CHAPTER III

CONCERNING THE VARIATION OF SIMPLE INTEGRAL FORMULAS INVOLVING TWO VARIABLES

DEFINITION 6.

70. *Here I call an integral formula simple, which is not involved with any other integral, but the integral refers simply to a formula of the differentials as well as including any differentials of these two variables.*

COROLLARY 1

71. Therefore if x and y shall be the two variables, the formula of the integral $\int W$ will be simple, if the expression W besides these variables shall contain only the differentials of these, of whatever order they should be, nor in addition should other formulas themselves be involved.

COROLLARY 2

72. But if now we may put in place $dy = pdx,\ dp = qdx,\ dq = rdx$ etc., so that the form of the differential is removed, because the integration requires a differential formula, thus the expression W may always be reduced to a form of this kind Vdx with the function V arising of the quantities x, y, p, q etc.

COROLLARY 3

73. Therefore since the simple formula of the integral shall be of this kind $\int Vdx$, where V is a function of the quantities x, y, p, q, r etc., the differential nature of this will be shown most conveniently, if we assert it to be

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + etc .$$

SCHOLIUM

74. Here I distinguish the simple integral formulas from the complicated, in which the integration of some kind of formulas of the differentials is proposed, which now themselves involve one or more integral formulas. Just as if the letter s should denote the integral

$$\int \sqrt{\left(dx^2 + dy^2\right)} = \int dx\sqrt{\left(1 + pp\right)}$$

and the quantity V besides these quantities also may contain this letter s, the formula of the integral $\int Vdx$ deservedly may be thought complicated; the variation of which then requires to be explained by individual precepts [ch. IV]. But first in this chapter I have put in place a method for treating the variations of simple integral formulas.

THEOREM 2

75. *The variation of the integral formula* $\int W$ *is equal always to the integral of the variation of the same differential formula, of which the integral is proposed, or there is*

$$\delta\int W=\int\delta W .$$

DEMONSTRATION

Since the variation shall be the excess, that the varied value shall be above the natural value of this, we may consider carefully the varied value of the proposed formula $\int W$, which may be obtained, if in place of the variables x and y the same values increased by their variations δx and δy may be substituted. Moreover since then the quantity W may change into $W+\delta W$ the varied value of the proposed form will be

$$\int(W+\delta W)=\int W+\int\delta W ;$$

from which since there shall be

$$\delta\int W=\int(W+\delta W)-\int W$$

we will have

$$\delta\int W=\int\delta W ,$$

from which it is apparent that the variation of the integral is equal to the integral of the variation.

Likewise also it can be shown in this manner. There is put $\int W=w$, thus so that the variation δw is required to be found. Therefore because with the differentials taken there is $dw=W$, now with the variations taken there will be

$$\delta dw=\delta W=d\delta w$$

on account of $\delta dw=d\delta w$. Now truly the equation $d\delta w=\delta W$ integrated anew will give

$$\delta W=\int\delta W=\delta\int W .$$

COROLLARY 1

76. Therefore with this proposed formula for the integral $\int Vdx$ the variation of this $\delta\int Vdx$ will be

$$=\int\delta(Vdx)=\int(V\delta dx+dx\delta V)$$

and hence on account of $\delta dx=d\delta x$ there may be considered

$$\delta\int Vdx=\int V\delta dx+\int dx\delta V\ .$$

COROLLARY 2

77. On putting $\delta x=\omega$, so that there shall be $d\delta x=d\omega$, because there becomes

$$\int Vd\omega=V\omega-\int\omega dV\ ,$$

in the first place, with the differential of the variation *dx* laid aside there becomes

$$\delta\int Vdx=V\delta x-\int dV\delta x+\int dx\delta V\ ,$$

where the first part is exempt from integration.

SCHOLIUM

78. Just as above [§ 37] we have shown that the sign for differentiation d with the sign of variation expressed by δ to be prefixed to whatever can be interchanged between themselves as wished, thus we may see that the sign of integration $\int$ can be permuted with the sign of the variation δ, since there shall be

$$\delta\int W=\int\delta W\ .$$

And this also is apparent for repeated integral, so that, if it should be proposed for such formulas $\iint W$, the variation of this may be shown in these ways

$$\delta\iint W=\int\delta\int W=\iint\delta W$$

and thus the variation of integral formulas can be reduced to the variations of expressions involving no further integration, with which now found the above precepts have been related.

PROBLEM 6

79. *For the proposed variations of the two variables x and y with the variations* δx *and* δy, if on putting

$$dy = pdx,\ dp = qdx,\ dq = rdx\ etc.$$

V should become some function of the quantities x, y, p, q, r etc., to investigate the variation of the formulas of the integral $\int Vdx$.

SOLUTION

But we have seen the variation of this integral formula (§77) to be expressed thus, so that there shall be

$$\delta\int Vdx = V\delta x - \int dV\delta x + \int dx\delta V .$$

Now towards eliminating δV, since *V* shall be a function of the quantities *x, y, p,q, r* etc., we may put in place the differential of this to be

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc}.$$

and in a similar manner the variation of this will be expressed thus

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.},$$

with which values substituted we may follow with the variation sought

$$\delta\int Vdx = V\delta x + \int dx\left(M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}\right)$$
$$- \int \delta x\left(Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}\right);$$

where since the parts depending on *M* cancel each other out, the variation from the separate parts following the letters *N, P, Q, R* etc. will be

$$\delta\int Vdx = V\delta x + \int N\left(dx\delta y - dy\delta x\right) + \int P\left(dx\delta p - dp\delta x\right)$$
$$+ \int Q\left(dx\delta q - dq\delta x\right) + \int R\left(dx\delta r - dr\delta x\right) + \text{etc.},$$

where there is, as we have found above [§ 56],

[recall : $\delta p = \frac{d\delta y}{dx} - \frac{pd\delta x}{dx}$ and $\delta q = \frac{d\delta p}{dx} - \frac{qd\delta x}{dx}$.]

$$dx\delta p = d\delta y - pd\delta x,\ dx\delta q = d\delta p - qd\delta x,\ dx\delta r = d\delta q - rd\delta x \quad \text{etc.},$$

with which values substituted on account of $dy = pdx$ there may be obtained

$$\delta\int Vdx = V\delta x + \int Ndx(\delta y - p\delta x) + \int Pd.(\delta y - p\delta x)$$
$$+\int Qd.(\delta p - q\delta x) + \int Rd.(\delta q - r\delta x) + \text{etc.},$$

Towards reducing this expression further it may be noted that

$$\delta p - q\delta x = \tfrac{d\delta y - pd\delta x - dp\delta x}{dx} = \tfrac{d.(\delta y - p\delta x)}{dx},$$
$$\delta q - r\delta x = \tfrac{d\delta p - qd\delta x - dq\delta x}{dx} = \tfrac{d.(\delta p - q\delta x)}{dx},$$
$$\delta r - s\delta x = \tfrac{d\delta q - rd\delta x - dr\delta x}{dx} = \tfrac{d.(\delta q - r\delta x)}{dx},$$
$$\text{etc.},$$

with which agreed upon whichever formula is reduced to the preceding ; from which , if we put for the sake of brevity, $\delta y - p\delta x = \omega$ there will be, as follows,

$$\delta y - p\delta x = \omega,$$
$$\delta p - q\delta x = \tfrac{d\omega}{dx},$$
$$\delta q - r\delta x = \tfrac{1}{dx}d.\tfrac{d\omega}{dx},$$
$$\delta r - s\delta x = \tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{d\omega}{dx}$$
$$\text{etc.},$$

and thus with the variations of the letters of the derivatives *p*, *q*, *r* etc. excluded from the calculation, we come upon

$$\delta\int Vdx = V\delta x + \int Ndx\omega + \int Pd\omega + \int Qd.\tfrac{d\omega}{dx} + \int Rd.\tfrac{1}{dx}d.\tfrac{d\omega}{dx}$$
$$+\int Sd.\tfrac{1}{dx}d\,\tfrac{1}{dx}d.\tfrac{d\omega}{dx} + \int Td.\tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{d\omega}{dx} + \text{etc.},$$

of which form the law of the progression is evident, of whatever order the differentials arise in the formula *V*.

COROLLARY 1

80. Therefore the first part $V\delta x$ of the variation is exempt from the integration sign and thus it involves the variation δx alone, truly the remaining parts may be retained each connected continually in the same manner, and handled through the letter $\omega = \delta y - p\delta x$

COROLLARY 2

81. The second part

$$\int Ndx \cdot w = \int N\omega dx$$

is unable to be expressed more conveniently, truly the third part $\int Pdw$ may be seen to be expressed more conveniently, so that it becomes

$$\int Pd\omega = P\omega - \int \omega dP$$

and now the quantity ω after the integral sign must itself be found.

COROLLARY 3

82. The fourth part $\int Qd.\frac{d\omega}{dx}$ may be reduced in a similar manner to

$$Q\frac{d\omega}{dx} - \int dQ\frac{d\omega}{dx},$$

and with this the latter part, since it shall be $\int \frac{dQ}{dx} d\omega$, again gives

$$\omega\frac{dQ}{dx} - \int \omega d.\frac{dQ}{dx},$$

thus so that the fourth part may be resolved into these parts

$$Q\frac{d\omega}{dx} - \frac{dQ}{dx}\omega + \int \omega d.\frac{dQ}{dx}.$$

COROLLARY 4

83. The fifth part

$$\int Rd.(\delta q - r\delta x)$$

at first is reduced according to

$$R\frac{1}{dx}d.\frac{d\omega}{dx}-\int\frac{dR}{dx}d.\frac{d\omega}{dx},$$

then truly the latter part to

$$\frac{dR}{dx}\frac{d\omega}{dx}-\int\frac{1}{dx}d.\frac{dR}{dx}\cdot d\omega$$

and this further finally to

$$\frac{1}{dx}d.\frac{dR}{dx}\cdot\omega-\int\omega d.\frac{1}{dx}d.\frac{dR}{dx},$$

so that this fifth part now may be expressed thus

$$R\frac{1}{dx}d.\frac{d\omega}{dx}-\frac{dR}{dx}\cdot\frac{d\omega}{dx}+\frac{1}{dx}d.\frac{dR}{dx}\cdot\omega-\int\omega d.\frac{1}{dx}d.\frac{dR}{dx}.$$

COROLLARY 5

84. The sixth part in a similar manner may be expressed thus

$$\int Sd.\frac{1}{dx}d.\frac{1}{dx}d.\frac{d\omega}{dx}$$

may be found expressed thus

$$S\frac{1}{dx}d.\frac{1}{dx}d.\frac{d\omega}{dx}-\frac{dS}{dx}\cdot\frac{1}{dx}d.\frac{d\omega}{dx}+\frac{1}{dx}d.\frac{dS}{dx}\cdot\frac{d\omega}{dx}-\frac{1}{dx}d.\frac{1}{dx}d.\frac{dS}{dx}\cdot\omega$$
$$+\int\omega d.\frac{1}{dx}d.\frac{1}{dx}d.\frac{dS}{dx}.$$

PROBLEM 7

85. *On putting* $dy=pdx$, $dp=qdx$, $dq=rdx$, $dr=sdx$ *etc . If V were some function of the quantities x, y, p, q, r,* s *etc., thus so that there should be*

$$dV=Mdx+Ndy+Pdp+Qdq+Rdr+Sds+\text{etc.},$$

to express the variation of the integral formula $\int Vdx$ *from each of the variables x and y arising from the variation, so that after the integral sign no variations of the differentials occur.*

SOLUTION

Now in the corollaries of the preceding problem all have been prepared according to this goal, so that there is no other need for anything else, except that the transformations of the individual parts may be reduced in order, with which agreed upon there may be obtained parts of two kinds, with the one containing integral formulas, which indeed one is permitted to gather into the same sum, with the other the resolved parts, which we will distribute thus into the member parts, so that the following variations δx and δy themselves and the differentials of these of any order may appear. But for the sake of brevity on putting the formula $\delta y - p\delta x = \omega$ the variation sought thus will be itself had :

$$\begin{aligned}
&\delta\int Vdx = \int \omega dx\left(N - \tfrac{dP}{dx} + \tfrac{1}{dx}d.\tfrac{dQ}{dx} - \tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{dR}{dx} + \tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{dS}{dx} - \text{etc.}\right)\\
&+V\delta x + \omega\left(P - \tfrac{dQ}{dx} + \tfrac{1}{dx}d.\tfrac{dR}{dx} - \tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{dS}{dx} + \text{etc.}\right)\\
&+\tfrac{d\omega}{dx}\left(Q - \tfrac{dR}{dx} + \tfrac{1}{dx}d.\tfrac{dS}{dx} - \text{etc.}\right)\\
&+\tfrac{1}{dx}d.\tfrac{d\omega}{dx}\left(R - \tfrac{dS}{dx} + \text{etc.}\right)\\
&+\tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{d\omega}{dx}\left(S - \text{etc.}\right)\\
&+\text{etc.},
\end{aligned}$$

the nature of the form of which is apparent immediately by inspection alone, so that there is no need for a more productive illustration.

COROLLARY 1

86. This expression is returned much simpler, if the element *dx* is taken constant, in which the magnitude of the expression is by no means restricted ; then indeed there becomes

$$\begin{aligned}
&\delta\int Vdx = \int \omega dx\left(N - \tfrac{dP}{dx} + \tfrac{ddQ}{dx^2} - \tfrac{d^3R}{dx^3} + \tfrac{d^4S}{dx^4} - \text{etc.}\right)\\
&+V\delta x + \omega\left(P - \tfrac{dQ}{dx} + \tfrac{ddR}{dx} - \tfrac{d^3S}{dx^3} + \text{etc.}\right)\\
&+\tfrac{d\omega}{dx}\left(Q - \tfrac{dR}{dx} + \tfrac{ddS}{dx^2} - \text{etc.}\right)\\
&+\tfrac{dd\omega}{dx}\left(R - \tfrac{dS}{dx} + \text{etc.}\right)\\
&+\tfrac{d^3\omega}{dx^3}\left(S - \text{etc.}\right)\\
&+\text{etc.}
\end{aligned}$$

COROLLARY 2

87. If the investigation is concerned with a curved line, the first part completing the value of the integral, along the whole curve from the beginning as far as to the end, where the coordinates *x* and *y* stop, also brings together all the variations made in the individual points of the curve, as long as the remaining resolved parts may be defined only by the variations made in the end of the curve.

COROLLARY 3

88. If we consider a curve defined as given by the coordinates x and y and another curve may be considered differing from that by an infinitely small amount, as long as in the individual points of the coordinates are each assigned some variation, the expression indicated will be found, however great the value of the integral formula $\int Vdx$ gathered from the varied curve exceeds the value of the same chosen from the given curve.

COROLLARY 4

89. Since there shall be $\omega = \delta y - p\delta x$, it is apparent that this quantity ω vanishes, if the variations of the individual points δx and δy may be taken thus, so that there shall be

$$\delta y : \delta x = p : 1 = dy : dx .$$

Therefore in this case the varied curve plainly does not disagree from the given, and the whole variation of the formula $\int Vdx$ is reduced to $V\delta x$.

SCHOLIUM 1

90. This variation found for the formula of the integral $\int Vdx$ at once satisfies the rule, which at one time I treated for finding the curve, in which the value of the same integral formula shall be a maximum or a minimum. [In Euler's *Methodus inveniendi lineas curvas*, Ch. II, §56.] For that rule postulates, that this expression

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.}$$

is established equal to nothing. Moreover here it is evident at once according to that, that the variation of the formula $\int Vdx$ should vanish, whenever the nature of the maximum or minimum demands to be required before all, so that the first part contained by the integral sign should vanish, from which there becomes

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0.$$

Now also besides it is required to equate the resolved parts to zero, by which the applied line [the y coordinate] is retained at each end of the curve. For the nature of the curve itself is expressed by that equation; with which on account of the differentials of higher orders it takes so many integrations and so many arbitrary constants, these parts are given over completely to looking after the determination of these constants, so that both at the beginning as well as at the end the curve responds to certain conditions, just as the given curved line may be terminated. And if that equation should be of the fourth order differential or to that extend higher, also the number of resolved parts must be augmented also, from which it can come about, that for the curve sought not only each may end according to given lines, but in that place also a certain direction, so that also, if it should rise to higher differentials, a certain law of the curvature has to be prescribed. But always the most beautiful applied line requiring to be made usually eventuates, so that the very nature of questions of this kind may involve conditions, by which it may be satisfied most suitably by the absolved parts.

SCHOLIUM 2

91. But how great the mysteries that may lie hidden in this form, as we have found from the variation of the integral formula $\int Vdx$, one can declare in a much more splendid manner in the application of this to maxima and minima; I note here that only a part by necessity enters into that variation. Since indeed we may embrace the thing in the widest sense, so that plainly we cannot attribute any variations of each point of the variables *x* and *y* to any law connecting the variables, it is entirely unable to come about, that an agreeing variation of the whole curve, does not likewise depend on all the intermediate variations, obviously from which it is necessary on being constituted otherwise, so that thence the variation of the whole curve is a perpetuated change. And in this variation of the integral formulas chiefly disagrees from the variation of expressions of this kind, such as we have considered in the above, which depends on a single given variation of the final elements. From which it follows clearly, if perhaps the quantity *V* should be prepared thus, so that the formula of the differential *Vdx* may admit an integration with no relation established between the variables *x* and *y* and thus the formula of the integral $\int Vdx$ shall be a resolved function of the quantities *x, y, p, q, r,* etc., then also the variation of this is able to depend only on the variation of the extreme elements and thus the integral part of the variation plainly must go to zero, from which the following significant theorem is deduced.

THEOREM 3

92. *On putting* $dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx$ *etc., if V were a function of x, y, p, q, r, s etc. themselves, so that on putting the differential of this*

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + \text{etc.}$$

there should be

$$N - \tfrac{dP}{dx} + \tfrac{ddQ}{dx^2} - \tfrac{d^3R}{dx^3} + \tfrac{d^4S}{dx^4} - \text{etc.} = 0,$$

on taking the element dx constant, then the formula of the differential Vdx will be integrable by itself, and in turn with no relation established between the variables x et y.

DEMONSTRATION

If there should be

$$N - \tfrac{dP}{dx} + \tfrac{ddQ}{dx^2} - \tfrac{d^3R}{dx^3} + \tfrac{d^4S}{dx^4} - \text{etc.} = 0,$$

then the variation of the integral formula $\int Vdx$ involves no integral formula and thus for whatever position of the coordinates *x* and *y* only depends on the variations which are given to the end term, because on no account can it happen, if the integration *Vdx* might be rejected, therefore because then the above variation likewise by necessity may depend on all the intermediate variations; from which it follows, as often as that equation needs to be considered, so also the formula *Vdx* is allowed to be integrated, thus so that *rVdx* shall become a sure and definite function of the quantities *x, y,p, q, r,* s etc. But in turn as often as the formula of the differential *Vdx* admits to integration and therefore the integral of this $\int Vdx$ is truly a function of the quantities *x, y, p, q, r, s* etc., so often too the variation of this depends only on the variations of the end values of *x* and *y* nor are they possible to be affected in any way by the intermediate variations. From which it is necessary, that the part of the integral variation found above must vanish, that which cannot happen, unless there should be

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0,$$

and thus the proposed theorem and also the inverse is agreed to be true.

COROLLARIUM 1

93. Behold therefore a significant criterion, with the help of which, a formula of the differential of two variables, the differentials of whatever order appear in that, can be decided as to whether or not it is integrable. Therefore it appears much wider than that well enough known criterion, by which it is customary to discern the integrability of a differential formula of the first order.

COROLLARY 2

94. Therefore in the first place if the quantity *V* shall be a function of *x* and *y* only involving no ratio of the differentials, so that there shall be

$$dV = Mdx + Ndy,$$

then the integration of the differential with the formula *Vdx* is not allowed, unless there shall be $N = 0$, that is, unless *V* shall be a function of *x* only; which indeed is evident by inspection.

COROLLARY 3

95. But a formula of the differential proposed of this kind $vdx + udy$, that prepared in the form *Vdx* on account of $dy = pdx$ gives $V = v + pu$ and thus

$$M = \left(\frac{dv}{dx}\right) + p\left(\frac{du}{dx}\right), \quad N = \left(\frac{dv}{dy}\right) + p\left(\frac{du}{dy}\right) \text{ et } P = u,$$

whenever the quantities *v* and *u* are taken not involved with differentials.

Hence there shall be

$$dP = du = dx\left(\frac{du}{dx}\right) + dy\left(\frac{du}{dy}\right).$$

Whereby since the criterion of integrability may be postulated, so that there shall be

$$N - \frac{dP}{dx} = 0,$$

for this case there will be

$$\left(\frac{dv}{dy}\right) + p\left(\frac{du}{dy}\right) - \left(\frac{du}{dx}\right) - p\left(\frac{du}{dy}\right) = 0$$

or

$$\left(\frac{dv}{dy}\right) = \left(\frac{du}{dx}\right),$$

which now is the commonly known criterion.

SCHOLIUM 1

96. The demonstration of this theorem is entirely unique, since it shall have been reached from the principles of variation, which yet is different by this straight forward argument; truly scarcely by another way is it apparent how the demonstration of this theorem may be reached. Then truly here a more precise knowledge of the functions is to be attended to carefully, from which we have shown the integral formula $\int Vdx$ in no manner can be regarded as a function of the quantities *x*, *y*, *p*, *q*, *r* etc., unless an actual integration may be allowed. Indeed the nature of the functions always has this added property, in order that, immediately from the quantities which enter that, the values to be determined are given, and the function itself determined from these forms gives the value ; just as this function *xy*, if we put $x = 2$ and $y = 3$, takes the value = 6. But by far it comes about in the formula of the integral $\int ydx$, the value of which for the case $x = 2$ and $y = 3$ on no account can be assigned, unless a certain relation is put in place between *y* and *x*; then moreover that formula will change into a function of a single variable. Therefore of the integral formulas, which are not able to be integrated, the nature must be distinguished carefully by the nature of the function, since functions, as soon as with the variable quantities, from which they are constructed, give the values to be determined, these determined values themselves are received also, even if the variables in no manner depend on each other; because it hardly ever comes about in integral formulas, obviously the determination of which clearly include likewise all the intermediate values. But in the first place for this to be distinguished, the general principles concerning maxima and minima, that we have attended to here, depends on this, where the formulas by which the maxima or minima property must be agreed upon, by necessity there are required to be integrals of this kind, which do not permit themselves to be integrated.

SCHOLIUM 2

97. To the greater illustration of a theorem of this kind we may consider the integral formula of this kind $\int Vdx$, which itself shall be integral, and we may put for the sake of an example

$$\int Vdx = \frac{xdy}{ydx} = \frac{xp}{y},$$

thus so that there shall be

$$V = \frac{p}{y} - \frac{xpp}{yy} + \frac{xq}{y}$$

and thus this differential formula

$$\left(\frac{p}{y} - \frac{xpp}{yy} + \frac{xq}{y}\right)dx$$

shall be completely integrable, and we may consider, or our theorem may declare this integrable.

Therefore we may differentiate the quantity V and with the differential compared with the form

$$dV = Mdx + Ndy + Pdp + Qdq$$

we will obtain

$$M = \frac{-pp}{yy} + \frac{q}{y}, \quad N = \frac{-p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy}, \quad P = \frac{1}{y} - \frac{2xp}{yy} \quad \text{and} \quad Q = \frac{x}{y}.$$

Now since following the theorem there must become

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} = 0,$$

in the first place we may deduce on differentiating

$$\frac{dP}{dx} = \frac{-3p}{yy} + \frac{4xpp}{y^3} - \frac{2xq}{yy} \quad \text{and} \quad \frac{dQ}{dx} = \frac{1}{y} - \frac{xp}{yy},$$

then truly

$$\frac{ddQ}{dx^2} = \frac{-2p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy}.$$

Hence

$$\frac{dP}{dx} - \frac{ddQ}{dx^2} = \frac{-p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy},$$

to which value the quantity N certainly is equal.

SCHOLIUM 3

98. Moreover when the formula of the differential Vdx allows itself to be integrated and thus on putting

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

following the theorem there is

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

hence other remarkable consequences are deduced. Indeed in the first place since on multiplying by dx and on integrating there becomes

$$\int Ndx - P + \frac{dQ}{dx} - \frac{ddR}{dx^2} + \frac{d^3S}{dx^3} - \text{etc.} = A,$$

it is apparent that also the formula Ndx is completely integrable. Then since again there becomes

$$\int dx \int (Ndx - P) + Q - \frac{dR}{dx^2} + \frac{dsS}{dx^2} - \text{etc.} = Ax + B,$$

also the formula

$$dx \int (Ndx - P)$$

is allowed to be integrated. Afterwards in the same manner there will be this form of integral

$$dx\left(\int dx \int (Ndx - P) + Q\right),$$

then indeed also

$$dx\left(\int dx\left(\int dx\left(\int Ndx - P\right) + Q\right) - R\right)$$

and thus so forth. From which we may deduce the following theorem no less noteworthy and most useful in practice.

THEOREM 4

99. *On putting* $dy = pdx$, $dp = qdx$, $dq = rdx$, $dr = sdx$ *etc., if* V *were a function of some kind of* x, y, p, q, r, s *etc. themselves, so that the formula of the differential Vdx shall itself be integrable, then on putting*

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + \text{etc} .$$

also the following differential formulas by themselves are allowed to be integrated:

I. *The formula N dx by itself will be integrable; then on putting* $P - \int Ndx = \mathfrak{P}$:

II. *The formula* $\mathfrak{P}dx$ *by itself will be integrable; again on putting* $Q - \int \mathfrak{P}dx = \mathfrak{Q}$:

III. The formula $\mathfrak{Q}dx$ *by itself will be integrable; then on putting* $R - \int \mathfrak{Q}dx = \mathfrak{R}$:

IV. *The formula* $\mathfrak{R}dx$ *by itself will be integrable; finally on putting* $S - \int \mathfrak{R}dx = \mathfrak{S}$:

V. *The formula* $\mathfrak{S}dx$ *by itself will be integrable and thus so forth.*

DEMONSTRATION

The truth of this theorem now has prevailed in the preceding paragraph, from which it is likewise apparent, if all these integral formulas are allowed, also the principal *Vdx* integral is to be resolved completely.

COROLLARY 1

100. Since V shall be a function of the quantities

$$x,\, y,\, p = \tfrac{dy}{dx},\, q = \tfrac{dp}{dx},\, r = \tfrac{dq}{dx} \;\; \text{etc.},$$

the quantities thence derived from differentiation *M, N, P, Q, R* etc. thus also can be shown, so that there shall be

$$M = \left(\tfrac{dV}{dx}\right), N = \left(\tfrac{dV}{dy}\right), P = \left(\tfrac{dV}{dp}\right), Q = \left(\tfrac{dV}{dq}\right) \text{ etc.},$$

from which on account of the first formula it is apparent, if the formula *Vdx* were integrable, then also the formula $\left(\tfrac{dV}{dx}\right)dx$ to be integrable.

COROLLARY 2

101. Therefore then also on account of the same reasoning this formula $\left(\frac{ddV}{dx^2}\right)dx$ and hence again these $\left(\frac{d^3V}{dx^3}\right)dx$, $\left(\frac{d^4V}{dx^4}\right)dx$ etc. are all themselves allowed to be integrated.

COROLLARIUM 3

102. Because only so many letters *P*, *Q*, *R* etc. are present, just so many differential orders are to be found in the formula *Vdx*, and all the following vanish, the derivative Gothic letters $\mathfrak{P}, \mathfrak{Q}, \mathfrak{R}, \mathfrak{S}$ etc. thence finally vanish or they must change into functions of *x*, because otherwise the following integrals cannot have a place.

EXAMPLE

103. *Let V be a function of this kind, so that there becomes*

$$\int Vdx = \frac{y\left(dx^2+dy^2\right)^{\frac{3}{2}}}{xdxddy}.$$

With the substitution made

$$dy = pdx,\ dp = qdx,\ dq = rdx$$

for this example the function *V* may thus be expressed

$$V = \frac{p(1+pp)^{\frac{3}{2}}}{xq} - \frac{y(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3yp\sqrt{(1+pp)}}{x} - \frac{yr(1+pp)^{\frac{3}{2}}}{xqq},$$

from which on differentiation we may elicit the following values

$$N = -\frac{(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3p\sqrt{(1+pp)}}{x} - \frac{r(1+pp)^{\frac{3}{2}}}{xqq},$$

$$P = \frac{(1+4pp)\sqrt{(1+pp)}}{xq} - \frac{3yp\sqrt{(1+pp)}}{xxq} + \frac{3y(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3ypr\sqrt{(1+pp)}}{xqq},$$

$$Q = -\frac{p(1+pp)^{\frac{3}{2}}}{xxq} + \frac{y(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2yr(1+pp)^{\frac{3}{2}}}{xxq^3},$$

$$R = -\frac{y(1+pp)^{\frac{3}{2}}}{xqq}.$$

Now therefore the first integral is required to be the formula *Ndx* or

$$-\frac{dx(1+pp)^{\frac{3}{2}}}{xxq}+\frac{3pdx\sqrt{(1+pp)}}{x}-\frac{dq(1+pp)^{\frac{3}{2}}}{xqq},$$

from which it is apparent at once the integral to be this

$$\int Ndx=\frac{(1+pp)^{\frac{3}{2}}}{xq}.$$

Now hence for the second formula we come upon

$$\mathfrak{P}=P-\int Ndx=\frac{3pp\sqrt{(1+pp)}}{xq}-\frac{3yp\sqrt{(1+pp)}}{xxq}+\frac{3y(1+2pp)}{x\sqrt{(1+pp)}}-\frac{3ypr\sqrt{(1+pp)}}{xqq}$$

thus so that on integrating there shall be this formula

$$\mathfrak{P}dx=\frac{3pdy\sqrt{(1+pp)}}{xq}-\frac{3ypdx\sqrt{(1+pp)}}{xxq}+\frac{3ydx(1+2pp)}{x\sqrt{(1+pp)}}-\frac{3ypdq\sqrt{(1+pp)}}{xqq},$$

of which the integral or at least a part of this is deduced from the last member evidently $\frac{3yp\sqrt{(1+pp)}}{xq}$; the differential of which with the total formula removed, will be

$$\int\mathfrak{P}dx=\frac{3yp\sqrt{(1+pp)}}{xq}.$$

Now for the fourth formula we will have

$$\mathfrak{Q}=Q-\int\mathfrak{P}dx=-\frac{p(1+pp)^{\frac{3}{2}}}{xqq}+\frac{y(1+pp)^{\frac{3}{2}}}{xxqq}+\frac{2yr(1+pp)^{\frac{3}{2}}}{xq^3}-\frac{3yp\sqrt{(1+pp)}}{xq},$$

from which being multiplied by dx on account of $dx=\frac{dp}{q}$ in the final part there becomes

$$\mathfrak{Q}dx=-\frac{dy(1+pp)^{\frac{3}{2}}}{xqq}+\frac{ydx(1+pp)^{\frac{3}{2}}}{xxqq}+\frac{2ydq(1+pp)^{\frac{3}{2}}}{xq^3}-\frac{3ypdp\sqrt{(1+pp)}}{xqq},$$

of which the penultimate member declares the integral

$$\int\mathfrak{Q}dx=-\frac{y(1+pp)^{\frac{3}{2}}}{xqq}.$$

Again the fourth formula may be prepared thus

$$\mathfrak{R} = R - \int \mathfrak{Q} dx = 0,$$

from which it is understood that not only this $\mathfrak{R}dx$, but also all the following are to be integrable.

SCHOLIUM

104. These theorems therefore may be considered prettier, because the demonstration of these depends on a principle of this kind, the account of which hence is completely different, therefore because in these truths no further trace of variation is apparent ; from which there is no doubt, why also a demonstration should not be taken from another more natural source [§96].

CAPUT III

DE VARIATIONE FORMULARUM INTEGRALIUM SIMPLICIUM DUAS VARIABILES INVOLVENTIUM

DEFINITIO 6.

70. *Formulam integralem simplicem hic appello, quae nulla alia integralia in se involvit, sed simpliciter integrale refert formulae differentialis praeter binas variabiles quaecunque earum differentialia complectentis.*

COROLLARIUM 1

71. Si ergo x et y sint binae variabiles, formula integralis $\int W$ erit simplex, si expressio W praeter has variabiles tantum earum differentialia, cuiuscunque fuerint ordinis, contineat neque praeterea alias formulas integrales in se implicet.

COROLLARIUM 2

72. Quodsi ergo statuamus $dy = pdx,\ dp = qdx,\ dq = rdx$ etc., ut species differentialium tollatur, quoniam integratio requirit formulam differentialem, expressio ita W semper reducetur ad huiusmodi formam Vdx existente V functione quantitatum x, y, p, q etc.

COROLLARIUM 3

73. Cum igitur formula integralis simplex sit huiusmodi $\int Vdx$, ubi V est functio quantitatum x, y, p, q, r etc., eius indolem commodissime eius differentiale repraesentabit, si dicamus esse

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + etc .$$

SCHOLION

74. Distinguo hic formulas integrales simplices a complicatis, in quibus integratio proponitur eiusmodi formularum differentialium, quae iam ipsae unam pluresve formulas integrales involvunt. Veluti si littera s denotet integrale

$$\int \sqrt{\left(dx^2 + dy^2\right)} = \int dx \sqrt{(1 + pp)}$$

atque quantitas V praeter illas quantitates etiam hanc s contineat, formula integralis $\int Vdx$ merito censetur complicata; cuius variatio singularia praecepta postulat deinceps [cap. IV] exponenda. Hoc autem capite primo methodum formularum integralium simplicium variationes inveniendi tradere constitui.

THEOREMA 2

75. *Variatio formulae integralis* $\int W$ *semper aequalis est integrali variationis eiusdem formulae differentialis, cuius integrale proponitur, seu est*

$$\delta\int W=\int\delta W .$$

DEMONSTRATIO

Cum variatio sit excessus, quo valor variatus cuiusque quantitatis superat eius valorem naturalem, perpendamus formulae propositae $\int W$ valorem variatum, quem obtinet, si loco variabilium x et y earundem valores suis variationibus δx et δy aucti substituantur. Cum autem tum quantitas W abeat in $W+\delta W$ formae propositae valor variatus erit

$$\int(W+\delta W)=\int W+\int\delta W ;$$

unde cum sit

$$\delta\int W=\int(W+\delta W)-\int W$$

habebimus

$$\delta\int W=\int\delta W ,$$

unde patet variationem integralis aequari integrali variationis.

Idem etiam hoc modo ostendi potest. Ponatur $\int W=w$, ita ut quaerenda sit variatio δw. Quia ergo sumtis differentialibus est $dw=W$, capiantur nunc variationes eritque

$$\delta dw=\delta W=d\delta w$$

ob $\delta dw=d\delta w$. Nunc vero aequatio $d\delta w=\delta W$ denuo integrata praebet

$$\delta W=\int\delta W=\delta\int W .$$

COROLLARIUM 1

76. Proposita ergo hac formula integrali $\int Vdx$ eius variatio $\delta\int Vdx$ erit

$$=\int\delta(Vdx)=\int(V\delta dx+dx\delta V)$$

hincque ob $\delta dx=d\delta x$ habebitur

$$\delta\int Vdx = \int V\delta dx + \int dx\delta V\ .$$

COROLLARIUM 2

77. Posito $\delta x = \omega$, ut sit $d\delta x = d\omega$, quia est

$$\int Vd\omega = V\omega - \int \omega dV\ ,$$

in priori membro differentiale variationis *dx* exuitur fietque

$$\delta\int Vdx = V\delta x - \int dV\delta x + \int dx\delta V\ ,$$

ubi prima pars ab integratione est immunis.

SCHOLION

78. Quemadmodum supra [§ 37] ostendimus signa differentiationis *d* cum signa variationis δ expressioni cuicunque praefixa inter se pro lubitu permutari posse, ita nunc videmus signum integrationis $\int$ cum signa variationis δ permutari posse, cum sit

$$\delta\int W = \int \delta W\ .$$

Atque hoc etiam ad integrationes repetitas patet, ut, si proposita fuerit talis formula $\int\int W$, eius variatio his modis exhiberi possit

$$\delta\int\int W = \int\delta\int W = \int\int\delta W$$

ideoque variatio formularum integralium ad variationes expressionum nullam amplius integrationem involventium reducatur, pro quibus inveniendis iam supra praecepta sunt tradita.

PROBLEMA 6

79. *Propositis binarum variabilium x et y variationibus* δx *et* δy, si *positis*

$$dy = pdx,\ dp = qdx,\ dq = rdx\ etc.$$

fuerit V functio quaecunque quantitatum x, y, p, q, r etc., formulae integralis $\int Vdx$ *variationem investigare.*

SOLUTIO

Modo vidimus (§77) huius formulae integralis variationem ita exprimi, ut sit

$$\delta\int Vdx = V\delta x - \int dV\delta x + \int dx\delta V .$$

Iam ad variationem δV elidendam, cum sit V functio quantitatum x, y, p, q, r etc., statuamus eius differentiale esse

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc}.$$

ac simili modo eius variatio ita erit expressa

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.},$$

quibus valoribus substitutis consequimur variationem quaesitam

$$\delta\int Vdx = V\delta x + \int dx\left(M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + \text{etc.}\right)$$
$$-\int \delta x\left(Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}\right);$$

ubi cum partes ab M pendentes se destruant, erit partibus secundum litteras
N, P, Q, R etc. separatis variatio

$$\delta\int Vdx = V\delta x + \int N\left(dx\delta y - dy\delta x\right) + \int P\left(dx\delta p - dp\delta x\right)$$
$$+\int Q\left(dx\delta q - dq\delta x\right) + \int R\left(dx\delta r - dr\delta x\right) + \text{etc.},$$

ubi est, uti supra [§ 56] invenimus,

$$dx\delta p = d\delta y - pd\delta x,\ dx\delta q = d\delta p - qd\delta x,\ dx\delta r = d\delta q - rd\delta x \quad \text{etc.},$$

quibus valoribus substitutis ob $dy = pdx$ obtinetur

$$\delta\int Vdx = V\delta x + \int Ndx\left(\delta y - p\delta x\right) + \int Pd.\left(\delta y - p\delta x\right)$$
$$+\int Qd.\left(\delta p - q\delta x\right) + \int Rd.\left(\delta q - r\delta x\right) + \text{etc.},$$

Ad hanc expressionem ulterius reducendam notetur esse

$$\delta p - q\delta x = \frac{d\delta y - pd\delta x - dp\delta x}{dx} = \frac{d.(\delta y - p\delta x)}{dx},$$
$$\delta q - r\delta x = \frac{d\delta p - qd\delta x - dq\delta x}{dx} = \frac{d.(\delta p - q\delta x)}{dx},$$
$$\delta r - s\delta x = \frac{d\delta q - rd\delta x - dr\delta x}{dx} = \frac{d.(\delta q - r\delta x)}{dx},$$
etc.,

quo pacto quaevis formula ad praecedentem reducitur; unde , si brevitatis gratia ponamus, $\delta y - p\delta x = \omega$ erit, ut sequitur,

$$\delta y - p\delta x = \omega,$$
$$\delta p - q\delta x = \frac{d\omega}{dx},$$
$$\delta q - r\delta x = \frac{1}{dx} d.\frac{d\omega}{dx},$$
$$\delta r - s\delta x = \frac{1}{dx} d.\frac{1}{dx} d.\frac{d\omega}{dx}$$
etc.,

sicque variationibus litterarum derivatarum *p*, *q*, *r* etc. ex calculo exclusis adipiscimur variationem quaesitam

$$\delta\int Vdx = V\delta x + \int Ndx\omega + \int Pd\omega + \int Qd.\frac{d\omega}{dx} + \int Rd.\frac{1}{dx}d.\frac{d\omega}{dx}$$
$$+ \int Sd.\frac{1}{dx}d\,\frac{1}{dx}d.\frac{d\omega}{dx} + \int Td.\frac{1}{dx}d.\frac{1}{dx}d.\frac{1}{dx}d.\frac{d\omega}{dx} + \text{ etc.},$$

cuius formae lex progressionis est manifesta, cuiuscunque gradus differentialia in formulam *V* ingrediantur.

COROLLARIUM 1

80. Huius igitur variationis pars prima $V\delta x$ a signa integrationis est immunis atque adeo solam variationem δx involvit, reliquae vero partes utramque perpetuo eodem modo iunctam et in littera $\omega = \delta y - p\delta x$ comprehensam continent.

COROLLARIUM 2

81. Secunda pars

$$\int Ndx \cdot w = \int N\omega dx$$

commodius exprimi nequit, tertia vero $\int Pdw$ commodius ita exprimi videtur, ut sit

$$\int Pd\omega = P\omega - \int \omega dP$$

ac post signum integrale iam ipsa quantitas ω reperiatur.

COROLLARIUM 3

82. Quarta pars $\int Qd.\frac{d\omega}{dx}$ simili modo reducitur ad

$$Q\frac{d\omega}{dx}-\int dQ\frac{d\omega}{dx},$$

hocque membrum posterius, cum sit $\int \frac{dQ}{dx}d\omega$, porro praebet

$$\omega\frac{dQ}{dx}-\int \omega d.\frac{dQ}{dx},$$

ita ut quarta pars resolvatur in haec membra

$$Q\frac{d\omega}{dx}-\frac{dQ}{dx}\omega+\int \omega d.\frac{dQ}{dx}.$$

COROLLARIUM 4

83. Quinta pars

$$\int Rd.(\delta q-r\delta x)$$

reducitur primo ad

$$R\frac{1}{dx}d.\frac{d\omega}{dx}-\int \frac{dR}{dx}d.\frac{d\omega}{dx},$$

tum vero posterius membrum ad

$$\frac{dR}{dx}\frac{d\omega}{dx}-\int \frac{1}{dx}d.\frac{dR}{dx}\cdot d\omega$$

hocque tandem ulterius ad

$$\frac{1}{dx}d.\frac{dR}{dx}\cdot\omega-\int \omega d.\frac{1}{dx}d.\frac{dR}{dx},$$

ita ut haec quinta pars iam ita exprimatur

$$R\frac{1}{dx}d.\frac{d\omega}{dx}-\frac{dR}{dx}\cdot\frac{d\omega}{dx}+\frac{1}{dx}d.\frac{dR}{dx}\cdot\omega-\int \omega d.\frac{1}{dx}d.\frac{dR}{dx}.$$

COROLLARIUM 5

84. Simili modo sexta pars ita reperitur expressa

$$\int Sd.\tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{d\omega}{dx}$$

ita reperitur expressa

$$S\tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{d\omega}{dx}-\tfrac{dS}{dx}\cdot\tfrac{1}{dx}d.\tfrac{d\omega}{dx}+\tfrac{1}{dx}d.\tfrac{dS}{dx}\cdot\tfrac{d\omega}{dx}-\tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{dS}{dx}\cdot\omega$$
$$+\int\omega d.\tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{dS}{dx}.$$

PROBLEMA 7

85. *Positis* $dy=pdx$, $dp=qdx$, $dq=rdx$, $dr=sdx$ *etc . Si V fuerit functio quaecunque quantitatum x, y, p, q, r,* s *etc., ita ut sit*

$$dV=Mdx+Ndy+Pdp+Qdq+Rdr+Sds+\text{etc.},$$

formulae integralis $\int Vdx$ *variationem ex utriusque variabilis x et y variatione natam ita exprimere, ut post signum integrale nulla occurrant variationum differentialia.*

SOLUTIO

In corollariis praecedentis problematis iam omnia ita sunt ad hunc scopum praeparata, ut nihil aliud opus sit, nisi ut transformationes singularum partium in ordinem redigantur, quo pacta duplicis generis partes obtinentur, uno continente formulas integrales, quas quidem omnes in eandem summam colligere licet, altero partes absolutas, quas ita in membra distribuemus, ut secundum ipsas variationes δx et δy earumque differentialia cuiusque gradus procedant. Posita autem brevitatis gratia formula $\delta y-p\delta x=\omega$ variatio quaesita ita se habebit

$$\delta\int Vdx=\int\omega dx\left(N-\tfrac{dP}{dx}+\tfrac{1}{dx}d.\tfrac{dQ}{dx}-\tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{dR}{dx}+\tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{dS}{dx}-\text{etc.}\right)$$
$$+V\delta x+\omega\left(P-\tfrac{dQ}{dx}+\tfrac{1}{dx}d.\tfrac{dR}{dx}-\tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{dS}{dx}+\text{etc.}\right)$$
$$+\tfrac{d\omega}{dx}\left(Q-\tfrac{dR}{dx}+\tfrac{1}{dx}d.\tfrac{dS}{dx}-\text{etc.}\right)$$
$$+\tfrac{1}{dx}d.\tfrac{d\omega}{dx}\left(R-\tfrac{dS}{dx}+\text{etc.}\right)$$
$$+\tfrac{1}{dx}d.\tfrac{1}{dx}d.\tfrac{d\omega}{dx}\left(S-\text{etc.}\right)$$
$$+\text{etc.},$$

cuius formae indoles ex sola inspectione statim est manifesta, ut uberiori illustratione non sit opus.

COROLLARIUM 1

86. Haec expressio multo simplicior redditur, si elementum dx capiatur constans, quo quidem amplitudo expressionis nequaquam restringitur; tum enim fiet

$$\begin{aligned}\delta\int Vdx = \int \omega dx\left(N - \tfrac{dP}{dx} + \tfrac{ddQ}{dx^2} - \tfrac{d^3R}{dx^3} + \tfrac{d^4S}{dx^4} - \text{etc.}\right)\\ + V\delta x + \omega\left(P - \tfrac{dQ}{dx} + \tfrac{ddR}{dx} - \tfrac{d^3S}{dx^3} + \text{etc.}\right)\\ + \tfrac{d\omega}{dx}\left(Q - \tfrac{dR}{dx} + \tfrac{ddS}{dx^2} - \text{etc.}\right)\\ + \tfrac{dd\omega}{dx}\left(R - \tfrac{dS}{dx} + \text{etc.}\right)\\ + \tfrac{d^3\omega}{dx^3}\left(S - \text{etc.}\right)\\ + \text{etc.}\end{aligned}$$

COROLLARIUM 2

87. Si quaestio sit de linea curva, prima pars integralis valorem per totam curvam ab initio usque ad terminum, ubi coordinatae x et y subsistunt, congregat simul omnes variationes in singulis curvae punctis factas complectens, dum reliquae partes absolutae tantum per variationes in extremitate curvae factas definiuntur.

COROLLARIUM 3

88. Si curvam coordinatis x et y definitam ut datam spectemus aliaque curva ab ea infinite parum discrepans consideretur, dum in singulis punctis utrique coordinatae variationes quaecunque tribuantur, expressio inventa indicat, quantum formulae integralis $\int Vdx$ valor ex curva variata collectus superat eiusdem valorem ex ipsa curva data desumtum.

COROLLARIUM 4

89. Cum sit $\omega = \delta y - p\delta x$, patet hanc quantitatem ω evanescere, si in singulis punctis variationes δx et δy ita accipiantur, ut sit

$$\delta y{:}\delta x = p{:}1 = dy{:}dx\,.$$

Hoc igitur casu curva variata plane non discrepat a data ac tota variatio formulae $\int Vdx$ reducitur ad $V\delta x$.

SCHOLION 1

90. Variatio haec pro formula integrali $\int Vdx$ inventa statim suppeditat regulam, quam olim tradidi pro curva invenienda, in qua valor eiusdem formulae integralis sit maximus vel minimus. Illa enim regula postulat, ut haec expressio

$$N - \tfrac{dP}{dx} + \tfrac{ddQ}{dx^2} - \tfrac{d^3R}{dx^3} + \tfrac{d^4S}{dx^4} - \text{etc.}$$

nihilo aequalis statuatur. Hic autem statim evidens est ad id, ut variatio formulae $\int V dx$ evanescat, quemadmodum natura maximorum et minimorum exigit, ante omnia requiri, ut prima pars signa integrali contenta evanescat, ex quo fit

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0.$$

Praeterea vero etiam partes absolutas nihilo aequari oportet, in quo applicatio ad utrumque curvae terminum continetur. Ipsa enim curvae natura per illam aequationem exprimitur; quae cum ob differentialia altioris gradus totidem integrationes totidemque constantes arbitrarias assumat, harum constantium determinationi illae partes absolutae inserviunt, ut tam in initio quam in fine quaesita curva certis conditionibus respondeat, veluti ad datas lineas curvas terminetur. Ac si aequatio illa fuerit differentialis ordinis quarti vel adeo altioris, partium quoque absolutarum numerus augetur, quibus effici potest, ut curva quaesita non solum utrinque ad datas lineas terminetur, sed ibidem quoque certa directio, quin etiam, si ad altiora differentialia assurgat, certa curvaminis lex praescribi queat. Semper autem applicationem faciendo pulcherrime evenire solet, ut ipsa quaestionum indoles eiusmodi conditiones involvat, quibus per partes absolutas commodissime satisfieri possit.

SCHOLION 2

91. Quanta autem mysteria in hac forma, quam pro variatione formulae integralis $\int V dx$ invenimus, lateant, in eius applicatione ad maxima et minima multo luculentius declarare licet; hic tantum observo partem integralem necessario in istam variationem ingredi. Cum enim rem in latissimo sensu simus complexi, ut in singulis curvae punctis utrique variabili x et y variationes quascunque nulla plane lege inter se connexas tribuerimus, fieri omnino nequit, ut variatio toti curvae conveniens non simul ab omnibus variationibus intermediis pendeat, quippe quibus aliter constitutis necesse est, ut inde totius curvae variatio mutationem perpetiatur. Atque in hoc variation formularum integralium potissimum dissidet a variatione eiusmodi expressionum, quales in superiori capite consideravimus, quae unice a variatione ultimis elementis tributa pendet. Ex quo lucelenter sequitur, si forte quantitas V ita fuerit comparata, ut formula differentialis $V dx$ integrationem admittat nulla stabilita relatione inter variables x et y sicque formula integralis $\int V dx$ sit functio absoluta quantitatum x, y, p, q, r, etc., tum etiam eius variationem tantum a variatione extremorum elementorum pendere posse sicque partem variationis integralem plane in nihilum abire debere, ex quo sequens insigne theoreme colligitur.

THEOREMA 3

92. *Posito* $dy = pdx$, $dp = qdx$, $dq = rdx$, $dr = sdx$ *etc* . si *V fuerit eiusmodi functio ipsarum x, y, p, q, r, s etc., ut posito eius differentiali*

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + \text{etc.}$$

fuerit

$$N - \tfrac{dP}{dx} + \tfrac{ddQ}{dx^2} - \tfrac{d^3R}{dx^3} + \tfrac{d^4S}{dx^4} - \text{etc.} = 0.$$

sumto elemento dx constante, tum formula differentialis Vdx per se erit integrabilis nulla stabilita relatione inter variabiles x et y, ac vicissim.

DEMONSTRATIO

Si fuerit

$$N - \tfrac{dP}{dx} + \tfrac{ddQ}{dx^2} - \tfrac{d^3R}{dx^3} + \tfrac{d^4S}{dx^4} - \text{etc.} = 0,$$

tum formulae integralis $\int Vdx$ variatio nullam implicat formulam integralem ideoque pro quovis situ coordinatarum *x* et *y* a solis variationibus, quae ipsis in extremo termino tribuuntur, pendet, quod fieri neutiquam posset, si formula *Vdx* integrationem respueret, propterea quod tum variatio insuper ab omnibus variationibus intermediis simul necessario penderet; unde sequitur, quoties aequatio illa locum habet, toties formulam *Vdx* integrationem admittere, ita ut *rVdx* futura sit certa ac definita functio quantitatum *x, y,p, q, r,* s etc. Vicissim autem quoties formula differentialis *Vdx* integrationem admittit eiusque propterea integrale $\int Vdx$ est vera functio quantitatum *x, y, p, q, r, s* etc., toties quoque eius variatio tantum ab extremis variationibus ipsarum *x* et *y* pendet neque variationes intermediae eam ullo modo afficere possunt. Ex quo necesse est, ut variationis pars integralis supra inventa evanescat, id quod fieri nequit, nisi fuerit

$$N - \tfrac{dP}{dx} + \tfrac{ddQ}{dx^2} - \tfrac{d^3R}{dx^3} + \tfrac{d^4S}{dx^4} - \text{etc.} = 0,$$

sicque theorema propositum etiam inversum veritati est consentaneum.

COROLLARIUM 1

93. En ergo insigne criterium, cuius ope formula differentialis duarum variabilium, cuiuscunque gradus differentialia in eam ingrediantur, diiudicari potest, utrum sit integrabilis necne. Multo latius ergo patet illo criterio satis noto, quo formularum differentialium primi gradus integrabilitas dignosci solet.

COROLLARIUM 2

94. Primo ergo si quantitas V sit tantum functio ipsarum x et y nullam differentialium rationem involvens, ut sit

$$dV = Mdx + Ndy\,,$$

tum formula differentialis Vdx integrationem non admittit, nisi sit $N = 0$, hoc est, nisi V sit functio ipsius x tantum; quod quidem per se est perspicuum.

COROLLARIUM 3

95. Proposita autem huiusmodi formula differentiali $vdx + udy$, ea cum forma Vdx ob $dy = pdx$ comparata dat $V = v + pu$ ideoque

$$M = \left(\tfrac{dv}{dx}\right) + p\left(\tfrac{du}{dx}\right), \quad N = \left(\tfrac{dv}{dy}\right) + p\left(\tfrac{du}{dy}\right) \text{ et } P = u,$$

quandoquidem quantitates v et u nulla differentialia implicare sumuntur.
Erit ergo

$$dP = du = dx\left(\tfrac{du}{dx}\right) + dy\left(\tfrac{du}{dy}\right).$$

Quare cum criterium integrabilitatis postulet, ut sit

$$N - \tfrac{dP}{dx} = 0,$$

erit pro hoc casu

$$\left(\tfrac{dv}{dy}\right) + p\left(\tfrac{du}{dy}\right) - \left(\tfrac{du}{dx}\right) - p\left(\tfrac{du}{dy}\right) = 0$$

seu

$$\left(\tfrac{dv}{dy}\right) = \left(\tfrac{du}{dx}\right),$$

quod est criterium iam vulgo cognitum.

SCHOLION 1

96. Demonstratio huius theorematis omnino est singularis, cum ex doctrina variationum sit petita, quae tamen ab hoc argumento prorsus est aliena; vix vero alia via patet ad eius demonstrationem pertingendi. Tum vero hic accuratior cognitio functionum diligenter est animadvertenda, qua ostendimus formulam integralem $\int Vdx$ neutiquam ut functionem quantitatum *x, y, p, q, r etc.* spectari posse, nisi revera integrationem admittat. Natura enim functionum semper hanc proprietatem habet adiunctam, ut, statim atque quantitatibus, quae eam ingrediuntur, valores determinati tribuuntur, ipsa functio ex iis formata determinatum adipiscatur valorem; veluti haec functio *xy*, si ponamus $x=2$ et $y=3$, valorem accipit = 6. Longe secus autem evenit in formula integrali $\int ydx$, cuius valor pro casu $x=2$ et $y=3$ neutiquam assignari potest, nisi inter *y* et *x* certa quaedam relatio statuatur; tum autem ea formula abit in functionem unicae variabilis. Formularum ergo integralium, quae integrari nequeunt, natura sollicite a natura functionum distingui debet, cum functiones, statim atque quantitatibus variabilibus, ex quibus conflantur, determinati valores tribuuntur, ipsae quoque determinatos valores recipiant, etiamsi variabiles nullo modo a se invicem pendeant; quod minime evenit in formulis integralibus, quippe quarum determinatio omnes plane valores intermedios simul includit. Imprimis autem huic discrimini universa doctrina de maximis et minimis, ad quam hic attendimus, innititur, ubi formulas, quibus maximi minimive proprietas conciliari debet, necessario eiusmodi integrales esse oportet, quae per se integrationem non admittant.

SCHOLION 2

97. Ad maiorem theorematis illustrationem eiusmodi formulam integralem $\int Vdx$ consideremus, quae per se sit integrabilis, ponamusque exempli gratia

$$\int Vdx=\frac{xdy}{ydx}=\frac{xp}{y},$$

ita ut sit

$$V=\frac{p}{y}-\frac{xpp}{yy}+\frac{xq}{y}$$

atque ideo haec formula differentialis

$$\left(\frac{p}{y}-\frac{xpp}{yy}+\frac{xq}{y}\right)dx$$

sit absolute integrabilis, ac videamus, an theorema nostrum hanc integrabilitatem declaret.

Quantitatem ergo *V* differentiemus et differentiali cum forma

$$dV=Mdx+Ndy+Pdp+Qdq$$

comparato obtinebimus

$$M=\frac{-pp}{yy}+\frac{q}{y},\quad N=\frac{-p}{yy}+\frac{2xpp}{y^3}-\frac{xq}{yy},P=\frac{1}{y}-\frac{2xp}{yy}\quad\text{et}\quad Q=\frac{x}{y}.$$

Cum nunc secundum theorema fieri debeat

$$N-\frac{dP}{dx}+\frac{ddQ}{dx^2}=0,$$

primo colligimus differentiando

$$\frac{dP}{dx} = \frac{-3p}{yy} + \frac{4xpp}{y^3} - \frac{2xq}{yy} \quad \text{et} \quad \frac{dQ}{dx} = \frac{1}{y} - \frac{xp}{yy},$$

tum vero

$$\frac{ddQ}{dx^2} = \frac{-2p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy}.$$

Ergo

$$\frac{dP}{dx} - \frac{ddQ}{dx^2} = \frac{-p}{yy} + \frac{2xpp}{y^3} - \frac{xq}{yy},$$

cui valori quantitas N utique est aequalis.

SCHOLION 3

98. Ceterum quando formula differentialis Vdx integrationem per se admittit ideoque posito

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}$$

secundum theorema est

$$N - \frac{dP}{dx} + \frac{ddQ}{dx^2} - \frac{d^3R}{dx^3} + \frac{d^4S}{dx^4} - \text{etc.} = 0$$

hinc alia insignia consectaria deducuntur. Primo enim Cum per dx multiplicando et integrando fiat

$$\int Ndx - P + \frac{dQ}{dx} - \frac{ddR}{dx^2} + \frac{d^3S}{dx^3} - \text{etc.} = A,$$

patet etiam formulam Ndx absolute esse integrabilem. Deinde cum hinc porro fiat

$$\int dx \int \left(Ndx - P\right) + Q - \frac{dR}{dx^2} + \frac{dsS}{dx^2} - \text{etc.} = Ax + B,$$

etiam formula

$$dx \int \left(Ndx - P\right)$$

haec integrationem admittit. Postea etiam simili modo integrabilis erit haec forma

$$dx\left(\int dx \int \left(Ndx - P\right) + Q\right),$$

tum vero etiam

$$dx\left(\int dx\left(\int dx\left(\int Ndx - P\right) + Q\right) - R\right)$$

et ita porro. Unde sequens theorema non minus notatu dignum et in praxi utilissimum colligimus.

THEOREMA 4

99. *Positis* $dy = pdx$, $dp = qdx$, $dq = rdx$, $dr = sdx$ *etc. si V eiusmodi fuerit functio ipsarum x, y, p, q, r, s etc., ut formula differentialis Vdx per se sit integrabilis, tum posito*

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + Sds + \text{etc}.$$

etiam sequentes formulae differentiales per se integrationem admittent:

I. *Formula N dx erit per se integrabilis; tum posito* $P - \int Ndx = \mathfrak{P}$:

II. *Formula* $\mathfrak{P}dx$ *erit per se integrabilis;porro posito* $Q - \int \mathfrak{P}dx = \mathfrak{Q}$:

III. Formula $\mathfrak{Q}dx$ *erit per se integrabilis; deinde posito* $R - \int \mathfrak{Q}dx = \mathfrak{R}$:

IV. *Formula* $\mathfrak{R}dx$ *erit per se integrabilis; ulterius posito* $S - \int \mathfrak{R}dx = \mathfrak{S}$:

V. *Formula* $\mathfrak{S}dx$ *erit per se integrabilis et ita porro.*

DEMONSTRATIO

Huius theorematis veritas iam in praecedente paragrapho est evicta, unde simul patet, si omnes hae formulae integrationem admittant, etiam principalem *Vdx* absolute fore integrabilem.

COROLLARIUM 1

100. Cum *V* sit functio quantitatum

$$x,\ y,\ p = \tfrac{dy}{dx},\ q = \tfrac{dp}{dx},\ r = \tfrac{dq}{dx} \ \text{etc.},$$

quantitates per differentiationem inde derivatae *M, N, P, Q, R* etc. etiam ita exhiberi possunt, ut sit

$$M = \left(\tfrac{dV}{dx}\right),\ N = \left(\tfrac{dV}{dy}\right),\ P = \left(\tfrac{dV}{dp}\right),\ Q = \left(\tfrac{dV}{dq}\right) \ \text{etc.},$$

unde ob primam formulam patet, si fuerit formula *Vdx* integrabilis, tum etiam formulam $\left(\tfrac{dV}{dx}\right)dx$ fore integrabilem.

COROLLARIUM 2

101. Deinde ergo quoque ob eandem rationem formula haec $\left(\tfrac{ddV}{dx^2}\right)dx$ hincque porro istae $\left(\tfrac{d^3V}{dx^3}\right)dx$, $\left(\tfrac{d^4V}{dx^4}\right)dx$ etc. omnes per se integrationmn admittent.

COROLLARIUM 3

102. Quia tot tantum litterae *P, Q, R* etc. adsunt, quoti gradus differentialia in formula *Vdx* reperiuntur, et sequentes omnes evanescunt, litterae germanicae inde derivatae $\mathfrak{P}$, $\mathfrak{Q}$, $\mathfrak{R}$, $\mathfrak{S}$ etc. tandem evanescere vel in functiones solius quantitatis *x* abire debent, quia alioquin sequentes integrabilitates locum habere non possent.

EXEMPLUM

103. *Sit V eiusmodi functio, ut fiat*

$$\int Vdx = \frac{y\left(dx^2+dy^2\right)^{\frac{3}{2}}}{xdxddy}.$$

Factis substitutionibus

$$dy = pdx,\ dp = qdx,\ dq = rdx$$

pro hoc exemplo functio *V* ita exprimetur

$$V = \frac{p(1+pp)^{\frac{3}{2}}}{xq} - \frac{y(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3yp\sqrt{(1+pp)}}{x} - \frac{yr(1+pp)^{\frac{3}{2}}}{xqq},$$

unde per differentiationem elicimus sequentes valores

$$N = -\frac{(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3p\sqrt{(1+pp)}}{x} - \frac{r(1+pp)^{\frac{3}{2}}}{xqq},$$

$$P = \frac{(1+4pp)\sqrt{(1+pp)}}{xq} - \frac{3yp\sqrt{(1+pp)}}{xxq} + \frac{3y(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3ypr\sqrt{(1+pp)}}{xqq},$$

$$Q = -\frac{p(1+pp)^{\frac{3}{2}}}{xxq} + \frac{y(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2yr(1+pp)^{\frac{3}{2}}}{xxq^3},$$

$$R = -\frac{y(1+pp)^{\frac{3}{2}}}{xqq}.$$

Iam igitur primo integrabilem esse oportet formulam *Ndx* seu

$$-\frac{dx(1+pp)^{\frac{3}{2}}}{xxq} + \frac{3pdx\sqrt{(1+pp)}}{x} - \frac{dq(1+pp)^{\frac{3}{2}}}{xqq},$$

unde statim patet integrale hoc fore

$$\int Ndx = \frac{(1+pp)^{\frac{3}{2}}}{xq}.$$

Iam pro secunda formula hinc nanciscimur

$$\mathfrak{P} = P - \int Ndx = \frac{3pp\sqrt{(1+pp)}}{xq} - \frac{3yp\sqrt{(1+pp)}}{xxq} + \frac{3y(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3ypr\sqrt{(1+pp)}}{xqq}$$

ita ut integranda sit haec formula

$$\mathfrak{P}dx = \frac{3pdy\sqrt{(1+pp)}}{xq} - \frac{3ypdx\sqrt{(1+pp)}}{xxq} + \frac{3ydx(1+2pp)}{x\sqrt{(1+pp)}} - \frac{3ypdq\sqrt{(1+pp)}}{xqq},$$

cuius integrale vel saltemn eius pars ex postremo membro manifesto colligitur $\frac{3yp\sqrt{(1+pp)}}{xq}$; cuius differentiale cum totam formulam exhauriat, erit

$$\int \mathfrak{P}dx = \frac{3yp\sqrt{(1+pp)}}{xq}.$$

Nunc pro tertia formula habebimus

$$\mathfrak{Q} = Q - \int \mathfrak{P}dx = -\frac{p(1+pp)^{\frac{3}{2}}}{xqq} + \frac{y(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2yr(1+pp)^{\frac{3}{2}}}{xq^3} - \frac{3yp\sqrt{(1+pp)}}{xq},$$

unde per *dx* multiplicando ob $dx = \frac{dp}{q}$ in ultimo membro fit

$$\mathfrak{Q}dx = -\frac{dy(1+pp)^{\frac{3}{2}}}{xqq} + \frac{ydx(1+pp)^{\frac{3}{2}}}{xxqq} + \frac{2ydq(1+pp)^{\frac{3}{2}}}{xq^3} - \frac{3ypdp\sqrt{(1+pp)}}{xqq},$$

cuius penultimum membrum declarat integrale

$$\int \mathfrak{Q}dx = -\frac{y(1+pp)^{\frac{3}{2}}}{xqq}.$$

Quarta porro formula ita erit comparata

$$\mathfrak{R} = R - \int \mathfrak{Q}dx = 0,$$

unde perspicuum est non solum hanc $\mathfrak{R}dx$, sed etiam sequentes omnes fore integrabiles.

SCHOLION

104. Theoremata haec eo pulcriora videntur, quod eorum demonstratio eiusmodi principio innititur, cuius ratio hinc prorsus est aliena, propterea quod in his veritatibus nullum amplius vestigium variationum apparet; ex quo nullum est dubium, quin demonstratio etiam ex alio fonte magis naturali hauriri queat [§ 96].