CHAPTER IV

CONCERNING THE VARIATION OF COMPLICATED INTEGRAL FORMULAS INVOLVING TWO VARIABLES

PROBLEM 8

105. *On putting* $v=\int\mathfrak{V}dx$ *with some function* $\mathfrak{V}$ *arising of the two variables x, y and of their differentials*

$$dy=pdx,\ dp=qdx,\ q=rdx\ etc.,$$

if V should denote some function of v, to investigate the variation of some complicated integral formula $\int Vdx$.

SOLUTION

Because the quantity v is the formula of the integral $\int\mathfrak{V}dx$, the formula $\int Vdx$ is certainly complicated. Therefore since the function V is put to involve the quantity v only, we may place $dV=Ldv$; then truly for the function $\mathfrak{V}$ the differential of this will be

$$d\mathfrak{V}=\mathfrak{M}dx+\mathfrak{N}dy+\mathfrak{P}dp+\mathfrak{Q}dq+\mathfrak{R}dr+\text{etc}.$$

With these in place since the variation sought shall be

$$\delta\int Vdx=\int\delta(Vdx)=\int(\delta Vdx+Vd\delta x),$$

there is, by the reduction used above [§ 77]

$$\delta\int Vdx=V\delta x+\int(dx\delta V-dV\delta x).$$

But since by hypotheses there shall be $dV=Ldv$, there will also be $\delta V=L\delta v$ for the variation; truly on account of $v=\int\mathfrak{V}dx$ in the first place there will be $dv=\mathfrak{V}dx$ and thus $dV=L\mathfrak{V}dx$, then truly

$$\delta v=\delta\int\mathfrak{V}dx=\mathfrak{V}\delta x+\int(dx\delta\mathfrak{V}-d\mathfrak{V}\delta x)$$

and therefore

$$\delta V=L\mathfrak{V}\delta x+L\int(dx\delta\mathfrak{V}-d\mathfrak{V}\delta x)$$

and hence

$$\delta\int Vdx = V\delta x + \int\left(L\mathfrak{B}dx\delta x + Ldx\int\left(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x\right) - L\mathfrak{B}dx\delta x\right)$$

or

$$\delta\int Vdx = V\delta x + \int Ldx\int\left(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x\right).$$

Moreover from the preceding chapter [§ 86], it is apparent that

$$\begin{aligned}&\int\left(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x\right) = \delta\int\mathfrak{B}dx - \mathfrak{B}\delta x\\ &= \int\omega dx\left(\mathfrak{N} - \tfrac{d\mathfrak{P}}{dx} + \tfrac{dd\mathfrak{Q}}{dx^2} - \tfrac{d^3\mathfrak{R}}{dx^3} + \tfrac{d^4\mathfrak{S}}{dx^4} - \text{etc.}\right)\\ &+\omega\left(\mathfrak{P} - \tfrac{d\mathfrak{Q}}{dx} + \tfrac{dd\mathfrak{R}}{dx^2} - \tfrac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\right)\\ &+\tfrac{d\omega}{dx}\left(\mathfrak{Q} - \tfrac{d\mathfrak{R}}{dx} + \tfrac{dd\mathfrak{S}}{dx^2} - \text{etc.}\right)\\ &+\tfrac{dd\omega}{dx^2}\left(\mathfrak{R} - \tfrac{d\mathfrak{S}}{dx} + \text{etc.}\right)\\ &+\quad \text{etc.}\end{aligned}$$

on taking the element dx constant, and therefore on putting for brevity $\omega = \delta y - p\delta x$.

Truly hence since the substitution may bring difficulties, it will be better to repeat everything from the first source. Therefore since from the differentiation and variation of the quantity $\mathfrak{B}$ there becomes

$$\begin{aligned}dx\delta\mathfrak{B} - d\mathfrak{B}\delta x &= dx\left(\mathfrak{M}\delta x + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{ etc.}\right)\\ &\quad - \delta x\left(\mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{ etc.}\right),\end{aligned}$$

on account of

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \quad \text{etc}.$$

there will be

$$dx\delta\mathfrak{B} - d\mathfrak{B}\delta x = \mathfrak{N}dx\left(\delta y - p\delta x\right) + \mathfrak{P}dx\left(\delta p - q\delta x\right) + \mathfrak{Q}dx\left(\delta q - r\delta x\right) + \text{etc}.$$

Now on account of constant dx from § 79 there becomes

$$\delta y - p\delta x = \omega,\quad \delta p - q\delta x = \tfrac{d\omega}{dx},\quad \delta q - r\delta x = \tfrac{dd\omega}{dx^2}\qquad \delta r - s\delta x = \tfrac{d^3\omega}{dx^3}r\ \text{ etc.}$$

and thus there will be had

$$dx\delta\mathfrak{B} - d\mathfrak{B}\delta x = \mathfrak{N}\omega dx + \mathfrak{P}d\omega + \mathfrak{Q}\tfrac{dd\omega}{dx} + \mathfrak{R}\tfrac{d^3\omega}{dx^2} + \mathfrak{S}\tfrac{d^4\omega}{dx^3} + \text{etc.},$$

the integral of which indeed gives the above expression. Now there is put the integral $\int Ldx = I$ and there shall be

$$\delta\int Vdx = V\delta x + I\int(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x) - \int I(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x).$$

Now indeed there is readily deduced to become [§ 81-85]

$$\begin{aligned}\int I(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x) = \int\omega dx\Big(I\mathfrak{N} - \tfrac{d.I\mathfrak{P}}{dx} + \tfrac{dd.I\mathfrak{Q}}{dx^2} - \tfrac{d^3.I\mathfrak{R}}{dx^3} - \text{etc.}\Big)\\ +\omega\Big(I\mathfrak{P} - \tfrac{d.I\mathfrak{Q}}{dx} + \tfrac{dd.I\mathfrak{R}}{dx^2} - \text{etc.}\Big)\\ +\tfrac{d\omega}{dx}\Big(I\mathfrak{Q} - \tfrac{d.I\mathfrak{R}}{dx} + \text{etc.}\Big)\\ + \quad \text{etc.}\end{aligned}$$

with which substitution done the variation sought is concluded

$$\begin{aligned}&\delta\int Vdx = V\delta x + I\int\omega dx\Big(\mathfrak{N} - \tfrac{d\mathfrak{P}}{dx} + \tfrac{dd\mathfrak{Q}}{dx^2} - \tfrac{d^3\mathfrak{R}}{dx^3} + \tfrac{d^4\mathfrak{S}}{dx^4} - \text{etc.}\Big)\\ &-\int\omega dx\Big(I\mathfrak{N} - \tfrac{d.I\mathfrak{P}}{dx} + \tfrac{dd.I\mathfrak{Q}}{dx^2} - \tfrac{d^3.I\mathfrak{R}}{dx^3} + \text{etc.}\Big)\\ &+I\omega\Big(\mathfrak{P} - \tfrac{d\mathfrak{Q}}{dx} + \tfrac{dd\mathfrak{R}}{dx^2} - \tfrac{d^3.\mathfrak{S}}{dx^3} + \text{etc.}\Big)\\ &-\omega\Big(I\mathfrak{P} - \tfrac{d.I\mathfrak{Q}}{dx} + \tfrac{dd.I\mathfrak{R}}{dx^2} - \tfrac{d^3.I\mathfrak{S}}{dx^3} + \text{etc.}\Big)\\ &+\tfrac{Id\omega}{dx}\Big(\mathfrak{Q} - \tfrac{d\mathfrak{R}}{dx} + \tfrac{dd\mathfrak{S}}{dx^2} - \text{etc.}\Big)\\ &-\tfrac{d\omega}{dx}\Big(I\mathfrak{Q} - \tfrac{d.I\mathfrak{R}}{dx} + \tfrac{dd.I\mathfrak{S}}{dx^2} - \text{etc.}\Big)\\ &+\tfrac{Idd\omega}{dx^2}\Big(\mathfrak{R} - \tfrac{d\mathfrak{S}}{dx} + \text{etc.}\Big)\\ &-\tfrac{dd\omega}{dx^2}\Big(I\mathfrak{R} - \tfrac{d.I\mathfrak{S}}{dx} + \text{etc.}\Big)\\ &+\tfrac{Id^3\omega}{dx^3}(\mathfrak{S} - \text{etc.})\\ &-\tfrac{d^3\omega}{dx^3}(I\mathfrak{S} - \text{etc.})\\ &+ \quad \text{etc.}\end{aligned}$$

Here if the two previous differentiated parts are integrated again, with the reduction made of eh remaining we will obtain the value *Ldx* in place of *dI* by restoring

$$\begin{aligned}\delta\int Vdx = V\delta x &+ \int Ldx\int \omega dx\left(\mathfrak{N} - \frac{d\mathfrak{P}}{dx} + \frac{dd\mathfrak{Q}}{dx^2} - \frac{d^3\mathfrak{R}}{dx^3} + \text{etc.}\right)\\ &+\int \omega dx\left(L\mathfrak{P} - \frac{Ld\mathfrak{Q}+d.L\mathfrak{Q}}{dx} + \frac{Ldd\mathfrak{R}+d.Ld\mathfrak{R}+dd.L\mathfrak{R}}{dx^2} - \text{etc.}\right)\\ &+\omega\left(L\mathfrak{Q} - \frac{Ld\mathfrak{R}+d.L\mathfrak{R}}{dx} + \frac{Ldd\mathfrak{S}+d.Ld\mathfrak{S}+dd.L\mathfrak{S}}{dx^2} - \text{etc.}\right)\\ &+\frac{d\omega}{dx}\left(L\mathfrak{R} - \frac{Ld\mathfrak{S}+d.L\mathfrak{S}}{dx} + \text{etc.}\right)\\ &+\frac{dd\omega}{dx^2}\left(L\mathfrak{S} - \text{etc.}\right)\\ &+\quad \text{etc.}\end{aligned}$$

which form may be considered the most simple and most convenient to use.

COROLLARY 1

106. If a relation of this kind is sought between *x* and *y*, so that the integral $\int Vdx$ emerges a maximum or a minimum, it is required to equate the integral parts of the variation to zeros ; because in general it cannot happen, but it is required to look to the limits [of integration] , as far as the integral $\int Vdx$ is extended ; for which if we put to become $I = \int Ldx = A$, from the first form we may deduce this equation

$$0 = (A-I)\mathfrak{N} - \frac{d.(A-I)\mathfrak{P}}{dx} + \frac{dd.(A-I)\mathfrak{Q}}{dx^2} - \frac{d^3.(A-I)\mathfrak{R}}{dx^3} + \text{etc.}$$

COROLLARIUM 2

107. But however this equation for whatever case brought forwards may be discussed, yet always it is reached from that, so that the integral formula must always be moved away by differentiating $I = \int Ldx$, from which operation likewise it is evident the quantity *A* is to be extracted ; and thus the resulting equation will no longer depend on the ends of the integration.

COROLLARY 3

108. But if in general with the variation of the formula of the integral $\int Vdx$ required to be found we may put the value $\int Ldx = I$ of the total integral corresponding $= A$, the variation sought we may express thus

$$\begin{aligned}\delta\int Vdx = V\delta x + \int \omega dx\Big(&(A-I)\mathfrak{N} - \frac{d.(A-I)\mathfrak{P}}{dx} + \frac{dd.(A-I)\mathfrak{Q}}{dx^2} - \frac{d^3.(A-I)\mathfrak{R}}{dx^3} + \text{etc.}\Big)\\ &+\omega\Big(L\mathfrak{Q} - \frac{Ld\mathfrak{R}+d.L\mathfrak{R}}{dx} + \frac{Ldd\mathfrak{S}+d.Ld\mathfrak{S}+dd.L\mathfrak{S}}{dx^2} - \text{etc.}\Big)\\ &+\frac{d\omega}{dx}\Big(L\mathfrak{R} - \frac{Ld\mathfrak{S}+d.L\mathfrak{S}}{dx} + \text{etc.}\Big)\\ &+\frac{dd\omega}{dx^2}\big(L\mathfrak{S} - \text{etc.}\big)\\ &+\quad \text{etc.},\end{aligned}$$

where $A-I$ is the value of the formula $\int Ldx$ from the outer end of the integration taken back to some indefinite middle place.

SCHOLIUM

109. In the solution of this problem a shortcut has presented itself, with which also the analysis in the above chapter cannot tolerably conclude. For since there (§ 79) we came upon

$$\delta\int Vdx = V\delta x + \int(dx\delta V - dV\delta x),$$

on account of

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

and

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}$$

there will be

$$dV = dx(M + Np + Pq + Qr + Rs + \text{etc.})$$

and hence it is deduced

$$dx\delta V - dV\delta x = dx\big(N(\delta y - p\delta x) + P(\delta p - q\delta x) + Q(\delta q - r\delta x) + \text{etc.}\big).$$

Now if for brevity there is put $\delta y - p\delta x = \omega$, there will be on differentiation

$$\delta(pdx) - qdx\delta x - p\delta dx = d\omega;$$

but

$$\delta(pdx) = pd\delta x + \delta pdx,$$

hence

$$\delta p - q\delta x = \tfrac{d\omega}{dx}.$$

In a like manner this formula on differentiation on account of $dp = qdx$ and $dq = rdx$ becomes

$$qd\delta x + \delta qdx - qd\delta x - dq\delta x = dx(\delta q - r\delta x) = d.\tfrac{d\omega}{dx},$$

from which it is evident on putting $\delta y - p\delta x = \omega$ to become

$$\delta p - q\delta x = \tfrac{d\omega}{dx}, \quad \delta q - r\delta x = \tfrac{1}{dx}d.\tfrac{d\omega}{dx} = \tfrac{dd\omega}{dx^2}$$

with *dx* assumed constant,

$$\delta r - s\delta x = \tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{d\omega}{dx} = \tfrac{d^3\omega}{dx^3}, \quad \text{etc.}$$

On account of which there becomes

$$dx\delta V - dV\delta x = dx\left(N\omega + P\tfrac{d\omega}{dx} + Q\tfrac{dd\omega}{dx^2} + R\tfrac{d^3\omega}{dx^3} + S\tfrac{d^4\omega}{dx^4} + \text{etc.}\right),$$

if indeed the differential *dx* is taken constant.

PROBLEM 9

110. *If there should be* $v = \int \mathfrak{B}dx$ *with*

$$d\mathfrak{B} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + etc. \ \textit{arising},$$

then truly V shall be some function not only of the quantities

$$x, \quad y, \quad p = \tfrac{dy}{dx}, \quad q = \tfrac{dp}{dx}, \quad r = \tfrac{dq}{dx} \quad etc.,$$

but also implicating the integral formula itself $v = \int \mathfrak{B}dx$, *to investigate the nature of the complicated integral formula* $\int Vdx$.

SOLUTION

Because *V* is a function of the quantities *v, x, y, p, q, r* etc., the differential of this is taken, which shall be

$$dV = Ldv + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

and the variation of V will be considered thus expressed

$$\delta V = L\delta v + M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.};$$

then truly there may be noted on account of

$$dv = \mathfrak{B}dx, \quad dy = pdx, \; dp = qdx \quad \text{etc.}$$

to become

$$dV = dx\left(L\mathfrak{B} + M + Np + Pq + Qr + Rs + \text{etc.}\right)$$

and

$$\delta\mathfrak{B} = \mathfrak{M}\delta x + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}$$

In addition we may consider

$$\delta v = \int\left(\mathfrak{B}\delta dx + dx\delta\mathfrak{B}\right) = \mathfrak{B}\delta x + \int\left(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x\right),$$

from which on putting $\delta y - p\delta x = \omega$ there will be by what has been found before

$$\delta v = \mathfrak{B}\delta x + \int dx\left(\mathfrak{N}\omega + \mathfrak{P}\tfrac{d\omega}{dx} + \mathfrak{Q}\tfrac{dd\omega}{dx^2} + \mathfrak{R}\tfrac{d^3\omega}{dx^3} + \mathfrak{S}\tfrac{d^4\omega}{dx^4} + \text{etc.}\right),$$

where hence for greater convenience we have taken dx constant.

With these prepared, since the variation sought shall be

$$\delta\int Vdx = V\delta x + \int\left(dx\delta V - dV\delta x\right),$$

so that we may be able to use as with the reduction found above, we put

$$dV = Ldv + dW$$

so that there shall be

$$\delta V = L\delta v + \delta W \quad \text{and} \quad dW = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc}.$$

On account of which we may arrive at this form

$$\delta\int Vdx = V\delta x + \int\left(Ldx\delta v - Ldv\delta x\right) + \int\left(dx\delta W - dW\delta x\right),$$

where there is

$$dx\delta W - dW\delta x = dx\left(N\omega + P\frac{d\omega}{dx} + Q\frac{dd\omega}{dx^2} + R\frac{d^3\omega}{dx^3} + \text{etc.}\right).$$

Then truly there shall be

$$dx\delta v - dv\delta x = dx\int dx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + \text{etc.}\right)$$

on account of $dv\delta x = \mathfrak{V}dx\delta x$. With which put in place the variation sought is deduced

$$\delta\int Vdx = V\delta x + \int Ldx\int dx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + \text{etc.}\right)$$
$$+\int dx\left(N + P\frac{d\omega}{dx} + Q\frac{dd\omega}{dx^2} + R\frac{d^3\omega}{dx^3} + \text{etc.}\right).$$

Now so that we may reduce this form further, we may put the integral $\int Ldx = I$ thus assumed, so that for the beginning from which the integral $\int Vdx$ is taken, it vanishes , but for the end, where the integral $\int Vdx$ is terminated, it becomes $I = A$; and thus there becomes

$$\delta\int Vdx = V\delta x + A\int dx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} + \text{etc.}\right)$$
$$-\int Idx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + \text{etc.}\right)$$
$$+\int dx\left(N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + \text{etc.}\right),$$

to which we may put in place the contracted form

$$N + (A - I)\mathfrak{N} = N',$$
$$P + (A - I)\mathfrak{P} = P',$$
$$Q + (A - I)\mathfrak{Q} = Q',$$
$$R + (A - I)\mathfrak{R} = R'$$
$$\text{etc.},$$

so that the form arises similar to that which we have treated above [§ 79-85]

$$\delta\int Vdx = V\delta x + \int dx\left(N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + \frac{R'd^3\omega}{dx^3} + \text{etc.}\right);$$

where hence if the differential ω may be eliminated after the integral sign, we may arrive at this expression following § 86 :

$$\begin{aligned}&\delta\int Vdx=\int\omega dx\left(N'-\tfrac{dP'}{dx}+\tfrac{ddQ'}{dx^2}-\tfrac{d^3R'}{dx^3}+\tfrac{d^4S'}{dx^4}-\text{etc.}\right)\\&+V\delta x+\omega\left(P'-\tfrac{dQ'}{dx}+\tfrac{ddR'}{dx}-\tfrac{d^3S'}{dx^3}+\text{etc.}\right)\\&\text{Const.}+\tfrac{d\omega}{dx}\left(Q'-\tfrac{dR'}{dx}+\tfrac{ddS'}{dx^2}-\text{etc.}\right)\\&+\tfrac{dd\omega}{dx}\left(R'-\tfrac{dS'}{dx}+\text{etc.}\right)\\&+\tfrac{d^3\omega}{dx^3}\left(S'-\text{etc.}\right)\\&+\text{etc.}\end{aligned}$$

Moreover a value must be attributed to the constant arising from an integration of this kind, so that for the start of the integration formula $\delta\int Vdx$ the parts may be reduced completely to zero, if indeed the first part of the integral is thus taken, so that for the same the beginning vanishes ; then truly the whole expression is required to be produced at the end of the integration, for which now we are able to put in place $\int Ldx = I = A$.

COROLLARY 1

111. The variability must be understood for the whole extent of the integration in the integral part, but in the completed parts it suffices to consider the start and finish of the integration, but for each end the conditions of the prescribed variation provide the prescribed values $\delta x,\ \omega,\ \frac{d\omega}{dx}, \frac{dd\omega}{dx^2}$, etc. And after the constant should be determined properly from the initial conditions, then there remains, that the individual parts be adapted at the end of the integration.

COROLLARY 2

112. Therefore for the start of the integration, where $I = 0$, in the first place there will be

$$N' = N + A\mathfrak{N},\quad P' = P + A\mathfrak{P},\quad Q' = Q + A\mathfrak{Q},\ \ R' = R + A\mathfrak{R}\ \ \text{etc.},$$

for the differential truly on account of $dI = Ldx$ there will be

$$\frac{dN'}{dx}=\frac{dN}{dx}+\frac{Ad\mathfrak{N}}{dx}-L\mathfrak{N}$$

and thus for the remaining in a like manner for the following differentials

$$\frac{ddN'}{dx^2}=\frac{ddN}{dx^2}+\frac{Add\mathfrak{N}}{dx^2}-\frac{2Ld\mathfrak{N}}{dx}-\frac{\mathfrak{N}dL}{dx}\,.$$

COROLLARY 3

113. But for the end of the integration, where $I = A$, there shall be

$$N' = N,\ P' = P,\ Q' = Q,\ R' = R \quad \text{etc.},$$

the differential values themselves will be considered thus

$$\frac{dN'}{dx} = \frac{dN}{dx} - L\mathfrak{N}, \quad \frac{dP'}{dx} = \frac{dP}{dx} - L\mathfrak{P}, \quad \frac{dQ'}{dx} = \frac{dQ}{dx} - L\mathfrak{Q}, \quad \text{etc.},$$

truly the second order in this manner

$$\frac{ddN'}{dx^2} = \frac{ddN}{dx^2} - \frac{2Ld\mathfrak{N}}{dx} - \frac{\mathfrak{N}dL}{dx},$$
$$\frac{ddP'}{dx^2} = \frac{ddP}{dx^2} - \frac{2Ld\mathfrak{P}}{dx} - \frac{\mathfrak{P}dL}{dx},$$

and thus so forth.

SCHOLIUM 1

114. Though the nature of the variation and also of the questions concerning that now has been explained well enough, yet both the worth as well as the novelty of this argument may be considered to require father illustration, indeed on the condition that being impressed more often it may become superfluous. Therefore since before we may have used geometry and with the application of this calculus to maxima and minima, here we will consider the more general situation for analysis alone to explain this principle further.

Therefore in the first place some relation may be considered between the two variables *x* and *y*, either that shall be known or requiring at last to be defined, and thence from the form the formula of some integral $\int Vdx$ may be considered, which taken to extend between certain limits or on integrating from a given initial point to a given final point certainly ought to record a sure value. Then that relation between *x* and *y*, whatever it should be, may be changed in some manner by an infinitely small amount, so that with the variations of the individual *x* increased by some amount δx now thus there may correspond the same *y* increased by some variation δy also, where indeed it is to be noted both at the beginning as well as the end an account of these variations by the conditions of the question are to be given, but in the middle these variations thus may be assumed generally, so that they may plainly not be connected by any rule. Then from this relation with the variation of the same integral formula $\int Vdx$ set out with the same beginning and the same end value, or considered to be defined contained between the same limits, and now everything sought in this may be turned around, so that the excess of the variation of this final value over that value of the formula $\int Vdx$ may be investigated. Which excess since it may be indicated by $\delta\int Vdx$, which form is the variation of the formula $\int Vdx$, the solution of this question we have given thus far set out widely, in which the quantity V is some function not only of these x, y, p, q, r, s etc., but also involving a certain integral formula $v = \int \mathfrak{V} dx$ above in some manner, that may be included.

SCHOLIUM 2

115. Because in the preceding chapter we have quietly assumed concerning the constant quantity requiring to be added to the variation, certainly as the part of the integral variation becomes involved by itself, this has been seen to be set out more carefully in the solution of this problem. Evidently since questions of this kind, which are reduced to integral formulas, shall always be with respect to the limits of integration, if indeed the integral is none other than the sum of the elements continued from a given boundary or beginning to another terminus or end, this consideration in a word is the essential of all integration, without which idea indeed it is not possible to consider the value of an integral. On account of which limits of the constituted integral, evidently the beginning and the end, immediately the part of the variation of the integral thus has been accepted, so that for the beginning nothing emerges, then it is required to add a constant of this kind, so that also the absolved parts for the same beginning may be destroyed and thus the entire expression of the variation may be reduced to nothing. Since with that done, one can at last proceed to the end of the integration, so that with this truth in place the variation of the formula of the integral from the beginning extended to the end may be obtained.

But this principle of variation can be applied to questions of two kinds ; while in one the relation between the variables x and y is assumed to be given and the variation of the formula of the integral likewise given $\int V dx$ may be investigated, afterwards some variations may be attributed to the variables x and y through the whole extent of the integration, but in the other kind that relation of the variables x et y is sought, so that the variation of the integral formula $\int V dx$ shall be endowed with a certain property ; just as if this formula must receive a maximum or a minimum, it is necessary that the variation goes to zero. Again when the two cases present themselves, as the maximum or minimum is required to be considered, or some variations may be attributed to x and y themselves, or if these variations are to be tied up in accordance with a certain rule. From which it is evident can be extended much wider, than that indeed called into use at this stage.

PROBLEM 10

116. *If the function V besides the two variables x, y with their differential values*

$$p = \frac{dy}{dx}, \quad q = \frac{dp}{dx}, \quad r = \frac{dq}{dx} \quad etc.$$

also involves the two or more integral formulas

$$v = \int \mathfrak{B}dx, \quad v' = \int \mathfrak{B}' dx \quad etc.,$$

so that there shall be

$$\delta\mathfrak{B} = \mathfrak{M}\delta x \ + \mathfrak{N}\delta y \ + \mathfrak{P}\delta p \ + \mathfrak{Q}\delta q \ + \mathfrak{R}\delta r \ + \text{etc.},$$
$$\delta\mathfrak{B}' = \mathfrak{M}'\delta x + \mathfrak{N}'\delta y + \mathfrak{P}'\delta p + \mathfrak{Q}'\delta q + \mathfrak{R}'\delta r + \text{etc.}$$

and on taking the differentials

$$dV = Ldv + L'dv' + Mdx + Ndy + Pdp + Qdq + \text{ etc.},$$

to find the variation of the integral formulas $\int Vdx$.

SOLUTION

If the solution of this problem and of the preceding may be put in place in the same manner, it will soon become apparent that the calculation is not to be disturbed by the two integral formulas

$$v = \int \mathfrak{B}dx, \quad \text{et} \quad v' = \int \mathfrak{B}' dx$$

nor also, if more of the same kind are involved. Whereby the whole solution will be returned finally to this, so that the first integral thus may be taken with the limits in place

$$\int Ldx = I \quad \text{and} \quad \int L'dx = I',$$

so that they vanish for the start of the integration, then truly for the end of the integration there becomes $I = A$ and $I' = A'$; with which quantities found there is put in place again

$$N+(A-I)\mathfrak{N}+(A'-I')\mathfrak{N}'=N',$$
$$P+(A-I)\mathfrak{P}+(A'-I')\mathfrak{P}'=P',$$
$$Q+(A-I)\mathfrak{Q}+(A'-I')\mathfrak{Q}'=Q',$$
$$R+(A-I)\mathfrak{R}+(A'-I')\mathfrak{R}'=R'$$
$$\text{etc.}$$

and the variation sought will be, provided each variable x and y are attributed some variation, from the preceding solutions [§ 110]

$$\begin{aligned}\delta\int Vdx = \int \omega dx\left(N'-\tfrac{dP'}{dx}+\tfrac{ddQ'}{dx^2}-\tfrac{d^3R'}{dx^3}+\tfrac{d^4S'}{dx^4}-\text{etc.}\right)\\
+V\delta x+\omega\left(P'-\tfrac{dQ'}{dx}+\tfrac{ddR'}{dx}-\tfrac{d^3S'}{dx^3}+\text{etc.}\right)\\
+\text{Const.}+\tfrac{d\omega}{dx}\left(Q'-\tfrac{dR'}{dx}+\tfrac{ddS'}{dx^2}-\text{etc.}\right)\\
+\tfrac{dd\omega}{dx}\left(R'-\tfrac{dS'}{dx}+\text{etc.}\right)\\
+\tfrac{d^3\omega}{dx^3}\left(S'-\text{etc.}\right)\\
+\text{etc.}\end{aligned}$$

where for the sake of convenience the element dx has been assumed constant.

COROLLARY

117. Therefore if also more integral formulas of this kind $\int\mathfrak{B}dx$ may be introduced in some manner into the function V, the expression of the variation sought thence is not changed, but only it is agreed to duly define the quantities N', P', Q', R' etc. from these.

SCHOLIUM

118. And if the integral formulas

$$I=\int Ldx,\quad I'=\int L'dx$$

involve two variables and thus may not seem possible to receive fixed values, yet in all questions of this kind it is require always to assess carefully the certain reliable relation supposed between the two variables x and y, either that is given completely or at last may be defined by the calculation. Therefore now with this relation to be called into use itself, so that the quantity y may be considered the image of the function x, these integral formulas certainly will choose the determined values.

PROBLEM 11

119. *If the function* $\mathfrak{V}$ *besides the variables x and y and the differential values p, q, r, s etc. of these, also involves the integral formula* $u=\int \mathfrak{v}dx$ *, so that the differential of this shall be*

$$d\mathfrak{V}=\mathfrak{L}du+\mathfrak{M}dx+\mathfrak{N}dy+\mathfrak{P}dp+\mathfrak{Q}dq+\mathfrak{R}dr+etc.$$

with

$$d\mathfrak{v}=\mathfrak{m}dx+\mathfrak{n}dy+\mathfrak{p}dp+\mathfrak{q}dq+\mathfrak{r}dr+etc.\ present,$$

then truly V shall be some function of x, y, p, q, r etc. and in the above integral formulas the integral $v=\int \mathfrak{V}dx$ *, so that there shall be*

$$dV=Ldv+Mdx+Ndy+Pdp+Qdq+Rdr+etc.,$$

to find the variation of the integral formula $\int Vdx$ *.*

SOLUTION

From Problem 9 we find immediately the variation of the integral formula $\int \mathfrak{V}dx=v$ *;* for with the limits of the integration in place and thus on taking the integral $\int \mathfrak{L}dx=\mathfrak{J}$, so that it vanished for the start of the integration, for the end there becomes $\mathfrak{J}=\mathfrak{A}$, then there becomes for the sake of brevity

$$\mathfrak{N}+(\mathfrak{A}-\mathfrak{J})\mathfrak{u}=\mathfrak{N}',\quad \mathfrak{P}+(\mathfrak{A}-\mathfrak{J})\mathfrak{p}=\mathfrak{P}',\quad \mathfrak{Q}+(\mathfrak{A}-\mathfrak{J})\mathfrak{q}=\mathfrak{Q}',\quad \text{etc.};$$

there will be from the solution of that problem

$$d\mathfrak{v}=\mathfrak{V}dx+\int dx\left(\mathfrak{N}'\omega+\frac{\mathfrak{P}'d\omega}{dx}+\frac{\mathfrak{Q}'dd\omega}{dx^2}+\frac{\mathfrak{R}'d^3\omega}{dx^3}+\text{etc}\right)$$

on putting $\omega=\delta y-p\delta x$ and on assuming dx constant.

Now indeed since there is sought $\delta\int Vdx$, on account of

$$\delta\int Vdx=V\delta x+\int(dx\delta V-dV\delta x)$$

therefore for brevity on putting

$$dV=Ldv+dW \quad \text{and} \quad \delta V=L\delta v+\delta W,$$

so that there shall be

$$dW = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

there will be, as we have seen in that place

$$\delta\int Vdx = V\delta x + \int\left(Ldx\delta v - Ldv\delta x\right) + dx\left(N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + \text{etc.}\right);$$

where if in place of dv and δv the values found in this manner are substituted, there will be

$$dx\delta v - dv\delta x = dx\int dx\left(\mathfrak{N}'\omega + \frac{\mathfrak{P}'d\omega}{dx} + \frac{\mathfrak{Q}'dd\omega}{dx^2} + \frac{\mathfrak{R}'d^3\omega}{dx^3} + \text{etc.}\right).$$

Now there is put $\int Ldx = I$ with the integral thus taken, so that it vanishes at the start of the integration, but at the end it becomes $I = A$, and we may have

$$\int L\left(dx\delta v - dv\delta x\right) = \int\left(A - I\right)dx\left(\mathfrak{N}'\omega + \frac{\mathfrak{P}'d\omega}{dx} + \frac{\mathfrak{Q}'dd\omega}{dx^2} + \frac{\mathfrak{R}'d^3\omega}{dx^3} + \text{etc.}\right).$$

The above values assumed for $\mathfrak{N}'$, $\mathfrak{P}'$, $\mathfrak{Q}'$, $\mathfrak{R}'$ etc. are restored but in order to contract the calculation there is put

$$\begin{aligned}
N + (A-I)\mathfrak{N} + (A-I)(\mathfrak{A}-\mathfrak{J})\mathfrak{u} &= N',\\
P + (A-I)\mathfrak{P} + (A-I)(\mathfrak{A}-\mathfrak{J})\mathfrak{p} &= P',\\
Q + (A-I)\mathfrak{Q} + (A-I)(\mathfrak{A}-\mathfrak{J})\mathfrak{q} &= Q',\\
R + (A-I)\mathfrak{R} + (A-I)(\mathfrak{A}-\mathfrak{J})\mathfrak{r} &= R'\\
&\text{etc.}
\end{aligned}$$

and it is evident the variation sought to become

$$\delta\int Vdx = V\delta x + \int dx\left(\mathfrak{N}'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + \frac{R'd^3\omega}{dx^3} + \text{etc.}\right),$$

which form again is produced in the same expression , as we have shown at the end of Problem 9 (§ 110), as hence it may be superfluous to add here anew.

COROLLARY 1

120. Therefore here the integral formula $\int V dx$, the variation of which we have assigned, thus has been prepared, so that not only does it involve a function V of the integral formula $\int \mathfrak{V} dx$, but also this function $\mathfrak{V}$ is itself included in another integral $\int \mathfrak{v} dx$, where indeed the function $\mathfrak{v}$ involves no further integral formula.

COROLLARY 2

121. But if this function $\mathfrak{v}$ and the above integral formula may itself be involved, now it has become evident enough, how then it may be necessary to put the solution in place, if indeed then the values N', P', Q', R' etc. take the above parts and depend on the last formula of the integral.

SCHOLIUM

122. Therefore in whatever way the formula of the integral $\int V dx$ should be put in place, the precepts set out at this stage generally suffice for the variation of this to be investigated, even if perhaps it should be infinitely complicated. Therefore since all the expressions involving two variables, the variations of which shall be required to be found at some time, either they shall be free from integral formulas or one or more included in themselves and these either simple or generally complicated, to this part of the calculus of variations, which is involved with two variables, it may be abundantly satisfactory, that scarcely anything further could be desired. On account of which we may progress to formulas of three variables and indeed the first such, the relation of which can be put in place by two equations to be defined, so that the two variables are able to be regarded as functions of the third, and either this twofold relation shall be known or requiring to be investigated from the nature of the variation itself.

CAPUT IV

DE VARIATIONE FORMULARUM INTEGRALIUM COMPLICATATARUM DUAS VARIABILES INVOLVENTIUM

PROBLEMA 8

105. *Posito* $v=\int\mathfrak{B}dx$ *existente* $\mathfrak{B}$ *functione quacunque binarum variabilium x, y earumque differentialium*

$$dy=pdx,\ dp=qdx,\ q=rdx\ etc.$$

si V denotet functionem quamcunque ipsius v, investigare variationem formulae integralis complicatae $\int Vdx$.

SOLUTIO

Quia quantitas *v* ipsa est formula integralis $\int\mathfrak{B}dx$, formula $\int Vdx$ est utique complicata. Cum igitur functio *V* solam quantitatem *v* involvere ponatur, statuamus $dV=Ldv$; tum vero pro functione $\mathfrak{B}$ sit eius differentiale

$$d\mathfrak{B}=\mathfrak{M}dx+\mathfrak{N}dy+\mathfrak{P}dp+\mathfrak{Q}dq+\mathfrak{R}dr+\text{etc}.$$

His positis cum variatio quaesita sit

$$\delta\int Vdx=\int\delta(Vdx)=\int(\delta Vdx+Vd\delta x),$$

est per reductionem supra [§ 77] adhibitam

$$\delta\int Vdx=V\delta x+\int(dx\delta V-dV\delta x).$$

Cum autem per hypothesin sit $dV=Ldv$, erit etiam pro variatione

$\delta V=L\delta v$; verum ob $v=\int\mathfrak{B}dx$ erit primo $dv=\mathfrak{B}dx$ ideoque $dV=L\mathfrak{B}dx$, tum vero

$$\delta v=\delta\int\mathfrak{B}dx=\mathfrak{B}\delta x+\int(dx\delta\mathfrak{B}-d\mathfrak{B}\delta x)$$

ac propterea

$$\delta V=L\mathfrak{B}\delta x+L\int(dx\delta\mathfrak{B}-d\mathfrak{B}\delta x)$$

hincque

$$\delta\int Vdx = V\delta x + \int\left(L\mathfrak{B}dx\delta x + Ldx\int\left(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x\right) - L\mathfrak{B}dx\delta x\right)$$

seu

$$\delta\int Vdx = V\delta x + \int Ldx\int\left(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x\right).$$

Ex praecedente autem capite [§ 86] patet esse

$$\begin{aligned}&\int\left(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x\right) = \delta\int\mathfrak{B}dx - \mathfrak{B}\delta x\\ &= \int\omega dx\left(\mathfrak{N} - \tfrac{d\mathfrak{P}}{dx} + \tfrac{dd\mathfrak{Q}}{dx^2} - \tfrac{d^3\mathfrak{R}}{dx^3} + \tfrac{d^4\mathfrak{S}}{dx^4} - \text{etc.}\right)\\ &+\omega\left(\mathfrak{P} - \tfrac{d\mathfrak{Q}}{dx} + \tfrac{dd\mathfrak{R}}{dx^2} - \tfrac{d^3\mathfrak{S}}{dx^3} + \text{etc.}\right)\\ &+\tfrac{d\omega}{dx}\left(\mathfrak{Q} - \tfrac{d\mathfrak{R}}{dx} + \tfrac{dd\mathfrak{S}}{dx^2} - \text{etc.}\right)\\ &+\tfrac{dd\omega}{dx^2}\left(\mathfrak{R} - \tfrac{d\mathfrak{S}}{dx} + \text{etc.}\right)\\ &+\quad \text{etc.}\end{aligned}$$

sumto elemento *dx* constante et posito brevitatis ergo $\omega = \delta y - p\delta x$.

Verum cum hinc substitutio molestias pariat, praestabit ex primo fonte rem repetere. Cum igitur ex differentiali et variatione quantitatis $\mathfrak{B}$ fiat

$$\begin{aligned}dx\delta\mathfrak{B} - d\mathfrak{B}\delta x = dx&\left(\mathfrak{M}\delta x + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{ etc.}\right)\\ -\delta x&\left(\mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{ etc.}\right),\end{aligned}$$

ob

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \quad \text{etc}.$$

erit

$$dx\delta\mathfrak{B} - d\mathfrak{B}\delta x = \mathfrak{N}dx\left(\delta y - p\delta x\right) + \mathfrak{P}dx\left(\delta p - q\delta x\right) + \mathfrak{Q}dx\left(\delta q - r\delta x\right) + \text{etc}.$$

Verum ob *dx* constans ex § 79 fit

$$\delta y - p\delta x = \omega,\quad \delta p - q\delta x = \tfrac{d\omega}{dx},\quad \delta q - r\delta x = \tfrac{dd\omega}{dx^2}\qquad \delta r - s\delta x = \tfrac{d^3\omega}{dx^3}r \ \text{ etc.}$$

sicque habebitur

$$dx\delta\mathfrak{B} - d\mathfrak{B}\delta x = \mathfrak{N}\omega dx + \mathfrak{P}d\omega + \mathfrak{Q}\tfrac{dd\omega}{dx} + \mathfrak{R}\tfrac{d^3\omega}{dx^2} + \mathfrak{S}\tfrac{d^4\omega}{dx^3} + \text{etc.},$$

cuius quidem integrale praebet superiorem expressionem. Ponatur nunc integrale $\int Ldx = I$ eritque

$$\delta\int Vdx = V\delta x + I\int\left(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x\right) - \int I\left(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x\right).$$

Nunc vero facile [§ 81-85] colligitur fore

$$\begin{aligned}\int I\left(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x\right) = \int \omega dx\left(I\mathfrak{N} - \tfrac{d.I\mathfrak{P}}{dx} + \tfrac{dd.I\mathfrak{Q}}{dx^2} - \tfrac{d^3.I\mathfrak{R}}{dx^3} - \text{etc.}\right)\\ +\omega\left(I\mathfrak{P} - \tfrac{d.I\mathfrak{Q}}{dx} + \tfrac{dd.I\mathfrak{R}}{dx^2} - \text{etc.}\right)\\ +\tfrac{d\omega}{dx}\left(I\mathfrak{Q} - \tfrac{d.I\mathfrak{R}}{dx} + \text{etc.}\right)\\ + \quad \text{etc.}\end{aligned}$$

unde facta substitutione concluditur variatio quaesita

$$\begin{aligned}&\delta\int Vdx = V\delta x + I\int \omega dx\left(\mathfrak{N} - \tfrac{d\mathfrak{P}}{dx} + \tfrac{dd\mathfrak{Q}}{dx^2} - \tfrac{d^3\mathfrak{R}}{dx^3} + \tfrac{d^4\mathfrak{S}}{dx^4} - \text{etc.}\right)\\ &-\int \omega dx\left(I\mathfrak{N} - \tfrac{d.I\mathfrak{P}}{dx} + \tfrac{dd.I\mathfrak{Q}}{dx^2} - \tfrac{d^3.I\mathfrak{R}}{dx^3} + \text{etc.}\right)\\ &+I\omega\left(\mathfrak{P} - \tfrac{d\mathfrak{Q}}{dx} + \tfrac{dd\mathfrak{R}}{dx^2} - \tfrac{d^3.\mathfrak{S}}{dx^3} + \text{etc.}\right)\\ &-\omega\left(I\mathfrak{P} - \tfrac{d.I\mathfrak{Q}}{dx} + \tfrac{dd.I\mathfrak{R}}{dx^2} - \tfrac{d^3.I\mathfrak{S}}{dx^3} + \text{etc.}\right)\\ &+\tfrac{Id\omega}{dx}\left(\mathfrak{Q} - \tfrac{d\mathfrak{R}}{dx} + \tfrac{dd\mathfrak{S}}{dx^2} - \text{etc.}\right)\\ &-\tfrac{d\omega}{dx}\left(I\mathfrak{Q} - \tfrac{d.I\mathfrak{R}}{dx} + \tfrac{dd.I\mathfrak{S}}{dx^2} - \text{etc.}\right)\\ &+\tfrac{Idd\omega}{dx^2}\left(\mathfrak{R} - \tfrac{d\mathfrak{S}}{dx} + \text{etc.}\right)\\ &-\tfrac{dd\omega}{dx^2}\left(I\mathfrak{R} - \tfrac{d.I\mathfrak{S}}{dx} + \text{etc.}\right)\\ &+\tfrac{Id^3\omega}{dx^3}\left(\mathfrak{S} - \text{etc.}\right)\\ &-\tfrac{d^3\omega}{dx^3}\left(I\mathfrak{S} - \text{etc.}\right)\\ &+ \quad \text{etc.}\end{aligned}$$

Si hic partes binae priores differentiatae iterum integrentur, reliquarum facta reductione impetrabimus loco *dI* valorem *Ldx* restituendo

$$\begin{aligned}\delta\int Vdx &= V\delta x+\int Ldx\int \omega dx\left(\mathfrak{N}-\frac{d\mathfrak{P}}{dx}+\frac{dd\mathfrak{Q}}{dx^2}-\frac{d^3\mathfrak{R}}{dx^3}+\text{etc.}\right)\\
&+\int \omega dx\left(L\mathfrak{P}-\frac{Ld\mathfrak{Q}+d.L\mathfrak{Q}}{dx}+\frac{Ldd\mathfrak{R}+d.Ld\mathfrak{R}+dd.L\mathfrak{R}}{dx^2}-\text{etc.}\right)\\
&+\omega\left(L\mathfrak{Q}-\frac{Ld\mathfrak{R}+d.L\mathfrak{R}}{dx}+\frac{Ldd\mathfrak{S}+d.Ld\mathfrak{S}+dd.L\mathfrak{S}}{dx^2}-\text{etc.}\right)\\
&+\frac{d\omega}{dx}\left(L\mathfrak{R}-\frac{Ld\mathfrak{S}+d.L\mathfrak{S}}{dx}+\text{etc.}\right)\\
&+\frac{dd\omega}{dx^2}\left(L\mathfrak{S}-\text{etc.}\right)\\
&+\quad \text{etc.}\end{aligned}$$

quae forma videtur simplicissima et ad usum maxime accommodata.

COROLLARIUM 1

106. Si eiusmodi relatio inter x et y quaeratur, ut integrale $\int Vdx$ maximum mlnlmumve evadat,. variationis partes integrales nihila aequari oportet; quod in genere fieri nequit, sed ad terminum, quousque integrale $\int Vdx$ extenditur, spectari oportet; pro quo si ponamus fieri $I=\int Ldx = A$, ex priori forma colligimus hanc aequationem

$$0=(A-I)\mathfrak{N}-\frac{d.(A-I)\mathfrak{P}}{dx}+\frac{dd.(A-I)\mathfrak{Q}}{dx^2}-\frac{d^3.(A-I)\mathfrak{R}}{dx^3}+\text{etc.}$$

COROLLARIUM 2

107. Quomodocunque autem haec aequatio pro quovis casu oblato tractetur, semper tandem eo est deveniendum, ut formula integralis $I=\int Ldx$ per differentiationem exturbari debeat, qua operatione simul quantitatem A inde extrudi evidens est; sicque aequatio resultans non amplius a termino integrationis pendebit.

COROLLARIUM 3

108. Quodsi in genere pro variatione formulae integralis $\int Vdx$ invenienda valorem $\int Ldx = I$ toti integrali respondentem ponamus $=A$, variatio quaesita ita exprimetur

$$\delta\int Vdx = V\delta x + \int \omega dx\left((A-I)\mathfrak{N} - \frac{d.(A-I)\mathfrak{P}}{dx} + \frac{dd.(A-I)\mathfrak{Q}}{dx^2} - \frac{d^3.(A-I)\mathfrak{R}}{dx^3} + \text{etc.}\right)$$
$$+\omega\left(L\mathfrak{Q} - \frac{Ld\mathfrak{R}+d.L\mathfrak{R}}{dx} + \frac{Ldd\mathfrak{S}+d.Ld\mathfrak{S}+dd.L\mathfrak{S}}{dx^2} - \text{etc.}\right)$$
$$+\frac{d\omega}{dx}\left(L\mathfrak{R} - \frac{Ld\mathfrak{S}+d.L\mathfrak{S}}{dx} + \text{etc.}\right)$$
$$+\frac{dd\omega}{dx^2}\left(L\mathfrak{S} - \text{etc.}\right)$$
$$+ \quad \text{etc.},$$

ubi $A-I$ est valor formulae $\int Ldx$ a termino integrationis extremo ad quemvis locum indefinitum medium retro sumtus.

SCHOLION

109. In solutione huius problematis compendium se obtulit, quo etiam analysis in superiori capite adhibita non mediocriter contrahi potest. Cum enim ibi (§ 79) pervenissemus ad

$$\delta\int Vdx = V\delta x + \int(dx\delta V - dV\delta x),$$

ob

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

et

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}$$

erit

$$dV = dx(M + Np + Pq + Qr + Rs + \text{etc.})$$

hincque colligitur

$$dx\delta V - dV\delta x = dx\left(N(\delta y - p\delta x) + P(\delta p - q\delta x) + Q(\delta q - r\delta x) + \text{etc.}\right).$$

Iam si brevitatis gratia ponatur $\delta y - p\delta x = \omega$, erit differentiando

$$\delta(pdx) - qdx\delta x - p\delta dx = d\omega;$$

at

$$\delta(pdx) = pd\delta x + \delta pdx,$$

ergo

$$\delta p - q\delta x = \frac{d\omega}{dx}.$$

Simili modo hanc formulam differentiando ob $dp = qdx$ et $dq = rdx$ fit

$$qd\delta x + \delta q dx - qd\delta x - dq\delta x = dx(\delta q - r\delta x) = d.\tfrac{d\omega}{dx},$$

unde perspicuum est posito $\delta y - p\delta x = \omega$ fore

$$\delta p - q\delta x = \tfrac{d\omega}{dx}, \quad \delta q - r\delta x = \tfrac{1}{dx}d.\tfrac{d\omega}{dx} = \tfrac{dd\omega}{dx^2}$$

sumto dx constante,

$$\delta r - s\delta x = \tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{d\omega}{dx} = \tfrac{d^3\omega}{dx^3}, \quad \text{etc.}$$

Quocirca erit

$$dx\delta V - dV\delta x = dx\left(N\omega + P\tfrac{d\omega}{dx} + Q\tfrac{dd\omega}{dx^2} + R\tfrac{d^3\omega}{dx^3} + S\tfrac{d^4\omega}{dx^4} + \text{etc.}\right),$$

siquidem differentiale dx constans accipiatur.

PROBLEMA 9

110. *Si fuerit* $v = \int \mathfrak{V}dx$ *existente*

$$d\mathfrak{V} = \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + etc.,$$

tum vero sit V functio quaecunque non solum quantitates

$$x, \quad y, \quad p = \tfrac{dy}{dx}, \quad q = \tfrac{dp}{dx}, \quad r = \tfrac{dq}{dx} \ etc.,$$

sed etiam ipsam formulam integralem $v = \int \mathfrak{V}dx$ *implicans, investigare variationem formulae integralis complicatae* $\int Vdx$.

SOLUTIO

Quoniam V est functio quantitatum v, x, y, p, q, r etc., sumatur eius differentiale, quod sit

$$dV = Ldv + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

ac habebitur variatio ipsius V ita expressa

$$\delta V = L\delta v + M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.};$$

tum vero notetur ob

$$dv = \mathfrak{V}dx, \quad dy = pdx, \quad dp = qdx \quad \text{etc.}$$

esse

$$dV = dx\left(L\mathfrak{B} + M + Np + Pq + Qr + Rs + \text{etc.}\right)$$

et

$$\delta\mathfrak{B} = \mathfrak{M}\delta x + \mathfrak{N}\delta y + \mathfrak{P}\delta p + \mathfrak{Q}\delta q + \mathfrak{R}\delta r + \text{etc.}$$

Praeterea habemus

$$\delta v = \int\left(\mathfrak{B}\delta dx + dx\delta\mathfrak{B}\right) = \mathfrak{B}\delta x + \int\left(dx\delta\mathfrak{B} - d\mathfrak{B}\delta x\right),$$

unde posito $\delta y - p\delta x = \omega$ erit per ante inventa

$$\delta v = \mathfrak{B}\delta x + \int dx\left(\mathfrak{N}\omega + \mathfrak{P}\tfrac{d\omega}{dx} + \mathfrak{Q}\tfrac{dd\omega}{dx^2} + \mathfrak{R}\tfrac{d^3\omega}{dx^3} + \mathfrak{S}\tfrac{d^4\omega}{dx^4} + \text{etc.}\right),$$

ubi commoditatis ergo sumsimus *dx* constans.

His praeparatis, cum variatio quaesita sit

$$\delta\int Vdx = V\delta x + \int\left(dx\delta V - dV\delta x\right),$$

ut reductione supra inventa uti possimus, ponamus

$$dV = Ldv + dW$$

ut sit

$$\delta V = L\delta v + \delta W \quad \text{et} \quad dW = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc}.$$

Quocirca nanciscemur hanc formam

$$\delta\int Vdx = V\delta x + \int\left(Ldx\delta v - Ldv\delta x\right) + \int\left(dx\delta W - dW\delta x\right),$$

ubi est

$$dx\delta W - dW\delta x = dx\left(N\omega + P\tfrac{d\omega}{dx} + Q\tfrac{dd\omega}{dx^2} + R\tfrac{d^3\omega}{dx^3} + \text{etc.}\right).$$

Tum vero est

$$dx\delta v - dv\delta x = dx\int dx\left(\mathfrak{N}\omega + \tfrac{\mathfrak{P}d\omega}{dx} + \tfrac{\mathfrak{Q}dd\omega}{dx^2} + \tfrac{\mathfrak{R}d^3\omega}{dx^3} + \text{etc.}\right)$$

ob $dv\delta x = \mathfrak{B}dx\delta x$. Quibus substitutis colligitur variatio quaesita

$$\delta\int Vdx = V\delta x + \int Ldx\int dx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + \text{etc.}\right)$$
$$+\int dx\left(N + P\frac{d\omega}{dx} + Q\frac{dd\omega}{dx^2} + R\frac{d^3\omega}{dx^3} + \text{etc.}\right).$$

Quo iam hanc formam ulterius reducamus, ponamus integrale $\int Ldx = I$ ita sumtum, ut pro initio, unde integrale $\int Vdx$ capitur, evanescat, pro fine antem, ubi integrale $\int Vdx$ terminatur, fiat $I = A$; sicque fiet

$$\delta\int Vdx = V\delta x + A\int dx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} + \text{etc.}\right)$$
$$-\int Idx\left(\mathfrak{N}\omega + \frac{\mathfrak{P}d\omega}{dx} + \frac{\mathfrak{Q}dd\omega}{dx^2} + \frac{\mathfrak{R}d^3\omega}{dx^3} + \text{etc.}\right)$$
$$+\int dx\left(N\omega + \frac{Pd\omega}{dx} + \frac{Qdd\omega}{dx^2} + \frac{Rd^3\omega}{dx^3} + \text{etc.}\right),$$

ad quam formam contrahendam statuamus

$$N + (A - I)\mathfrak{N} = N',$$
$$P + (A - I)\mathfrak{P} = P',$$
$$Q + (A - I)\mathfrak{Q} = Q',$$
$$R + (A - I)\mathfrak{R} = R'$$
$$\text{etc.},$$

ut prodeat forma illi, quam supra [§ 79-85] tractavimus, similis

$$\delta\int Vdx = V\delta x + \int dx\left(N'\omega + \frac{P'd\omega}{dx} + \frac{Q'dd\omega}{dx^2} + \frac{R'd^3\omega}{dx^3} + \frac{d^4\mathfrak{S}}{dx^4} + \text{etc.}\right);$$

ubi ergo si post signum integrale differentialia ipsius *w* eliminentur, perveniemus secundum § 86 ad hanc expressionem

$$\delta\int Vdx = \int \omega dx\left(N' - \frac{dP'}{dx} + \frac{ddQ'}{dx^2} - \frac{d^3R'}{dx^3} + \frac{d^4S'}{dx^4} - \text{etc.}\right)$$
$$+V\delta x + \omega\left(P' - \frac{dQ'}{dx} + \frac{ddR'}{dx} - \frac{d^3S'}{dx^3} + \text{etc.}\right)$$
$$\text{Const.} + \frac{d\omega}{dx}\left(Q' - \frac{dR'}{dx} + \frac{ddS'}{dx^2} - \text{etc.}\right)$$
$$+\frac{dd\omega}{dx}\left(R' - \frac{dS'}{dx} + \text{etc.}\right)$$
$$+\frac{d^3\omega}{dx^3}\left(S' - \text{etc.}\right)$$
$$+\text{etc.}$$

Constanti autem per integrationem invectae eiusmodi valor tribui debet, ut pro initio integrationis formulae $\delta\int Vdx$ partes absolutae ad nihilum redigantur, siquidem prima pars integralis ita sumatur, ut pro eodem initio evanescat; tum vero universam expressionem ad finem integrationis produci oportet, pro quo iam posuimus fieri $\int Ldx = I = A$.

COROLLARIUM 1

111. In parte integrali variabilitas per totam integrationis extensionem debet comprehendi, in partibus autem absolutis sufficit respexisse ad initium ac finem integrationis, pro utroque autem termino conditiones variationis praescriptae suppeditant valores $\delta x,\ \omega,\ \frac{d\omega}{dx}, \frac{dd\omega}{dx^2}$, etc. Ac postquam ex conditionibus initii constans rite fuerit determinata, tum superest, ut singula membra ad finem integrationis accommodentur.

COROLLARIUM 2

112. Pro initio igitur integrationis, ubi $I = 0$, erit primo

$$N' = N + A\mathfrak{N},\quad P' = P + A\mathfrak{P},\quad Q' = Q + A\mathfrak{Q},\ \ R' = R + A\mathfrak{R}\ \ \text{etc.},$$

pro differentialibus vero ob $dI = Ldx$ erit

$$\frac{dN'}{dx} = \frac{dN}{dx} + \frac{Ad\mathfrak{N}}{dx} - L\mathfrak{N}$$

et ita de reliquis similique modo pro differentialibus secundis

$$\frac{ddN'}{dx^2} = \frac{ddN}{dx^2} + \frac{Add\mathfrak{N}}{dx^2} - \frac{2Ld\mathfrak{N}}{dx} - \frac{\mathfrak{N}dL}{dx}.$$

COROLLARIUM 3

113. Pro fine autem integrationis, ubi $I = A$, fit

$$N' = N,\ P' = P,\ Q' = Q,\ R' = R \quad \text{etc.},$$

valores vero differentiales ita se habebunt

$$\frac{dN'}{dx} = \frac{dN}{dx} - L\mathfrak{N}, \quad \frac{dP'}{dx} = \frac{dP}{dx} - L\mathfrak{P}, \quad \frac{dQ'}{dx} = \frac{dQ}{dx} - L\mathfrak{Q}, \quad \text{etc.},$$

secundi vero gradus hoc modo

$$\frac{ddN'}{dx^2} = \frac{ddN}{dx^2} - \frac{2Ld\mathfrak{N}}{dx} - \frac{\mathfrak{N}dL}{dx},$$
$$\frac{ddP'}{dx^2} = \frac{ddP}{dx^2} - \frac{2Ld\mathfrak{P}}{dx} - \frac{\mathfrak{P}dL}{dx},$$

et ita porro.

SCHOLION 1

114. Quanquam natura variationum atque etiam quaestionum eo pertinentium iam satis est explicata, tamen huius argumenti tam dignitas quam novitas ampliorem illustrationem requirere videntur, cum ne superfluum quidem foret eadem saepius inculcari. Cum igitur ante Geometria et huius calculi applicatione ad maxima et minima usi simus, ad hanc doctrinam magis explanandam hic rem generalius pro sola Analysi contemplabimur.

Primo igitur spectatur relatio quaecunque inter binas variabiles *x* et *y*, sive ea sit cognita sive demum definienda, indeque formata consideratur formula integralis quaecunque $\int Vdx$, quae intra certos terminos comprehensa seu integratione a dato initio ad datum finem extensa utique certum quendam valorem recipere debet. Tum illa relatio inter *x* et *y*, quaecunque fuerit, quomodocunque infinite parum immutetur, ut singulis *x* variationibus quibuscunque δx auctis iam respondeant eaedem *y* variationibus quoque quibuscunque δy auctae, ubi quidem observandum est tam in initio quam fine rationem harum variationum per conditiones quaestionum dari, in medio autem istas variationes ita generaliter assumi, ut nulla plane lege inter se connectantur. Tum ex hac relatione variata eiusdem formulae integralis $\int Vdx$ ab eodem initio ad eundem finem expansus seu intra eosdem terminos contentus definiri concipitur ac tota iam quaestio in hoc versatur, ut huius postremi valoris variati excessus supra priorem illum valorem formulae $\int Vdx$ investigetur. Qui excessus cum per $\delta\int Vdx$, quae forma ipsa est variatio formulae $\int Vdx$, indicetur, huius quaestionis solutionem hactenus dedimus ita late patentem, ut omnes casus, quibus quantitas *V* est functio quaecunque non solum ipsarum *x, y, p, q, r, s* etc., sed etiam insuper formulam quandam integralem $v = \int \mathfrak{V}dx$ utcunque involvens, in se complectatur.

SCHOLION 2

115. Quod in praecedente capite tacite assumsimus de quantitate constante variationi inventae adiicienda, quippe quam pars integralis variationis sponte involvit, hoc in istius problematis solutione accuratius exponere est visum. Cum scilicet in huiusmodi quaestionibus, quae ad formulas integrales reducuntur, perpetuo ad terminos integrationis sit respiciendum, siquidem integrale nihil aliud est nisi summa elementorum a termino dato seu initio ad alium terminum seu finem continuatorum, haec consideratio prorsus essentialis est omni integrationi, sine qua idea valoris integralis ne consistere quidem potest. Quamobrem constitutis integrationis terminis, initio scilicet et fine, statim ac variationis pars integralis ita est accepta, ut pro initio evadat nulla, tum eiusmodi constantem adiici oportet, ut etiam partes absolutae pro eodem initio destruantur sicque universa variationis expressio ad nihilum redigatur. Quod cum fuerit factum, ad finem integrationis demum progredi licet, ut hoc pacto vera variatio formulae integralis propositae ab initio ad finem extensae obtineatur.

Haec autem variationum doctrina ad duplicis generis quaestiones accommodari potest; dum in altero relatio inter variabiles x et y data assumitur et formulae integralis itidem datae $\int Vdx$ variatio investigatur, postquam per totam integrationis extensionem variabilibus x et y variationes quaecunque fuerint tributae, in altero autem genere ipsa illa variabilium x et y relatio quaeritur, ut formulae integralis $\int Vdx$ variatio certa proprietate sit praedita; quemadmodum si ea formula maximum minimumve valorem recipere debeat, hanc variationem in nihilum abire necesse est. Ubi iterum duo casus se offerunt, prout maximum minimumve locum habere debet, vel quaecunque yariationes ipsis x et y tribuantur, vel si tantum hae variationes certae cuidam legi adstringantur. Ex quo manifestum est hanc theoriam multo latius patere, quam quidem ea adhuc in usum est vocata.

PROBLEMA 10

116. *Si functio V praeter binas variabiles x, y cum suis valoribus differentialibus*

$$p=\frac{dy}{dx},\quad q=\frac{dp}{dx},\quad r=\frac{dq}{dx}\quad etc.$$

etiam duas pluresve formulas integrales

$$v=\int \mathfrak{V}dx,\quad v'=\int \mathfrak{V}'dx\quad etc.$$

involvat, ut sit

$$\delta\mathfrak{V}=\mathfrak{M}\delta x+\mathfrak{N}\delta y+\mathfrak{P}\delta p+\mathfrak{Q}\delta q+\mathfrak{R}\delta r+\text{etc.},$$
$$\delta\mathfrak{V}'=\mathfrak{M}'\delta x+\mathfrak{N}'\delta y+\mathfrak{P}'\delta p+\mathfrak{Q}'\delta q+\mathfrak{R}'\delta r+\text{etc.}$$

atque differentiali sumto

$$dV=Ldv+L'dv'+Mdx+Ndy+Pdp+Qdq+\text{ etc.},$$

invenire variationem formulae integralis $\int Vdx$.

SOLUTIO

Si huius problematis solutio eodem modo instituatur ac praecedentis, mox patebit calculum a geminata formula integrali

$$v=\int \mathfrak{V}dx, \quad \text{et} \quad v'=\int \mathfrak{V}'\,dx$$

non turbari neque etiam, si plures eiusmodi involverentur. Quare tota solutio tandem huc redibit, ut constitutis integrationis terminis primo integralia

$$\int Ldx=I \quad \text{et} \quad \int L'\,dx=I'$$

ita sint capienda, ut pro initio integrationis evanescant, tum vero pro fine integrationis fiat $I=A$ et $I'=A'$; quibus quantitatibus inventis statuatur porro

$$\begin{aligned}
&N+(A-I)\mathfrak{N}+(A'-I')\mathfrak{N}'=N',\\
&P+(A-I)\mathfrak{P}+(A'-I')\mathfrak{P}'=P',\\
&Q+(A-I)\mathfrak{Q}+(A'-I')\mathfrak{Q}'=Q',\\
&R+(A-I)\mathfrak{R}+(A'-I')\mathfrak{R}'=R'\\
&\qquad\text{etc.}
\end{aligned}$$

eritque variatio quaesita, dum utrique variabili *x* et *y* variationes quaecunque tribuuntur, ex praecedentibus solutionibus [§ 110]

$$\begin{aligned}
&\delta\int Vdx=\int \omega dx\left(N'-\tfrac{dP'}{dx}+\tfrac{ddQ'}{dx^2}-\tfrac{d^3R'}{dx^3}+\tfrac{d^4S'}{dx^4}-\text{etc.}\right)\\
&+V\delta x+\omega\left(P'-\tfrac{dQ'}{dx}+\tfrac{ddR'}{dx}-\tfrac{d^3S'}{dx^3}+\text{etc.}\right)\\
&+\text{Const.}+\tfrac{d\omega}{dx}\left(Q'-\tfrac{dR'}{dx}+\tfrac{ddS'}{dx^2}-\text{etc.}\right)\\
&+\tfrac{dd\omega}{dx}\left(R'-\tfrac{dS'}{dx}+\text{etc.}\right)\\
&+\tfrac{d^3\omega}{dx^3}\left(S'-\text{etc.}\right)\\
&+\text{etc.}
\end{aligned}$$

ubi commoditatis gratia elementum *dx* constans est assumtum.

COROLLARIUM

117. Si ergo etiam plures huiusmodi formulae integrales $\int \mathfrak{V}dx$ in functionem V quomodocunque ingrediantur, expressio variationis quaesitae inde non mutatur, sed tantum quantitates N', p', Q', R' etc. ex iis rite definiri convenit.

SCHOLION

118. Etsi formulae integrales

$$I = \int Ldx, \quad I' = \int L'dx$$

binas variabiles involvunt ideoque valores fixos recipere non posse videntur, tamen perpendendum est in omnibus huiusmodi quaestionibus semper certam quandam relationem inter binas variabiles x et y supponi, sive ea absolute detur sive demum per calculum definiri debeat. Hac igitur ipsa relatione iam in usum vocata, ut quantitas y instar functionis ipsius x spectari possit, formulae illae integrales utique determinatos valores sortientur.

PROBLEMA 11

119. *Si functio $\mathfrak{V}$ praeter variabiles x et y earumque valores differentiales p, q, r, s etc. ipsam quoque formulam integralem $u = \int \mathfrak{v}dx$ involvat, ut eius differentiale sit*

$$d\mathfrak{V} = \mathfrak{L}du + \mathfrak{M}dx + \mathfrak{N}dy + \mathfrak{P}dp + \mathfrak{Q}dq + \mathfrak{R}dr + \text{etc}.$$

existente

$$d\mathfrak{v} = \mathfrak{m}dx + \mathfrak{n}dy + \mathfrak{p}dp + \mathfrak{q}dq + \mathfrak{r}dr + \text{etc.},$$

tum vero sit V functio quaecunque ipsarum x, y, p, q, r etc. insuperque formulae integralis $v = \int \mathfrak{V}dx$, ut sit

$$dV = Ldv + Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

invenire variationem formulae integralis $\int Vdx$.

SOLUTIO

Ex Problemate 9 statim invenimus variationem formulae integralis $\int \mathfrak{V}dx = v$; constitutis enim integrationis terminis sumtoque integrali $\int \mathfrak{L}dx = \mathfrak{I}$ ita, ut evanescat pro integrationis initio, pro fine fiat $\mathfrak{I} = \mathfrak{A}$, tum fiat brevitatis gratia

$$\mathfrak{N}+(\mathfrak{A}-\mathfrak{I})\mathfrak{u}=\mathfrak{N}', \quad \mathfrak{P}+(\mathfrak{A}-\mathfrak{I})\mathfrak{p}=\mathfrak{P}', \quad \mathfrak{Q}+(\mathfrak{A}-\mathfrak{I})\mathfrak{q}=\mathfrak{Q}', \quad \text{etc.}$$

erit ex illius problematis solutione

$$d\mathfrak{v}=\mathfrak{V}dx+\int dx\left(\mathfrak{N}'\omega+\frac{\mathfrak{P}'d\omega}{dx}+\frac{\mathfrak{Q}'dd\omega}{dx^2}+\frac{\mathfrak{R}'d^3\omega}{dx^3}+\text{etc}\right)$$

posito $\omega=\delta y-p\delta x$ et sumto dx constante.

Iam vero cum quaeratur $\delta\int Vdx$, ob

$$\delta\int Vdx=V\delta x+\int(dx\delta V-dV\delta x)$$

posito brevitatis ergo

$$dV=Ldv+dW \quad \text{et} \quad \delta V=L\delta v+\delta W,$$

ut sit

$$dW=Mdx+Ndy+Pdp+Qdq+Rdr+\text{etc.},$$

erit, ut ibidem vidimus,

$$\delta\int Vdx=V\delta x+\int(Ldx\delta v-Ldv\delta x)+dx\left(N\omega+\frac{Pd\omega}{dx}+\frac{Qdd\omega}{dx^2}+\frac{Rd^3\omega}{dx^3}+\text{etc.}\right);$$

ubi si loco dv et δv valores modo inventi substituantur, erit

$$dx\delta v-dv\delta x=dx\int dx\left(\mathfrak{N}'\omega+\frac{\mathfrak{P}'d\omega}{dx}+\frac{\mathfrak{Q}'dd\omega}{dx^2}+\frac{\mathfrak{R}'d^3\omega}{dx^3}+\text{etc.}\right).$$

Nunc ponatur $\int Ldx=I$ integrali ita sumto, ut evanescat in integrationis initio, in fine autem fiat $I=A$, et habebimus

$$\int L(dx\delta v-dv\delta x)=\int(A-I)dx\left(\mathfrak{N}'\omega+\frac{\mathfrak{P}'d\omega}{dx}+\frac{\mathfrak{Q}'dd\omega}{dx^2}+\frac{\mathfrak{R}'d^3\omega}{dx^3}+\text{etc.}\right).$$

Restituantur pro $\mathfrak{N}', \mathfrak{P}', \mathfrak{Q}', \mathfrak{R}'$ etc. valores supra assumti at ad calculum contrahendum ponatur

$$N+(A-I)\mathfrak{N}+(A-I)(\mathfrak{A}-\mathfrak{I})\mathfrak{n}=N',$$
$$P+(A-I)\mathfrak{P}+(A-I)(\mathfrak{A}-\mathfrak{I})\mathfrak{p}=P',$$
$$Q+(A-I)\mathfrak{Q}+(A-I)(\mathfrak{A}-\mathfrak{I})\mathfrak{q}=Q',$$
$$R+(A-I)\mathfrak{R}+(A-I)(\mathfrak{A}-\mathfrak{I})\mathfrak{r}=R'$$
etc.

ac manifestum est fore variationem quaesitam

$$\delta\int Vdx=V\delta x+\int dx\left(\mathfrak{N}'\omega+\frac{P'd\omega}{dx}+\frac{Q'dd\omega}{dx^2}+\frac{R'd^3\omega}{dx^3}+\text{etc.}\right),$$

quae forma porro evolvitur in eandem expressionem, quam sub finem Problematis 9 (§ 110) exhibuimus, quam ergo hic denuo apponere foret superfluum.

COROLLARIUM 1

120. Hic ergo formula integralis $\int Vdx$, cuius variationem assignavimus, ita est comparata, ut non solum functio V formulam integralem $\int\mathfrak{V}dx$ involvat, sed etiam haec functio $\mathfrak{V}$ aliam formulam integralem $\int\mathfrak{v}dx$ in se complectatur, ubi quidem functio $\mathfrak{v}$ nullam amplius formulam integralem implicat.

COROLLARIUM 2

121. Sin autem et haec functio $\mathfrak{v}$ insuper formulam integralem in se involvat, iam satis perspicuum est, quomodo tum solutionem institui oporteat, siquidem tum valores N', P', Q', R' etc. partes insuper recipient a postrema formula integrali pendentes.

SCHOLION

122. Quomodocunque ergo formula integralis $\int Vdx$ fuerit complicata, praecepta hactenus exposita omnino sufficiunt ad eius variationem investigandam, etiamsi forte complicatio fuerit infinita. Cum igitur omnes expressiones binas variabiles implicantes, quarum variationes unquam sint investigandae, vel a formulis integralibus sint liberae vel unam pluresve in se complectantur easque vel simplices vel complicatas utcunque, huic calculi variationum parti, quae circa duas variabiles versatur, abunde satisfactum videtur, ut vix quicquam amplius desiderari queat. Quamobrem ad formulas trium variabilium progrediamur ac primo quidem tales, quarum relatio per geminam aequationem definiri ponitur, ut binae variabiles tanquam functiones tertiae spectari queant, sive haec duplex relatio sit cognita sive ex ipsa variationis indole investiganda.