CHAPTER V

CONCERNING THE VARIATION OF INTEGRAL FORMULAS INVOLVING THREE VARIABLES AND INCLUDING TWO RELATIONS

PROBLEM 12

123. *With any proposed formula involving the three variables x, y, z with the differential of this of any order, to define the variation of this from the variations arising from all three variables.*

SOLUTION

Let *W* be this proposed formula, of which initially the value of the variation $W+\delta W$ is sought which arises, in place of *x*, *y*, *z* there may be written the values of these to be varied

$$x+\delta x,\ y+\delta y,\ z+\delta z$$

and thus also likewise for the differential of these

$$dx+d\delta x,\ dy+d\delta y,\ dz+d\delta z\ ;$$

from which if the formula *W* itself may be taken away, there will remain the variation of this δW. From which it is understood that this variation is to be obtained by the usual differentiation, if in place of the sign of differentiation *d* there may be written the sign of the variation δ. Only it will be helpful to note, if it shall be required to take the variations of differentials, likewise the sign of the variation δ may be put in place anywhere among the differentiation signs, just as we have shown above [§ 37, 40]; from which the variation sign can always be put in the final place, because, when they progress to integral formulas, it may be seen most conveniently, so that thus it is clear enough with these, which have been treated at this stage concerning integral formulas involving two variables.

COROLLARY 1

124. Because *z* and *y* likewise can be considered as a function of *x*, if there is put $\frac{dy}{dx}=p$ and $\frac{dz}{dx}=\mathfrak{p}$, there will be

$$\delta p=\frac{d\delta y-pd\delta x}{dx} \quad\text{and}\quad \delta\mathfrak{p}=\frac{d\delta z-\mathfrak{p}d\delta x}{dx}$$

and in a similar manner hence the derived formulas do not disagree with those above.

COROLLARY 2

125. We may put $\delta y - p\delta x = \omega$ and $\delta z - \mathfrak{p}\delta x = \mathfrak{w}$ and there will be

$$d\delta y - pd\delta x - qdx\delta x = d\omega \quad \text{and} \quad d\delta z - \mathfrak{p}d\delta x - \mathfrak{q}dx\delta x = d\mathfrak{w},$$

if Cleary we put in place

$$\tfrac{dp}{dx} = q \text{ and } \tfrac{d\mathfrak{p}}{dx} = \mathfrak{q},$$

from which it is apparent

$$\delta p - q\delta x = \tfrac{d\omega}{dx} \quad \text{and} \quad \delta\mathfrak{p} - \mathfrak{q}\delta x = \tfrac{d\mathfrak{w}}{dx}$$

COROLLARY 3

126. If further we may put in place

$$\tfrac{dq}{dx} = r, \ \tfrac{d\mathfrak{q}}{dx} = \mathfrak{r}, \ \tfrac{dr}{dx} = s, \ \tfrac{d\mathfrak{r}}{dx} = \mathfrak{s} \quad \text{etc.,}$$

in a similar manner there will be on taking *dx* constant

$$\delta q - r\delta x = \tfrac{dd\omega}{dx^2}, \ \delta\mathfrak{q} - \mathfrak{r}\delta x = \tfrac{dd\mathfrak{w}}{dx^2},$$
$$\delta r - s\delta x = \tfrac{d^3\omega}{dx^3}, \ \delta\mathfrak{r} - \mathfrak{s}\delta x = \tfrac{d^3\mathfrak{w}}{dx^3}$$

and thus henceforth.

SCHOLIUM 1

127. Therefore if the formula to be varied should have either a finite or infinite value or should vanish, with the aid of these precepts the variation of this and likewise the above can be found ; nor indeed do these precepts disagree with the above, except that here with differential values of two kinds, they may be introduced on the one hand by Latin letters *p, q, r, s* etc., and on the other indicated by Gothic letters $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}$ etc. ; an account of which matter has been put in place there above, because here each variable y and z can be regarded as a function of *x*. But since a single equation may be given or sought between the three coordinates, the letters *p* and $\mathfrak{p}$ introduced here may not have certain values, since with these valid equations the fractions $\tfrac{dy}{dx}$ and $\tfrac{dz}{dx}$ generally may take all values. But with these letters and with their differentials in the calculation set aside, in this case also, the rule set out in the variation solution will be indicated.

SCHOLIUM 2

128. Now above [§12-24] I have noted this case of three variables, the relation of which is defined by a twin equation, carefully to be distinguished from that, where the relation is assumed to be defined by a single equation. This distinction is most clearly shown from geometry, where the three variables in turn give rise to three coordinates ; moreover it is required to use an equivalent number in the calculation, not only when a question concerning a surface is considered, but also when curved lines are to be investigated which are not in the same plane. And indeed in this last case the determination of curved lines demands two equations between the three coordinates, thus so that any two can be regarded as functions of the third. But the nature of a surface is defined by a single equation between the three coordinates, thus so that it can be regarded as a single function of two variables, from which a huge distinction in the treatment from that other case arises. Therefore the present chapter will be concerned with investigating curved lines of this kind, which not being in the same plane enjoy a certain property of maxima or minima.

PROBLEM 13

129. *If V should be some function of the three variables x, y, z, the differentials of these above of any order being included, and these undertake some variations, to find the variation of the integral formula* $\int V dx$.

SOLUTION

Whichever differentials may arise in the function V, those may be removed from these substitutions

$$dy = pdx,\ dp = qdx,\ dq = rdr,\ dr = sdx \text{ etc.},$$
$$dz = \mathfrak{p}dx,\ d\mathfrak{p} = \mathfrak{q}dx,\ d\mathfrak{q} = \mathfrak{r}dx,\ d\mathfrak{r} = \mathfrak{s}dx \text{ etc.}$$

and the quantity V will be a function of the finite quantities x, y, z, p, q, r, s etc., $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}$ etc. Therefore the differential of this will have a form of this kind

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + \text{etc.}$$
$$+ \mathfrak{N}dz + \mathfrak{P}d\mathfrak{p} + \mathfrak{Q}d\mathfrak{q} + \mathfrak{R}d\mathfrak{r} + \mathfrak{S}d\mathfrak{s} + \text{etc.},$$

from which with the change of the differentiation sign d into δ likewise the variation δV will be had. But from the above demonstration also for this case of the three variables there will be had

$$\delta \int V dx = \int (V d\delta x + dx \delta V) = V\delta x + \int (dx\delta V - dV\delta x).$$

But with the substitution made there becomes

$$\begin{aligned}\tfrac{dx\delta V-dV\delta x}{dx}=&M\delta x+N\delta y+P\delta p+Q\delta q+R\delta r+\text{etc.}\\&+\mathfrak{N}\delta z+\mathfrak{P}\delta\mathfrak{p}+\mathfrak{Q}\delta\mathfrak{q}+\mathfrak{R}\delta\mathfrak{r}+\text{etc.}\\&-M\delta x-Np\delta x-Pq\delta x-Qr\delta x-Rs\delta x-\text{etc.}\\&-\mathfrak{N}\mathfrak{p}\delta x-\mathfrak{P}\mathfrak{q}\delta x-\mathfrak{Q}\mathfrak{r}\delta x-\mathfrak{R}\mathfrak{s}\delta x-\text{etc.}\end{aligned}$$

Because if now for the sake of brevity we should put

$$\delta y-p\delta x=\omega \text{ and } \delta z-\mathfrak{p}\delta x=\mathfrak{w},$$

with the element *dx* assumed constant from §125 and §126 there will be

$$\begin{aligned}&\delta p-q\delta x=\tfrac{d\omega}{dx},\quad \delta\mathfrak{p}-\mathfrak{q}\delta x=\tfrac{d\mathfrak{w}}{dx},\\&\delta q-r\delta x=\tfrac{dd\omega}{dx^2},\quad \delta\mathfrak{q}-\mathfrak{r}\delta x=\tfrac{dd\mathfrak{w}}{dx^2},\\&\delta r-s\delta x=\tfrac{d^3\omega}{dx^3},\quad \delta\mathfrak{r}-\mathfrak{s}\delta x=\tfrac{d^3\mathfrak{w}}{dx^3},\\&\text{etc.,}\end{aligned}$$

from which the variation sought may be expressed conveniently in this manner

$$\delta\int Vdx=V\delta x+\int dx\left\{\begin{aligned}&N\omega+\tfrac{Pd\omega}{dx}+\tfrac{Qdd\omega}{dx^2}+\tfrac{Rd^3\omega}{dx^3}+\text{etc.}\\&\mathfrak{N}\mathfrak{w}+\tfrac{\mathfrak{P}d\mathfrak{w}}{dx}+\tfrac{\mathfrak{Q}dd\mathfrak{w}}{dx^2}+\tfrac{\mathfrak{R}d^3\mathfrak{w}}{dx^3}+\text{etc.}\end{aligned}\right\},$$

which as above [§ 80-85] is reduced to this form

$$\delta\int Vdx = \int \omega dx\left(N - \tfrac{dP}{dx} + \tfrac{ddQ}{dx^2} - \tfrac{d^3R}{dx^3} + \tfrac{d^4S}{dx^4} - \text{etc.}\right)$$
$$+\int \mathfrak{w} dx\left(\mathfrak{N} - \tfrac{d\mathfrak{P}}{dx} + \tfrac{dd\mathfrak{Q}}{dx^2} - \tfrac{d^3\mathfrak{R}}{dx^3} + \tfrac{d^4\mathfrak{S}}{dx^4} - \text{etc.}\right)$$
$$+V\delta x + \omega\left(P - \tfrac{dQ}{dx} + \tfrac{ddR}{dx} - \tfrac{d^3S}{dx^3} + \text{etc.}\right)$$
$$+\text{ Const.} + \mathfrak{w}\left(\mathfrak{P} - \tfrac{d\mathfrak{Q}}{dx} + \tfrac{dd\mathfrak{R}}{dx} - \tfrac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\right)$$
$$+\tfrac{d\omega}{dx}\left(Q - \tfrac{dR}{dx} + \tfrac{ddS}{dx^2} - \text{etc.}\right)$$
$$+\tfrac{d\omega}{dx}\left(\mathfrak{Q} - \tfrac{d\mathfrak{R}}{dx} + \tfrac{dd\mathfrak{S}}{dx^2} - \text{etc.}\right)$$
$$+\tfrac{dd\omega}{dx}\left(R - \tfrac{dS}{dx} + \text{etc.}\right)$$
$$+\tfrac{dd\mathfrak{w}}{dx}\left(\mathfrak{R} - \tfrac{d\mathfrak{S}}{dx} + \text{etc.}\right)$$
$$+\tfrac{d^3\omega}{dx^3}\left(S - \text{etc.}\right)$$
$$+\tfrac{d^3\mathfrak{w}}{dx^3}\left(\mathfrak{S} - \text{etc.}\right)$$
$$+\text{etc.},$$

the nature of which from the above is clear enough, and the same about the addition of the constant are to be noted [§ 115].

COROLLARY 1

130. In this solution both the variables y and z may be considered as functions of x, either now they shall be known or in the end to be defined from the nature of the variation. Nor also is it certain that the formula of the integral $\int Vdx$ should be considered to have a value, unless both y as well as z may be conceived to be determined by x.

COROLLARY 2

131. If the formula Vdx shall be integrable by itself with no relation assumed between the three variables, the variation of the integral $\int Vdx$ can involve no formulas too and thus it is necessary that there shall then be both

$$N - \tfrac{dP}{dx} + \tfrac{ddQ}{dx^2} - \tfrac{d^3R}{dx^3} + \tfrac{d^4S}{dx^4} - \text{etc.} = 0$$

and

$$\mathfrak{N} - \tfrac{d\mathfrak{P}}{dx} + \tfrac{dd\mathfrak{Q}}{dx^2} - \tfrac{d^3\mathfrak{R}}{dx^3} + \tfrac{d^4\mathfrak{S}}{dx^4} - \text{etc.} = 0.$$

COROLLARY 3

132. Also in turn if these two equations should have a place, this will be a sure criterion that the differential formula can be integrated by itself *Vdx*, with no relation established between the variables.

EXAMPLE

133. *So that we may illustrate this criterion further, we may take a formula of this kind integral by itself and let it be* $\int Vdx = \frac{zdy}{xdz} = \frac{pz}{x\mathfrak{p}}$, *from which there shall be*

$$V = \frac{-pz}{xx\mathfrak{p}} + \frac{p}{x} + \frac{zq}{x\mathfrak{p}} - \frac{zp\mathfrak{q}}{x\mathfrak{p}\mathfrak{p}}.$$

From the differentiation of which we may deduce

$$N = 0 \quad \text{and} \quad P = \frac{-z}{xx\mathfrak{p}} + \frac{1}{x} - \frac{z\mathfrak{q}}{x\mathfrak{p}\mathfrak{p}}, \quad Q = \frac{z}{x\mathfrak{p}},$$

again

$$\mathfrak{N} = \frac{-p}{xx\mathfrak{p}} + \frac{q}{x\mathfrak{p}} - \frac{p\mathfrak{q}}{x\mathfrak{p}\mathfrak{p}}, \quad \mathfrak{P} = \frac{pz}{xx\mathfrak{p}\mathfrak{p}} - \frac{zq}{x\mathfrak{p}\mathfrak{p}} + \frac{2zp\mathfrak{q}}{x\mathfrak{p}^3} \quad \text{and} \quad \mathfrak{Q} = \frac{-zp}{x\mathfrak{p}\mathfrak{p}}.$$

Now for the first equation on account of $N = 0$ [as V is not a function of y] there must become

$$-\frac{dP}{dx} + \frac{ddQ}{dx^2} = 0 \ \text{ or } \ P - \frac{dQ}{dx} = \text{Const.},$$

the truth of this will be immediately obvious from the differentiation of Q itself.
For the other equation

$$\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} = 0,$$

because hence there is $\int \mathfrak{N}dx = \mathfrak{P} - \frac{d\mathfrak{Q}}{dx}$, at first it is necessary that this integral formula exists

$$\mathfrak{N}dx = \frac{-pdx}{xx\mathfrak{p}} + \frac{qdx}{x\mathfrak{p}} - \frac{p\mathfrak{q}dx}{x\mathfrak{p}\mathfrak{p}},$$

from which on account of $qdx = dp$ clearly there shall be $\int \mathfrak{N}dx = \frac{p}{x\mathfrak{p}}$. Hence there remains, so that there shall be

$$\frac{d\mathfrak{Q}}{dx} = \mathfrak{P} - \int \mathfrak{N}dx = \frac{pz}{xx\mathfrak{p}\mathfrak{p}} - \frac{zq}{x\mathfrak{p}\mathfrak{p}} + \frac{2zp\mathfrak{q}}{x\mathfrak{p}^3} - \frac{p}{x\mathfrak{p}}.$$

Truly on differentiating $\mathfrak{Q} = \frac{-zp}{x\mathfrak{p}\mathfrak{p}}$ on both sides, perfect equality results.

SCHOLIUM 1

134. But if therefore the question may be reduced to this, so that the maximum or minimum value of the integral formula $\int Vdx$ shall be required to be brought about, then before all in the variation of this it is required that both integral parts and that [absolved part] be equated to zero, therefore because, in whatever manner the variations may be put in place, the variation $\delta\int Vdx$ must always vanish; from which these two equations emerge

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

and

$$\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} - \text{etc.} = 0,$$

from which a twofold relation between the three variables *x*, *y*, *z* thus is expressed, so that in turn both *y* as well as *z* are to be regarded correctly as a function of *x*. But when these equations are differentials and that of higher order, just as many arbitrary constants through integration must be introduced into the calculation, as often as each order should be of the differential. It is required thus that these constants be defined successively, so that both the starting conditions as well as the conditions for the end of the integration formula $\int Vdx$ prescribed should be satisfied, which returns the task there, so that in addition the absolute parts of the variation may be reduced to zero. Clearly in the first place the constant must be defined thus, so that it satisfies the prescribed conditions for the start, where indeed from the nature of the question the individual defined values

$$\omega, \quad \mathfrak{w}, \quad \frac{d\omega}{dx}, \quad \frac{d\mathfrak{w}}{dx}, \quad \frac{dd\omega}{dx^2}, \quad \frac{dd\mathfrak{w}}{dx^2} \text{ etc.}$$

are accustomed to be chosen. Then truly, since the same may come about in use about the upper end of the integration, they will be determined from the individual constants entering through integration.

SCHOLIUM 2

135. It will be most conducive here to observe that the members, by which the variation, $\delta\int Vdx$ is expressed, are separated at once into two classes, in the one of which only these letters are conspicuous, which are referred to the variability of *y* or to the expression of this with respect to *x* and also that thus, if the quantity *z* should be assumed constant, truly the other class will contain letters depending on the variability of *z* only, as if the quantity *y* should be constant. From which one is allowed to deduce, if also a fourth variable *v* should be present, which can be regarded also as a function of *x*, then to those two classes a third is required to be added above, which includes similar members depending on the variability of *v* only. On account of which the solution given here can be considered, as if it may be extended to any number of variables, as long as just as many equations are considered to be given between these, so that they can be considered as functions of the one variable [x]. And if therefore only three variables present themselves in this chapter, yet it is to be understood to pertain to any number of variables, but only if conditions of this kind are proposed, so that by one variable all the rest may be determined. But the integral formulas of this form $\int Vdx$ by necessity involve such a condition ; for however many variables there may be present in the quantity V, the expression $\int Vdx$ cannot obtain a definite value entirely, unless all the variables are able to be regarded as functions of the one x. Moreover otherwise an account of these integral formulas is to be prepared at length, which refer to two or more variables depending minimally on each other in turn.

PROBLEM 14

136. *If the function V above besides the three variables x, y, z and the differentials of these of any order should involve an integral formula* $v=\int \mathfrak{V}dx$ *, where* $\mathfrak{V}$ *shall be some function of the same three variables x, y, z with the differentials of these, to investigate the variation of the integral formula* $\int Vdx$.

SOLUTION

In order that the types of differentials can be removed from the calculation at any rate, we may put as before

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \ \text{ etc.},$$
$$dz = \mathfrak{p}dx,\ d\mathfrak{p} = \mathfrak{q}dx,\ d\mathfrak{q} = \mathfrak{r}dx,\ d\mathfrak{r} = \mathfrak{s}dx \ \text{ etc.}$$

and with the function V differentiated there shall be produced

$$dV = Ldv + Mdx + Ndy + Pdp + Qdq + Rdr + \text{ etc.}$$
$$+\mathfrak{N}dz + \mathfrak{P}d\mathfrak{p} + \mathfrak{Q}d\mathfrak{q} + \mathfrak{R}d\mathfrak{r} + \text{ etc.};$$

then truly on account of $dv = \mathfrak{V}dx$ there shall be

$$d\mathfrak{V} = M'dx + N'dy + P'dp + Q'dq + R'dr + \text{ etc.}$$
$$+\mathfrak{N}'dz + \mathfrak{P}'d\mathfrak{p} + \mathfrak{Q}'d\mathfrak{q} + \mathfrak{R}'d\mathfrak{r} + \text{ etc.},$$

where on account of the lack of letters I use the same distinguished by an accent. Hence moreover likewise variations of the same quantities V and $\mathfrak{V}$ may be considered. Now since the variation $\delta\int Vdx$ is sought, in the first place indeed we will consider as before

$$\delta\int Vdx = V\delta x + \int\left(dx\delta V - dV\delta x\right);$$

where since the value of V itself does not disagree with the preceding, except that here to the differential of this dV there may be added the part $Ldv = L\delta\mathfrak{V}dx$ and to the variation δV this part $L\delta v = L\delta\int\mathfrak{V}dx$, also the variation sought $\delta\int Vdx$ may be expressed from the form found before, but only if this member may be added to that :

$$\int L\left(dx\delta\int\mathfrak{V}dx - \mathfrak{V}dx\delta x\right) = \int Ldx\left(\delta\int\mathfrak{V}dx - \mathfrak{V}\delta x\right).$$

Because truly the integral formula $\int\mathfrak{V}dx$ is the same, which was treated in the preceding problem if, as we did there , we may put in place,

$$\delta y - p\delta x = \omega \text{ and } \delta z - \mathfrak{p}\delta x = \mathfrak{w},$$

we will have, with the element assumed constant dx

$$\delta\int\mathfrak{V}dx - \mathfrak{V}\delta x = \int dx\left\{\begin{matrix} N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + \frac{R'd^3\omega}{dx^3} + \text{etc.} \\ \mathfrak{N}'\mathfrak{w} + \frac{\mathfrak{P}'d\mathfrak{w}}{dx} + \frac{\mathfrak{Q}'dd\mathfrak{w}}{dx^2} + \frac{\mathfrak{R}'d^3\mathfrak{w}}{dx^3} + \text{etc.}\end{matrix}\right\},$$

Now we may put the integral $\int Ldx = I$, clearly if it may be taken thus, so that for the start of the integration it vanishes, then truly for the final term of the integration it becomes $I = A$, with which put in place, there will be for the whole extent of the integration

$$\int Ldx\left(\delta\int\mathfrak{V}dx - \mathfrak{V}\delta x\right) = \int(A-I)dx\left\{\begin{matrix} N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + \frac{R'd^3\omega}{dx^3} + \text{etc.} \\ \mathfrak{N}'\mathfrak{w} + \frac{\mathfrak{P}'d\mathfrak{w}}{dx} + \frac{\mathfrak{Q}'dd\mathfrak{w}}{dx^2} + \frac{\mathfrak{R}'d^3\mathfrak{w}}{dx^3} + \text{etc.}\end{matrix}\right\},$$

Now therefore we can introduce the following abbreviations

$$\begin{aligned} N + (A-I)N' &= N^0, & \mathfrak{N} + (A-I)\mathfrak{N}' &= \mathfrak{N}^0, \\ P + (A-I)P' &= P^0, & \mathfrak{P} + (A-I)\mathfrak{P}' &= \mathfrak{P}^0, \\ Q + (A-I)Q' &= Q^0, & \mathfrak{Q} + (A-I)\mathfrak{Q}' &= \mathfrak{Q}^0, \\ R + (A-I)R' &= R^0, & \mathfrak{R} + (A-I)\mathfrak{R}' &= \mathfrak{R}^0, \\ &\text{etc.} & &\text{etc.} \end{aligned}$$

and it is evident that the variation sought thus becomes the expression

$$\delta\int Vdx = V\delta x + \int dx\begin{cases} N^0\omega + \frac{P^0 d\omega}{dx} + \frac{Q^0 dd\omega}{dx^2} + \frac{R^0 d^3\omega}{dx^3} + \text{etc.} \\ \mathfrak{N}^0\mathfrak{w} + \frac{\mathfrak{P}^0 d\mathfrak{w}}{dx} + \frac{\mathfrak{Q}^0 dd\mathfrak{w}}{dx^2} + \frac{\mathfrak{R}^0 d^3\mathfrak{w}}{dx^3} + \text{etc.} \end{cases},$$

which also as before can be spread out in this form

$$\begin{aligned}\delta\int Vdx = &\int \omega dx\left(N^0 - \frac{dP^0}{dx} + \frac{ddQ^0}{dx^2} - \frac{d^3R^0}{dx^3} + \frac{d^4S^0}{dx^4} - \text{etc.}\right)\\ &+\int \mathfrak{w}dx\left(\mathfrak{N}^0 - \frac{d\mathfrak{P}^0}{dx} + \frac{dd\mathfrak{Q}^0}{dx^2} - \frac{d^3\mathfrak{R}^0}{dx^3} + \frac{d^4\mathfrak{S}^0}{dx^4} - \text{etc.}\right)\\ &+V\delta x + \omega\left(P^0 - \frac{dQ^0}{dx} + \frac{ddR^0}{dx} - \frac{d^3S^0}{dx^3} + \text{etc.}\right)\\ &+\text{Const.} + \mathfrak{w}\left(\mathfrak{P}^0 - \frac{d\mathfrak{Q}^0}{dx} + \frac{dd\mathfrak{R}^0}{dx} - \frac{d^3\mathfrak{S}^0}{dx^3} + \text{etc.}\right)\\ &+\frac{d\omega}{dx}\left(Q^0 - \frac{dR^0}{dx} + \frac{ddS^0}{dx^2} - \text{etc.}\right)\\ &+\frac{d\omega}{dx}\left(\mathfrak{Q}^0 - \frac{d\mathfrak{R}^0}{dx} + \frac{dd\mathfrak{S}^0}{dx^2} - \text{etc.}\right)\\ &+\frac{dd\omega}{dx}\left(R^0 - \frac{dS^0}{dx} + \text{etc.}\right)\\ &+\frac{dd\mathfrak{w}}{dx}\left(\mathfrak{R}^0 - \frac{d\mathfrak{S}^0}{dx} + \text{etc.}\right)\\ &+\frac{d^3\omega}{dx^3}\left(S^0 - \text{etc.}\right)\\ &+\frac{d^3\mathfrak{w}}{dx^3}\left(\mathfrak{S}^0 - \text{etc.}\right)\\ &+\text{etc.},\end{aligned}$$

where the suffixed sign of zero offends nobody, if indeed it may not denote an exponent, but may be used only for these letters to be distinguished from the same naked letters.

COROLLARY 1

137. If therefore the formula of the integral $\int Vdx$ must have a maximum or minimum value, it is required at once to put in place the two members of the variation found at first equal to zero, from which two differential equations result, from which an indefinite relation of each of the variables y and z is defined in terms of x.

COROLLARY 2

138. Even if here of the conditions, which perhaps may be proposed for the beginning and end of the integration, an account may not yet be had, yet these now may enter hidden in the calculation, because the letters I and A consider the ends of the integration. Yet meanwhile these again must be expelled from the treatment of the differential equation itself; while indeed the formula of the integral $\int Ldx = I$ is removed, and likewise the quantity A may disappear.

COROLLARY 3

139. But with the equations brought out from these two differential equations and that most generally, so that just as many arbitrary constants may be introduced in the calculation, as the number of integrations it is required to put in place, then at last according to the conditions each of the limits of the integration formula $\int Vdx$ is required to be attended, since hence from the remaining absolved members, these constants of the variation must be determined.

SCHOLIUM

140. The solution of this problem thus has been prepared, so that now it may be evident enough, to the extent that also it may be convenient to put in place more complicated formulas, just as if the function V may involve several integral formulas or if also the function $\mathfrak{B}$ may involve new integrals. So that even now it is evident, if integral formulas of this kind may contain more than three variables, how it may then be required to find the variations, and accordingly it would not only be tedious but also superfluous, if I should wish to pursue this argument in more detail. I advance therefore to another much more abstruse part of this instruction, where also with the relations put in place between two or more variables, depending minimally on each other and in turn relinquished from the calculation.

CAPUT V

DE VARIATIONE FORMULARUM INTEGRALIUM TRES VARIABILES INVOLVENTIUM ET DUPLICEM RELATIONEM IMPLICANTIUM

PROBLEMA 12

123. *Proposita formula quacunque ternas variabiles x, y, z cum suis differentialibus cuiuscunque gradus involvente eius variationem definire ex variationibus omnium trium variabilium oriundam.*

SOLUTIO

Sit W formula ista proposita, cuius primo quaeratur valor variatus $W+\delta W$ qui oritur, si loco x, y, z scribantur ipsarum valores variati

$$x+\delta x,\ y+\delta y,\ z+\delta z$$

similiterque pro earum differentialibus

$$dx+d\delta x,\ dy+d\delta y,\ dz+d\delta z$$

et ita porro; a quo si ipsa formula W auferatur, remanebit eius variatio δW. Ex quo intelligitur hanc variationem per consuetam differentiationem obtineri, si modo loco signi differentiationis d signum variationis δ scribatur. Tantum notasse iuvabit, si differentialium variationes capi oporteat, perinde esse, in quonam loco inter differentiationis signa signum variationis δ collocetur, quemadmodum supra [§ 37, 40] demonstravimus; unde signum variationis perpetuo in postremo loco poni poterit, quod, cum ad formulas integrales progredientur, commodissimum videtur, sicut ex iis, quae hactenus de formulis integralibus binas variabiles involventibus sunt tradita, satis est manifestum.

COROLLARIUM 1

124. Quoniam z perinde ac y tanquam functio ipsius x spectari potest, si ponatur $\frac{dy}{dx}=p$ et $\frac{dz}{dx}=\mathfrak{p}$, erit

$$\delta p=\frac{d\delta y-pd\delta x}{dx} \quad \text{et} \quad \delta\mathfrak{p}=\frac{d\delta z-\mathfrak{p}d\delta x}{dx}$$

similique modo formulae hinc derivatae ae superioribus non discrepant.

COROLLARIUM 2

125. Ponamus $\delta y-p\delta x=\omega$ et $\delta z-\mathfrak{p}\delta x=\mathfrak{w}$ eritque

$$d\delta y - pd\delta x - qdx\delta x = d\omega \quad \text{et} \quad d\delta z - \mathfrak{p}d\delta x - \mathfrak{q}dx\delta x = d\mathfrak{w},$$

si scilicet statuamus

$$\frac{dp}{dx} = q \quad \text{et} \quad \frac{d\mathfrak{p}}{dx} = \mathfrak{q},$$

unde patet fore

$$\delta p - q\delta x = \frac{d\omega}{dx} \quad \text{et} \quad \delta\mathfrak{p} - \mathfrak{q}\delta x = \frac{d\mathfrak{w}}{dx}$$

COROLLARIUM 3

126. Si ulterius statuamus

$$\frac{dq}{dx} = r, \quad \frac{d\mathfrak{q}}{dx} = \mathfrak{r}, \quad \frac{dr}{dx} = s, \quad \frac{d\mathfrak{r}}{dx} = \mathfrak{s} \quad \text{etc.},$$

erit simili modo sumto dx constante

$$\delta q - r\delta x = \frac{dd\omega}{dx^2}, \ \delta\mathfrak{q} - \mathfrak{r}\delta x = \frac{dd\mathfrak{w}}{dx^2},$$
$$\delta r - s\delta x = \frac{d^3\omega}{dx^3}, \ \delta\mathfrak{r} - \mathfrak{s}\delta x = \frac{d^3\mathfrak{w}}{dx^3}$$

sicque deinceps.

SCHOLION 1

127. Sive ergo formula varianda habuerit valorem finitum sive infinitum sive evanescentem, ope horum praeceptorum eius variatio perinde ac supra inveniri potest; neque enim haec praecepta a superioribus discrepant, nisi quod hic duplicis generis valores differentiales, alteri litteris latinis p, q, r, s etc., alteri germanicis $\mathfrak{p}$, $\mathfrak{q}$, $\mathfrak{r}$, $\mathfrak{s}$ etc. indicati, introduci debeant; cuius rei ratio in eo est sita, quod hic utraque variabilis y et z tanquam functio ipsius x spectari potest. Sin autem unica aequatio inter ternas coordinatas daretur vel quaereretur, litterae hic introductae p et $\mathfrak{p}$ nullos habiturae essent valores certos, cum salva illa aequatione fractiones $\frac{dy}{dx}$ et $\frac{dz}{dx}$ omnes omnino valores recipere possent. Omissis autem his litteris ipsisque differentialibus in calculo relictis, etiam pro hoc casu regula in solutione exposita variationem declarabit.

SCHOLION 2

128. Supra [§ 12-24] iam notavi hunc casum trium variabilium, quarum relatio gemina aequatione definitur, sollicite esse distinguendum ab eo, ubi relatio unica aequatione definiri assumitur. Discrimen hoc ex Geometria clarissime illustratur, ubi ternae variabiles vicem ternarum coordinatarum gerunt; totidem autem in calculo adhiberi oportet, non solum quando quaestio circa superficies versatur, sed etiam quando lineae curvae non in eodem plano sitae sunt explorandae. Atque hoc quidem casu posteriori determinatio lineae curvae duas aequationes inter ternas coordinatas postulat, ita ut binae quaevis tanquam functiones tertiae spectari possint. Superficiei autem natura iam unica aequatione inter ternas coordinatas definitur, ita ut unaquaeque tanquam functio binarum reliquarum spectari queat, unde ingens discrimen in ipsa tractatione oritur. Praesens igitur caput inservire poterit eiusmodi lineis curvis indagandis, quae non in eodem plano sitae maximi minimive quapiam gaudeant proprietate.

PROBLEMA 13

129. *Si V fuerit functio quaecunque trium variabilium x, y, z, earum insuper differentialia cuiuscunque ordinis implicans, eaeque variabiles variationes quascunque recipiant, invenire variationem formulae integralis* $\int Vdx$.

SOLUTIO

Quaecunque differentialia in functionem *V* ingrediantur, ea his factis substitutionibus

$$dy = pdx,\ dp = qdx,\ dq = rdr,\ dr = sdx \text{ etc.},$$
$$dz = \mathfrak{p}dx,\ d\mathfrak{p} = \mathfrak{q}dx,\ d\mathfrak{q} = \mathfrak{r}dx,\ d\mathfrak{r} = \mathfrak{s}dx \text{ etc.}$$

tollentur et quantitas *V* erit functio quantitatum finitarum *x, y, z, p, q, r, s* etc., $\mathfrak{p}, \mathfrak{q}, \mathfrak{r}, \mathfrak{s}$ etc. Eius ergo differentiale huiusmodi habebit formam

$$\begin{aligned} dV = Mdx + Ndy &+ Pdp + Qdq + Rdr + Sds + \text{etc.} \\ &+ \mathfrak{N}dz + \mathfrak{P}d\mathfrak{p} + \mathfrak{Q}d\mathfrak{q} + \mathfrak{R}d\mathfrak{r} + \mathfrak{S}d\mathfrak{s} + \text{etc.}, \end{aligned}$$

unde mutatis signis differentiationis *d* in δ simul habebitur variatio δV. Ex supra autem demonstratis etiam pro hoc casu trium variabilium habebitur

$$\delta\int Vdx = \int\left(Vd\delta x + dx\delta V\right) = V\delta x + \int\left(dx\delta V - dV\delta x\right).$$

At facta substitutione fiet

$$\begin{aligned} \tfrac{dx\delta V - dV\delta x}{dx} = M\delta x + N\delta y &+ P\delta p + Q\delta q + R\delta r + \text{etc.} \\ &+ \mathfrak{N}\delta z + \mathfrak{P}\delta\mathfrak{p} + \mathfrak{Q}\delta\mathfrak{q} + \mathfrak{R}\delta\mathfrak{r} + \text{etc.} \\ - M\delta x &- Np\delta x - Pq\delta x - Qr\delta x - Rs\delta x - \text{etc.} \\ &- \mathfrak{N}\mathfrak{p}\delta x - \mathfrak{P}\mathfrak{q}\delta x - \mathfrak{Q}\mathfrak{r}\delta x - \mathfrak{R}\mathfrak{s}\delta x - \text{etc.} \end{aligned}$$

Quodsi iam brevitatis gratia statuamus

$$\delta y - p\delta x = \omega \ \text{ et } \ \delta z - \mathfrak{p}\delta x = \mathfrak{w},$$

sumto elemento *dx* constante ex § 125 et 126 erit

$$\delta p - q\delta x = \tfrac{d\omega}{dx}, \quad \delta\mathfrak{p} - \mathfrak{q}\delta x = \tfrac{d\mathfrak{w}}{dx},$$
$$\delta q - r\delta x = \tfrac{dd\omega}{dx^2}, \quad \delta\mathfrak{q} - \mathfrak{r}\delta x = \tfrac{dd\mathfrak{w}}{dx^2},$$
$$\delta r - s\delta x = \tfrac{d^3\omega}{dx^3}, \quad \delta\mathfrak{r} - \mathfrak{s}\delta x = \tfrac{d^3\mathfrak{w}}{dx^3},$$
$$\text{etc.},$$

unde variatio quaesita hoc modo commode exprimetur

$$\delta\int Vdx = V\delta x + \int dx \begin{Bmatrix} N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + \text{etc.} \\ \mathfrak{N}\mathfrak{w} + \frac{\mathfrak{P}d\mathfrak{w}}{dx} + \frac{\mathfrak{Q}dd\mathfrak{w}}{dx^2} + \frac{\mathfrak{R}d^3\mathfrak{w}}{dx^3} + \text{etc.} \end{Bmatrix},$$

quae ut supra [§ 80-85] ad hanc formam reducitur

$$\begin{aligned}\delta\int Vdx = &\int \omega dx\left(N - \tfrac{dP}{dx} + \tfrac{ddQ}{dx^2} - \tfrac{d^3R}{dx^3} + \tfrac{d^4S}{dx^4} - \text{etc.}\right)\\
&+\int \mathfrak{w}dx\left(\mathfrak{N} - \tfrac{d\mathfrak{P}}{dx} + \tfrac{dd\mathfrak{Q}}{dx^2} - \tfrac{d^3\mathfrak{R}}{dx^3} + \tfrac{d^4\mathfrak{S}}{dx^4} - \text{etc.}\right)\\
&+V\delta x + \omega\left(P - \tfrac{dQ}{dx} + \tfrac{ddR}{dx} - \tfrac{d^3S}{dx^3} + \text{etc.}\right)\\
&+\text{Const.} + \mathfrak{w}\left(\mathfrak{P} - \tfrac{d\mathfrak{Q}}{dx} + \tfrac{dd\mathfrak{R}}{dx} - \tfrac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\right)\\
&+\tfrac{d\omega}{dx}\left(Q - \tfrac{dR}{dx} + \tfrac{ddS}{dx^2} - \text{etc.}\right)\\
&+\tfrac{d\omega}{dx}\left(\mathfrak{Q} - \tfrac{d\mathfrak{R}}{dx} + \tfrac{dd\mathfrak{S}}{dx^2} - \text{etc.}\right)\\
&+\tfrac{dd\omega}{dx}\left(R - \tfrac{dS}{dx} + \text{etc.}\right)\\
&+\tfrac{dd\mathfrak{w}}{dx}\left(\mathfrak{R} - \tfrac{d\mathfrak{S}}{dx} + \text{etc.}\right)\\
&+\tfrac{d^3\omega}{dx^3}\left(S - \text{etc.}\right)\\
&+\tfrac{d^3\mathfrak{w}}{dx^3}\left(\mathfrak{S} - \text{etc.}\right)\\
&+\text{etc.},\end{aligned}$$

cuius indoles ex superioribus satis est manifesta, eademque circa constantis additionem sunt observanda [§ 115].

COROLLARIUM 1

130. In hac solutione ambae variabiles y et z tanquam functiones ipsius x spectantur, sive iam sint cognitae sive demum ex variationis indole definiendae. Neque etiam formula integralis $\int Vdx$ certum esset habitura valorem, nisi tam y quam z per x determinari conciperetur.

COROLLARIUM 2

131. Si formula Vdx per se sit integrabilis nulla assumta relatione inter ternas variabiles, variatio integralis $\int Vdx$ nullas quoque formulas integrales involvere potest ideoque necesse est, ut tum sit

$$\text{et}\quad N-\frac{dP}{dx}+\frac{ddQ}{dx^2}-\frac{d^3R}{dx^3}+\frac{d^4S}{dx^4}-\text{etc.}=0$$
$$\text{et}\quad \mathfrak{N}-\frac{d\mathfrak{P}}{dx}+\frac{dd\mathfrak{Q}}{dx^2}-\frac{d^3\mathfrak{R}}{dx^3}+\frac{d^4\mathfrak{S}}{dx^4}-\text{etc.}=0.$$

COROLLARIUM 3

132. Vicissim etiam si hae duae aequationes locum habeant, hoc certum erit criterium formulam differentialem Vdx per se, integrationem admittere nulla inter variabiles stabilita relatione.

EXEMPLUM

133. *Quo hoc criterium magis illustremus, sumamus eiusmodi formulam per se integrabilem sitque* $\int Vdx=\frac{zdy}{xdz}=\frac{pz}{x\mathfrak{p}}$, *unde fit*

$$V=\frac{-pz}{xx\mathfrak{p}}+\frac{p}{x}+\frac{zq}{x\mathfrak{p}}-\frac{zp\mathfrak{q}}{x\mathfrak{p}\mathfrak{p}}.$$

Ex cuius differentiatione colligimus

$$N=0\quad\text{et}\quad P=\frac{-z}{xx\mathfrak{p}}+\frac{1}{x}-\frac{z\mathfrak{q}}{x\mathfrak{p}\mathfrak{p}},\quad Q=\frac{z}{x\mathfrak{p}},$$

porro

$$\mathfrak{N}=\frac{-p}{xx\mathfrak{p}}+\frac{q}{x\mathfrak{p}}-\frac{p\mathfrak{q}}{x\mathfrak{p}\mathfrak{p}},\quad \mathfrak{P}=\frac{pz}{xx\mathfrak{p}\mathfrak{p}}-\frac{zq}{x\mathfrak{p}\mathfrak{p}}+\frac{2zp\mathfrak{q}}{x\mathfrak{p}^3}\quad\text{et}\quad \mathfrak{Q}=\frac{-zp}{x\mathfrak{p}\mathfrak{p}}.$$

Iam pro prima aequatione ob $N=0$ fieri oportet

$$-\frac{dP}{dx}+\frac{ddQ}{dx^2}=0\quad\text{seu}\quad P-\frac{dQ}{dx}=\text{Const.},$$

cuius veritas ex differentiatione ipsius Q statim fit perspicua.
Pro altera aequatione

$$\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} = 0,$$

quia hinc est $\int \mathfrak{N}dx = \mathfrak{P} - \frac{d\mathfrak{Q}}{dx}$, primo necesse est, ut integrabilis existat haec formula

$$\mathfrak{N}dx = \frac{-pdx}{xx\mathfrak{p}} + \frac{qdx}{x\mathfrak{p}} - \frac{p\mathfrak{q}dx}{x\mathfrak{p}\mathfrak{p}},$$

unde ob $qdx = dp$ manifesto fit $\int \mathfrak{N}dx = \frac{p}{x\mathfrak{p}}$. Superest ergo, ut sit

$$\frac{d\mathfrak{Q}}{dx} = \mathfrak{P} - \int \mathfrak{N}dx = \frac{pz}{xx\mathfrak{p}\mathfrak{p}} - \frac{zq}{x\mathfrak{p}\mathfrak{p}} + \frac{2zp\mathfrak{q}}{x\mathfrak{p}^3} - \frac{p}{x\mathfrak{p}}.$$

Verum differentiando $\mathfrak{Q} = \frac{-zp}{x\mathfrak{p}\mathfrak{p}}$ utrinque perfecta aequalitas resultat.

SCHOLION 1

134. Quodsi ergo quaestio huc redeat, ut formulae integrali $\int Vdx$ valor maximus minimusve sit conciliandus, tum ante omnia in eius variatione ambas partes integrales idque seorsim nihilo aequari oportet, propterea quod, utcunque variationes constituantur, variatio $\delta \int Vdx$ semper debeat evanescere; unde duae emergunt aequationes istae

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

et

$$\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} - \text{etc.} = 0,$$

quibus duplex relatio inter ternas variabiles *x, y, z* ita exprimitur, ut deinceps tam *y* quam *z* recte tanquam functio ipsius *x* spectari possit. Quando autem hae aequationes sunt differentiales idque altioris gradus, totidem utrinque constantes arbitrariae per integrationes in calculum invehuntur, quoti gradus utraque fuerit differentialis. Has vero constantes deinceps ita definiri oportet, ut conditionibus tam pro initio quam pro fine integrationis formulae $\int Vdx$ praescriptis satisfiat, quod negotium eo redit, ut praeterea variationis partes absolutae ad nihilum redigantur. Primo scilicet Constans ita definiri debet, ut conditionibus pro initio praescriptis satisfiat, ubi quidem ex quaestionis indole particulae

$$\omega, \quad \mathfrak{w}, \quad \frac{d\omega}{dx}, \quad \frac{d\mathfrak{w}}{dx}, \quad \frac{dd\omega}{dx^2}, \quad \frac{dd\mathfrak{w}}{dx^2} \quad \text{etc.}$$

definitos valores sortiri solent. Tum vero, cum idem circa finem integrationis usu veniat, ex singulis constantes per integrationem ingressae determinabuntur.

SCHOLION 2

135. Plurimum conducet hic observasse membra, quibus variatio $\delta\int Vdx$ exprimitur, sponte in duas classes dispesci, in quarum altera litterae tantum eae conspiciuntur, quae ad varibilitatem ipsius y seu ad eius habitum respectu x referuntur idque ita, ac si quantitas z constans esset assumta, altera vero classis similes litteras a variabilitate ipsius z tantum pendentes continet, quasi quantitas y esset constans. Ex quo colligere licet, si etiam quarta variabilis v accedat, quae ut functio ipsius x quoque spectari queat, tum ad illas duas classes tertiam insuper esse adiiciendam, quae similia membra a variabilitate solius v pendentia complectatur. Quocirca solutio hic data spectari potest, quasi ad quotcunque variabiles extendatur, dummodo tot inter eas aequationes dari concipiantur, ut omnes pro functionibus unius haberi queant. Etsi ergo hoc caput tantum tres variabiles prae se fert, tamen ad quotcunque pertinere est intelligendum, si modo eiusmodi conditiones proponantur, ut tandem per unam reliquae omnes determinentur. Talem autem conditionem formulae integrales huius formae $\int Vdx$ necessario involvunt; quotcunque enim variabiles in quantitatem V ingrediantur, expressio $\int Vdx$ certum valorem definitum omnino obtinere nequit, nisi omnes variabiles tanquam functiones unius x spectari queant. Longe aliter autem est comparata ratio earum formularum integralium, quae ad duas pluresve variabiles a se invicem minime pendentes referuntur.

PROBLEMA 14

136. *Si functio V praeter tres variabiles x, y, z earumque differentialia cuiuscunque gradus insuper involvat formulam integralem $v=\int\mathfrak{V}dx$, ubi $\mathfrak{V}$ sit functio quaecunque earundem variabilium x, y, z cum suis differentialibus, investigare variationem formulae integralis $\int Vdx$.*

SOLUTIO

Ut species saltem differentialium e calculo tollatur, ponamus ut ante

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \text{ etc.},$$
$$dz = \mathfrak{p}dx,\ d\mathfrak{p} = \mathfrak{q}dx,\ d\mathfrak{q} = \mathfrak{t}dx,\ d\mathfrak{t} = \mathfrak{s}dx \text{ etc.}$$

ac functione V differentiata prodeat

$$\begin{aligned} dV = Ldv + Mdx + Ndy + Pdp + Qdq + Rdr + \text{ etc.} \\ + \mathfrak{N}dz + \mathfrak{P}d\mathfrak{p} + \mathfrak{Q}d\mathfrak{q} + \mathfrak{R}d\mathfrak{r} + \text{ etc.;} \end{aligned}$$

tum vero ob $dv = \mathfrak{V}dx$ sit

$$\begin{aligned} d\mathfrak{V} = M'dx + N'dy + P'dp + Q'dq + R'dr + \text{ etc.} \\ + \mathfrak{N}'dz + \mathfrak{P}'d\mathfrak{p} + \mathfrak{Q}'d\mathfrak{q} + \mathfrak{R}'d\mathfrak{r} + \text{ etc.,} \end{aligned}$$

ubi ob defectum litterarum iisdem accentu distinctis utor. Hinc autem simul earundem quantitatum V et $\mathfrak{B}$ variationes habentur. Iam cum quaeratur variatio $\delta\int Vdx$, habebimus primo quidem ut ante

$$\delta\int Vdx = V\delta x + \int\left(dx\delta V - dV\delta x\right);$$

ubi cum valor ipsius V non discrepet a praecedente, nisi quod hic ad eius differentiale dV accedat pars $Ldv = L\delta\mathfrak{B}dx$ et ad variationem δV haec pars $L\delta v = L\delta\int\mathfrak{B}dx$, etiam variatio quaesita $\delta\int Vdx$ forma ante inventa exprimetur, si modo ad eam adiiciatur hoc membrum

$$\int L\left(dx\delta\int\mathfrak{B}dx - \mathfrak{B}dx\delta x\right) = \int Ldx\left(\delta\int\mathfrak{B}dx - \mathfrak{B}\delta x\right).$$

Quia vero formula integralis $\int\mathfrak{B}dx$ eadem est, quae in problemate praecedente est tractata, si, ut ibi fecimus, statuamus

$$\delta y - p\delta x = \omega \text{ et } \delta z - \mathfrak{p}\delta x = \mathfrak{w},$$

elemento dx constante assumto habebimus

$$\delta\int\mathfrak{B}dx - \mathfrak{B}\delta x = \int dx\begin{Bmatrix} N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + \frac{R'd^3\omega}{dx^3} + \text{etc.} \\ \mathfrak{N}'\mathfrak{w} + \frac{\mathfrak{P}'d\mathfrak{w}}{dx} + \frac{\mathfrak{Q}'dd\mathfrak{w}}{dx^2} + \frac{\mathfrak{R}'d^3\mathfrak{w}}{dx^3} + \text{etc.} \end{Bmatrix},$$

Ponamus iam integrale $\int Ldx = I$, si scilicet ita capiatur, ut pro initio integrationis evanescat, tum vero pro termino finali integrationis fiat $I = A$, quo facto pro tota integrationis extensione erit

$$\int Ldx\left(\delta\int\mathfrak{B}dx - \mathfrak{B}\delta x\right) = \int\left(A - I\right)dx\begin{Bmatrix} N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + \frac{R'd^3\omega}{dx^3} + \text{etc.} \\ \mathfrak{N}'\mathfrak{w} + \frac{\mathfrak{P}'d\mathfrak{w}}{dx} + \frac{\mathfrak{Q}'dd\mathfrak{w}}{dx^2} + \frac{\mathfrak{R}'d^3\mathfrak{w}}{dx^3} + \text{etc.} \end{Bmatrix},$$

Nunc igitur introducamus sequentes abbreviationes

$$N+(A-I)N'=N^0,\quad \mathfrak{N}+(A-I)\mathfrak{N}'=\mathfrak{N}^0,$$
$$P+(A-I)P'=P^0,\quad \mathfrak{P}+(A-I)\mathfrak{P}'=\mathfrak{P}^0,$$
$$Q+(A-I)Q'=Q^0,\quad \mathfrak{Q}+(A-I)\mathfrak{Q}'=\mathfrak{Q}^0,$$
$$R+(A-I)R'=R^0,\quad \mathfrak{R}+(A-I)\mathfrak{R}'=\mathfrak{R}^0,$$
$$\text{etc.}\qquad\qquad \text{etc.}$$

atque manifestum est variationem quaesitam ita expressam iri

$$\delta\int Vdx=V\delta x+\int dx\left\{\begin{matrix} N^0\omega+\frac{P^0d\omega}{dx}+\frac{Q^0dd\omega}{dx^2}+\frac{R^0d^3\omega}{dx^3}+\text{etc.}\\ \mathfrak{N}^0\mathfrak{w}+\frac{\mathfrak{P}^0d\mathfrak{w}}{dx}+\frac{\mathfrak{Q}^0dd\mathfrak{w}}{dx^2}+\frac{\mathfrak{R}^0d^3\mathfrak{w}}{dx^3}+\text{etc.}\end{matrix}\right\},$$

quae etiam ut ante evolvitur in hanc formam

$$\begin{aligned}\delta\int Vdx=&\int\omega dx\left(N^0-\frac{dP^0}{dx}+\frac{ddQ^0}{dx^2}-\frac{d^3R^0}{dx^3}+\frac{d^4S^0}{dx^4}-\text{etc.}\right)\\ &+\int\mathfrak{w}dx\left(\mathfrak{N}^0-\frac{d\mathfrak{P}^0}{dx}+\frac{dd\mathfrak{Q}^0}{dx^2}-\frac{d^3\mathfrak{R}^0}{dx^3}+\frac{d^4\mathfrak{S}^0}{dx^4}-\text{etc.}\right)\\ &+V\delta x+\omega\left(P^0-\frac{dQ^0}{dx}+\frac{ddR^0}{dx}-\frac{d^3S^0}{dx^3}+\text{etc.}\right)\\ &+\text{Const.}+\mathfrak{w}\left(\mathfrak{P}^0-\frac{d\mathfrak{Q}^0}{dx}+\frac{dd\mathfrak{R}^0}{dx}-\frac{d^3\mathfrak{S}^0}{dx^3}+\text{etc.}\right)\\ &+\frac{d\omega}{dx}\left(Q^0-\frac{dR^0}{dx}+\frac{ddS^0}{dx^2}-\text{etc.}\right)\\ &+\frac{d\omega}{dx}\left(\mathfrak{Q}^0-\frac{d\mathfrak{R}^0}{dx}+\frac{dd\mathfrak{S}^0}{dx^2}-\text{etc.}\right)\\ &+\frac{dd\omega}{dx}\left(R^0-\frac{dS^0}{dx}+\text{etc.}\right)\\ &+\frac{dd\mathfrak{w}}{dx}\left(\mathfrak{R}^0-\frac{d\mathfrak{S}^0}{dx}+\text{etc.}\right)\\ &+\frac{d^3\omega}{dx^3}\left(S^0-\text{etc.}\right)\\ &+\frac{d^3\mathfrak{w}}{dx^3}\left(\mathfrak{S}^0-\text{etc.}\right)\\ &+\text{etc.},\end{aligned}$$

ubi neminem offendat signum nihili litteris suffixum, siquidem non exponentem denotat, sed tantum ad has litteras ab iisdem nude positis distinguendas adhibetur.

COROLLARIUM 1

137. Si igitur formula integralis $\int Vdx$ habere debeat valorem maximum vel minimum, variationis inventae bina membra priora statim nihilo aequalia statui oportet, unde duae resultant aequationes differentiales, quibus indefinita relatio utriusque variabilis y et z ad x definitur.

COROLLARIUM 2

138. Etiamsi hic conditionum, quae forte pro initio et fine integrationis proponantur, nondum ratio habetur, tamen eae iam occulte in calculum ingrediuntur, quia litterae I et A terminos integrationis respiciunt. Interim tamen eae in ipsa aequationum differentialium traetatione iterum ex calculo expelluntur; dum enim formula integralis $\int Ldx = I$ eliditur, simul quantitas constans A egreditur.

COROLLARIUM 3

139. Expeditis autem aequationibus his duabus differentialibus idque generalissime, ut totidem constantes arbitrariae in calculum invehantur, quot integrationes institui oportuit, tum demum ad conditiones utriusque termini integrationis formulae $\int Vdx$ est attendendum, quandoquidem hinc ex reliquis variationis membris absolutis illae constantes determinari debent.

SCHOLION

140. Solutio huius problematis ita est comparata, ut iam satis sit perspicuum, ,quemadmodum etiam formulas magis complicatas, veluti si functio V plures formulas integrales involvat vel si quoque functio $\mathfrak{V}$ formulas novas integrales complectatur, expediri conveniat. Quin etiamnunc est manifestum, si huiusmodi formulae integrales plures tribus variabiles contineant, quomodo, tum variationes inveniri oporteat, atque adeo non solum taediosum, sed etiam superfluum foret, si copiosius hoc argumentum persequi vellem. Ad partem igitur huius doctrinae alteram multo abstrusiorem progredior, ubi etiam relationibus inter variabiles constitutis duae pluresve a se invicem minime pendentes in calculo relinquuntur.