CONCERNING THE VALUE OF THE INTEGRAL FORMULA

$$\int \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}} \cdot \frac{dz}{z} (lz)^{\mu}$$

IN THE CASE WHERE $z = 1$ AFTER THE INTEGRATION.

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1. From a consideration of the innumerable circular arcs which both the sine or tangent have in common , I deduced now some time ago to be the sum of two infinite series, which on account of the generality of the sum may be considered to be most worthy to be mentioned. Indeed if the letters *m* and *n* may denote some numbers, with the diameter in the ratio to the periphery as 1 to π, these two summations may themselves be had in this manner [see E59 in these translations]

$$\frac{1}{m} + \frac{1}{n-m} - \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} + \frac{1}{3n-m} - \text{etc.} = \frac{\pi}{n \sin . \frac{m\pi}{n}},$$

and

$$\frac{1}{m} - \frac{1}{n-m} + \frac{1}{n+m} - \frac{1}{2n-m} + \frac{1}{2n+m} - \frac{1}{3n-m} + \text{etc.} = \frac{\pi}{n \,\text{tang.} \frac{m\pi}{n}},$$

and now moreover from these two series of the time I was deducing the sums of all of these series, of which the denominators progress following the powers of natural numbers, just as I have established in the *Introductione in analysin infinitorum* [Vol. I, Ch. X.] and set out further elsewhere. But now the same series have led me to the integral of the formula expressed in the title, which therefore may be seen to be more noteworthy, since integrations of this kind by no means may be permitted to follow from other methods.

2. But it is apparent at once that the two infinite series arise from the expansion of certain integral formulas, if after the integration the value of the variable quantity may be granted a certain value, such as unity; thus the first series is deduced from the expansion of this integral formula:

$$\int \frac{z^{m-1} + z^{n-m-1}}{1 + z^{n}} \cdot dz,$$

truly the latter from the expansion of this :

$$\int \frac{z^{m-1} - z^{n-m-1}}{1 - z^{n}} \cdot dz,$$

if indeed after the integration there may be put $z = 1$. But following from these principles of the integral calculus I have shown the value of the first integral of these two formulas, if indeed there may be put $z = 1$, to be reduced to this simple form :

$$\frac{\pi}{n \sin. \frac{m\pi}{n}},$$

moreover the integral of the latter in the same case $z = 1$ to be reduced to that same :

$$\frac{\pi}{n \operatorname{tang.} \frac{m\pi}{n}},$$

thus so that from the principles of the integral calculus [see E59], there shall certainly become

$$\int \frac{z^{m-1} + z^{n-m-1}}{1 + z^n} \cdot dz = \frac{\pi}{n \sin. \frac{m\pi}{n}},$$

if indeed after the integration thus put in place, so that the integral may vanish on putting $z = 0$, there may be put $z = 1$.

3. Now where we may reduce this two-fold integration to the proposed form, we may set $n = 2\lambda$ and $m = \lambda - \omega$, from which these two infinite series will adopt this form

$$\frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega} - \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} + \frac{1}{5\lambda+\omega} - \text{etc.}$$

and

$$\frac{1}{\lambda-\omega} - \frac{1}{\lambda+\omega} + \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} - \frac{1}{5\lambda+\omega} + \text{etc.}$$

therefore the sum of the first series will be

$$\frac{\pi}{2\lambda \sin. \frac{\pi(\lambda-\omega)}{2\lambda}} = \frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}},$$

truly of the latter the sum will be

$$\frac{\pi}{2\lambda \operatorname{tang.} \frac{\pi(\lambda-\omega)}{2\lambda}} = \frac{\pi}{2\lambda \operatorname{cotang.} \frac{\pi\omega}{2\lambda}} = \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda}$$

But if for the sake of brevity we may put

$$\frac{\pi}{2\lambda \cos. \frac{\pi\omega}{2\lambda}} = S \text{ and } \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda} = T,$$

we will have the two following integrations

$$\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} = S \text{ and } \int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} = T.$$

4. Concerning these two integrations I observe before everything these likewise have to noted, whole numbers or fractions may be taken for the letters λ and ω. Indeed λ and ω shall be any fractions, which may become whole, if they may be multiplied by α, with which in place there may become $z = x^{\alpha}$ and there will be $\frac{dz}{z} = \frac{\alpha dx}{x}$ and for any power $z^{\theta} = x^{\alpha\theta}$; therefore the first formula will become :

$$\int \frac{x^{\alpha(\lambda-\omega)}+x^{\alpha(\lambda+\omega)}}{1+x^{2\alpha\lambda}} \cdot \frac{\alpha dx}{x};$$

where since now all the exponents shall be whole numbers, the value of the formula on putting $x=1$ after the integration, since then also there shall be $z=1$, this will only differ from the preceding, because here we shall have $\alpha\lambda$ and $\alpha\omega$ in place of λ and ω and besides here the factor α shall be present, wherefore the value of this formula will be

$$\alpha \frac{\pi}{2\alpha\lambda \cos.\frac{\pi\omega}{2\lambda}} = \frac{\pi}{2\lambda \cos.\frac{\pi\omega}{2\lambda}},$$

which therefore is the value $= S$, exactly as before; which identity is evident in the other formula also, from which it is apparent, even if any fractions may be taken for λ and ω, the integration shown here is going none the less to be present; which circumstance deserves to be noted properly, because in the following we are going to treat the letter ω as the variable.

5. Therefore after these two integral formulas indicated by the letters S and T will have been integrated thus, so that they may vanish on putting $z=0$, the integrals can be considered not only as functions of the quantity z, but also as functions of the two variables z et ω, since the number ω may be treated as a variable quantity ; there is no reason why the exponent λ may not be allowed to be had as a variable quantity also ; but hence because integral formulas of another kind are going to be produced, and I have established this to be considered [i.e. as an option] , I am going to treat here only the quantity ω besides the variable z as a variable quantity.

6. Therefore since there shall be

$$S = \int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z},$$

in which integration z alone may be considered variable, certainly there will be following the custom of designating now well enough received in use

$$\left(\frac{dS}{dz}\right) = \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{1}{z};$$

now this formula may be differentiated anew with the letter ω only to be variable, and there will become :

$$\left(\frac{ddS}{dzd\omega}\right) = \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{1}{z} lz,$$

which formula multiplied by dz and integrated anew with z only for the variable will give :

$$\int dz \left(\frac{ddS}{dzd\omega}\right) = \int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} lz,$$

where it may be observed to be :

$$S = \frac{\pi}{2\lambda \cos.\frac{\pi\omega}{2\lambda}},$$

thus so that hence we may deduce :

$$\left(\frac{dS}{d\omega}\right) = \frac{\pi\pi \sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda \cos^2.\frac{\pi\omega}{2\lambda}};$$

therefore with this value substituted, we arrive at this integration :

$$\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} lz = \frac{\pi\pi \sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda \cos^2.\frac{\pi\omega}{2\lambda}}.$$

7. But if now the other formula may be treated in a similar manner, since there shall be :

$$T = \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda},$$

there will be

$$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{4\lambda\lambda \cos^2.\frac{\pi\omega}{2\lambda}},$$

but from the formula of the integral there will be

$$\left(\frac{dT}{d\omega}\right) = \int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz,$$

from which we deduce the following integration :

$$\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz = \frac{-\pi\pi}{4\lambda\lambda \cos^2.\frac{\pi\omega}{2\lambda}}.$$

8. Because we have given the letters S and T also expressed by series , also by similar series there will become :

$$\left(\frac{dS}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} - \frac{1}{(\lambda+\omega)^2} - \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} - \text{etc.}$$
$$= \frac{\pi\pi \sin . \frac{\pi\omega}{2\lambda}}{4\lambda\lambda \cos^2 . \frac{\pi\omega}{2\lambda}}.$$

And in a similar manner for the other series also :

$$\left(\frac{dT}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} + \frac{1}{(\lambda+\omega)^2} + \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} + \text{etc.}$$
$$= \frac{\pi\pi}{4\lambda\lambda \cos^2 . \frac{\pi\omega}{2\lambda}},$$

and thus we have represented the sums of these series in a two-fold manner also, clearly by the formula expanded out involving the quantity π , then also truly by an integral formula, which has been prepared thus, so that its integral cannot be accustomed to be designated by any other method at present.

9. We may apply these integrations to some particular cases ; and indeed initially we take $\omega = 0$, where certainly in the first case the integration comes to mind at once, but the latter produces

$$\int \frac{2z^{\lambda}}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz = -\frac{\pi\pi}{4\lambda\lambda}$$

or

$$\int \frac{z^{\lambda-1} dz lz}{1-z^{2\lambda}} = -\frac{\pi\pi}{8\lambda\lambda}$$

and hence likewise we arrive at that same summation

$$\frac{1}{\lambda\lambda} + \frac{1}{\lambda\lambda} + \frac{1}{9\lambda\lambda} + \frac{1}{25\lambda\lambda} + \frac{1}{25\lambda\lambda} + \text{etc.} = \frac{\pi\pi}{4\lambda\lambda}$$

or

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \frac{1}{121} + \frac{1}{169} + \text{etc.} = \frac{\pi\pi}{8},$$

that which now some time ago was demonstrated by me. [See E 41]

10. This therefore is apparent at once, whatever the number may be taken for λ ; therefore there may be put $\lambda = 1$ and this equation will be found:

$$\int \frac{dzlz}{1-z^2} = -\frac{\pi\pi}{8},$$

from which the simpler integrals

$$\int \frac{dzlz}{1-z} \text{ and } \int \frac{dzlz}{1+z}$$

will be permitted to be derived with the aid of this reasoning ; there may be put

$$\int \frac{zdzlz}{1-zz} = P$$

and on putting $zz = v$, so that there shall become $zdz = \frac{dv}{2}$ and $lz = \frac{1}{2} lv$, there will be produced :

$$\frac{1}{4} \int \frac{dvlv}{1-v} = P,$$

if evidently after the integration there may become $v = 1$, certainly from which case also there becomes $z = 1$; thus there will therefore become :

$$\int \frac{dvlv}{1-v} = 4P;$$

now that first formula may be added to that found and there will become :

$$\int \frac{dzlz + zdzlz}{1-zz} = P - \frac{\pi\pi}{8},$$

but that formula is reduced at once to this :

$$\int \frac{dzlz}{1-z} = P - \frac{\pi\pi}{8};$$

but as we have seen just now to be $\int \frac{dvlv}{1-v}$ or $\int \frac{dzlz}{1-z} = 4P$, thus so that there shall be

$$4P = P - \frac{\pi\pi}{8}$$

from which there becomes evidently

$$P = -\frac{\pi\pi}{24},$$

from which there follows to become

$$\int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6};$$

in a similar manner there will be

$$\int \frac{dzlz - zdzlz}{1-zz} = -P - \frac{\pi\pi}{8} = -\frac{\pi\pi}{12},$$

which on dividing above and below by $1-z$ gives

$$\int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$

whereby now we have come upon three most noteworthy integrations :

$$\text{I.} \quad \int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$

$$\text{II.} \quad \int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6},$$

$$\text{III.} \quad \int \frac{dzlz}{1-zz} = -\frac{\pi\pi}{8},$$

to which there can be added :

$$\text{IV.} \quad \int \frac{zdzlz}{1-zz} = -\frac{\pi\pi}{24}.$$

11. Therefore just as these formulas have been deduced from the principles of the calculus themselves, thus also the truth of these by resolution into series is proven easily ; since indeed there shall be :

$$\frac{1}{1+z} = 1 - z + zz - z^3 + z^4 - z^5 + \text{etc.}$$

and in general:

$$\int z^n dzlz = \frac{z^{n+1}}{n+1} lz - \frac{z^{n+1}}{(n+1)^2},$$

which value on putting $z=1$ is reduced to $-\frac{1}{(n+1)^2}$, it is evident to become

$$\int \frac{dzlz}{1+z} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \text{etc.} = -\frac{\pi\pi}{12}$$

or

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.} = \frac{\pi\pi}{12};$$

but in a similar manner

$$\frac{1}{1-z} = 1 + z + zz + z^3 + z^4 + z^5 + \text{etc.}$$

there will be

$$\int \frac{dzlz}{1-z} = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \frac{1}{25} - \text{etc.} = -\frac{\pi\pi}{6}$$

or

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \frac{\pi\pi}{6};$$

then truly on account of

$$\frac{1}{1-zz} = 1 + zz + z^4 + z^6 + z^8 + \text{etc.}$$

there will become

$$\int \frac{dzlz}{1-zz} = -1 - \frac{1}{9} - \frac{1}{25} - \frac{1}{49} - \frac{1}{81} - \text{etc.} = -\frac{\pi\pi}{8}$$

or

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \text{etc.} = \frac{\pi\pi}{8}.$$

In the same manner also

$$\int \frac{zdzlz}{1-zz} = -\frac{1}{4} - \frac{1}{16} - \frac{1}{36} - \frac{1}{64} - \text{etc.} = -\frac{\pi\pi}{24}$$

or

$$\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \frac{1}{64} + \text{etc.} = \frac{\pi\pi}{24},$$

indeed which summations now are most noteworthy. Yet nor by any other method at this time is it to be shown directly that

$$\int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12}.$$

12. Now we may put $\omega = 1$ and our integrations will adopt these forms

$$1^0. \int \frac{-z^{\lambda-2}(1-zz)dzlz}{1+z^{2\lambda}} = \frac{\pi\pi \sin.\frac{\pi}{2\lambda}}{4\lambda\lambda \cos.^2 \frac{\pi}{2\lambda}}$$

and

$$2^0. \int \frac{-z^{\lambda-2}(1+zz)dzlz}{1-z^{2\lambda}} = +\frac{\pi\pi}{4\lambda\lambda \cos.^2 \frac{\pi}{2\lambda}},$$

from which for different values of λ, which indeed are not allowed to be taken smaller than two, the following integrations are obtained:

I. If $\lambda = 2$, there will be

$$1^0.\ \int \frac{-(1-zz)dzlz}{1+z^4} = \frac{\pi\pi}{8\sqrt{2}},$$

$$2^0.\ \int \frac{-(1+zz)dzlz}{1-z^4} = +\frac{\pi\pi}{8} \text{ or } \int \frac{-dzlz}{1-zz} = +\frac{\pi\pi}{8}.$$

II. If $\lambda = 3$, we will have

$$1^0.\ \int \frac{-z(1-zz)dzlz}{1+z^6} = \frac{\pi\pi}{54} \text{ and } 2^0.\ \int \frac{-z(1+zz)dzlz}{1-z^6} = +\frac{\pi\pi}{27}.$$

But these two formulas by putting $zz = v$ will be changed into the following

$$1^0.\ \int \frac{-dv(1-v)lv}{1+v^3} = \frac{2\pi\pi}{27} \text{ and } 2^0.\ \int \frac{-dv(1+v)lv}{1-v^3} = \frac{4\pi\pi}{27}.$$

III. Let $\lambda = 4$ and we will follow with

$$1^0.\ \int \frac{-zz(1-zz)dzlz}{1+z^8} = \frac{\pi\pi\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}}{16(2+\sqrt{2})} = \frac{\pi\pi\sqrt{(2-\sqrt{2})}}{32(2+\sqrt{2})}$$

and

$$2^0.\ \int \frac{-zz(1+zz)dzlz}{1-z^8} = \int \frac{-zzdzlz}{(1-zz)(1+z^4)} = \frac{\pi\pi}{16(2+\sqrt{2})},$$

which latter form is reduced to this :

$$\int \frac{-dzlz}{1-zz} + \int \frac{(1-zz)dzlz}{1+z^4} = \frac{\pi\pi}{8(2+\sqrt{2})};$$

truly there is $\int \frac{-dzlz}{1-zz} = \frac{\pi\pi}{8}$, from which there is found :

$$\int \frac{dzlz(1-zz)}{1+z^4} = -\frac{\pi\pi(1+\sqrt{2})}{8(2+\sqrt{2})} = -\frac{\pi\pi}{8\sqrt{2}},$$

which value has been found now in the above case $\lambda = 2$.

13. Moreover nothing prevents, where we may diminish also and make $\lambda = 1$, as long as thus the integrals may be taken, so that they vanish on putting $z = 0$; but then we will find

$$1^0.\ \int \frac{-(1-zz)dzlz}{z(1+zz)} = \infty \text{ and } 2^0.\ \int \frac{-(1+zz)dzlz}{z(1-zz)} = \infty,$$

[*i.e.* here the integrals diverge on the lower limit approaching zero] from which hence nothing can be concluded. Moreover our series found above also evidently indicate their sums to be infinite, whenever the first term of each $\frac{1}{(\lambda-\omega)^2}$ shall be indicate on taking, as we have done, $\lambda=1$ and $\omega=1$.

14. With these cases expanded out we may progress further and put the integral formulas found :

$$\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}lz=S' \text{ and } \int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}lz=T',$$

thus so that there becomes:

$$S'=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\cos^2.\frac{\pi\omega}{2\lambda}} \text{ et } T'=\frac{\pi\pi}{4\lambda\lambda\cos^2.\frac{\pi\omega}{2\lambda}},$$

now as before we may differentiate with the number ω had for the variable ; with which done we obtain the following integrations :

$$\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}(lz)^2=\left(\frac{dS'}{d\omega}\right) \text{ and } \int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}(lz)^2=\left(\frac{dT'}{d\omega}\right).$$

Therefore in the end we may put this angle $\frac{\pi\omega}{2\lambda}=\varphi$, so that there shall be

$$S'=\frac{\pi\pi\sin.\varphi}{4\lambda\lambda\cos^2.\varphi}=\frac{\pi\pi}{4\lambda\lambda}\cdot\frac{\sin.\varphi}{\cos^2.\varphi} \text{ and } T'=\frac{\pi\pi}{4\lambda\lambda}\cdot\frac{1}{\cos^2.\varphi},$$

and we will find:

$$d.\frac{\sin.\varphi}{\cos^2.\varphi}=\left(\frac{\cos^2.\varphi+2\sin^2.\varphi}{\cos^3.\varphi}\right)d\varphi=\left(\frac{1+\sin^2.\varphi}{\cos^3.\varphi}\right)d\varphi,$$

where there is $d\varphi=\frac{\pi d\omega}{2\lambda}$; from which we deduce

$$\left(\frac{dS'}{d\omega}\right)=\frac{\pi^3}{8\lambda^3}\cdot\left(\frac{1+\sin^2.\frac{\pi\omega}{2\lambda}}{\cos^2.\frac{\pi\omega}{2\lambda}}\right)=\frac{\pi^3}{8\lambda^3}\cdot\left(\frac{2}{\cos^3.\frac{\pi\omega}{2\lambda}}-\frac{1}{\cos.\frac{\pi\omega}{2\lambda}}\right);$$

in a similar manner since $T'=\frac{\pi\pi}{4\lambda\lambda}\cdot\frac{1}{\cos^2.\varphi}$, there will be

$$d\cdot\frac{1}{\cos^2.\varphi}=\frac{2d\varphi\sin.\varphi}{\cos^3.\varphi}$$

and hence

$$\left(\frac{dT'}{d\omega}\right) = \frac{\pi^3}{8\lambda^3} \cdot \frac{2\sin.\frac{\pi\omega}{2\lambda}}{\cos^3.\frac{\pi\omega}{2\lambda}};$$

consequently the integrations hence will arise :

$$\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \frac{\pi^3}{8\lambda^3} \cdot \left(\frac{2}{\cos^3.\frac{\pi\omega}{2\lambda}} - \frac{1}{\cos.\frac{\pi\omega}{2\lambda}} \right),$$

$$\int \frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \frac{\pi^3}{8\lambda^3} \cdot \frac{2\sin.\frac{\pi\omega}{2\lambda}}{\cos^3.\frac{\pi\omega}{2\lambda}}.$$

15. Now if in the same manner we may differentiate anew the series found in § 8 with ω taken to be the only variable, we may come upon the following summations

$$\frac{\pi^3}{8\lambda^3} \cdot \left(\frac{2}{\cos^3.\frac{\pi\omega}{2\lambda}} - \frac{1}{\cos.\frac{\pi\omega}{2\lambda}} \right)$$
$$= \frac{2}{(\lambda-\omega)^3} + \frac{2}{(\lambda+\omega)^3} - \frac{2}{(3\lambda-\omega)^3} - \frac{2}{(3\lambda+\omega)^3} + \frac{2}{(5\lambda-\omega)^3} + \frac{2}{(5\lambda+\omega)^3} - \text{etc.},$$
$$\frac{\pi^3}{8\lambda^3} \cdot \frac{2\sin.\frac{\pi\omega}{2\lambda}}{\cos^3.\frac{\pi\omega}{2\lambda}} = \frac{2}{(\lambda-\omega)^3} - \frac{2}{(\lambda+\omega)^3} + \frac{2}{(3\lambda-\omega)^3} - \frac{2}{(3\lambda+\omega)^3} + \frac{2}{(5\lambda-\omega)^3} + \text{etc.}$$

16. If now we may suppose here $\omega = 0$ and $\lambda = 1$, the first integration adopts this form:

$$\int \frac{2dz(lz)^2}{1+zz} = \frac{\pi^3}{8} = \frac{2}{1^3} + \frac{2}{1^3} - \frac{2}{3^3} - \frac{2}{3^3} + \frac{2}{5^3} + \frac{2}{5^3} - \frac{2}{7^3} - \frac{2}{7^3} + \text{etc.}$$

so that thus there shall be

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \text{etc.} = \frac{\pi^3}{32},$$

just as I have shown formerly. But the other integration in this case will go to zero. Truly from the first integral:

$$\int \frac{dz(lz)^2}{1+zz} = \frac{\pi^3}{16}.$$

it is not allowed to derive others, as we have done above from the formula $\int \frac{dzlz}{1-zz} = -\frac{\pi\pi}{8}$, because here the denominator $1+zz$ has no real factors.

17. Therefore we may take $\lambda = 2$ and $\omega = 1$ and the first integral will give

$$\int \frac{(1+zz)dz(lz)^2}{1+z^4} = \frac{3\pi^3}{32\sqrt{2}};$$

moreover the series hence arising will be

$$\frac{2}{1^3} + \frac{2}{3^3} - \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} + \frac{2}{11^3} - \text{etc.},$$

thus so that there shall become :

$$\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - \text{etc.} = \frac{3\pi^3}{64\sqrt{2}},$$

which added to the above gives

$$\frac{1}{1^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{23^3} + \text{etc.} = \frac{\pi^3(3+2\sqrt{2})}{128\sqrt{2}}.$$

Truly the other integration in this case gives :

$$\int \frac{dz(lz)^2}{1+zz} = \frac{\pi^3}{16},$$

which agrees perfectly with the preceding paragraph, just as also the series hence arising is :

$$\frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.}$$

18. But where we may prevail to elicit the following integrations more easily by continued differentiation, we may represent these in general; and since for the first there shall be

$$S = \frac{\pi}{2\lambda \cos.\frac{\pi\omega}{2\lambda}},$$

the integrations hence arising will proceed in the order thus:

I. $\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} = S,$

II. $\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz = \left(\frac{dS}{d\omega}\right),$

III. $\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \left(\frac{ddS}{d\omega^2}\right),$

IV. $\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^3 = \left(\frac{d^3S}{d\omega^3}\right),$

V. $\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^4 = \left(\frac{d^4S}{d\omega^4}\right),$

VI. $\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^5 = \left(\frac{d^5S}{d\omega^5}\right),$

VII. $\int \frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^6 = \left(\frac{d^6S}{d\omega^6}\right)$

etc.

19. For these continued differentials requiring to be resolved easily we may put for brevity therefore $\frac{\pi}{2\lambda} = \alpha$, so that there shall become

$$S = \frac{\alpha}{\cos.\alpha\omega};$$

then truly there shall be

$$\sin.\alpha\omega = p \text{ and } \cos.\alpha\omega = q$$

and there will be

$$dp = \alpha q d\omega \text{ and } dq = -\alpha p d\omega.$$

Truly besides there may be noted to be

$$d.\frac{p^n}{q^{n+1}} = \alpha d\omega\left(\frac{np^{n-1}}{q^n} + \frac{(n+1)p^{n+1}}{q^{n+2}}\right).$$

With these in place on account of $S = \alpha \cdot \frac{1}{q}$ there will become

$$\left(\frac{dS}{d\omega}\right) = \alpha\alpha \frac{p}{qq},$$

from which

$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3\left(\frac{1}{1} + \frac{2pp}{q^3}\right),$$

again

$$\left(\frac{d^3S}{d\omega^3}\right) = \alpha^4\left(\frac{5p}{qq} + \frac{6p^3}{q^4}\right),$$
$$\left(\frac{d^4S}{d\omega^4}\right) = \alpha^5\left(\frac{5}{q} + \frac{28pp}{q^3} + \frac{24p^4}{q^5}\right),$$
$$\left(\frac{d^5S}{d\omega^5}\right) = \alpha^6\left(\frac{61p}{qq} + \frac{180p^3}{q^4} + \frac{120p^5}{q^6}\right),$$
$$\left(\frac{d^6S}{d\omega^6}\right) = \alpha^7\left(\frac{61}{q} + \frac{662pp}{q^3} + \frac{1320p^4}{q^5} + \frac{720p^6}{q^7}\right),$$
$$\left(\frac{d^7S}{d\omega^7}\right) = \alpha^8\left(\frac{1385}{qq} + \frac{7266pp}{q^4} + \frac{10920p^4}{q^6} + \frac{5040p^7}{q^8}\right);$$

but these values on account of $pp = 1 - qq$ are reduced to the following :

$$S = \alpha \cdot \frac{1}{q},$$
$$\left(\frac{dS}{d\omega}\right) = \alpha\alpha p \frac{1}{qq},$$
$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3\left(\frac{1 \cdot 2}{q^3} - \frac{1}{q}\right),$$
$$\left(\frac{d^3S}{d\omega^3}\right) = \alpha^4 p\left(\frac{1 \cdot 2 \cdot 3}{q^4} - \frac{1}{qq}\right),$$
$$\left(\frac{d^4S}{d\omega^4}\right) = \alpha^5 p\left(\frac{1 \cdot 2 \cdot 3 \cdot 4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right),$$
$$\left(\frac{d^5S}{d\omega^5}\right) = \alpha^6 p\left(\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{q^6} - \frac{60}{q^4} + \frac{1}{qq}\right),$$
$$\left(\frac{d^6S}{d\omega^6}\right) = \alpha^7 p\left(\frac{1 \cdots\cdots 6}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right).$$

20. These latter forms are able to be found with the aid of these two lemmas :

$$\text{I. } d.\frac{1}{q^{n+1}} = \alpha d\omega \frac{(n+1)p}{q^{n+2}}, \text{ II. } d.\frac{p}{q^{n+1}} = \alpha d\omega \left(\frac{n+1}{q^{n+2}} - \frac{n}{q^n}\right);$$

hence also we will find :

$$S = \alpha \cdot \frac{1}{q},$$

$$\left(\frac{dS}{d\omega}\right) = \alpha\alpha \frac{p}{qq},$$

$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3 \left(\frac{2}{q^3} - \frac{1}{q}\right),$$

$$\left(\frac{d^3S}{d\omega^3}\right) = \alpha^4 \left(\frac{2 \cdot 3p}{q^4} - \frac{p}{qq}\right),$$

$$\left(\frac{d^4S}{d\omega^4}\right) = \alpha^5 \left(\frac{2 \cdot 3 \cdot 4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right),$$

$$\left(\frac{d^5S}{d\omega^5}\right) = \alpha^6 \left(\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5p}{q^6} - \frac{60p}{q^4} + \frac{p}{qq}\right),$$

$$\left(\frac{d^6S}{d\omega^6}\right) = \alpha^7 \left(\frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right),$$

$$\left(\frac{d^7S}{d\omega^7}\right) = \alpha^8 \left(\frac{2 \cdots 7p}{q^8} - \frac{5 \cdot 840p}{q^6} + \frac{3 \cdot 182p}{q^4} - \frac{p}{qq}\right).$$

21. But these same series correspond to these formulas

$$S = \frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega} - \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} + \frac{1}{5\lambda+\omega} - \text{etc.}$$

$$\left(\frac{dS}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} - \frac{1}{(\lambda+\omega)^2} - \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} - \frac{1}{(5\lambda+\omega)^2} - \text{etc},$$

$$\left(\frac{ddS}{d\omega^2}\right) = \frac{1 \cdot 2}{(\lambda-\omega)^3} + \frac{1 \cdot 2}{(\lambda+\omega)^3} - \frac{1 \cdot 2}{(3\lambda-\omega)^3} - \frac{1 \cdot 2}{(3\lambda+\omega)^3} + \frac{1 \cdot 2}{(5\lambda-\omega)^3} + \text{etc},$$

$$\left(\frac{d^3S}{d\omega^3}\right) = \frac{1 \cdot 2 \cdot 3}{(\lambda-\omega)^4} - \frac{1 \cdot 2 \cdot 3}{(\lambda+\omega)^4} - \frac{1 \cdot 2 \cdot 3}{(3\lambda-\omega)^4} + \frac{1 \cdot 2 \cdot 3}{(3\lambda+\omega)^4} + \frac{1 \cdot 2 \cdot 3}{(5\lambda-\omega)^4} - \text{etc},$$

$$\left(\frac{d^4S}{d\omega^4}\right) = \frac{1 \cdot 2 \cdot 3 \cdot 4}{(\lambda-\omega)^5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{(\lambda+\omega)^5} - \frac{1 \cdot 2 \cdot 3 \cdot 4}{(3\lambda-\omega)^5} - \frac{1 \cdot 2 \cdot 3 \cdot 4}{(3\lambda+\omega)^5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{(5\lambda-\omega)^5} + \text{etc},$$

$$\left(\frac{d^5S}{d\omega^5}\right) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(\lambda-\omega)^6} - \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(\lambda+\omega)^6} - \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(3\lambda-\omega)^6} + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(3\lambda+\omega)^6} + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(5\lambda-\omega)^6} - \text{etc},$$

$$\left(\frac{d^6S}{d\omega^6}\right) = \frac{1 \cdots 6}{(\lambda-\omega)^7} + \frac{1 \cdots 6}{(\lambda+\omega)^7} - \frac{1 \cdots 6}{(3\lambda-\omega)^7} - \frac{1 \cdots 6}{(3\lambda+\omega)^7} + \frac{1 \cdots 6}{(5\lambda-\omega)^7} + \text{etc},$$

$$\left(\frac{d^7S}{d\omega^7}\right) = \frac{1 \cdots 7}{(\lambda-\omega)^8} - \frac{1 \cdots 7}{(\lambda+\omega)^8} - \frac{1 \cdots 7}{(3\lambda-\omega)^8} + \frac{1 \cdots 7}{(3\lambda+\omega)^8} + \frac{1 \cdots 7}{(5\lambda-\omega)^8} + \text{etc},$$

etc.

But it is required to be remembered properly about these values :

$$\alpha = \frac{\pi}{2\lambda},\ p = \sin.\alpha\omega = \sin.\frac{\pi\omega}{2\lambda} \text{ and } q = \cos.\alpha\omega = \cos.\frac{\pi\omega}{2\lambda}.$$

22. We can establish the values or integral formulas of the other kind in the same way, for which there is

$$T = \frac{\pi}{2\lambda} \text{tang.} \frac{\pi\omega}{2\lambda},$$

from which by continued differentiation the following integrals arise :

$$\text{I.} \quad \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} = T,$$

$$\text{II.} \quad \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} lz = \left(\frac{dT}{d\omega}\right),$$

$$\text{III.} \quad \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \left(\frac{ddT}{d\omega^2}\right),$$

$$\text{IV.} \quad \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^3 = \left(\frac{d^3T}{d\omega^3}\right),$$

$$\text{V.} \quad \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^4 = \left(\frac{d^4T}{d\omega^4}\right),$$

$$\text{VI.} \quad \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z} (lz)^5 = \left(\frac{d^5T}{d\omega^5}\right),$$

$$\text{VII.} \quad \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^6 = \left(\frac{d^6T}{d\omega^6}\right).$$

23. Again, there may be put $\alpha = \frac{\pi}{2\lambda}$, $\sin.\alpha\omega = p$ and $\cos.\alpha\omega = q$, so that there shall be

$$T = \frac{\alpha p}{q},$$

which formula following the lemma in § 20 by continued differentiation will give :

$$T = \frac{\alpha p}{q},$$
$$\left(\frac{dT}{d\omega}\right) = \alpha\alpha \frac{1}{qq},$$
$$\left(\frac{ddT}{d\omega^2}\right) = \alpha^3 \frac{2p}{q^3},$$
$$\left(\frac{d^3T}{d\omega^3}\right) = \alpha^4 \left(\frac{6}{q^4} - \frac{4}{qq}\right),$$
$$\left(\frac{d^4T}{d\omega^4}\right) = \alpha^5 \left(\frac{24p}{q^5} - \frac{8}{q^3}\right),$$
$$\left(\frac{d^5T}{d\omega^5}\right) = \alpha^6 \left(\frac{120}{q^6} - \frac{120}{q^4} + \frac{16}{qq}\right),$$
$$\left(\frac{d^6T}{d\omega^6}\right) = \alpha^7 \left(\frac{720}{q^7} - \frac{480}{q^5} + \frac{32}{q^3}\right),$$
$$\left(\frac{d^7T}{d\omega^7}\right) = \alpha^8 \left(\frac{5040}{q^8} - \frac{6720}{q^6} + \frac{2016}{q^4} - \frac{64}{qq}\right).$$

24. Moreover the infinite series, which hence arise, will be

$$T = \frac{1}{\lambda-\omega} - \frac{1}{\lambda+\omega} + \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} - \frac{1}{5\lambda+\omega} + \text{etc.}$$
$$\left(\frac{dT}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} + \frac{1}{(\lambda+\omega)^2} + \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} + \text{etc,}$$
$$\left(\frac{ddT}{d\omega^2}\right) = \frac{1\cdot2}{(\lambda-\omega)^3} - \frac{1\cdot2}{(\lambda+\omega)^3} + \frac{1\cdot2}{(3\lambda-\omega)^3} - \frac{1\cdot2}{(3\lambda+\omega)^3} + \frac{1\cdot2}{(5\lambda-\omega)^3} + \text{etc,}$$
$$\left(\frac{d^3T}{d\omega^3}\right) = \frac{1\cdot2\cdot3}{(\lambda-\omega)^4} + \frac{1\cdot2\cdot3}{(\lambda+\omega)^4} + \frac{1\cdot2\cdot3}{(3\lambda-\omega)^4} + \frac{1\cdot2\cdot3}{(3\lambda+\omega)^4} + \frac{1\cdot2\cdot3}{(5\lambda-\omega)^4} + \text{etc,}$$
$$\left(\frac{d^4T}{d\omega^4}\right) = \frac{1\cdot2\cdot3\cdot4}{(\lambda-\omega)^5} - \frac{1\cdot2\cdot3\cdot4}{(\lambda+\omega)^5} + \frac{1\cdot2\cdot3\cdot4}{(3\lambda-\omega)^5} - \frac{1\cdot2\cdot3\cdot4}{(3\lambda+\omega)^5} + \frac{1\cdot2\cdot3\cdot4}{(5\lambda-\omega)^5} - \text{etc,}$$
$$\left(\frac{d^5T}{d\omega^5}\right) = \frac{1\cdot2\cdot3\cdot4\cdot5}{(\lambda-\omega)^6} + \frac{1\cdot2\cdot3\cdot4\cdot5}{(\lambda+\omega)^6} + \frac{1\cdot2\cdot3\cdot4\cdot5}{(3\lambda-\omega)^6} + \frac{1\cdot2\cdot3\cdot4\cdot5}{(3\lambda+\omega)^6} + \frac{1\cdot2\cdot3\cdot4\cdot5}{(5\lambda-\omega)^6} + \text{etc,}$$
$$\left(\frac{d^6T}{d\omega^6}\right) = \frac{1\cdots\cdots6}{(\lambda-\omega)^7} - \frac{1\cdots\cdots6}{(\lambda+\omega)^7} + \frac{1\cdots\cdots6}{(3\lambda-\omega)^7} - \frac{1\cdots\cdots6}{(3\lambda+\omega)^7} + \frac{1\cdots\cdots6}{(5\lambda-\omega)^7} - \frac{1\cdots\cdots6}{(5\lambda+\omega)^7} + \text{etc.}$$

25. Hence it will be worth the effort to set out the most simple cases, which arise by putting $\lambda = 1$ and $\omega = 0$, thus so that there shall be $a = \frac{\pi}{2}$, $p = 0$ and $q = 1$, from which we will have :

For the prior order	For the posterior order
$S = \frac{\pi}{2}$	$T = 0$
$\left(\frac{dS}{d\omega}\right) = 0$	$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{4}$
$\left(\frac{ddS}{d\omega^2}\right) = \frac{\pi^3}{8}$	$\left(\frac{ddT}{d\omega^2}\right) = 0$
$\left(\frac{d^3S}{d\omega^3}\right) = 0$	$\left(\frac{d^3T}{d\omega^3}\right) = \frac{\pi^4}{8}$
$\left(\frac{d^4S}{d\omega^4}\right) = \frac{5\pi^5}{32}$	$\left(\frac{d^4T}{d\omega^4}\right) = 0$
$\left(\frac{d^5S}{d\omega^5}\right) = 0$	$\left(\frac{d^5T}{d\omega^5}\right) = \frac{\pi^6}{4}$
$\left(\frac{d^6S}{d\omega^6}\right) = \frac{61\pi^7}{128}$	$\left(\frac{d^6T}{d\omega^6}\right) = 0$
$\left(\frac{d^7S}{d\omega^7}\right) = 0$	$\left(\frac{d^7T}{d\omega^7}\right) = \frac{17\pi^8}{16}$
etc.	etc.

26. Hence therefore with the vanishing orders omitted from the first order we will have the following integral formulas with the series thence arising :

$$\int \frac{dz}{1+zz} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.},$$

$$\int \frac{dz(lz)^2}{1+zz} = \frac{\pi^2}{16} = \frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.},$$

$$\int \frac{dz(lz)^4}{1+zz} = \frac{5\pi^5}{64} = \frac{24}{1^5} - \frac{24}{3^5} + \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} - \frac{24}{11^5} + \frac{24}{13^5} - \text{etc.},$$

$$\int \frac{dz(lz)^6}{1+zz} = \frac{61\pi^5}{256} = \frac{720}{1^7} - \frac{720}{3^7} + \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} - \frac{720}{11^7} + \frac{720}{13^7} - \text{etc.},$$

etc.

27. Moreover from the other order for the same case there arise :

$$\int \frac{-dzlz}{1-zz} = \frac{\pi\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.},$$

$$\int \frac{-dz(lz)^3}{1-zz} = \frac{\pi^4}{16} = \frac{6}{1^4} + \frac{6}{3^4} + \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} + \frac{6}{11^4} + \frac{6}{13^4} + \text{etc.},$$

$$\int \frac{-dz(lz)^5}{1-zz} = \frac{\pi^6}{8} = \frac{120}{1^6} + \frac{120}{3^6} + \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} + \frac{120}{11^6} + \frac{120}{13^6} + \text{etc.},$$

etc.

28. Just as from the first integral we have deduced these formulas from the second order

$$\int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6} \text{ et } \int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$

also similar integral formulas can be deduced from the following; since indeed there shall be $\int \frac{dz(lz)^3}{1-zz} = -\frac{\pi^4}{16}$, we may put to be

$$\int \frac{zdz(lz)^3}{1-zz} = P$$

and there will become

$$\int \frac{dz(lz)^3}{1-z} = P - \frac{\pi^4}{16} \text{ and } \int \frac{dz(lz)^3}{1+z} = -P - \frac{\pi^4}{16};$$

now truly there may be put in place $zz = v$, so that there shall be $zdz = \frac{1}{2}dv$ and $lz = \frac{1}{2}lv$ and thus $(lz)^3 = \frac{1}{8}(lv)^3$, with which substituted there will be

$$P = \frac{1}{16}\int \frac{dv(lv)^3}{1-v} = \frac{1}{16}\left(P - \frac{\pi^4}{16}\right),$$

from which there becomes

$$P = \frac{1}{16}\left(P - \frac{\pi^4}{16}\right) \text{ and thus } P = -\frac{\pi^4}{240},$$

and thus we will have these two new integrations :

$$\int \frac{dz(lz)^3}{1-z} = -\frac{\pi^4}{15} \text{ and } \int \frac{dz(lz)^3}{1+z} = -\frac{7\pi^4}{120};$$

moreover, hence from the series there will become

$$\int \frac{-dz(lz)^3}{1-z} = +\frac{\pi^4}{15} = 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} + \text{etc.}\right)$$

and

$$\int \frac{-dz(lz)^3}{1+z} = +\frac{7\pi^4}{120} = 6\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} - \frac{1}{8^4} + \text{etc.}\right)$$

29. Again $\int \frac{dz(lz)^5}{1-zz} = -\frac{\pi^6}{8}$; we may put

$$\int \frac{zdz(lz)^5}{1-zz} = P,$$

so that hence we may obtain

$$\int\frac{dz(lz)^5}{1-z}=P-\frac{\pi^6}{8} \text{ and } \int\frac{dz(lz)^5}{1+z}=-P-\frac{\pi^6}{8};$$

therefore now we may put $zz=v$ and there will be

$$P=\frac{1}{64}\int\frac{dv(lv)^5}{1-v}=\frac{1}{64}\left(P-\frac{\pi^6}{8}\right),$$

from which there becomes

$$P=-\frac{\pi^6}{504},$$

and hence the new integrations have been deduced:

$$\int\frac{dz(lz)^5}{1-z}=-\frac{8\pi^6}{63} \text{ and } \int\frac{dz(lz)^5}{1+z}=-\frac{31\pi^6}{252},$$

but truly by series there is found :

$$\int\frac{dz(lz)^5}{1-z}=-\frac{8\pi^6}{63}=-120\left(1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\frac{1}{7^6}+\text{etc.}\right)$$

and

$$\int\frac{dz(lz)^5}{1+z}=-\frac{31\pi^6}{252}=-120\left(1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\frac{1}{7^6}-\text{etc.}\right),$$

thus so that there shall become:

$$1+\frac{1}{2^6}+\frac{1}{3^6}+\frac{1}{4^6}+\frac{1}{5^6}+\frac{1}{6^6}+\frac{1}{7^6}+\text{etc}=\frac{\pi^6}{945}$$

and

$$1-\frac{1}{2^6}+\frac{1}{3^6}-\frac{1}{4^6}+\frac{1}{5^6}-\frac{1}{6^6}+\frac{1}{7^6}-\frac{1}{8^6}+\text{etc}=\frac{31\pi^6}{40240}=\frac{31\pi^6}{32\cdot 945}.$$

30. We may consider the case also, in which $\lambda=2$ and $\omega=1$, thus so that there shall be $\alpha=\frac{\pi}{4}$ and $\alpha\omega=\frac{\pi}{4}$, hence $p=q=\frac{1}{\sqrt{2}}$, from which for each order we will have the following values :

For the prior order	For the posterior order
$S = \frac{\pi}{2\sqrt{2}}$	$T = \frac{\pi}{4}$
$\left(\frac{dS}{d\omega}\right) = \frac{\pi\pi}{8\sqrt{2}}$	$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{8}$
$\left(\frac{ddS}{d\omega^2}\right) = \frac{3\pi^3}{32\sqrt{2}}$	$\left(\frac{ddT}{d\omega^2}\right) = \frac{\pi^3}{16}$
$\left(\frac{d^3S}{d\omega^3}\right) = \frac{11\pi^4}{128\sqrt{2}}$	$\left(\frac{d^3T}{d\omega^3}\right) = \frac{\pi^4}{16}$
$\left(\frac{d^4S}{d\omega^4}\right) = \frac{57\pi^5}{512\sqrt{2}}$	$\left(\frac{d^4T}{d\omega^4}\right) = \frac{5\pi^5}{64}$
$\left(\frac{d^5S}{d\omega^5}\right) = \frac{361\pi^6}{2048\sqrt{2}}$	$\left(\frac{d^5T}{d\omega^5}\right) = \frac{\pi^6}{8}$
$\left(\frac{d^6S}{d\omega^6}\right) = \frac{2763\pi^7}{8192\sqrt{2}}$	$\left(\frac{d^6T}{d\omega^6}\right) = \frac{61\pi^7}{256\sqrt{2}}$
$\left(\frac{d^7S}{d\omega^7}\right) = \frac{24611\pi^8}{32768\sqrt{2}}$	$\left(\frac{d^7T}{d\omega^7}\right) = \frac{17\pi^8}{32}$
etc.	etc.

31. Therefore the following integrations hence arise with the corresponding series ; and with the first indeed from the first order :

$$\int \frac{(1+zz)dz}{1+z^4} = \frac{\pi}{2\sqrt{2}} \qquad = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \text{etc.},$$

$$\int \frac{-(1-zz)dzlz}{1+z^4} = \frac{\pi\pi}{2\sqrt{2}} \qquad = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \text{etc.},$$

$$\int \frac{dz(1+zz)(lz)^2}{1+z^4} = \frac{3\pi^3}{32\sqrt{2}} \qquad = \frac{2}{1^3} + \frac{2}{3^3} - \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} + \frac{2}{11^3} - \frac{2}{13^3} - \text{etc.},$$

$$\int \frac{-dz(1-zz)(lz)^3}{1+z^4} = \frac{11\pi^4}{128\sqrt{2}} \qquad = \frac{6}{1^4} - \frac{6}{3^4} - \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} - \frac{6}{11^4} - \frac{6}{13^4} + \text{etc.},$$

$$\int \frac{dz(1+zz)(lz)^4}{1+z^4} = \frac{57\pi^5}{512\sqrt{2}} \qquad = \frac{24}{1^5} + \frac{24}{3^5} - \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} + \frac{24}{11^5} - \frac{24}{13^5} - \text{etc.},$$

$$\int \frac{-dz(1-zz)(lz)^5}{1+z^4} = \frac{361\pi^6}{2048\sqrt{2}} \qquad = \frac{120}{1^6} - \frac{120}{3^6} - \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} - \frac{120}{11^6} - \frac{120}{13^6} + \text{etc.},$$

$$\int \frac{dz(1+zz)(lz)^6}{1+z^4} = \frac{2763\pi^7}{8192\sqrt{2}} \qquad = \frac{720}{1^7} + \frac{720}{3^7} - \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} + \frac{720}{11^7} - \frac{720}{13^7} + \text{etc.},$$

$$\int \frac{-dz(1-zz)(lz)^7}{1+z^4} = \frac{24611\pi^8}{32768\sqrt{2}} = \frac{5040}{1^8} - \frac{5040}{3^8} - \frac{5040}{5^8} + \frac{5040}{7^8} + \frac{5040}{9^8} - \frac{5040}{11^8} - \frac{5040}{13^8} + \text{etc.}$$

etc.

32. In the same manner the other integrations of the other order with the series will be :

$$\int \frac{dz}{1+zz} = \frac{\pi}{4} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \text{etc.},$$

$$\int \frac{-dzlz}{1-zz} = \frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.},$$

$$\int \frac{dz(lz)^2}{1+zz} = \frac{\pi^3}{16} = \frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.},$$

$$\int \frac{-dz(lz)^3}{1-zz} = \frac{\pi^4}{16} = \frac{6}{1^4} + \frac{6}{3^4} + \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} + \frac{6}{11^4} + \frac{6}{13^4} + \text{etc.},$$

$$\int \frac{dz(lz)^4}{1+zz} = \frac{5\pi^5}{64} = \frac{24}{1^5} - \frac{24}{3^5} + \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} - \frac{24}{11^5} - \frac{24}{13^5} - \text{etc.},$$

$$\int \frac{-dz(lz)^5}{1-zz} = \frac{\pi^6}{8} = \frac{120}{1^6} + \frac{120}{3^6} + \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} + \frac{120}{11^6} + \frac{120}{13^6} + \text{etc.},$$

$$\int \frac{dz(lz)^6}{1+zz} = \frac{61\pi^7}{256} = \frac{720}{1^7} - \frac{720}{3^7} + \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} - \frac{720}{11^7} + \frac{720}{13^7} - \text{etc.},$$

$$\int \frac{-dz(lz)^7}{1-zz} = \frac{17\pi^8}{32} = \frac{5040}{1^8} + \frac{5040}{3^8} + \frac{5040}{5^8} + \frac{5040}{7^8} + \frac{5040}{9^8} + \frac{5040}{11^8} + \frac{5040}{13^8} + \text{etc.}$$

etc.

But these series are these themselves, which we have attended to now above in (§ 26 and 27).

33. But besides these, the cases deserve to be noted especially, in which the integral formulas can be resolved into simpler forms. But this resolution is considered only for the fraction

$$\pm \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}},$$

with the factor $\frac{dz}{z}(lz)^{\mu}$ omitted, for which we may take initially to be shown with $\lambda = 3$ and $\omega = 1$, from which there becomes $\alpha = \frac{\pi}{6}$, $p = \sin.\frac{\pi}{6}$ and $q =$ $\cos.\frac{\pi}{6}$; moreover as in the prior order the following alternating fractions occur:

$$\text{I.} \quad \frac{zz(1+zz)}{1+z^6} = \frac{zz}{1-zz+z^4},$$

which on putting $zz = v$ will be changed into $\frac{v}{1-v+vv}$; therefore since there shall be $\frac{dz}{z} = \frac{1}{2}\frac{dv}{v}$ and $lz = \frac{1}{2}lv$, hence such a form will be able to be integrated:

$$\frac{1}{2^{2i+1}} \int \frac{dv(lv)^{2i}}{1-v+vv},$$

evidently in the case $v=1$;

$$\text{II.} \quad -\frac{zz(1-zz)}{1+z^6} = +\frac{2}{3(1+zz)} - \frac{2-zz}{3\left(1-zz+z^4\right)},$$

which on putting $zz = v$ will change into

$$\frac{2}{3(1+v)} - \frac{2-v}{3(1-v+vv)},$$

therefore which form multiplied by $\frac{dz}{z}(lz)^{2i+1}$ or by $\frac{1}{2^{2i+1}}\frac{dv}{v}(lv)^{2i+1}$ can be integrated always on putting $v=1$.

34. From the same case the posterior order supplies the following resolutions :

$$\text{I.} \quad \frac{zz(1-zz)}{1-z^6} = \frac{zz}{1+zz+z^4} = \frac{vv}{1+v+vv},$$

which multiplied by $\frac{dz}{z}(lz)^{2i}$, or by $\frac{1}{2^{2i+1}}\frac{dv}{v}(lv)^{2i}$, is integrable always;

$$\text{II.} \quad \frac{-zz(1+zz)}{1-z^6} = \frac{-2}{3(1-zz)} + \frac{2+zz}{3\left(1+zz+z^4\right)},$$

which on making $zz = v$ becomes

$$\frac{-2}{3(1-v)} + \frac{2+v}{3(1+v+vv)},$$

which formulas therefore multiplied by $\frac{dv}{v}(lv)^{2i+1}$ become integrable; but since in that resolution the numerators cannot be divided by *z* or *v*, there is a need for another resolution, which is found :

$$\frac{-zz(1+zz)}{1-z^6} = \frac{-2zz}{3(1-zz)} - \frac{zz(2+zz)}{3\left(1+zz+z^4\right)}$$

or

$$\frac{-2v}{3(1-v)} - \frac{v(2+v)}{3(1+v+vv)},$$

which formulas multiplied by $\frac{dz}{z}(lz)^{2i+1}$ or by $\frac{1}{2^{2i+2}}\frac{dv}{v}(lv)^{2i+1}$ also can be integrated .

35. Again with $\lambda = 3$ there may be taken $\omega = 2$, so that there shall be

$$\alpha = \tfrac{\pi}{6},\ p = \sin.\tfrac{\pi}{3} \text{ and } q = \cos.\tfrac{\pi}{3},$$

and from the first order the following reductions may arise

$$\text{I.}\quad \frac{z(1+z^4)}{1+z^6} = \frac{2z}{3(1+zz)} + \frac{z(1+zz)}{3(1-zz+z^4)},$$

from which on multiplying by $\frac{dz}{z}(lz)^{2i}$ allowing the following integral formulas to arise in the case $z = 1$;

$$\text{II.}\quad \frac{-z(1-z^4)}{1+z^6} = -\frac{z(1-zz)}{1-zz+z^4},$$

which multiplied by $\frac{dz}{z}(lz)^{2i+1}$ will be able to be integrated in the case $z = 1$.

Truly from the latter order the following reductions will be produced :

$$\text{I.}\quad \frac{z(1-z^4)}{1-z^6} = \frac{z(1+zz)}{1+zz+z^4},$$

which multiplied by $\frac{dz}{z}(lz)^{2i}$ becomes integrable;

$$\text{II.}\quad \frac{-z(1+z^4)}{1-z^6} = \frac{-2z}{3(1-zz)} - \frac{z(1-zz)}{3(1+zz+z^4)},$$

which formulas multiplied by $\frac{dz}{z}(lz)^{2i+1}$ shall become integrable.

36. Now it will be worth the effort to actually set out this integration, whereby from §. 33 and from its number I, we arrive at the following integrations :

$$1^0.\quad \tfrac{1}{2}\int \frac{dv}{1-v+vv} = \alpha \frac{1}{q} = \frac{\pi}{3\sqrt{3}},$$

$$2^0.\quad \tfrac{1}{8}\int \frac{dv(lv)^2}{1-v+vv} = \alpha^3 \left(\frac{2}{q^3} \frac{1}{-q} \right) = \frac{5\pi^3}{324\sqrt{3}},$$

then truly from number II from the same paragraph, where this reduction can be found also:

$$-\frac{zz(1-zz)}{1+z^6} = -\frac{zz}{3(1+zz)} - \frac{zz(1-2zz)}{3\left(1-zz+z^4\right)} = -\frac{v}{3(1+v)} - \frac{v(1-2v)}{3(1-v+vv)},$$

which multiplied by $\frac{1}{4} \cdot \frac{dv}{v} lv$ will give :

$$-\frac{1}{6}\int \frac{dvlv}{(1+v)} - \frac{1}{12}\int \frac{dv(1-2v)lv}{1-v+vv} = \alpha\alpha \frac{p}{qq} = \frac{\pi\pi}{54},$$

of which formulas the first order allows the integration; indeed there is :

$$\int \frac{dvlv}{(1+v)} = -\frac{\pi\pi}{12},$$

from the latter order there is found

$$\int \frac{dv(1-2v)lv}{1-v+vv} = -\frac{\pi\pi}{18}.$$

37. From number I of § 34 there follows:

$$1^0. \ \frac{1}{2}\int \frac{dv}{1+v+vv} = \alpha \frac{p}{q} = \frac{\pi}{6\sqrt{3}},$$

$$2^0. \ \frac{1}{8}\int \frac{dv(lv)^2}{1+v+vv} = \alpha^3 \frac{2p}{q^3} = \frac{\pi^3}{81\sqrt{3}},$$

then truly from number II there becomes

$$-\frac{1}{6}\int \frac{dvlv}{1-v} - \frac{1}{12}\int \frac{dv(1+2v)lv}{1+v+vv} = \alpha\alpha \frac{1}{qq} = \frac{\pi\pi}{27};$$

but above we have found to be

$$\int \frac{dvlv}{1-v} = -\frac{\pi\pi}{6},$$

with which value substituted there becomes

$$\int \frac{dv(1+2v)lv}{1+v+vv} = -\frac{\pi\pi}{9};$$

therefore it has been observed to be well worth the effort to have set out these last integrations.

38. But if both the integral formulas

$$\int \frac{dv(1-2v)lv}{1-v+vv} \quad \text{and} \quad \int \frac{dv(1+2v)lv}{1+v+vv}$$

may be changed into series, there is found :

$$\int \frac{dv(1-2v)lv}{1-v+vv} = -1 + \frac{1}{4} + \frac{2}{9} + \frac{1}{16} - \frac{1}{25} - \frac{2}{36} - \frac{1}{49} + \text{etc.}$$

and

$$\int \frac{dv(1+2v)lv}{1+v+vv} = -1 - \frac{1}{4} + \frac{2}{9} - \frac{1}{16} - \frac{1}{25} + \frac{2}{36} - \frac{1}{49} + \text{etc.},$$

we proceed from these two summations worthy of our attention:

$$\text{I.} \quad 1 - \frac{1}{4} - \frac{2}{9} - \frac{1}{16} + \frac{1}{25} + \frac{2}{36} + \frac{1}{49} - \frac{1}{64} - \frac{2}{81} - \frac{1}{100} + \text{etc.} = \frac{\pi\pi}{18},$$

$$\text{II.} \; 1 + \frac{1}{4} - \frac{2}{9} + \frac{1}{16} + \frac{1}{25} - \frac{2}{36} + \frac{1}{49} + \frac{1}{64} - \frac{2}{81} + \frac{1}{100} + \text{etc.} = \frac{\pi\pi}{9},$$

of which the first taken from the second produces

$$\frac{2}{4} + \frac{2}{16} - \frac{4}{36} + \frac{2}{64} + \frac{2}{100} - \text{etc.} = \frac{\pi\pi}{18},$$

of which the double leads to this

$$1 + \frac{1}{4} - \frac{2}{9} + \frac{1}{16} + \frac{1}{25} - \frac{2}{36} + \text{etc.} = \frac{\pi\pi}{9};$$

which since it agrees with the second, the truth of each summation is confirmed well enough ; but if truly the second may be taken from twice the first, this memorable series will remain

$$1 - \frac{3}{4} - \frac{2}{9} - \frac{3}{16} + \frac{1}{25} + \frac{6}{36} + \frac{1}{49} - \frac{3}{64} - \frac{2}{81} - \frac{3}{100} + \text{etc.} = 0.$$

which distributed in 6 term periods clearly establishes the order of the numerators, which are clearly $1, \; -3, \; -2, \; -3, \; +1, \; +6.$

ADDENDUM

39. Just as we have deduced the above integrations by continued differentiations of the formulas S and T, thus also by integration we will obtain other and certainly singular integrations; if indeed as above [§ 3] there were

$$S = \int P \frac{dz}{z}$$

with P being that formula present

$$\frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}},$$

which besides z also is considered to involve the variable exponent ω, will become by the nature of integrations involving two variables

$$\int S d\omega = \int \frac{dz}{z} \int P d\omega,$$

where in the first formula of the integral $\int S d\omega$, where z is taken as constant, at once can be written $z = 1$; therefore with this lemma established, since there shall be

$$\int P d\omega = \frac{-z^{\lambda-\omega} \pm z^{\lambda+\omega}}{\left(1 \pm z^{2\lambda}\right) lz},$$

we will set out both the formulas treated above in this manner, evidently S and T, and because we have given each in three ways, evidently the first by an infinite series, the second by a finite formula, and the third by an integral formula, also the quantities which will result for the integrals $\int S d\omega$ and $\int T d\omega$, will be equal between themselves.

40. We may begin from the formula S, and since from the series there would have been

$$S = \frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega} - \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} + \frac{1}{5\lambda+\omega} + \text{etc.}$$

there will be

$$\int S d\omega = -l(\lambda - \omega) + l(\lambda + \omega) + l(3\lambda - \omega) - l(3\lambda + \omega) - \text{etc.} + C,$$

as it is fitting to define the constant thus, so that the integral may vanish on putting $\omega = 0$, with which done there will become :

$$\int S d\omega = l \frac{\lambda+\omega}{\lambda-\omega} + l \frac{3\lambda-\omega}{3\lambda+\omega} + l \frac{5\lambda+\omega}{5\lambda-\omega} + l \frac{7\lambda-\omega}{7\lambda+\omega} + \text{etc.},$$

which expression is reduced to the following :

$$\int S d\omega = l \frac{(\lambda+\omega)(3\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega)(9\lambda+\omega)\cdot \text{etc.}}{(\lambda-\omega)(3\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega)(9\lambda-\omega)\cdot \text{etc.}}.$$

Then since by the finite formula there was $S = \frac{\pi}{2\lambda \cos.\frac{\pi\omega}{2\lambda}}$, there will be

$$\int S d\omega = \int \frac{\pi d\omega}{2\lambda \cos.\frac{\pi\omega}{2\lambda}},$$

where if for the sake of brevity there may be put $\frac{\pi\omega}{2\lambda} = \varphi$, so that there shall be $d\omega = \frac{2\lambda d\varphi}{\pi}$, there will be :

$$\int S d\omega = \int \frac{d\varphi}{\cos.\varphi};$$

therefore since we know to be

$$\int \frac{d\theta}{\sin.\theta} = l\text{tang.}\tfrac{1}{2}\theta,$$

we may take $\sin.\theta = \cos.\varphi$ or $\theta = 90^0 - \varphi = \frac{\pi}{2} - \varphi$ and there will be $d\theta = -d\varphi$, from which there shall be

$$\int \frac{-d\varphi}{\cos.\varphi} = l\,\text{tang.}\left(\frac{\pi}{4} - \frac{1}{2}\varphi\right);$$

but since there is $\varphi = \frac{\pi\omega}{2\lambda}$, there will be

$$\frac{\pi}{4} - \frac{1}{2}\varphi = \frac{\pi(\lambda-\omega)}{4\lambda},$$

from which our integral is

$$\int S d\omega = -l\,\text{tang.}\frac{\pi(\lambda-\omega)}{4\lambda} = +l\,\text{tang.}\frac{\pi(\lambda+\omega)}{4\lambda};$$

but from the third formula of the integral :

$$S = \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}$$

there is gathered to be

$$\int S d\omega = \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{zlz},$$

which integral is assumed to extend from the limit $z = 0$ as far as to the limit $z = 1$; and thus these same three values found will be equal to each other. And lest perhaps on account of the constants requiring to be added, any doubt will be overcome, these individual expressions vanish at once in the case $\omega = 0$.

41. Hence we will consider initially the equation between the first and second formulas, and because each is a logarithm, there will be

$$\text{tang.} \frac{\pi(\lambda-\omega)}{4\lambda} = \frac{(\lambda+\omega)(3\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega) \cdot \text{etc.}}{(\lambda-\omega)(3\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega) \cdot \text{etc.}};$$

therefore since the numerator of this fraction shall vanish in the cases where either $\omega = -\lambda$, $\omega = +3\lambda$, $\omega = -5\lambda$, or $\omega = +7\lambda$ etc., it is evident in the same cases the tangent becomes $= 0$ also ; truly the denominator will vanish in the cases $\omega = \lambda$, $\omega = -3\lambda$, $\omega = 5\lambda$, $\omega = -7\lambda$ etc., therefore in which cases the tangent must increase to infinity, which also eventuates most beautifully. Moreover, this expression agrees with that, as I have found and explained now some time ago in the *Introductione*.

42. But this same product can be reduced to integral formulas from principles established elsewhere [see E59] with the aid of this lemma of the widest extent

$$\frac{a(c+b)(a+k)(c+b+k)(a+2k)(c+b+2k) \text{ etc.}}{b(c+a)(b+k)(c+a+k)(b+2k)(c+a+2k) \text{ etc.}} = \frac{\int z^{c-1} dz \left(1-z^k\right)^{\frac{b-k}{k}}}{\int z^{c-1} dz \left(1-z^k\right)^{\frac{a-k}{k}}},$$

if indeed after each integration there may be become $z = 1$. Therefore in our case there will become

$$a = \lambda + \omega,\ b = \lambda - \omega,\ c = 2\lambda \text{ and } k = 4\lambda,$$

from which value of our product will become

$$\frac{\int z^{2\lambda-1} dz \left(1-z^{4\lambda}\right)^{\frac{-3\lambda-\omega}{4\lambda}}}{\int z^{2\lambda-1} dz \left(1-z^{4\lambda}\right)^{\frac{-3\lambda+\omega}{4\lambda}}} = \text{tang.} \frac{\pi(\lambda+\omega)}{4\lambda};$$

but these integral formulas will emerge more concisely by setting $z^{2\lambda} = y$; then indeed here will be

$$\text{tang.} \frac{\pi(\lambda+\omega)}{4\lambda} = \frac{\int dy (1-yy)^{\frac{-3\lambda-\omega}{4\lambda}}}{\int dy (1-yy)^{\frac{-3\lambda+\omega}{4\lambda}}},$$

which expression certainly may be considered to be worthy of all attention. And finally from the integral formula there will be found also :

$$\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{zlz} = \text{tang.} \frac{\pi(\lambda+\omega)}{4\lambda}.$$

43. It will be worthwhile also to set out some particular cases. Therefore initially let $\lambda = 2$ and $\omega = 1$, and by the infinite expression there will be

$$\int S d\omega = l \frac{3 \cdot 5}{1 \cdot 7} \cdot \frac{11 \cdot 13}{9 \cdot 15} \cdot \frac{19 \cdot 21}{17 \cdot 23} \cdot \frac{27 \cdot 29}{25 \cdot 31} \cdot \frac{35 \cdot 37}{33 \cdot 39} \cdot \text{etc.},$$

then by the finite expression we will have

$$\int S d\omega = l \, \text{tang.} \frac{3\pi}{8}$$

and by the integral formula :

$$\int S d\omega = \int \frac{-(1-zz)}{1+z^4} \cdot \frac{dz}{lz},$$

then truly from the equality of these two first expressions :

$$\text{tang.} \frac{3\pi}{8} = \frac{3 \cdot 5}{1 \cdot 7} \cdot \frac{11 \cdot 13}{9 \cdot 15} \cdot \frac{19 \cdot 21}{17 \cdot 23} \cdot \text{etc.},$$

and hence by the two integral formulas,

$$\text{tang.} \frac{3\pi}{8} = \frac{\int dy (1-yy)^{-\frac{7}{8}}}{\int dy (1-yy)^{-\frac{5}{8}}}.$$

44. We may now put to be $\lambda = 3$ and $\omega = 1$ and by the infinite expression there will become

$$\int S d\omega = l \frac{2 \cdot 4}{1 \cdot 5} \cdot \frac{8 \cdot 10}{7 \cdot 11} \cdot \frac{14 \cdot 16}{13 \cdot 17} \cdot \frac{20 \cdot 22}{19 \cdot 23} \cdot \text{etc.},$$

secondly by the finite expression:

$$\int S d\omega = l \, \text{tang.} \frac{\pi}{3} = l\sqrt{3} = \frac{1}{2} l3,$$

thus so that there shall become:

$$\sqrt{3} = \frac{2 \cdot 4}{1 \cdot 5} \cdot \frac{8 \cdot 10}{7 \cdot 11} \cdot \frac{14 \cdot 16}{13 \cdot 17} \cdot \text{etc.},$$

and the value of this product by the integral formulas will become :

$$\frac{\int dy(1-yy)^{-\frac{5}{6}}}{\int dy(1-yy)^{-\frac{2}{3}}}.$$

And finally the integral formula will produce

$$\int Sd\omega=\int\frac{-(1-zz)}{1+z^6}\cdot\frac{dz}{lz}.$$

45. In the same way we may set out the other formula T also, the value of which by the series was :

$$T=\frac{1}{\lambda-\omega}-\frac{1}{\lambda+\omega}+\frac{1}{3\lambda-\omega}-\frac{1}{3\lambda+\omega}+\frac{1}{5\lambda-\omega}-\frac{1}{5\lambda+\omega}+\text{etc.};$$

from which there becomes

$$\int Td\omega=-l(\lambda-\omega)-l(\lambda+\omega)-l(3\lambda-\omega)-l(3\lambda+\omega)-\text{etc.};$$

so that which expression may vanish on putting $w=0$, there will be

$$\int Td\omega=l\frac{\lambda\lambda}{\lambda\lambda-\omega\omega}\cdot\frac{\lambda\lambda}{9\lambda-\omega\omega}\cdot\frac{\lambda\lambda}{25\lambda-\omega\omega}\cdot\text{etc.};$$

then truly since by the finite formula there would become $T=\frac{\pi}{2\lambda}\text{tang.}\frac{\pi\omega}{2\lambda}$, there will be

$$\int Td\omega=\int\frac{\pi d\omega}{2\lambda}\text{tang.}\frac{\pi\omega}{2\lambda},$$

where on putting $\frac{\pi\omega}{2\lambda}=\varphi$, there will be

$$\int Td\omega=\int d\varphi\text{tang.}\varphi=-l\cos\varphi,$$

thus so that there shall be

$$\int Td\omega=-l\cos\frac{\pi\omega}{2\lambda},$$

of which the value in the case $\omega=0$ at once shall be $=0$; and finally by the integral formula we will have :

$$\int Td\omega=\int\frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{zlz};$$

likewise the integral must be extended from the limit $z=0$ as far as to the limit $z=1$.

46. Now from the comparison of the first two values, this equation is produced :

$$\frac{1}{\cos.\frac{\pi\omega}{2\lambda}} = \frac{\lambda\lambda}{\lambda\lambda-\omega\omega} \cdot \frac{9\lambda\lambda}{9\lambda-\omega\omega} \cdot \frac{25\lambda\lambda}{25\lambda-\omega\omega} \cdot \frac{49\lambda\lambda}{49\lambda-\omega\omega} \cdot \text{etc.},$$

$$\cos.\frac{\pi\omega}{2\lambda} = \left(1-\frac{\omega\omega}{\lambda\lambda}\right) \cdot \left(1-\frac{\omega\omega}{9\lambda\lambda}\right) \cdot \left(1-\frac{\omega\omega}{25\lambda\lambda}\right) \cdot \left(1-\frac{\omega\omega}{49\lambda\lambda}\right) \cdot \text{etc.},$$

or again if the individual factors may be set out simply, there will become :

$$\cos.\frac{\pi\omega}{2\lambda} = \frac{\lambda+\omega}{\lambda} \cdot \frac{\lambda-\omega}{\lambda} \cdot \frac{3\lambda+\omega}{3\lambda} \cdot \frac{3\lambda-\omega}{3\lambda} \cdot \frac{5\lambda+\omega}{5\lambda} \cdot \frac{5\lambda-\omega}{5\lambda} \cdot \text{etc.},$$

which formula compared with the general reduction provided above [§ 42] gives $a = \lambda+\omega,\ b = \lambda,\ c = -\omega$ and $k = 2\lambda,$ from which we deduce

$$\cos.\frac{\pi\omega}{2\lambda} = \int \frac{z^{-\omega-1} dz \left(1-z^{2\lambda}\right)^{-\frac{1}{2}}}{z^{-\omega-1} dz \left(1-z^{2\lambda}\right)^{\frac{\omega-\lambda}{2\lambda}}}.$$

But as we may discount the negative exponents $z^{-\omega-1}$, we may represent the above product thus :

$$\cos.\frac{\pi\omega}{2\lambda} = \frac{\lambda-\omega}{\lambda} \cdot \frac{\lambda+\omega}{\lambda} \cdot \frac{3\lambda-\omega}{\lambda} \cdot \frac{3\lambda+\omega}{\lambda} \cdot \text{etc.}$$

and with the comparison made there shall be $a = \lambda-\omega,\ b = \lambda,\ c = +\omega$ and $k = 2\lambda$, and thus by the integral formulas there will be :

$$\cos.\frac{\pi\omega}{2\lambda} = \int \frac{z^{-\omega-1} dz \left(1-z^{2\lambda}\right)^{-\frac{1}{2}}}{z^{\omega-1} dz \left(1-z^{2\lambda}\right)^{\frac{-\lambda-\omega}{2\lambda}}},$$

which expression cannot be reduced to a simpler form.

47. Now also, let $\lambda = 2$ and $\omega = 1$ and our three expressions will become :

I. $\int T d\omega = l \frac{4}{3} \cdot \frac{36}{35} \cdot \frac{100}{99} \cdot \frac{196}{195} \cdot \text{etc.},$ or $\int T d\omega = l \frac{2\cdot2}{1\cdot3} \cdot \frac{6\cdot6}{5\cdot7} \cdot \frac{10\cdot10}{9\cdot11} \cdot \frac{14\cdot14}{13\cdot15} \cdot \text{etc.};$

II. $\int T d\omega = -l \cos.\frac{\pi}{4} = +\frac{1}{2} l2,$

so that there shall become $\frac{1}{2}\sqrt{2} = \frac{2\cdot2}{1\cdot3} \cdot \frac{6\cdot6}{5\cdot7} \cdot \frac{10\cdot10}{9\cdot11} \cdot \frac{14\cdot14}{13\cdot15} \cdot \text{etc.}$

so that the product is expressed thus by the integral formulas

$$\frac{\int dz\left(1-z^4\right)^{-\frac{3}{4}}}{\int dz\left(1-z^4\right)^{-\frac{1}{2}}} = \sqrt{2};$$

$$\text{III.} \int Td\omega = \int \frac{-(1+zz)}{\left(1-z^4\right)} \cdot \frac{dz}{lz} = \int \frac{-dz}{(1-zz)lz},$$

which therefore extended from the limit $z = 0$ as far as to $z = 1$ will present the same value $+\frac{1}{2}l2$, the equality of which ratio certainly may appear most difficult.

48. And finally as above let $\lambda = 3$ and $\omega = 1$ and the third formula thus may itself be had :

$$\text{I.} \int Td\omega = l\,\frac{9}{8} \cdot \frac{81}{80} \cdot \frac{225}{224} \cdot \text{etc.} = l\,\frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{9 \cdot 9}{8 \cdot 10} \cdot \frac{15 \cdot 15}{14 \cdot 16} \cdot \frac{21 \cdot 21}{20 \cdot 22} \cdot \text{etc.};$$

$$\text{II.} \int Td\omega = -l\cos.\frac{\pi}{6} = -l\,\frac{\sqrt{3}}{2} = +l\,\frac{2}{\sqrt{3}},$$

thus so that there shall be $\frac{2}{\sqrt{3}} = \frac{3 \cdot 3}{2 \cdot 4} \cdot \frac{9 \cdot 9}{8 \cdot 10} \cdot \frac{15 \cdot 15}{14 \cdot 16} \cdot \frac{21 \cdot 21}{20 \cdot 22} \cdot \text{etc.}$

and thus by the two integral formulas :

$$\frac{2}{\sqrt{3}} = \frac{\int dz\left(1-z^6\right)^{-\frac{1}{2}}}{\int dz\left(1-z^6\right)^{-\frac{2}{3}}};$$

$$\text{III.} \int Td\omega = \int \frac{-(1+zz)}{\left(1-z^6\right)} \cdot \frac{dz}{lz},$$

which on putting $zz = v$ will change into this

$$\text{III.} \int Td\omega = \int \frac{-dv(1+v)}{\left(1-v^3\right)lv}.$$

Hence therefore it is plainly apparent to come upon new integral formulas by this method, which cannot be set out in any other way by methods known at this stage, or at any rate be allowed to be compared with each other.

DE VALORE FORMULAE INTEGRALIS
$$\int \frac{z^{\lambda-\omega}\pm z^{\lambda+\omega}}{1\pm z^{2\lambda}}\cdot\frac{dz}{z}(lz)^{\mu}$$
CASU QUO POST INTEGRATIONEM PONITUR $z=1$

Commentatio 463 indicis ENESTROEMIANI

Novi commentarii academiae scientiarum Petropolitanae 19 (1774), 1775, p. 30-65

1. Ex consideratione innumerabilium arcuum circularium, qui communem habent vel sinum vel tangentem, iam olim summationem duarum serierum infinitarum deduxi, quae ob summam generalitatem maxime memoratu dignae videbantur. Si enim litterae m et n numeros quoscunque denotant, posita diametri ratione ad peripheriam ut 1 ad π illae duae summationes hoc modo se habebant

$$\frac{1}{m}+\frac{1}{n-m}-\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}+\frac{1}{3n-m}-\text{etc.}=\frac{\pi}{n\sin.\frac{m\pi}{n}},$$

et

$$\frac{1}{m}-\frac{1}{n-m}+\frac{1}{n+m}-\frac{1}{2n-m}+\frac{1}{2n+m}-\frac{1}{3n-m}+\text{etc.}=\frac{\pi}{n\,\text{tang.}\frac{m\pi}{n}},$$

atque ex his duabus seriebus iam tum temporis elicueram summationes omnium serierum illarum, quarum denominatores secundum potestates numerorum naturalium progrediuntur, quemadmodum in *Introductione in analysin infinitorum* et alibi fusius exposui. Nunc autem eaedem series me perduxerunt ad integrationem formulae in titulo expressae, quae eo magis attentione digna videtur, quod huiusmodi integrationes aliis methodis neutiquam ex sequi liceat.

2. Statim autem patet has duas series infinitas oriri ex evolutione quarundam formularum integralium, si post integrationem quantitati variabili certus valor, veluti unitas, tribuatur; ita prior series deducitur ex evolutione huius formulae integralis

$$\int \frac{z^{m-1}+z^{n-m-1}}{1+z^{n}}\cdot dz,$$

posterior vero ex evolutione istius

$$\int \frac{z^{m-1}-z^{n-m-1}}{1-z^{n}}\cdot dz$$

siquidem post integrationem statuatur $z=1$. Deinceps autem ex ipsis principiis calculi integralis demonstravi valorem integralis prioris harum duarum formularum, siquidem ponatur $z=1$, reduci ad hanc formulam simplicem

$$\frac{\pi}{n \sin . \frac{m\pi}{n}}$$

integrale autem posterius eodem casu $z = 1$ ad istam

$$\frac{\pi}{n \text{tang.} \frac{m\pi}{n}},$$

ita ut ex ipsis calculi integralis principiis certum sit esse

$$\int \frac{z^{m-1} + z^{n-m-1}}{1 + z^n} \cdot dz = \frac{\pi}{n \sin . \frac{m\pi}{n}},$$

siquidem post integrationem ita institutam, ut integrale evanescat posito $z = 0$, statuatur $z = 1$.

3. Quo iam hanc duplicem integrationem ad formam propositam reducamus, faciamus $n = 2\lambda$ et $m = \lambda - \omega$, unde binae illae series infinitae hanc induent formam

$$\frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega} - \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} + \frac{1}{5\lambda+\omega} - \text{etc.}$$

et

$$\frac{1}{\lambda-\omega} - \frac{1}{\lambda+\omega} + \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} - \frac{1}{5\lambda+\omega} + \text{etc.}$$

harum igitur serierum prioris summa erit

$$\frac{\pi}{2\lambda \sin . \frac{\pi(\lambda-\omega)}{2\lambda}} = \frac{\pi}{2\lambda \cos . \frac{\pi\omega}{2\lambda}},$$

posterioris vero summa erit

$$\frac{\pi}{2\lambda \text{tang.} \frac{\pi(\lambda-\omega)}{2\lambda}} = \frac{\pi}{2\lambda \text{cotang.} \frac{\pi\omega}{2\lambda}} = \frac{\pi}{2\lambda} \text{tang.} \frac{\pi\omega}{2\lambda}$$

Quodsi ergo brevitatis gratia ponamus

$$\frac{\pi}{2\lambda \cos . \frac{\pi\omega}{2\lambda}} = S \text{ et } \frac{\pi}{2\lambda} \text{tang.} \frac{\pi\omega}{2\lambda} = T,$$

habebimus sequentes duas integrationes

$$\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z} = S \text{ et } \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} = T.$$

4. Circa has binas integrationes ante omnia observo eas perinde locum habere, sive pro litteris λ et ω accipiantur numeri integri sive fracti. Sint enim λ et ω numeri fracti

quicunque, qui evadant integri, si multiplicentur per α, quo posito fiat $z = x^{\alpha}$ eritque $\frac{dz}{z} = \frac{\alpha dx}{x}$ et potestas quaecunque $z^{\theta} = x^{\alpha\theta}$; prior igitur formula erit

$$\int \frac{x^{\alpha(\lambda-\omega)} + x^{\alpha(\lambda+\omega)}}{1 + x^{2\alpha\lambda}} \cdot \frac{\alpha dx}{x};$$

ubi cum iam omnes exponentes sint numeri integri, valor huius formulae posito post integrationem $x = 1$, quandoquidem tunc etiam sit $z = 1$, a praecedente eo tantum differt, quod hic habeamus $\alpha\lambda$ et $\alpha\omega$ loco λ et ω ac praeterea hic adsit factor α, quocirca valor istius formulae erit

$$\alpha \frac{\pi}{2\alpha\lambda \cos.\frac{\pi\omega}{2\lambda}} = \frac{\pi}{2\lambda \cos.\frac{\pi\omega}{2\lambda}},$$

qui ergo valor est $= S$ prorsus ut ante; quae identitas etiam manifesto est in altera formula, unde patet, etiamsi pro λ et ω fractiones quaecunque accipiantur, integrationem hic exhibitam nihilo minus locum esse habituram; quae circumstantia probe notari meretur, quoniam in sequentibus litteram ω tanquam variabilem sumus tractaturi.

5. Postquam igitur binae istae formulae integrales litteris *S* et *T* indicatae fuerint integratae ita, ut evanescant posito $z = 0$, integralia spectari poterunt non solum ut functiones quantitatis *z*, sed etiam ut functiones binarum variabilium *z* et ω, quandoquidem numerum ω tanquam quantitatem variabilem tractare licet; quin etiam exponentem λ pro quantitate variabili habere liceret; sed quia hinc formulae integrales alius generis essent prodituræ, atque hic contemplari constitui, solam quantitatem ω praeter ipsam variabilem *z* hic ut quantitatem variabilem sum tractaturus.

6. Cum igitur sit

$$S = \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z},$$

in qua integratione sola *z* ut variabilis spectatur, erit utique secundum signandi morem iam satis usu receptum

$$\left(\frac{dS}{dz}\right) = \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{1}{z};$$

haec iam formula denuo differentietur posita sola littera ω variabili eritque

$$\left(\frac{ddS}{dzd\omega}\right) = \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{1}{z} lz,$$

quae formula ducta in *dz* ac denuo integrata sola *z* habita pro variabili dabit

$$\int dz \left(\frac{ddS}{dzd\omega}\right) = \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z} lz,$$

ubi notetur esse

$$S = \frac{\pi}{2\lambda \cos . \frac{\pi\omega}{2\lambda}},$$

ita ut hinc deducamus

$$\left(\frac{dS}{d\omega}\right) = \frac{\pi\pi \sin . \frac{\pi\omega}{2\lambda}}{4\lambda\lambda \cos^2 . \frac{\pi\omega}{2\lambda}};$$

hoc igitur valore substituto nanciscimur hanc integrationem

$$\int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z} lz = \frac{\pi\pi \sin . \frac{\pi\omega}{2\lambda}}{4\lambda\lambda \cos^2 . \frac{\pi\omega}{2\lambda}}.$$

7. Quodsi iam altera formula simili modo tractetur, cum sit

$$T = \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda},$$

erit

$$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{4\lambda\lambda \cos^2 . \frac{\pi\omega}{2\lambda}},$$

ex formula autem integrali erit

$$\left(\frac{dT}{d\omega}\right) = \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} lz,$$

unde colligimus sequentem integrationem

$$\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} lz = \frac{-\pi\pi}{4\lambda\lambda \cos^2 . \frac{\pi\omega}{2\lambda}}.$$

8. Quoniam litteras *S* et *T* etiam per series expressas dedimus, erit etiam per similes series

$$\left(\frac{dS}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} - \frac{1}{(\lambda+\omega)^2} - \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} - \text{etc.}$$
$$= \frac{\pi\pi \sin . \frac{\pi\omega}{2\lambda}}{4\lambda\lambda \cos^2 . \frac{\pi\omega}{2\lambda}}.$$

Similique modo etiam pro altera serie

$$\left(\frac{dT}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} + \frac{1}{(\lambda+\omega)^2} + \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} + \text{etc.}$$
$$= \frac{\pi\pi}{4\lambda\lambda \cos^2 \cdot \frac{\pi\omega}{2\lambda}},$$

sicque summas harum serierum quoque duplici modo repraesentavimus, scilicet per formulam evolutam quantitatem π involventem, tum vero etiam per formulam integralem, quae ita est comparata, ut eius integrale nulla methodo adhuc consueta assignari possit.

9. Applicemus has integrationes ad aliquot casus particulares; ac primo quidem sumamus $\omega = 0$, quo quidem casu prior integratio sponte in oculos incurrit, at posterior praebet

$$\int \frac{2z^{\lambda}}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz = -\frac{\pi\pi}{4\lambda\lambda}$$

sive

$$\int \frac{z^{\lambda-1} dz lz}{1-z^{2\lambda}} = -\frac{\pi\pi}{8\lambda\lambda}$$

hincque simul istam summationem adipiscimur

$$\frac{1}{\lambda\lambda} + \frac{1}{\lambda\lambda} + \frac{1}{9\lambda\lambda} + \frac{1}{25\lambda\lambda} + \frac{1}{25\lambda\lambda} + \text{etc.} = \frac{\pi\pi}{4\lambda\lambda}$$

sive

$$1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \frac{1}{81} + \frac{1}{121} + \frac{1}{169} + \text{etc.} = \frac{\pi\pi}{8},$$

id quod iam dudum a me est demonstratum.

10. Hic statim patet perinde esse, quinam numerus pro λ accipiatur; sit igitur $\lambda = 1$ et habebitur ista integratio

$$\int \frac{dzlz}{1-z^2} = -\frac{\pi\pi}{8},$$

ex qua sequentia integralia simpliciora

$$\int \frac{dzlz}{1-z} \text{ et } \int \frac{dzlz}{1+z}$$

derivare licet ope huius ratiocinii; statuatur

$$\int \frac{zdzlz}{1-zz} = P$$

et posito $zz = v$, ut sit $zdz = \frac{dv}{2}$ et $lz = \frac{1}{2} lv$, prodibit

$$\frac{1}{4} \int \frac{dvlv}{1-v} = P,$$

si scilicet post integrationem fiat $v = 1$, quippe quo casu etiam fit $z = 1$; sic igitur erit

$$\int \frac{dvlv}{1-v} = 4P;$$

nunc prior illa formula addatur ad inventam eritque

$$\int \frac{dzlz + zdzlz}{1-zz} = P - \frac{\pi\pi}{8},$$

haec autem formula sponte reducitur ad hanc

$$\int \frac{dzlz}{1-z} = P - \frac{\pi\pi}{8};$$

modo autem vidimus esse $\int \frac{dvlv}{1-v}$ sive $\int \frac{dzlz}{1-z} = 4P$, ita ut sit

$$4P = P - \frac{\pi\pi}{8}$$

unde manifesto fit

$$P = -\frac{\pi\pi}{24},$$

ex quo sequitur fore

$$\int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6};$$

simili modo erit

$$\int \frac{dzlz - zdzlz}{1-zz} = -P - \frac{\pi\pi}{8} = -\frac{\pi\pi}{12},$$

quae supra et infra per $1-z$ dividendo praebet

$$\int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$

quare iam adepti sumus tres integrationes memoratu maxime dignas

$$\text{I.} \quad \int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$

$$\text{II.} \quad \int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6},$$

$$\text{III.} \quad \int \frac{dzlz}{1-zz} = -\frac{\pi\pi}{8},$$

quibus adiungi potest

$$\text{IV.} \quad \int \frac{zdzlz}{1-zz} = -\frac{\pi\pi}{24}.$$

11. Quemadmodum igitur hae formulae ex ipsis calculi integralis principiis sunt deductae, ita etiam earum veritas per resolutionem in series facile comprobatur; cum enim sit

$$\frac{1}{1+z} = 1 - z + zz - z^3 + z^4 - z^5 + \text{etc.}$$

et in genere

$$\int z^n dzlz = \frac{z^{n+1}}{n+1} lz - \frac{z^{n+1}}{(n+1)^2},$$

qui valor posito $z = 1$ reducitur ad $-\frac{1}{(n+1)^2}$, patet fore

$$\int \frac{dzlz}{1+z} = -1 + \frac{1}{4} - \frac{1}{9} + \frac{1}{16} - \frac{1}{25} + \text{etc.} = -\frac{\pi\pi}{12}$$

sive

$$1 - \frac{1}{4} + \frac{1}{9} - \frac{1}{16} + \frac{1}{25} - \text{etc.} = \frac{\pi\pi}{12};$$

simili modo ob

$$\frac{1}{1-z} = 1 + z + zz + z^3 + z^4 + z^5 + \text{etc.}$$

erit

$$\int \frac{dzlz}{1-z} = -1 - \frac{1}{4} - \frac{1}{9} - \frac{1}{16} - \frac{1}{25} - \text{etc.} = -\frac{\pi\pi}{6}$$

seu

$$1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \frac{1}{25} + \text{etc.} = \frac{\pi\pi}{6};$$

tum vero ob

$$\frac{1}{1-zz} = 1 + zz + z^4 + z^6 + z^8 + \text{etc.}$$

erit

$$\int \frac{dzlz}{1-zz} = -1 - \frac{1}{9} - \frac{1}{25} - \frac{1}{49} - \frac{1}{81} - \text{etc.} = -\frac{\pi\pi}{8}$$

sive

$$1+\frac{1}{9}+\frac{1}{25}+\frac{1}{49}+\frac{1}{81}+\text{etc.}=\frac{\pi\pi}{8}.$$

Eodem modo etiam

$$\int \frac{zdzlz}{1-zz}=-\frac{1}{4}-\frac{1}{16}-\frac{1}{36}-\frac{1}{64}-\text{etc.}=-\frac{\pi\pi}{24}$$

sive

$$\frac{1}{4}+\frac{1}{16}+\frac{1}{36}+\frac{1}{64}+\text{etc.}=\frac{\pi\pi}{24},$$

quae quidem summationes iam sunt notissimae. Neque tamen quisquam adhuc methodo directa ostendit esse

$$\int \frac{dzlz}{1+z}=-\frac{\pi\pi}{12}.$$

12. Ponamus nunc $\omega=1$ et nostrae integrationes has induent formas

$$1^0. \int \frac{-z^{\lambda-2}(1-zz)dzlz}{1+z^{2\lambda}}=\frac{\pi\pi \sin.\frac{\pi}{2\lambda}}{4\lambda\lambda \cos.^2 \frac{\pi}{2\lambda}}$$

et

$$2^0. \int \frac{-z^{\lambda-2}(1+zz)dzlz}{1-z^{2\lambda}}=+\frac{\pi\pi}{4\lambda\lambda \cos.^2 \frac{\pi}{2\lambda}},$$

unde pro diversis valoribus ipsius λ, quos quidem binario non minores accipere licet, sequentes obtinentur integrationes:

I. Si $\lambda=2$, erit

$$1^0. \int \frac{-(1-zz)dzlz}{1+z^4}=\frac{\pi\pi}{8\sqrt{2}},$$

$$2^0. \int \frac{-(1+zz)dzlz}{1-z^4}=+\frac{\pi\pi}{8} \text{ sive } \int \frac{-dzlz}{1-zz}=+\frac{\pi\pi}{8}.$$

II. Si $\lambda=3$, habebimus

$$1^0. \int \frac{-z(1-zz)dzlz}{1+z^6}=\frac{\pi\pi}{54} \text{ et } 2^0. \int \frac{-z(1+zz)dzlz}{1-z^6}=+\frac{\pi\pi}{27}.$$

Hae autem duae formulae ponendo $zz=v$ abibunt in sequentes

$$1^0. \int \frac{-dv(1-v)lv}{1+v^3}=\frac{2\pi\pi}{27} \text{ et } 2^0. \int \frac{-dv(1+v)lv}{1-v^3}=\frac{4\pi\pi}{27}.$$

III. Sit $\lambda=4$ et consequemur

$$1^0.\ \int\frac{-zz(1-zz)dzlz}{1+z^8}=\frac{\pi\pi\sqrt{\frac{\sqrt{2}-1}{2\sqrt{2}}}}{16\left(2+\sqrt{2}\right)}=\frac{\pi\pi\sqrt{\left(2-\sqrt{2}\right)}}{32\left(2+\sqrt{2}\right)}$$

et

$$2^0.\ \int\frac{-zz(1+zz)dzlz}{1-z^8}=\int\frac{-zzdzlz}{(1-zz)\left(1+z^4\right)}=\frac{\pi\pi}{16\left(2+\sqrt{2}\right)},$$

quae postrenia forma reducitur ad hanc

$$\int\frac{-dzlz}{1-zz}+\int\frac{(1-zz)dzlz}{1+z^4}=\frac{\pi\pi}{8\left(2+\sqrt{2}\right)};$$

est vero $\int\frac{-dzlz}{1-zz}=\frac{\pi\pi}{8}$, unde reperitur

$$\int\frac{dzlz(1-zz)}{1+z^4}=-\frac{\pi\pi\left(1+\sqrt{2}\right)}{8\left(2+\sqrt{2}\right)}=-\frac{\pi\pi}{8\sqrt{2}},$$

qui valor iam in superiori casu $\lambda=2$ est inventus.

13. Nihil autem impedit, quominus etiam faciamus $\lambda=1$, dummodo integralia ita capiantur, ut evanescant posito $z=0$; tum autem reperiemus

$$1^0.\ \int\frac{-(1-zz)dzlz}{z(1+zz)}=\infty \text{ et } 2^0.\ \int\frac{-(1+zz)dzlz}{z(1-zz)}=\infty,$$

unde hinc nihil concludere licet. Ceterum etiam nostrae series supra inventae manifesto declarant earum summas esse infinitas, quandoquidem primus terminus utriusque $\frac{1}{(\lambda-\omega)^2}$ fit infinitus sumto, uti fecimus, $\lambda=1$ et $\omega=1$.

14. His casibus evolutis ulterius progrediamur ac ponamus formulas integrales inventas

$$\int\frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}lz=S' \text{ et } \int\frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}lz=T',$$

ita ut sit

$$S'=\frac{\pi\pi\sin.\frac{\pi\omega}{2\lambda}}{4\lambda\lambda\cos^2.\frac{\pi\omega}{2\lambda}} \quad\text{et}\quad T'=\frac{\pi\pi}{4\lambda\lambda\cos^2.\frac{\pi\omega}{2\lambda}},$$

ut ante iam differentiemus solo numero ω pro variabili habito; quo facto sequentes nanciscimur integrationes

$$\int\frac{z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}}\cdot\frac{dz}{z}(lz)^2=\left(\frac{dS'}{d\omega}\right) \text{ et } \int\frac{z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{z}(lz)^2=\left(\frac{dT'}{d\omega}\right).$$

Hunc in finem ponamus brevitatis ergo angulum $\frac{\pi\omega}{2\lambda} = \varphi$, ut sit

$$S' = \frac{\pi\pi \sin.\varphi}{4\lambda\lambda \cos^2.\varphi} = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{\sin.\varphi}{\cos^2.\varphi} \quad \text{et} \quad T' = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{1}{\cos^2.\varphi},$$

ac reperiemus

$$d.\frac{\sin.\varphi}{\cos^2.\varphi} = \left(\frac{\cos^2.\varphi + 2\sin^2.\varphi}{\cos^3.\varphi} \right) d\varphi = \left(\frac{1+\sin^2.\varphi}{\cos^3.\varphi} \right) d\varphi,$$

ubi est $d\varphi = \frac{\pi d\omega}{2\lambda}$; unde colligimus

$$\left(\frac{dS'}{d\omega} \right) = \frac{\pi^3}{8\lambda^3} \cdot \left(\frac{1+\sin^2.\frac{\pi\omega}{2\lambda}}{\cos^2.\frac{\pi\omega}{2\lambda}} \right) = \frac{\pi^3}{8\lambda^3} \cdot \left(\frac{2}{\cos^3.\frac{\pi\omega}{2\lambda}} - \frac{1}{\cos.\frac{\pi\omega}{2\lambda}} \right);$$

simili modo ob $T' = \frac{\pi\pi}{4\lambda\lambda} \cdot \frac{1}{\cos^2.\varphi}$, erit

$$d \cdot \frac{1}{\cos^2.\varphi} = \frac{2d\varphi \sin.\varphi}{\cos^3.\varphi}$$

hincque

$$\left(\frac{dT'}{d\omega} \right) = \frac{\pi^3}{8\lambda^3} \cdot \frac{2\sin.\frac{\pi\omega}{2\lambda}}{\cos^3.\frac{\pi\omega}{2\lambda}};$$

consequenter integrationes hinc natae erunt

$$\int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \frac{\pi^3}{8\lambda^3} \cdot \left(\frac{2}{\cos^3.\frac{\pi\omega}{2\lambda}} - \frac{1}{\cos.\frac{\pi\omega}{2\lambda}} \right),$$

$$\int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \frac{\pi^3}{8\lambda^3} \cdot \frac{2\sin.\frac{\pi\omega}{2\lambda}}{\cos^3.\frac{\pi\omega}{2\lambda}}.$$

15. Si iam eodem modo series § 8 inventas denuo differentiemus sumta sola ω variabili, perveniamus ad sequentes summationes

$$\frac{\pi^3}{8\lambda^3} \cdot \left(\frac{2}{\cos^3.\frac{\pi\omega}{2\lambda}} - \frac{1}{\cos.\frac{\pi\omega}{2\lambda}} \right)$$
$$= \frac{2}{(\lambda-\omega)^3} + \frac{2}{(\lambda+\omega)^3} - \frac{2}{(3\lambda-\omega)^3} - \frac{2}{(3\lambda+\omega)^3} + \frac{2}{(5\lambda-\omega)^3} + \frac{2}{(5\lambda+\omega)^3} - \text{etc.},$$

$$\frac{\pi^3}{8\lambda^3} \cdot \frac{2\sin.\frac{\pi\omega}{2\lambda}}{\cos^3.\frac{\pi\omega}{2\lambda}} = \frac{2}{(\lambda-\omega)^3} - \frac{2}{(\lambda+\omega)^3} + \frac{2}{(3\lambda-\omega)^3} - \frac{2}{(3\lambda+\omega)^3} + \frac{2}{(5\lambda-\omega)^3} + \text{etc.}$$

16. Si iam hic sumamus $\omega = 0$ et $\lambda = 1$, prior integratio hanc induit formam

$$\int \frac{2dz(lz)^2}{1+zz} = \frac{\pi^3}{8} = \frac{2}{1^3} + \frac{2}{1^3} - \frac{2}{3^3} - \frac{2}{3^3} + \frac{2}{5^3} + \frac{2}{5^3} - \frac{2}{7^3} - \frac{2}{7^3} + \text{etc.}$$

ita ut sit

$$\frac{1}{1^3} - \frac{1}{3^3} + \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{11^3} + \text{etc.} = \frac{\pi^3}{32}$$

quemadmodum iam dudum demonstravi. Altera autem integratio hoc casu in nihilum abit. Ex priori vero integrali

$$\int \frac{dz(lz)^2}{1+zz} = \frac{\pi^3}{16}.$$

alia derivare non licet, uti supra fecimus ex formula $\int \frac{dzlz}{1-zz} = -\frac{\pi\pi}{8}$, quod hic denominator $1+zz$ non habet factores reales.

17. Sumamus igitur $\lambda = 2$ et $\omega = 1$ ac prior integratio dabit

$$\int \frac{(1+zz)dz(lz)^2}{1+z^4} = \frac{3\pi^3}{32\sqrt{2}};$$

series autem hinc nata erit

$$\frac{2}{1^3} + \frac{2}{3^3} - \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} + \frac{2}{11^3} - \text{etc.},$$

ita ut sit

$$\frac{1}{1^3} + \frac{1}{3^3} - \frac{1}{5^3} - \frac{1}{7^3} + \frac{1}{9^3} + \frac{1}{11^3} - \text{etc.} = \frac{3\pi^3}{64\sqrt{2}}$$

quae superiori addita praebet

$$\frac{1}{1^3} - \frac{1}{7^3} + \frac{1}{9^3} - \frac{1}{15^3} + \frac{1}{17^3} - \frac{1}{23^3} + \text{etc.} = \frac{\pi^3\left(3+2\sqrt{2}\right)}{128\sqrt{2}}.$$

Altera vero integratio hoc casu dat

$$\int \frac{dz(lz)^2}{1+zz} = \frac{\pi^3}{16}.$$

quae cum paragrapho praecedenti perfecte congruit, quemadmodum etiam series hinc nata est

$$\frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.}$$

18. Quo autem facilius sequentes integrationes per continuam differentiationem elicere valeamus, eas in genere repraesentemus; et cum pro priore sit

$$S = \frac{\pi}{2\lambda \cos.\frac{\pi\omega}{2\lambda}},$$

integrationes hinc ortae ita ordine procedent:

$$\text{I.} \quad \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z} = S,$$

$$\text{II.} \quad \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1-z^{2\lambda}} \cdot \frac{dz}{z} lz = \left(\frac{dS}{d\omega}\right),$$

$$\text{III.} \quad \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^2 = \left(\frac{ddS}{d\omega^2}\right),$$

$$\text{IV.} \quad \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^3 = \left(\frac{d^3S}{d\omega^3}\right),$$

$$\text{V.} \quad \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^4 = \left(\frac{d^4S}{d\omega^4}\right),$$

$$\text{VI.} \quad \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^5 = \left(\frac{d^5S}{d\omega^5}\right),$$

$$\text{VII.} \quad \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}(lz)^6 = \left(\frac{d^6S}{d\omega^6}\right)$$

etc.

19. Pro his differentiationibus continuis facilius absolvendis ponamus brevitatis ergo $\frac{\pi}{2\lambda} = \alpha$, ut sit

$$S = \frac{\alpha}{\cos.\alpha\omega};$$

tum vero sit

$$\sin.\alpha\omega = p \text{ et } \cos.\alpha\omega = q$$

eritque

$$dp = \alpha q d\omega \text{ et } dq = -\alpha p d\omega.$$

Praeterea vero notetur esse

$$d.\frac{p^n}{q^{n+1}} = \alpha d\omega\left(\frac{np^{n-1}}{q^n} + \frac{(n+1)p^{n+1}}{q^{n+2}}\right).$$

His praemissis ob $S = \alpha \cdot \frac{1}{q}$ erit

$$\left(\frac{dS}{d\omega}\right) = \alpha\alpha\frac{p}{qq},$$

deinde

$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3 \left(\frac{1}{1} + \frac{2pp}{q^3}\right),$$

porro

$$\left(\frac{d^3S}{d\omega^3}\right) = \alpha^4 \left(\frac{5p}{qq} + \frac{6p^3}{q^4}\right),$$
$$\left(\frac{d^4S}{d\omega^4}\right) = \alpha^5 \left(\frac{5}{q} + \frac{28pp}{q^3} + \frac{24p^4}{q^5}\right),$$
$$\left(\frac{d^5S}{d\omega^5}\right) = \alpha^6 \left(\frac{61p}{qq} + \frac{180p^3}{q^4} + \frac{120p^5}{q^6}\right),$$
$$\left(\frac{d^6S}{d\omega^6}\right) = \alpha^7 \left(\frac{61}{q} + \frac{662pp}{q^3} + \frac{1320p^4}{q^5} + \frac{720p^6}{q^7}\right),$$
$$\left(\frac{d^7S}{d\omega^7}\right) = \alpha^8 \left(\frac{1385}{qq} + \frac{7266pp}{q^4} + \frac{10920p^4}{q^6} + \frac{5040p^7}{q^8}\right);$$

hi autem valores ob $pp = 1 - qq$ ad sequentes reducuntur

$$S = \alpha \cdot \frac{1}{q},$$
$$\left(\frac{dS}{d\omega}\right) = \alpha\alpha p \frac{1}{qq},$$
$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3 \left(\frac{1 \cdot 2}{q^3} - \frac{1}{q}\right),$$
$$\left(\frac{d^3S}{d\omega^3}\right) = \alpha^4 p \left(\frac{1 \cdot 2 \cdot 3}{q^4} - \frac{1}{qq}\right),$$
$$\left(\frac{d^4S}{d\omega^4}\right) = \alpha^5 p \left(\frac{1 \cdot 2 \cdot 3 \cdot 4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right),$$
$$\left(\frac{d^5S}{d\omega^5}\right) = \alpha^6 p \left(\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{q^6} - \frac{60}{q^4} + \frac{1}{qq}\right),$$
$$\left(\frac{d^6S}{d\omega^6}\right) = \alpha^7 p \left(\frac{1 \cdots\cdots 6}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right).$$

20. Has posteriores formas reperire licet ope horum duorum lemmatum

$$\text{I. } d.\frac{1}{q^{n+1}} = \alpha d\omega \frac{(n+1)p}{q^{n+2}}, \text{ II. } d.\frac{p}{q^{n+1}} = \alpha d\omega \left(\frac{n+1}{q^{n+2}} - \frac{n}{q^n}\right);$$

hinc enim reperiemus

$$S = \alpha \cdot \frac{1}{q},$$
$$\left(\frac{dS}{d\omega}\right) = \alpha\alpha \frac{p}{qq},$$
$$\left(\frac{ddS}{d\omega^2}\right) = \alpha^3 \left(\frac{2}{q^3} - \frac{1}{q}\right),$$
$$\left(\frac{d^3S}{d\omega^3}\right) = \alpha^4 \left(\frac{2 \cdot 3p}{q^4} - \frac{p}{qq}\right),$$
$$\left(\frac{d^4S}{d\omega^4}\right) = \alpha^5 \left(\frac{2 \cdot 3 \cdot 4}{q^5} - \frac{20}{q^3} + \frac{1}{q}\right),$$
$$\left(\frac{d^5S}{d\omega^5}\right) = \alpha^6 \left(\frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5p}{q^6} - \frac{60p}{q^4} + \frac{p}{qq}\right),$$
$$\left(\frac{d^6S}{d\omega^6}\right) = \alpha^7 \left(\frac{2 \cdot 3 \cdot 4 \cdot 5 \cdot 6}{q^7} - \frac{840}{q^5} + \frac{182}{q^3} - \frac{1}{q}\right),$$
$$\left(\frac{d^7S}{d\omega^7}\right) = \alpha^8 \left(\frac{2 \cdots 7p}{q^8} - \frac{5 \cdot 840p}{q^6} + \frac{3 \cdot 182p}{q^4} - \frac{p}{qq}\right).$$

21. Ipsae autem series his formulis respondentes

$$S = \frac{1}{\lambda-\omega} + \frac{1}{\lambda+\omega} - \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} + \frac{1}{5\lambda+\omega} - \text{etc.}$$
$$\left(\frac{dS}{d\omega}\right) = \frac{1}{(\lambda-\omega)^2} - \frac{1}{(\lambda+\omega)^2} - \frac{1}{(3\lambda-\omega)^2} + \frac{1}{(3\lambda+\omega)^2} + \frac{1}{(5\lambda-\omega)^2} - \frac{1}{(5\lambda+\omega)^2} - \text{etc},$$
$$\left(\frac{ddS}{d\omega^2}\right) = \frac{1 \cdot 2}{(\lambda-\omega)^3} + \frac{1 \cdot 2}{(\lambda+\omega)^3} - \frac{1 \cdot 2}{(3\lambda-\omega)^3} - \frac{1 \cdot 2}{(3\lambda+\omega)^3} + \frac{1 \cdot 2}{(5\lambda-\omega)^3} + \text{etc},$$
$$\left(\frac{d^3S}{d\omega^3}\right) = \frac{1 \cdot 2 \cdot 3}{(\lambda-\omega)^4} - \frac{1 \cdot 2 \cdot 3}{(\lambda+\omega)^4} - \frac{1 \cdot 2 \cdot 3}{(3\lambda-\omega)^4} + \frac{1 \cdot 2 \cdot 3}{(3\lambda+\omega)^4} + \frac{1 \cdot 2 \cdot 3}{(5\lambda-\omega)^4} - \text{etc},$$
$$\left(\frac{d^4S}{d\omega^4}\right) = \frac{1 \cdot 2 \cdot 3 \cdot 4}{(\lambda-\omega)^5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{(\lambda+\omega)^5} - \frac{1 \cdot 2 \cdot 3 \cdot 4}{(3\lambda-\omega)^5} - \frac{1 \cdot 2 \cdot 3 \cdot 4}{(3\lambda+\omega)^5} + \frac{1 \cdot 2 \cdot 3 \cdot 4}{(5\lambda-\omega)^5} + \text{etc},$$
$$\left(\frac{d^5S}{d\omega^5}\right) = \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(\lambda-\omega)^6} - \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(\lambda+\omega)^6} - \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(3\lambda-\omega)^6} + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(3\lambda+\omega)^6} + \frac{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5}{(5\lambda-\omega)^6} - \text{etc},$$
$$\left(\frac{d^6S}{d\omega^6}\right) = \frac{1 \cdots 6}{(\lambda-\omega)^7} + \frac{1 \cdots 6}{(\lambda+\omega)^7} - \frac{1 \cdots 6}{(3\lambda-\omega)^7} - \frac{1 \cdots 6}{(3\lambda+\omega)^7} + \frac{1 \cdots 6}{(5\lambda-\omega)^7} + \text{etc},$$
$$\left(\frac{d^7S}{d\omega^7}\right) = \frac{1 \cdots 7}{(\lambda-\omega)^8} - \frac{1 \cdots 7}{(\lambda+\omega)^8} - \frac{1 \cdots 7}{(3\lambda-\omega)^8} + \frac{1 \cdots 7}{(3\lambda+\omega)^8} + \frac{1 \cdots 7}{(5\lambda-\omega)^8} + \text{etc},$$
etc.

Circa hos autem valores probe meminisse oportet esse

$$\alpha = \frac{\pi}{2\lambda},\ p = \sin.\alpha\omega = \sin.\frac{\pi\omega}{2\lambda} \text{ et } q = \cos.\alpha\omega = \cos.\frac{\pi\omega}{2\lambda}.$$

22. Eodem modo expediamus valores seu formulas integrales alterius generis, pro quibus est

$$T = \frac{\pi}{2\lambda} \operatorname{tang.} \frac{\pi\omega}{2\lambda},$$

unde continuo differentiando oriuntur sequentes integrationes:

$$\text{I.} \quad \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} = T,$$

$$\text{II.} \quad \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} lz = \left(\frac{dT}{d\omega}\right),$$

$$\text{III.} \quad \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^2 = \left(\frac{ddT}{d\omega^2}\right),$$

$$\text{IV.} \quad \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^3 = \left(\frac{d^3T}{d\omega^3}\right),$$

$$\text{V.} \quad \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^4 = \left(\frac{d^4T}{d\omega^4}\right),$$

$$\text{VI.} \quad \int \frac{-z^{\lambda-\omega} - z^{\lambda+\omega}}{1 + z^{2\lambda}} \cdot \frac{dz}{z} (lz)^5 = \left(\frac{d^5T}{d\omega^5}\right),$$

$$\text{VII.} \quad \int \frac{z^{\lambda-\omega} - z^{\lambda+\omega}}{1 - z^{2\lambda}} \cdot \frac{dz}{z} (lz)^6 = \left(\frac{d^6T}{d\omega^6}\right).$$

23. Ponatur iterum $\alpha = \frac{\pi}{2\lambda}, \sin.\alpha\omega = p$ et $\cos.\alpha\omega = q,$ ut sit

$$T = \frac{\alpha p}{q},$$

quae formula secundum lemmata § 20 continuo differentiata dabit

$$T = \frac{\alpha p}{q},$$
$$\left(\frac{dT}{d\omega}\right) = \alpha\alpha \frac{1}{qq},$$
$$\left(\frac{ddT}{d\omega^2}\right) = \alpha^3 \frac{2p}{q^3},$$
$$\left(\frac{d^3T}{d\omega^3}\right) = \alpha^4 \left(\frac{6}{q^4} - \frac{4}{qq}\right),$$
$$\left(\frac{d^4T}{d\omega^4}\right) = \alpha^5 \left(\frac{24p}{q^5} - \frac{8}{q^3}\right),$$
$$\left(\frac{d^5T}{d\omega^5}\right) = \alpha^6 \left(\frac{120}{q^6} - \frac{120}{q^4} + \frac{16}{qq}\right),$$
$$\left(\frac{d^6T}{d\omega^6}\right) = \alpha^7 \left(\frac{720}{q^7} - \frac{480}{q^5} + \frac{32}{q^3}\right),$$
$$\left(\frac{d^7T}{d\omega^7}\right) = \alpha^8 \left(\frac{5040}{q^8} - \frac{6720}{q^6} + \frac{2016}{q^4} - \frac{64}{qq}\right).$$

24. Series autem infinitae, quae hinc nascuntur, erunt

$$\int \frac{-dzlz}{1-zz} = \frac{\pi\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.},$$
$$\int \frac{-dz(lz)^3}{1-zz} = \frac{\pi^4}{16} = \frac{6}{1^4} + \frac{6}{3^4} + \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} + \frac{6}{11^4} + \frac{6}{13^4} + \text{etc.},$$
$$\int \frac{-dz(lz)^5}{1-zz} = \frac{\pi^6}{8} = \frac{120}{1^6} + \frac{120}{3^6} + \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} + \frac{120}{11^6} + \frac{120}{13^6} + \text{etc.},$$

etc.

25. Operae pretium erit hinc casus simplicissimos evolvere, qui oriuntur ponendo $\lambda = 1$ et $\omega = 0$, ita ut sit $a = \frac{\pi}{2}$, $p = 0$ et $q = 1$, unde habebimus:

Pro ordine priore	Pro ordine posteriore
$S = \frac{\pi}{2}$	$T = 0$
$\left(\frac{dS}{d\omega}\right) = 0$	$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{4}$
$\left(\frac{ddS}{d\omega^2}\right) = \frac{\pi^3}{8}$	$\left(\frac{ddT}{d\omega^2}\right) = 0$
$\left(\frac{d^3S}{d\omega^3}\right) = 0$	$\left(\frac{d^3T}{d\omega^3}\right) = \frac{\pi^4}{8}$
$\left(\frac{d^4S}{d\omega^4}\right) = \frac{5\pi^5}{32}$	$\left(\frac{d^4T}{d\omega^4}\right) = 0$

$\left(\frac{d^5 S}{d\omega^5}\right) = 0$	$\left(\frac{d^5 T}{d\omega^5}\right) = \frac{\pi^6}{4}$
$\left(\frac{d^6 S}{d\omega^6}\right) = \frac{61\pi^7}{128}$	$\left(\frac{d^6 T}{d\omega^6}\right) = 0$
$\left(\frac{d^7 S}{d\omega^7}\right) = 0$	$\left(\frac{d^7 T}{d\omega^7}\right) = \frac{17\pi^8}{16}$
etc.	etc.

26. Hinc ergo omissis valoribus evanescentibus ex priore ordine habebimus sequentes formulas integrales cum seriebus inde natis

$$\int \frac{dz}{1+zz} = \frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \text{etc.},$$

$$\int \frac{dz(lz)^2}{1+zz} = \frac{\pi^2}{16} = \frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.},$$

$$\int \frac{dz(lz)^4}{1+zz} = \frac{5\pi^5}{64} = \frac{24}{1^5} - \frac{24}{3^5} + \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} - \frac{24}{11^5} + \frac{24}{13^5} - \text{etc.},$$

$$\int \frac{dz(lz)^6}{1+zz} = \frac{61\pi^5}{256} = \frac{720}{1^7} - \frac{720}{3^7} + \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} - \frac{720}{11^7} + \frac{720}{13^7} - \text{etc.},$$

etc.

27. Ex altero autem ordine pro eodem casu oriuntur

$$\int \frac{-dzlz}{1-zz} = \frac{\pi\pi}{8} = \frac{1}{1^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.},$$

$$\int \frac{-dz(lz)^3}{1-zz} = \frac{\pi^4}{16} = \frac{6}{1^4} + \frac{6}{3^4} + \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} + \frac{6}{11^4} + \frac{6}{13^4} + \text{etc.},$$

$$\int \frac{-dz(lz)^5}{1-zz} = \frac{\pi^6}{8} = \frac{120}{1^6} + \frac{120}{3^6} + \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} + \frac{120}{11^6} + \frac{120}{13^6} + \text{etc.},$$

$$\int \frac{-dz(lz)^7}{1-zz} = \frac{61\pi^5}{256} = \frac{720}{1^7} - \frac{720}{3^7} + \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} - \frac{720}{11^7} + \frac{720}{13^7} - \text{etc.},$$

etc.

28. Quemadmodum ex primo integrali ordinis posterioris deduximus has formulas

$$\int \frac{dzlz}{1-z} = -\frac{\pi\pi}{6} \quad \text{et} \quad \int \frac{dzlz}{1+z} = -\frac{\pi\pi}{12},$$

similes quoque formulae integrales ex sequentibus deduci possunt; cum enim sit $\int \frac{dz(lz)^3}{1-zz} = -\frac{\pi^4}{16}$, ponamus esse

$$\int \frac{zdz(lz)^3}{1-zz} = P$$

eritque

$$\int \frac{dz(lz)^3}{1-z} = P - \frac{\pi^4}{16} \quad \text{et} \quad \int \frac{dz(lz)^3}{1+z} = -P - \frac{\pi^4}{16};$$

nunc vero statuatur $zz = v$, ut sit $zdz = \frac{1}{2}dv$ et $lz = \frac{1}{2}lv$ ideoque $(lz)^3 = \frac{1}{8}(lv)^3$, quibus substitutis erit

$$P = \frac{1}{16} \int \frac{dv(lv)^3}{1-v} = \frac{1}{16}\left(P - \frac{\pi^4}{16}\right),$$

unde fit

$$P = \frac{1}{16}\left(P - \frac{\pi^4}{16}\right) \text{ ideoque } P = -\frac{\pi^4}{240},$$

sicque has duas habebimus integrationes novas

$$\int \frac{dz(lz)^3}{1-z} = -\frac{\pi^4}{15} \quad \text{et} \quad \int \frac{dz(lz)^3}{1+z} = -\frac{7\pi^4}{120};$$

hinc autem per series erit

$$\int \frac{-dz(lz)^3}{1-z} = +\frac{\pi^4}{15} = 6\left(1 + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \frac{1}{6^4} + \frac{1}{7^4} + \frac{1}{8^4} + \text{etc.}\right)$$

et

$$\int \frac{-dz(lz)^3}{1+z} = +\frac{7\pi^4}{120} = 6\left(1 - \frac{1}{2^4} + \frac{1}{3^4} - \frac{1}{4^4} + \frac{1}{5^4} - \frac{1}{6^4} + \frac{1}{7^4} - \frac{1}{8^4} + \text{etc.}\right)$$

29. Porro $\int \frac{dz(lz)^5}{1-zz} = -\frac{\pi^6}{8}$; ponamus esse

$$\int \frac{zdz(lz)^5}{1-zz} = P,$$

ut hinc obtineamus

$$\int \frac{dz(lz)^5}{1-z} = P - \frac{\pi^6}{8} \quad \text{et} \quad \int \frac{dz(lz)^5}{1+z} = -P - \frac{\pi^6}{8};$$

nunc igitur statuamus $zz = v$ eritque

$$P = \frac{1}{64} \int \frac{dv(lv)^5}{1-v} = \frac{1}{64}\left(P - \frac{\pi^6}{8}\right),$$

unde fit

$$P = -\frac{\pi^6}{504},$$

novaeque integrationes hinc deductae sunt

$$\int \frac{dz(lz)^5}{1-z} = -\frac{8\pi^6}{63} \quad \text{et} \quad \int \frac{dz(lz)^5}{1+z} = -\frac{31\pi^6}{252},$$

at vero per series reperitur

$$\int \frac{dz(lz)^5}{1-z} = -\frac{8\pi^6}{63} = -120\left(1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \frac{1}{7^6} + \text{etc.}\right)$$

et

$$\int \frac{dz(lz)^5}{1+z} = -\frac{31\pi^6}{252} = -120\left(1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \frac{1}{7^6} - \text{etc.}\right),$$

ita ut sit

$$1 + \frac{1}{2^6} + \frac{1}{3^6} + \frac{1}{4^6} + \frac{1}{5^6} + \frac{1}{6^6} + \frac{1}{7^6} + \text{etc} = \frac{\pi^6}{945}$$

et

$$1 - \frac{1}{2^6} + \frac{1}{3^6} - \frac{1}{4^6} + \frac{1}{5^6} - \frac{1}{6^6} + \frac{1}{7^6} - \frac{1}{8^6} + \text{etc} = \frac{31\pi^6}{40240} = \frac{31\pi^6}{32 \cdot 945}.$$

30. Consideremus etiam casus, quibus $\lambda = 2$ et $\omega = 1$, ita ut sit $\alpha = \frac{\pi}{4}$ et $\alpha\omega = \frac{\pi}{4}$, hinc $p = q = \frac{1}{\sqrt{2}}$, unde pro utroque ordine sequentes habebimus valores:

Pro ordine priore	Pro ordine posteriore
$S = \frac{\pi}{2\sqrt{2}}$	$T = \frac{\pi}{4}$
$\left(\frac{dS}{d\omega}\right) = \frac{\pi\pi}{8\sqrt{2}}$	$\left(\frac{dT}{d\omega}\right) = \frac{\pi\pi}{8}$
$\left(\frac{ddS}{d\omega^2}\right) = \frac{3\pi^3}{32\sqrt{2}}$	$\left(\frac{ddT}{d\omega^2}\right) = \frac{\pi^3}{16}$
$\left(\frac{d^3S}{d\omega^3}\right) = \frac{11\pi^4}{128\sqrt{2}}$	$\left(\frac{d^3T}{d\omega^3}\right) = \frac{\pi^4}{16}$
$\left(\frac{d^4S}{d\omega^4}\right) = \frac{57\pi^5}{512\sqrt{2}}$	$\left(\frac{d^4T}{d\omega^4}\right) = \frac{5\pi^5}{64}$
$\left(\frac{d^5S}{d\omega^5}\right) = \frac{361\pi^6}{2048\sqrt{2}}$	$\left(\frac{d^5T}{d\omega^5}\right) = \frac{\pi^6}{8}$
$\left(\frac{d^6S}{d\omega^6}\right) = \frac{2763\pi^7}{8192\sqrt{2}}$	$\left(\frac{d^6T}{d\omega^6}\right) = \frac{61\pi^7}{256\sqrt{2}}$

$$\left(\frac{d^7S}{d\omega^7}\right) = \frac{24611\pi^8}{32768\sqrt{2}} \qquad \left(\frac{d^7T}{d\omega^7}\right) = \frac{17\pi^8}{32}$$

etc. | etc.

31. Hinc igitur sequentes integrationes cum seriebus respondentibus resultant; ac primo quidem ex ordine primo

$$\int \frac{(1+zz)dz}{1+z^4} = \frac{\pi}{2\sqrt{2}} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \text{etc.},$$

$$\int \frac{-(1-zz)dzlz}{1+z^4} = \frac{\pi\pi}{2\sqrt{2}} = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \text{etc.},$$

$$\int \frac{dz(1+zz)(lz)^2}{1+z^4} = \frac{3\pi^3}{32\sqrt{2}} = \frac{2}{1^3} + \frac{2}{3^3} - \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} + \frac{2}{11^3} - \frac{2}{13^3} - \text{etc.},$$

$$\int \frac{-dz(1-zz)(lz)^3}{1+z^4} = \frac{11\pi^4}{128\sqrt{2}} = \frac{6}{1^4} - \frac{6}{3^4} - \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} - \frac{6}{11^4} - \frac{6}{13^4} + \text{etc.},$$

$$\int \frac{dz(1+zz)(lz)^4}{1+z^4} = \frac{57\pi^5}{512\sqrt{2}} = \frac{24}{1^5} + \frac{24}{3^5} - \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} + \frac{24}{11^5} - \frac{24}{13^5} - \text{etc.},$$

$$\int \frac{-dz(1-zz)(lz)^5}{1+z^4} = \frac{361\pi^6}{2048\sqrt{2}} = \frac{120}{1^6} - \frac{120}{3^6} - \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} - \frac{120}{11^6} - \frac{120}{13^6} + \text{etc.},$$

$$\int \frac{dz(1+zz)(lz)^6}{1+z^4} = \frac{2763\pi^7}{8192\sqrt{2}} = \frac{720}{1^7} + \frac{720}{3^7} - \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} + \frac{720}{11^7} - \frac{720}{13^7} + \text{etc.},$$

$$\int \frac{-dz(1-zz)(lz)^7}{1+z^4} = \frac{24611\pi^8}{32768\sqrt{2}} = \frac{5040}{1^8} - \frac{5040}{3^8} - \frac{5040}{5^8} + \frac{5040}{7^8} + \frac{5040}{9^8} - \frac{5040}{11^8} - \frac{5040}{13^8} + \text{etc.}$$

etc.

32. Eodem modo integrationes alterius ordinis cum seriebus erunt

$$\int \frac{dz}{1+zz} = \frac{\pi}{4} = 1 + \frac{1}{3} - \frac{1}{5} - \frac{1}{7} + \frac{1}{9} + \frac{1}{11} - \frac{1}{13} - \text{etc.},$$

$$\int \frac{-dzlz}{1-zz} = \frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.},$$

$$\int \frac{dz(lz)^2}{1+zz} = \frac{\pi^3}{16} = \frac{2}{1^3} - \frac{2}{3^3} + \frac{2}{5^3} - \frac{2}{7^3} + \frac{2}{9^3} - \frac{2}{11^3} + \frac{2}{13^3} - \text{etc.},$$

$$\int \frac{-dz(lz)^3}{1-zz} = \frac{\pi^4}{16} = \frac{6}{1^4} + \frac{6}{3^4} + \frac{6}{5^4} + \frac{6}{7^4} + \frac{6}{9^4} + \frac{6}{11^4} + \frac{6}{13^4} + \text{etc.},$$

$$\int \frac{dz(lz)^4}{1+zz} = \frac{5\pi^5}{64} = \frac{24}{1^5} - \frac{24}{3^5} + \frac{24}{5^5} - \frac{24}{7^5} + \frac{24}{9^5} - \frac{24}{11^5} - \frac{24}{13^5} - \text{etc.},$$

$$\int \frac{-dz(lz)^5}{1-zz} = \frac{\pi^6}{8} = \frac{120}{1^6} + \frac{120}{3^6} + \frac{120}{5^6} + \frac{120}{7^6} + \frac{120}{9^6} + \frac{120}{11^6} + \frac{120}{13^6} + \text{etc.},$$

$$\int \frac{dz(lz)^6}{1+zz} = \frac{61\pi^7}{256} = \frac{720}{1^7} - \frac{720}{3^7} + \frac{720}{5^7} - \frac{720}{7^7} + \frac{720}{9^7} - \frac{720}{11^7} + \frac{720}{13^7} - \text{etc.},$$

$$\int \frac{-dz(lz)^7}{1-zz} = \frac{17\pi^8}{32} = \frac{5040}{1^8} + \frac{5040}{3^8} + \frac{5040}{5^8} + \frac{5040}{7^8} + \frac{5040}{9^8} + \frac{5040}{11^8} + \frac{5040}{13^8} + \text{etc.}$$

etc.

Hae autem series sunt eae ipsae, quas iam supra (§ 26 et 27) sumus consecuti.

33. Praeterea autem ii casus imprimis notari merentur, quibus formulae integrales in formas simpliciores resolvi possunt. Haec autem resolutio tantum spectat ad fractionem

$$\pm \frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}}$$

omisso factore $\frac{dz}{z}(lz)^{\mu}$ ad quod ostendendum sumamus primo $\lambda = 3$ et $\omega = 1$, unde fit $\alpha = \frac{\pi}{6}$, $p = \sin.\frac{\pi}{6}$ et $q = \cos.\frac{\pi}{6}$; tum autem in priori ordine occurrunt alternatim sequentes fractiones

$$\text{I.} \quad \frac{zz(1+zz)}{1+z^6} = \frac{zz}{1-zz+z^4},$$

quae posito $zz = v$ abit in $\frac{v}{1-v+vv}$; ergo cum sit $\frac{dz}{z} = \frac{1}{2}\frac{dv}{v}$ et $lz = \frac{1}{2}lv$, hinc talis forma

$$\frac{1}{2^{2i+1}} \int \frac{dv(lv)^{2i}}{1-v+vv}$$

integrari poterit, casu scilicet $v = 1$;

$$\text{II.} \quad -\frac{zz(1-zz)}{1+z^6} = +\frac{2}{3(1+zz)} - \frac{2-zz}{3\left(1-zz+z^4\right)},$$

quae posito $zz = v$ abit in

$$\frac{2}{3(1+v)} - \frac{2-v}{3(1-v+vv)},$$

quae ergo forma ducta in $\frac{dz}{z}(lz)^{2i+1}$ vel in $\frac{1}{2^{2i+1}}\frac{dv}{v}(lv)^{2i+1}$ semper integrari potest posito $v = 1$.

34. Eodem casu ordo posterior sequentes suppeditat resolutiones

$$\text{I.} \quad \frac{zz(1-zz)}{1-z^6} = \frac{zz}{1+zz+z^4} = \frac{vv}{1+v+vv},$$

quae in $\frac{dz}{z}(lz)^{2i}$ vel in $\frac{1}{2^{2i+1}}\frac{dv}{v}(lv)^{2i}$ ducta semper est integrabilis;

$$\text{II.} \quad \frac{-zz(1+zz)}{1-z^6} = \frac{-2}{3(1-zz)} + \frac{2+zz}{3(1+zz+z^4)},$$

quae facto $zz = v$ fit

$$\frac{-2}{3(1-v)} + \frac{2+v}{3(1+v+vv)},$$

quae ergo formulae in $\frac{dv}{v}(lv)^{2i+1}$ ductae fiunt integrabiles; quia autem in hac resolutione numeratores per z vel v dividere non licet, alia resolutione est opus, quae reperitur

$$\frac{-zz(1+zz)}{1-z^6} = \frac{-2zz}{3(1-zz)} - \frac{zz(2+zz)}{3(1+zz+z^4)}$$

sive

$$\frac{-2v}{3(1-v)} - \frac{v(2+v)}{3(1+v+vv)},$$

quae formulae ductae in $\frac{dz}{z}(lz)^{2i+1}$ vel in $\frac{1}{2^{2i+2}}\frac{dv}{v}(lv)^{2i+1}$ integrationem quoque admittunt.

35. Porro manente $\lambda = 3$ sumatur $\omega = 2$, ut sit

$$\alpha = \frac{\pi}{6}, p = \sin.\frac{\pi}{3} \text{ et } q = \cos.\frac{\pi}{3},$$

et ex ordine priore orientur sequentes reductiones

$$\text{I.}\quad \frac{z(1+z^4)}{1+z^6} = \frac{2z}{3(1+zz)} + \frac{z(1+zz)}{3(1-zz+z^4)},$$

unde multiplicando per $\frac{dz}{z}(lz)^{2i}$ oriuntur formulae integrationem admittentes casu $z=1$;

$$\text{II.}\quad \frac{-z(1-z^4)}{1+z^6} = -\frac{z(1-zz)}{1-zz+z^4},$$

quae per $\frac{dz}{z}(lz)^{2i+1}$ multiplicata integrari poterit casu $z=1$.

Ex ordine vero posteriore sequentes prodibunt reductiones

$$\text{I.}\quad \frac{z(1-z^4)}{1-z^6} = \frac{z(1+zz)}{1+zz+z^4},$$

quae ducta in $\frac{dz}{z}(lz)^{2i}$ fit integrabilis;

$$\text{II.}\quad \frac{-z(1+z^4)}{1-z^6} = \frac{-2z}{3(1-zz)} - \frac{z(1-zz)}{3(1+zz+z^4)},$$

quae formulae in $\frac{dz}{z}(lz)^{2i+1}$ ductae fiunt integrabiles.

36. Operae iam erit pretium haec integralia actu evolvere, quare ex § 33 eiusque numero I nanciscimur sequentes integrationes

$$1^0.\quad \frac{1}{2}\int \frac{dv}{1-v+vv} = \alpha\frac{1}{q} = \frac{\pi}{3\sqrt{3}},$$

$$2^0.\quad \frac{1}{8}\int \frac{dv(lv)^2}{1-v+vv} = \alpha^3\left(\frac{2}{q^3}\frac{1}{-q}\right) = \frac{5\pi^3}{324\sqrt{3}},$$

deinde vero ex eiusdem paragraphi numero II, ubi etiam haec reductio locum habet

$$-\frac{zz(1-zz)}{1+z^6} = -\frac{zz}{3(1+zz)} - \frac{zz(1-2zz)}{3(1-zz+z^4)} = -\frac{v}{3(1+v)} - \frac{v(1-2v)}{3(1-v+vv)},$$

quae ducta in $\frac{1}{4}\cdot\frac{dv}{v}lv$ dabit

$$-\frac{1}{6}\int \frac{dvlv}{(1+v)} - \frac{1}{12}\int \frac{dv(1-2v)lv}{1-v+vv} = \alpha\alpha \frac{p}{qq} = \frac{\pi\pi}{54},$$

quarum formularum prior integrationem admittit; est enim

$$\int \frac{dvlv}{(1+v)} = -\frac{\pi\pi}{12},$$

unde invenitur posterior

$$\int \frac{dv(1-2v)lv}{1-v+vv} = -\frac{\pi\pi}{18}.$$

37. Ex § 34 eiusque numero I sequitur

$$1^0.\ \ \frac{1}{2}\int \frac{dv}{1+v+vv} = \alpha \frac{p}{q} = \frac{\pi}{6\sqrt{3}},$$

$$2^0.\ \ \frac{1}{8}\int \frac{dv(lv)^2}{1+v+vv} = \alpha^3 \frac{2p}{q^3} = \frac{\pi^3}{81\sqrt{3}},$$

deinde vero ex numero II fit

$$-\frac{1}{6}\int \frac{dvlv}{1-v} - \frac{1}{12}\int \frac{dv(1+2v)lv}{1+v+vv} = \alpha\alpha \frac{1}{qq} = \frac{\pi\pi}{27};$$

supra autem invenimus esse

$$\int \frac{dvlv}{1-v} = -\frac{\pi\pi}{6},$$

quo valore substituto fit

$$\int \frac{dv(1+2v)lv}{1+v+vv} = -\frac{\pi\pi}{9};$$

maxime igitur operae pretium est visum has postremas integrationes evolvisse.

38. Quodsi ambae formulae integrales

$$\int \frac{dv(1-2v)lv}{1-v+vv} \quad \text{et} \quad \int \frac{dv(1+2v)lv}{1+v+vv}$$

in series convertantur, reperitur

$$\int \frac{dv(1-2v)lv}{1-v+vv} = -1 + \frac{1}{4} + \frac{2}{9} + \frac{1}{16} - \frac{1}{25} - \frac{2}{36} - \frac{1}{49} + \text{etc.}$$

et

$$\int \frac{dv(1+2v)lv}{1+v+vv} = -1 - \frac{1}{4} + \frac{2}{9} - \frac{1}{16} - \frac{1}{25} + \frac{2}{36} - \frac{1}{49} + \text{etc.},$$

unde has duas summationes attentione nostra non indignas assequimur

$$\text{I.}\quad 1 - \frac{1}{4} - \frac{2}{9} - \frac{1}{16} + \frac{1}{25} + \frac{2}{36} + \frac{1}{49} - \frac{1}{64} - \frac{2}{81} - \frac{1}{100} + \text{etc.} = \frac{\pi\pi}{18},$$

$$\text{II.}\ 1 + \frac{1}{4} - \frac{2}{9} + \frac{1}{16} + \frac{1}{25} - \frac{2}{36} + \frac{1}{49} + \frac{1}{64} - \frac{2}{81} + \frac{1}{100} + \text{etc.} = \frac{\pi\pi}{9},$$

quarum prior a posteriore ablata praebet

$$\frac{2}{4} + \frac{2}{16} - \frac{4}{36} + \frac{2}{64} + \frac{2}{100} - \text{etc.} = \frac{\pi\pi}{18},$$

cuius duplum perducit ad hanc

$$1 + \frac{1}{4} - \frac{2}{9} + \frac{1}{16} + \frac{1}{25} - \frac{2}{36} + \text{etc.} = \frac{\pi\pi}{9};$$

quae quoniam cum secunda congruit, veritas utriusque summationis satis confirmatur; quodsi vero secunda a duplo primae subtrahatur, remanebit ista series memorabilis

$$1 - \frac{3}{4} - \frac{2}{9} - \frac{3}{16} + \frac{1}{25} + \frac{6}{36} + \frac{1}{49} - \frac{3}{64} - \frac{2}{81} - \frac{3}{100} + \text{etc.} = 0.$$

quae in periodos 6 terminos complectentes distributa manifestum ordinem in numeratoribus declarat, quippe qui sunt $1,\ -3,\ -2,\ -3,\ +1,\ +6$.

ADDITAMENTUM

39. Quemadmodum superiores integrationes per continuam differentiationem formularum S et T deduximus, ita etiam per integrationem alias et prorsus singulares integrationes impetrabimus; si enim ut supra [§ 3] fuerit

$$S = \int P \frac{dz}{z}$$

existente P formula illa

$$\frac{z^{\lambda-\omega} \pm z^{\lambda+\omega}}{1 \pm z^{2\lambda}},$$

quae praeter *z* etiam exponentem variabilem *m* involvere concipitur, erit per naturam integralium duas variabiles involventium

$$\int Sd\omega=\int\frac{dz}{z}\int Pd\omega,$$

ubi in priore formula integrali $\int Sd\omega$, ubi *z* pro constanti habetur, statim scribi potest $z=1$; hoc igitur lemmate praemisso, quia est

$$\int Pd\omega=\frac{-z^{\lambda-\omega}\pm z^{\lambda+\omega}}{\left(1\pm z^{2\lambda}\right)lz},$$

ambas formulas supra tractatas, nempe *S* et *T,* hoc modo evolvamus, et quia utramque triplici modo expressam dedimus, primo scilicet per seriem infinitam, secundo per formulam finitam ac tertio per formulam integralem, etiam quantitates, quae pro integralibus $\int Sd\omega$ et $\int Td\omega$ resultabunt, erunt inter se aequales.

40. Incipiamus a formula *S,* et cum per seriem fuerit

$$S=\frac{1}{\lambda-\omega}+\frac{1}{\lambda+\omega}-\frac{1}{3\lambda-\omega}-\frac{1}{3\lambda+\omega}+\frac{1}{5\lambda-\omega}+\frac{1}{5\lambda+\omega}+\text{etc.}$$

erit

$$\int Sd\omega=-l(\lambda-\omega)+l(\lambda+\omega)+l(3\lambda-\omega)-l(3\lambda+\omega)-\text{etc.}+C,$$

quam constantem ita definire decet, ut integrale evanescat posito $\omega=0$, quo facto erit

$$\int Sd\omega=l\frac{\lambda+\omega}{\lambda-\omega}+l\frac{3\lambda-\omega}{3\lambda+\omega}+l\frac{5\lambda+\omega}{5\lambda-\omega}+l\frac{7\lambda-\omega}{7\lambda+\omega}+\text{etc.},$$

quae expressio reducitur ad sequentem

$$\int Sd\omega=l\frac{(\lambda+\omega)(3\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega)(9\lambda+\omega)\cdot\text{etc.}}{(\lambda-\omega)(3\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega)(9\lambda-\omega)\cdot\text{etc.}}.$$

Deinde quia per formulam finitam erat $S=\frac{\pi}{2\lambda\cos.\frac{\pi\omega}{2\lambda}}$, erit

$$\int Sd\omega=\int\frac{\pi d\omega}{2\lambda\cos.\frac{\pi\omega}{2\lambda}},$$

ubi si brevitatis gratia ponatur $\frac{\pi\omega}{2\lambda}=\varphi$, ut sit $d\omega=\frac{2\lambda d\varphi}{\pi}$, erit

$$\int S d\omega = \int \frac{d\varphi}{\cos.\varphi};$$

quia igitur novimus esse

$$\int \frac{d\theta}{\sin.\theta} = l\text{tang.}\tfrac{1}{2}\theta,$$

sumamus $\sin.\theta = \cos.\varphi$ sive $\theta = 90^0 - \varphi = \frac{\pi}{2} - \varphi$ erit $d\theta = -d\varphi,$ unde fit

$$\int \frac{-d\varphi}{\cos.\varphi} = l\,\text{tang.}\left(\tfrac{\pi}{4} - \tfrac{1}{2}\varphi\right);$$

quoniam autem est $\varphi = \frac{\pi\omega}{2\lambda}$, erit

$$\tfrac{\pi}{4} - \tfrac{1}{2}\varphi = \frac{\pi(\lambda-\omega)}{4\lambda},$$

unde nostrum integrale erit

$$\int S d\omega = -l\,\text{tang.}\frac{\pi(\lambda-\omega)}{4\lambda} = +l\,\text{tang.}\frac{\pi(\lambda+\omega)}{4\lambda};$$

ex tertia autum formula integrali

$$S = \int \frac{z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{z}$$

colligitur fore

$$\int S d\omega = \int \frac{-z^{\lambda-\omega} + z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{zlz},$$

quod integrale a termino $z = 0$ usque ad terminum $z = 1$ extendi assumitur; sicque tres isti valores inventi inter se erunt aequales. Ac ne ob constantes forte addendas ullum dubium supersit, singulae istae expressiones sponte evanescunt casu $\omega = 0$.

41. Consideremus hinc primo aequalitatem inter formulam primam et secundam, et quia utraque est logarithmus, erit

$$\text{tang.}\frac{\pi(\lambda-\omega)}{4\lambda} = \frac{(\lambda+\omega)(3\lambda-\omega)(5\lambda+\omega)(7\lambda-\omega)\cdot\text{etc.}}{(\lambda-\omega)(3\lambda+\omega)(5\lambda-\omega)(7\lambda+\omega)\cdot\text{etc.}};$$

cum igitur huius fractionis numerator evanescat casibus vel $\omega = -\lambda$ vel $\omega = +3\lambda$ vel $\omega = -5\lambda$ vel $\omega = +7\lambda$ etc., evidens est iisdem casibus quoque tangentem fieri $= 0$; denominator vero evanescit casibus vel $\omega = \lambda$ vel $\omega = -3\lambda$ vel $\omega = 5\lambda$ vel $\omega = -7\lambda$ etc., quibus ergo casibus tangens in infinitum excrescere debet, id quod etiam pulcherrime evenit. Ceterum haec expressio congruit cum ea, quam iam dudum inveni et in *Introductione* exposui.

42. Productum autem istud infinitum per principia alibi stabilita ad formulas integrales reduci potest ope huius lemmatis latissime patentis

$$\frac{a(c+b)(a+k)(c+b+k)(a+2k)(c+b+2k) \text{ etc.}}{b(c+a)(b+k)(c+a+k)(b+2k)(c+a+2k) \text{ etc.}} = \frac{\int z^{c-1}dz\left(1-z^{k}\right)^{\frac{b-k}{k}}}{\int z^{c-1}dz\left(1-z^{k}\right)^{\frac{a-k}{k}}},$$

siquidem post utramque integrationem fiat $z=1$. Nostro igitur casu erit

$$a=\lambda+\omega,\ b=\lambda-\omega,\ c=2\lambda \text{ et } k=4\lambda,$$

unde valor nostri producti erit

$$\frac{\int z^{2\lambda-1}dz\left(1-z^{4\lambda}\right)^{\frac{-3\lambda-\omega}{4\lambda}}}{\int z^{2\lambda-1}dz\left(1-z^{4\lambda}\right)^{\frac{-3\lambda+\omega}{4\lambda}}} = \text{tang.}\frac{\pi(\lambda+\omega)}{4\lambda};$$

formulae autem istae integrales concinniores evadunt statuendo $z^{2\lambda}=y;$ tum enim erit

$$\text{tang.}\frac{\pi(\lambda+\omega)}{4\lambda} = \frac{\int dy(1-yy)^{\frac{-3\lambda-\omega}{4\lambda}}}{\int dy(1-yy)^{\frac{-3\lambda+\omega}{4\lambda}}},$$

quae expressio utique omni attentione digna videtur. Denique ex formula integrali inventa erit quoque

$$\int \frac{-z^{\lambda-\omega}+z^{\lambda+\omega}}{1+z^{2\lambda}} \cdot \frac{dz}{zlz} = \text{tang.}\frac{\pi(\lambda+\omega)}{4\lambda}.$$

43. Operae erit pretium etiam aliquot casus particulares evolvere. Sit igitur primo $\lambda=2$ et $\omega=1$ ac per expressionem infinitam erit

$$\int Sd\omega = l\,\frac{3\cdot5}{1\cdot7}\cdot\frac{11\cdot13}{9\cdot15}\cdot\frac{19\cdot21}{17\cdot23}\cdot\frac{27\cdot29}{25\cdot31}\cdot\frac{35\cdot37}{33\cdot39}\cdot\text{etc.},$$

deinde per expressionem finitam habebimus

$$\int Sd\omega = l\,\text{tang.}\frac{3\pi}{8}$$

ac per formulam integralem

$$\int Sd\omega = \int \frac{-(1-zz)}{1+z^4} \cdot \frac{dz}{lz},$$

tum vero ex aequalitate duarum priorum expressionum

$$\text{tang.}\frac{3\pi}{8} = \frac{3\cdot 5}{1\cdot 7} \cdot \frac{11\cdot 13}{9\cdot 15} \cdot \frac{19\cdot 21}{17\cdot 23} \cdot \text{etc. },$$

hincque per binas formulas integrales

$$\text{tang.}\frac{3\pi}{8} = \frac{\int dy(1-yy)^{-\frac{7}{8}}}{\int dy(1-yy)^{-\frac{5}{8}}}.$$

44. Ponamus nunc esse $\lambda = 3$ et $\omega = 1$ ac per expressionem infinitam erit

$$\int Sd\omega = l\,\frac{2\cdot 4}{1\cdot 5} \cdot \frac{8\cdot 10}{7\cdot 11} \cdot \frac{14\cdot 16}{13\cdot 17} \cdot \frac{20\cdot 22}{19\cdot 23} \cdot \text{etc. },$$

secundo per expressionem finitam

$$\int Sd\omega = l\,\text{tang.}\frac{\pi}{3} = l\sqrt{3} = \frac{1}{2}l3,$$

ita ut futurum sit

$$\sqrt{3} = \frac{2\cdot 4}{1\cdot 5} \cdot \frac{8\cdot 10}{7\cdot 11} \cdot \frac{14\cdot 16}{13\cdot 17} \cdot \text{etc.},$$

huiusque producti valor per formulas integrales erit

$$\frac{\int dy(1-yy)^{-\frac{5}{6}}}{\int dy(1-yy)^{-\frac{2}{3}}}.$$

Denique formula integralis praebebit

$$\int Sd\omega = \int \frac{-(1-zz)}{1+z^6} \cdot \frac{dz}{lz}.$$

45. Eodem modo etiam evolvamus alteram formulam *T*, cuius valor per seriem erat

$$T = \frac{1}{\lambda-\omega} - \frac{1}{\lambda+\omega} + \frac{1}{3\lambda-\omega} - \frac{1}{3\lambda+\omega} + \frac{1}{5\lambda-\omega} - \frac{1}{5\lambda+\omega} + \text{etc.};$$

unde fit

$$\int Td\omega = -l(\lambda-\omega) - l(\lambda+\omega) - l(3\lambda-\omega) - l(3\lambda+\omega) - \text{etc.};$$

quae expressio ut evanescat posito $w=0$, erit

$$\int Td\omega = l\,\frac{\lambda\lambda}{\lambda\lambda-\omega\omega}\cdot\frac{\lambda\lambda}{9\lambda-\omega\omega}\cdot\frac{\lambda\lambda}{25\lambda-\omega\omega}\cdot\text{etc.}$$

deinde vero cum per formulam finitam fuerit $T=\frac{\pi}{2\lambda}\,\text{tang.}\frac{\pi\omega}{2\lambda}$, erit

$$\int Td\omega = \int \frac{\pi d\omega}{2\lambda}\,\text{tang.}\frac{\pi\omega}{2\lambda},$$

ubi posito $\frac{\pi\omega}{2\lambda}=\varphi$ erit

$$\int Td\omega = \int d\varphi\,\text{tang.}\varphi = -l\cos\varphi,$$

ita ut sit

$$\int Td\omega = -l\cos\frac{\pi\omega}{2\lambda},$$

cuius valor casu $\omega=0$ fit sponte $=0$; denique per formulam integralem habebimus

$$\int Td\omega = \int \frac{-z^{\lambda-\omega}-z^{\lambda+\omega}}{1-z^{2\lambda}}\cdot\frac{dz}{zlz};$$

integrale itidem a termino $z=0$ usque ad terminum $z=1$ extendi debet.

46. Iam comparatio duorum priorum valorum hanc praebet aequationem

$$\frac{1}{\cos.\frac{\pi\omega}{2\lambda}} = \frac{\lambda\lambda}{\lambda\lambda-\omega\omega}\cdot\frac{9\lambda\lambda}{9\lambda-\omega\omega}\cdot\frac{25\lambda\lambda}{25\lambda-\omega\omega}\cdot\frac{49\lambda\lambda}{49\lambda-\omega\omega}\cdot\text{etc.},$$

$$\cos.\frac{\pi\omega}{2\lambda} = \left(1-\frac{\omega\omega}{\lambda\lambda}\right)\cdot\left(1-\frac{\omega\omega}{9\lambda\lambda}\right)\cdot\left(1-\frac{\omega\omega}{25\lambda\lambda}\right)\cdot\left(1-\frac{\omega\omega}{49\lambda\lambda}\right)\cdot\text{etc.},$$

vel si factores singuli iterum in simplices evolvantur, erit

$$\cos.\frac{\pi\omega}{2\lambda} = \frac{\lambda+\omega}{\lambda}\cdot\frac{\lambda-\omega}{\lambda}\cdot\frac{3\lambda+\omega}{\lambda}\cdot\frac{3\lambda-\omega}{\lambda}\cdot\frac{5\lambda+\omega}{\lambda}\cdot\frac{5\lambda-\omega}{\lambda}\cdot\text{etc.},$$

quae formula cum reductione generali supra [§ 42] allata comparata dat $a=\lambda+\omega,\ b=\lambda,\ c=-\omega$ et $k=2\lambda,$ unde colligimus

$$\cos.\frac{\pi\omega}{2\lambda} = \int \frac{z^{-\omega-1}dz\left(1-z^{2\lambda}\right)^{-\frac{1}{2}}}{z^{-\omega-1}dz\left(1-z^{2\lambda}\right)^{\frac{\omega-\lambda}{2\lambda}}}.$$

Ut autem exponentes negativos $z^{-\omega-1}$ evitemus, superius productum ita repraesentemus

$$\cos.\frac{\pi\omega}{2\lambda} = \frac{\lambda+\omega}{\lambda} \cdot \frac{\lambda-\omega}{\lambda} \cdot \frac{3\lambda+\omega}{3\lambda} \cdot \frac{3\lambda-\omega}{3\lambda} \cdot \frac{5\lambda+\omega}{5\lambda} \cdot \frac{5\lambda-\omega}{5\lambda} \cdot \text{etc.},$$

eritque facta comparatione $a = \lambda - \omega,\ b = \lambda,\ c = +\omega$ et $k = 2\lambda$ sicque per formulas integrales erit

$$\cos.\frac{\pi\omega}{2\lambda} = \int \frac{z^{-\omega-1} dz \left(1 - z^{2\lambda}\right)^{-\frac{1}{2}}}{z^{\omega-1} dz \left(1 - z^{2\lambda}\right)^{\frac{-\lambda-\omega}{2\lambda}}},$$

quae expressio ad simpliciorem formam reduci nequit.

47. Sit nunc etiam $\lambda = 2$ et $\omega = 1$ eruntque ternae nostrae expressiones

$$\text{I. } \int T d\omega = l\,\frac{4}{3} \cdot \frac{36}{35} \cdot \frac{100}{99} \cdot \frac{196}{195} \cdot \text{etc. sive } \int T d\omega = l\,\frac{2\cdot2}{1\cdot3} \cdot \frac{6\cdot6}{5\cdot7} \cdot \frac{10\cdot10}{9\cdot11} \cdot \frac{14\cdot14}{13\cdot15} \cdot \text{etc.};$$

$$\text{II. } \int T d\omega = -l \cos.\frac{\pi}{4} = +\frac{1}{2} l2,$$

ita ut sit $\frac{1}{2}\sqrt{2} = \frac{2\cdot2}{1\cdot3} \cdot \frac{6\cdot6}{5\cdot7} \cdot \frac{10\cdot10}{9\cdot11} \cdot \frac{14\cdot14}{13\cdot15} \cdot \text{etc.}$

quod productum per formulas integrales ita exprimitur

$$\frac{\int dz \left(1 - z^4\right)^{-\frac{3}{4}}}{\int dz \left(1 - z^4\right)^{-\frac{1}{2}}} = \sqrt{2};$$

$$\text{III. } \int T d\omega = \int \frac{-(1+zz)}{\left(1 - z^4\right)} \cdot \frac{dz}{lz} = \int \frac{-dz}{(1-zz) lz},$$

quod ergo integrale a termino $z = 0$ usque ad $z = 1$ extensum praebet eundem valorem $+\frac{1}{2} l2$, cuius aequalitatis ratio utique difficillime patet.

48. Sit denique ut supra $\lambda = 3$ et $\omega = 1$ ac ternae formulae ita se habebunt

$$\text{I. } \int T d\omega = l\,\frac{9}{8} \cdot \frac{81}{80} \cdot \frac{225}{224} \cdot \text{etc.} = l\,\frac{3\cdot3}{2\cdot4} \cdot \frac{9\cdot9}{8\cdot10} \cdot \frac{15\cdot15}{14\cdot16} \cdot \frac{21\cdot21}{20\cdot22} \cdot \text{etc.};$$

$$\text{II. } \int T d\omega = -l \cos.\frac{\pi}{6} = -l\,\frac{\sqrt{3}}{2} = +l\,\frac{2}{\sqrt{3}},$$

ita ut sit $\frac{2}{\sqrt{3}} = \frac{3\cdot3}{2\cdot4} \cdot \frac{9\cdot9}{8\cdot10} \cdot \frac{15\cdot15}{14\cdot16} \cdot \frac{21\cdot21}{20\cdot22} \cdot \text{etc.}$

ideoque per binas formulas integrales

$$\frac{2}{\sqrt{3}} = \frac{\int dz\left(1-z^{6}\right)^{-\frac{1}{2}}}{\int dz\left(1-z^{6}\right)^{-\frac{2}{3}}};$$

$$\text{III.} \quad \int T d\omega = \int \frac{-(1+zz)}{\left(1-z^{6}\right)} \cdot \frac{dz}{lz},$$

quae posito $zz = v$ abit in hanc

$$\text{III.} \quad \int T d\omega = \int \frac{-dv(1+v)}{\left(1-v^{3}\right) lv}.$$

Hinc igitur patet hac methodo plane nova perveniri ad formulas integrales, quas per methodos adhuc cognitas nullo modo evolvere vel saltem inter se comparare licuit.