CHAPTER II

CONCERNING THE VARIATION OF DIFFERENTIAL FORMULAS INVOLVING TWO VARIABLES

THEOREM 1

37. *The variation of a differential is equal always to the differential of the variation or there is* $\delta dV = d\delta V$ *whatever the quantity V should be, provided it has increased by the differential, also by the variation it has received.*

DEMONSTRATION

The variable quantity V can be considered as the applied line of some curve, which may be proceeding by its own differentials along the same curve, truly which may jump across by its variations to another curve nearest to that. But provided that it may be moved into the nearest point of the closest curve, the value of this shall become $=V+dV$ which shall be $=V'$, and thus $dV=V'-V$; from which the variation of dV itself, that is δdV, will be $=\delta V'-\delta V$. Now $\delta V'$ is the closest value, into which δV will become increased by its own differential, thus so that there shall be $\delta V'=\delta V+d\delta V$ or $\delta V'-\delta V=d\delta V$, from which it is apparent that $\delta dV=d\delta V$ or the variation of the differential is equal to the differential of the variation, precisely as the theorem confirms.

COROLLARY 1

38. Hence the variation of the second differential ddV is defined thus, so that there shall be

$$\delta ddV=d\delta .dV,$$

but since there shall be $\delta dV=d\delta V$, there will be equality between these formulas

$$\delta ddV=d\delta dV=dd\delta V$$

COROLLARY 2

39. In the same manner for differentials of the third order there will be

$$\delta d^3V=d\delta ddV=dd\delta dV=d^3\delta V$$

and for the differentials of the fourth order the variation will be had thus, so that there becomes

$$\delta d^4V=d\delta d^3V=dd\delta ddV=d^3\delta dV=d^4\delta V,$$

and in a similar manner for higher orders.

COROLLARY 3

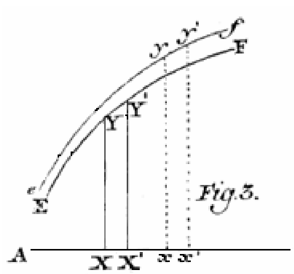
40. Therefore if a variation should be desired of some order or other, the variation sign δ, can be inserted between the signs of the differentials *d*, wherever it pleases ; but at the end place the position indicates that the variation of the differentials of any order to be equal to the differential of the variation of the same order.

COROLLARY 4

41. Therefore since there shall be $\delta d^n V = d^n \delta V$, the matter always is reduced to that, so that the variation of the quantity *V* or the differentials of this δV of any order may be taken, and in this reduction the strength of this new calculus is to be established.

SCHOLIUM 1

42. The strength of the demonstration is centred chiefly in this, so that δV may change into $\delta V'$, if the quantity *V* has increased by its own differential, which indeed is evident from the nature of the differentials ; yet meanwhile it will be helpful to illustrate that by geometry. For some curve *EF* (Fig. 3) the coordinates shall be $AX = x$ and $XY = y$; on which we may progress by an infinitely small interval *YY'*, there will be in the differentials



$$AX' = x + dx \text{ and } X'Y' = y + dy$$

and thus

$$dx = AX' - AX \quad \text{and} \quad dy = X'Y' - XY$$

Now we may consider the other curve *ef* closest to that, the points *y* and *y'* may be compared with the points *Y* and *Y'*, and which therefore through variations a crossing is made, and with the coordinates assumed in a similar manner there will be

$$Ax = x + \delta x \text{ and } xy = y + \delta y$$

and thus [Note Euler's occasionally confusing habit of calling points and variables by the same name.]

$$\delta x = Ax - AX \quad \text{and} \quad \delta y = xy - XY \text{ ;}$$

then indeed there will be

$$Ax' = x + dx + \delta(x + dx) \quad \text{and} \quad x'y' = y + dy + \delta(y + dy),$$

in as much as we have jumped from the point *Y'* to the point *y'* by the variation. Truly we may arrive at the same point *y'* from the point *y* by differentiation too, from which it is deduced

$$Ax' = x + \delta x + d(x + \delta x) \quad \text{and} \quad x'y' = y + \delta y + d(y + \delta y).$$

Now from these values since it may these taken together there is produced

$$x+dx+\delta x+\delta dx=x+\delta x+dx+d\delta x$$

and

$$y+dy+\delta y+\delta dy=y+\delta y+dy+d\delta y\,,$$

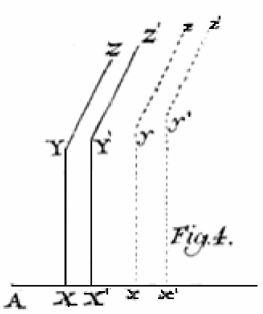
from which clearly there follows to become

$$\delta dx=d\delta x \quad \text{and} \quad \delta dy=d\delta y.$$

Which principle on which this demonstration depends, if we should examine it more carefully, we learn here returns to this, that if the variable quantity be advanced first by differentiation, then truly by a variation, in the inverse order likewise it arises if first it should be advanced by a variation and then by differentiation. Just as in the figure, from the first point *Y* by differentiation it may arrive at *Y'*, hence truly by variation in *y';* but in the inverse order first by variation from the point *Y* it arrives at *y*, hence truly by differentiation at the point *y'*, as likewise before.
[Thus, in this case the two operations are seen to commute.]

SCHOLIUM 2

43. This theorem can be extended more widely ; nor indeed is it restricted to the case of two variables only, moreover it is amenable to the truth however many variables may enter into the calculation, since it may be considered in the demonstration of that variable only, of which both by differentiation as well as by variation, an account may be had without any regard to the remaining variables. But lest any doubt should remain in place, we may consider some surface, some point of which *Z* may be defined by the three coordinates (Fig. 4) $AX=x,\ XY=y$ and $YZ=z$; from which if we may progress to another nearby point *Z'* in the same surface, these coordinates will increase by their differentials. Then truly we may consider some other nearby surface, the points of which *z* and *z'* may be brought together with these *Z* and *Z'*, which arises from variation. With these in position it is evident that it is possible to arrive at the point *Z'* in a twofold manner, with one by the variation from the point *Z'*, and with the other by differentiation from the point *z*, and thus there shall be



$$\begin{aligned} Ax' &= AX'+\delta.AX' = Ax+d.Ax,\\ x'y' &= X'Y'+\delta.X'Y' = xy+d.xy,\\ y'z' &= Y'Z'+\delta.Y'Z' = yz+d.yz, \end{aligned}$$

because it serves also for all the other variable quantities to be referred to this point. Hence moreover it follows very well to become :

$$\delta dx = d\delta x,\ \delta dy = d\delta y,\ \delta dz = d\delta z.$$

SCHOLIUM 3

44. To sum up it is remarkable, that in the case of differentials of higher order the variation sign δ can be inscribed as it pleases between the differential signs *d*, and hence one is allowed to understand this position changeability also needs to be considered, even if the variation sign δ likewise may be repeated a number of times with the differentiations, which perhaps may be able to find a use in other explorations. Now in the present circumstances established a repetition of the variation δ in no manner can be considered to have a place, because we may compare a line or surface only with a single other one close to itself; and even if this is considered generally, so that all possible closest ones are included likewise, yet as it is regarded with a single one, from the principle, after we have jumped across to the closest one, a new transition to another is conceded. Hence therefore speculations of this kind, in which variations of the variations are required to be sought, are entirely excluded. But in turn a differential of the variation of any order must be admitted here, and since in formulas with differentials, which have a finite number [of differentials] requiring to be signified, an account of the differentials only may be considered, which, if the two variables shall be *x* and *y*, from the positions of this kind

$$dy = pdx,\ dp = qdx,\ dq = rdx \quad \text{etc.}$$

which are accustomed to be recalled to finite forms, the variations of which quantities *p, q, r* etc. it is necessary to assign.

PROBLEM 1

45. *With two of the given variables x and y with the variations δx and δy, to define the variation of the differential formula $p = \frac{dy}{dx}$.*

SOLUTION

Since there shall be $\delta dy = d\delta y$ and $\delta dx = d\delta x$, the variation sought δp is found by the rules of differentiation noted, as long as in place of the sign of differentiation *d* there is prescribed the sign of the variation δ; from which since there arises

$$\delta p = \frac{dx\delta dy - dy\delta dx}{dx^2},$$

there will be by the conversion shown

$$\delta p = \frac{dxd\delta y - dyd\delta x}{dx^2};$$

where since δx and δy shall be the variations of *x* and *y*, and hence $\delta x + d\delta x$ and $\delta y + d\delta y$ are the variations of $x + dx$ and $y + dy$, there is required to be noted, as we have now observed [§ 37], to become

$$d\delta x = \delta\left(x + dx\right) - \delta x \quad \text{and} \quad d\delta y = \delta\left(y + dy\right) - \delta y .$$

Likewise it is found from first principles ; for since the value of p varied shall be $p + \delta p$ and that may arise, if in place of x and y the values of these may be substituted which are to be varied, which are $x + \delta x$ and $y + \delta y$, there will be

$$p + \delta p = \frac{d\left(y + \delta y\right)}{d\left(x + \delta x\right)} = \frac{dy + d\delta y}{dx + d\delta x},$$

from which on account of $p = \frac{dy}{dx}$, there becomes

$$\delta p = \delta . \frac{dy}{dx} = \frac{dy + d\delta y}{dx + d\delta x} - \frac{dy}{dx} = \frac{dxd\delta y - dyd\delta x}{dx^2},$$

since in the denominator the small part $dxd\delta x$ vanishes before dx^2.

COROLLARY 1

46. If, as long as we progress by differentials, we may designate the variables x et y immediately increased by x', x'', x''' etc., y', y'', y''' etc., so that there shall be

$$x' = x + dx' \text{ and } y' = y + dy ,$$

there will become

$$d\delta x = \delta x' - \delta x \quad \text{and} \quad d\delta y = \delta y' - \delta y$$

and hence

$$\delta p = \delta . \frac{dy}{dx} = \frac{dx\left(\delta y' - \delta y\right) - dy\left(\delta x' - \delta x\right)}{dx^2}.$$

COROLLARY 2

47. Because the variations of both the variables x and y in turn by no means depend on each other, but in a word are left by our choice, if we may give for x no variation, so that there shall be

$$\delta x = 0 \text{ and } \delta x' = 0,$$

there will be

$$\delta p = \frac{d\delta y}{dx} = \frac{\delta y' - \delta y}{dx}.$$

COROLLARIUM 3

48. If in addition we may attribute δy for the variation of the single variable *y*, so that there shall be $\delta y' = 0$, there will be $\delta p = -\frac{\delta y}{dx}$, which acts counter to the nature of the hypothesis minimally, because one is allowed to assume thus it is required to be in agreement with the principal curve, so that it disagrees with that at a single point only.

SCHOLIUM

49. Generally in the solution of isoperimetric problems and of others pertaining to that kind it is the custom to put in place a variation thus agreeing, so that it may disagree only as if at a single point. Thus if the curve *EF* (Fig. 5) shall be sought enjoying a certain property of the maximum or minimum, with a single point *Y* accustomed to be transferred to a place nearest to *y*, so that with the curve varied *EMyY'F* deviates only in the smallest interval *MY'* from that sought, thus so that on putting $AX = x$ and $XY = y$ there shall be for the varied curve

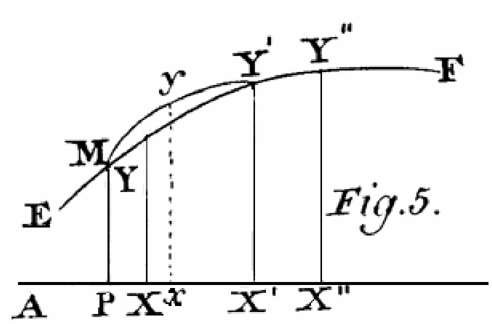


Fig.5.

$$Ax = x + \delta x \text{ and } xy - y + \delta y$$

or

$$\delta x = Ax - AX \quad \text{and} \quad \delta y = xy - XY \ ,$$

truly for the following points, to which the differentials lead, shall be everywhere

$$\delta x' = 0, \quad \delta y' = 0, \quad \delta x'' = 0, \quad \delta y'' = 0 \text{ etc..}$$

and likewise for what goes before. So that also it suffices for the variation $Xx = \delta x$ to be taken conveniently as zero, so that all the variation may be produced by a single element δy, in which case there will be considered everywhere $\delta p = -\frac{\delta y}{dx}$, and this variation by a single variable certainly is sufficient for problems of this kind, which indeed if they were treated, may be resolved.

If now, as we have established here, we extend these problems further, so that the curve sought is able to receive certain beginning and end boundaries, certainly the calculus of variations is necessary as the most general to be resolved and in all the points of the curve indefinite variations are to be attributed to the coordinates. Since also it is especially necessary, if we wish to apply investigations of this kind to non–continuous curved lines.

PROBLEM 2

50. *With the variations δx and δy of two given variables x and y, if there is put $dy = pdx$ and $dp = qdx$, to find the variation of the quantity q or the value of δq.*

SOLUTION

Since there shall be $q = \frac{dp}{dx}$ there will be from the value the variation

$$q + \delta q = \frac{d(p+\delta p)}{d(x+\delta x)} = \frac{dp+d\delta p}{dx+d\delta x},$$

from which on taking away the quantity q there is left

$$\delta q = \frac{dxd\delta p - dpd\delta x}{dx^2},$$

which variation hence also results from the differentiation of the formula $q = \frac{dp}{dx}$, if the differentiation is put in place in the usual manner, truly in place of the differential sign d there is written the variation sign δ; where indeed it will be of help to bear in mind

$$\delta dx = d\delta x \quad \text{and} \quad \delta dp = d\delta p.$$

But above we have found on account of $p = \frac{dy}{dx}$ there shall be

$$\delta p = \frac{dxd\delta y - dyd\delta x}{dx^2}$$

from which again by the customary differentiation the value of $d\delta p$ is deduced, clearly from the differential of δp itself.

COROLLARY 1

51. Since there shall be $\frac{dy}{dx} = p$ and $\frac{dp}{dx} = q$, in the first place there shall be

$$\delta p = \frac{d\delta y}{dx} - \frac{pd\delta x}{dx},$$

then truly

$$\delta q = \frac{d\delta p}{dx} - \frac{qd\delta x}{dx}.$$

Moreover for future use the small part kept here $d\delta p$ is to be left as the value of this has be elicited from the preceding formula.

COROLLARY 2

52. Meanwhile since yet the first by differentiation will give

$$d\delta p = \frac{dd\delta y}{dx} - \frac{ddxd\delta y}{dx^2} - \frac{pdd\delta x}{dx} - qd\delta x + \frac{pddxd\delta x}{dx^2},$$

with this value substituted there emerges

$$\delta q = \frac{dd\delta y}{dx^2} - \frac{ddxd\delta y}{dx^3} - \frac{pdd\delta x}{dx^2} - \frac{2qd\delta x}{dx} + \frac{pddxd\delta x}{dx^3}.$$

COROLLARY 3

53. But if variations are attributed to *y* only, so that the small parts δx and which thence are derived vanish, we will have

$$\delta p = \frac{d\delta y}{dx} \quad \text{and} \quad \delta q = \frac{d\delta p}{dx} = \frac{ddx\delta y}{dx^2} - \frac{ddxd\delta y}{dx^3},$$

and if with the differential *dx* a constant is taken, there will be $\delta q = \frac{dd\delta y}{dx^2}$.

SCHOLIUM 1

54. So that these may be easily understood, we may consider, through a relation between the variables $AX = x$ and $XY = y$ on the curve *EF* (Fig. 5), several points *Y*, *Y'*, *Y''* etc. continually advanced following differentiation, so that there shall be

$$AX = x, \quad AX' = x + dx, \; AX'' = x + 2dx + ddx,$$
$$AX''' = x + 3dx + 3ddx + d^3x,$$
$$XY = y, \; X'Y' = y + dy, \quad X''Y'' = y + 2dy + ddy,$$
$$X'''Y''' = y + 3dy + 3ddy + d^3y,$$

which differentials indicating of which order may be represented for the sake of brevity thus:

$$AX = x, \; AX' = x', \quad AX'' = x'', \quad AX''' = x''' \quad \text{etc.},$$
$$Xy = y, \; X'Y' = y', \; X''Y'' = y'', \; X'''y''' = y''' \quad \text{etc.},$$

for which single [points], their variations in no manner are considered to be given depending on each other in turn, thus so that all these variations

$$\delta x, \; \delta x', \delta x'', \; \delta x''' \quad \text{etc.},$$
$$\delta y, \delta y', \delta y'', \; \delta y''' \quad \text{etc.}$$

can be regarded as known depending on our pleasing. Now from these thus constituted the differentials of any order of the variation will be represented in this way, so that there shall be

$$d\delta x = \delta x' - \delta x, \quad dd\delta x = \delta x'' - 2\delta x' + \delta x, \ d^3\delta x = \delta x''' - 3\delta x'' + 3\delta x' - \delta x,$$
$$d\delta y = \delta' y - \delta y, \ dd\delta y = \delta y'' - 2\delta y' + \delta y, \ d^3\delta y = \delta y''' - 3\delta y'' + 3\delta y' - \delta y.$$

Because if now we assume a single point of the curve Y to be varied, there will be

$$d\delta x = -\delta x, \ dd\delta x = +\delta x, \ d^3\delta x = -\delta x \quad \text{etc.},$$
$$d\delta y = -\delta y, \ dd\delta y = +\delta y, \ d^3\delta y = -\delta y \quad \text{etc.}$$

and hence

$$\delta p = -\frac{dy}{dx} + \frac{p\delta x}{dx}$$

and

$$\delta q = \frac{\delta y}{dx^2} + \frac{ddx\delta y}{dx^3} - \frac{p\delta x}{dx^2} + \frac{2q\delta x}{dx} - \frac{pddx\delta x}{dx^3},$$

where with regard to the vanishing parts omitted there will be

$$\delta q = \delta y \cdot \frac{1}{dx^2} - \delta x \cdot \frac{p}{dx^2}.$$

And then if the variation is attributed to the applied line $XY = y$ only, there will be had

$$\delta p = -\frac{1}{dx} \text{ et } \delta q = \frac{1}{dx^2}\delta y.$$

SCHOLION 2

55. Hence it is apparent, if a variation is put in place at a single point of a curve, it impinges markedly against the received principles of differentiation, since the higher order differentials of the variations by no means vanish before the lower orders, but continually maintain the same value and thus the variations of the quantities p and q increase indefinitely, if indeed with infinitely small δx and δy so that the differentials dx and dy are taken from the same order. So that also hence in the calculation one must beware especially, lest we may fall into huge errors, since the precepts of the calculation depend on the law of continuity, where curved lines are considered to be described from the continual flow of a point, thus so that a jump is nowhere acknowledged in the curvature of these. But if moreover a single point of the curve Y (Fig. 5) is led astray at y with the remainder of the curve drawn with the remainder unchanged besides as it were the two elements My and yY', it is evident that a huge irregularity has been induced into the nature of the curve, since the usual rules of the calculus are unable to be applied further. For which inconvenience as we encounter it, the safest remedy will be, that at least with the variations attributed with the individual points of the curve in mind, which may be continued by a certain law of continuity, and neither before or after may an irregularity be admitted into the calculation, so that all the differentiations and integrations may be performed, and in this manner at least a kind of continuity may be retained in the calculation. Therefore although the differentials of the variations

$$d\delta y,\quad dd\delta y,\quad d^3\delta y \quad \text{etc., item}$$
$$d\delta x,\quad dd\delta x,\quad d^3\delta x \quad \text{etc.,}$$

one can perhaps from the hypothesis made recall these to simpler variations, yet it is arranged to retain these forms in the calculation and the following integrations are to be adapted according to these; and here also the operations may be returned, which at some time I had shown how to arrange, since I may have discussed the same argument concerned with finding the maxima and minima for curves with the given property.

PROBLEM 3

56. *With two given variables x and y with the variations δx and δy, to investigate the variations between the differentials of any order.*

SOLUTIO

Here the question corresponds, so that on putting continually

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \quad \text{etc.}$$

the variations of the quantities *p, q, r, s* etc. may be assigned, since all the accounts may be reduced to these magnitudes of the differentials of any order, which may indeed be contained by any finite values. And indeed we may consider now from the two first values of these *p* and *q* to become

$$\delta p = \tfrac{d\delta y}{dx} - \tfrac{pd\delta x}{dx} \quad \text{and} \quad \delta q = \tfrac{d\delta p}{dx} - \tfrac{qd\delta x}{dx}.$$

Therefore because again there is

$$r = \tfrac{dq}{dx} \ \text{ and } \ s = \tfrac{dr}{dx} \quad \text{etc.,}$$

the variations of these may be found in a similar manner through the rules of differentiation

$$\delta r = \tfrac{d\delta q}{dx} - \tfrac{rd\delta x}{dx},\quad \delta s = \tfrac{d\delta r}{dx} - \tfrac{sd\delta x}{dx} \quad \text{etc.,}$$

where, if it should please, in place of $d\delta p,\ d\delta q,\ d\delta r$ etc. the differentials of the variations $\delta p,\ \delta q \quad \delta r$ etc. found before can be substituted. But this may lead not only to exceedingly prolix formulas, but also, as will become apparent from the following, nor indeed is it necessary, since hence all are able to be put in place by much easier reductions, by which there will be a need.

COROLLARY 1

57. If variations are attributed to the variable *y* only, or with the abscissas *x* remaining as the applied line of this *y* may be increased by its variation , we will have

$$\delta p = \frac{d\delta y}{dx}, \quad \delta q = \frac{d\delta p}{dx}, \quad \delta r = \frac{d\delta q}{dx}, \quad \delta s = \frac{d\delta r}{dx}, \quad \text{etc.}$$

COROLLARY 2

58. But if besides all the increments *dx* of *x* may be taken equal or the element *dx* is decided to be constant, there will be obtained with the differentials of the preceding formulas substituted in the following

$$\delta p = \frac{d\delta y}{dx}, \quad \delta q = \frac{dd\delta y}{dx^2}, \quad \delta r = \frac{d^3\delta y}{dx^3}, \quad \delta s = \frac{d^4\delta r}{dx^4} \quad \text{etc.}$$

COROLLARY 3

59. If the variations of the abscissas *x* alone are attributed, so that the variation δy with all its derivatives vanish, and likewise the element *dx* may be taken constant, these individual variations these themselves will be had

$$\delta p = \frac{-pd\delta x}{dx}, \quad \delta q = \frac{-pdd\delta y}{dx^2} - \frac{2qd\delta x}{dx},$$

$$\delta r = \frac{-pd^3\delta x}{dx^3} - \frac{3qdd\delta x}{dx^2} - \frac{3rd\delta x}{dx},$$

$$\delta s = \frac{-pd^4\delta x}{dx^4} - \frac{4qd^3\delta x}{dx^3} - \frac{6rdd\delta x}{dx^2} - \frac{4sd\delta x}{dx},$$

etc.

COROLLARY 4

60. Even if hence in this case the element *dx* may be taken as constant, here still the differentials of each order occur with the variations δx*;* concerning which the reason is, because the variations of the values of x continually moved further *x', x"* etc. on no account depend on the differentials put in place.

SCHOLIUM

61. But when it pleased to attribute the variations to the variable *x* only, then the variables *x* and *y* prevail entirely between themselves and rather with the positions to be interchanged in order that

$$dx = pdy,\ dp = qdy,\ dq = rdy \ \text{ etc.},$$

from which the type of differentials is removed ; then truly with the element *dy* taken constant like simpler formulas may be found for the variations of the quantities *p, q, r* etc., and with Corollary 2. Moreover so that the calculus can be adapted to all cases, it is arranged always that for each variable I have attributed its variations ; for if then many complex forms may be produced, especially if they are to be set out, yet the calculation may be described in detail as they offer exceptional shortcuts, so that in the end the calculation scarcely becomes more laborious, nor is one tired with the extent of this. Therefore we may progress to more general problems pertaining to this chapter.

PROBLEM 4

62. *With two given variables x and y with the variations* δx *and* δy *of some finite formula* V, *to find the variation assembled from the variations of these as well as from the differentials of these of any order.*

SOLUTION

Since V shall be a quantity having a finite value, on putting

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \quad \text{etc.}$$

the differentials may be removed thence and the function V will emerge from the finite quantities formed x, y, p, q, r, s etc. Therefore whatever shall be the method of the composition, the differential of this always shall have a form of this kind

$$dV = Mdx + Ndy + Pdp + Qdq + Bdr + Sds + \text{etc},$$

with the number of members arising there increased, so that higher differentials are introduced in V. But if truly a variation δV of this formula V were to be investigated, that may be obtained, if in place of the magnitudes of the variables x, y, p, q, r etc., the same increased by their variations may be substituted and from the resulting form the quantity V is taken away, from which it is understood that the variation is to found with the help of the customary differentials with only the sign of the differential d changed into the sign of the variation δ. Whereby since the above differential now shall be shown, we will obtain the variation sought :

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + S\delta s + \text{etc.};$$

just as moreover the variations $\delta p,\ \delta q,\ \delta r,\ \delta s$ etc. may be determined by the variations assumed δx and δy, now shown above [§ 56].

COROLLARY 1

63. If here we may substitute the values found before, we will obtain the variation sought expressed thus

$$\delta V = M\delta x + N\delta y + \tfrac{1}{dx}\left(Pd\delta p + Qd\delta q + Rd\delta r + Sd\delta s + \text{etc.}\right)$$
$$- \tfrac{d\delta x}{dx}\left(Pp + Qq + Rr + Ss + \text{etc.}\right).$$

COROLLARY 2

64. If plainly no variation is attributed to the variable x and the above element dx is taken as constant, then the variation of the proposed quantity V will be expressed thus :

$$\delta V = N\delta y + \frac{Pd\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \frac{Sd^4\delta y}{dx^4} + \text{ etc.}$$

SCHOLIUM

65. At least in these forms a kind of homogeneity is seen in the differentials, if indeed δx and δy may be referred to the order of differentiation; because it may come about otherwise by far, in that case, in which a single point of the curve is varied, we may wish to substitute at once the above values shown (§ 54) of the variation in place of the values of the differentials, which obviously may be excluded in accordance with the idea of integration, which these formulas etc need. Moreover it is apparent, how finding the variation may be recalled according to customary differentiation, provided that the whole distinction has been established in this to such an extent, that in place of the variations $\delta p,\ \delta q,\ \delta r$ etc., the values are to be substituted now as assigned before, which indeed we have elucidated by customary differentiation. Moreover it will be appropriate to illustrate this operation with some examples, from which the whole nature of this treatment may be perceived more clearly.

EXAMPLE 1

66. *To find the variation of the formula* $\frac{ydx}{dy}$ *expressing the subtangent.*

On account of $dy = pdx$ this formula becomes $\frac{y}{p}$, from which the variation of this $\frac{\delta y}{p} - \frac{y\delta p}{pp}$, with the value substituted everywhere in place of δp becomes this [Recall that $\delta p = \frac{d\delta y}{dx} - \frac{pd\delta x}{dx}$.]

$$\frac{\delta y}{p} - \frac{yd\delta y}{ppdx} + \frac{yd\delta x}{pdx} = \frac{dx}{dy}\delta y - \frac{ydx}{dy^2}d\delta y + \frac{y}{dy}d\delta x\,,$$

which latter form arise at once from the differentiation of the formula.

EXAMPLE 2

67. *To find the variation of the formula* $\frac{y\sqrt{\left(dx^2+dy^2\right)}}{dy}$ *expressing the tangent itself .*

On putting $dy = pdx$ this finite form will be given $\frac{y}{p}\sqrt{\left(1+pp\right)}$, from which the variation sought is

$$\frac{\delta y}{p}\sqrt{\left(1+pp\right)} - \frac{y\delta p}{pp\sqrt{\left(1+pp\right)}}\ ,$$

which is transformed into this

$$\frac{\sqrt{(dx^2+dy^2)}}{dy}\delta y-\frac{y\delta x}{dy^2\sqrt{(dx^2+dy^2)}}(dxd\delta y-dyd\delta x).$$

EXAMPLE 3

68. *To define the radius of curvature expression* $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy}$ *variation.*

On putting $dy = pdx$ and $dp = qdx$ this formula passes into this $\frac{(1+pp)^{\frac{3}{2}}}{q}$, therefore the variation of this is

$$\frac{3p\delta p}{q}\sqrt{(1+pp)}-\frac{\delta q}{qq}(1+pp)^{\frac{3}{2}},$$

where indeed I will not dwell on substituting the values found before.

PROBLEM 5

69. *With the variations given* δx and δy *of two given variable quantities x and y, to investigate the variation of the formula constructed both from these variables as well as from the differentials of any order, whether it should be boundless or infinitely small.*

SOLUTION

As thus far on putting $dy = pdx,\ dp = qdx,\ dq = rdx$ etc. a formula may always be reduced to a form of this kind Vdx^n, where *V* shall be a finite function of the quantities *x, y, p, q, r* etc., truly the exponent *n* either positive or negative, thus so that in the first case the formula shall be infinitely small, truly in the latter infinitely large. Therefore we may put the differentiation of the orders to give

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

from which likewise the variation of this may be had. Therefore since the variation of the proposed form shall be

$$nVdx^{n-1}d\delta x + dx^n\delta V,$$

certainly there will be this variation, as we sought,

$$nVdx^{n-1}d\delta x + dx^n(Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}),$$

where form the earlier work [§ 56] it is required to substitute these values

$$\delta p = \frac{d\delta y - pd\delta x}{dx}, \quad \delta q = \frac{d\delta p - qd\delta x}{dx},$$
$$\delta r = \frac{d\delta q - rd\delta x}{dx}, \quad \delta s = \frac{d\delta r - sd\delta x}{dx}$$
etc.

Which since they shall be evident by themselves, they are in need of no further explanation and likewise this chapter may be considered completely absolved.

CAPUT II

DE VARIATIONE FORMULARUM DIFFERENTIALIUM DUAS VARIABILES INVOLVENTIUM

THEOREMA 1

37. *Variatio differentialis semper aequalis est differentiali variationis seu est* $\delta dV = d\delta V$ *quaecunque fuerit quantitas V, quae, dum per differentialia crescit, etiam variationem recipit.*

DEMONSTRATIO

Quantitas variabilis V spectari potest tanquam applicata curvae cuiuspiam, quae suis differentialibus per eandem curvam progrediatur, suis variationibus vero in aliam curvam illi proximam transiliat. Dum autem in eiusdem curvae punctum proximum promovetur, fit eius valor $=V+dV$ qui sit $=V'$, ideoque $dV=V'-V$; ex quo variatio ipsius dV hoc est δdV, erit $=\delta V'-\delta V$. Verum $\delta V'$ est valor proximus, in quem δV suo differentiali auctum abit, ita ut sit $\delta V'=\delta V+d\delta V$ seu $\delta V'-\delta V=d\delta V$, unde evidens est fore $\delta dV=d\delta V$ seu variationem differentialis esse aequalem differentiali variationis, prorsus uti theorema affirmat.

COROLLARIUM 1

38. Hinc variatio differentialis secundi ddV ita definitur, ut sit

$$\delta ddV = d\delta .dV,$$

at cum sit $\delta dV = d\delta V$, aequalitas erit inter has formulas

$$\delta ddV = d\delta dV = dd\delta V$$

COROLLARIUM 2

39. Eodem modo pro differentialibus tertii ordinis erit

$$\delta d^3V = d\delta ddV = dd\delta dV = d^3\delta V$$

et pro differentialibus quarti ordinis variatio ita se habebit, ut sit

$$\delta d^4V = d\delta d^3V = dd\delta ddV = d^3\delta dV = d^4\delta V,$$

similique modo pro altioribus gradibus.

COROLLARIUM 3

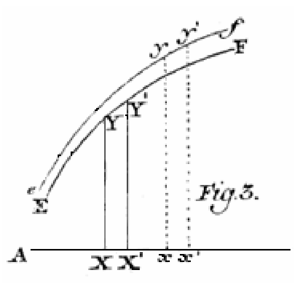
40. Si igitur variatio desideretur differentialis cuiuscunque gradus, signum variationis δ, ubicunque libuerit, inter signa differentiationis *d* inseri potest; in ultimo autem loco positum declarat variationem differentialis cuiusvis gradus aequalem esse differentiali eiusdem gradus ipsius variationis.

COROLLARIUM 4

41. Cum igitur sit $\delta d^n V = d^n \delta V$, res semper eo reducitur, ut variationis quantitatis *V* seu ipsius δV differentialia cuiusque gradus capi possint, atque in hac reductione praecipua vis huius novi calculi est constituenda.

SCHOLION 1

42. Vis demonstrationis in hoc potissimum est sita, quod δV abeat in $\delta V'$, si quantitas *V* suo differentiali increscit, quod quidem ex natura differentialium per se est manifestum; interim tamen



iuvabit id per Geometriam illustrasse. Pro curva quacunque *EF* (Fig. 3) sint coordinatae $AX = x$ et $XY = y$; in qua si per intervallum infinite parvum *YY'* progrediamur, erit in differentialibus

$$AX' = x + dx \quad \text{et} \quad X'Y' = y + dy$$

ideoque

$$dx = AX' - AX \quad \text{et} \quad dy = X'Y' - XY$$

Nunc concipiamus aliam curvam *ef* illi proximam, cuius puncta *y* et *y'* cum illius punctis *Y* et *Y'* comparentur, ad quae propterea per variationes transitus fiat, ac sumtis simili modo coordinatis erit

$$Ax = x + \delta x \quad \text{et} \quad xy = y + \delta y$$

ideoque

$$\delta x = Ax - AX \quad \text{et} \quad \delta y = xy - XY\,;$$

tum vero erit

$$Ax' = x + dx + \delta(x + dx) \quad \text{et} \quad x'y' = y + dy + \delta(y + dy),$$

quatenus a puncto *Y'* per variationem in punctum *y'* transilimus. Verum ad idem punctum *y'* quoque ex puncto *y* per differentiationem pervenimus, unde colligitur

$$Ax' = x + \delta x + d(x + \delta x) \quad \text{et} \quad x'y' = y + \delta y + d(y + \delta y).$$

His iam valoribus cum illis collatis prodit

$$x + dx + \delta x + \delta dx = x + \delta x + dx + d\delta x$$

et

$$y + dy + \delta y + \delta dy = y + \delta y + dy + d\delta y,$$

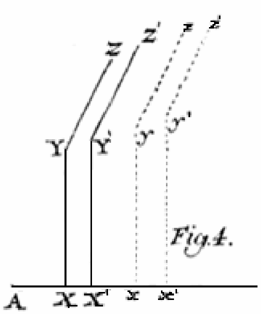
unde manifesto sequitor fore

$$\delta dx = d\delta x \quad \text{et} \quad \delta dy = d\delta y.$$

Quae si attentius consideremus, principium, cui demonstratio innititur, huc redire comperimus, ut, si quantitas variabilis primo per differentiationem, deinde vero per variationem proferatur, idem proveniat, ac si ordine inverso primo per variationem, tum vero per differentiationem promoveretur. Veluti in figura ex puncto *Y* primo per differentiationem pervenitur in *Y'*, hinc vero per variationem in *y';* inverso autem ordine primum ex puncto *Y* per variationem pervenitur in *y*, hinc vero per differentiationem in punctum *y'*, idem quod ante.

SCHOLION 2

43. Theorema hoc latissime patet; neque enim ad casum duarum variabilium tantum restringitur, sed veritati est etiam consentaneum, quotcunque variabiles in calculum ingrediantur, quandoquidem in demonstratione solius illius variabilis, cuius tam differentiale quam variatio consideratur, ratio habetur sine ullo respectu ad reliquas variabiles. Ne autem hic ulli dubio locus relinquatur, consideremus superficiem quamcunque, cuius punctum quodvis *Z* (Fig. 4) per coordinatas ternas $AX = x$, $XY = y$ et $YZ = z$ definiatur; a quo si ad aliud punctum proximum *Z'* in eadem superficie progrediamur, hae coordinatae suis differentialibus increscent. Tum vero aliam quamcunque superficiem concipiamus proximam, cuius puncta *z* et *z'* cum illis *Z* et *Z'* conferantur, quod fit per variationem. His positis perspicuum est duplici modo ad punctum *Z'* perveniri posse, altero per variationem ex puncto *Z'*, altero per differentiale ex puncto *z*, sicque fore



$$Ax' = AX' + \delta.AX' = Ax + d.Ax,$$
$$x'y' = X'Y' + \delta.X'Y' = xy + d.xy,$$
$$y'z' = Y'Z' + \delta.Y'Z' = yz + d.yz,$$

quod etiam de omnibus aliis quantitatibus variabilibus ad haec puncta referendis valet. Hinc autem luculenter sequitur fore

$$\delta dx = d\delta x,\ \delta dy = d\delta y,\ \delta dz = d\delta z.$$

SCHOLION 3

44. Memorabile prorsus est, quod casu differentialium altioris ordinis signum variationis δ pro lubitu inter signa differentiationis *d* inscribi possit, atque hinc intelligere licet hanc permutabilitatem locum quoque esse habituram, etiamsi signum variationis δ perinde ac differentiationis *d* aliquoties repetatur, quod fortasse in aliis speculationibus usu venire posset. Verum in praesenti instituto repetitio variationis δ nullo modo locum habere potest, quoniam lineam vel superficiem tantum cum unica alia sibi proxima comparamus; etsi enim haec generalissime consideratur, ut omnes possibiles itidem proximas in se complectatur, tamen tanquam unica spectatur neque, postquam e principali in proximam transiliverimus, novus transitus in aliam conceditur. Hinc ergo eiusmodi speculationes, quibus variationum variationes essent quaerendae, omnino excluduntur. Vicissim autem variationum differentialia cuiusque ordinis hic admitti debent, et cum in formulis differentialibus, quae quidem significatum habent finitum, ratio differentialium tantum spectetur, quae, si binae variabiles sint *x* et *y*, huiusmodi positionibus

$$dy = pdx,\ dp = qdx,\ dq = rdx \quad \text{etc.}$$

ad formas finitas revocari solent, harum quantitatum *p*, *q*, *r* etc. variationes potissimum assignari necesse est.

PROBLEMA 1

45. *Datis binarum variabilium x et y variationibus* δx *et* δy *formulae differentialis* $p = \frac{dy}{dx}$ *variationem definire.*

SOLUTIO

Cum sit $\delta dy = d\delta y$ et $\delta dx = d\delta x$, variatio quaesita δp per notas differentiationis regulas reperitur, dummodo loco signi differentiationis *d* scribatur signum variationis δ*;* unde cum oriatur

$$\delta p = \frac{dx\delta dy - dy\delta dx}{dx^2},$$

erit per conversionem ante demonstratam

$$\delta p = \frac{dxd\delta y - dyd\delta x}{dx^2};$$

ubi cum δx et δy sint variationes ipsarum *x* et *y* hincque $\delta x + d\delta x$ et $\delta y + d\delta y$ variationes ipsarum $x + dx$ et $y + dy$, notandum est fore, uti iam observavimus [§ 37],

$$d\delta x = \delta(x + dx) - \delta x \quad \text{et} \quad d\delta y = \delta(y + dy) - \delta y.$$

Idem invenitur ex primis principiis; cum enim valor ipsius *p* variatus sit $p+\delta p$ isque prodeat, si loco *x* et *y* earum valores variati, qui sunt $x+\delta x$ et $y+\delta y$, substituantur, erit

$$p+\delta p=\frac{d(y+\delta y)}{d(x+\delta x)}=\frac{dy+d\delta y}{dx+d\delta x},$$

unde ob $p=\frac{dy}{dx}$ fit

$$\delta p=\delta.\frac{dy}{dx}=\frac{dy+d\delta y}{dx+d\delta x}-\frac{dy}{dx}=\frac{dxd\delta y-dyd\delta x}{dx^2},$$

quoniam in denominatore particula $dxd\delta x$ prae dx^2 evanescit.

COROLLARIUM 1

46. Si, dum per differentialia progredimur, variabiles *x* et *y* continuo auctas designemus per x', x'', x''' etc., y', y'', y''' etc., ut sit

$$x'=x+dx' \text{ et } y'=y+dy,$$

erit

$$d\delta x=\delta x'-\delta x \quad \text{et} \quad d\delta y=\delta y'-\delta y$$

hincque

$$\delta p=\delta.\frac{dy}{dx}=\frac{dx(\delta y'-\delta y)-dy(\delta x'-\delta x)}{dx^2}.$$

COROLLARIUM 2

47. Quoniam variationes ambarum variabilium *x* et *y* neutiquam a se invicem pendent, sed prorsus arbitrio nostro relinquuntur, si ipsi *x* nullas tribuamus variationes, ut sit

$$\delta x=0 \quad \text{et} \quad \delta x'=0,$$

erit

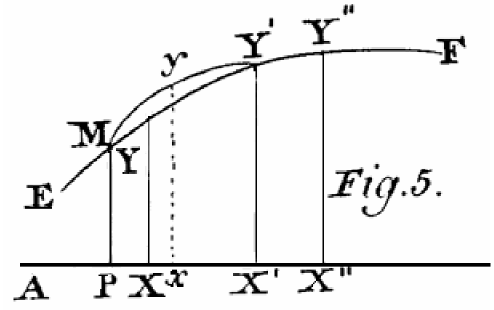
$$\delta p=\frac{d\delta y}{dx}=\frac{\delta y'-\delta y}{dx}.$$

COROLLARIUM 3

48. Si praeterea unicae variabili *y* variationem δy tribuamus, ut sit $\delta y'=0$, erit $\delta p=-\frac{\delta y}{dx}$, quae hypothesis minime naturae refragatur, quia curvam proximam ita cum principali congruentem assumi licet, ut in unico tantum puncto ab ea discrepet.

SCHOLION

49. Vulgo in solutione problematum isoperimetricorum aliorumque ad id genus pertinentium curva variata ita congruens statui solet, ut tantum in uno quasi elemento discrepet. Ita si quaerenda sit curva *EF* (Fig. 5) certa quadam maximi minimive proprietate gaudens, unicum punctum *Y* in locum proximum *y* transferri solet, ut curva variata *EMyY'F* tantum in intervallo minimo *MY'* a quaesita deflectat, ita ut positis $AX = x$ et $XY = y$ sit pro variata curva



$$Ax = x + \delta x \quad \text{et} \quad xy - y + \delta y$$

seu

$$\delta x = Ax - AX \quad \text{et} \quad \delta y = xy - XY\,,$$

pro sequentibus vero punctis, ad quae differentialia ducunt, sit ubique

$$\delta x' = 0, \quad \delta y' = 0, \quad \delta x'' = 0, \quad \delta y'' = 0 \ \text{etc.}.$$

itemque pro antecedentibus. Quin etiam ad calculi commodum variatio $Xx = \delta x$ nulla sumi solet, ut omnis variatio ad solum elementum δy perducatur, quo casu utique habebitur $\delta p = -\frac{\delta y}{dx}$, haecque unica variatio utique sufficit ad problemata huius generis, quae quidem fuerint tractata, resolvenda.

Verum si, uti hic instituimus, haec problemata latius extendimus, ut curva quaesita circa initium et finem certas determinationes recipere queat, utique necessarium est calculum variationum quam generalissime absolvere atque in omnibus curvae punctis variationes indefinitas coordinatis tribuere. Quod etiam maxime est necessarium, si huiusmodi investigationes ad lineas curvas non continuas accommodare velimus.

PROBLEMA 2

50. *Datis binarum variabilium x et y variationibus δx et δy, si ponatur $dy = pdx$ et $dp = qdx$, invenire variationem quantitatis q seu valorem ipsius δq.*

SOLUTIO

Cum sit $q = \frac{dp}{dx}$ erit pro valore variato

$$q + \delta q = \frac{d(p+\delta p)}{d(x+\delta x)} = \frac{dp+d\delta p}{dx+d\delta x},$$

unde auferendo quantitatem *q* relinquitur

$$\delta q = \frac{dxd\delta p - dpd\delta x}{dx^2},$$

quae variatio ergo etiam ex differentiatione formulae $q = \frac{dp}{dx}$ resultat, si more consueto differentiatio instituatur, loco vero signi differentialis *d* scribatur signum variationis δ; ubi quidem meminisse iuvabit esse

$$\delta dx = d\delta x \quad \text{et} \quad \delta dp = d\delta p.$$

Supra autem invenimus ob $p = \frac{dy}{dx}$ esse

$$\delta p = \frac{dxd\delta y - dyd\delta x}{dx^2}$$

unde porro per consuetam differentiationem valor ipsius $d\delta p$, scilicet differentiale ipsius δp colligitur.

COROLLARIUM 1

51. Cum sit $\frac{dy}{dx} = p$ et $\frac{dp}{dx} = q$, erit primo

$$\delta p = \frac{d\delta y}{dx} - \frac{pd\delta x}{dx},$$

tum vero

$$\delta q = \frac{d\delta p}{dx} - \frac{qd\delta x}{dx}.$$

Pro usu autem futuro praestat hic particulam $d\delta p$ relinqui quam eius valorem ex praecedente formula erui.

COROLLARIUM 2

52. Interim tamen cum prior per differentiationem det

$$d\delta p = \frac{dd\delta y}{dx} - \frac{ddxd\delta y}{dx^2} - \frac{pdd\delta x}{dx} - qd\delta x + \frac{pddxd\delta x}{dx^2},$$

hoc valore substituto prodit

$$\delta q = \frac{dd\delta y}{dx^2} - \frac{ddxd\delta y}{dx^3} - \frac{pdd\delta x}{dx^2} - \frac{2qd\delta x}{dx} + \frac{pddxd\delta x}{dx^3}.$$

COROLLARIUM 3

53. Quodsi soli variabili y variationes tribuantur, ut particulae δx et quae inde derivantur evanescant, habebimus

$$\delta p = \frac{d\delta y}{dx} \quad \text{et} \quad \delta q = \frac{d\delta p}{dx} = \frac{ddx\delta y}{dx^2} - \frac{ddxd\delta y}{dx^3},$$

ac si differentiale dx constans accipiatur, erit $\delta q = \frac{dd\delta y}{dx^2}$.

SCHOLION 1

54. Quo haec facilius intelligantur, consideremus in curva EF (Fig. 5) per relationem inter variabiles $AX = x$ et $XY = y$ plura puncta Y, Y', Y'' etc. secundum differentialia continuo promota, ut sit

$$AX = x,\quad AX' = x + dx,\ AX'' = x + 2dx + ddx,$$
$$AX''' = x + 3dx + 3ddx + d^3x,$$
$$XY = y,\ X'Y' = y + dy,\quad X''Y'' = y + 2dy + ddy,$$
$$X'''Y''' = y + 3dy + 3ddy + d^3y,$$

quae differentialia cuiusque ordinis indicantes ita brevitatis gratia repraesententur

$$AX = x,\ AX' = x',\quad AX'' = x'',\quad AX''' = x''' \quad \text{etc.},$$
$$Xy = y,\ X'Y' = y',\ X''Y'' = y'',\ X'''y''' = y''' \quad \text{etc.},$$

quibus singulis suae variationes nullo modo a se invicem pendentes tribui concipiantur, ita ut omnes istae variationes

$$\delta x,\ \delta x',\ \delta x'',\ \delta x''' \quad \text{etc.},$$
$$\delta y,\ \delta y',\ \delta y'',\ \delta y''' \quad \text{etc.}$$

a lubitu nostro pendentes tanquam cognitae spectari queant. His iam ita constitutis differentialia cuiusque ordinis variationum in hunc modum repraesentabuntur, ut sit

$$d\delta x = \delta x' - \delta x,\quad dd\delta x = \delta x'' - 2\delta x' + \delta x,\ d^3\delta x = \delta x''' - 3\delta x'' + 3\delta x' - \delta x,$$
$$d\delta y = \delta' y - \delta y,\ dd\delta y = \delta y'' - 2\delta y' + \delta y,\ d^3\delta y = \delta y''' - 3\delta y'' + 3\delta y' - \delta y.$$

Quodsi iam unicum punctum curvae Y variari sumamus, erit

$$d\delta x = -\delta x,\ \ dd\delta x = +\delta x,\ d^3\delta x = -\delta x \quad \text{etc.,}$$
$$d\delta y = -\delta y,\ dd\delta y = +\delta y,\ d^3\delta y = -\delta y \quad \text{etc.}$$

hincque

$$\delta p = -\tfrac{dy}{dx} + \tfrac{p\delta x}{dx}$$

et

$$\delta q = \tfrac{\delta y}{dx^2} + \tfrac{ddx\delta y}{dx^3} - \tfrac{p\delta x}{dx^2} + \tfrac{2q\delta x}{dx} - \tfrac{pddx\delta x}{dx^3},$$

ubi omissis partibus reliquarum respectu evanescentibus erit

$$\delta q = \delta y \cdot \tfrac{1}{dx^2} - \delta x \cdot \tfrac{p}{dx^2}.$$

Denique si soli applicatae $XY = y$ variatio tribuatur, habebitur

$$\delta p = -\tfrac{1}{dx} \text{ et } \delta q = \tfrac{1}{dx^2}\delta y.$$

SCHOLION 2

55. Hinc patet, si in unico curvae puncto variatio statuatur, insigniter contra recepta differentialium principia impingi, cum variationum differentialia superiora neutiquam prae inferioribus evanescant, sed iugiter eundem valorem retineant atque adeo variationes quantitatum *p* et *q* in infinitum excrescant, siquidem infinite parva δx et δy ex eodem ordine quo differentialia *dx* et *dy* assumantur. Quin etiam hinc in calculo maxime cavendum est, ne in enormes errores praecipitemur, cum calculi praecepta legi continuitatis innitantur, qua lineae curvae continuo puncti fluxu describi concipiuntur, ita ut in earum curvatura nusquam saltus agnoscatur. Quodsi autern unicum curvae punctum *Y* (Fig. 5) in *y* diducatur reliquo curvae tractu praeter bina quasi elementa *My* et *yY'* invariato relicto, evidens est curvaturae ingentem irregularitatenl induci, cum vulgares calculi regulae non amplius applicari queant. Cui incommodo ut occurramus, tutissimum erit remedium, ut singulis curvae punctis mente saltern variationes tribuantur, quae continuitatis quapiam lege contineantur, neque ante irregularitas in calculo admittatur, quam omnes differentiationes et integrationes fuerint peractae, hocque modo saltem species continuitatis in calculo retineatur. Quamvis ergo variationum differentialia

$$d\delta y,\quad dd\delta y,\quad d^3\delta y \quad \text{etc., item}$$
$$d\delta x,\quad dd\delta x,\quad d^3\delta x \quad \text{etc.,}$$

forte in facta hypothesi ad simplices variationes revocare liceat, tamen expedit illas formas in calculo retineri ad easque sequentes integrationes accomodari; atque huc etiam redeunt operationes, quas olim, cum idem argumentum de inveniendis curvis maximi minimive proprietate praeditis tractassem, expedire docueram.

PROBLEMA 3

56. *Datis binarum variabilium x et y variationibus δx et δy rationum inter differentialia cuiuscunque gradus variationes investigare.*

SOLUTIO

Quaestio huc redit, ut positis continuo

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \quad \text{etc.}$$

quantitatum *p, q, r, s* etc. variationes assignentur, cum ad has quantitates omnes differentialium cuiuscunque ordinis rationes, quae quidem finitis valoribus continentur, reducantur. Ac de harum quidem duabus primis *p* et *q* iam vidimus esse

$$\delta p = \frac{d\delta y}{dx} - \frac{pd\delta x}{dx} \quad \text{et} \quad \delta q = \frac{d\delta p}{dx} - \frac{qd\delta x}{dx}.$$

Quoniam igitur porro est

$$r = \frac{dq}{dx} \ \text{ et } \ s = \frac{dr}{dx} \quad \text{etc.,}$$

harum variationes simili modo per differentiationis regulas inveniuntur

$$\delta r = \frac{d\delta q}{dx} - \frac{rd\delta x}{dx}, \quad \delta s = \frac{d\delta r}{dx} - \frac{sd\delta x}{dx} \quad \text{etc.,}$$

ubi, si lubuerit, loco $d\delta p,\ d\delta q,\ d\delta r$ etc. differentialia variationum $\delta p,\ \delta q\ \ \delta r$ etc. ante inventarum substitui possunt. Hoc autem non solum in formulas nimis prolixas induceret, sed etiam, uti ex sequentibus patebit, ne quidem est necessarium, cum hinc multo facilius omnes reductiones, quibus opus erit institui queant.

COROLLARIUM 1

57. Si soli variabili *y* variationes tribuantur seu manentibus abscissis *x* tantum applicatae *y* suis variationibus augeantur, habebimus

$$\delta p = \frac{d\delta y}{dx}, \quad \delta q = \frac{d\delta p}{dx}, \quad \delta r = \frac{d\delta q}{dx}, \quad \delta s = \frac{d\delta r}{dx}, \quad \text{etc.}$$

COROLLARIUM 2

58. Quodsi praeterea omnia ipsius *x* incrementa *dx* aequalia capiantur seu elementum *dx* constans statuatur, substitutis differentialibus praecedentium formularum in sequentibus obtinebitur

$$\delta p = \frac{d\delta y}{dx}, \quad \delta q = \frac{dd\delta y}{dx^2}, \quad \delta r = \frac{d^3\delta y}{dx^3}, \quad \delta s = \frac{d^4\delta r}{dx^4} \quad \text{etc.}$$

COROLLARIUM 3

59. Si solis abscissis x variationes tribuantur, ut variatio δy cum omnibus derivatis evanescat, simulque elementum dx constans capiatur, singulae hae variationes ita se habebunt

$$\delta p = \frac{-pd\delta x}{dx}, \quad \delta q = \frac{-pdd\delta y}{dx^2} - \frac{2qd\delta x}{dx},$$
$$\delta r = \frac{-pd^3\delta x}{dx^3} - \frac{3qdd\delta x}{dx^2} - \frac{3rd\delta x}{dx},$$
$$\delta s = \frac{-pd^4\delta x}{dx^4} - \frac{4qd^3\delta x}{dx^3} - \frac{6rdd\delta x}{dx^2} - \frac{4sd\delta x}{dx},$$
etc.

COROLLARIUM 4

60. Etiamsi ergo hoc casu elementum dx constans accipiatur, tamen hic occurrunt differentialia cuiusque ordinis variationis δx *;* cuius rei ratio est, quod variationes valorum ipsius x continuo ulterius promotorum x', x'' etc. neutiquam a differentialibus pendere statuuntur.

SCHOLION

61. Quando autem placuerit soli variabili x variationes tribuere, tum omnino praestat variabiles x et y inter se permutari atque huiusmodi potius positionibus uti

$$dx = pdy,\ dp = qdy,\ dq = rdy \text{ etc.},$$

quibus species differentialium tollatur; tum vero sumto elemento dy constante similes formulae simpliciores pro variationibus quantitatum p, q, r etc. reperiuntur atque Corollario 2. Ceterum quo calculus ad omnes casus accommodari queat, semper expedit utrique variabili suas variationes tribui; etsi enim tum formae multo perplexiores prodeant, praecipue si evolvantur, tamen calculum prosequendo tam egregia se offerunt compendia, ut in fine calculus vix fiat operosior neque huius prolixitatis taedeat. Ad problemata ergo magis generalia ad hoc caput pertinentia progrediamur.

PROBLEMA 4

62. *Datis duarum variabilium x et y variationibus δx et δy formulae cuiuscunque finitae V tam ex illis variabilibus ipsis quam earum differentialibus cuiuscunque ordinis conflatae variationem invenire.*

SOLUTIO

Cum V sit quantitas valorem habens finitum, ponendo

$$dy = pdx,\ dp = qdx,\ dq = rdx,\ dr = sdx \quad \text{etc.}$$

differentialia inde tollentur prodibitque pro V functio ex quantitatibus finitis formata x, y, p, q, r, s etc. Quaecunque ergo sit ratio compositionis, eius differentiale semper huiusmodi habebit formam

$$dV = Mdx + Ndy + Pdp + Qdq + Bdr + Sds + \text{etc}.$$

horum membrorum numero existente eo maiore, quo altiora differentialia ingrediuntur in *V*. Quodsi vero huius formulae *V* variatio δV fuerit indaganda, ea obtinetur, si loco quantitatum variabilium *x, y, p, q, r* etc. eaedem suis variationibus auctae substituantur et a forma resultante ipsa quantitas *V* auferatur, ex quo intelligitur variationem ope consuetae differentiationis inveniri signa tantum differentialis *d* in signum variationis δ mutato. Quare cum differentiale supra iam sit exhibitum, impetrabimus variationem quaesitam

$$\delta V = M\delta x + N\delta y + P\delta p + Q\delta q + R\delta r + S\delta s + \text{etc.};$$

quemadmodum autem variationes $\delta p,\ \delta q,\ \delta r,\ \delta s$ etc. per variationes sumtas δx et δy determinentur, iam supra [§ 56] est ostensum.

COROLLARIUM 1

63. Si hic substituamus valores ante inventos, obtinebimus variationem quaesitam ita expressam

$$\delta V = M\delta x + N\delta y + \tfrac{1}{dx}\left(Pd\delta p + Qd\delta q + Rd\delta r + Sd\delta s + \text{etc.}\right)$$
$$- \tfrac{d\delta x}{dx}\left(Pp + Qq + Rr + Ss + \text{etc.}\right).$$

COROLLARIUM 2

64. Si variabili *x* nulla plane tribuatur variatio atque insuper elementum *dx* constans accipiatur, tum quantitatis propositae *V* variatio ita prodibit expressa

$$\delta V = N\delta y + \frac{Pd\delta y}{dx} + \frac{Qdd\delta y}{dx^2} + \frac{Rd^3\delta y}{dx^3} + \frac{Sd^4\delta y}{dx^4} + \text{ etc.}$$

SCHOLION

65. In his formis saltem species homogeneitatis in differentialibus spectatur, siquidem δx et δy ad ordinem differentialium referantur; quod longe secus eveniret, si eo casu, quo unicum curvae punctum variatur, statim vellemus loco differentialiurn variationum valores supra (§ 54) exhibitos substituere, quo quippe pacto idea integrationis, qua hae formulae deinceps indigent, excluderetur. Ceterum patet, quomodo inventio variationum ad consuetam differentiationem revocetur, dum totum discrimen in hoc tantum est situm, ut loco variationum $\delta p,\ \delta q,\ \delta r$ etc. valores iam ante assignati, quos quidem ipsos quoque per consuetam differentiationem elicuimus, substituantur. Conveniet autem hanc operationem aliquot exemplis illustrari, quo clarius indoles totius huius tractationis percipiatur.

EXEMPLUM 1

66. *Formulae subtangentem exprimentis* $\frac{ydx}{dy}$ *variationem invenire.*

Ob $dy = pdx$ haec formula fit $\frac{y}{p}$, unde eius variatio $\frac{\delta y}{p} - \frac{y\delta p}{pp}$, ubi loco δp valore substituto fit ea

$$\frac{\delta y}{p} - \frac{yd\delta y}{ppdx} + \frac{yd\delta x}{pdx} = \frac{dx}{dy}\delta y - \frac{ydx}{dy^2}d\delta y + \frac{y}{dy}d\delta x,$$

quae postrema forma immediate ex differentiatione formulae propositae nascitur.

EXEMPLUM 2

67. *Formulae ipsam tangentem exprimentis* $\frac{y\sqrt{(dx^2+dy^2)}}{dy}$ *variationem invenire.*

Positio $dy = pdx$ praebet hanc formam finitam $\frac{y}{p}\sqrt{(1+pp)}$, unde variatio quaesita est

$$\frac{\delta y}{p}\sqrt{(1+pp)} - \frac{y\delta p}{pp\sqrt{(1+pp)}},$$

quae transformatur in hanc

$$\frac{\sqrt{(dx^2+dy^2)}}{dy}\delta y - \frac{y\delta x}{dy^2\sqrt{(dx^2+dy^2)}}(dxd\delta y - dyd\delta x).$$

EXEMPLUM 3

68. *Formulae radium curvedinis exprimentis* $\frac{(dx^2+dy^2)^{\frac{3}{2}}}{dxddy}$ *variationem definire.*

Posito $dy = pdx$ et $dp = qdx$ haee formula transit in hanc $\frac{(1+pp)^{\frac{3}{2}}}{q}$, cuius propterea variatio est

$$\frac{3p\delta p}{q}\sqrt{(1+pp)} - \frac{\delta q}{qq}(1+pp)^{\frac{3}{2}},$$

ubi quidem substitutioni valorum ante inventorum non immoror.

PROBLEMA 5

69. *Datis duarum quantitatum variabilium x et y variationibus* δx et δy *formulae tam ex illis variabilibus quam earum differentialibus cuiuscunque ordinis conflatae, sive fuerit infinita sive infinite parva, variationem investigare.*

SOLUTIO

Positis ut hactenus $dy = pdx,\ dp = qdx,\ dq = rdx$ etc. formula semper reducetur ad huiusmodi formam Vdx^n, ubi V sit functio finita quantitatum x, y, p, q, r etc., exponens vero n sive positivus sive negativus, ita ut priori casu formula sit infinite parva, posteriori vero infinite magna. Ponamus igitur differentiationem ordinariam dare

$$dV = Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.},$$

unde simul eius variatio habetur. Cum igitur formae propositae variatio sit

$$nVdx^{n-1}d\delta x + dx^n\delta V\ ,$$

erit utique haec variatio, quam quaerimus,

$$nVdx^{n-1}d\delta x + dx^n\left(Mdx + Ndy + Pdp + Qdq + Rdr + \text{etc.}\right),$$

ubi ex superioribus [§ 56] hos valores substitui oportet

$$\delta p = \tfrac{d\delta y - pd\delta x}{dx},\quad \delta q = \tfrac{d\delta p - qd\delta x}{dx},$$
$$\delta r = \tfrac{d\delta q - rd\delta x}{dx},\quad \delta s = \tfrac{d\delta r - sd\delta x}{dx}$$
$$\text{etc.}$$

Quae cum per se sint perspicua, nulla ampliori explicatione indigent simulque hoc caput penitus absolutum videtur.