CHAPTER VII

CONCERNING THE VARIATION OF INTEGRAL FORMULAS INVOLVING THREE VARIABLES OF WHICH ONE IS CONSIDERED AS A FUNCTION OF THE REMAINING TWO

PROBLEM 19

159. *To set out the method and to explain here the nature of the related integral formulas, by which it is convenient to investigate the variations of these.*

SOLUTION

Since there may be considered three variables x, y and z, of which the one z is required to be considered as a function of the remaining two x and y, even if in the investigation of the variation an account of this function must be considered as for an unknown, the integral formulas, which generally occur in this kind of calculation, most frequently disagree with these, which are usually proposed concerning only two variables. Just as indeed with such a form of the integral $\int Vdx$, where V may be considered to involve the two variables x and y, of which it is considered that y depends on x, as if the sum of all the values of the elements Vdx can be considered through all the values of x gathered together, thus, when three variables x, y and z may be considered, of which this z is likewise considered to depend on x and y, here the related integrals involve a collection of all the related elements both of x as well as of y and thus they require a double integral, the one through all the values of x, and truly with the elements of the other y gathered together. From which integrals of such a form $\iint Vdxdy$ must be put in place, from which clearly a two-fold integration may be expressed, of which the setting out may thus usually be put in place, so that first the other variable y may be considered as constant and the value of the formula $\int Vdx$ may be sought through the limits of the integration ; in which now since x either may obtain or be given a value depending on y, this integral $\int Vdx$ will be changed into a function of y, which remains multiplied by dy, so that the integral $\int dy\int Vdx$ may be investigated; therefore which form $\int dy\int Vdx$ treated in this manner is required to be considered equivalent to this $\iint Vdxdy$. And if in the inverse order the first quantity x may be taken constant and the integral $\int Vdy$ may be extended through the prescribed limits, that henceforth may be considered as a function of x and the integral $\int dx\int Vdy$ sought can be found. But it is the same, by whatever manner we have used the value of the double integral $\iint Vdxdy$ to be set out.

Therefore since in generally other integral formulas are unable to occur except of this kind $\iint Vdxdy$, the whole matter corresponds to this, so that, just as we may show, it is required to fins a variation of this kind of form. But since we have assumed x and y lacking in variation, from these, which have been shown in the beginning [§ 75], it is readily deduced that

$$\delta\iint Vdxdy = \iint \delta Vdxdy,$$

where δV denotes the variation of *V*; and here there is a need for this double integral, evidently as in the manner we have expressed before.

COROLLARY 1

160. If we may put the integral $\iint Vdxdy = W$ since there shall be $\int dx\int Vdy = W$ there will be on differentiating by *x* only $\int Vdy = \left(\frac{dW}{dx}\right)$ and hence on differentiating again by *y* there will be $V = \left(\frac{ddW}{dydx}\right)$, from which it is apparent the integral *W* thus is to be prepared, so that there becomes $V = \left(\frac{ddW}{dxdy}\right)$.

COROLLARY 2

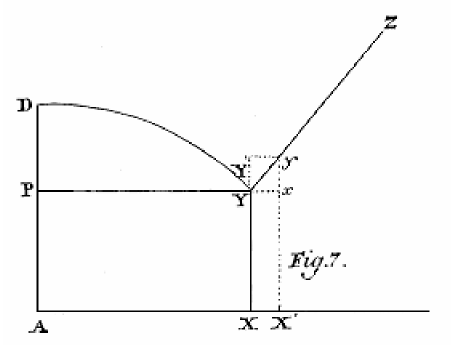
161. Since a twofold integration shall be required to be put in place, with each an arbitrary quantity is introduced; moreover the one integration in place of a constant there shall be some function of *x* brought in, which shall be *X*, and the other some function of *y* brought in, which shall be *Y*, thus so that the complete integral shall be $\iint Vdxdy = W + X + Y$.

COROLLARY 3

162. This also may be confirmed from the same resolution; for in the first place let there be $\int Vdy = \left(\frac{dW}{dx}\right) + \left(\frac{dX}{dx}\right)$ on account of $\left(\frac{dY}{dx}\right) = 0$, then truly there shall be $V = \left(\frac{ddW}{dxdy}\right)$, because neither *X* nor $\frac{dX}{dx}$ depends on *y*. Whereby if there should be $\left(\frac{ddW}{dxdy}\right) = V$, the complete integral will be $\iint Vdxdy = W + X + Y$.

SCHOLION 1

163. But generally it is by necessity, that the nature of double integrals of this kind be subjected to a more careful examination, which most conveniently will be best done by the theory of surfaces. Therefore let there be at this point the two orthogonal coordinates *x* and *y* so that for the base assumed $AX = x,\ XY = y$ (Fig. 7), to which at *Y* there stands normally the third ordinate $YZ = z$ extended as far as the surface. If now those two coordinates *x* and *y* may increase by their differentials $XX' = dx$ and $YY' = dy$, thence on the base there arises the element of the parallelogram $YxyY' = dxdy$, to which the element of the integral formula agrees. Thus if from the volume shall be sought from the included surface, the element of this will be $= zdxdy$ and thus the whole volume $= \int zdxdy$; if the surface itself may be sought, on putting $dz = pdx + p'dy$ there will be the element of this projecting for the rectangle $dxdy = dxdy\sqrt{(1+pp+p'p')}$ and thus the surface itself $= \int dxdy\sqrt{(1+pp+p'p')}$, from which generally an account of the double integral formula $\int\int Vdxdy$ is understood. But if now the value of such a formula is sought, which corresponds to the given area in the base such as *ADYX*, in the first place the simple integral $\int Vdy$ may be investigated with *x* taken constant and then the magnitude *XY* of *y* itself may be assigned extended to the curve *DY*, which from the nature of this curve is equal to a certain function of *x*. Thus therefore $dx\int Vdy$ will express the element of the proposed formula agreeing with the rectangle $XYxX' = ydx$, the integral of which taken anew $\int dx\int Vdy$ and from the variable *x* finally will give the constant value corresponding to the whole area *ADYX*, if indeed the integration with the value of the constant duly added may be determined.



SCHOLIUM 2

164. Thus the explanation of double integrals of this kind must itself be considered, if it should be applied to the figure on the given base such as *AD*Y*X*; but if we wish to set out each indefinite integral, as initially on assuming *x* constant we may search for the integral $\int Vdy$, because it is understand to agree with the rectangular element $XYx\ X' = ydx$, if indeed it is taken by *dx*, from that indeed in the integration of the formula $\int dx\int Vdy$ we may consider the quantity $y = XY$ in the same to remain with *x* only assumed to be variable, then the value will be produced by the corresponding indefinite rectangle $APYX = xy$, if indeed the constants due may be defined entering each integration. But if the remaining terms of this interval besides the lines *XY* and *PY* may be considered indefinite, the integral $\iint V dxdy$ may receive two indefinite functions $X + Y$, that of *x*, this indeed of *y*. But if hence we wish to adapt this henceforth to the calculation of maximas and minimas, because the maxima or minima property, which must be agreed in a certain given area *ADYX*, likewise also for any indefinite area as you please *APYX* it is necessary to agree, that double integral here set out in this way may be agreed to be changed to the indefinite.

PROBLEMA 20

165. *If V shall be some formula composed from the three variables x, y, z and from the differentials of these, to find the variation of the double integral formula* $\iint V dxdy$ *, as long as any variations are attributed to the quantity z, which may be considered as a function of the two x and y.*

SOLUTION

Towards removing the types of differentials we may put in place

$$p = \left(\tfrac{dz}{dx}\right), \quad p' = \left(\tfrac{dz}{dy}\right),$$

$$q = \left(\tfrac{dp}{dx}\right), \quad q' = \left(\tfrac{dp}{dy}\right) = \left(\tfrac{dp'}{dx}\right), \quad q'' = \left(\tfrac{dp'}{dy}\right),$$

$$r = \left(\tfrac{dq}{dx}\right), \quad r' = \left(\tfrac{dq}{dy}\right) = \left(\tfrac{dq'}{dx}\right), \quad r'' = \left(\tfrac{dq'}{dy}\right) = \left(\tfrac{dq''}{dx}\right), \quad r''' = \left(\tfrac{dq''}{dy}\right)$$

etc.,

so that *V* may be made a function of the finite quantities $x, y, z, p, p', q, q', q'', r, r', r'', r'''$ etc. Then there is put the differential of this

$$\begin{aligned} dV = Ldx + Mdy + Ndz + Pdp & + Qdq + Rdr \\ + P'dp' & + Q'dq' + R'dr' \\ & + Q''dq'' + R''dr'' \\ & + R'''dr''' \end{aligned}$$

etc.

from which since likewise the variation of this may be unknown, from the preceding problem the variation sought may be deduced

$$\delta \iint V dxdy = \iint dxdy \left\{ \begin{aligned} N\delta z + P\delta p & + Q\delta q + R\delta r \\ + P'\delta p' & + Q'\delta q' + R'\delta r' \\ & + Q''\delta q'' + R''\delta r'' \\ & + R'''\delta r''' \\ & \text{etc.} \end{aligned} \right\}$$

But if now, as we accomplished in §154, we may put the variation $\delta z = \omega$, as one is allowed to regard that as some function of the two variables *x* and *y*, from that same place we may consider the variation to be

$$\delta\iint Vdxdy = \iint dxdy \left\{ \begin{aligned} &N\omega + P\left(\frac{d\omega}{dx}\right) + Q\left(\frac{dd\omega}{dx^2}\right) + R\left(\frac{d^3\omega}{dx^3}\right) \\ &\quad + P'\left(\frac{d\omega}{dy}\right) + Q'\left(\frac{dd\omega}{dxdy}\right) + R'\left(\frac{d^3\omega}{dx^2dy}\right) \\ &\quad\quad + Q''\left(\frac{dd\omega}{dy^2}\right) + R''\left(\frac{d^3\omega}{dxdy^2}\right) \\ &\quad\quad\quad + R'''\left(\frac{d^3\omega}{dy^3}\right) \\ &\quad\quad \text{etc.,} \end{aligned} \right\}$$

COROLLARY 1

166. Therefore if the nature of each of the functions *z* and $\delta z = \omega$ or an account of the composition of the two variables *x* and *y* should be given, then by the precepts explained before the variation of the twofold integral $\iint Vdxdy$ may be possible to be assigned, just as the quantity *V* was constructed from the variables *x*, *y*, *z* and from the differentials of these.

COROLLARY 2

167. Clearly the whole work will be reduced to the setting out of the twofold integration found ; which since it may be constructed from several parts, it may be agreed to treat the individual parts through a double integration, as explained before.

SCHOLIUM

168. But when an account of the function *z* is not in place and must be elicited finally from the condition of the variation, thus so that the variation itself $\delta z = \omega$ is not permitted any clear determination, just as happens, if the formula $\iint Vdxdy$ should obtain a maximum or minimum value, then generally it is of necessity, that the individual members of the variation found $\delta\iint Vdxdy$ may be reduced thus, so that everywhere after the double integral sign no differential values of the variation except this value itself $\delta z = \omega$ may occur; now in a reduction of this kind above we have used only the two variables in the formulas involved. But such a reduction, since it shall be less suited for the twofold integral formulas, demands a more thorough explanation. Which I observe to arrive in the end at simpler integral formulas by a reduction of this kind, in which with only the one quantity *x* or *y* to be regarded as variable as the other considered as constant, towards indicated which, lest the signs be multiplied beyond necessity, such a form $\int Tdx$ will denote the integral of the differential formula *Tdx*, while the quantity *y* may be considered to be constant; and in a similar manner it may be understood in this formula $\int Tdy$ only the quantity y is to be considered to be a variable, because with that it becomes more evident, with this condition omitted these formulas clearly are unable to have a meaning. Hence nor in the latter work has it been declared, if T should involve both the variables x and y, each of these in the simple integral formulas $\int Tdx$ or $\int Tdy$ is taken either constant or variable, since that alone, the differential of which is expressed, is required to be considered as the variable. But in the duplicate formulas $\iint Vdxdy$ it is always to be understood that the one integration in accordance of x only, truly the other to be restricted to the variable y only and likewise to be [in the order], whichever is put in place first.

PROBLEM 21

169. *Thus to transform the variation of the double integral formula $\iint Vdxdy$ found in the previous problem, so that after the double integral sign the variation $\delta z = \omega$ itself may occur with the differentials of this removed.*

SOLUTION

So that this transformation may be extended wider, T and v shall be any two functions of the two variables x and y and this duplicate formula may be considered $\iint Tdxdy\left(\frac{dv}{dx}\right)$, which with the individual variation of the integrals separated may be represented thus $\int dy\int Tdx\left(\frac{dv}{dx}\right)$, so that in the integration $\int Tdx\left(\frac{dv}{dx}\right)$ only the quantity x may be regarded as the variable. But then there will be $dx\left(\frac{dv}{dx}\right) = dv$, because y may be considered as constant, and thus there becomes

$$\int Tdv = Tv - \int vdT\ ;$$

where since in the differential of dT only a ratio of the variable x may be had, towards making that clear it has been agreed in place of dT to write $dx\left(\frac{dT}{dx}\right)$, thus so that there shall be

$$\int Tdx\left(\frac{dv}{dx}\right) = Tv - \int vdx\left(\frac{dT}{dx}\right)$$

and hence our reduced formula thus leads to

$$\iint Tdxdy\left(\frac{dv}{dx}\right) = \int Tvdy - \iint vdxdy\left(\frac{dT}{dx}\right).$$

In a similar manner with the variables interchanged we may follow with

$$\iint Tdxdy\left(\frac{dv}{dy}\right) = \int Tvdx - \iint vdxdy\left(\frac{dT}{dy}\right).$$

Now here as if with a lemma established now the variation found in the preceding problem reduced thus there will be had

$$\iint Pdxdy\left(\frac{d\omega}{dx}\right)=\int P\omega dy-\iint \omega dxdy\left(\frac{dP}{dx}\right),$$
$$\iint P'dxdy\left(\frac{d\omega}{dy}\right)=\int P'\omega dx-\iint \omega dxdy\left(\frac{dP'}{dy}\right).$$

Again for the following parts, there shall be for the first $\left(\frac{d\omega}{dx}\right)=v$ and thus $\left(\frac{dd\omega}{dx^2}\right)=\left(\frac{dv}{dx}\right)$, from which it is deduced :

$$\iint Qdxdy\left(\frac{dd\omega}{dx^2}\right)=\int Qdy\left(\frac{d\omega}{dx}\right)-\iint dxdy\left(\frac{dQ}{dx}\right)\left(\frac{d\omega}{dx}\right),$$

and for the latter part similarly reduced there will be

$$\iint Qdxdy\left(\frac{dd\omega}{dx^2}\right)=\int Qdy\left(\frac{d\omega}{dx}\right)-\int \omega dy\left(\frac{dQ}{dx}\right)+\iint \omega dxdy\left(\frac{ddQ}{dx^2}\right).$$

By the same substitution we will have $\left(\frac{dd\omega}{dxdy}\right)=\left(\frac{dv}{dy}\right)$ and hence

$$\iint Q'dxdy\left(\frac{dd\omega}{dxdy}\right)=\int Q'dx\left(\frac{d\omega}{dx}\right)-\iint dxdy\left(\frac{d\omega}{dx}\right)\left(\frac{dQ'}{dy}\right).$$

or

$$\iint Q'dxdy\left(\frac{dd\omega}{dxdy}\right)=\int Q'dx\left(\frac{d\omega}{dx}\right)-\int \omega dy\left(\frac{dQ'}{dy}\right)+\iint \omega dxdy\left(\frac{ddQ'}{dxdy}\right),$$

which form on account of

$$\int Q'dx\left(\frac{d\omega}{dx}\right)=Q'\omega-\int \omega dx\left(\frac{dQ'}{dx}\right)$$

will change into this

$$\iint Q'dxdy\left(\frac{dd\omega}{dxdy}\right)=Q'\omega-\int \omega dx\left(\frac{dQ'}{dx}\right)+\iint \omega dxdy\left(\frac{ddQ'}{dxdy}\right)-\int \omega dy\left(\frac{dQ'}{dy}\right).$$

Then indeed for the third form of this order we come upon

$$\iint Q''dxdy\left(\frac{dd\omega}{dy^2}\right)=\int Q''dx\left(\frac{d\omega}{dy}\right)-\int \omega dx\left(\frac{dQ''}{dy}\right)+\iint \omega dxdy\left(\frac{ddQ''}{dy^2}\right).$$

Again on account of $\left(\frac{d^3\omega}{dx^3}\right)=\left(\frac{ddv}{dx^2}\right)$ with $v=\left(\frac{d\omega}{dx}\right)$ remaining there becomes

$$\iint Rdxdy\left(\frac{ddv}{dx^2}\right)=\int Rdy\left(\frac{dv}{dx}\right)-\int vdy\left(\frac{dR}{dx}\right)+\iint vdxdy\left(\frac{ddR}{dx^2}\right)$$

and

$$\iint vdxdy\left(\frac{ddR}{dx^2}\right)=\int \omega dy\left(\frac{ddR}{dx^2}\right)-\iint \omega dxdy\left(\frac{d^3R}{dx^3}\right),$$

thus so that there shall be

$$\iint Rdxdy\left(\frac{d^3v}{dx^3}\right)=\int Rdy\left(\frac{dd\omega}{dx^2}\right)-\int dy\left(\frac{d\omega}{dx}\right)\left(\frac{dR}{dx}\right)+\int \omega dy\left(\frac{ddR}{dx^2}\right)-\iint \omega dxdy\left(\frac{d^3R}{dx^3}\right).$$

Accordingly on account of $\left(\frac{d^3\omega}{dx^2dy}\right)=\left(\frac{ddv}{dxdy}\right)$ there will be

$$\iint R'\,dxdy\left(\frac{ddv}{dxdy}\right)=R'v-\int vdx\left(\frac{dR'}{dx}\right)+\iint vdxdy\left(\frac{ddR'}{dxdy}\right)-\int vdy\left(\frac{dR'}{dy}\right),$$

and because here

$$\iint vdxdy\left(\frac{ddR'}{dxdy}\right)=\int \omega dy\left(\frac{ddR'}{dxdy}\right)-\iint \omega dxdy\left(\frac{d^3R'}{dx^2dy}\right),$$

we will conclude to become

$$\iint R'\,dxdy\left(\frac{d^3\omega}{dx^2dy}\right)=R'\left(\frac{d\omega}{dx}\right)-\int\left(\frac{d\omega}{dx}\right)dx\left(\frac{dR'}{dx}\right)+\int \omega dy\left(\frac{ddR'}{dxdy}\right)-\int\left(\frac{d\omega}{dx}\right)dy\left(\frac{dR'}{dy}\right)-\iint \omega dxdy\left(\frac{d^3R'}{dx^2dy}\right).$$

Finally with *x* and *y* interchanged we conclude hence

$$\iint R''\,dxdy\left(\frac{d^3\omega}{dxdy^2}\right)=R''\left(\frac{d\omega}{dy}\right)-\int\left(\frac{d\omega}{dy}\right)dy\left(\frac{dR''}{dy}\right)+\int \omega dx\left(\frac{ddR''}{dxdy}\right)-\int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR''}{dx}\right)-\iint \omega dxdy\left(\frac{d^3R''}{dxdy^2}\right).$$

and

$$\iint R'''\,dxdy\left(\frac{d^3\omega}{dy^3}\right)=\int R'''\,dx\left(\frac{dd\omega}{dy^2}\right)-\int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR'''}{dy}\right)+\int \omega dx\left(\frac{ddR'''}{dy^2}\right)-\iint \omega dxdy\left(\frac{d^3R'''}{dy^3}\right).$$

If we substitute which values, we find

$$\delta\iint Vdxdy = \iint \omega dxdy \left\{ \begin{array}{l} N-\left(\frac{dP}{dx}\right) + \left(\frac{ddQ}{dx^2}\right) - \left(\frac{d^3R}{dx^3}\right) \\ \quad -\left(\frac{dP'}{dy}\right) + \left(\frac{ddQ'}{dxdy}\right) - \left(\frac{d^3R'}{dx^2dy}\right) \\ \qquad + \left(\frac{ddQ''}{dy^2}\right) - \left(\frac{d^3R''}{dxdy^2}\right) \\ \qquad\quad - \left(\frac{d^3R'''}{dy^3}\right) \end{array} \right\}$$

$$+\int P\omega dy + \int P'\omega dx + \int Qdy\left(\frac{d\omega}{dx}\right) - \int \omega dx\left(\frac{dQ'}{dx}\right) + Q'\omega + \int Q''dx\left(\frac{d\omega}{dy}\right)$$

$$-\int \omega dy\left(\frac{dQ}{dx}\right) - \int \omega dy\left(\frac{dQ'}{dy}\right) \qquad - \int \omega dx\left(\frac{dQ''}{dy}\right)$$

$$+\int Rdy\left(\frac{dd\omega}{dx^2}\right) - \int\left(\frac{d\omega}{dx}\right)dx\left(\frac{dR'}{dx}\right) + R'\left(\frac{d\omega}{dx}\right) - \int\left(\frac{d\omega}{dy}\right)dy\left(\frac{dR''}{dy}\right) + \int R'''dx\left(\frac{dd\omega}{dy^2}\right)$$

$$-\int\left(\frac{d\omega}{dx}\right)dy\left(\frac{dR}{dx}\right) - \int\left(\frac{d\omega}{dx}\right)dy\left(\frac{dR'}{dy}\right) \qquad - \int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR''}{dx}\right) - \int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR'''}{dy}\right)$$

$$+\int \omega dy\left(\frac{ddR}{dx^2}\right) \quad + \int \omega dy\left(\frac{ddR'}{dxdy}\right) + R''\left(\frac{d\omega}{dy}\right) + \int \omega dx\left(\frac{ddR''}{dxdy}\right) \quad + \int \omega dx\left(\frac{ddR'''}{dy^2}\right)$$

etc.

COROLLARY 1

170. The first part of this expression is clear enough, truly the remaining parts are set out conveniently thus, so that an account of these can be understood :

$$\int \omega dy \left\{ \begin{array}{l} P-\left(\frac{dQ}{dx}\right) + \left(\frac{ddR}{dx^2}\right) - \text{etc.} \\ \quad -\left(\frac{dQ'}{dy}\right) + \left(\frac{ddR'}{dxdy}\right) \\ \qquad + \left(\frac{d^3R''}{dy^3}\right) \end{array} \right\} + \int \omega dx \left\{ \begin{array}{l} P'-\left(\frac{dQ''}{dy}\right) + \left(\frac{ddR'''}{dy^2}\right) - \text{etc.} \\ \quad -\left(\frac{dQ'}{dx}\right) + \left(\frac{ddR''}{dxdy}\right) \\ \qquad + \left(\frac{ddR'}{dx^2}\right) \end{array} \right\}$$

$$+\int\left(\frac{d\omega}{dx}\right)dy \left\{ \begin{array}{l} \mathrm{Q}-\left(\frac{dR}{dx}\right) + \text{etc.} \\ \quad -\left(\frac{dR'}{dy}\right) \end{array} \right\} + \int\left(\frac{d\omega}{dy}\right)dx \left\{ \begin{array}{l} Q''-\left(\frac{dR''}{dy}\right) + \text{etc.} \\ \quad -\left(\frac{dR''}{dx}\right) \end{array} \right\}$$

$$+\int\left(\frac{dd\omega}{dx^2}\right)dy\left(R-\text{etc.}\right) + \int\left(\frac{dd\omega}{dy^2}\right)dx\left(R'''-\text{etc.}\right) + \text{etc.}$$

$$+\omega \left\{ \begin{array}{l} Q'-\left(\frac{dR'}{dx}\right) + \text{etc.} \\ \quad -\left(\frac{dR''}{dy}\right) \end{array} \right\} + \left(\frac{d\omega}{dx}\right)\left(R'-\text{etc.}\right) + \left(\frac{d\omega}{dy}\right)\left(R''-\text{etc.}\right) + \text{etc.}$$

COROLLARY 2

171. Here with a little attention summoned it will soon be apparent, how these parts ought to be continued further, if perhaps the quantity V may include differentials of higher orders.

COROLLARY 3

172. In the others of these integral formulas, which have been affected by the differential dy, the quantity x is assumed constant, to which an convenient value is attributed at the end of the integration ; indeed with the others, which have been affected by the differential dx, y is constant and equal to the end of the integration, from which it is apparent in the limits of the integrations that both x and y take a constant value.

SCHOLIUM 1

173. Hence this formula of the variation has been adapted to that case, in which both limits of the integration attribute constant values both of x itself as well as y itself. Just as if from a surface the formula of the integral $\iint Vdxdy$ should be sought, according to the rectangle $APYX$ (Fig. 7) taken on the base, and the value of this being referred to must be defined thus, so that with $x=0$ and $y=0$ taken, which are the initial values, it may vanish, with which done on putting in place $x=AX$ and $y=AP$, which are the final values; and according to the same law the variation found is to be arranged. But if now that surface is sought, in which the value of the formula $\iint Vdxdy$ defined in this manner becomes a maximum or a minimum, everything before is necessary, so that the first part of the variation involving the double integration may be reduced to nothing, in whatever manner the variation $d'z=\omega$ may be taken, from which this equation may arise

$$\begin{aligned}0=N&-\left(\frac{dP}{dx}\right)+\left(\frac{ddQ}{dx^2}\right)-\left(\frac{d^3R}{dx^3}\right)+\text{etc.},\\&-\left(\frac{dP'}{dy}\right)+\left(\frac{ddQ'}{dxdy}\right)-\left(\frac{d^3R'}{dx^2dy}\right)\\&\qquad+\left(\frac{ddQ''}{dy^2}\right)-\left(\frac{d^3R''}{dxdy^2}\right)\\&\qquad\qquad-\left(\frac{d^3R'''}{dy^3}\right)\end{aligned}$$

from which the nature of the surface provided here with this character may be expressed. Moreover the constants entering into the double integration thus must be determined, so that the remaining parts of the variation may be satisfied.

SCHOLIUM 2

174. So that this investigation in itself very abstruse may be illustrated by an example, we may put in place a surface of this kind that ought to be investigated, which among all the same volumes included shall be a minimum. Hence in the end it is required to be brought about , so that the twofold formula of the integral emerges a maximum or a minimum [This equation is developed by Euler elsewhere, see *Methodus inveniendi lineas curvas*, ch. V.] :

$$\iint dxdy\left(z+a\sqrt{(1+pp+p'\,p')}\right).$$

Therefore since there shall be

$$V=z+a\sqrt{(1+pp+p'\,p')}\,,$$

there will be

$$L=0,\ M=0,\ N=1$$

and

$$P=\frac{ap}{\sqrt{(1+pp+p'\,p')}}\ \text{ and }\ P'=\frac{ap'}{\sqrt{(1+pp+p'\,p')}}$$

and thus

$$dV=Ndz+Pdp+P'\,dp'$$

with arising

$$dz=pdx+p'\,dy.$$

Where the nature of the surface sought may be expressed by this equation

$$N-\left(\tfrac{dP}{dx}\right)-\left(\tfrac{dP'}{dy}\right)=0\ \text{ or }\ 1=\left(\tfrac{dP}{dx}\right)+\left(\tfrac{dP'}{dy}\right).$$

Indeed there is

$$\left(\tfrac{dP}{dx}\right)=\frac{a}{(1+pp+p'\,p')^{\frac{3}{2}}}\left((1+p'\,p')\left(\tfrac{dp}{dx}\right)-pp'\left(\tfrac{dp'}{dx}\right)\right),$$

$$\left(\tfrac{dP'}{dy}\right)=\frac{a}{(1+pp+p'\,p')^{\frac{3}{2}}}\left((1+pp)\left(\tfrac{dp'}{dx}\right)-pp'\left(\tfrac{dp}{dx}\right)\right),$$

where it may be noted $\left(\tfrac{dp}{dy}\right)=\left(\tfrac{dp'}{dx}\right)$. From which this equation may be obtained :

$$\frac{(1+pp+p'\,p')^{\frac{3}{2}}}{a}=(1+p'\,p')\left(\tfrac{dp}{dx}\right)-2pp'\left(\tfrac{dp}{dy}\right)+(1+pp)\left(\tfrac{dp'}{dy}\right);$$

but as it is required to be treated in this manner, it may not be apparent, even if it may be easily understood, the equation for the surface from the sphere $zz=cc-xx-yy$ be contained in that, so also the cylinder $zz=\text{cc}-yy$ is to be contained.

CAPUT VII

DE VARIATIONE FORMULARUM INTEGRALIUM TRES VARIABILES INVOLVENTIUM QUARUM UNA UT FUNCTIO BINARUM RELIQUARUM SPECTATUR

PROBLEMA 19

159. *Formularum integralium huc pertinentium naturam evolvere ac rationem, qua earum variationes investigari conveniat, explicare.*

SOLUTIO

Cum tres habeantur variabiles x, y et z, quarum una z ut functio binarum reliquarum x et y est spectanda, etiamsi in ipsa variationis investigatione ratio huius functionis pro incognita haberi debet, formulae integrales, quae in hoc calculi genere occurrunt, plurimum discrepant ab iis, quae circa binas tantum variabiles proponi solent. Quemadmodum enim talis forma integralis $\int Vdx$, ubi V duas variabiles x et y implicare censetur, quarum y ab x pendere concipitur, quasi summa omnium valorum elementarium Vdx per omnes valores ipsius x collectorum considerari potest, ita, quando tres variabiles x, y et z habentur, quarum haec z a binis x et y simul pendere concipitur, integralia huc pertinentia collectionem omnium elementorum ad omnes valores tam ipsius x quam ipsius y relatorum involvunt ideoque duplicem integrationem, alteram per omnes valores ipsius x, alteram vero ipsius y elementa congregantem requirunt. Ex quo huiusmodi integralia tali forma $\iint Vdxdy$ contineri debent, qua scilicet duplex integratio innuatur, cuius evolutio ita institui solet, ut primo altera variabilis y ut constans spectetur et formulae $\int Vdx$ valor per terminos integrationis extensus quaeratur; in quo cum iam x obtineat valorem vel datum vel ab y pendentem, hoc integrale $\int Vdx$ in functionem ipsius y tantum abibit, qua in dy ducta superest, ut integrale $\int dy\int Vdx$ investigetur; quae ergo forma $\int dy\int Vdx$ hoc modo tractata illi $\iint Vdxdy$ aequivalere est censenda. Ac si ordine inverso primo quantitas x constans accipiatur et integrale $\int Vdy$ per terminos praescriptos extendatur, id deinceps ut functio ipsius x spectari et integrale quaesitum $\int dx\int Vdy$ inveniri poterit. Perinde autem est, utro modo valorem integralis formulae duplicatae $\iint Vdxdy$ evolvendi utamur.

Cum igitur in hoc genere aliae formulae integrales nisi huiusmodi $\iint Vdxdy$ occurrere nequeant, totum negotium huc redit, ut, quemadmodum huiusmodi formae variationem inveniri oporteat, ostendamus. Quoniam autem quantitates x et y variationis expertes assumimus, ex iis, quae initio sunt demonstrata [§ 75], facile colligitur fore

$$\delta\iint Vdxdy = \iint \delta Vdxdy,$$

ubi δV variationem ipsius V denotat; hicque integratione pariter duplici est opus, prorsus ut modo ante innuimus.

COROLLARIUM 1

160. Si ponamus integrale $\iint Vdxdy = W$ cum sit $\int dx\int Vdy = W$ erit per solam x differentiando $\int Vdy = \left(\frac{dW}{dx}\right)$ hincque porro per y differentiando $V = \left(\frac{ddW}{dydx}\right)$, unde patet integrale W ita comparatunl esse, ut fiat $V = \left(\frac{ddW}{dxdy}\right)$.

COROLLARIUM 2

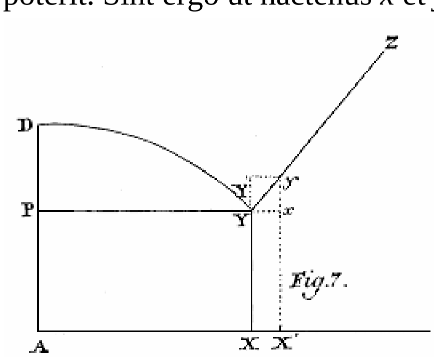
161. Cum duplex integratio sit instituenda, utraque quantitas arbitraria introducitur; altera autem integratio loco constantis functionem quamcunque ipsius x, quae sit X, altera autem functionem quamcunque ipsius y, quae sit Y invehit, ita ut completum integrale sit $\iint Vdxdy = W + X + Y$.

COROLLARIUM 3

162. Hoc etiam per ipsam resolutionem confirmatur; fit enim primo $\int Vdy = \left(\frac{dW}{dx}\right) + \left(\frac{dX}{dx}\right)$ ob $\left(\frac{dY}{dx}\right) = 0$, tum vero fit $V = \left(\frac{ddW}{dxdy}\right)$, quia neque X neque $\frac{dX}{dx}$ ab y pendet. Quare si fuerit $\left(\frac{ddW}{dxdy}\right) = V$, erit integrale completum $\iint Vdxdy = W + X + Y$.

SCHOLION 1

163. Omnino autem necessarium est, ut indoles huiusmodi formularum integralium duplicatarum accuratius examini subiiciatur, quod commodissime per theoriam superficierum praestari poterit. Sint ergo ut hactenus x et y binae coordinatae orthogonales in basi assumtae $AX = x$, $XY = y$ (Fig. 7) 1), cui in Y normaliter insistat tertia ordinata $YZ = z$ ad superficiem usque porrecta. Si iam binae illae coordinatae x et y suis differentialibus crescant $XX' = dx$ et $YY' = dy$, inde in basi oritur parallelogrammum elementare $YxyY' = dxdy$, cui elementum formulae integralis convenit. Ita si de soliditate a superficie inclusa sit quaestio, eius elementum erit $= zdxdy$ ideoque tota soliditas $= \int zdxdy$; si superficies ipsa quaeratur, posito $dz = pdx + p'dy$ erit eius elementum huic rectangulo $dxdy$ imminens



$= dxdy\sqrt{(1+pp+p'p')}$ ideoque ipsa superficies $=\int dxdy\sqrt{(1+pp+p'p')}$, ex quo generatim intelligitur ratio formulae integralis duplicatae $\iint Vdxdy$. Quodsi iam talis formulae valor quaeratur, qui dato spatio in basi veluti *ADYX* respondeat, primo sumta *x* constante investigetur integrale simplex $\int Vdy$ ac tum ipsi *y* assignetur magnitudo *XY* ad curvam *DY* porrecta, quae ex huius curvae natura aequabitur certae functioni ipsius *x*. Sic igitur $dx\int Vdy$ exprimet formulae propositae elementum rectangulo $XYxX' = ydx$ conveniens, cuius integrale denuo sumtum $\int dx\int Vdy$ et ex sola variabili *x* constans tandem dabit valorem toti spatio *ADYX* respondentem, siquidem utraque integratio adiectione constantis rite determinetur.

SCHOLION 2

164. Ita se habere debet evolutio huiusmodi formularum integralium duplicatarum, si ad figuram in basi datam veluti *AD*Y*X* fuerit accommodanda; sin autem utramque integrationem indefinite expedire velimus, ut primo sumta *x* constante quaeramus integrale $\int Vdy$, quod rectangulo elementari $XYxX' = ydx$ convenire est intelligendum, siquidem in *dx* ducatur, deinde vero in integratione formulae $\int dx\int Vdy$ quantitatem $y = XY$ eandem manere concipiamus sola *x* pro variabili sumta, tum valor prodibit rectangulo indefinito $APYX = xy$ respondens, siquidem constantes per utramque integrationem ingressae debite definiantur. At si spatii istius reliqui termini praeter lineas *XY* et *PY* ut indefiniti spectentur, integrale $\iint Vdxdy$ recipiet binas functiones $X+Y$ indefinitas, illam ipsius *x*, hanc vero ipsius *y*. Quodsi ergo ad calculum maximorum et minimorum haec deinceps accommodare velimus, quoniam maximi minimive proprietas, quae in spatium quodpiam datum *ADYX* competere debet, simul quoque cuivis spatio indefinito veluti *APYX* conveniat necesse est, duplicem illam integrationem modo hic exposito indefinito administrari conveniet.

PROBLEMA 20

165. *Si V sit formula quaecunque ex ternis variabilibus x, y, z earumque differentialibus composita, invenire variationem formulae integralis duplicatae* $\iint Vdxdy$ *, dum quantitati z, quae ut functio binarum x et y spectatur, variationes quaecunque tribuuntur.*

SOLUTIO

Ad speciem differentialium tollendam statuamus

$$p=\left(\tfrac{dz}{dx}\right),\quad p'=\left(\tfrac{dz}{dy}\right),$$
$$q=\left(\tfrac{dp}{dx}\right),\quad q'=\left(\tfrac{dp}{dy}\right)=\left(\tfrac{dp'}{dx}\right),\quad q''=\left(\tfrac{dp'}{dy}\right),$$
$$r=\left(\tfrac{dq}{dx}\right),\quad r'=\left(\tfrac{dq}{dy}\right)=\left(\tfrac{dq'}{dx}\right),\quad r''=\left(\tfrac{dq'}{dy}\right)=\left(\tfrac{dq''}{dx}\right),\quad r'''=\left(\tfrac{dq''}{dy}\right)$$
$$\text{etc.},$$

ut V fiat functio quantitatum finitarum $x, y, z, p, p', q, q', q'', r, r', r'', r'''$ etc. Tum ponitur eius differentiale

$$\begin{aligned} dV = Ldx + Mdy + Ndz + Pdp &+ Qdq + Rdr \\ &+ P'dp' + Q'dq' + R'dr' \\ &+ Q''dq'' + R''dr'' \\ &+ R'''dr''' \quad \text{etc.} \end{aligned}$$

ex quo cum simul eius variatio δV innotescat, ex problemate praecedente colligitur variatio quaesita

$$\delta\iint Vdxdy = \iint dxdy\left\{\begin{aligned} &N\delta z + P\delta p + Q\delta q + R\delta r \\ &\quad + P'\delta p' + Q'\delta q' + R'\delta r' \\ &\qquad + Q''\delta q'' + R''\delta r'' \\ &\qquad\quad + R'''\delta r''' \\ &\qquad\quad \text{etc.} \end{aligned}\right\}$$

Quodsi iam, uti § 154 fecimus, ponamus variationem $\delta z=\omega$, quam ut functionem quamcunque binarum variabilium x et y spectare licet, indidem istam variationem concludimus fore

$$\delta\iint Vdxdy = \iint dxdy\left\{\begin{array}{l} N\omega + P\left(\frac{d\omega}{dx}\right) + Q\left(\frac{dd\omega}{dx^2}\right) + R\left(\frac{d^3\omega}{dx^3}\right) \\ \qquad + P'\left(\frac{d\omega}{dy}\right) + Q'\left(\frac{dd\omega}{dxdy}\right) + R'\left(\frac{d^3\omega}{dx^2dy}\right) \\ \qquad\qquad + Q''\left(\frac{dd\omega}{dy^2}\right) + R''\left(\frac{d^3\omega}{dxdy^2}\right) \\ \qquad\qquad\qquad + R'''\left(\frac{d^3\omega}{dy^3}\right) \\ \qquad\qquad \text{etc.,} \end{array}\right\}$$

COROLLARIUM 1

166. Si ergo utriusque functionis *z* et $\delta z = \omega$ indoles seu ratio compositionis ex binis variabilibus *x* et *y* esset data, tum per praecepta ante exposita variatio formulae integralis duplicatae $\iint Vdxdy$ assignari posset, quomodocunque quantitas *V* ex variabilibus *x*, *y*, *z* earumque differentialibus fuerit conflata.

COROLLARIUM 2

167. Totum scilicet negotium redibit ad evolutionem formulae integralis duplicatae inventae; quae cum pluribus constet partibus, singulas partes per duplicem integrationem, uti ante explicatum, tractari conveniet.

SCHOLION

168. Quando autem ratio functionis *z* non constat aaque demum ex conditione variationis elici debet, ita ut ipsa variatio $\delta z = \omega$ nullam plane determinationem patiatur, quemadmodum fit, si formula $\iint Vdxdy$ valorem maximum minimumve obtinere debeat, tum omnino necessarium est, ut singula variationis inventae $\delta\iint Vdxdy$ membra ita reducantur, ut ubique post signum integrationis duplicatum non valores differentiales variationis $\delta z = \omega$, sed haec ipsa variatio occurrat; cuiusmodi reductione iam supra in formulis binas tantum variabiles involventibus sumus usi. Talis autem reductio, cum pro formulis integralibus duplicatis minus sit consueta, accuratiorem explicationem postulat. Quem in finem observo huiusmodi reductione perveniri ad formulas simpliciter integrales, in quibus alterutra tantum quantitatum *x* et *y* pro variabili habeatur altera ut constante spectata, ad quod indicandum, ne signa praeter necessitatem multiplicentur, talis forma $\int Tdx$ denotabit integrale formulae differentialis *Tdx,* dum quantitas *y* pro constanti habetur; similique modo intelligendum est in hac forma $\int Tdy$ solam quantitatem *y* ut variabilem considerari, quod eo magis perspicuum est, cum hac conditione omissa hae formulae nullum plane significatum essent habiturae. Neque ergo in posterum opus est declarari, si *T* ambas variabiles *x* et *y* complectatur, utra earum in formulis integralibus simplicibus $\int Tdx$ vel $\int Tdy$ sive constans sive variabilis accipiatur, cum ea sola, cuius differentiale exprimitur, pro variabili sit habenda. In formulis autem duplicatis $\iint Vdxdy$ perpetuo tenendum est alteram integrationem ad solius *x*, alteram vero ad solius *y* variabilitatem adstringi perindeque esse, utra integratio prior instituatur.

PROBLEMA 21

169. *Variationem formulae integralis duplicatae* $\iint Vdxdy$ *in praecedente problemate inventam ita transformare, ut post signum integrale duplicatum ubique ipsa variatio* $\delta z = \omega$ *occurrat exturbatis eius differentialibus.*

SOLUTIO

Quo haec transformatio latius pateat, sint *T* et *v* functiones quaecunque binarum variabilium *x* et *y* et consideretur haec formula duplicata $\iint Tdxdy\left(\frac{dv}{dx}\right)$, quae separata integrationum varietate ita repraesentetur $\int dy\int Tdx\left(\frac{dv}{dx}\right)$, ut in integratione $\int Tdx\left(\frac{dv}{dx}\right)$ sola quantitas *x* ut variabilis spectetur. Tum autem erit $dx\left(\frac{dv}{dx}\right) = dv$, quia *y* pro constante habetur, ideoque fiet

$$\int Tdv = Tv - \int vdT\ ;$$

ubi cum in differentiali *dT* solius variabilis *x* ratio habetur, ad hoc declarandum loco *dT* scribi convenit $dx\left(\frac{dT}{dx}\right)$, ita ut sit

$$\int Tdx\left(\frac{dv}{dx}\right) = Tv - \int vdx\left(\frac{dT}{dx}\right)$$

hincque nostra formula ita prodeat reducta

$$\iint Tdxdy\left(\frac{dv}{dx}\right) = \int Tvdy - \iint vdxdy\left(\frac{dT}{dx}\right).$$

Simili modo permutatis variabilibus consequemur

$$\iint Tdxdy\left(\frac{dv}{dy}\right) = \int Tvdx - \iint vdxdy\left(\frac{dT}{dy}\right).$$

Hoc iam quasi lemmate praemisso variationis in praecedente problemate inventae reductio ita se habebit

$$\iint Pdxdy\left(\frac{d\omega}{dx}\right)=\int P\omega dy-\iint \omega dxdy\left(\frac{dP}{dx}\right),$$
$$\iint P'dxdy\left(\frac{d\omega}{dy}\right)=\int P'\omega dx-\iint \omega dxdy\left(\frac{dP'}{dy}\right).$$

Porro pro sequentibus membris sit primo $\left(\frac{d\omega}{dx}\right)=v$ ideoque $\left(\frac{dd\omega}{dx^2}\right)=\left(\frac{dv}{dx}\right)$, unde colligitur

$$\iint Qdxdy\left(\frac{dd\omega}{dx^2}\right)=\int Qdy\left(\frac{d\omega}{dx}\right)-\iint dxdy\left(\frac{dQ}{dx}\right)\left(\frac{d\omega}{dx}\right),$$

ac postremo membro similiter reducto fit

$$\iint Qdxdy\left(\frac{dd\omega}{dx^2}\right)=\int Qdy\left(\frac{d\omega}{dx}\right)-\int \omega dy\left(\frac{dQ}{dx}\right)+\iint \omega dxdy\left(\frac{ddQ}{dx^2}\right).$$

Per eandem substitutionem habebimus $\left(\frac{dd\omega}{dxdy}\right)=\left(\frac{dv}{dy}\right)$ hincque

$$\iint Q'dxdy\left(\frac{dd\omega}{dxdy}\right)=\int Q'dx\left(\frac{d\omega}{dx}\right)-\iint dxdy\left(\frac{d\omega}{dx}\right)\left(\frac{dQ'}{dy}\right).$$

seu

$$\iint Q'dxdy\left(\frac{dd\omega}{dxdy}\right)=\int Q'dx\left(\frac{d\omega}{dx}\right)-\int \omega dy\left(\frac{dQ'}{dy}\right)+\iint \omega dxdy\left(\frac{ddQ'}{dxdy}\right),$$

quae forma ob

$$\int Q'dx\left(\frac{d\omega}{dx}\right)=Q'\omega-\int \omega dx\left(\frac{dQ'}{dx}\right)$$

abit in hanc

$$\iint Q'dxdy\left(\frac{dd\omega}{dxdy}\right)=Q'\omega-\int \omega dx\left(\frac{dQ'}{dx}\right)+\iint \omega dxdy\left(\frac{ddQ'}{dxdy}\right)-\int \omega dy\left(\frac{dQ'}{dy}\right).$$

Tum vero pro tertia forma huius ordinis nanciscimur

$$\iint Q''dxdy\left(\frac{dd\omega}{dy^2}\right)=\int Q''dx\left(\frac{d\omega}{dy}\right)-\int \omega dx\left(\frac{dQ''}{dy}\right)+\iint \omega dxdy\left(\frac{ddQ''}{dy^2}\right).$$

Porro ob $\left(\frac{d^3\omega}{dx^3}\right)=\left(\frac{ddv}{dx^2}\right)$ manente $v=\left(\frac{d\omega}{dx}\right)$ fiet

$$\iint Rdxdy\left(\frac{ddv}{dx^2}\right)=\int Rdy\left(\frac{dv}{dx}\right)-\int vdy\left(\frac{dR}{dx}\right)+\iint vdxdy\left(\frac{ddR}{dx^2}\right)$$

et

$$\iint vdxdy\left(\frac{ddR}{dx^2}\right)=\int \omega dy\left(\frac{ddR}{dx^2}\right)-\iint \omega dxdy\left(\frac{d^3R}{dx^3}\right),$$

ita ut sit

$$\iint Rdxdy\left(\frac{d^3v}{dx^3}\right)=\int Rdy\left(\frac{dd\omega}{dx^2}\right)-\int dy\left(\frac{d\omega}{dx}\right)\left(\frac{dR}{dx}\right)+\int \omega dy\left(\frac{ddR}{dx^2}\right)-\iint \omega dxdy\left(\frac{d^3R}{dx^3}\right).$$

Deinde ob $\left(\frac{d^3\omega}{dx^2dy}\right)=\left(\frac{ddv}{dxdy}\right)$ erit

$$\iint R'dxdy\left(\frac{ddv}{dxdy}\right)=R'v-\int vdx\left(\frac{dR'}{dx}\right)+\iint vdxdy\left(\frac{ddR'}{dxdy}\right)-\int vdy\left(\frac{dR'}{dy}\right),$$

et quia hic

$$\iint vdxdy\left(\frac{ddR'}{dxdy}\right)=\int \omega dy\left(\frac{ddR'}{dxdy}\right)-\iint \omega dxdy\left(\frac{d^3R'}{dx^2dy}\right),$$

concludimus fore

$$\iint R'dxdy\left(\frac{d^3\omega}{dx^2dy}\right)=R'\left(\frac{d\omega}{dx}\right)-\int\left(\frac{d\omega}{dx}\right)dx\left(\frac{dR'}{dx}\right)+\int \omega dy\left(\frac{ddR'}{dxdy}\right)-\int\left(\frac{d\omega}{dx}\right)dy\left(\frac{dR'}{dy}\right)-\iint \omega dxdy\left(\frac{d^3R'}{dx^2dy}\right).$$

Tandem permutandis x et y hinc colligimus

$$\iint R''dxdy\left(\frac{d^3\omega}{dxdy^2}\right)=R''\left(\frac{d\omega}{dy}\right)-\int\left(\frac{d\omega}{dy}\right)dy\left(\frac{dR''}{dy}\right)+\int \omega dx\left(\frac{ddR''}{dxdy}\right)-\int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR''}{dx}\right)-\iint \omega dxdy\left(\frac{d^3R''}{dxdy^2}\right).$$

et

$$\iint R'''dxdy\left(\frac{d^3\omega}{dy^3}\right)=\int R'''dx\left(\frac{dd\omega}{dy^2}\right)-\int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR'''}{dy}\right)+\int \omega dx\left(\frac{ddR'''}{dy^2}\right)-\iint \omega dxdy\left(\frac{d^3R'''}{dy^3}\right).$$

Quos valores si substituamus, reperimus

$$\delta\iint Vdxdy = \iint \omega dxdy \left\{ \begin{array}{l} N-\left(\frac{dP}{dx}\right) +\left(\frac{ddQ}{dx^2}\right) -\left(\frac{d^3R}{dx^3}\right) \\ \quad -\left(\frac{dP'}{dy}\right) +\left(\frac{ddQ'}{dxdy}\right) -\left(\frac{d^3R'}{dx^2dy}\right) \\ \qquad + \left(\frac{ddQ''}{dy^2}\right) -\left(\frac{d^3R''}{dxdy^2}\right) \\ \qquad\qquad -\left(\frac{d^3R'''}{dy^3}\right) \end{array} \right\}$$

$$+\int P\omega dy+\int P'\omega dx+\int Qdy\left(\frac{d\omega}{dx}\right)-\int \omega dx\left(\frac{dQ'}{dx}\right)+Q'\omega+\int Q''dx\left(\frac{d\omega}{dy}\right)$$
$$-\int \omega dy\left(\frac{dQ}{dx}\right)-\int \omega dy\left(\frac{dQ'}{dy}\right) \qquad -\int \omega dx\left(\frac{dQ''}{dy}\right)$$
$$+\int Rdy\left(\frac{dd\omega}{dx^2}\right)-\int\left(\frac{d\omega}{dx}\right)dx\left(\frac{dR'}{dx}\right)+R'\left(\frac{d\omega}{dx}\right)-\int\left(\frac{d\omega}{dy}\right)dy\left(\frac{dR''}{dy}\right)+\int R'''dx\left(\frac{dd\omega}{dy^2}\right)$$
$$-\int\left(\frac{d\omega}{dx}\right)dy\left(\frac{dR}{dx}\right)-\int\left(\frac{d\omega}{dx}\right)dy\left(\frac{dR'}{dy}\right) \qquad -\int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR''}{dx}\right)-\int\left(\frac{d\omega}{dy}\right)dx\left(\frac{dR'''}{dy}\right)$$
$$+\int \omega dy\left(\frac{ddR}{dx^2}\right) \quad +\int \omega dy\left(\frac{ddR'}{dxdy}\right)+R''\left(\frac{d\omega}{dy}\right)+\int \omega dx\left(\frac{ddR''}{dxdy}\right) \quad +\int \omega dx\left(\frac{ddR'''}{dy^2}\right)$$

etc.

COROLLARIUM 1

170. Huius expressionis pars prima satis est perspicua, reliquae vero partes commode ita disponi possunt, ut earum ratio comprehendatur:

$$\int \omega dy \left\{ \begin{array}{l} P-\left(\frac{dQ}{dx}\right) +\left(\frac{ddR}{dx^2}\right) -\text{etc.} \\ \quad -\left(\frac{dQ'}{dy}\right)+\left(\frac{ddR'}{dxdy}\right) \\ \qquad +\left(\frac{d^3R''}{dy^3}\right) \end{array} \right\} + \int \omega dx \left\{ \begin{array}{l} P'-\left(\frac{dQ''}{dy}\right) +\left(\frac{ddR'''}{dy^2}\right)-\text{etc.} \\ \quad -\left(\frac{dQ'}{dx}\right) +\left(\frac{ddR''}{dxdy}\right) \\ \qquad +\left(\frac{ddR'}{dx^2}\right) \end{array} \right\}$$

$$+\int\left(\frac{d\omega}{dx}\right)dy \left\{ \begin{array}{l} \mathrm{Q}-\left(\frac{dR}{dx}\right) +\text{etc.} \\ \quad -\left(\frac{dR'}{dy}\right) \end{array} \right\} + \int\left(\frac{d\omega}{dy}\right)dx \left\{ \begin{array}{l} Q''-\left(\frac{dR''}{dy}\right)+\text{etc.} \\ \quad -\left(\frac{dR''}{dx}\right) \end{array} \right\}$$

$$+\int\left(\frac{dd\omega}{dx^2}\right)dy\left(R-\text{etc.}\right)+\int\left(\frac{dd\omega}{dy^2}\right)dx\left(R'''-\text{etc.}\right)+\text{etc.}$$

$$+\omega \left\{ \begin{array}{l} Q'-\left(\frac{dR'}{dx}\right) +\text{etc.} \\ \quad -\left(\frac{dR''}{dy}\right) \end{array} \right\} + \left(\frac{d\omega}{dx}\right)\left(R'-\text{etc.}\right)+\left(\frac{d\omega}{dy}\right)\left(R''-\text{etc.}\right)+\text{etc.}$$

COROLLARIUM 2

171. Hic levi attentione adhibita mox patebit, quomodo istae partes ulterius continuari debeant, si forte quantitas V differentialia altiorum graduum complectatur.

COROLLARIUM 3

172. In harum formularum integralium aliis,quae differentiali dy sunt affectae, quantitas x constans sumitur, cui tribuitur valor termino integrationis conveniens; aliis vero, quae differentiali dx sunt affectae, y est constans et termino integrationis aequalis, unde patet in terminis integrationum tam x quam y recipere valorem constantem.

SCHOLION 1

173. Haec ergo variationis formula ad eum casum est accommodata, quo utriusque integrationis termini tribuunt tam ipsi x quam ipsi y valores constantes. Veluti si de superficie fuerit quaestio, formula integralis $\iint Vdxdy$ ad rectangulum $APYX$ (Fig. 7) in basi assumtum est referenda eiusque valor ita definiri debet, ut sumtis $x = 0$ et $y = 0$, qui sunt valores initiales, evanescat, quo facto statui oportet $x = AX$ et $y = AP$, qui sunt valores finales; atque ad eandem legem ipsa variatio inventa est expedienda. Quodsi iam ea quaeratur superficies, in qua formulae $\iint Vdxdy$ hoc modo definitae valor fiat maximus vel minimus, ante omnia necesse est, ut pars variationis prima duplicem integrationem involvens ad nihilum redigatur, quomodocunque variatio $d'z = \omega$ accipiatur, unde haec nascetur aequatio

$$\begin{aligned}0 = N &- \left(\frac{dP}{dx}\right) + \left(\frac{ddQ}{dx^2}\right) - \left(\frac{d^3R}{dx^3}\right) + \text{etc.},\\ &- \left(\frac{dP'}{dy}\right) + \left(\frac{ddQ'}{dxdy}\right) - \left(\frac{d^3R'}{dx^2dy}\right)\\ &\qquad + \left(\frac{ddQ''}{dy^2}\right) - \left(\frac{d^3R''}{dxdy^2}\right)\\ &\qquad\qquad - \left(\frac{d^3R'''}{dy^3}\right)\end{aligned}$$

qua natura superficiei hac indole praeditae exprimetur. Constantes autem per duplicem integrationem ingressae ita determinari debent, ut reliquis variationis partibus satisfiat.

SCHOLION 2

174. Quo haec jnvestigatio in se maxime abstrusa exemplo illustretur, ponamus eiusmodi superficiem investigari debere, quae inter omnes alias eandem soliditatem includentes sit minima. Hunc in finem efficiendum est, ut haec formula integralis duplicata

$$\iint dxdy\left(z + a\sqrt{(1 + pp + p'p')}\right)$$

maximum minimumve evadat. Cum ergo sit

$$V = z + a\sqrt{(1 + pp + p'p')}\,,$$

erit

$$L = 0,\ M = 0,\ N = 1$$

atque

$$P = \frac{ap}{\sqrt{(1+pp+p'p')}} \text{ et } P' = \frac{ap'}{\sqrt{(1+pp+p'p')}}$$

ideoque

$$dV = Ndz + Pdp + P'dp'$$

existente

$$dz = pdx + p'dy.$$

Quare superficiei quaesitae natura hac aequatione exprimetur

$$N - \left(\frac{dP}{dx}\right) - \left(\frac{dP'}{dy}\right) = 0 \text{ seu } 1 = \left(\frac{dP}{dx}\right) + \left(\frac{dP'}{dy}\right).$$

Est vero

$$\left(\frac{dP}{dx}\right) = \frac{a}{(1+pp+p'p')^{\frac{3}{2}}}\left((1+p'p')\left(\frac{dp}{dx}\right) - pp'\left(\frac{dp'}{dx}\right)\right),$$

$$\left(\frac{dP'}{dy}\right) = \frac{a}{(1+pp+p'p')^{\frac{3}{2}}}\left((1+pp)\left(\frac{dp'}{dx}\right) - pp'\left(\frac{dp}{dx}\right)\right),$$

ubi notetur esse $\left(\frac{dp}{dy}\right) = \left(\frac{dp'}{dx}\right)$. Ex quo ista obtinetur aequatio

$$\frac{(1+pp+p'p')^{\frac{3}{2}}}{a} = (1+p'p')\left(\frac{dp}{dx}\right) - 2pp'\left(\frac{dp}{dy}\right) + (1+pp)\left(\frac{dp'}{dy}\right);$$

quam autem quomodo tractari oporteat, haud patet, etiamsi facile perspiciatur in ea aequationem pro superficie sphaerica $zz = cc - xx - yy$. quin etiam cylindrica $zz = cc - yy$ contineri.