THEOREMS
CONCERNED WITH THE REDUCTION OF INTEGRAL FORMULAS TO THE QUADRATURE OF THE CIRCLE.

E 59
Miscellanea Berolinensia 7, 143, p. 91-129

LEMMA 1

1. *In a circle, of which the radius is* $=1$ *and the semi circumference* $=\pi$, *the sine of some of the angle of some arc* $s = x$ *;the sines of all the arcs contained in this infinite series shall be the same quantity* x *:*

$$s,\ \pi-s,\ 2\pi+s,\ 3\pi-s,\ 4\pi+s,\ 5\pi-\ s \ etc.$$

Truly in addition, x *will be the sine of all the negative angles contained in this series*

$$-\pi-s,\ -2\pi+s,\ -3\pi-s,\ -4\pi+s,\ -5\pi-s \ etc.$$

COROLLARY 1

2. Therefore if i may indicate some positive integer, then the same common sine x will be contained by all the arcs in this expression $\pm 2i\pi+s$ as of the arcs contained in this expression $\pm(2i+1)\pi-s$.

COROLLARY 2

3. Since the sines of negative angles taken shall become negative, the sines of all the angles contained in this form $\pm 2i\pi-s$ shall be $=-x$, and of the angles contained in this form $\pm(2i+1)\pi+s$ of which the sine $=-x$, if indeed the sine of the angle were $=+x$.

LEMMA 2

4. *In the circle, of which the radius* $=1$ *and of semi perimeter* $=\pi$, *the cosine of any angle s shall be* y*; the same quantity* y *shall be the cosine of all the positive angles contained in this series*

$$s,\ 2\pi-s,\ 2\pi+s,\ 4\pi-s,\ 4\pi+s,\ 6\pi-s \ etc.$$

And likewise the same quantity y *will be the cosine of all the negative angles contained in this series*

$$-s,\ -2\pi+s,\ -2\pi-s,\ -4\pi+s,\ -4\pi-s,\ -6\pi+s \ etc.$$

COROLLARY 1

5. Therefore if i shall denote some positive integer, the same common cosine y will contained in all the arcs of this general expression $\pm 2i\pi\pm s$.

COROLLARY

6. Because the cosine of two right angles or of the arc π either increased or diminished shall be negative, the same cosine $=-y$ will be of all the angles contained in this form $\pm(2i+1)\pi \pm s$, if indeed the cosine of the angle *s* were $+y$.

LEMMA 3

7. *With the same in place, if t shall be the tangent of the angle s, t will be the tangent of all the angles both positive as well as negative contained in these two series*

$$s,\ \pi+s,\ 2\pi+s,\ 3\pi+s,\ 4\pi+s,\ 5\pi+s\ etc.,$$
$$-\pi+s,\ -2\pi+s,\ -3\pi+\ s,\ -4\pi+\ s, -5\pi+s\ etc.$$

COROLLARY

8. Therefore with *i* denoting some positive number, the same common tangent $=t$ will be retained in this expression $\pm i\pi+s$ of all the angles *t;* truly the tangent of the negative angles $\pm i\pi-s$ will be $=-t$, if indeed the tangent of the angle *s* were $=+t$.

PROBLEM 1

9. *To find the roots of this infinite equation*

$$x = z - \frac{z^3}{1\cdot2\cdot3} + \frac{z^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{z^7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} + \text{etc.}$$

SOLUTION

If *z* may indicate the arc of a circle of radius $=1$, *x* will be its sine. Therefore we may put the arc to be *s*, of which the sine shall be $=x$; there will be an infinitude of values of *z* itself retained by the two series

$$s,\ \pi-s,\ 2\pi+s,\ 3\pi-s,\ 4\pi+\ s,\ 5\pi-s\ \ etc.,$$
$$-\pi-s,\ -2\pi+s,\ -3\pi-s,\ 4\pi+s,\ -5\pi-s\ etc.$$

Q. E. I.

COROLLARY 1

10. Therefore if the equation may be changed into this form :

$$0 = 1 - \frac{z}{1x} + \frac{z^3}{1\cdot2\cdot3x} - \frac{z^5}{1\cdot2\cdot3\cdot4\cdot5x} + \frac{z^7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7x} + \text{etc.}$$

the following factors from its infinite number may be had

$$\left(1-\tfrac{z}{s}\right)\left(1-\tfrac{z}{\pi-s}\right)\left(1+\tfrac{z}{\pi+s}\right)\left(1+\tfrac{z}{2\pi-s}\right)\left(1-\tfrac{z}{2\pi+s}\right)\left(1-\tfrac{z}{3\pi-s}\right)\left(1+\tfrac{z}{3\pi+s}\right) \text{ etc.,}$$

in which factors the law of the progression is readily observed.

COROLLARY 2

11. Therefore since the coefficient of the second term in the product shall be equal to the sum of the coefficients of z in all the factors, there will be

$$\frac{1}{x}=\frac{1}{s}+\frac{1}{\pi-s}-\frac{1}{\pi+s}-\frac{1}{2\pi-s}+\frac{1}{2\pi+s}+\frac{1}{3\pi-s}-\text{etc.,}$$

COROLLARY 3

12. Let $s=\frac{m}{n}\pi$, thus so that there may become $x=\sin.\text{A}.\frac{m}{n}\pi;$ it will be with the series multiplied by $\frac{m}{n}$:

$$\frac{\pi}{nx}=\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\text{etc.,}$$

COROLLARY 4

13. Because the coefficient of z^2 itself in the product, which is $=0$, shall be equal to the sum of the factors from the second order coefficients of z in the factors, truly thus sum taken from these is equal to the square of the sum of the same coefficients removed from the sum of the squares of the same, there will become:

$$\frac{1}{xx}=\frac{1}{ss}+\frac{1}{(\pi-s)^2}+\frac{1}{(\pi+s)^2}+\frac{1}{(2\pi-s)^2}+\frac{1}{(2\pi+s)^2}+\text{etc.}$$

COROLLARY 5

14. Therefore again on putting $s=\frac{m}{n}\pi$, so that there shall be $x=\sin.\text{A}.\frac{m}{n}\pi;$ this same summation will be produced:

$$\frac{\pi\pi}{nnxx}=\frac{1}{m^2}+\frac{1}{(n-m)^2}+\frac{1}{(n+m)^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}+\text{etc.}$$

SCHOLIUM

15. I could progress further in this manner and determine the sums of the higher powers ; truly since I have done this already elsewhere [E61; see e.g. Series I, Vol. 14, *O.O.*] and for our present situation these powers suffice, I will forgo further investigations.

PROBLEM 2

16. *To find the roots of this infinite equation:*

$$y=1-\frac{z^2}{1\cdot 2}+\frac{z^4}{1\cdot 2\cdot 3\cdot 4}-\frac{z^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}+\frac{z^8}{1\cdot 2\cdot 3\cdots 8}-\text{etc.}$$

SOLUTION

If z shall denote the arc from the circle, of which the radius $=1$, y will be the cosine of this arc. But if therefore an arc s may be taken, of which the cosine shall be $=y$, innumerable values of z will be obtained from the two series following :

$$s,\ 2\pi-s,\ 2\pi+s,\ 4\pi-s,\ 4\pi+s \text{ etc.},$$
$$-s,\ -2\pi+s,\ -2\pi-s,\ -4\pi+s,\ -4\pi-s \text{ etc.}$$

Q. E. I.

COROLLARY 1

17. Therefore if the proposed equation may be changed into this form :

$$0=1-\frac{z^2}{1\cdot 2(1-y)}+\frac{z^4}{1\cdot 2\cdot 3\cdot 4(1-y)}-\frac{z^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6(1-y)}+\text{etc.},$$

the following factors of that will be found from the infinite number

$$\left(1-\frac{zz}{ss}\right)\left(1-\frac{zz}{(2\pi-s)^2}\right)\left(1-\frac{zz}{(2\pi+s)^2}\right)\left(1-\frac{zz}{(4\pi-s)^2}\right)\left(1-\frac{zz}{(4\pi+s)^2}\right)\text{etc.}$$

COROLLARY 2

18. Therefore since in the product the coefficient of zz shall be equal to the sum of the coefficients of zz itself from the factors, the sum of the following series will be had :

$$\frac{1}{2(1-y)}=\frac{1}{ss}+\frac{1}{(2\pi-s)^2}+\frac{1}{(2\pi+s)^2}+\frac{1}{(4\pi-s)^2}+\frac{1}{(4\pi+s)^2}+\text{etc.}$$

COROLLARY 3

19. Putting $s = \frac{m}{n}\pi$, so that there shall become

$$y = \cos.\mathrm{A}.\tfrac{m}{n}\pi \text{ and } 1-y = 2\left(\sin.\mathrm{A}.\tfrac{m}{2n}\pi\right)^2;$$

there will be

$$\frac{\pi\pi}{2nn(1-y)} = \frac{\pi\pi}{4nn\left(\sin.\mathrm{A}.\frac{m}{2n}\pi\right)^2} = \frac{1}{mm} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n-m)^2} + \frac{1}{(4n+m)^2} + \text{etc.}$$

which agrees with § 14, but only if here n may be written in place of $2n$.

PROBLEM 3

20. *To find the roots of* z *of this infinite equation*

$$t = \frac{z - \frac{z^3}{1\cdot2\cdot3} + \frac{z^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{z^7}{1\cdot2\cdot3\cdots7} + \text{etc}}{1 - \frac{z^2}{1\cdot2} + \frac{z^4}{1\cdot2\cdot3\cdot4} - \frac{z^6}{1\cdot2\cdot3\cdots6} + \text{etc}}.$$

SOLUTION

If z may denote an arc of the circle, of which the radius $=1$, t will be the tangent of this arc ; therefore if the arc s may be taken in this circle, of which the tangent shall be $=t$, there will be the following boundless values of z

$$s,\ \pi+s,\ 2\pi+s,\ 3\pi+s,\ 4\pi+s,\ 5\pi+s \text{ etc.},$$
$$-\pi+s,\ -2\pi+s,\ -3\pi+s,\ -4\pi+s,\ -5\pi+s \text{ etc.}$$

Q. E. I.

COROLLARY 1

21. The proposed equation may be reduced to this form :

$$0 = 1 - \frac{z}{t} - \frac{z^2}{1\cdot2} + \frac{z^3}{1\cdot2\cdot3t} + \frac{z^4}{1\cdot2\cdot3\cdot4} - \frac{z^5}{1\cdot2\cdot3\cdot4\cdot5t} - \text{etc.}$$

and the simple factors of this will be the following :

$$\left(1-\frac{z}{s}\right)\left(1+\frac{z}{\pi-s}\right)\left(1-\frac{z}{\pi+s}\right)\left(1+\frac{z}{2\pi-s}\right)\left(1-\frac{z}{2\pi+s}\right)\left(1+\frac{z}{3\pi-s}\right) \text{ etc.},$$

COROLLARY 2

22. Therefore since the coefficient of z in the equation shall be equal to the sum of the coefficients of z from the individual factors, there will be

$$\frac{1}{t}=\frac{1}{s}-\frac{1}{\pi-s}+\frac{1}{\pi+s}-\frac{1}{2\pi-s}+\frac{1}{2\pi+s}-\frac{1}{3\pi-s}+\text{etc.},$$

COROLLARY 3

23. There may be put $s=\frac{m}{n}\pi$, so that there shall be $t=\text{tang.A.}\frac{m}{n}\pi=\frac{x}{y}$, if there shall be $x=\sin.\text{A.}\frac{m}{n}\pi$ and $y=\cos.\text{A.}\frac{m}{n}\pi$, and the summation of the following sequence will produce

$$\frac{\pi}{nt}=\frac{\pi y}{nx}=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{etc.}$$

COROLLARY 4

24. The sum of the individual terms of the series § 22 is equal to the square of the sum of the series with $\frac{1}{tt}$ taken away twice from both, that is -1; therefore the sum of the squares $=\frac{1}{tt}+1=\frac{yy}{xx}+1=\frac{1}{xx}$. Therefore this summation will be produced as in § 13 :

$$\frac{1}{xx}=\frac{1}{ss}+\frac{1}{(\pi-s)^2}+\frac{1}{(\pi+s)^2}+\frac{1}{(2\pi-s)^2}+\frac{1}{(2\pi+s)^2}+\text{etc.}$$

THEOREM 1

25. *With π denoting the semi-circumference of the circle, of which the radius* $=1$, *if there were* $x=\sin.\text{A.}\frac{m}{n}\pi$, there will become :

$$\frac{\pi(p+qy)}{nx}=\frac{p+q}{m}+\frac{p-q}{n-m}-\frac{p-q}{n+m}-\frac{p+q}{2n-m}+\frac{p+q}{2n+m}+\frac{p-q}{3n-m}-\frac{p-q}{3n+m}-\text{etc.}$$

DEMONSTRATION

If the series found in § 12 were multiplied by p, there becomes :

$$\frac{\pi p}{nx}=\frac{p}{m}+\frac{p}{n-m}-\frac{p}{n+m}-\frac{p}{2n-m}+\frac{p}{2n+m}+\text{etc.},$$

and if the series found in § 23 were multiplied by q, there will become :

$$\frac{\pi qy}{nx}=\frac{q}{m}-\frac{q}{n-m}+\frac{q}{n+m}-\frac{q}{2n-m}+\frac{q}{2n+m}-\text{etc.}$$

These two series may be added and the proposed series will be produced, of which therefore the sum is $=\frac{\pi(p+qy)}{nx}$. Q. E. D.

COROLLARY 1

26. There may be taken $p=q$; this same sum will be produced :

$$\frac{\pi(1+y)}{2nx}=\frac{1}{m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{4n-m}+\frac{1}{4n+m}-\text{etc.};$$

but we have $\frac{x}{1+y}=\text{tang.A.}\frac{m}{2n}\pi$, which therefore agrees with the series found in § 23 on putting n in place of $2n$.

COROLLARY 2

27. It may be assumed that $p=-q$; this same sum will be produced :

$$\frac{\pi(1-y)}{2nx}=\frac{1}{n-m}-\frac{1}{n+m}+\frac{1}{3n-m}-\frac{1}{3n+m}+\text{etc.};$$

but $\frac{1-y}{x}=\text{tang.A.}\frac{m}{2n}\pi$ and hence

$$\frac{x}{1-y}=\text{tang.A.}\left(\frac{-m}{2n}\pi+\frac{1}{2}\pi\right)=\text{tang.A.}\frac{(n-m)}{2n}\pi,$$

which series, with m again written in place of $n-m$, is reduced to the preceding.

PROBLEM 4

28. *To find the sum of this series*

$$\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\text{etc.}$$

by an integral formula.

SOLUTION

The numerators are provided with individual fractions, which shall be the powers of z, of which the exponents shall be equal to the denominators, and this series will be obtained :

$$\frac{z^m}{m}+\frac{z^{n-m}}{n-m}-\frac{z^{n+m}}{n+m}-\frac{z^{2n-m}}{2n-m}+\frac{z^{2n+m}}{2n+m}+\text{etc.}$$

which on making $z=1$ will change into that. The sum of this series may be put $=s$ and with the differentials taken it will become

$$\frac{zds}{dz}=z^m+z^{n-m}-z^{n+m}-z^{2n-m}+z^{2n+m}+z^{3n+m}-\text{etc.}$$

which series has been composed from two geometric series, and thus its sum

$$=\frac{z^m}{1+z^n}+\frac{z^{n-m}}{1+z^n}.$$

Therefore we will have:

$$\frac{zds}{dz}=\frac{z^m+z^{n-m}}{1+z^n} \text{ and } ds=\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz,$$

consequently

$$s=\int\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz;$$

and thus the value of this integral in the case where $z=1$, will give the sum of the proposed series. Q. E. I.

COROLLARY 1

29. Therefore since the sum of the proposed series is $\frac{\pi}{nx}=\frac{\pi}{n\sin.\text{A}.\frac{m}{n}\pi}$, there will become :

$$\frac{\pi}{n\sin.\text{A}.\frac{m}{n}\pi}=\int\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz$$

if there may be put $z=1$ after the integration . In this case the integral of the formula $\int\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz$ can be shown with the help of the circle.

COROLLARY 2

30. Therefore by putting definite numbers in place of m and n the following integrations will be found in the case $z=1$:
If $m=1$, $n=2$, there will become :

$$\frac{\pi}{2}=\int\frac{2}{1+zz}dz;$$

if $m = 1,\ n = 3$, there is

$$\frac{2\pi}{3\sqrt{3}} = \int \frac{1+z}{1+z^3} dz = \int \frac{dz}{1-z+zz};$$

if $m = 1,\ n = 4$, there becomes

$$\frac{\pi}{2\sqrt{2}} = \int \frac{1+zz}{1+z^4} dz;$$

if $m = 1, n = 6$, there becomes+

$$\frac{\pi}{3} = \int \frac{1+z^4}{1+z^6} dz;$$

which all are taken to be in agreement with the truth of the actual integration put in place.

PROBLEM 5

31. *To find the sum of this series*

$$\frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \frac{1}{3n+m} - \text{etc.}$$

by an integral formula.

SOLUTION

Since suitable numerators may be taken together with these fractions, as we done before, and there may be put

$$s = \frac{z^m}{m} - \frac{z^{n-m}}{n-m} + \frac{z^{n+m}}{n+m} - \frac{z^{2n-m}}{2n-m} + \frac{z^{2n+m}}{2n+m} - \text{etc.}$$

certainly on making $z = 1$ the value of s itself will be the sum of the proposed series. Now it may be set up by differentiation and there will be produced :

$$\frac{zds}{dz} = z^m - z^{n-m} + z^{n+m} - z^{2n-m} + z^{2n+m} - \text{etc.};$$

because the sum of which series can be shown, it will become

$$\frac{zds}{dz} = \frac{z^m - z^{n-m}}{1-z^n},$$

from which there becomes

$$s = \int \frac{z^{m-1} - z^{n-m-1}}{1-z^n} dz.$$

Therefore the value of this integral formula in the case $z = 1$ will give the sum of the proposed series. Q. E. I.

COROLLARY 1

32. By § 23, the sum of the proposed series is $=\frac{\pi}{nt}=\frac{\pi y}{nx}=\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}$.

On account of which there will be

$$\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=\int\frac{z^{m-1}-z^{n-m-1}}{1-z^{n}}dz$$

in the case where there is put $z=1$ after the integration.

COROLLARY 2

33. Therefore the simpler cases thus will be considered :

If $m=1,\ n=3$, there will become $\frac{\pi}{3\sqrt{3}}=\int\frac{(1-z)dz}{1-z^{3}}=\int\frac{dz}{1+z+zz}$;

if $m=1,\ n=4$, there will be $\frac{\pi}{4}=\int\frac{\left(1-z^{2}\right)dz}{1-z^{4}}=\int\frac{dz}{1+z^{2}}$;

if $m=1$, $n=6$, there will be $\frac{\pi}{2\sqrt{3}}=\int\frac{\left(1-z^{4}\right)dz}{1-z^{6}}=\int\frac{(1+zz)dz}{1+zz+z^{4}}$;

of which it is at once apparent that there is agreement of the formulas with the true value after the integration actually put in place.

SCHOLIUM

34. Now if in both the series examined the two terms may be combined into one, the following summations arise :

$$\frac{\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=\frac{1}{m}+\frac{2m}{n^{2}-m^{2}}-\frac{2m}{4n^{2}-m^{2}}+\frac{2m}{9n^{2}-m^{2}}-\frac{2m}{16n^{2}-m^{2}}+\text{etc.}$$

$$\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=\frac{1}{m}-\frac{2m}{n^{2}-m^{2}}-\frac{2m}{4n^{2}-m^{2}}-\frac{2m}{9n^{2}-m^{2}}-\frac{2m}{16n^{2}-m^{2}}-\text{etc.}$$

Therefore from these ordinates we will have these two most noteworthy following series :

$$\frac{\pi}{2mn\sin.\mathrm{A}.\frac{m}{n}\pi}-\frac{1}{2mn}=\frac{1}{n^{2}-m^{2}}-\frac{1}{4n^{2}-m^{2}}+\frac{1}{9n^{2}-m^{2}}-\frac{1}{16n^{2}-m^{2}}+\text{etc.}$$

$$\frac{1}{2mn}-\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=\frac{1}{n^2-m^2}+\frac{1}{4n^2-m^2}+\frac{1}{9n^2-m^2}+\frac{1}{16n^2-m^2}+\text{etc.}$$

which if they may be added in turn, will give

$$\frac{\pi\left(1-\cos.\mathrm{A}.\frac{m}{n}\pi\right)}{4mn\sin.\mathrm{A}.\frac{m}{n}\pi}=\frac{\pi\sin.\mathrm{A}.\frac{m}{2n}\pi}{4mn\cos.\mathrm{A}.\frac{m}{2n}\pi}=\frac{1}{n^2-m^2}+\frac{1}{9n^2-m^2}+\frac{1}{25n^2-m^2}+\frac{1}{49n^2-m^2}+\text{etc.}$$

And I have attended to this series some years ago from quite diverse principles [See E130: *De seriebus quibusdam considerationes*; Series I, vol. 14, *O.O.*].

THEOREM. 2

35. *The sum of this series of squares*

$$\frac{1}{m^2}-\frac{1}{(n-m)^2}-\frac{1}{(n+m)^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}-\frac{1}{(3n-m)^2}-\text{etc.}$$

is

$$=\frac{\pi^2\cos.\mathrm{A}.\frac{m}{n}\pi}{nn\left(\sin.\mathrm{A}.\frac{m}{n}\pi\right)^2}.$$

DEMONSTRATION

In § 12 we have seen to be

$$\frac{\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\text{etc.};$$

since which equality shall always be valid, whatever m may be, the differentials also must be equal. Therefore we may differentiate both the series as well as the its sum, with m put to be variable, and the differentials also will be equal. Moreover both sides divided by dm will give :

$$\frac{-\pi^2\cos.\mathrm{A}.\frac{m}{n}\pi}{nn\left(\sin.\mathrm{A}.\frac{m}{n}\pi\right)^2}=-\frac{1}{m^2}+\frac{1}{(n-m)^2}+\frac{1}{(n+m)^2}-\frac{1}{(2n-m)^2}-\frac{1}{(2n+m)^2}+\text{etc.}$$

The signs may be changed and the sum of the proposed series will be given. Q. E. D.

COROLLARY 1

36. We may expand some of the simpler cases, and there shall become $m=1,\ n=2$; there will become $\sin.A.\tfrac{m}{n}\pi=1$ and $\cos.A.\tfrac{m}{n}\pi=0$,
from which

$$0=1-1-\tfrac{1}{9}+\tfrac{1}{9}+\tfrac{1}{25}-\tfrac{1}{25}-\tfrac{1}{49}+\text{etc.}$$

Let $m=1,\ n=3$; there will become

$$\sin.A.\tfrac{m}{n}\pi=\tfrac{\sqrt{3}}{2}\ \text{and}\ \cos.A.\tfrac{m}{n}\pi=\tfrac{1}{2},$$

from which

$$\tfrac{2\pi\pi}{27}=1-\tfrac{1}{4}-\tfrac{1}{16}+\tfrac{1}{25}+\tfrac{1}{49}-\tfrac{1}{64}-\tfrac{1}{100}+\text{etc.}$$

Let $m=1, n=4$; there will become

$$\sin.A.\tfrac{m}{n}\pi=\cos.A.\tfrac{m}{n}\pi=\tfrac{1}{\sqrt{2}},$$

from which

$$\tfrac{\pi\pi}{8\sqrt{2}}=1-\tfrac{1}{9}-\tfrac{1}{25}+\tfrac{1}{49}+\tfrac{1}{81}-\tfrac{1}{121}-\tfrac{1}{169}+\text{etc.}$$

COROLLARY 2

37. We may multiply the series

$$\tfrac{2\pi\pi}{27}=1-\tfrac{1}{4}-\tfrac{1}{16}+\tfrac{1}{25}+\tfrac{1}{49}-\tfrac{1}{64}-\tfrac{1}{100}+\text{etc.}$$

in which the squares divisible by three are absent, by

$$\tfrac{9}{8}=1+\tfrac{1}{9}+\tfrac{1}{81}+\tfrac{1}{729}+\text{etc.},$$

so that all the squares may occur, and there will become :

$$\tfrac{\pi\pi}{12}=1-\tfrac{1}{4}+\tfrac{1}{9}-\tfrac{1}{16}+\tfrac{1}{25}-\tfrac{1}{36}+\tfrac{1}{49}-\text{etc.},$$

the truth of which has been shown be me elsewhere [E61].

COROLLARY 3

38. Since there shall be from § 14,

$$\frac{\pi\pi}{nnxx}=\frac{\pi\pi}{nn\left(\sin.A.\tfrac{m}{n}\pi\right)^2}=\frac{1}{m^2}+\frac{1}{(n-m)^2}+\frac{1}{(n+m)^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}+\text{etc.}$$

from these series added there will become

$$\frac{\pi\pi\left(1+\cos.A.\frac{m}{n}\pi\right)}{2nn\left(\sin.A.\frac{m}{n}\pi\right)^2}=\frac{1}{m^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}+\frac{1}{(4n-m)^2}+\frac{1}{(4n+m)^2}+\text{etc.}$$

which series is reduced to that by writing n in place of $2n$; indeed there is:

$$\frac{1+\cos.A.\frac{m}{n}\pi}{\left(\sin.A.\frac{m}{n}\pi\right)^2}=\frac{1}{2\left(\sin.A.\frac{m}{2n}\pi\right)^2}.$$

SCHOLIUM

39. Therefore the summation demonstrated in this proposition shall be able to be deduced directly from the summation of the series § 14 gave. Indeed since there shall be :

$$\frac{\pi\pi}{nn\left(\sin.A.\frac{m}{n}\pi\right)^2}=\frac{1}{m^2}+\frac{1}{(n-m)^2}+\frac{1}{(n+m)^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}+\text{etc.},$$

also there will become on writing $2n$ in place of n :

$$\frac{\pi\pi}{4nn\left(\sin.A.\frac{m}{2n}\pi\right)^2}=\frac{\pi\pi\left(1+\cos.A.\frac{m}{n}\pi\right)}{2nn\left(\sin.A.\frac{m}{n}\pi\right)^2}=\frac{1}{m^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}+\frac{1}{(4n-m)^2}+\frac{1}{(4n+m)^2}+\text{etc.},$$

$$[c.f.\ 2\sin^2\tfrac{\theta}{2}=\frac{\sin^2\theta}{1+\cos\theta}=\frac{4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{1+2\cos^2\frac{\theta}{2}-1}.]$$

from which, if twice that may be subtracted, the proposed will remain :

$$\frac{\pi\pi\cos.A.\frac{m}{n}\pi}{nn\left(\sin.A.\frac{m}{n}\pi\right)^2}=\frac{1}{m^2}-\frac{1}{(n-m)^2}-\frac{1}{(n+m)^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}-\text{etc.},$$

moreover in a like manner from the series § 23, which on account of the alternating signs may be seen to be most regular, the series § 12 showed can be deduced. Indeed since there shall be

$$\frac{\pi\cos.A.\frac{m}{n}\pi}{n\sin.A.\frac{m}{n}\pi}=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{etc.},$$

there will become, if $2n$ may be written in place of n,

$$\frac{\pi\cos.\mathrm{A}.\frac{m}{2n}\pi}{2n\sin.\mathrm{A}.\frac{m}{2n}\pi}=\frac{\pi\left(1+\cos.\mathrm{A}.\frac{m}{n}\pi\right)}{2n\sin.\mathrm{A}.\frac{m}{n}\pi}=\frac{1}{m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{4n-m}+\frac{1}{4n+m}-\text{etc.},$$

That series may be subtracted from twice this one and there will be :

$$\frac{\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\text{etc.},$$

which is that same series found in § 12.

Moreover in a similar manner the integral formulas, which were found for these sums, in turn will be reduced to the same. Indeed since (§. 32) there shall be

$$\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=\int\frac{z^{m-1}-z^{n-m-1}}{1-z^{n}}dz\,,$$

there will be also

$$\frac{\pi\cos.\mathrm{A}.\frac{m}{2n}\pi}{2n\sin.\mathrm{A}.\frac{m}{2n}\pi}=\frac{\pi\left(1+\cos.\mathrm{A}.\frac{m}{n}\pi\right)}{2n\sin.\mathrm{A}.\frac{m}{n}\pi}=\int\frac{z^{m-1}-z^{2n-m-1}}{1-z^{2n}}dz;$$

from twice which the first may be subtracted ; there will become :

$$\frac{\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=\int\frac{2z^{m-1}-2z^{2n-m-1}}{1-z^{2n}}dz-\int\frac{z^{m-1}-z^{2n-m-1}}{1-z^{2n}}dz$$
$$=\int\frac{z^{m-1}-z^{n+m-1}+z^{n-m-1}-z^{2n-m-1}}{1-z^{2n}}dz=\int\frac{z^{m-1}+z^{n-m-1}}{1+z^{n}}dz,$$

which is that same integration found in § 29 . From which it is evident that everything, which have been elicited at this point, shall be able to be deduced from this summation :

$$\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=\int\frac{z^{m-1}-z^{n-m-1}}{1-z^{n}}dz=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\frac{1}{3n+m}-\text{etc.},$$

From which by differentiation this is obtained :

$$\frac{\pi\pi}{nn\left(\sin.\mathrm{A}.\frac{m}{n}\pi\right)^{2}}=\frac{1}{m^{2}}+\frac{1}{(n-m)^{2}}+\frac{1}{(n+m)^{2}}+\frac{1}{(2n-m)^{2}}+\frac{1}{(2n+m)^{2}}+\text{etc.},$$

which now has been found in § 14.

PROBLEM. 5

40. *To find the differentials of the first, second, and of the following higher orders of this quantity*

$$\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi},$$

with m taken to be variable.

SOLUTION

For brevity we shall put

$$\sin.\mathrm{A}.\tfrac{m}{n}\pi = x \text{ and } \cos.\mathrm{A}.\tfrac{m}{n}\pi = y$$

initially there will be $y = \sqrt{(1-xx)}$ *;* then truly there will become

$$dx = \tfrac{\pi dm}{n} y = \tfrac{\pi y}{n} dm \text{ and } dy = -\tfrac{\pi x}{n} dm.$$

Also the proposed quantity, of which the differentials are sought,

$$\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi} = V;$$

there will be $V = \frac{\pi y}{nx}$. Hence therefore there becomes :

$$dV = \frac{\pi(xdy - ydx)}{nxx} = \frac{-\pi\pi dm}{nnxx};$$

on account of $xx + yy = 1$ therefore

$$\frac{dV}{dm} = \frac{-\pi\pi}{nn}\cdot\frac{1}{xx};$$

of this again with the differential taken, and there will be

$$\frac{ddV}{dm} = +\frac{\pi\pi}{nn}\cdot\frac{2dx}{x^3} = \frac{2\pi^3}{n^3}\cdot\frac{ydm}{x^3}$$

and thus

$$\frac{d^2V}{dm^2} = \frac{\pi^3}{n^3}\cdot\frac{2y}{x^3}.$$

But if the following differentials may be computed in a similar manner, thus there will be found :

$$V = +\tfrac{\pi}{nx}\cdot y$$
$$\tfrac{dV}{dm} = -\tfrac{\pi^2}{n^2x^2}\cdot 1,$$
$$\tfrac{ddV}{dm^2} = +\tfrac{\pi^3}{n^3x^3}\cdot 2y,$$
$$\tfrac{d^3V}{dm^3} = -\tfrac{\pi^4}{n^4x^4}\cdot(4yy+2),$$
$$\tfrac{d^4V}{dm^4} = +\tfrac{\pi^5}{n^5x^5}\cdot(8y^3+16y),$$
$$\tfrac{d^5V}{dm^5} = -\tfrac{\pi^6}{n^6x^6}\cdot(16y^4+88y^2+16),$$
$$\tfrac{d^6V}{dm^6} = +\tfrac{\pi^7}{n^7x^7}\cdot(32y^5+416y^3+272y),$$
$$\tfrac{d^7V}{dm^7} = -\tfrac{\pi^8}{n^8x^8}\cdot(64y^6+824y^4+2880y^2+272)$$

etc.

The law of the progression thus can be considered, so that, if there were

$$\tfrac{d^vV}{dm^v} = \pm\tfrac{\pi^{v+1}}{n^{v+1}x^{v+1}}\cdot\left(\alpha y^{v-1}+\beta y^{v-3}+\gamma y^{v-5}+\delta y^{v-7}+\varepsilon y^{v-9}+\text{etc.}\right),$$

the following order of the differential shall become :

$$\tfrac{d^{v+1}V}{dm^{v+1}} = \mp\tfrac{\pi^{v+2}}{n^{v+2}x^{v+2}}\cdot\left\{\begin{array}{l}2\alpha y^v+\left(4\beta+(v-1)\alpha\right)y^{v-2}+\left(6\gamma+(v-3)\beta\right)y^{v-4}\\+\left(8\delta+(v-5)\gamma\right)y^{v-6}+\left(10\varepsilon+(v-7)\delta\right)y^{v-8}+\text{etc}\end{array}\right\}.$$

Therefore the differentials of any order will be determined from the preceding order.
Q. E. I.

PROBLEM 6

41. *To find the sum of this series*

$$\frac{1}{m^v}+\frac{1}{(m-n)^v}+\frac{1}{(m+n)^v}+\frac{1}{(m-2n)^v}+\frac{1}{(m+2n)^v}+\frac{1}{(m-3n)^v}+\text{etc.}$$

from the individual terms of the series found in § 23, *raised to some power.*

SOLUTION

If we may put $\sin.A.\tfrac{m}{n}\pi = x$, $\cos.A.\tfrac{m}{n}\pi = y$ and $\tfrac{\pi y}{nx} = V$, from § 23 there will become :

$$V=\frac{1}{m}+\frac{1}{m-n}+\frac{1}{m+n}+\frac{1}{m-2n}+\frac{1}{m+2n}+\frac{1}{m-3n}+\text{etc.}$$

Because if now with *m* put for the variable, the differentials may be taken, and the following summations will be produced:

$$-\frac{dV}{1dm}=\frac{1}{m^2}+\frac{1}{(m-n)^2}+\frac{1}{(m+n)^2}+\frac{1}{(m-2n)^2}+\frac{1}{(m+2n)^2}+\frac{1}{(m-3n)^2}+\text{etc.}$$

$$+\frac{ddV}{1\cdot 2dm^2}=\frac{1}{m^3}+\frac{1}{(m-n)^3}+\frac{1}{(m+n)^3}+\frac{1}{(m-2n)^3}+\frac{1}{(m+2n)^3}+\frac{1}{(m-3n)^3}+\text{etc.}$$

$$-\frac{ddV}{1\cdot 2\cdot 3dm^3}=\frac{1}{m^4}+\frac{1}{(m-n)^4}+\frac{1}{(m+n)^4}+\frac{1}{(m-2n)^4}+\frac{1}{(m+2n)^4}+\frac{1}{(m-3n)^4}+\text{etc.}$$

$$+\frac{d^4V}{1\cdot 2\cdot 3\cdot 4dm^4}=\frac{1}{m^5}+\frac{1}{(m-n)^5}+\frac{1}{(m+n)^5}+\frac{1}{(m-2n)^5}+\frac{1}{(m+2n)^5}+\frac{1}{(m-3n)^5}+\text{etc.}$$

etc.

Therefore the sum of the indefinite order of the series proposed

$$\frac{1}{m^v}+\frac{1}{(m-n)^v}+\frac{1}{(m+n)^v}+\frac{1}{(m-2n)^v}+\frac{1}{(m+2n)^v}+\frac{1}{(m-3n)^v}+\text{etc.}$$

will become

$$\frac{\pm d^{v-1}V}{1\cdot 2\cdot 3\cdots(v-1)dm^{v-1}}.$$

But we have shown the value of $\frac{d^{v-1}V}{dm^{v-1}}$ from the preceding problem ; on account of which also the sums of the powers of these series will be able to be defined. Q. E. I.

PROBLEM 7

42. *To show the sine of any angle $\frac{m}{n}\pi$ by a product from an infinitude of factors.*

SOLUTION

Since there shall be

$$\frac{\pi\cos.\text{A}.\frac{m}{n}\pi}{\text{nsin.A}.\frac{m}{n}\pi}=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\text{etc.},$$

m may be treated as the variable quantity and multiplied everywhere by *dm;* there will be

$$\frac{\pi dm}{n}\cos.\mathrm{A}.\tfrac{m}{n}\pi = d.\sin.\mathrm{A}.\tfrac{m}{n}\pi$$

and on account of this matter, there will be

$$\frac{\pi dm\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi} = \frac{d.\sin.\mathrm{A}.\frac{m}{n}\pi}{\sin.\mathrm{A}.\frac{m}{n}\pi} = \frac{dm}{m} - \frac{dm}{n-m} + \frac{dm}{n+m} - \frac{dm}{2n-m} + \frac{dm}{2n+m} - \text{etc.}$$

from which with each side integrated completely there will be :

$$l\sin.\mathrm{A}.\tfrac{m}{n}\pi = lm - l(n-m) + l(n+m) + l(2n-m) + l(2n+m) + C.$$

Thus the constant C must be prepared, so that on making $m=\frac{1}{2}n$, the logarithm if the sine shall become $=0$, certainly in which case the whole sine will be considered. Therefore with the constant C determined in this manner there will be :

$$l\sin.\mathrm{A}.\tfrac{m}{n}\pi = l\,\tfrac{2m}{n} + l\,\tfrac{2n-2m}{n} + l\,\tfrac{2n+2m}{3n} + l\,\tfrac{4n-2m}{3n} + l\,\tfrac{4n+2m}{5n} + \text{etc.}$$

From which, if we may pass over to numbers, we will have

$$\sin.\mathrm{A}.\tfrac{m}{n}\pi = \tfrac{2m}{n}\cdot\tfrac{2n-2m}{n}\cdot\tfrac{2n+2m}{3n}\cdot\tfrac{4n-2m}{3n}\cdot\tfrac{4n+2m}{5n}\cdot\text{etc.}$$

Or if we may multiply the two numbers together, there will be

$$\sin.\mathrm{A}.\tfrac{m}{n}\pi = \tfrac{2m}{n}\cdot\tfrac{4nn-4mm}{3nn}\cdot\tfrac{16nn-4mm}{15nn}\cdot\tfrac{36nn-4mm}{35nn}\cdot\text{etc.}$$

Q. E. I.

COROLLARY 1

43. If in place of $2m$ we may write *m,* we will have

$$\sin.\mathrm{A}.\tfrac{m}{2n}\pi = \tfrac{m}{n}\cdot\tfrac{2n-m}{n}\cdot\tfrac{2n+m}{3n}\cdot\tfrac{4n-m}{3n}\cdot\tfrac{4n+m}{5n}\cdot\text{etc.}$$

or

$$\sin.\mathrm{A}.\tfrac{m}{2n}\pi = \tfrac{m}{n}\cdot\tfrac{4nn-mm}{4nn-nn}\cdot\tfrac{16nn-mm}{16nn-nn}\cdot\tfrac{36nn-mm}{36nn-nn}\cdot\text{etc.}$$

COROLLARY 2

44. Since $\sin.\mathrm{A}.\frac{m}{2n}\pi=\cos.\mathrm{A}.\frac{(n-m)}{2n}\pi$, if we may write $n-m$ in place of *m,* from the series found there will be

$$\cos.\mathrm{A}.\frac{m}{2n}\pi=\frac{n-m}{n}\cdot\frac{n+m}{n}\cdot\frac{3n-m}{3n}\cdot\frac{3n+m}{3n}\cdot\text{etc.}$$

or

$$\cos.\mathrm{A}.\frac{m}{2n}\pi=\frac{nn-mm}{nn}\cdot\frac{9nn-mm}{9nn}\cdot\frac{25nn-mm}{25nn}\cdot\text{etc.}$$

COROLLARY 3

45 . Because there is $\sin.\mathrm{A}.\frac{m}{n}\pi=2\sin.\mathrm{A}.\frac{m}{2n}\pi\cdot\cos.\mathrm{A}.\frac{m}{2n}\pi$, if we may divide by $2\sin.\mathrm{A}.\frac{m}{2n}\pi$, we will have

$$\cos.\mathrm{A}.\frac{m}{2n}\pi=\frac{2n-2m}{2n-m}\cdot\frac{2n+2m}{2n+m}\cdot\frac{4n-2m}{4n-m}\cdot\frac{4n+2m}{4n+m}\cdot\text{etc.}$$

and with $\sin.\mathrm{A}.\frac{m}{n}\pi$ divided by $2\cos.\mathrm{A}.\frac{m}{2n}\pi$, we will have

$$\sin.\mathrm{A}.\frac{m}{2n}\pi=\frac{m}{n-m}\cdot\frac{2n-2m}{n+m}\cdot\frac{2n+2m}{3n-m}\cdot\frac{4n-2m}{3n+m}\cdot\frac{4n+2m}{5n-m}\cdot\text{etc.}$$

COROLLARY 4

46. With these duplicate expressions of the sines and cosines equated to each other they will give :

$$1=\frac{nn}{nn-mm}\cdot\frac{4nn-4mm}{4nn-mm}\cdot\frac{9nn}{9nn-mm}\cdot\frac{16nn-4mm}{16nn-mm}\cdot\frac{25nn}{25nn-mm}\cdot\text{etc.}$$

COROLLARY 5

47. If *n* may be taken to be infinite or *m* infinitely small, there will be $\sin.\mathrm{A}.\frac{m}{2n}\pi=\frac{m}{2n}\pi$ and in this case from each series the same value of π is obtained given by WALLIS

$$\pi=\ 2\,\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10.\text{ etc.}}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdot11\cdot\text{ etc.}}$$

LEMMA 4

48. *The value of this product from the infinite constant factors*

$$\frac{a(c+b)(a+k)(c+b+k)(a+2k)(c+b+2k)\ etc.}{b(c+a)(b+k)(c+a+k)(b+2k)(c+a+2k)\ etc.}$$

is

$$= \frac{\int z^{c-1}dz\left(1-z^k\right)^{\frac{b-k}{k}}}{\int z^{c-1}dz\left(1-z^k\right)^{\frac{a-k}{k}}},$$

if after each integration there may be put $z = 1$. [E122. *De productis ex infinitis factoribus ortis*, Series I, vol. 14, *O.O.*]

PROBLEM 8

49. *To express the sine of the angle* $\frac{m}{2n}\pi$ *by integral formulas.*

SOLUTION

Since there shall be

$$\sin.A.\frac{m}{2n}\pi = \frac{m}{n}\cdot\frac{2n-m}{n}\cdot\frac{2n+m}{3n}\cdot\frac{4n-m}{3n}\cdot\frac{4n+m}{5n}\cdot\text{etc.}$$

by § 43, this infinite product may be compared with the preceding lemma and there will be $a = m,\ b = n,\ k = 2n$ and $c + m = n$ or $c + n = 2n - m$; each gives $c = n - m$. Hence therefore there will become

$$\sin.A.\frac{m}{2n}\pi = \frac{\int z^{n-m-1}dz\left(1-z^{2n}\right)^{-\frac{1}{2}}}{\int z^{n-m-1}dz\left(1-z^{2n}\right)^{\frac{-2n+m}{2n}}},$$

if after each integration thus put in place, so that the integrals vanish on putting $z = 0,$ there may be put $z = 1$.

Truly since also by § 45 there shall be

$$2\sin.A.\frac{m}{2n}\pi = \frac{2m}{n-m}\cdot\frac{2n-2m}{n+m}\cdot\frac{2n+2m}{3n-m}\cdot\frac{4n-2m}{3n+m}\cdot\text{etc.},$$

it will be by comparing with the lemma put in place
$a = 2m,\ b = n - m,\ c = n - m$ and $k = 2n$, from which there will be obtained :

$$2\sin.A.\frac{m}{2n}\pi = \frac{\int z^{n-m-1}dz\left(1-z^{2n}\right)^{\frac{-n-m}{2n}}}{\int z^{n-m-1}dz\left(1-z^{2n}\right)^{\frac{m-n}{n}}},$$

if after the integrations there may be put $z = 1$. Q. E. I.

COROLLARY 1

50. Therefore the following comparisons of different integral formulas are obtained :

$$\sin.\text{A}.\tfrac{m}{2n}\pi \cdot \int \frac{z^{n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}} = \int \frac{z^{n-m-1}dz}{\sqrt{\left(1-z^{2n}\right)}}$$

and

$$2\sin.\text{A}.\tfrac{m}{2n}\pi \cdot \int \frac{z^{n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{n-m}{n}}} = \int \frac{z^{n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}.$$

COROLLARY 2

51. Then truly with the ratio had without the sine, this comparison of the same integrals can be put in place :

$$2\int \frac{z^{n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{n-m}{n}}} : \int \frac{z^{n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}} = \int \frac{z^{n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}} : \int \frac{z^{n-m-1}dz}{\sqrt{\left(1-z^{2n}\right)}}$$

COROLLARY 3

52. We may put $m=1$ and $n=1$; there will be $\sin.\text{A}.\tfrac{m}{2n}\pi = 1$ and the comparisons thus may be had

$$\int \frac{dz}{z\sqrt{(1-zz)}} = \int \frac{dz}{z\sqrt{(1-zz)}} \text{ and } 2\int \frac{dz}{z} = \int \frac{dz}{z(1-zz)}$$

of which latter equations, two infinite hyperbolic areas are compared with each other .

COROLLARY 4

53. We may consider $m=2$ and $n=3$, there becomes $\sin.\text{A}.\tfrac{m}{2n}\pi = \frac{\sqrt{3}}{2}$, from which the following comparisons arise :

$$\frac{\sqrt{3}}{2}\int \frac{dz}{\left(1-z^6\right)^{\frac{2}{3}}} = \int \frac{dz}{\sqrt{\left(1-z^6\right)}} \text{ and } \sqrt{3}\int \frac{dz}{\left(1-z^6\right)^{\frac{1}{3}}} = \int \frac{dz}{\left(1-z^6\right)^{\frac{5}{6}}};$$

from these this proportion is obtained :

$$\frac{1}{2}\int \frac{dz}{\left(1-z^6\right)^{\frac{2}{3}}} : \int \frac{dz}{\left(1-z^6\right)^{\frac{1}{3}}} = \int \frac{dz}{\left(1-z^6\right)^{\frac{1}{2}}} : \int \frac{dz}{\left(1-z^6\right)^{\frac{5}{6}}}.$$

COROLLARY 5

54. Let $m=1,\ n=2$, so that there shall be $\sin.\text{A}.\frac{m}{2n}\pi=\frac{1}{\sqrt{2}}$; there will be

$$\frac{1}{\sqrt{2}}\int\frac{dz}{\left(1-z^4\right)^{\frac{3}{4}}}=\int\frac{dz}{\left(1-z^4\right)^{\frac{1}{2}}}\ \text{ and }\ \sqrt{2}\int\frac{dz}{\left(1-z^4\right)^{\frac{1}{2}}}=\int\frac{dz}{\left(1-z^4\right)^{\frac{3}{4}}},$$

which two equations are in agreement.

COROLLARY 6

55. Let $m=1, n=3$, so that there shall be $\sin.\text{A}.\frac{m}{2n}\pi=\frac{1}{2}$; there will be

$$\frac{1}{2}\int\frac{zdz}{\left(1-z^6\right)^{\frac{5}{6}}}=\int\frac{zdz}{\left(1-z^6\right)^{\frac{1}{2}}}\ \text{ and }\ \int\frac{zdz}{\left(1-z^6\right)^{\frac{2}{3}}}=\int\frac{zdz}{\left(1-z^6\right)^{\frac{2}{3}}},$$

of which the latter is identical, but the former gives

$$\int\frac{zdz}{\left(1-z^6\right)^{\frac{5}{6}}}=2\int\frac{zdz}{\left(1-z^6\right)^{\frac{1}{2}}}$$

on putting $z=1$ after the integration, which condition is understood to be added always.

PROBLEM. 9

56. *To reduce the infinite expressions, which we have found for the cosine of the angle* $\frac{m}{2n}\pi$ *, to integral formulas.*

SOLUTION

At first in § 44 we found

$$\cos.\text{A}.\frac{m}{2n}\pi=\frac{n-m}{n}\cdot\frac{n+m}{n}\cdot\frac{3n-m}{3n}\cdot\frac{3n+m}{3n}\cdot\text{etc.}$$

which expression compared with the lemma § 48 gives

$$a=n-m,\ b=n,\ c=m \text{ and } k=2n,$$

with which substituted, there becomes :

$$\cos.\text{A}.\frac{m}{2n}\pi=\frac{\int z^{m-1}dz\left(1-z^{2n}\right)^{-\frac{1}{2}}}{\int z^{m-1}dz\left(1-z^{2n}\right)^{\frac{-n-m}{2n}}}.$$

Then from § 45 we have seen

$$\cos.A.\tfrac{m}{2n}\pi = \tfrac{2n-2m}{2n-m}\cdot\tfrac{2n+2m}{2n+m}\cdot\tfrac{4n-2m}{4n-m}\cdot\tfrac{4n+2m}{4n+m}\cdot\text{etc.},$$

from which expression compared with the lemma there is found
$a = 2n-2m,\ b = 2n-m,\ c = 3m$ et $k = 2n,$ from which there will become :

$$\cos.A.\tfrac{m}{2n}\pi = \frac{\int z^{3m-1}dz\left(1-z^{2n}\right)^{-\frac{m}{2n}}}{\int z^{3m-1}dz\left(1-z^{2n}\right)^{\frac{-m}{n}}},$$

if there may be put $z=1$ after the integration. Q. E. I.

COROLLARY 1

57. Hence again the sine of the angle $\tfrac{m}{2n}\pi$ can be expressed by putting $n-m$ in place of *m*; indeed the first expression gives that the same as we have now found, but from the latter there arises :

$$\sin.A.\tfrac{m}{2n}\pi = \frac{\int z^{3n-3m-1}dz\left(1-z^{2n}\right)^{\frac{-n+m}{2n}}}{\int z^{3n-3m-1}dz\left(1-z^{2n}\right)^{\frac{-n+m}{n}}}.$$

COROLLARY 2

58. Since we have three expressions for the sine, thus for the two expessions for the cosine found a third may be added from the following expression of the sine [§ 49], which will give :

$$2\cos.A.\tfrac{m}{2n}\pi = \frac{\int z^{m-1}dz\left(1-z^{2n}\right)^{\frac{-2n+m}{2n}}}{\int z^{m-1}dz\left(1-z^{2n}\right)^{\frac{-m}{n}}}.$$

COROLLARY 3

59. Hence therefore innumerable pairs of integral formulas in the case where $z=1$, can be compared with each other, and these comparisons depend on the multi-sections of the angle.

PROBLEM 10

60. *To find integral expressions, which may show the tangent of the angle $\frac{m}{2n}\pi$ in the case where $z=1$.*

SOLUTION

Since the tangent of the angle shall be the quotient arising from the division of the sine by the cosine of the angle, from § 43 and 44 there will be

$$\text{tang.A.}\frac{m}{2n}\pi = \frac{m}{n-m}\cdot\frac{2n-m}{n+m}\cdot\frac{2n+m}{3n-m}\cdot\frac{4n-m}{3n+m}\cdot\text{etc. };$$

this expression may be compared with the lemma § 48 and there will be
$a=m,\ b=n-m,\ c=n$ et $k=2n$, from which there will become

$$\text{tang.A.}\frac{m}{2n}\pi = \frac{\int z^{n-1}dz\left(1-z^{2n}\right)^{\frac{-n-m}{2n}}}{\int z^{n-1}dz\left(1-z^{2n}\right)^{\frac{m-2n}{2n}}}$$

with $z=1$ put in place after each integration. Thence from § 45 this same expression is elicited for the tangent

$$\text{tang.A.}\frac{m}{2n}\pi = \frac{m}{n-m}\cdot\frac{2n-m}{n+m}\cdot\frac{2n+m}{3n-m}\cdot\frac{4n-m}{3n+m}\cdot\text{etc.},$$

from which the same expression of the integral is found as before. Q. E. I.

COROLLARY 1

61. We may put $m=2$ and $n=3$; there will be $\text{tang.A.}\frac{m}{2n}\pi=\sqrt{3}$ and hence

$$\sqrt{3}\int\frac{zzdz}{\left(1-z^6\right)^{\frac{2}{3}}}=\int\frac{zzdz}{\left(1-z^6\right)^{\frac{5}{6}}};$$

if we may put $z^3=v$, there will become $zzdz=\frac{1}{3}dv$ and therefore

$$\int\frac{\sqrt{3}dv}{\left(1-v^2\right)^{\frac{2}{3}}}=\int\frac{dv}{\left(1-v^2\right)^{\frac{5}{6}}}.$$

COROLLARY 2

62. Generally if we may put $z^n=v$, there will be obtained :

$$\text{tang.A.}\frac{m}{2n}\pi = \frac{\int dv(1-vv)^{\frac{-n-m}{2n}}}{\int dv(1-vv)^{\frac{m-2n}{2n}}}$$

or

$$\text{tang.A.}\frac{m}{2n}\pi \cdot \int \frac{dv}{(1-vv)^{\frac{2n-m}{2n}}} = \int \frac{dv}{(1-vv)^{\frac{n+m}{2n}}}.$$

COROLLARY 3

63. Let $m=1,\ n=2$; there becomes $\text{tang.A.}\frac{m}{2n}\pi = 1$ and hence

$$\int \frac{dv}{(1-vv)^{\frac{3}{4}}} = \int \frac{dv}{(1-vv)^{\frac{3}{4}}},$$

which identical equation serves to confirm the truth of the calculation.

SCHOLIUM

64. More comparisons of this kind will be able to be put in place, if they may be called in to aid the comparisons of integral formulas shown be me elsewhere [E122], from which I shall bring forth certain lemmas as examples.

LEMMA 5

65. *If on integrating there may be put* $z=1$ *everywhere*, *there will become*

$$\int \frac{z^{a-1}dz}{\left(1-z^b\right)^{1-c}} \cdot \int \frac{z^{a+bc-1}dz}{\left(1-z^b\right)^{1-\gamma}} = \int \frac{z^{a-1}dz}{\left(1-z^b\right)^{1-\gamma}} \cdot \int \frac{z^{a+b\gamma-1}dz}{\left(1-z^b\right)^{1-c}}$$

LEMMA 6

66. *If on integrating there may be put* $z=1$ *everywhere*, *there will become*

$$\frac{b+1}{c+1} = \frac{\int z^{b\left(\frac{1}{2}-k\right)-1}dz\left(1-z^b\right)^c \cdot \int z^{b\left(\frac{3}{2}+c-k\right)-1}dz\left(1-z^b\right)^{-\frac{1}{2}+k}}{\int z^{b\left(\frac{1}{2}-k\right)-1}dz\left(1-z^b\right)^b \cdot \int z^{b\left(\frac{3}{2}+b+k\right)-1}dz\left(1-z^b\right)^{-\frac{1}{2}-k}}.$$

LEMMA 7

67. *If on integrating there may be put* $z=1$ *everywhere, there will become*

$$\frac{c}{a}=\frac{\int z^{a-1}dz\left(1-z^{b}\right)^{-\frac{1}{2}+k}\cdot\int z^{a+\left(\frac{1}{2}+k\right)b-1}dz\left(1-z^{b}\right)^{-\frac{1}{2}-k}}{\int z^{c-1}dz\left(1-z^{b}\right)^{-\frac{1}{2}-k}\cdot\int z^{c+\left(\frac{1}{2}-k\right)b-1}dz\left(1-z^{b}\right)^{-\frac{1}{2}+k}}.$$

LEMMA 8

68. *If on integrating there may be put* $z=1$ *everywhere, there will become*

$$\frac{(a+1)(a-k+1)}{(c+1)(c+k+1)}=\frac{\int z^{b(1+k)-1}dz\left(1-z^{b}\right)^{c}\cdot\int z^{b(1-k)-1}dz\left(1-z^{b}\right)^{c+k}}{\int z^{b(1-k)-1}dz\left(1-z^{b}\right)^{a}\cdot\int z^{b(1+k)-1}dz\left(1-z^{b}\right)^{a-k}}.$$

THEOREM 3

69. *If on integrating there may be put* $z=1$ *everywhere, there will become*

$$\cos.\mathrm{A}.\tfrac{m}{2n}\pi=\frac{\int\frac{z^{m-1}dz}{\left(1-z^{2n}\right)^{\frac{1}{2}}}\cdot\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{1-c}}}{\int\frac{z^{m-1}dz}{\left(1-z^{2n}\right)^{1-c}}\cdot\int\frac{z^{m+2nc-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}}.$$

DEMONSTRATION

If indeed in lemma 5, we may put $a=m,\ b=2n$ et $\gamma=\frac{n-m}{2n}$, there shall become

$$\int\frac{z^{a-1}dz}{\left(1-z^{b}\right)^{1-\gamma}}=\int\frac{z^{m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}.$$

But by § 56, there is

$$\int\frac{z^{m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}=\frac{1}{\cos.\mathrm{A}.\frac{m}{2n}\pi}\int\frac{z^{m-1}dz}{\left(1-z^{2n}\right)^{\frac{1}{2}}},$$

which value substituted into the lemma will give the equation, which it was required to demonstrate.

COROLLARY 1

70. The indefinite exponent c is present in this equation, which is allowed to be determined arbitrarily; therefore if $c=\frac{1}{2}$, and because the numerator and denominator have a common factor, there will become :

$$\cos.\mathrm{A}.\tfrac{m}{2n}\pi\cdot\int\frac{z^{n+m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}=\int\frac{z^{n-1}dz}{\sqrt{\left(1-z^{2n}\right)}}.$$

COROLLARY 2

71. If we may put $z^{n}=v$ into the formula $\int\frac{z^{n-1}dz}{\sqrt{\left(1-z^{2n}\right)}}$, that will change into $\frac{1}{n}\int\frac{dv}{\sqrt{(1-vv)}}$

of which the integral on putting $z=1$ or $v=1$ will be $\frac{\pi}{2n}$. Hence by this reason there will become

$$\int\frac{z^{n+m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}=\frac{\pi}{2n\cos.\mathrm{A}.\frac{m}{2n}\pi}$$

on putting $z=1$.

COROLLARY 3

72. If we may put $z=\frac{u}{\left(1+u^{2n}\right)^{\frac{1}{2n}}}$, so that in place of the variable z we may introduce u,

there will be $z=0,$ if $u=0,$ but if $u=\infty$ then $z=1$. On account of which with the substitution made, there will be

$$\int\frac{u^{n+m-1}du}{1+u^{2n}}=\frac{\pi}{2n\cos.\mathrm{A}.\frac{m}{2n}\pi}$$

with $u=\infty$ after the integration.

COROLLARY 4

73. If in § 29 in place of n we may put $2n$, there will be

$$\frac{\pi}{2n\sin.\mathrm{A}.\frac{m}{2n}\pi}=\int\frac{z^{m-1}+z^{2n-m-1}}{1+z^{2n}}dz,$$

if after integrating there may be made $z=1$. Because if therefore in place of m there may be written $n-m$, there will become :

$$\frac{\pi}{2n\cos.\text{A}.\frac{m}{2n}\pi}=\int\frac{z^{n-m-1}+z^{n+m-1}}{1+z^{2n}}dz\,,$$

putting $z=1$ after the integration, since therefore the integral may be equal to this $\int\frac{u^{n+m-1}du}{1+u^{2n}}$, if there may be put $u=\infty$.

THEOREM 4

74. *If after the integrations there may be put* $z=1$, *there will be*

$$2\cos.\text{A}.\tfrac{m}{2n}\pi\cdot\int\frac{z^{2m-1}dz}{\left(1-z^{2n}\right)^{\frac{m}{n}}}=\int\frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}.$$

DEMONSTRATION

In §58 we have obtained this cosine expression

$$2\cos.\text{A}.\tfrac{m}{2n}\pi=\frac{\int z^{m-1}dz\left(1-z^{2n}\right)^{\frac{-2n+m}{2n}}}{\int z^{m-1}dz\left(1-z^{2n}\right)^{\frac{-m}{n}}}.$$

If now from lemma 5 we may make $a=m,\ b=2n,\ c=\frac{m}{2n}$ and $\gamma=\frac{n-m}{n}$, the integral formulas of the two lemmas shall be changed into these, which express $2\cos.\text{A}.\frac{m}{2n}\pi$; in place of which if there may be written $2\cos.\text{A}.\frac{m}{2n}\pi$,there will be produced

$$2\cos.\text{A}.\tfrac{m}{2n}\pi\cdot\int\frac{z^{2m-1}dz}{\left(1-z^{2n}\right)^{\frac{m}{n}}}=\int\frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}.$$

Q. E. D.

COROLLARY 1

75. If hence in lemma 6 there may be put $b=2n,\ c=\frac{-m}{n}$ and $k=\frac{n-2m}{2n}$, the formula $\int z^{b\left(\frac{1}{2}-k\right)-1}dz\left(1-z^{b}\right)^{c}$ will go into this $\int\frac{z^{2m-1}dz}{\left(1-z^{2n}\right)^{\frac{m}{n}}}$, in place of which if we may write

$$\frac{1}{2\cos.\text{A}.\frac{m}{2n}\pi}\int\frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}.$$

and we may make $b=0$, we will obtain this reduction

$$2\cos.\text{A}.\tfrac{m}{2n}\pi=\frac{\int\frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}}{\int\frac{z^{2n-2m-1}dz}{\left(1-z^{2n}\right)^{\frac{n-m}{n}}}}\ \text{ or } \int\frac{z^{2m-1}dz}{\left(1-z^{2n}\right)^{\frac{m}{n}}}=\int\frac{z^{2n-2m-1}dz}{\left(1-z^{2n}\right)^{\frac{n-m}{n}}}.$$

COROLLARY 2

76. If we may put $m=2$ and $n=3$, there will be $\cos.\text{A}.\frac{m}{2n}\pi=\frac{1}{2}$, from which the equation of the theorem will give

$$\int\frac{z^3dz}{\left(1-z^6\right)^{\frac{2}{3}}}=\int\frac{z^3dz}{\left(1-z^6\right)^{\frac{2}{3}}},$$

but the equation of the preceding corollary will give

$$\int\frac{zdz}{\left(1-z^6\right)^{\frac{1}{3}}}=\int\frac{z^3dz}{\left(1-z^6\right)^{\frac{2}{3}}}$$

or this by putting z in place of zz

$$\int\frac{dz}{\left(1-z^3\right)^{\frac{1}{3}}}=\int\frac{zdz}{\left(1-z^3\right)^{\frac{2}{3}}},$$

on putting $z=1$.

COROLLARY 3

77. Let $m=1$ and $n=2$; there becomes $\cos.\text{A}.\frac{m}{2n}\pi=\frac{1}{\sqrt{2}}$ and thus

$$\int\frac{zdz\sqrt{2}}{\left(1-z^4\right)^{\frac{1}{2}}}=\int\frac{zzdz}{\left(1-z^4\right)^{\frac{3}{4}}}=\frac{\pi}{2\sqrt{2}},$$

on account of $\int\frac{zdz}{\sqrt{\left(1-z^4\right)}}=\frac{\pi}{4}$. Truly from Corollary 1 there will become

$$\int\frac{zdz\sqrt{2}}{\left(1-z^4\right)^{\frac{1}{2}}}=\int\frac{zzdz}{\left(1-z^4\right)^{\frac{3}{4}}}$$

which is the same equality.

THEOREM 5

78. *If after the integrations there may be put* $z=1$, *there will be*

$$\text{tang.A.}\tfrac{m}{2n}\pi\cdot\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}\cdot\int\frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{1-\gamma}}=\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{1-\gamma}}.\int\frac{z^{n+2n\gamma-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}.$$

DEMONSTRATION

In § 60 we have found to be

$$\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+2m}{2n}}}=\text{tang.A.}\tfrac{m}{2n}\pi\cdot\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}.$$

Now in lemma 5, there may become $a=n,\ b=2n$ and $c=\frac{n-m}{2n}$ and with the substitution made there will be

$$\text{tang.A.}\tfrac{m}{2n}\pi\cdot\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}.\int\frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{1-\gamma}}=\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{1-\gamma}}\cdot\int\frac{z^{n+2n\gamma-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}.$$

Q. E. D.

COROLLARY 1

79. If there may be put $\gamma=1$, from the two integral parts there shall become :

$$\frac{n}{2n-m}\,\text{tang.A.}\tfrac{m}{2n}\pi\cdot\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}=\int\frac{z^{3n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}=\frac{n}{2n-m}\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}};$$

on account of this there will become

$$\text{tang.A.}\tfrac{m}{2n}\pi\cdot\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}=\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}},$$

which is that same found in § 60.

COROLLARY 2

80. Let $\gamma=\frac{m}{2n}$; there will be

$$\text{tang.A.}\tfrac{m}{2n}\pi\cdot\int\frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}=\int\frac{z^{n+m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}.$$

and if there may be put $\gamma = \frac{1}{2}$, the quadrature of the circle will be entered into and there will be

$$\int \frac{z^{2n-m-1}dz}{\sqrt{\left(1-z^{2n}\right)}} \cdot \int \frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}} = \frac{\pi}{\text{tang.A.}\frac{m}{2n}\pi} \int \frac{z^{2n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}} = \frac{\pi}{2n(n-m)\text{tang.A.}\frac{m}{2n}\pi}$$

or

$$\frac{\pi\text{tang.A.}\frac{m}{2n}\pi}{2mn} = \int \frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}} \cdot \int \frac{z^{n+m-1}dz}{\sqrt{\left(1-z^{2n}\right)}}.$$

COROLLARY 3

81. Therefore since there shall become

$$\frac{\pi\text{tang.A.}\frac{m}{2n}\pi}{2mn} = \int \frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}} \cdot \int \frac{z^{n+m-1}dz}{\sqrt{\left(1-z^{2n}\right)}}.$$

and from § 60 there shall be

$$\int \frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}} = \text{tang.A.}\frac{m}{2n}\pi \cdot \int \frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}},$$

there will become

$$\frac{\pi}{2mn} = \int \frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}} \cdot \int \frac{z^{n+m-1}dz}{\sqrt{\left(1-z^{2n}\right)}}.$$

Therefore the product of these two integral formulas in the case, where $z = 1$, can be shown by the periphery of the circle.

COROLLARY 4

82. Let $m = 1$ and $n = 1$; from the preceding corollary there will be :

$$\frac{\pi}{2} = \int \frac{dz}{\sqrt{(1-zz)}} \cdot \int \frac{zdz}{\sqrt{(1-zz)}} = \frac{\pi}{2}\left(1-\sqrt{(1-zz)}\right),$$

in which case, if there shall become $z = 1$, the quality is seen at once.

COROLLARY 5

83. Let $m = 1$ and $n = 2$; there will be $\text{tang.A.}\frac{m}{2n}\pi = 1$, hence there will become from corollary 2 :

$$\frac{\pi}{4}=\int\frac{zdz}{\left(1-z^{4}\right)^{\frac{3}{4}}}\cdot\int\frac{zzdz}{\sqrt{\left(1-z^{4}\right)}},$$

but from the third there will become :

$$\frac{\pi}{4}=\int\frac{zdz}{\left(1-z^{4}\right)^{\frac{3}{4}}}\cdot\int\frac{zzdz}{\sqrt{\left(1-z^{4}\right)}},$$

which two equations agree between themselves.

COROLLARY 6

84. Let $m=2$ and $n=3$; there will become $\text{tang.A.}\frac{m}{2n}\pi=\sqrt{3}$, hence from corollary 2 there becomes :

$$\frac{\pi}{4\sqrt{3}}=\int\frac{zzdz}{\left(1-z^{6}\right)^{\frac{5}{6}}}\cdot\int\frac{z^{4}dz}{\sqrt{\left(1-z^{6}\right)}},$$

truly from the third this equation is obtained :

$$\frac{\pi}{12}=\int\frac{zzdz}{\left(1-z^{6}\right)^{\frac{2}{3}}}\cdot\int\frac{z^{4}dz}{\sqrt{\left(1-z^{6}\right)}}.$$

SCHOLION

85. A great many theorems of this kind can be deduced with the aid of lemmas 5, 6, 7, and 8 from the integral formulas for the sine, cosine and tangent, so that with these being considered a whole book would not suffice. But for any source uncovered, however much it may have please, thence theorems can be drawn. Indeed many cases occur, as we have seen, for which either identical equations or equations of such a kind, which can readily be reduced to that, and these cases arrived at confirm the truth of the remaining theorems therefore even more, in which the ratio of equality is not seen. Thus in equation § 80 if there may be put $m=0$ and $n=1$, there becomes $\text{tang.A.}\frac{m}{2n}\pi=\frac{m}{2n}\pi$, since there the tangent of the vanishing arc may be equal to the arc itself ; hence therefore there will become

$$\frac{\pi\pi}{4}=\int\frac{dz}{\sqrt{(1-zz)}}\cdot\int\frac{dz}{\sqrt{(1-zz)}},$$

of which the truth will be apparent at once, since there shall be $\int \frac{dz}{\sqrt{(1-zz)}} = \frac{\pi}{2}$, in the case where $z = 1$. Otherwise comparisons of integral formulas of this kind are more noteworthy there, which neither are able to be integrated nor to be reduced mutually, where a short-cut may appear to be seen towards proving these. Thus the first theorem of this kind, for which some time ago now I had made a deduction, simply recommended itself, where the product of these two integral formulas is to be found

$$\int \frac{dz}{\sqrt{(1-z^4)}} \text{ and } \int \frac{z^2 dz}{\sqrt{(1-z^4)}},$$

and the one is an arc, and the other expresses an ordinate on an elastic curve, in the case $z = 1$, to be equal to the area of the circle, of which the diameter shall be $= 1$.

THEOREMATA
CIRCA REDUCTIONEM FORMULARUM INTEGRALIUM
AD QUADRATURAM CIRCULI

Commentatio 59 indicis ENESTROEMIANI
Miscellanea Berolinensia 7, 143, p. 91-129

LEMMA 1

1. *In circulo, cuius radius est* $=1$ *et semiperipheria* $=\pi$ *, sit anguli cuiusvis s sinus = x; erit eadem quantitas x sinus omnium arcuum in hac serie infinita contentorum*

$$s,\ \pi-s,\ 2\pi+s,\ 3\pi-s,\ 4\pi+s,\ 5\pi-\ s\ etc.$$

Praeterea vero x erit sinus omnium arcuum negativorum in hac serie contentorum

$$-\pi-s,\ -2\pi+s,\ -3\pi-s,\ -4\pi+s,\ -5\pi-s\ etc.$$

COROLLARIUM 1

2. Si igitur *i* denotet numerum quemcunque integrum affirmativum, erit tam arcuum omnium in hac expressione $\pm 2i\pi+s$ contentorum quam arcuum in hac expressione $\pm(2i+1)\pi-s$ contentorum idem sinus communis *x*.

COROLLARIUM 2

3. Cum angulorum negative sumtorum sinus fiant negativi, erit angulorum in hac forma $\pm 2i\pi-s$ contentorum sinus $=x$ angulorumque in hac forma $\pm(2i+1)\pi+s$ contentorum sinus $=-x$, siquidem anguli *s* fuerit sinus $=+x$.

LEMMA 2

4. *In circulo, cuius radius* $=1$ *et semiperipheria* $=\pi$ *, sit anguli cuiusvis s cosinus y; erit eadem quantitas y cosinus omnium angulorum affirmativorum in hac serie contentorum*

$$s,\ 2\pi-s,\ 2\pi+s,\ 4\pi-s,\ 4\pi+s,\ 6\pi-s\ etc.$$

pariterque eadem quantitas y erit cosinus omnium angulorum negativorum in hac serie contentorum

$$-s,\ -2\pi+s,\ -2\pi-s,\ -4\pi+s,\ -4\pi-s,\ -6\pi+s\ etc.$$

COROLLARIUM 1

5. Si igitur *i* denotet numerum quemcunque integrum affirmativum, erit omnium arcuum in hac expressione generali $\pm 2i\pi \pm s$ contentorum idem communis cosinus *y*.

COROLLARIUM 2

6. Quoniam anguli duobus rectis seu arcu *n* aucti sive minuti cosinus fit negativus, erit omnium angulorum in hac forma $\pm(2i+1)\pi \pm s$ [contentorum] idem cosinus $=-y$, siquidem anguli *s* cosinus fuerit + *y*.

LEMMA 3

7. *Iisdem positis si t sit tangens anguli s, erit quoque t tangens omnium angulorum tam affirmativorum quam negativorum in his duabus seriebus contentorum*

$$s,\ \pi+s,\ 2\pi+s,\ 3\pi+s,\ 4\pi+s,\ 5\pi+s\ etc.,$$
$$-\pi+s,\ -2\pi+s,\ -3\pi+\ s,\ -4\pi+\ s, -5\pi+s\ etc.$$

COROLLARIUM

8. Denotante ergo *i* numerum quemcunque affirmativum, erit omnium angulorum in hac expressione $\pm i\pi + s$ contentorum eadem communis tangens *t;* angulorum vero $\pm i\pi - s$ tangens erit $=-t$, siquidem anguli *s* tangens sit $=+t$.

PROBLEMA 1

9. *Invenire radices huius aequationis infinitae*

$$x = z - \frac{z^3}{1\cdot2\cdot3} + \frac{z^5}{1\cdot2\cdot3\cdot4\cdot5} - \frac{z^7}{1\cdot2\cdot3\cdot4\cdot5\cdot6\cdot7} + \text{etc.}$$

SOLUTIO

Si *z* denotet arcum circuli radii $=1$, erit *x* ipsius sinus. Ponamus ergo arcum, cuius sinus sit $= x$, esse *s;* erunt infiniti ipsius *z* valores in his duabus seriebus contenti

$$s,\ \pi-s,\ 2\pi+s,\ 3\pi-s,\ 4\pi+\ s,\ 5\pi-s\ \ etc.,$$
$$-\pi-s,\ -2\pi+s,\ -3\pi-s,\ 4\pi+s,\ -5\pi-s\ etc.$$

Q. E. I.

COROLLARIUM 1

10. Si igitur aequatio proposita in hanc formam transmutetur

$$0=1-\frac{z}{1x}+\frac{z^3}{1\cdot 2\cdot 3x}-\frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5x}+\frac{z^7}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7x}+\text{etc.}$$

eius habebuntur factores numero infiniti sequentes

$$\left(1-\frac{z}{s}\right)\left(1-\frac{z}{\pi-s}\right)\left(1+\frac{z}{\pi+s}\right)\left(1+\frac{z}{2\pi-s}\right)\left(1-\frac{z}{2\pi+s}\right)\left(1-\frac{z}{3\pi-s}\right)\left(1+\frac{z}{3\pi+s}\right)\text{ etc.,}$$

in quibus factoribus lex progressionis facile perspicitur.

COROLLARIUM 2

11. Cum igitur coefficiens termini secundi in producto aequetur summae coefficientium ipsius z in omnibus factoribus, erit

$$\frac{1}{x}=\frac{1}{s}+\frac{1}{\pi-s}-\frac{1}{\pi+s}-\frac{1}{2\pi-s}+\frac{1}{2\pi+s}+\frac{1}{3\pi-s}-\text{etc.,}$$

COROLLARIUM 3

12. Sit $s=\frac{m}{n}\pi$, ita ut fiat $x=\sin.A.\frac{m}{n}\pi;$ erit serie per $\frac{m}{n}$ multiplicata

$$\frac{\pi}{nx}=\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\text{etc.,}$$

COROLLARIUM 4

13. Quoniam coefficiens ipsius z^2 in producto, qui est $=0$, aequatur summae factorum ex binis coefficientibus ipsius z in factoribus, haec vero summa his sumta aequalis est quadrato summae istorum coefficientium demta summa quadratorum eorundem, erit

$$\frac{1}{xx}=\frac{1}{ss}+\frac{1}{(\pi-s)^2}+\frac{1}{(\pi+s)^2}+\frac{1}{(2\pi-s)^2}+\frac{1}{(2\pi+s)^2}+\text{etc.}$$

COROLLARIUM 5

14. Posito ergo iterum $s=\frac{m}{n}\pi$, ut sit $x=\sin.A.\frac{m}{n}\pi;$ prodibit ista summatio

$$\frac{\pi\pi}{nnxx}=\frac{1}{m^2}+\frac{1}{(n-m)^2}+\frac{1}{(n+m)^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}+\text{etc.}$$

SCHOLION

15. Possem hoc modo ultra progredi atque summas altiorum potestatum determinare; quoniam vero hoc alibi iam feci atque ad institutum nostrum hae series sufficiunt, ulteriori investigationi supersedebo.

PROBLEMA 2

16. *Invenire radices huius aequationis infinitae*

$$y = 1 - \frac{z^2}{1\cdot 2} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4} - \frac{z^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6} + \frac{z^8}{1\cdot 2\cdot 3\cdots 8} - \text{etc.}$$

SOLUTIO

Si z denotet arcum in circulo, cuius radius $=1$, erit y huius arcus cosinus. Quodsi ergo capiatur arcus s, cuius cosinus sit $=y$, continebuntur innumerabiles ipsius z valores in binis sequentibus seriebus

$$s,\ 2\pi - s,\ 2\pi + s,\ 4\pi - s,\ 4\pi + s \text{ etc.},$$
$$-s,\ -2\pi + s,\ -2\pi - s,\ -4\pi + s,\ -4\pi - s \text{ etc.}$$

Q. E. I.

COROLLARIUM 1

17. Si igitur aequatio proposita in hanc formam transmutetur

$$0 = 1 - \frac{z^2}{1\cdot 2(1-y)} + \frac{z^4}{1\cdot 2\cdot 3\cdot 4(1-y)} - \frac{z^6}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6(1-y)} + \text{etc.}$$

eius habebuntur factores numero infiniti sequentes

$$\left(1 - \frac{zz}{ss}\right)\left(1 - \frac{zz}{(2\pi - s)^2}\right)\left(1 - \frac{zz}{(2\pi + s)^2}\right)\left(1 - \frac{zz}{(4\pi - s)^2}\right)\left(1 - \frac{zz}{(4\pi + s)^2}\right)\text{etc.}$$

COROLLARIUM 2

18. Cum igitur in producto coefficiens ipsius zz aequalis sit summae coefficientium ipsius zz in factoribus, habebitur sequentis seriei summatio

$$\frac{1}{2(1-y)} = \frac{1}{ss} + \frac{1}{(2\pi - s)^2} + \frac{1}{(2\pi + s)^2} + \frac{1}{(4\pi - s)^2} + \frac{1}{(4\pi + s)^2} + \text{etc.}$$

COROLLARIUM 3

19. Ponatur $s=\frac{m}{n}\pi$, ut sit $y=\cos.A.\frac{m}{n}\pi$ et $1-y=2\left(\sin.A.\frac{m}{2n}\pi\right)^2$; erit

$$\frac{\pi\pi}{2nn(1-y)}=\frac{\pi\pi}{4nn\left(\sin.A.\frac{m}{2n}\pi\right)^2}=\frac{1}{mm}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}+\frac{1}{(4n-m)^2}+\frac{1}{(4n+m)^2}+\text{etc.}$$

quae congruit cum § 14, si modo hic loco $2n$ scribatur n.

PROBLEMA 3

20. *Invenire radices ipsius z huius aequationis infinitae*

$$t=\frac{z-\frac{z^3}{1\cdot 2\cdot 3}+\frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{z^7}{1\cdot 2\cdot 3\cdots 7}+\text{etc}}{1-\frac{z^2}{1\cdot 2}+\frac{z^4}{1\cdot 2\cdot 3\cdot 4}-\frac{z^6}{1\cdot 2\cdot 3\cdots 6}+\text{etc}}.$$

SOLUTIO

Si z denotet arcum circuli, cuius radius $=1$, erit t tangens huius arcus; si ergo sumatur arcus s in hoc circulo, cuius tangens sit $=t$, erunt infiniti ipsius z valores sequentes

$$s,\ \pi+s,\ 2\pi+s,\ 3\pi+s,\ 4\pi+s,\ 5\pi+s \text{ etc.},$$
$$-\pi+s,\ -2\pi+s,\ -3\pi+s,\ -4\pi+s,\ -5\pi+s \text{ etc.}$$

Q. E. I.

COROLLARIUM 1

21. Reducatur aequatio proposita ad hanc formam

$$0=1-\frac{z}{t}-\frac{z^2}{1\cdot 2}+\frac{z^3}{1\cdot 2\cdot 3t}+\frac{z^4}{1\cdot 2\cdot 3\cdot 4}-\frac{z^5}{1\cdot 2\cdot 3\cdot 4\cdot 5t}-\text{etc.}$$

eiusque factores simplices erunt sequentes

$$\left(1-\frac{z}{s}\right)\left(1+\frac{z}{\pi-s}\right)\left(1-\frac{z}{\pi+s}\right)\left(1+\frac{z}{2\pi-s}\right)\left(1-\frac{z}{2\pi+s}\right)\left(1+\frac{z}{3\pi-s}\right)\text{ etc.},$$

COROLLARIUM 2

22. Cum igitur coefficiens ipsius z in aequatione aequalis sit summae coefficientium ipsius z in singulis factoribus, erit

$$\frac{1}{t}=\frac{1}{s}-\frac{1}{\pi-s}+\frac{1}{\pi+s}-\frac{1}{2\pi-s}+\frac{1}{2\pi+s}-\frac{1}{3\pi-s}+\text{etc.},$$

COROLLARIUM 3

23. Ponatur $s=\frac{m}{n}\pi$, ut sit $t=\text{tang.A.}\frac{m}{n}\pi=\frac{x}{y}$, si sit $x=\text{sin.A.}\frac{m}{n}\pi$ et $y=\text{cos.A.}\frac{m}{n}\pi$, atque prodibit sequentis seriei summatio

$$\frac{\pi}{nt}=\frac{\pi y}{nx}=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{etc.}$$

COROLLARIUM 4

24. Summa quadratorum singulorum terminorum seriei § 22 aequalis est quadrato summae ipsorum $\frac{1}{tt}$ demta duplici, summa factorum ex binis, hoc est -1; erit ergo summa quadratorum $=\frac{1}{tt}+1=\frac{yy}{xx}+1=\frac{1}{xx}$. Prodibit ergo uti in § 13 haec summatio

$$\frac{1}{xx}=\frac{1}{ss}+\frac{1}{(\pi-s)^2}+\frac{1}{(\pi+s)^2}+\frac{1}{(2\pi-s)^2}+\frac{1}{(2\pi+s)^2}+\text{etc.}$$

THEOREMA 1

25. *Denotante π semiperipheriam circuli, cuius radius* $=1$, *si fuerit* $x=\text{sin.A.}\frac{m}{n}\pi$, erit

$$\frac{\pi(p+qy)}{nx}=\frac{p+q}{m}+\frac{p-q}{n-m}-\frac{p-q}{n+m}-\frac{p+q}{2n-m}+\frac{p+q}{2n+m}+\frac{p-q}{3n-m}-\frac{p-q}{3n+m}-\text{etc.}$$

DEMONSTRATIO

Si series § 12 inventa multiplicetur per *p,* erit

$$\frac{\pi p}{nx}=\frac{p}{m}+\frac{p}{n-m}-\frac{p}{n+m}-\frac{p}{2n-m}+\frac{p}{2n+m}+\text{etc.},$$

et si series § 23 multiplicetur per *q,* erit

$$\frac{\pi qy}{nx}=\frac{q}{m}-\frac{q}{n-m}+\frac{q}{n+m}-\frac{q}{2n-m}+\frac{q}{2n+m}-\text{etc.}$$

Addantur hae duae series prodibitque series proposita, cuius propterea summa est $=\frac{\pi(p+qy)}{nx}$. Q. E. D.

COROLLARIUM 1

26. Sumatur $p = q$; prodibit ista summatio

$$\frac{\pi(1+y)}{2nx} = \frac{1}{m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{4n-m} + \frac{1}{4n+m} - \text{etc.};$$

at est $\frac{x}{1+y} = \text{tang.A.}\frac{m}{2n}\pi$, quae ergo posito n loco $2n$ congruit cum serie § 23 inventa.

COROLLARIUM 2

27. Sumatur $p = -q$; prodibit ista summatio

$$\frac{\pi(1-y)}{2nx} = \frac{1}{n-m} - \frac{1}{n+m} + \frac{1}{3n-m} - \frac{1}{3n+m} + \text{etc.};$$

at est $\frac{1-y}{x} = \text{tang.A.}\frac{m}{2n}\pi$ hincque

$$\frac{x}{1-y} = \text{tang.A.}\left(\frac{-m}{2n}\pi + \frac{1}{2}\pi\right) = \text{tang.A.}\frac{(n-m)}{2n}\pi,$$

quae series denuo scripto m loco $n-m$ ad praecedentem reducitur.

PROBLEMA 4

28. *Invenire summam huius seriei*

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.}$$

per formulam integralem.

SOLUTIO

Tribuantur singulis fractionibus numeratores, qui sint potestates ipsius z, quarum exponentes denominatoribus aequentur, habebiturque haec series

$$\frac{z^m}{m} + \frac{z^{n-m}}{n-m} - \frac{z^{n+m}}{n+m} - \frac{z^{2n-m}}{2n-m} + \frac{z^{2n+m}}{2n+m} + \text{etc.}$$

quae facto $z = 1$ in illam abibit. Ponatur summa huius seriei $= s$ ac sumtis differentialibus erit

$$\frac{zds}{dz} = z^m + z^{n-m} - z^{n+m} - z^{2n-m} + z^{2n+m} + z^{3n+m} - \text{etc.}$$

quae series composita est ex duabus geometricis, eritque ideo eius summa

$$=\frac{z^m}{1+z^n}+\frac{z^{n-m}}{1+z^n}.,$$

Habemus ergo

$$\frac{zds}{dz}=\frac{z^m+z^{n-m}}{1+z^n} \text{ et } ds=\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz,$$

consequenter

$$s=\int\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz;$$

huius itaque integralis valor casu, quo $z=1$, dabit summam seriei propositae.
Q. E. I.

COROLLARIUM 1

29. Quoniam igitur seriei propositae summa est $\frac{\pi}{nx}=\frac{\pi}{n\sin.\text{A}.\frac{m}{n}\pi}$, erit

$$\frac{\pi}{n\sin.\text{A}.\frac{m}{n}\pi}=\int\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz$$

si post integrationem ponatur $z=1$. Hoc ergo casu integrale formulae $\int\frac{z^{m-1}+z^{n-m-1}}{1+z^n}dz$ ope circuli potest exhiberi.

COROLLARIUM 2

30. Ponendo ergo loco *m* et *n* numeros definitos habebuntur sequentes integrationes casu $z=1$:
Si $m=1,\ n=2$, erit

$$\frac{\pi}{2}=\int\frac{2}{1+zz}dz$$

si $m=1,\ n=3$, erit

$$\frac{2\pi}{3\sqrt{3}}=\int\frac{1+z}{1+z^3}dz=\int\frac{dz}{1-z+zz};$$

si $m=1,\ n=4$, erit

$$\frac{\pi}{2\sqrt{2}}=\int\frac{1+zz}{1+z^4}dz;$$

si $m=1, n=6$, erit

$$\frac{\pi}{3}=\int\frac{1+z^4}{1+z^6}dz;$$

quae omnia integratione actu instituta veritati consentanea deprehenduntur.

PROBLEMA 5

31. *Invenire summam huius seriei*

$$\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\frac{1}{3n+m}-\text{etc.}$$

per formulam integralem.

SOLUTIO

Coniungantur cum his fractionibus numeratores idonei, ut ante fecimus, ac ponatur

$$s=\frac{z^m}{m}-\frac{z^{n-m}}{n-m}+\frac{z^{n+m}}{n+m}-\frac{z^{2n-m}}{2n-m}+\frac{z^{2n+m}}{2n+m}-\text{etc.}$$

erit utique facto $z=1$ valor ipsius s summa seriei propositae. Instituatur iam differentiatio ac prodibit

$$\frac{zds}{dz}=z^m-z^{n-m}+z^{n+m}-z^{2n-m}+z^{2n+m}-\text{etc.};$$

cuius seriei summa quia exhiberi potest, habebitur

$$\frac{zds}{dz}=\frac{z^m-z^{n-m}}{1-z^n},$$

unde fit

$$s=\int\frac{z^{m-1}-z^{n-m-1}}{1-z^n}dz.$$

Huius ergo formulae integralis valor casu $z=1$ dabit summam seriei propositae.

Q. E. I.

COROLLARIUM 1

32. Per § 23 est seriei propositae summa $=\frac{\pi}{nt}=\frac{\pi y}{nx}=\frac{\pi\cos.\text{A}.\frac{m}{n}\pi}{n\sin.\text{A}.\frac{m}{n}\pi}$

Quamobrem erit

$$\frac{\pi\cos.\text{A}.\frac{m}{n}\pi}{n\sin.\text{A}.\frac{m}{n}\pi}=\int\frac{z^{m-1}-z^{n-m-1}}{1-z^n}dz$$

casu, quo post integrationem ponitur $z=1$.

COROLLARIUM 2

33. Casus ergo simpliciores ita se habebunt:

Si $m=1,\ n=3$, erit $\frac{\pi}{3\sqrt{3}}=\int\frac{(1-z)dz}{1-z^3}=\int\frac{dz}{1+z+zz}$;

si $m=1,\ n=4$, erit $\frac{\pi}{4}=\int\frac{\left(1-z^2\right)dz}{1-z^4}=\int\frac{dz}{1+z^2}$;

si $m=1$, $n=6$, erit $\frac{\pi}{2\sqrt{3}}=\int\frac{\left(1-z^4\right)dz}{1-z^6}=\int\frac{(1+zz)dz}{1+zz+z^4}$;

quarum formularum congruentia cum veritate post integrationem actu institutam sponte patet.

SCHOLION

34. Si in binis seriebus nunc tractatis bini termini in unum colligantur, orientur sequentes summationes:

$$\frac{\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=\frac{1}{m}+\frac{2m}{n^2-m^2}-\frac{2m}{4n^2-m^2}+\frac{2m}{9n^2-m^2}-\frac{2m}{16n^2-m^2}+\text{etc.}$$

$$\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=\frac{1}{m}-\frac{2m}{n^2-m^2}-\frac{2m}{4n^2-m^2}-\frac{2m}{9n^2-m^2}-\frac{2m}{16n^2-m^2}-\text{etc.}$$

His ergo ordinatis habebimus duas sequentes series notatu maxime dignas

$$\frac{\pi}{2mn\sin.\mathrm{A}.\frac{m}{n}\pi}-\frac{1}{2mn}=\frac{1}{n^2-m^2}-\frac{1}{4n^2-m^2}+\frac{1}{9n^2-m^2}-\frac{1}{16n^2-m^2}+\text{etc.}$$

$$\frac{1}{2mn}-\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=\frac{1}{n^2-m^2}+\frac{1}{4n^2-m^2}+\frac{1}{9n^2-m^2}+\frac{1}{16n^2-m^2}+\text{etc.}$$

quae si invicem addantur, dabunt

$$\frac{\pi\left(1-\cos.\mathrm{A}.\frac{m}{n}\pi\right)}{4mn\sin.\mathrm{A}.\frac{m}{n}\pi}=\frac{\pi\sin.\mathrm{A}.\frac{m}{2n}\pi}{4mn\cos.\mathrm{A}.\frac{m}{2n}\pi}=\frac{1}{n^2-m^2}+\frac{1}{9n^2-m^2}+\frac{1}{25n^2-m^2}+\frac{1}{49n^2-m^2}+\text{etc.}$$

Hasque series ante aliquot annos ex principiis longe diversis sum consecutus.

THEOREMA. 2

35. *Summa huius seriei quadratorum*

$$\frac{1}{m^2}-\frac{1}{(n-m)^2}-\frac{1}{(n+m)^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}-\frac{1}{(3n-m)^2}-\text{etc.}$$

est

$$=\frac{\pi^2\cos.A.\frac{m}{n}\pi}{nn\left(\sin.A.\frac{m}{n}\pi\right)^2}.$$

DEMONSTRATIO

In § 12 vidimus esse

$$\frac{\pi}{n\sin.A.\frac{m}{n}\pi}=\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\text{etc.};$$

quae aequalitas cum semper habeat locum, quicquid sit *m,* differentialia quoque aequalia esse debent. Differentiemus ergo tam seriem quam ipsius summam posita *m* variabili eruntque differentialia quoque aequalia. Habebitur autem utrinque per *dm* diviso

$$\frac{-\pi^2\cos.A.\frac{m}{n}\pi}{nn\left(\sin.A.\frac{m}{n}\pi\right)^2}=-\frac{1}{m^2}+\frac{1}{(n-m)^2}+\frac{1}{(n+m)^2}-\frac{1}{(2n-m)^2}-\frac{1}{(2n+m)^2}+\text{etc.}$$

Mutentur signa atque habebitur summa seriei quadratorum propositae. Q. E. D.

COROLLARIUM 1

36. Evolvamus aliquot casus simpliciores sitque $m=1,\ n=2$; erit
$\sin.A.\frac{m}{n}\pi=1$ et $\cos.A.\frac{m}{n}\pi=0$,
unde

$$0=1-1-\tfrac{1}{9}+\tfrac{1}{9}+\tfrac{1}{25}-\tfrac{1}{25}-\tfrac{1}{49}+\text{etc.}$$

Sit $m=1,\ n=3$; erit

$$\sin.A.\tfrac{m}{n}\pi=\tfrac{\sqrt{3}}{2}\ \text{ et }\cos.A.\tfrac{m}{n}\pi=\tfrac{1}{2},$$

unde

$$\tfrac{2\pi\pi}{27}=1-\tfrac{1}{4}-\tfrac{1}{16}+\tfrac{1}{25}+\tfrac{1}{49}-\tfrac{1}{64}-\tfrac{1}{100}+\text{etc.}$$

Sit $m=1, n=4$ erit

$$\sin.A.\tfrac{m}{n}\pi=\cos.A.\tfrac{m}{n}\pi=\tfrac{1}{\sqrt{2}},$$

unde

$$\tfrac{\pi\pi}{8\sqrt{2}}=1-\tfrac{1}{9}-\tfrac{1}{25}+\tfrac{1}{49}+\tfrac{1}{81}-\tfrac{1}{121}-\tfrac{1}{169}+\text{etc.}$$

COROLLARIUM 2

37. Multiplicemus seriem

$$\tfrac{2\pi\pi}{27}=1-\tfrac{1}{4}-\tfrac{1}{16}+\tfrac{1}{25}+\tfrac{1}{49}-\tfrac{1}{64}-\tfrac{1}{100}+\text{etc.}$$

in qua quadrata per ternarium divisibilia desunt, per

$$\tfrac{9}{8} = 1 + \tfrac{1}{9} + \tfrac{1}{81} + \tfrac{1}{729} + \text{etc.},$$

ut omnia quadrata occurrant, eritque

$$\tfrac{\pi\pi}{12} = 1 - \tfrac{1}{4} + \tfrac{1}{9} - \tfrac{1}{16} + \tfrac{1}{25} - \tfrac{1}{36} + \tfrac{1}{49} - \text{etc.},$$

cuius veritas iam alibi a me est demonstrata.

COROLLARIUM 3

38. Cum sit ex § 14

$$\frac{\pi\pi}{nnxx} = \frac{\pi\pi}{nn\left(\sin.\text{A}.\frac{m}{n}\pi\right)^2} = \frac{1}{m^2} + \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \text{etc.}$$

erit his seriebus additis

$$\frac{\pi\pi\left(1+\cos.\text{A}.\frac{m}{n}\pi\right)}{2nn\left(\sin.\text{A}.\frac{m}{n}\pi\right)^2} = \frac{1}{m^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n-m)^2} + \frac{1}{(4n+m)^2} + \text{etc.}$$

quae series scribendo *n* loco 2*n* ad illam reducitur; est enim

$$\frac{1+\cos.\text{A}.\frac{m}{n}\pi}{\left(\sin.\text{A}.\frac{m}{n}\pi\right)^2} = \frac{1}{2\left(\sin.\text{A}.\frac{m}{2n}\pi\right)^2}.$$

SCHOLION

39. Summatio ergo in hac propositione demonstrata directe deduci potuisset ex summatione seriei § 14 datae. Cum enim sit

$$\frac{\pi\pi}{nn\left(\sin.\text{A}.\frac{m}{n}\pi\right)^2} = \frac{1}{m^2} + \frac{1}{(n-m)^2} + \frac{1}{(n+m)^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \text{etc.},$$

erit quoque scribendo 2*n* loco *n*

$$\frac{\pi\pi}{4nn\left(\sin.\text{A}.\frac{m}{2n}\pi\right)^2} = \frac{\pi\pi\left(1+\cos.\text{A}.\frac{m}{n}\pi\right)}{2nn\left(\sin.\text{A}.\frac{m}{n}\pi\right)^2} = \frac{1}{m^2} + \frac{1}{(2n-m)^2} + \frac{1}{(2n+m)^2} + \frac{1}{(4n-m)^2} + \frac{1}{(4n+m)^2} + \text{etc.},$$

$$[c.f.\ 2\sin^2\tfrac{\theta}{2} = \tfrac{\sin^2\theta}{1+\cos\theta} = \tfrac{4\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}}{1+2\cos^2\frac{\theta}{2}-1}.]$$

a cuius duplo si illa subtrahatur, remanebit proposita

$$\frac{\pi\pi}{4nn\left(\sin.A.\frac{m}{2n}\pi\right)^2}=\frac{\pi\pi\left(1+\cos.A.\frac{m}{n}\pi\right)}{2nn\left(\sin.A.\frac{m}{n}\pi\right)^2}=\frac{1}{m^2}+\frac{1}{(2n-m)^2}+\frac{1}{(2n+m)^2}+\frac{1}{(4n-m)^2}+\frac{1}{(4n+m)^2}+\text{etc.},$$

simili autem modo ex serie § 23, quae ob signa alternantia maxime videtur regularis, deduci potest series § 12 exhibita. Cum enim sit

$$\frac{\pi\cos.A.\frac{m}{n}\pi}{n\sin.A.\frac{m}{n}\pi}=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{etc.},$$

erit, si $2n$ loco n scribatur,

$$\frac{\pi\cos.A.\frac{m}{2n}\pi}{2n\sin.A.\frac{m}{2n}\pi}=\frac{\pi\left(1+\cos.A.\frac{m}{n}\pi\right)}{2n\sin.A.\frac{m}{n}\pi}=\frac{1}{m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{4n-m}+\frac{1}{4n+m}-\text{etc.},$$

Ab huius duplo subtrahatur illa series eritque

$$\frac{\pi}{n\sin.A.\frac{m}{n}\pi}=\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\text{etc.},$$

quae est ipsa series § 12 inventa.

Simili autem modo formulae integrales, quae pro his summis sunt inventae, ad se invicem reducuntur. Cum enim (§. 32) sit

$$\frac{\pi\cos.A.\frac{m}{n}\pi}{n\sin.A.\frac{m}{n}\pi}=\int\frac{z^{m-1}-z^{n-m-1}}{1-z^{n}}dz\,,$$

erit quoque

$$\frac{\pi\cos.A.\frac{m}{2n}\pi}{2n\sin.A.\frac{m}{2n}\pi}=\frac{\pi\left(1+\cos.A.\frac{m}{n}\pi\right)}{2n\sin.A.\frac{m}{n}\pi}=\int\frac{z^{m-1}-z^{2n-m-1}}{1-z^{2n}}dz;$$

ab huius duplo subtrahatur prior; erit

$$\frac{\pi}{n\sin.A.\frac{m}{n}\pi}=\int\frac{2z^{m-1}-2z^{2n-m-1}}{1-z^{2n}}dz-\int\frac{z^{m-1}-z^{2n-m-1}}{1-z^{2n}}dz$$
$$=\int\frac{z^{m-1}-z^{n+m-1}+z^{n-m-1}-z^{2n-m-1}}{1-z^{2n}}dz=\int\frac{z^{m-1}+z^{n-m-1}}{1+z^{n}}dz,$$

quae est ipsa integratio § 29 inventa. Ex quibus perspicuum est omnia, quae hactenus sunt eruta, deduci potuisse ex hac summatione

$$\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=\int\frac{z^{m-1}-z^{n-m-1}}{1-z^{n}}dz=\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\frac{1}{3n+m}-\text{etc.},$$

Ex qua per differentiationem nascitur haec

$$\frac{\pi\pi}{nn\left(\sin.\mathrm{A}.\frac{m}{n}\pi\right)^{2}}=\frac{1}{m^{2}}+\frac{1}{(n-m)^{2}}+\frac{1}{(n+m)^{2}}+\frac{1}{(2n-m)^{2}}+\frac{1}{(2n+m)^{2}}+\text{etc.},$$

quae § 14 iam est inventa.

PROBLEMA. 5

40. *Invenire differentialia primi, secundi sequentiumque altiorum ordinum huius quantitatis*

$$\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}$$

posito m variabili.

SOLUTIO

Ponamus brevitatis gratia

$$\sin.\mathrm{A}.\tfrac{m}{n}\pi=x \text{ et } \cos.\mathrm{A}.\tfrac{m}{n}\pi=y$$

erit primo $y=\sqrt{(1-xx)}$; tum vero erit

$$dx=\frac{\pi dm}{n}y=\frac{\pi y}{n}dm \;\text{ et }\; dy=-\frac{\pi x}{n}dm.$$

Vocetur quoque quantitas proposita, cuius differentialia quaeruntur,

$$\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi}=V;$$

erit $V=\frac{\pi y}{nx}$. Hinc ergo erit

$$dV=\frac{\pi(xdy-ydx)}{nxx}=\frac{-\pi\pi dm}{nnxx};$$

ob $xx+yy=1$ ideoque

$$\frac{dV}{dm}=\frac{-\pi\pi}{nn}\cdot\frac{1}{xx}$$

huius porro sumatur differentiale eritque

$$\frac{ddV}{dm}=+\frac{\pi\pi}{nn}\cdot\frac{2dx}{x^{3}}=\frac{2\pi^{3}}{n^{3}}\cdot\frac{ydm}{x^{3}}$$

ideoque

$$\frac{d^2V}{dm^2}=\frac{\pi^3}{n^3}\cdot\frac{2y}{x^3}.$$

Quodsi simili modo sequentia differentialia computentur, ita se habebunt:

$$\begin{aligned}
&V=+\frac{\pi}{nx}\cdot y\\
&\frac{dV}{dm}=-\frac{\pi^2}{n^2x^2}\cdot 1,\\
&\frac{ddV}{dm^2}=+\frac{\pi^3}{n^3x^3}\cdot 2y,\\
&\frac{d^3V}{dm^3}=-\frac{\pi^4}{n^4x^4}\cdot(4yy+2),\\
&\frac{d^4V}{dm^4}=+\frac{\pi^5}{n^5x^5}\cdot\left(8y^3+16y\right),\\
&\frac{d^5V}{dm^5}=-\frac{\pi^6}{n^6x^6}\cdot\left(16y^4+88y^2+16\right),\\
&\frac{d^6V}{dm^6}=+\frac{\pi^7}{n^7x^7}\cdot\left(32y^5+416y^3+272y\right),\\
&\frac{d^7V}{dm^7}=-\frac{\pi^8}{n^8x^8}\cdot\left(64y^6+824y^4+2880y^2+272\right)
\end{aligned}$$

etc.

Lex progressionis ita se habet, ut, si fuerit

$$\frac{d^vV}{dm^v}=\pm\frac{\pi^{v+1}}{n^{v+1}x^{v+1}}\cdot\left(\alpha y^{v-1}+\beta y^{v-3}+\gamma y^{v-5}+\delta y^{v-7}+\varepsilon y^{v-9}+\text{etc.}\right),$$

futurus sit sequens differentiationis ordo

$$\frac{d^{v+1}V}{dm^{v+1}}=\mp\frac{\pi^{v+2}}{n^{v+2}x^{v+2}}\cdot\begin{Bmatrix}2\alpha y^v+\left(4\beta+(v-1)\alpha\right)y^{v-2}+\left(6\gamma+(v-3)\beta\right)y^{v-4}\\+\left(8\delta+(v-5)\gamma\right)y^{v-6}+\left(10\varepsilon+(v-7)\delta\right)y^{v-8}+\text{etc}\end{Bmatrix}.$$

Differentialia igitur cuiuscunque ordinis ex praecedentibus determinabuntur.
Q. E. I.

PROBLEMA 6

41. *Invenire summam huius seriei*

$$\frac{1}{m^v}+\frac{1}{(m-n)^v}+\frac{1}{(m+n)^v}+\frac{1}{(m-2n)^v}+\frac{1}{(m+2n)^v}+\frac{1}{(m-3n)^v}+\text{etc.}$$

singulis terminis seriei § 23 *inventae ad dignitatem quamcunque elevatis.*

SOLUTIO

Si ponamus $\sin.\mathrm{A}.\frac{m}{n}\pi = x$, $\cos.\mathrm{A}.\frac{m}{n}\pi = y$ atque $\frac{\pi y}{nx} = V$, erit ex § 23

$$V = \frac{1}{m} + \frac{1}{m-n} + \frac{1}{m+n} + \frac{1}{m-2n} + \frac{1}{m+2n} + \frac{1}{m-3n} + \text{etc.}$$

Quodsi iam posito *m* variabili differentialia capiantur, prodibunt sequentes summationes:

$$-\frac{dV}{1dm} = \frac{1}{m^2} + \frac{1}{(m-n)^2} + \frac{1}{(m+n)^2} + \frac{1}{(m-2n)^2} + \frac{1}{(m+2n)^2} + \frac{1}{(m-3n)^2} + \text{etc.}$$

$$+\frac{ddV}{1\cdot 2dm^2} = \frac{1}{m^3} + \frac{1}{(m-n)^3} + \frac{1}{(m+n)^3} + \frac{1}{(m-2n)^3} + \frac{1}{(m+2n)^3} + \frac{1}{(m-3n)^3} + \text{etc.}$$

$$-\frac{ddV}{1\cdot 2\cdot 3dm^3} = \frac{1}{m^4} + \frac{1}{(m-n)^4} + \frac{1}{(m+n)^4} + \frac{1}{(m-2n)^4} + \frac{1}{(m+2n)^4} + \frac{1}{(m-3n)^4} + \text{etc.}$$

$$+\frac{d^4V}{1\cdot 2\cdot 3\cdot 4dm^4} = \frac{1}{m^5} + \frac{1}{(m-n)^5} + \frac{1}{(m+n)^5} + \frac{1}{(m-2n)^5} + \frac{1}{(m+2n)^5} + \frac{1}{(m-3n)^5} + \text{etc.}$$

etc.

Erit ergo seriei gradus indefiniti propositae

$$\frac{1}{m^v} + \frac{1}{(m-n)^v} + \frac{1}{(m+n)^v} + \frac{1}{(m-2n)^v} + \frac{1}{(m+2n)^v} + \frac{1}{(m-3n)^v} + \text{etc.}$$

summa

$$\frac{\pm d^{v-1}V}{1\cdot 2\cdot 3\cdots(v-1)dm^{v-1}}.$$

At problemate praecedente valorem ipsius $\frac{d^{v-1}V}{dm^{v-1}}$ exhibuimus; quamobrem quoque summae harum serierum potestatum poterunt definiri. Q. E. I.

PROBLEMA 7

42. *Sinum anguli cuiuscunque* $\frac{m}{n}\pi$ *per productum ex infinitis factoribus exhibere.*

SOLUTIO

Cum sit

$$\frac{\pi\cos.\mathrm{A}.\frac{m}{n}\pi}{n\sin.\mathrm{A}.\frac{m}{n}\pi} = \frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \text{etc.},$$

tractetur *m* uti quantitas variabilis ac multiplicetur ubique per *dm;* erit

$$\frac{\pi dm}{n}\cos.A.\tfrac{m}{n}\pi = d.\sin.A.\tfrac{m}{n}\pi$$

et hanc ob rem erit

$$\frac{\pi dm\cos.A.\frac{m}{n}\pi}{n\sin.A.\frac{m}{n}\pi} = \frac{d.\sin.A.\frac{m}{n}\pi}{\sin.A.\frac{m}{n}\pi} = \frac{dm}{m} - \frac{dm}{n-m} + \frac{dm}{n+m} - \frac{dm}{2n-m} + \frac{dm}{2n+m} - \text{etc.}$$

unde integratione utrinque absoluta erit

$$l\sin.A.\tfrac{m}{n}\pi = lm - l(n-m) + l(n+m) + l(2n-m) + l(2n+m) + C.$$

Constans *C* ita esse debet comparata, ut facto $m = \frac{1}{2}n$ logarithmus sinus fiat $= 0$, quippe quo casu habetur sinus totus. Hoc ergo modo constante *C* determinata erit

$$l\sin.A.\tfrac{m}{n}\pi = l\tfrac{2m}{n} + l\tfrac{2n-2m}{n} + l\tfrac{2n+2m}{3n} + l\tfrac{4n-2m}{3n} + l\tfrac{4n+2m}{5n} + \text{etc.}$$

Unde, si transeamus ad numeros, habebimus

$$\sin.A.\tfrac{m}{n}\pi = \tfrac{2m}{n}\cdot\tfrac{2n-2m}{n}\cdot\tfrac{2n+2m}{3n}\cdot\tfrac{4n-2m}{3n}\cdot\tfrac{4n+2m}{5n}\cdot\text{etc.}$$

Vel si binos factores in se ducamus, erit

$$\sin.A.\tfrac{m}{n}\pi = \tfrac{2m}{n}\cdot\tfrac{4nn-4mm}{3nn}\cdot\tfrac{16nn-4mm}{15nn}\cdot\tfrac{36nn-4mm}{35nn}\cdot\text{etc.}$$

Q. E. I.

COROLLARIUM 1

43. Si loco 2*m* scribamus *m,* habebimus

$$\sin.A.\tfrac{m}{2n}\pi = \tfrac{m}{n}\cdot\tfrac{2n-m}{n}\cdot\tfrac{2n+m}{3n}\cdot\tfrac{4n-m}{3n}\cdot\tfrac{4n+m}{5n}\cdot\text{etc.}$$

sive

$$\sin.A.\tfrac{m}{2n}\pi = \tfrac{m}{n}\cdot\tfrac{4nn-mm}{4nn-nn}\cdot\tfrac{16nn-mm}{16nn-nn}\cdot\tfrac{36nn-mm}{36nn-nn}\cdot\text{etc.}$$

COROLLARIUM 2

44. Quia est $\sin.A.\frac{m}{2n}\pi = \cos.A.\frac{(n-m)}{2n}\pi$, erit , si $n-m$ scribamus loco *m,* ex serie inventa

$$\cos.\mathrm{A}.\tfrac{m}{2n}\pi = \tfrac{n-m}{n}\cdot\tfrac{n+m}{n}\cdot\tfrac{3n-m}{3n}\cdot\tfrac{3n+m}{3n}\cdot\text{etc.}$$

sive

$$\cos.\mathrm{A}.\tfrac{m}{2n}\pi = \tfrac{nn-mm}{nn}\cdot\tfrac{9nn-mm}{9nn}\cdot\tfrac{25nn-mm}{25nn}\cdot\text{etc.}$$

COROLLARIUM 3

45 . Quoniam est $\sin.\mathrm{A}.\tfrac{m}{n}\pi = 2\sin.\mathrm{A}.\tfrac{m}{2n}\pi\cdot\cos.\mathrm{A}.\tfrac{m}{2n}\pi$, si dividamus per $2\sin.\mathrm{A}.\tfrac{m}{2n}\pi$, habebimus

$$\cos.\mathrm{A}.\tfrac{m}{2n}\pi = \tfrac{2n-2m}{2n-m}\cdot\tfrac{2n+2m}{2n+m}\cdot\tfrac{4n-2m}{4n-m}\cdot\tfrac{4n+2m}{4n+m}\cdot\text{etc.}$$

et diviso $\sin.\mathrm{A}.\tfrac{m}{n}\pi$ per $2\cos.\mathrm{A}.\tfrac{m}{2n}\pi$ habebimus

$$\sin.\mathrm{A}.\tfrac{m}{2n}\pi = \tfrac{m}{n-m}\cdot\tfrac{2n-2m}{n+m}\cdot\tfrac{2n+2m}{3n-m}\cdot\tfrac{4n-2m}{3n+m}\cdot\tfrac{4n+2m}{5n-m}\cdot\text{etc.}$$

COROLLARIUM 4

46. Duplices istae sinuum et cosinuum expressiones inter se aequatae dabunt

$$1 = \tfrac{nn}{nn-mm}\cdot\tfrac{4nn-4mm}{4nn-mm}\cdot\tfrac{9nn}{9nn-mm}\cdot\tfrac{16nn-4mm}{16nn-mm}\cdot\tfrac{25nn}{25nn-mm}\cdot\text{etc.}$$

COROLLARIUM 5

47. Si n capiatur infinitum seu m infinite parvum, erit $\sin.\mathrm{A}.\tfrac{m}{2n}\pi = \tfrac{m}{2n}\pi$ hocque casu ex utraque serie nascitur idem valor ipsius π a WALLISIO datus

$$\pi = 2\,\frac{2\cdot2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10.\ \text{etc.}}{1\cdot3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdot11\cdot\ \text{etc.}}$$

LEMMA 4

48. *Valor huius producti ex infinitis factoribus constantis*

$$\frac{a(c+b)(a+k)(c+b+k)(a+2k)(c+b+2k)\ etc.}{b(c+a)(b+k)(c+a+k)(b+2k)(c+a+2k)\ etc.}$$

est

$$=\frac{\int z^{c-1}dz\left(1-z^{k}\right)^{\frac{b-k}{k}}}{\int z^{c-1}dz\left(1-z^{k}\right)^{\frac{a-k}{k}}},$$

si post utramque integrationem ponatur $z=1$.

PROBLEMA 8

49. *Sinum anguli* $\frac{m}{2n}\pi$ *per formulas integrales exprimere.*

SOLUTIO

Cum sit

$$\sin.\mathrm{A}.\frac{m}{2n}\pi=\frac{m}{n}\cdot\frac{2n-m}{n}\cdot\frac{2n+m}{3n}\cdot\frac{4n-m}{3n}\cdot\frac{4n+m}{5n}\cdot\text{etc.}$$

per § 43, comparetur hoc productum infinitum cum lemmate praecedente eritque $a=m,\ b=n,\ k=2n$ et $c+m=n$ vel $c+n=2n-m$; utrumque dat $c=n-m$. Hinc igitur fiet

$$\sin.\mathrm{A}.\frac{m}{2n}\pi=\frac{\int z^{n-m-1}dz\left(1-z^{2n}\right)^{-\frac{1}{2}}}{\int z^{n-m-1}dz\left(1-z^{2n}\right)^{\frac{-2n+m}{2n}}},$$

si post utramque integrationem ita institutam, ut integralia evanescant posito $z=0,$ ponatur $z=1$.

Cum vero etiam sit per § 45

$$2\sin.\mathrm{A}.\frac{m}{2n}\pi=\frac{2m}{n-m}\cdot\frac{2n-2m}{n+m}\cdot\frac{2n+2m}{3n-m}\cdot\frac{4n-2m}{3n+m}\cdot\text{etc.},$$

erit comparatione cum lemmate instituta $a=2m, b=n-m,\ c=n-m$ et $k=2n$, unde obtinebitur

$$2\sin.\mathrm{A}.\frac{m}{2n}\pi=\frac{\int z^{n-m-1}dz\left(1-z^{2n}\right)^{\frac{-n-m}{2n}}}{\int z^{n-m-1}dz\left(1-z^{2n}\right)^{\frac{m-n}{n}}},$$

si post integrationes ponatur $z=1$. Q. E. I.

COROLLARIUM 1

50. Sequentes ergo nanciscimur diversarum formularum integralium comparationes

$$\sin.\mathrm{A}.\frac{m}{2n}\pi\cdot\int\frac{z^{n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}=\int\frac{z^{n-m-1}dz}{\sqrt{\left(1-z^{2n}\right)}}$$

et

$$2\sin.\mathrm{A}.\tfrac{m}{2n}\pi\cdot\int\frac{z^{n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{n-m}{n}}}=\int\frac{z^{n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}.$$

COROLLARIUM 2

51. Tum vero sine sinus ratione habita institui potest ista integralium comparatio

$$2\int\frac{z^{n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{n-m}{n}}}:\int\frac{z^{n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}=\int\frac{z^{n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}:\int\frac{z^{n-m-1}dz}{\sqrt{\left(1-z^{2n}\right)}}$$

COROLLARIUM 3

52. Ponamus esse $m=1$ et $n=1$; erit $\sin.\mathrm{A}.\tfrac{m}{2n}\pi=1$ atque comparationes ita se habebunt

$$\int\frac{dz}{z\sqrt{(1-zz)}}=\int\frac{dz}{z\sqrt{(1-zz)}}\ \text{ et }\ 2\int\frac{dz}{z}=\int\frac{dz}{z(1-zz)}$$

quarum aequationum posteriori duo spatia hyperbolica infinita inter se comparantur.

COROLLARIUM 4

53 . Ponamus esse $m=2$ et $n=3$, erit $\sin.\mathrm{A}.\tfrac{m}{2n}\pi=\frac{\sqrt{3}}{2}$, unde orientur sequentes comparationes

$$\frac{\sqrt{3}}{2}\int\frac{dz}{\left(1-z^{6}\right)^{\frac{2}{3}}}=\int\frac{dz}{\sqrt{\left(1-z^{6}\right)}}\ \text{ et }\ \sqrt{3}\int\frac{dz}{\left(1-z^{6}\right)^{\frac{1}{3}}}=\int\frac{dz}{\left(1-z^{6}\right)^{\frac{5}{6}}};$$

ex his nascitur ista proportio

$$\frac{1}{2}\int\frac{dz}{\left(1-z^{6}\right)^{\frac{2}{3}}}:\int\frac{dz}{\left(1-z^{6}\right)^{\frac{1}{3}}}=\int\frac{dz}{\left(1-z^{6}\right)^{\frac{1}{2}}}:\int\frac{dz}{\left(1-z^{6}\right)^{\frac{5}{6}}}.$$

COROLLARIUM 5

54. Sit $m=1,\ n=2$, ut sit $\sin.\mathrm{A}.\tfrac{m}{2n}\pi=\frac{1}{\sqrt{2}}$; erit

$$\frac{1}{\sqrt{2}}\int\frac{dz}{\left(1-z^{4}\right)^{\frac{3}{4}}}=\int\frac{dz}{\left(1-z^{4}\right)^{\frac{1}{2}}}\ \text{ et }\ \sqrt{2}\int\frac{dz}{\left(1-z^{4}\right)^{\frac{1}{2}}}=\int\frac{dz}{\left(1-z^{4}\right)^{\frac{3}{4}}},$$

quae duae aequationes inter se congruunt.

COROLLARIUM 6

55. Sit $m=1, n=3$, ut sit sit $\sin.A.\frac{m}{2n}\pi=\frac{1}{2}$; erit

$$\frac{1}{2}\int\frac{zdz}{\left(1-z^6\right)^{\frac{5}{6}}}=\int\frac{zdz}{\left(1-z^6\right)^{\frac{1}{2}}} \text{ et } \int\frac{zdz}{\left(1-z^6\right)^{\frac{2}{3}}}=\int\frac{zdz}{\left(1-z^6\right)^{\frac{2}{3}}},$$

quarum posterior est identica, prior autem dat

$$\int\frac{zdz}{\left(1-z^6\right)^{\frac{5}{6}}}=2\int\frac{zdz}{\left(1-z^6\right)^{\frac{1}{2}}}$$

posito post integrationem $z=1$, quae conditio semper adiuncta est intelligenda.

PROBLEMA. 9

56. *Expressiones infinitas, quas pro cosinu anguli* $\frac{m}{2n}\pi$ *invenimus, ad formulas integrales reducere.*

SOLUTIO

Primum § 44 invenimus esse

$$\cos.A.\frac{m}{2n}\pi=\frac{n-m}{n}\cdot\frac{n+m}{n}\cdot\frac{3n-m}{3n}\cdot\frac{3n+m}{3n}\cdot\text{etc.}$$

quae expressio cum lemmate § 48 comparata dat $a=n-m,\ b=n,\ c=m$ et $k=2n$, quibus substitutis oritur

$$\cos.A.\frac{m}{2n}\pi=\frac{\int z^{m-1}dz\left(1-z^{2n}\right)^{-\frac{1}{2}}}{\int z^{m-1}dz\left(1-z^{2n}\right)^{\frac{-n-m}{2n}}}.$$

Deinde § 45 vidimus esse

$$\cos.A.\frac{m}{2n}\pi=\frac{2n-2m}{2n-m}\cdot\frac{2n+2m}{2n+m}\cdot\frac{4n-2m}{4n-m}\cdot\frac{4n+2m}{4n+m}\cdot\text{etc.},$$

qua expressione cum lemmate comparata reperietur
$a=2n-2m,\ b=2n-m,\ c=3m$ et $k=2n,$ unde erit

$$\cos.A.\frac{m}{2n}\pi=\frac{\int z^{3m-1}dz\left(1-z^{2n}\right)^{-\frac{m}{2n}}}{\int z^{3m-1}dz\left(1-z^{2n}\right)^{\frac{-m}{n}}},$$

si post integrationem ponatur $z=1$. Q. E. I.

COROLLARIUM 1

57. Hinc iterum sinus anguli $\frac{m}{2n}\pi$ exprimi potest ponendo $n-m$ loco m; prior quidem expressio dat eam ipsam, quam iam invenimus, at ex posteriori nascitur

$$\sin.\mathrm{A}.\frac{m}{2n}\pi=\frac{\int z^{3n-3m-1}dz\left(1-z^{2n}\right)^{\frac{-n+m}{2n}}}{\int z^{3n-3m-1}dz\left(1-z^{2n}\right)^{\frac{-n+m}{n}}}.$$

COROLLARIUM 2

58. Quemadmodum tres expressiones pro sinu habemus, ita ad duas expressiones pro cosinu inventas tertia accedet ex secunda expressione sinus [§ 49], quae dabit

$$2\cos.\mathrm{A}.\frac{m}{2n}\pi=\frac{\int z^{m-1}dz\left(1-z^{2n}\right)^{\frac{-2n+m}{2n}}}{\int z^{m-1}dz\left(1-z^{2n}\right)^{\frac{-m}{n}}}.$$

COROLLARIUM 3

59. Hinc igitur innumerabilia paria formularum integralium casu, quo $z=1$, inter se comparari poterunt haeque comparationes pendebunt a multisectione anguli.

PROBLEMA 10

60. *Invenire expressiones integrales, quae casu* $z=1$ *tangentem anguli* $\frac{m}{2n}\pi$ *exhibeant.*

SOLUTIO

Cum tangens anguli sit quotus ex divisione sinus per cosinum ortus, erit ex § 43 et 44

$$\mathrm{tang}.\mathrm{A}.\frac{m}{2n}\pi=\frac{m}{n-m}\cdot\frac{2n-m}{n+m}\cdot\frac{2n+m}{3n-m}\cdot\frac{4n-m}{3n+m}\cdot\text{etc.}$$

comparetur haec expressio cum lemmate § 48 eritque $a=m,\ b=n-m,\ c=n$ et $k=2n$, unde orietur

$$\text{tang.A.}\tfrac{m}{2n}\pi = \frac{\int z^{n-1}dz\left(1-z^{2n}\right)^{\frac{-n-m}{2n}}}{\int z^{n-1}dz\left(1-z^{2n}\right)^{\frac{m-2n}{2n}}}$$

posito post utramque integrationem $z=1$. Deinde ex § 45 elicitur pro tangente ista expressio

$$\text{tang.A.}\tfrac{m}{2n}\pi = \tfrac{m}{n-m}\cdot\tfrac{2n-m}{n+m}\cdot\tfrac{2n+m}{3n-m}\cdot\tfrac{4n-m}{3n+m}\cdot\text{etc.},$$

ex qua eadem expressio integralis quae ante invenitur. Q. E. I.

COROLLARIUM 1

61. Ponamus $m=2$ et $n=3$; erit $\text{tang.A.}\tfrac{m}{2n}\pi=\sqrt{3}$ hincque

$$\sqrt{3}\int\frac{zzdz}{\left(1-z^{6}\right)^{\frac{2}{3}}}=\int\frac{zzdz}{\left(1-z^{6}\right)^{\frac{5}{6}}};$$

si ponamus $z^{3}=v,$ erit $zzdz=\tfrac{1}{3}dv$ ac proinde

$$\int\frac{\sqrt{3}dv}{\left(1-v^{2}\right)^{\frac{2}{3}}}=\int\frac{dv}{\left(1-v^{2}\right)^{\frac{5}{6}}}.$$

COROLLARIUM 2

62. Si generaliter ponamus $z^{n}=v$, habebitur

$$\text{tang.A.}\tfrac{m}{2n}\pi=\frac{\int dv(1-vv)^{\frac{-n-m}{2n}}}{\int dv(1-vv)^{\frac{m-2n}{2n}}}$$

sive

$$\text{tang.A.}\tfrac{m}{2n}\pi\cdot\int\frac{dv}{(1-vv)^{\frac{2n-m}{2n}}}=\int\frac{dv}{(1-vv)^{\frac{n+m}{2n}}}.$$

COROLLARIUM 3

63. Sit $m=1,\ n=2$; erit $\text{tang.A.}\tfrac{m}{2n}\pi=1$ hincque

$$\int\frac{dv}{(1-vv)^{\frac{3}{4}}}=\int\frac{dv}{(1-vv)^{\frac{3}{4}}},$$

quae aequatio identica inservit veritati calculi comprobandae.

SCHOLION

64. Plures huius generis comparationes institui poterunt, si in subsidium vocentur theoremata circa comparationes formularum integralium alibi a me demonstrata [E122], unde quaedam instar lemmatum depromam.

LEMMA 5

65. *Si post integrationes ubique ponatur* $z=1$, *erit*

$$\int\frac{z^{a-1}dz}{\left(1-z^{b}\right)^{1-c}}\cdot\int\frac{z^{a+bc-1}dz}{\left(1-z^{b}\right)^{1-\gamma}}=\int\frac{z^{a-1}dz}{\left(1-z^{b}\right)^{1-\gamma}}\cdot\int\frac{z^{a+b\gamma-1}dz}{\left(1-z^{b}\right)^{1-c}}$$

LEMMA 6

66. *Si post integrationes ponatur* $z=1$, *erit*

$$\frac{b+1}{c+1}=\frac{\int z^{b\left(\frac{1}{2}-k\right)-1}dz\left(1-z^{b}\right)^{c}\cdot\int z^{b\left(\frac{3}{2}+c-k\right)-1}dz\left(1-z^{b}\right)^{-\frac{1}{2}+k}}{\int z^{b\left(\frac{1}{2}-k\right)-1}dz\left(1-z^{b}\right)^{b}\cdot\int z^{b\left(\frac{3}{2}+b+k\right)-1}dz\left(1-z^{b}\right)^{-\frac{1}{2}-k}}.$$

LEMMA 7

67. *Si post integrationes ponatur* $z=1$, *erit*

$$\frac{c}{a}=\frac{\int z^{a-1}dz\left(1-z^{b}\right)^{-\frac{1}{2}+k}\cdot\int z^{a+\left(\frac{1}{2}+k\right)b-1}dz\left(1-z^{b}\right)^{-\frac{1}{2}-k}}{\int z^{c-1}dz\left(1-z^{b}\right)^{-\frac{1}{2}-k}\cdot\int z^{c+\left(\frac{1}{2}-k\right)b-1}dz\left(1-z^{b}\right)^{-\frac{1}{2}+k}}.$$

LEMMA 8

68. *Si post integrationes ponatur* $z=1$, *erit*

$$\frac{(a+1)(a-k+1)}{(c+1)(c+k+1)}=\frac{\int z^{b(1+k)-1}dz\left(1-z^{b}\right)^{c}\cdot\int z^{b(1-k)-1}dz\left(1-z^{b}\right)^{c+k}}{\int z^{b(1-k)-1}dz\left(1-z^{b}\right)^{a}\cdot\int z^{b(1+k)-1}dz\left(1-z^{b}\right)^{a-k}}.$$

THEOREMA 3

69. *Si post integrationes ponatur* $z=1$, *erit*

$$\cos.\mathrm{A}.\frac{m}{2n}\pi=\frac{\int\frac{z^{m-1}dz}{\left(1-z^{2n}\right)^{\frac{1}{2}}}\cdot\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{1-c}}}{\int\frac{z^{m-1}dz}{\left(1-z^{2n}\right)^{1-c}}\cdot\int\frac{z^{m+2nc-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}}.$$

DEMONSTRATIO

Si enim in Lemmate 5 ponamus $a=m$, $b=2n$ et $\gamma=\frac{n-m}{2n}$, fit

$$\int\frac{z^{a-1}dz}{\left(1-z^{b}\right)^{1-\gamma}}=\int\frac{z^{m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}.$$

At per § 56 est

$$\int\frac{z^{m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}=\frac{1}{\cos.\mathrm{A}.\frac{m}{2n}\pi}\int\frac{z^{m-1}dz}{\left(1-z^{2n}\right)^{\frac{1}{2}}},$$

qui valor in lemmate substitutus dabit aequalitatem, quam demonstrari oportebat.

COROLLARIUM 1

70. Inest in hac aequalitate exponens indefinitus *c,* quem pro arbitrio determinare licet; sit igitur $c=\frac{1}{2}$, et quia numerator et denominator factorem habent communem, erit

$$\cos.\mathrm{A}.\frac{m}{2n}\pi\cdot\int\frac{z^{n+m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}=\int\frac{z^{n-1}dz}{\sqrt{\left(1-z^{2n}\right)}}.$$

COROLLARIUM 2

71 Si in formula $\int\frac{z^{n-1}dz}{\sqrt{\left(1-z^{2n}\right)}}$ ponamus $z^{n}=v$, abit ea in $\frac{1}{n}\int\frac{dv}{\sqrt{(1-vv)}}$ cuius integrale posito $z=1$ seu $v=1$ erit $\frac{\pi}{2n}$. Hanc ob rem erit

$$\int\frac{z^{n+m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}=\frac{\pi}{2n\cos.\mathrm{A}.\frac{m}{2n}\pi}$$

posito $z=1$.

COROLLARIUM 3

72. Si ponamus $z = \frac{u}{\left(1+u^{2n}\right)^{\frac{1}{2n}}}$, ut loco variabilis z introducamus u, erit $z=0$, si $u=0$, at fit $z=1$, si $u=\infty$. Quamobrem facta substitutione erit

$$\int \frac{u^{n+m-1}du}{1+u^{2n}} = \frac{\pi}{2n\cos.\mathrm{A}.\frac{m}{2n}\pi}$$

posito post integrationem $u=\infty$.

COROLLARIUM 4

73. Si in § 29 loco n ponamus $2n$, erit

$$\frac{\pi}{2n\sin.\mathrm{A}.\frac{m}{2n}\pi} = \int \frac{z^{m-1}+z^{2n-m-1}}{1+z^{2n}}dz,$$

si post integrationem fiat $z=1$. Quodsi ergo pro m scribatur $n-m$, fiet

$$\frac{\pi}{2n\cos.\mathrm{A}.\frac{m}{2n}\pi} = \int \frac{z^{n-m-1}+z^{n+m-1}}{1+z^{2n}}dz$$

posito post integrationem $z=1$, quod ergo integrale aequatur huic $\int \frac{u^{n+m-1}du}{1+u^{2n}}$, si ponatur $u=\infty$.

THEOREMA 4

74. *Si post integrationes ponatur* $z=1$, *erit*

$$2\cos.\mathrm{A}.\frac{m}{2n}\pi \cdot \int \frac{z^{2m-1}dz}{\left(1-z^{2n}\right)^{\frac{m}{n}}} = \int \frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}.$$

DEMONSTRATIO

In §58 nacti sumus hanc cosinus expressionem

$$2\cos.\mathrm{A}.\frac{m}{2n}\pi = \frac{\int z^{m-1}dz\left(1-z^{2n}\right)^{\frac{-2n+m}{2n}}}{\int z^{m-1}dz\left(1-z^{2n}\right)^{\frac{-m}{n}}}.$$

Si iam Lemmate 5 faciamus $a=m,\ b=2n,\ c=\frac{m}{2n}$ et $\gamma=\frac{n-m}{n}$, duae lemmatis formulae integrales in has, quae $2\cos.\mathrm{A}.\frac{m}{2n}\pi$ exprimunt, transmutantur; quarum loco si scribatur $2\cos.\mathrm{A}.\frac{m}{2n}\pi$, prodibit

$$2\cos.\mathrm{A}.\frac{m}{2n}\pi\cdot\int\frac{z^{2m-1}dz}{\left(1-z^{2n}\right)^{\frac{m}{n}}}=\int\frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}.$$

Q. E. D.

COROLLARIUM 1

75. Si hinc in Lemmate 6 ponatur $b=2n,\ c=\frac{-m}{n}$ et $k=\frac{n-2m}{2n}$, formula $\int z^{b\left(\frac{1}{2}-k\right)-1}dz\left(1-z^{b}\right)^{c}$ abit in hanc $\int\frac{z^{2m-1}dz}{\left(1-z^{2n}\right)^{\frac{m}{n}}}$, cuius loco si scribamus

$$\frac{1}{2\cos.\mathrm{A}.\frac{m}{2n}\pi}\int\frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}.$$

faciamusque $b=0$, obtinebimus hanc reductionem

$$2\cos.\mathrm{A}.\frac{m}{2n}\pi=\frac{\int\frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}}{\int\frac{z^{2n-2m-1}dz}{\left(1-z^{2n}\right)^{\frac{n-m}{n}}}}\text{ seu }\int\frac{z^{2m-1}dz}{\left(1-z^{2n}\right)^{\frac{m}{n}}}=\int\frac{z^{2n-2m-1}dz}{\left(1-z^{2n}\right)^{\frac{n-m}{n}}}$$

COROLLARIUM 2

76. Si ponamus $m=2$ et $n=3$, erit $\cos.\mathrm{A}.\frac{m}{2n}\pi=\frac{1}{2}$, unde aequatio theorematis dabit

$$\int\frac{z^{3}dz}{\left(1-z^{6}\right)^{\frac{2}{3}}}=\int\frac{z^{3}dz}{\left(1-z^{6}\right)^{\frac{2}{3}}},$$

at aequatio corollarii praecedentis dabit

$$\int\frac{zdz}{\left(1-z^{6}\right)^{\frac{1}{3}}}=\int\frac{z^{3}dz}{\left(1-z^{6}\right)^{\frac{2}{3}}}$$

seu posito z loco zz hanc

$$\int \frac{dz}{\left(1-z^3\right)^{\frac{1}{3}}} = \int \frac{zdz}{\left(1-z^3\right)^{\frac{2}{3}}}$$

posito $z=1$.

COROLLARIUM 3

77. Sit $m=1$ et $n=2$; fiet $\cos.\text{A}.\frac{m}{2n}\pi = \frac{1}{\sqrt{2}}$ ideoque

$$\int \frac{zdz\sqrt{2}}{\left(1-z^4\right)^{\frac{1}{2}}} = \int \frac{zzdz}{\left(1-z^4\right)^{\frac{3}{4}}} = \frac{\pi}{2\sqrt{2}}$$

ob $\int \frac{zdz}{\sqrt{\left(1-z^4\right)}} = \frac{\pi}{4}$. Ex Corollario 1 vero erit

$$\int \frac{zdz\sqrt{2}}{\left(1-z^4\right)^{\frac{1}{2}}} = \int \frac{zzdz}{\left(1-z^4\right)^{\frac{3}{4}}}$$

quae est eadem aequalitas.

THEOREMA 5

78. *Si post integrationes ponatur* $z=1$, *erit*

$$\tan.\text{A}.\frac{m}{2n}\pi \cdot \int \frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}} \cdot \int \frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{1-\gamma}} = \int \frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{1-\gamma}} . \int \frac{z^{n+2n\gamma-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}.$$

DEMONSTRATIO

In § 60 invenimus esse

$$\int \frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+2m}{2n}}} = \text{tang.A}.\frac{m}{2n}\pi \cdot \int \frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}.$$

Fiat iam in Lemmate 5 $a=n,\ b=2n$ et $c=\frac{n-m}{2n}$ atque facta substitutione erit

$$\text{tang.A}.\frac{m}{2n}\pi \cdot \int \frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}} . \int \frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{1-\gamma}} = \int \frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{1-\gamma}} \cdot \int \frac{z^{n+2n\gamma-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}.$$

Q. E. D.

COROLLARIUM 1

79. Si ponatur $\gamma = 1$, ob duo membra integrabilia fiet

$$\frac{n}{2n-m}\,\text{tang.A.}\frac{m}{2n}\pi\cdot\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}=\int\frac{z^{3n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}=\frac{n}{2n-m}\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}};$$

hanc ob rem erit

$$\text{tang.A.}\frac{m}{2n}\pi\cdot\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}=\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}},$$

quae est ipsa in § 60 inventa.

COROLLARIUM 2

80. Sit $\gamma = \frac{m}{2n}$; erit

$$\text{tang.A.}\frac{m}{2n}\pi\cdot\int\frac{z^{2n-m-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}=\int\frac{z^{n+m-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}.$$

ac si ponatur $\gamma = \frac{1}{2}$ ingredietur quadratura circuli eritque

$$\int\frac{z^{2n-m-1}dz}{\sqrt{\left(1-z^{2n}\right)}}\cdot\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}=\frac{\pi}{\text{tang.A.}\frac{m}{2n}\pi}\int\frac{z^{2n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}=\frac{\pi}{2n(n-m)\text{tang.A.}\frac{m}{2n}\pi}$$

seu

$$\frac{\pi\,\text{tang.A.}\frac{m}{2n}\pi}{2mn}=\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}\cdot\int\frac{z^{n+m-1}dz}{\sqrt{\left(1-z^{2n}\right)}}.$$

COROLLARIUM 3

81. Cum igitur sit

$$\frac{\pi}{2mn}\,\text{tang.A.}\frac{m}{2n}\pi=\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}\cdot\int\frac{z^{n-m-1}dz}{\sqrt{\left(1-z^{2n}\right)}}$$

atque ex § 60 sit

$$\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{n+m}{2n}}}=\text{tang.A.}\frac{m}{2n}\pi\cdot\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}},$$

erit

$$\frac{\pi}{2mn}=\int\frac{z^{n-1}dz}{\left(1-z^{2n}\right)^{\frac{2n-m}{2n}}}\cdot\int\frac{z^{n+m-1}dz}{\sqrt{\left(1-z^{2n}\right)}}.$$

Productum ergo harum duarum formularum integralium casu, quo $z = 1$, per peripheriam circuli exhiberi potest .

COROLLARIUM 4

82. Sit $m=1$ et $n=1$; erit ex corollario praecedente

$$\frac{\pi}{2}=\int\frac{dz}{\sqrt{(1-zz)}}\cdot\int\frac{zdz}{\sqrt{(1-zz)}}=\frac{\pi}{2}\left(1-\sqrt{(1-zz)}\right),$$

quo casu, si fiat $z=1$, aequalitas sponte perspicitur.

COROLLARIUM 5

83. Sit $m=1$ et $n=2$; erit $\text{tang.A.}\frac{m}{2n}\pi=1$, hinc ex Corollario 2 erit

$$\frac{\pi}{4}=\int\frac{zdz}{\left(1-z^4\right)^{\frac{3}{4}}}\cdot\int\frac{zzdz}{\sqrt{\left(1-z^4\right)}},$$

ex tertio autem oritur

$$\frac{\pi}{4}=\int\frac{zdz}{\left(1-z^4\right)^{\frac{3}{4}}}\cdot\int\frac{zzdz}{\sqrt{\left(1-z^4\right)}},$$

quae duae aequationes inter se congruunt.

COROLLARIUM 6

84. Sit $m=2$ et $n=3$; erit $\text{tang.A.}\frac{m}{2n}\pi=\sqrt{3}$, hinc ex Corollario 2 oritur

$$\frac{\pi}{4\sqrt{3}}=\int\frac{zzdz}{\left(1-z^6\right)^{\frac{5}{6}}}\cdot\int\frac{z^4dz}{\sqrt{\left(1-z^6\right)}},$$

ex tertio vero nascitur haec aequatio

$$\frac{\pi}{12}=\int\frac{zzdz}{\left(1-z^6\right)^{\frac{2}{3}}}\cdot\int\frac{z^4dz}{\sqrt{\left(1-z^6\right)}}.$$

SCHOLION

85. Huiusmodi theorematum ex formulis integralibus pro sinu, cosinu et tangente inventis ope Lemmatum 5, 6, 7 et 8 tanta multitudo deduci potest, ut iis capiendis integrum

volumen non sufficeret. Aperto autem fonte quilibet, quantum libuerit, inde haurire poterit. Complures quidem occurrunt casus, uti vidimus, quibus vel ad aequationes identicas vel eiusmodi, quae facile eo reducuntur, pervenitur hique casus veritatem reliquorum theorematum eo magis confirmant, in quibus ratio aequalitatis non perspicitur. Sic in aequatione § 80 si ponatur $m=0$ et $n=1$, fiet $\text{tang.A.}\frac{m}{2n}\pi=\frac{m}{2n}\pi$, eo quod tangens arcus evanescentis ipsi arcui aequatur; hinc igitur fiet

$$\frac{\pi\pi}{4}=\int\frac{dz}{\sqrt{(1-zz)}}\cdot\int\frac{dz}{\sqrt{(1-zz)}},$$

cuius veritas, cum si $\int\frac{dz}{\sqrt{(1-zz)}}=\frac{\pi}{2}$ casu, quo $z=1$, sponte apparet. Ceterum huiusmodi formularum integralium, quae neque integrari neque ad se mutuo reduci possunt, comparationes eo magis sunt notatu dignae, quo minus via ad eas comprobandas patere videatur. Sic primum huius generis theorema, ad quod iam pridem fui deductus, simplicitate se commendabat, quo inveni esse productum harum duarum formularum integralium

$$\int\frac{dz}{\sqrt{\left(1-z^4\right)}}\quad\text{et}\quad\int\frac{z^2dz}{\sqrt{\left(1-z^4\right)}},$$

et quarum altera arcum, altera ordinatam in curva elastica exprimit, casu $z=1$ aequale areae circuli, cuius diameter sit $=1$.