CHAPTER IV

CONCERNING THE RAISING OF WATER WITH THE AID OF PUMPS

PROBLEM 57

101. *If the cylindrical tube* (Fig. 57) *BbCc greatly enlarged below to Aa shall be immersed in still water Ee, and in that tube the piston POo is drawn upwards by a given force, so that on account of the atmospheric pressure water shall enter continually, to define the motion of the ascent of this water.*

SOLUTION

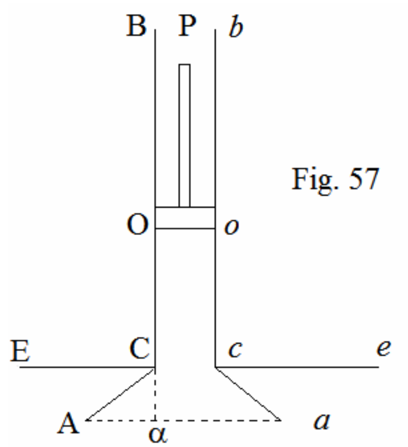


Fig. 57

In the elapsed time t the piston with water now shall be raised to the height $CO = x$, and the cross-section of the tube shall be $Oo = ff$ and the speed of the ascent both of the piston and the water shall be $= \upsilon$. Moreover the force of the piston drawing upwards shall be $= ffu$, and since the piston shall be depressed by the atmospheric pressure k, the pressure at $Oo = k - u$, if no motion may be present, but since the motion shall introduce a change, that may be put $= \pi$, until it may be determined from the following. Again the widest cross-section Aa may be put $= cc$ and for its depth below the surface of the water $C\alpha = \alpha$, and the pressure at $Aa = k + \alpha$. With these in place the solution of the problem 55 will be adapted to this, if we may put $L = k + \alpha$, $\mathfrak{a} = -\alpha$, $\mathfrak{o} = x$, and since cc is very great, in place of $\mathfrak{O} = \int \frac{ds}{\omega}$ we will have $\frac{x}{ff}$, but since there it was k, for us here it is π; from which there becomes $L - k + \mathfrak{a} - \mathfrak{o} = k - \pi - x$.
And thus we arrive at this equation:

$$4g\left(k - \pi - x\right)dt - \upsilon\upsilon dt = 2xd\upsilon;$$

moreover the height of the element gives $dt = \frac{dx}{\upsilon}$ and there becomes:

$$2x\upsilon d\upsilon + \upsilon\upsilon dx = 4g\left(k - \pi - x\right)dx$$

and on integrating [w.r.t. x],

$$\upsilon\upsilon x = 4g\int\left(k - \pi - x\right)dx,$$

therefore there becomes hence

$$\upsilon\upsilon x = 4g\left(kx - \tfrac{1}{2}xx - \int \pi dx\right)$$

and now finally it remains at this stage, that we may investigate the unknown pressure π ; the value of which it will be necessary to find from the motion of the piston. Therefore we may put the mass of the whole piston to be equal to the mass of the water, of which the volume is $= ffh,$ by which likewise its weight is expressed [as the density of water is taken to be 1], and since the friction of the piston especially opposes the motion, that may be put $= \delta ffh$. Now on account of the pressure between the piston and the water $= \pi$ from that the piston will be forced upwards by a motive force $= \pi ff$, to which may be added the actual force lifting upwards *ffu*; truly from the sum the atmospheric pressure may be subtracted ffk, thus so that the force driving upwards shall be

$$= ff\left(\pi + u - k\right),$$

from which again the resistance arising both from the weight of the piston as well as that arising from the friction must be taken away, which is $= (1+\delta)\, ffh,$ from which on account of the mass being moved = *ffh*, the accelerating force produced

$$= \tfrac{\pi + u - k - (1+\delta)h}{h}.$$

Now since the speed of the piston directed upwards is υ, from which in the time increment *dt* it may be raised through the distance increment *dx*, therefore the acceleration $= \frac{d\upsilon}{dt} = \frac{\upsilon d\upsilon}{dx},$ from which this equation arises

$$\tfrac{\upsilon d\upsilon}{dx} = \tfrac{4g}{h}\left(\pi + u - k - (1+\delta)h\right),$$

which multiplied by $2hdx$ and integrated produces

$$h\upsilon\upsilon = 4g\left(\int \pi dx + \int udx - kx - (1+\delta)hx\right),$$

truly before we have found:

$$x\upsilon\upsilon = 4g\left(kx - \tfrac{1}{2}xx - \int \pi dx\right);$$

from the addition of which equation the unknown formula $\int \pi dx$ is elicited and there arises

$$(h+x)\upsilon\upsilon = 4g\left(\int udx - (1+\delta)hx - \tfrac{1}{2}xx\right),$$

from which equation the speed at some height $CO = x$ is determined. But if we may eliminate $\upsilon\upsilon$ from these two equations, we will arrive at this equation:

$$(h+x)\int \pi dx - k(h+x)x + x\int udx - \left(\tfrac{1}{2}+\delta\right)hxx = 0$$

or

$$\int \pi dx = kx + \frac{\left(\frac{1}{2}+\delta\right)hxx}{h+x} - \frac{x\int udx}{h+x},$$

which difference shows that unknown pressure

$$\pi = k + \frac{\left(\frac{1}{2}+\delta\right)h(2hx+xx)}{(h+x)^2} - \frac{h\int udx}{(h+x)^2} - \frac{ux}{h+x},$$

which thus it will be required to know the extent, so that, when that becomes negative, we may know the piston to follow the water to further, but an empty space continually to be left between that and the water, on which the calculation depends, removed from the middle.

COROLLARY 1

102. Therefore since we may have found to be :

$$\upsilon\upsilon = \frac{4g\left(\int udx - (1+\delta)hx - \frac{1}{2}xx\right)}{h+x},$$

unless this quantity shall be positive, no motion shall be produced; therefore now at the start of the first motion, where $x = 0$, there must be $u > (1+\delta)h$. Truly there is a need for a greater force after the start with x increasing continually.

COROLLARY 2

103. If the lifting force u shall be constant, there will become :

$$\upsilon\upsilon = \frac{4gx\left(u - (1+\delta)hx - \frac{1}{2}xx\right)}{h+x},$$

from which it is apparent the speed υ, which will increase from the start, to decrease again and finally to vanish, as it will vanish $x = 2u - 2(1+\delta)h$, but then the pressure produced

$$\pi = k + \left(\frac{1}{2}+\delta\right)h - u + \frac{hh}{4\left(u-\left(\frac{1}{2}+\delta\right)h\right)},$$

which while it may not become negative, there the water will follow as far as the piston.

COROLLARY 3

104. Therefore so that we may know, to how great a height it may be allowed to raise the water, we shall make that pressure to be vanishing and for the sake of brevity putting

$$u-\left(\tfrac{1}{2}+\delta\right)h=r$$

there will become:

$$4kr-4rr+hh=0 \text{ and hence } r=\tfrac{1}{2}k+\tfrac{1}{2}\sqrt{(hh-kk)}$$

and

$$u=\left(\tfrac{1}{2}+\delta\right)h+\tfrac{1}{2}k+\tfrac{1}{2}\sqrt{(hh+kk)}.$$

And by this force the water will be raised to the height

$$u=k+\sqrt{(hh+kk)}-h,$$

from which it is apparent, when the mass of the piston shall be smaller, this height to become greater there, which thus may be able to be increased as far as to $2k$, if there shall be $h=0$. But it deserves to be noted that no effect of friction has been introduced.

SCHOLIUM 1

105. Thus we join the wider part *ccAa* below to the tube *BbCc*, so that there would be no need for water to be entering the tube suddenly with a finite speed, as before it would have been at rest; but a straight forwards account of this has been passed over in the calculation, we understand the same water to be raised as if a whole cylindrical tube were used, and its lower opening *Cc* were immersed in still water. Nor truly is it required to demand then to accept water entering suddenly with a finite speed through *Cc*, but rather motion of this kind is generated in the water around the external opening *Cc*, as if such a wider part *CcAa* were connected. Then at first it was necessary with the generation of the motion to include the motion of the piston both of its inertia as well as to have an account of the friction, because a notable part of the forces acting is expended with these obstacles requiring to be overcome, which provides the greatest moment in the initial motion. If indeed, because it is unable to happen, the inertia of the piston as well as the friction may vanish, so that the lifting force plainly may be able to be put together without any mass being moved, on account of $h=0$ there would become $vv=\frac{4g\left(\int udx-\frac{1}{2}xx\right)}{x}$, and thus if the force u were finite, on putting $x=0$ at once from the beginning the speed in the water thus will arise infinite, indeed soon requiring to be diminished; truly this inconvenience of the calculation also at no time can be found in the world, since no forces are given, which may not themselves be joined together having a certain proper mass requiring to be moved.

SCHOLIUM 2

106. A cylinder of this kind constructed with a piston may be called a pump, with the aid of which while the piston is drawn upwards, with the lower opening *Cc* immersed in still water, likewise water is raised in the cylinder, or rather is forced in by the atmospheric pressure. Indeed even if the atmospheric pressure *k* is not found in the formula found for the speed υ , yet evidently it is involved in that condition, because water may not follow the piston rising in the tube, unless the pressure π between the piston and the water shall be positive; for if the atmospheric pressure *k* were zero, from the initial position $x=0$ at once the pressure π must be zero, nor therefore will the water follow the piston and likewise the whole calculation for the following motion continually being raised itself may fail. From which it is apparent only the atmospheric pressure *k* to be the cause, why water will be raised in these pumps. But here contrary to popular opinion our calculation declares it may be possible so that water with a length beyond the height *k*, which is estimated to be 32 feet, and thus may be raised almost twice as great, only if the inertia of the piston *h* may be small enough and the raising force large enough. Moreover from Corollary 3 it is necessary for this, that the force raising the piston shall be :

$$u=\left(\tfrac{1}{2}+\delta\right)h+\tfrac{1}{2}k+\tfrac{1}{2}\sqrt{(hh+kk)},$$

where the water may be able to be raised to the height $k+\sqrt{(hh+kk)}-h$, then truly at some height less than x for motion with the speed υ, there will become

$$\upsilon\upsilon=\frac{2gx\left(k+\sqrt{(hh+kk)}-h-x\right)}{h+x};$$

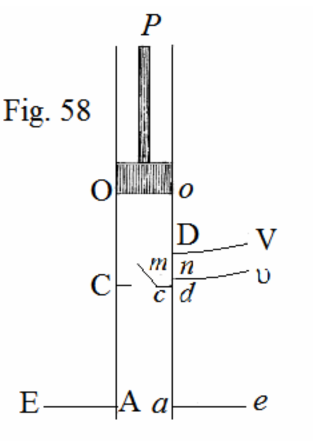
which speed becomes a maximum, when $x=-h+\sqrt{\left(hk+h\sqrt{(hh+kk)}\right)}$,
and that maximum speed will be

$$=\left(\sqrt{\left(hk+h\sqrt{(hh+kk)}\right)}-h\right)\sqrt{\tfrac{2g}{h}}.$$

If, for the sake of an example, there may be $h=\tfrac{3}{4}k$, water will be able to be raised to the height $=\tfrac{3}{2}k$ by the force acting $u=\tfrac{3}{2}k+\tfrac{3}{4}\delta k$, and the maximum speed will become $=\tfrac{3-\sqrt{3}}{2}\sqrt{2gk}$, by which in one second it will run through a distance of $19\tfrac{1}{2}$ feet.

SCHOLIUM 3

Fig. 58



107. Moreover in the use of pumps of this kind (Fig. 58), where water will be accustomed to be sent to a much greater height after it has been raised in the manner shown from the lowering of the piston, generally the height of the pump is accustomed to be small enough, thus so that this case of so great a height by no means may occur, nor much less shall it be true, that the water may follow the piston. Such pumps have a diaphragm with a hole bored through at the section *Cc*, which valve *m* thus is closed, so that, while the piston may draw the water, this valve is open, opening up a path for the water below to be rising. Then truly generally this section *Cc* is not placed on the surface of the still water *Ee*, but at some height *AC* above that, as the circumstances are seen to demand, thus so that this lower part *CA* of the tube may remain always filled with water and thus it shall be required to be raised to the cavity above. But on this account the adjoining tube, if we may put its height above the still water $AC = \alpha$ with the distance above remaining $CO = x$, the preceding determination demands some change, where there becomes :

$$\upsilon\upsilon = \frac{4g\left(\int udx - \alpha x - (1+\delta)hx - \frac{1}{2}xx\right)}{h+\alpha+x},$$

if indeed the lower tube CA shall be full of water and the above tube *BC*, if it may not be large, in the denominator some small quantity α must be taken, since this part arises from the quantity $\mathfrak{O} = \int \frac{ds}{\omega}$, in the numerator truly α always specifies the same height AC. Indeed when the piston *Oo* will have been raised to a certain height, and then again depressed and likewise the valve *m* is closed, but below another of the pumps *DdV*υ has been put in place, the opening *Dd* of which the valve *n* was closed so far, now truly it is open with the piston depressed, so that water raised before may be sent through the tube *DdV*υ, we will investigate how this motion may arise in the following problem.

PROBLEM 58

108. *Since the pump BbCc* (Fig. 59) *will have been filled with water as far as to Bb, then truly the piston may have pushed down with a given force and water may be expelled through some tube DdZz, which tube now we assume to have been filled with water from the beginning, hence to investigate this motion, by which water may be ejected through the opening Zz.*

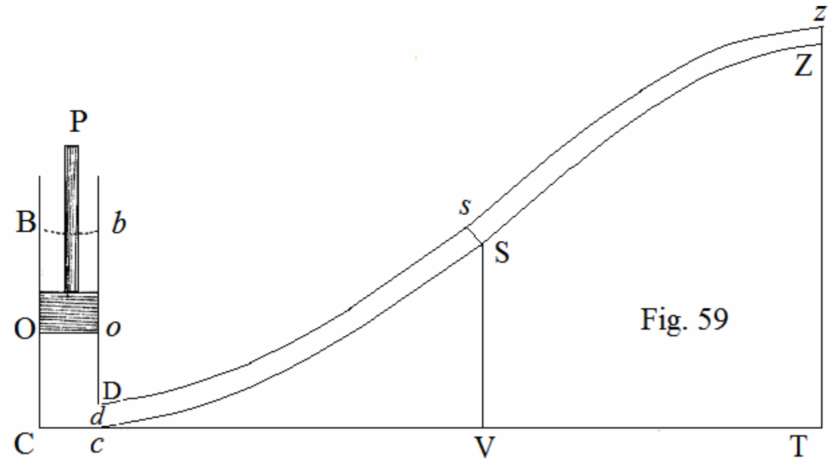


Fig. 59

SOLUTION

As before the cross-section will be *ff*, the height $BC = b$, and with the laps of time *t* the piston now shall be pressing down at *Oo*, where the speed of the piston downwards shall be $= \upsilon$ and the height $CO = x$. Again the force of the piston pressing down shall be $= ffu$, from which the pressure $= \pi$ arises on the surface *Oo*, then by comparison the motion of the piston is required to be defined. Now in the adjoining tube *Dz* the opening shall be *Zz* with the cross-section $Zz = ee$ and the height $TZ = a$, and the speed of the efflux through this opening $= \frac{ff\upsilon}{ee}$. Then at the place of any middle section *Ss* the length of the tube will be $DS = s$, the cross-section $Ss = \omega$ and the height $VS = z$, moreover the pressure at $Ss = p$, with which in place, since at *Ss* the speed shall be $\mathfrak{T} = \frac{ff\upsilon}{\omega}$, the motion principle provides this equation

$$2gp = \Delta : t - 2gz - \frac{f^4\upsilon\upsilon}{2\omega\omega} - \frac{ffd\upsilon}{dt}\int \frac{ds}{\omega},$$

where $\int \frac{ds}{\omega}$ must be extended from *Oo* as far as to *Ss*. But the value of this in the tube *OC* is $= \frac{x}{ff}$, then the value arising through the whole length of the adjoining tube *DZ* may be called $=D$. Hence, since the pressure at *Oo* is $=\pi$, there will be $2gp=\varDelta:t-2gx-\frac{1}{2}\upsilon\upsilon$ and on account of the pressure at *Zz* being equal to the atmospheric pressure $=k$, there will become here:

$$2gk=\varDelta:t-2ga-\tfrac{f^4\upsilon\upsilon}{2e^4}-\tfrac{ffd\upsilon}{dt}\left(D+\tfrac{x}{ff}\right),$$

of which this equation taken from that gives

$$2g(\pi-k)=2g(a-x)+\tfrac{1}{2}\upsilon\upsilon\left(\tfrac{f^4}{e^4}-1\right)+\tfrac{ffd\upsilon}{dt}\left(D+\tfrac{x}{ff}\right).$$

Truly since in the time increment *dt* the height *x* is diminished by the element *dx* with the speed υ, there will become $dt=-\frac{dx}{\upsilon}$, there will become

$$4g(\pi-k-a+x)dx=\left(\tfrac{f^4}{e^4}-1\right)\upsilon\upsilon dx-2ff\upsilon d\upsilon\left(D+\tfrac{x}{ff}\right).$$

We may put $D=\frac{m}{ff}$ and $\frac{f^4}{e^4}-1=\lambda$, so that there shall be

$$4g(\pi-k-a+x)dx=\lambda\upsilon\upsilon dx-2(m+x)\upsilon d\upsilon,$$

which divided by $(m+x)^{\lambda+1}$ and integrated provides

$$4g\int\tfrac{\pi-k-a+x}{(m+x)^{\lambda+1}}dx=\text{Const.}-\tfrac{\upsilon\upsilon}{(m+x)^{\lambda}},$$

where it will be necessary to define the constant thus, so that on putting $x=b$ the speed υ vanishes. Now for the motion of the piston put in place, with its weight $=ffh$, friction $=\delta ffh$, this is pushed downwards by the force

$$=ff(k+u+h-\pi-\delta h),$$

$$\upsilon\upsilon=\text{Const.}-\frac{4g\left(kx+(1-\delta)hx+\int udx-\int\pi dx\right)}{h},$$

truly for the elimination of the pressure π from the differential equations we may use rather :

$$4g(\pi-k-a+x)dx=\lambda\upsilon\upsilon dx-2(m+x)\upsilon d\upsilon$$

and

$$4g(k+u+h-\delta h-\pi)dx=-2h\upsilon d\upsilon,$$

and from which the sum is deduced :

$$4g\int\frac{(u-a+(1-\delta)h+x)dx}{(h+m+x)^{\lambda+1}}=\text{Const.}-\frac{\upsilon\upsilon}{(h+m+x)^{\lambda}},$$

where the value of the constant of the integral formula is required to be attributed, which it receives on making $x=b$, if indeed that may be integrated thus, so that it may vanish on putting $x=0$.

COROLLARY 1

109. If the interval may be put to be $BO=y$, on account of $x=b-y$ the equation of motion being defined will become :

$$\frac{\upsilon\upsilon}{(h+m+b-x)^{\lambda}}=4g\int\frac{(u-a+(1-\delta)h+b-y)}{(h+m+b-y)^{\lambda+1}}dy$$

thus with the integral taken, so that it may vanish on putting $y=0$.

COROLLARY 2

110. Therefore so that the piston at least may be able to impress a motion on the water, it is necessary, the force acting shall become $u>a+\delta h-h-b$; then truly the motion will be accelerated from the beginning, and the maximum will arise, when there becomes $y=u-a+(1-\delta)h+b$. Therefore if there were $u<a+\delta h-h$ thus it may be contained between the limits $a-(1-\delta)h$ and $a-(1-\delta)h-b$, but the motion, when it follows after the maximum speed, again will be retarded.

COROLLARY 3

111. If the height $TZ=a$ to which the water must be raised, were very great besides the height of the pump $BO=b$, also the quantity $m=Dff=ff\int\frac{ds}{\omega}$ will be very large, because, if the cross-section of the tube everywhere were $=ff$, there may become $m=DSZ$. Therefore in this case on account of the constant $h+m+b-y$, there will be had

$$(h+m+b)\upsilon\upsilon=4g\int\big(u-a+b+(1-\delta)h\big)dy$$

or

$$(h+m+b)\upsilon\upsilon=4g\int\Big(udy-\big(ab-(1-\delta)h\big)dy\Big):$$

nor here does the largest cross-section of the orifice $Zz=ee$ enter into the computation.

SCHOLIUM

112. Here I have not defined the time, in which the whole height BC is depressed by the piston and the volume of the water contained in that $= bff$ is ejected by the opening Zz, since the force of the piston acting ffu or the magnitude u is not yet known; nor indeed may that be allowed to be chosen arbitrarily, since the effectiveness of any particular speed will be agreed on from the nature of the nature forces. But if this effect must be obtained with the aid of the weight of the same piston imposed, there is no need for a particular investigation, since that same weight can be shown jointly by ffh; then truly, since friction alone affects the piston, the number δ emerges smaller by so much, so that δffh may represent the magnitude of the friction. Therefore with this agreed on the magnitude u may be present in h, there will become :

$$\frac{\upsilon\upsilon}{(h+m+b-x)^{\lambda}} = 4g\int\frac{(1-\delta)hu-a+b-y}{(h+m+b-y)^{\lambda+1}}dy$$
$$= -\frac{4g(h+m+b-y)^{1-\lambda}}{1-\lambda} - \frac{4g(a+m+\delta h)}{\lambda}(h+m+b-y)^{-\lambda}$$
$$+\frac{4g(h+m+b)^{1-\lambda}}{1-\lambda} + \frac{4g(a+m+\delta h)}{\lambda}(h+m+b)^{-\lambda},$$

and with the calculation set out:

$$\upsilon\upsilon = -\frac{4g}{\lambda(1-\lambda)}\Big(\lambda(h+b-y)+m+(1-\lambda)(a+\delta h)\Big)$$
$$+\frac{4g(h+m+b-y)^{\lambda}}{\lambda(1-\lambda)(h+m+b)^{\lambda}}\Big(\lambda(h+b)+m+(1-\lambda)(a+\delta h)\Big).$$

From which indeed it is apparent there must be $h > a-b+\delta h$, because otherwise indeed no motion may begin, then, if y is very small with respect to $h+m+b$, there will be approximately

$$\upsilon\upsilon = \frac{2g}{h+m+b}\left(2(h+a-\delta h+b)y + \frac{(1-\lambda)(h+a-\delta h+b)}{h+m+b}yy - yy\right).$$

But because it cannot be agreed to be the only force to be considered, by which the water now drawn off is driven forwards, but for that to be the first force, by which water is drawn off into the pump, and now we are going to examine both the preceding problems treated together: the one pump drawing the water in, and at the same time the other driving it forwards.

PROBLEM 59

113. *If the two similar pumps* (Fig. 60) *BbCc and B'b'C'c' may be acted on by a similar force, of which that one may draw off the water, this truly may be driven through the opening D' to the height* $= a$*, as we have established in the preceding problem, to define the motion in each pump.*

SOLUTION

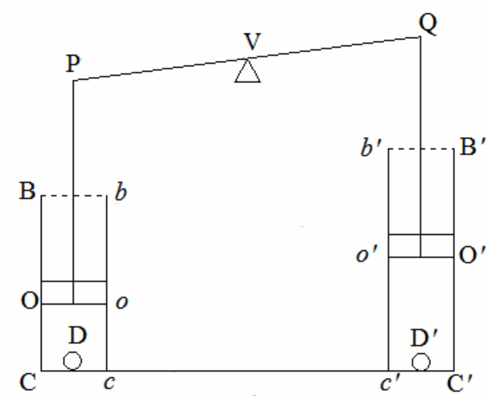


Fig. 60

As before the height of each pump shall be $BC = b$ and the cross-section $= ff$, the mass of each piston $= ffh$ and the friction $= \delta ffh$. At the same time the piston will begin to be raised from the base *Cc*, and the other to be pressed down descending from the top *B'b'*, but the two pestles shall be joined above by the lever *PQ* moveable about the middle V, thus so that for any time, however much the piston *Oo* has risen, the other $O'o'$ shall depress below $B'b'$ just as much, and each motion shall be performed with an equal speed. Now we may put the lever to be depressed at Q by the force $= Vff$, that we may consider as known, and from that arises the force of the piston *Oo* upwards $= Pff$, truly from the other part the force of the piston *O'o'* pressing downwards $= Qff$, thus so that there shall become $P+Q=V$. Again the distance may be called $CO = B'O' = x$ and the speed of each piston $= \upsilon$. And for the motion of the former, from paragraph 107 on account of $u = P$, we will have :

$$\upsilon\upsilon = \frac{4g\left(\int udx - \alpha x - (1+\delta)hx - \frac{1}{2}xx\right)}{h+\alpha+x},$$

truly from paragraph 109, for the motion of the latter, which here becomes $u = Q$ and $y = x$, there will become:

$$\frac{\upsilon\upsilon}{(h+m+b-x)^{\lambda}} = 4g\int\frac{Q-a+(1-\delta)h+b-x}{(h+m+b-y)^{\lambda+1}}dx.$$

But rather the differential equations of these may be considered, which are

$$2\upsilon d\upsilon\left(h+\alpha+x\right)+\upsilon\upsilon dx=4gdx\left(P-\alpha-\left(1+\delta\right)-x\right),$$
$$2\upsilon d\upsilon\left(h+m+b-x\right)+\lambda vvdx=4gdx\ (Q-a+\left(1-\delta\right)h+b-x),$$

which added in turn on account of $P+Q=V$ provide for each motion

$$2\upsilon d\upsilon\left(2h+m+b+\alpha\right)+\left(\lambda+1\right)\upsilon\upsilon dx=4gdx\left(V-\alpha-a-2\delta h+b-2x\right).$$

Therefore by putting for the sake of brevity $\frac{\lambda+1}{2h+m+b+\alpha}=\mu$, on integrating there will become:

$$\upsilon\upsilon=\tfrac{4g\varepsilon^{-\mu x}}{2h+m+b+\alpha}\int\varepsilon^{\mu x}dx\left(V-\alpha-a-2\delta h+b-2x\right),$$

thus with the integral taken, so that it may vanish on putting $x=0$, where the letters α and *m* indicate the same, as in the preceding problems, and with *ee* denoting the cross-section of the upper opening *Zz* there became $\lambda=\frac{f^4}{e^4}-1.$

COROLLARY 1

114. Therefore with whatever movement of the lever *PVQ* , by which the arm *VP* may be raised, truly the other *VQ* is depressed, the pump *BC* is refilled with water, truly the pump *B'C'* is emptied, while the water contained in that is raised to a higher altitude *a* ; but the mass of each water is $=bff$.

COROLLARY 2

115. With this movement finished, if the following arm *VQ* may be raised in the same way and the other *VP* is depressed, the pump *B'C'* is refilled with water again, truly with the water from the other *BC*, which had been refilled, is ejected, and that to the same height, but if the tubes from each pump at *D* and *D'* to be inserted into the tube of water below they may be united in raising the water upwards.

COROLLARY 3

116. Therefore with such reciprocal motion of the lever *PVQ* the water rises up continually and with the individual movements a volume of water $=bff$ carried up higher is ejected from the orifice *Zz* ; and this effect is produced by the force from the moving lever, which is equal to the weight of the water, the volume of which is $=Vff$.

SCHOLIUM

117. It is understood from the formula found for the acceleration, how that force *Vff*, by which we understand the lever to be moved, may be able to be prepared, so that it will be required for this effect to be produced equal. Clearly it is evident from any kind of disturbance to start there must become $V > \alpha + a + 2\delta h - b$, where α is the depth of the still water, from which the water is raised, below the level *Cc* of each pump and the height *a* likewise above, to which the water may be raised, thus so that $\alpha + a$ may show the whole height of the elevation, which were it were greater, certainly that demands a greater force. Then truly $2\delta h$ expresses the friction, what each piston encounters in its motion, which equally must be overcome by the force acting above. Finally the height of the pump *b* must be subtracted from the sum $\alpha + a + 2\delta h$, since the water contained in that helps with its own weight. Truly again for the determination of this motion the quantities 2*h* and *m* concur, of which 2*h* contains the inertia of each piston, which also contains the inertia of the lever *PVQ* as well that which is appropriate for the force acting, it will be required to take jointly, truly the magnitude *m* from the length of the outlet tube, then from its cross-section there is defined, so that there shall become $m = \int ff \frac{ds}{\omega}$, with *s* denoting the indefinite length of this tube *Ds* (Fig. 59) and ω its cross-section *Ss* in this place. Finally also the opening of the upper outlet tube $Zz = ee$ enters into the calculation, and it is present in the number $\lambda = \frac{f^4}{e^4} - 1$; from which it is understood the determination of the motion to be most difficult, hence since it may be unable to be treated in the general formula of the time $dt = \frac{dx}{\upsilon}$. Moreover we will encounter these difficulties , if we may adapt the action of any machine to be determined only more by this motion it is required produce.

PROBLEM 60

118. *If the lever (Fig. 61) PVQ, which we have used in the preceding problem for operating the two pumps, alternately may be depressed and raised, or with the aid of the handle of the curved axis MFN to be turning uniformly in a vertical circle: to define the forces, by which this curved axis will require to be moved, so that the effect described before may be produced.*

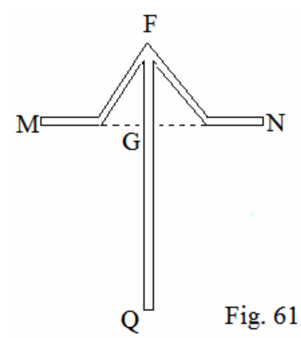


Fig. 61

SOLUTION

The reciprocal motion of a lever of this kind *PVQ*, such as we have described is accustomed to be effected with the help of a horizontal axis *MN* bent at *F*, which bears at *F* the rigid rod *FQ* with the other end *Q* connected thus, so that, while this axis turns in a vertical circle [about the centre *G*], so that the first end *Q* with the help of the rod *FQ* may be depressed through the distance $= 2FG$, then truly it is raised in turn through the whole distance; and thus in some manner each piston of the pump is depressed and raised by the rotation of the axis MFN . Whereby, so that (Fig. 62) each piston will be acted on through the whole height of the pump $BC = b$, it shall be required that $FG = \frac{1}{2}b$, and the upper end of the rod FQ will be moved along the periphery of the vertical circle FSH , which motion I assume to be uniform, henceforth may be able to be applied to the machine more conveniently. Initially the rigid rod will hold the upper position at *F*, so that the end *Q* of the rod would be at *I*, and we may put the length of the rod $FI = l$, and the motion, so that its end *F* is carried along the periphery of the circle, the speed shall be $= c$. Now in the elapsed time t the upper end of the rod will arrive at *S*, and the angle shall be $FGS = \varphi$; the arc will be $FS = \frac{1}{2}b\varphi$ and the element of the time $dt = \frac{bd\varphi}{2c}$; truly the rod now will hold the position *SQ*, so that there shall become $SQ = l$ and the distance $IQ = x$, since now each piston we may put now to have moved forwards through the distance *x*. Therefore since there shall be $GI = l - \frac{1}{2}b$, there will become

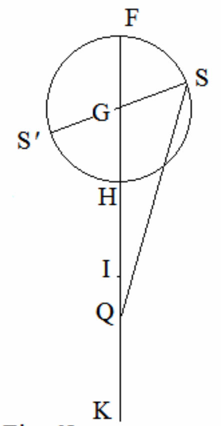


Fig. 62

$$GQ = l - \tfrac{1}{2}b + x \text{ and } \cos\varphi = \frac{bl - \frac{1}{2}bb - (2l-b)x - xx}{bl - \frac{1}{2}bb + bx},$$

and from the angle φ the interval *x* is defined thus, so that there shell become

$$x = \sqrt{(ll - \tfrac{1}{4}bbsin^2\varphi)} - l + \tfrac{1}{2}b - \tfrac{1}{2}bcos\varphi,$$

hence by differentiation there shall become

$$dx = \frac{-\frac{1}{4}bbd\varphi sin\varphi cos\varphi}{\sqrt{(ll - \frac{1}{4}bbsin^2\varphi)}} + \tfrac{1}{2}bd\varphi\sin\varphi,$$

But there is $dt = \frac{dx}{\upsilon} = \frac{bd\varphi}{2c}$ and thus $\upsilon = \frac{2c}{b}\cdot\frac{dx}{d\varphi}$ or

$$\upsilon = c\sin\varphi - \frac{\frac{1}{2}bcsin\varphi cos\varphi}{\sqrt{(ll - \frac{1}{4}bbsin^2\varphi)}}.$$

Now we may put the force needed $= Sff$ required to be moving the end of the rod *S* in a circle, the direction of which since it shall be normal to the radius GS , will give for the direction *SQ* a force $= \frac{Sff}{\sin GSQ}$, by which the rod is propelled following its direction, therefore there the point *Q* is depressed by a force

$$= \frac{Sff\cos GQS}{\sin GSQ},$$

which is that same force, which I have called *Vff* in the preceding problem, since it shall become $V = \frac{S\cos GQS}{\sin GSQ}$.
Truly there is

$$\sin GSQ = \frac{GQ\cdot\sin\varphi}{l}$$

and

$$\cos GQS = \frac{GQ+\frac{1}{2}bcos\varphi}{l} \text{ and thus } V = \frac{S\left(GQ+\frac{1}{2}bcos\varphi\right)}{GQ\cdot\sin\varphi}:$$

and on account of $GQ = \sqrt{(ll-\frac{1}{4}bbsin^2\varphi)} - \frac{1}{2}bcos\varphi,$ there will become

$$V = \frac{S\sqrt{(ll-\frac{1}{4}bbsin^2\varphi)}}{\sin\varphi\sqrt{(ll-\frac{1}{4}bbsin^2\varphi)}-\frac{1}{2}b\sin\varphi cos\varphi}.$$

With these defined we will consider the differential equation, where the solution of the preceding problem is held,

$$2\upsilon d\upsilon\left(2h+m+b+\alpha\right)+\left(\lambda+1\right)\upsilon\upsilon dx = 4gdx\left(V-\alpha-a-2\delta h+b-2x\right).$$

We will consider the length of the rod *l* as much greater besides the radius of the circle $\frac{1}{2}b$, and there will become

$$x = \frac{1}{2}b\left(1-\cos\varphi\right),\ \frac{dx}{d\varphi} = \frac{1}{2}b\sin\varphi,\ \upsilon\upsilon = cc\sin^2\varphi,\ \frac{2\upsilon d\upsilon}{d\varphi} = 2cc\, sin\varphi cos\varphi$$

and

$$V = \frac{Sl}{l\sin\varphi-\frac{1}{2}b\sin\varphi\,\cos\varphi} = \frac{S}{\sin\varphi};$$

with which substitution made, there will become

$$\begin{aligned}&\left(2h+m+b+\alpha\right)2cc\, sin\varphi\, cos\varphi+\left(\lambda+1\right)cc\, sin^2\varphi\cdot\tfrac{1}{2}bsin\varphi\\ &= 2gb\left(S-\left(\alpha+a+2\delta h-b\right)sin\varphi\right)-2gb\ sin\varphi\cdot b\left(1-cos\varphi 1\right)\\ &= 2gb\left(S-\left(\alpha+a+2\delta h-bcos\varphi\right)sin\ \varphi\right)\end{aligned}$$

and hence we elicit the force required force for the uniform motion of the axis:

$$S = \left(\alpha+a+2\delta h-bcos\varphi\right)sin\varphi+\frac{cc}{gb}\left(2h+m+b+\alpha\right)sin\varphi cos\varphi+\frac{(\lambda+1)cc}{4g}sin^3\varphi.$$

With which force it is effected, so that, while the axis carries out a half revolution, this is in the time $=\frac{\pi b}{2c}$ sec., the mass of water $= bff$ raised through the tube above *DZ* may be ejected [Fig. 59]. But when the curvature if the axis were produced at the lowest point *H*, then by the continued rotation of the lever the end Q will be raised, and in a similar manner, when it will have arrived at S' now with the angle being $HGS' = \varphi$, for the force, with which the axis must be rotated, there is found as before :

$$S = (\alpha + a + 2\delta h - b\cos\varphi)\sin\varphi + \tfrac{cc}{gb}(2h + m + b + \alpha)\sin\varphi\cos\varphi + \tfrac{(\lambda+1)cc}{4g}\sin^3\varphi,$$

thus so that whither the elbow of the axis F shall be at S or from the region S', the same force is required for its rotation.

COROLLARY 1

119. Therefore while the axis of the elbow is turning either in the highest place F or in the lowest H, there becomes $S = 0$, or clearly there is no need for a force for maintaining a uniform rate of rotation. But hence in the places with 90° distance, for the force here there will become :

$$S = \alpha + a + 2\delta h + \tfrac{(\lambda+1)cc}{4g}.$$

COROLLARY 2

120. If the angle $FGS = \varphi$ were 45° or 225°, because $\sin\varphi = \frac{1}{\sqrt{2}}$ and $\cos\varphi = \frac{1}{\sqrt{2}}$ there will become

$$S = \tfrac{\alpha + a + 2\delta h}{\sqrt{2}} - \tfrac{b}{\sqrt{2}} + \tfrac{cc}{2gb}(2h + m + b + \alpha) + \tfrac{(\lambda+1)cc}{8g\sqrt{2}}.$$

truly in the other semi quadrant, where $\varphi = 135°$ or $\varphi = 315°$, on account of $\sin\varphi = \frac{1}{\sqrt{2}}$ and $\cos\varphi = -\frac{1}{\sqrt{2}}$ there becomes

$$S = \tfrac{\alpha + a + 2\delta h}{\sqrt{2}} + \tfrac{b}{\sqrt{2}} - \tfrac{cc}{2gb}(2h + m + b + \alpha) + \tfrac{(\lambda+1)cc}{8g\sqrt{2}}.$$

COROLLARY 3

121. Therefore so that if the axis MN may have two elbows of this kind between the perpendiculars themselves, with which four similar pumps may be made to move, so that in the time $\frac{\pi b}{2c}$ sec., a volume of water $= 2bff$ may flow out on top then, while the other elbow may be passing through the top position For the bottom H , the force required is

$$S = \alpha + a + 2\delta h + \tfrac{(\lambda+1)cc}{4g},$$

but while both may be inclined to the vertical FH by the 45°, there will become :

$$S = (\alpha + a + 2\delta h)\sqrt{2} + \tfrac{(\lambda+1)cc}{4g\sqrt{2}},$$

which two forces will be equal to each other, if there were

$$\alpha + a + 2\delta h = \frac{(\lambda+1)cc}{4g\sqrt{2}}.$$

SCHOLIUM 1

122. It is concerned most with all the action of machines, that the motion of these is uniform and that they may be acted on always by an equal force, however great the motion may become, from which at least it is agreed, that in the manner described it may only be applied minimally to these two pumps, since for this force the greatest inequality shall be required ; but if nevertheless the forces may be applied to two pumps equally, so that the axis of the elbows shall be normal to each other, the requisite forces will be agreed to be required to be changed around much more for equality: yet at least it is agreed for the greater equality requiring to be obtained, only the value of the formula $\frac{(\lambda+1)cc}{4g}$ to be agreed on, to be as great as we have found. Moreover it may be agreed instead that this formula to be returned always just as small, as the circumstances permit, since in this way the effect requiring to be produced may be diminished. Therefore with the cross-section of the upper orifice $Zz = ee$ there shall become $\lambda + 1 = \frac{f^4}{e^4}$, certainly it will be agreed this orifice to be the widest to be used, but concerning the speed c nothing is left to our choice ; indeed since in the time $\frac{\pi b}{2c}$ sec. the amount of water = $2bff$ is ejected above, the amount ejected per second is $= \frac{4}{\pi} cff$, as we may compare with the action of the force used.
Therefore between the two values of the S the mean shall be required to be taken, as if there shall be

$$S = \tfrac{5}{4}\left(\alpha + a + \delta h\right) + \tfrac{5}{6}\frac{(\lambda+1)cc}{4g},$$

since this force itself $= Sff$ and it acts with the speed c, its action [*i.e.* power] will be

$$= cff\left(\tfrac{5}{4}\left(\alpha + a + \delta h\right) + \tfrac{5}{6}\frac{(\lambda+1)cc}{4g}\right).$$

Therefore so that if the principal total force acting on the machine shall be $= V$ and for that working with the speed $= u$, its action will be $= Vu$, to which that may be placed equal to the amount of water raised to the height $\alpha + a$ per second

$$\frac{\frac{16}{5\pi}Vu}{\alpha+a+\delta h+\frac{(\lambda+1)cc}{6g}}.$$

where the coefficient $\frac{16}{5\pi}$ may be equated almost to unity.

SCHOLIUM 2

123. But if, as in the manner we have assumed, only two equal pumps may be applied to the machine, even if the elbows moving these shall be placed at right angles, yet a notable inequality in the forces is required to be understood at this stage ; but which will be enabled to be diminished much more, if four equal pumps may be applied and the four axis of the elbows acting there may be arranged at half a right angle between themselves, then indeed there will be almost always

$$S=\tfrac{5}{2}(\alpha+a+\delta h)+\tfrac{17}{10}\cdot\frac{(\lambda+1)cc}{4g},$$

and thus the amount of water raised per second shall be $\frac{8}{\pi}cff$. Whereby, if as before we may call the principal force moving the machine *V* and the speed *u* by which that is performed, the amount of water raised per second will be

$$\frac{\frac{16}{5\pi}Vu}{\alpha+a+\delta h+\frac{17}{100}\cdot\frac{(\lambda+1)cc}{g}},$$

which hardly differs from the preceding force. Hence it is understood how to extricate the speed *c* always so that it may be a minimum may be put in place, which now can be done as it pleases, indeed then the cross-section of the pump may be defined thus, so that thee shall become

$$ff=\frac{\frac{2}{5c}Vu}{\alpha+a+\delta h+\frac{(\lambda+1)cc}{6g}}:$$

whereby it always leads to these widest pumps being constructed, so that thence the speed *c* from that may arise smaller ; then truly the height of the pumps *b* is determined from the extension of the elbows from the axis, since there shall become $b=2FG$, which we are allowed to choose.