc methodum aliquot exemplis illustremus.

EXEMPLUM 1

74. *Proposita formula differentiali* $\frac{(A+Bx)dx}{\alpha+\beta x+\gamma xx}$ *definire eius integrale.*

Cum in numeratore variabilis x pauciores habeat dimensiones quam in denominatore, haec fractio nullas partes integras complectitur. Hinc denominatoris indoles perpendatur, utrum habeat duos factores simplices reales necne, ac priori casu, num factores sint aequales; ex quo tres habebimus casus evolvendos.

1. Habeat denominator ambos factores aequales sitque $=(a+bx)^2$ et fractio $\frac{(A+Bx)dx}{(a+bx)^2}$ resolvitur in has duas

$$\frac{Ab-Ba}{b(a+bx)^2}+\frac{B}{b(a+bx)},$$

unde fit

$$\int\frac{(A+Bx)dx}{(a+bx)^2}=\frac{Ba-Ab}{bb(a+bx)}+\frac{B}{bb}l(a+bx)+\text{Const.};$$

si integrale ita determinetur, ut evanescat posito $x = 0$, reperitur

$$\int\frac{(A+Bx)dx}{(a+bx)^2}=\frac{(Ab-Ba)x}{ab(a+bx)}+\frac{B}{bb}l\,\frac{a+bx}{a}.$$

II. Habeat denominator duos factores inaequales sitque proposita haec formula $\frac{A+Bx}{(a+bx)(f+gx)}dx$ et haec fractio resolvitur in has partiales

$$\frac{Ab-Ba}{bf-ag}\cdot\frac{dx}{a+bx}+\frac{Ag-Bf}{ag-bf}\cdot\frac{dx}{f+gx},$$

unde obtinetur integrale quaesitum

$$\int\frac{(A+Bx)dx}{(a+bx)(f+gx)}=\frac{Ab-Ba}{b(bf-ag)}l\,\frac{a+bx}{a}+\frac{Ag-Bf}{g(ag-bf)}l\,\frac{f+gx}{f}+\text{Const.}$$

Ponatur

$$\frac{Ab-Ba}{b(bf-ag)}=m+n \quad \text{et} \quad \frac{Ag-Bf}{g(ag-bf)}=m-n,$$

ut integrale fiat

$$ml\,\frac{(a+bx)(f+gx)}{af}+nl\,\frac{f(a+bx)}{a(f+gx)}$$

erit

$$2m=\frac{B(bf-ag)}{bg(bf-ag)}=\frac{B}{bg} \quad \text{et} \quad 2n=\frac{2Abg-Bag-Bbf}{bg(bf-ag)};$$

erit ergo

$$\int\frac{(A+Bx)dx}{(a+bx)(f+gx)}=\frac{B}{2bg}l\frac{(a+bx)(f+gx)}{af}+\frac{2Abg-B(ag+bf)}{2bg(bf\text{ -}ag)}l\frac{f(a+bx)}{a(f+gx)}.$$

III. Sint denominatoris factores simplices ambo imaginarii, quo casu formam habebit $aa-2abx\cos.\zeta+bbxx$; qui casus cum supra [§ 64] iam sit tractatus, erit

$$\int\frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$
$$=\frac{B}{bb}l\frac{\sqrt{(aa-2abx\cos.\zeta+bbxx)}}{a}+\frac{Ab+Ba\cos.\zeta}{abb\sin.\zeta}\text{Arc.tang.}\frac{bx\sin.\zeta}{a-bx\cos.\zeta}$$

COROLLARIUM 1

75. Casu secundo, quo $f=a$ et $g=-b$, erit

$$\int\frac{(A+Bx)dx}{aa-bbxx}=\frac{-B}{2bb}l\frac{aa-bbxx}{aa}+\frac{A}{2ab}l\frac{a+bx}{a-bx}$$

hinc seorsim sequitur

$$\int\frac{Adx}{aa-bbxx}=\frac{A}{2ab}l\frac{a+bx}{a-bx}+C$$

et

$$\int\frac{Bxdx}{aa-bbxx}=\frac{-B}{2bb}l\frac{aa-bbxx}{aa}=\frac{B}{bb}l\frac{a}{\sqrt{(aa-bbxx)}}+C.$$

COROLLARIUM 2

76. Casu tertio, si ponamus $\cos.\zeta=0$, habemus

$$\int\frac{(A+Bx)dx}{aa+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa+bbxx)}}{a}+\frac{A}{ab}\text{Arc.tang.}\frac{bx}{a}+c$$

hincque singillatim

$$\int\frac{Adx}{aa+bbxx}=\frac{A}{ab}\text{Arc.tang.}\frac{bx}{a}+c$$

et

$$\int\frac{Bxdx}{aa+bbxx}=\frac{B}{bb}l\frac{\sqrt{(aa+bbxx)}}{a}+c$$

EXEMPLUM 2

77. *Proposita formula differentiali* $\frac{x^{m-1}dx}{1+x^n}$, *siquidem exponens m* – 1 *minor sit quam n, integrale definire.*

In capite ultimo *Institutionum Calculi Differentialis* [L. EULERI *Institutiones calculus differentialis cum eius usu in analysi finitorum ac doctrina serierum*, Petropoli 1755] invenimus fractiones simplices, in quas haec fractio $\frac{x^{m-1}dx}{1+x^n}$ resolvitur, sumto π pro mensura duorum angulorum rectorum in hac forma generali contineri

$$\frac{2\,sin.\frac{(2k-1)\pi}{n}\,sin.\frac{(2k-1)\pi}{n}-2\,cos.\frac{m(2k-1)\pi}{n}\left(x-cos.\frac{(2k-1)\pi}{n}\right)}{n\left(1-2x\,cos.\frac{(2k-1)\pi}{n}+xx\right)},$$

ubi pro *k* successive omnes nnmeros 1, 2, 3 etc. substitui convenit, quoad *2k* - 1 numerum *n* superare incipiat. Hac ergo forma in *dx* ducta et cum generali nostra

$$\frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$$

comparata fit

$$a=1,\quad b=1,\quad \zeta=\frac{(2k-1)\pi}{n}$$

et

$$A=\tfrac{2}{n}\,sin.\frac{(2k-1)\pi}{n}\,sin.\frac{m(2k-1)\pi}{n}+\tfrac{2}{n}\,cos.\frac{(2k-1)\pi}{n}\,cos.\frac{m(2k-1)\pi}{n}$$

seu

$$A=\tfrac{2}{n}\,cos.\frac{(m-1)(2k-1)\pi}{n}\quad \text{et}$$

$$B=-\tfrac{2}{n}\,cos.\frac{m(2k-1)\pi}{n},$$

unde fit

$$Ab+Ba\,cos.\zeta=\tfrac{2}{n}\,sin.\frac{(2k-1)\pi}{n}\,sin.\frac{m(2k-1)\pi}{n}$$

ac propterea huius partis integrale erit

$$-\frac{2}{n}cos.\frac{m(2k-1)\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{(2k-1)\pi}{n}+xx\right)}$$

$$+\frac{2}{n}sin.\frac{m(2k-1)\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{(2k-1)\pi}{n}}{1-x\,cos.\frac{(2k-1)\pi}{n}}.$$

Ac si n sit numerus impar, praeterea accedit fractio $\frac{\pm dx}{n(1+x)}$, cuius integraIe est $\pm\frac{1}{n}l(1+x)$, ubi signum superius valet, si m impar, inferius vero, si m par. Quocirca integrale quaesitum $\int\frac{x^{m-1}dx}{1+x^n}$ sequenti modo exprimetur

$$-\frac{2}{n}cos.\frac{m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{\pi}{n}+xx\right)}+\frac{2}{n}sin.\frac{m\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{n}}{1-x\,cos.\frac{\pi}{n}}$$

$$-\frac{2}{n}cos.\frac{3m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{3\pi}{n}+xx\right)}+\frac{2}{n}sin.\frac{3m\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{3\pi}{n}}{1-x\,cos.\frac{3\pi}{n}}$$

$$-\frac{2}{n}cos.\frac{5m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{5\pi}{n}+xx\right)}+\frac{2}{n}sin.\frac{5m\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{5\pi}{n}}{1-x\,cos.\frac{5\pi}{n}}$$

$$-\frac{2}{n}cos.\frac{7m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{7\pi}{n}+xx\right)}+\frac{2}{n}sin.\frac{7m\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{7\pi}{n}}{1-x\,cos.\frac{7\pi}{n}}$$

etc.

secundum numeros impares ipso n minores; sicque totum obtinetur integrale, si n fuerit numerus par, sin autem n sit numerus impar, insuper accedit haec pars $\pm\frac{1}{n}l(1+x)$, prout m sit numerus vel impar vel par; unde si $m = 1$, accedit insuper $+\frac{1}{n}l(1+x)$.

COROLLARIUM 1

78. Sumamus $m = 1$, ut habeatur forma $\int\frac{dx}{1+x^n}$, et pro variis casibus ipsius n adipiscimur

I. $\int \frac{dx}{1+x} = l(1+x)$

II. $\int \frac{dx}{1+x^2} = \text{Arc.tang.}x$

III. $\int \frac{dx}{1+x^3} = -\frac{2}{3}cos.\frac{\pi}{3}\, l\sqrt{\left(1-2x\,cos.\frac{\pi}{3}+xx\right)}$

$$+\frac{2}{3}sin.\frac{\pi}{3}\,\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{3}}{1-x\,cos.\frac{\pi}{3}}+\frac{1}{3}l(1+x)$$

IV. $\int \frac{dx}{1+x^4} = -\frac{2}{4}cos.\frac{\pi}{4}\, l\sqrt{\left(1-2x\,cos.\frac{\pi}{4}+xx\right)} \quad +\frac{2}{4}sin.\frac{\pi}{4}\,\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{4}}{1-x\,cos.\frac{\pi}{4}}$

$$-\frac{2}{4}cos.\frac{3\pi}{4}\, l\sqrt{\left(1-2x\,cos.\frac{3\pi}{4}+xx\right)} \ +\frac{2}{4}sin.\frac{3\pi}{4}\,\text{Arc.tang.}\frac{x\,sin.\frac{3\pi}{4}}{1-x\,cos.\frac{3\pi}{4}}$$

V. $\int \frac{dx}{1+x^5} = -\frac{2}{5}cos.\frac{\pi}{5}\, l\sqrt{\left(1-2x\,cos.\frac{\pi}{5}+xx\right)} \quad +\frac{2}{5}sin.\frac{\pi}{5}\,\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{5}}{1-x\,cos.\frac{\pi}{5}}$

$$-\frac{2}{5}cos.\frac{3\pi}{5}\, l\sqrt{\left(1-2x\,cos.\frac{3\pi}{5}+xx\right)} \ +\frac{2}{5}sin.\frac{3\pi}{5}\,\text{Arc.tang.}\frac{x\,sin.\frac{3\pi}{5}}{1-x\,cos.\frac{3\pi}{5}}+\frac{1}{5}l(1+x)$$

VI. $\int \frac{dx}{1+x^6} = -\frac{2}{6}cos.\frac{\pi}{6}\, l\sqrt{\left(1-2x\,cos.\frac{\pi}{6}+xx\right)} \quad +\frac{2}{6}sin.\frac{\pi}{6}\,\text{Arc.tang.}\frac{x\,sin.\frac{\pi}{6}}{1-x\,cos.\frac{\pi}{6}}$

$$-\frac{2}{6}cos.\frac{3\pi}{6}\, l\sqrt{\left(1-2x\,cos.\frac{3\pi}{5}+xx\right)} \ +\frac{2}{6}sin.\frac{3\pi}{6}\,\text{Arc.tang.}\frac{x\,sin.\frac{3\pi}{6}}{1-x\,cos.\frac{3\pi}{6}}$$

$$-\frac{2}{6}cos.\frac{5\pi}{6}\, l\sqrt{\left(1-2x\,cos.\frac{5\pi}{5}+xx\right)} \ +\frac{2}{6}sin.\frac{5\pi}{6}\,\text{Arc.tang.}\frac{x\,sin.\frac{5\pi}{6}}{1-x\,cos.\frac{5\pi}{6}}$$

COROLLARIUM 2

79. Loco sinuum et cosinuum valores, ubi commode fieri potest, substituendo obtinemus

$$\int \frac{dx}{1+x^3} = -\frac{1}{3}l\sqrt{\left(1-x+xx\right)} \ +\frac{1}{\sqrt{3}}\text{Arc.tang.}\frac{x\sqrt{3}}{2-x}+\frac{1}{3}l(1+x)$$

seu

$$\int \frac{dx}{1+x^3} = -\frac{1}{3}l\frac{1+x}{\sqrt{(1-x+xx)}} \ +\frac{1}{\sqrt{3}}\text{Arc.tang.}\frac{x\sqrt{3}}{2-x}$$

Deinde ob $sin.\frac{\pi}{4}=cos.\frac{\pi}{4}=\frac{1}{\sqrt{2}}=sin.\frac{3\pi}{4}=-cos.\frac{3\pi}{4}$ fit

$$\int \frac{dx}{1+x^4} = +\frac{1}{2\sqrt{2}}\,l\frac{\sqrt{\left(1+x\sqrt{2}+xx\right)}}{\sqrt{\left(1-x\sqrt{2}+xx\right)}} \quad +\frac{1}{2\sqrt{2}}\text{Arc.tang.}\frac{x\sqrt{2}}{1-xx},$$

tum vero

$$\int \frac{dx}{1+x^6} = +\frac{1}{2\sqrt{3}} l \frac{\sqrt{\left(1+x\sqrt{3}+xx\right)}}{\sqrt{\left(1-x\sqrt{3}+xx\right)}} \quad +\frac{1}{6}\text{Arc.tang.}\frac{3x(1-xx)}{1-4xx+x^4}$$

EXEMPLUM 3

80. *Proposita formula differentiali* $\frac{x^{m-1}dx}{1+x^n}$*, siquidem exponens* $m-1$ *sit minor quam n, eius integrale definire.*

Functionis fractae $\frac{x^{m-1}}{1+x^n}$ pars ex factore quocunque oriunda hac forma continetur

$$\frac{2\sin.\frac{2k\pi}{n}\sin.\frac{2mk\pi}{n}-2\cos.\frac{2mk\pi}{n}\left(x-\cos.\frac{2k\pi}{n}\right)}{n\left(1-2x\cos.\frac{2k\pi}{n}+xx\right)}$$

quae cum forma nostra $\frac{(A+Bx)dx}{aa-2abx\cos.\zeta+bbxx}$ comparata dat

$$a=1,\quad b=1,\quad \zeta=\frac{2k\pi}{n},$$

$$A=\frac{2}{n}\sin.\frac{2k\pi}{n}\sin.\frac{2mk\pi}{n}+\frac{2}{n}\cos.\frac{2k\pi}{n}\cos.\frac{2mk\pi}{n},\quad B=-\frac{2}{n}\cos.\frac{2mk\pi}{n}$$

hincque

$$Ab+Ba\cos.\zeta=\frac{2}{n}\sin.\frac{2k\pi}{n}\sin.\frac{2mk\pi}{n}.$$

Ex quo integrale hinc oriundum erit

$$-\frac{2}{n}\cos.\frac{2mk\pi}{n}l\sqrt{\left(1-2x\cos.\frac{2k\pi}{n}+xx\right)}+\frac{2}{n}\sin.\frac{2mk\pi}{n}\text{Arc.tang.}\frac{x\sin.\frac{2k\pi}{n}}{1-x\cos.\frac{2k\pi}{n}},$$

ubi pro k successive omnes numeri 1, 2, 3 etc. substitui debent, quamdiu $2k$ minor est quam n. Accedunt insuper hae ex fractione $\frac{1}{n(1-x)}$ et, si n est numerus par, ex fractione $\frac{\mp 1}{n(1+x)}$ oriundae integralis partes $-\frac{1}{n}l(1-x)$ et $\mp\frac{1}{n}l(1+x)$, ubi signum superius valet, si m est par, inferius vero, si m impar.

Quocirca integrale $\int\frac{x^{m-1}dx}{1+x^n}$ hoc modo exprimitur

$$-\tfrac{1}{n}l\left(1-x\right)$$

$$-\tfrac{2}{n}cos.\frac{2m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{\pi}{n}+xx\right)}+\tfrac{2}{n}sin.\frac{2m\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{2\pi}{n}}{1-x\,cos.\frac{2\pi}{n}}$$

$$-\tfrac{2}{n}cos.\frac{4m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{4\pi}{n}+xx\right)}+\tfrac{2}{n}sin.\frac{4m\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{4\pi}{n}}{1-x\,cos.\frac{4\pi}{n}}$$

$$-\tfrac{2}{n}cos.\frac{6m\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{6\pi}{n}+xx\right)}+\tfrac{2}{n}sin.\frac{6m\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{6\pi}{n}}{1-x\,cos.\frac{6\pi}{n}}$$

etc.

COROLLARIUM

81. Sit $m = 1$ et pro n successive numeri 1, 2, 3 etc. substituantur, ut nanciscamur sequentes integrationes

I. $\int\frac{dx}{1-x}=-l\left(1-x\right)$

II. $\int\frac{dx}{1-xx}=-\frac{1}{2}l\left(1-x\right)+\frac{1}{2}l\left(1+x\right)=\frac{1}{2}l\frac{(1+x)}{(1-x)}$

III. $\int\frac{dx}{1-x^3}=-\frac{1}{3}l\left(1-x\right)-\frac{2}{3}cos.\frac{2\pi}{3}l\sqrt{\left(1-2x\,cos.\frac{2\pi}{3}+xx\right)}$

$\qquad +\frac{2}{3}sin.\frac{2\pi}{3}\text{Arc.tang.}\frac{x\,sin.\frac{2\pi}{3}}{1-x\,cos.\frac{2\pi}{3}}$

IV. $\int\frac{dx}{1-x^4}=-\frac{1}{4}l\left(1-x\right)-\frac{2}{4}cos.\frac{2\pi}{4}l\sqrt{\left(1-2x\,cos.\frac{2\pi}{4}+xx\right)}\quad+\frac{2}{4}sin.\frac{2\pi}{4}\text{Arc.tang.}\frac{x\,sin.\frac{2\pi}{4}}{1-x\,cos.\frac{2\pi}{4}}$

$\qquad +\frac{2}{4}l\left(1+x\right)$

V. $\int\frac{dx}{1-x^5}=-\frac{1}{5}l\left(1-x\right)-\frac{2}{5}cos.\frac{2\pi}{5}l\sqrt{\left(1-2x\,cos.\frac{2\pi}{5}+xx\right)}\quad+\frac{2}{5}sin.\frac{2\pi}{5}\text{Arc.tang.}\frac{x\,sin.\frac{2\pi}{5}}{1-x\,cos.\frac{2\pi}{5}}$

$\qquad -\frac{2}{5}cos.\frac{4\pi}{5}l\sqrt{\left(1-2x\,cos.\frac{4\pi}{5}+xx\right)}\ +\frac{2}{5}sin.\frac{4\pi}{5}\text{Arc.tang.}\frac{x\,sin.\frac{4\pi}{5}}{1-x\,cos.\frac{4\pi}{5}}$

VI. $\int\frac{dx}{1-x^6}=-\frac{1}{6}l\left(1-x\right)-\frac{2}{6}cos.\frac{2\pi}{6}l\sqrt{\left(1-2x\,cos.\frac{2\pi}{6}+xx\right)}\quad+\frac{2}{6}sin.\frac{2\pi}{6}\text{Arc.tang.}\frac{x\,sin.\frac{2\pi}{6}}{1-x\,cos.\frac{2\pi}{6}}$

$\qquad -\frac{2}{6}cos.\frac{4\pi}{6}l\sqrt{\left(1-2x\,cos.\frac{4\pi}{5}+xx\right)}\ +\frac{2}{6}sin.\frac{4\pi}{6}\text{Arc.tang.}\frac{x\,sin.\frac{4\pi}{6}}{1-x\,cos.\frac{4\pi}{6}}$

$\qquad +\frac{1}{6}l\left(1+x\right)$

EXEMPLUM 4

82. *Proposita formula differentiali* $\frac{\left(x^{m-1}+x^{n-m-1}\right)dx}{1+x^n}$ *existente* $n > m-1$ *eius integrale definire.*

Ex exemplo 2 patet integralis partem quamcunque In genere esse, sumto *i* pro numero quocunque impare non maiore quam *n*,

$$-\frac{2}{n}cos.\frac{im\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{i\pi}{n}+xx\right)}+\frac{2}{n}sin.\frac{im\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{i\pi}{n}}{1-x\,cos.\frac{i\pi}{n}},$$

$$-\frac{2}{n}cos.\frac{i(n-m)\pi}{n}l\sqrt{\left(1-2x\,cos.\frac{i\pi}{n}+xx\right)}+\frac{2}{n}sin.\frac{i(n-m)\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{i\pi}{n}}{1-x\,cos.\frac{i\pi}{n}}.$$

Verum est

$$cos.\frac{i(n-m)\pi}{n}=cos.\left(i\pi-\frac{im\pi}{n}\right)=-cos.\frac{im\pi}{n}$$

et

$$sin.\frac{i(n-m)\pi}{n}=sin.\left(i\pi-\frac{im\pi}{n}\right)=+sin.\frac{im\pi}{n},$$

unde partes logarithmicae se destruent, eritque pars integralis In genere

$$\frac{4}{n}sin.\frac{im\pi}{n}\text{Arc.tang.}\frac{x\,sin.\frac{i\pi}{n}}{1-x\,cos.\frac{i\pi}{n}}.$$

Ponatur commoditatis ergo angulus $\frac{\pi}{n}=\omega$ eritque

$$\begin{aligned}\int\frac{\left(x^{m-1}+x^{n-m-1}\right)dx}{1+x^n}=&+\frac{4}{n}sin.m\omega\text{Arc.tang.}\frac{x\,sin.\omega}{1-x\,cos\,\omega}\\&+\frac{4}{n}sin.3m\omega\text{Arc.tang.}\frac{x\,sin.3\omega}{1-x\,cos\,3\omega}\\&+\frac{4}{n}sin.5m\omega\text{Arc.tang.}\frac{x\,sin.5\omega}{1-x\,cos\,5\omega}\\&\quad\vdots\\&+\frac{4}{n}sin.im\omega\text{Arc.tang.}\frac{x\,sin.i\omega}{1-x\,cos\,i\omega}\end{aligned}$$

sumto pro *i* maximo numero impare exponentem *n* non excedente. Si ipse numerus *n* sit impar, pars ex positione i = *n* oriunda ob sin. *mn* = 0 evanescet. Notetur ergo hic totum integrale per meros angulos exprimi.

COROLLARIUM

83. Simili modo sequens integrale elicitur, ubi soli logarithmi relinquentur, manente $\frac{\pi}{n}=\omega$:

$$\int\frac{\left(x^{m-1}-x^{n-m-1}\right)dx}{1+x^n}=-\frac{4}{n}cos.m\omega\, l\sqrt{\left(1-2x\,cos.\omega+xx\right)}$$
$$-\frac{4}{n}cos.3m\omega\, l\sqrt{\left(1-2x\,cos.3\omega+xx\right)}$$
$$-\frac{4}{n}cos.5m\omega\, l\sqrt{\left(1-2x\,cos.5\omega+xx\right)}$$
$$\vdots$$
$$-\frac{4}{n}cos.im\omega\, l\sqrt{\left(1-2x\,cos.i\omega+xx\right)},$$

donec scilicet numerus impar *i* non superet exponentem *n.*

EXEMPLUM 5

84. *Proposita formula differentiali* $\frac{\left(x^{m-1}-x^{n-m-1}\right)dx}{1+x^n}$ *existente* $n>m-1$ *eius integrale definire.*

Ex exemplo 3 integralis pars quaecunque concluditur, siquidem brevitatis gratia $\frac{\pi}{n}=\omega$ statuamus,

$$-\frac{2}{n}cos.2mk\omega\, l\sqrt{\left(1-2x\,cos.2k\omega+xx\right)}+\frac{2}{n}sin.2mk\omega\text{Arc.tang.}\frac{x\,sin.2mk\omega}{1-x\,cos.2mk\omega}$$
$$+\frac{2}{n}cos.2k\left(n-m\right)\omega\, l\sqrt{\left(1-2x\,cos.2k\omega+xx\right)}+\frac{2}{n}sin.2k\left(n-m\right)\omega\text{Arc.tang.}\frac{x\,sin.2mk\omega}{1-x\,cos.2mk\omega}$$

At est

$$cos.2k\left(n-m\right)\omega=cos.\left(2k\pi-2km\omega\right)=cos.2km\omega$$

et

$$sin.2k\left(n-m\right)\omega=sin.\left(2k\pi-2km\omega\right)=-sin.2km\omega,$$

unde ista pars generalis abit in

$$\frac{4}{n}sin.2km\omega\ \text{Arc.tang.}\frac{x\,sin.2k\omega}{1-x\,cos.2m\omega},$$

quare hinc ista integratio colligitur

$$\int\frac{\left(x^{m-1}-x^{n-m-1}\right)dx}{1+x^{n}}=+\frac{4}{n}\sin.2m\omega\,\text{Arc.tang.}\frac{x\sin.2\omega}{1-x\cos 2\omega}$$
$$+\frac{4}{n}\sin.4m\omega\,\text{Arc.tang.}\frac{x\sin.4\omega}{1-x\cos 4\omega}$$
$$+\frac{4}{n}\sin.6m\omega\,\text{Arc.tang.}\frac{x\sin.6\omega}{1-x\cos 6\omega}$$

numeris paribus tamdiu ascendendo, quoad exponentem n non superent.

COROLLARIUM

85. Indidem etiam haec integratio absolvitur manente $\frac{\pi}{n}=\omega$

$$\int\frac{\left(x^{m-1}+x^{n-m-1}\right)dx}{1-x^{n}}=-\frac{2}{n}l\left(1-x\right)$$
$$-\frac{4}{n}\cos.2m\omega\, l\sqrt{\left(1-2x\cos.2\omega+xx\right)}$$
$$-\frac{4}{n}\cos.4m\omega\, l\sqrt{\left(1-2x\cos.4\omega+xx\right)}$$
$$-\frac{4}{n}\cos.6m\omega\, l\sqrt{\left(1-2x\cos.6\omega+xx\right)},$$

ubi etiam numeri pares non ultra terminum n sunt continuandi.

EXEMPLUM 6

86. *Proposita formula differentiali* $dy=\frac{dx}{x^{3}(1+x)\left(1-x^{4}\right)}$ *eius integrale invenire.*

Functio fracta per dx affecta secundum denominatoris factores est

$$\frac{1}{x^{3}(1+x)^{2}(1-x)(1+xx)}$$

quae in has fractiones simplices resolvitur

$$\frac{1}{x^{3}}-\frac{1}{x^{2}}+\frac{1}{x}-\frac{1}{4(1+x)^{2}}-\frac{9}{8(1+x)}+\frac{1}{8(1-x)}+\frac{1+x}{4(1+xx)}=\frac{dy}{dx},$$

unde per integrationem elicitur

$$y=-\frac{1}{2x^{2}}+\frac{1}{x}+lx+\frac{1}{4(1+x)}-\frac{9}{8}l\left(1+x\right)-\frac{1}{8}l\left(1-x\right)$$
$$+\frac{1}{8}l\left(1+xx\right)+\frac{1}{4}\text{Arc.tang.}x,$$

quae expressio in hanc formam transmutatur

$$y=C+\frac{-2+2x+5xx}{4xx(1+x)}-l\frac{1+x}{x}+\frac{1}{8}l\left(\frac{1+xx}{1-xx}\right)+\frac{1}{4}\text{Arc.tang.}x,$$

SCHOLION

87. Hoc igitur caput ita pertractare licuit, ut nihil amplius in hoc genere desiderari possit. Quoties ergo eiusmodi functio y ipsius x quaeritur, ut $\frac{dy}{dx}$ aequetur functioni rationali ipsius x, toties integratio nihil habet difficultatis, nisi forte ad denominatoris singulos

factores eliciendos Algebrae praecepta non sufficiant; verum tum defectus ipsi Algebrae, non vero methodo integrandi, quam hic tractamns, est tribuendus. Deinde etiam potissimum notari convenit semper, cum $\frac{dy}{dx}$ functioni rationali ipsius x aequale ponitur, functionem y, nisi sit algebraica, alias quantitates transcendentes non involvere praeter logarithmos et angulos; ubi quidem observandum est hic perpetuo logarithmos hyperbolicos intelligi oportere, cum ipsius lx differentiale non sit = $\frac{1}{x}$, nisi logarithmus hyperbolicus sumatur; at horum reductio ad vulgares est facillima, ita ut hinc applicatio calculi ad praxin nulli impedimento sit obnoxia. Quare progrediamur ad eos casus, quibus formula $\frac{dy}{dx}$ functioni irrationali ipsius x aequatur, ubi quidem primo notandum est, quoties ista functio per idoneam substitutionem ad rationalitatem perduci poterit, casum ad hoc caput revolvi. Veluti si fuerit

$$dy = \frac{1+\sqrt{x}-\sqrt[3]{xx}}{1+\sqrt[3]{x}}dx,$$

evidens est ponendo $x = z^6$, unde fit $dx = 6z^5dz$, fore

$$dy = \frac{\left(1+z^3-z^4\right)}{1+zz}\cdot 6z^5dz$$

ideoque

$$\frac{dy}{dz} = -6z^7 + 6z^6 + 6z^5 - 6z^4 + 6zz - 6 + \frac{6}{1+zz},$$

unde integrale

$$y = -\frac{3}{4}z^8 + \frac{6}{7}z^7 + z^6 - \frac{6}{5}z^5 + 2z^3 - 6z + 6\text{Arc.tang.}z$$

et restituto valore

$$y = -\frac{3}{4}x\sqrt[3]{x} + \frac{6}{7}x\sqrt[6]{x} + x - \frac{6}{5}\sqrt[6]{x^5} + 2\sqrt{x} - 6\sqrt[6]{x} + 6\text{Arc.tang.}\sqrt[6]{x} + C.$$