CHAPTER III

CONCERNING THE FLOW OF WATER IN TUBES OF UNEQUAL SECTIONS

PROBLEM 53

64. *If the given quantity of water may be moving in a tube, the cross-section of which is variable in some manner, and each end may be pressed on by [gravitational] forces in some manner, to determine its motion and pressure at individual points.*

SOLUTION

Whatever shape the directrix of the tube may have become (Fig. 50), that will be considered so that the right line *AO*, to which moreover the curved line $\alpha\omega$ may be added,

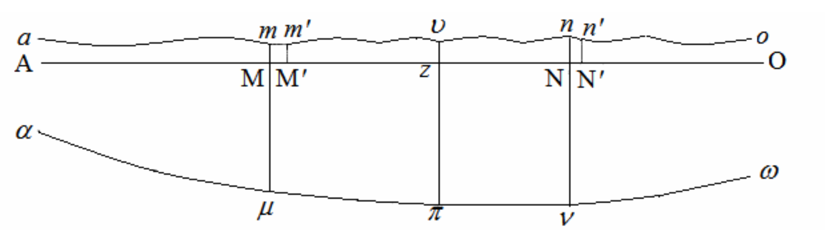


Fig. 50

of which the individual points z of the applied lines $z\pi$ will denote the heights beyond the fixed horizontal plane. Now in the elapsed time *t* some small particle of the water may be considered, which will be moving along the tube at the point *z*, with the length of the directrix being $Az = s$ reckoned from the fixed point *A*, and where the cross-section of the tube shall be $z\upsilon = \omega$ and the height $z\pi = z$, which are assumed for a given *s*. The density of water may be expressed by unity, so that there shall be $q = 1$, truly the pressure $= p$ may be called equally to be the water pressure at z, and the speed of this particle in the tube towards *O* shall be $= \mathfrak{T}$, which shall be functions of the two variables *s* and *t*. With which in place, problem 46 supplied this equation to us $\left(\frac{d\cdot\mathfrak{T}\omega}{ds}\right) = 0$, from which $\mathfrak{T}\omega$ by necessity is a function of the time only, therefore since with this time the speed everywhere shall be inversely as the cross-section, we may consider the given cross-section somewhere to be given $= ff$, at which the speed shall be $= \upsilon$, the value for the time *t,* and there will become

$$\mathfrak{T}\omega = ff\upsilon \quad \text{and} \quad \mathfrak{T} = \frac{ff\upsilon}{\omega};$$

thus so that, if the speed *v* were defined for this time appropriate for given cross-section *ff* , from that the speed may become known at any other cross-section ω for the same time ; and thus the first determination may be contained in this formula $\mathfrak{T} = \frac{ff\upsilon}{\omega}$, where it is to be understood properly the speed υ to be a function of the time t only, truly the cross-section ω of the distance *s* only. Now we may move on to the other equation, from which the pressure *p* is defined, and since we put water to be moved by gravity alone, there will become $P=0$, $Q=0$, and $R=-1$, then truly, since in this equation the time *t* is taken constant, there will be :

$$d\mathfrak{T}=\frac{-ff\upsilon d\omega}{\omega\omega} \text{ and } \left(\frac{d\mathfrak{T}}{dt}\right)=\frac{ffd\upsilon}{\omega dt},$$

and thus the latter equation will be changed into this form:

$$2gdp=-2gdz+\frac{f^4\upsilon\upsilon d\omega}{\omega^3}-\frac{ffds}{\omega}\cdot\frac{d\upsilon}{dt},$$

which, since υ and $\frac{d\upsilon}{dt}$ may be considered as constants, by integration gives :

$$2gp=\varDelta:t-2gz+\frac{f^4\upsilon\upsilon}{2\omega\omega}-\frac{ffd\upsilon}{dt}\int\frac{ds}{\omega},$$

where since ω shall be a function of s only, the integral $\int\frac{ds}{\omega}$ can be considered as a known quantity.

Now we may consider for both ends of our mass of water which shall be present at *M* and N: $AM=m$, $AN=n$, with the cross-section at $M=\mu$, and at $N=\nu$, with the height $M\mu=\mathfrak{m}$, and at $N\nu=\mathfrak{n}$; with the value of the integral $\int\frac{ds}{\omega}$ at $M=\mathfrak{M}$, at $N=\mathfrak{N}$; then truly with the pressure at $M=M$ and at $N=N$. Therefore since the speed at M sit $=\frac{ff\upsilon}{\mu}$, in $N=\frac{ff\upsilon}{\nu}$, in the time increment *dt* both ends *M* and *N* will be moved forwards to *M', N',* so that there shall be

$$MM'=\frac{ff\upsilon}{\mu}dt \text{ and } NN'=\frac{ff\upsilon dt}{\nu},$$

from which, since *m* and *n* shall be functions of the time *t* only, there will be

$$dm=\frac{ff\upsilon dt}{\mu},\; dn=\frac{ff\upsilon dt}{\nu}$$

and hence $\mu dm=\nu dn$. Moreover from the known pressures at *M* and *N,* we will obtain these two equations:

$$2gM=\varDelta:t-2g\mathfrak{m}-\frac{f^4\upsilon\upsilon}{2\mu\mu}-\frac{ffd\upsilon}{dt}\cdot\mathfrak{M},$$

$$2gN=\varDelta:t-2g\mathfrak{n}-\frac{f^4\upsilon\upsilon}{2\nu\nu}-\frac{ffd\upsilon}{dt}\cdot\mathfrak{N},$$

from which we deduce :

$$2g\left(M-N\right)=2g\left(\mathfrak{n}-\mathfrak{m}\right)-\frac{f^4\upsilon\upsilon}{2}\left(\frac{1}{\nu\nu}-\frac{1}{\mu\mu}\right)+\frac{ffd\upsilon}{dt}\left(\mathfrak{N}-\mathfrak{M}\right),$$

which equation involves only functions of the time t, and therefore thence will be able to define the speed υ. Then truly there is found for the pressure:

$$2g\left(M-p\right)=2g\left(z-\mathfrak{m}\right)-\frac{f^4\upsilon\upsilon}{2}\left(\frac{1}{\omega\omega}-\frac{1}{\mu\mu}\right)+\frac{ffd\upsilon}{dt}\left(\int\frac{ds}{\omega}-\mathfrak{M}\right),$$

which with the formula $\frac{ffd\upsilon}{dt}$ removed provides this equation:

$$2g\left(p+z\right)+\frac{f^4\upsilon\upsilon}{2\omega\omega}\left(\mathfrak{N}-\mathfrak{M}\right)=\begin{cases}+2g\left(M+\mathfrak{m}\right)+\frac{f^4\upsilon\upsilon}{2\mu\mu}\left(\mathfrak{N}-\int\frac{ds}{\omega}\right)\\ +2g\left(N+\mathfrak{n}\right)+\frac{f^4\upsilon\upsilon}{2vv}\left(\int\frac{ds}{\omega}-\mathfrak{M}\right).\end{cases}$$

COROLLARY 1

65. Since the mass of the fluid contained in the tube is given, from the given distance $AM=m$, where likewise the quantities μ, $\mathfrak{m}$ and $\mathfrak{M}=\int\frac{dm}{\mu}$ may be determined , the distance $AN=n$ may be defined, since $\int vdn-\int\mu dm$ produces that mass, and thus also n with $v,\mathfrak{n}$ and $\mathfrak{N}=\int\frac{dn}{v}$ will be required to be considered as functions of the quantity m only.

COROLLARY 2

66. Because the whole matter depends on the resolution of the differential equation found and there is $dt=\frac{\mu dm}{ff\upsilon}$, if that equation may be multiplied by $\mu dm=ff\upsilon dt$, there will be had :

$$2g(M-N+\mathfrak{m}-\mathfrak{n})\mu dm=\tfrac{1}{2}f^4\upsilon\upsilon\mu dm\left(\frac{1}{vv}-\frac{1}{\mu\mu}\right)+f^4\upsilon d\upsilon\left(\mathfrak{N}-\mathfrak{M}\right),$$

which on putting $f^4\upsilon\upsilon=V$ will be changed into this :

$$2g(M-N+\mathfrak{m}-\mathfrak{n})\frac{\mu dm}{\mathfrak{N}-\mathfrak{M}}=dV+\frac{V\mu dm}{\mathfrak{N}-\mathfrak{M}}\left(\frac{1}{vv}-\frac{1}{\mu\mu}\right),$$

from which the quantity V will be able to be deduced, with which found in the first place the speed is found $\upsilon=\frac{\sqrt{V}}{ff}$, and thence again the time $t=\int\frac{\mu dm}{\sqrt{V}}$.

COROLLARY 3

67. Indeed if the pressures M and N either may be constants, or may depend on the distances m and n , since n is determined by m, it is required to agree that equation to

contain only the two variables *m* and *V* and is rendered integrable, if it may be multiplied by e^Q with there being :

$$Q=\int\frac{\mu dm}{\mathfrak{N}-\mathfrak{M}}\left(\frac{1}{vv}-\frac{1}{\mu\mu}\right).$$

Truly since there is $\mu dm = vdn$ and $\frac{dn}{v}=d\mathfrak{N}$ likewise $\frac{dm}{\mu}=d\mathfrak{M}$, there becomes

$$Q=\int\frac{d\mathfrak{N}-d\mathfrak{M}}{\mathfrak{N}-\mathfrak{M}}$$

and hence the multiplier $e^Q=\mathfrak{N}-\mathfrak{M}$.

COROLLARY 4

68. On account of which the integral of this equation is :

$$(\mathfrak{N}-\mathfrak{M})V=f^4\upsilon\upsilon(\mathfrak{N}-\mathfrak{M})=4g\int\mu dm(M-N+\mathfrak{m}-\mathfrak{n}),$$

where it is to be noted, since the speed of the water at $M=\frac{ff\upsilon}{\mu}$ and at $N=\frac{ff\upsilon}{v}$, the expression

$$f^4\upsilon\upsilon(\mathfrak{N}-\mathfrak{M})=\int\frac{f^4\upsilon\upsilon}{vv}\cdot vdn-\int\frac{f^4\upsilon\upsilon}{\mu\mu}\cdot\mu dm$$

to designate the living force of the mass of water *MmNn,* whenever vdn is the element of *NnN'n',* and that is multiplied by the square of the speed.

SCHOLIUM

69. It is most noteworthy, that the differential equation found shall have been able to be integrated so easily and its integral shall lead to the living force of the water contained in the tube, from which it is observed most clearly, that the greatest use has been made of the principle of the conservation of living forces, with which previously the most celebrated Bernoulli had used with the greatest success in his *Hydrodynamics*. Hence evidently we understand , if the forces *M* and *N* pressing on each end were equal and the directrix of the tube were horizontal, that no forces shall be present either accelerating or retarding the motion of the fluid, then the mass of the fluid is going to be conserving the same living force always, for on putting $M=N$ and $\mathfrak{m}=0$ and $\mathfrak{n}=0$ or generally $z=0$, the living force produced

$$f^4\upsilon\upsilon(\mathfrak{N}-\mathfrak{M})=\text{Const.},$$

but if the heights $\mathfrak{m}$ and $\mathfrak{n}$ may not vanish, the equation found on account of $\mu dm = vdn$

can be represented thus :

$$f^4\upsilon\upsilon\left(\mathfrak{N}-\mathfrak{M}\right)=4g\int(M+\mathfrak{m})\mu dm-4g\int(N+\mathfrak{n})\nu dm,$$

from which it is evident, how great the increment of the living force may be taken for the accelerating force ; indeed since the pressure M accelerates the motion, truly the pressure N retards, and besides either the ascent or descent is defined from the heights $\mathfrak{m}$ and $\mathfrak{n}$ of the individual elements. Moreover here it is especially noteworthy, so that the differential equation found at once may be rendered integrable only on being multiplied by

$$2\mu dm=2\nu dn=2ff\upsilon dt,$$

whereby it produces

$$4g(M-N+\mathfrak{m}-\mathfrak{n})\mu dm=f^4\upsilon\upsilon\left(\tfrac{dn}{\nu}-\tfrac{dm}{\mu}\right)+2f^4\upsilon d\upsilon\left(\mathfrak{N}-\mathfrak{M}\right),$$

the integrability of which is evident at once on account of $\frac{dn}{\nu}=d\mathfrak{N}$ and $\frac{dm}{\mu}=d\mathfrak{M}$; thus so that now the whole business may be reduced to the integration of the first parts. Therefore it will suffice for greater elucidation, that we may present some examples.

EXAMPLE 1

70. *If the tube* (Fig. 51), *in which the mass of water ACc may fall freely, shall be conical and its directrix AO vertical, to define its motion.*

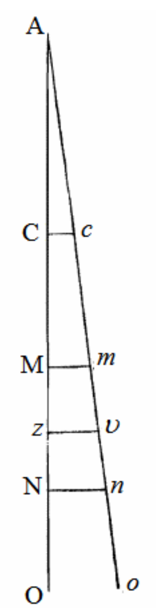


Fig. 51

There shall be $AC=c$ and the cross-section of the tube at C certainly $Cc=\alpha cc$, so that the total mass of water may become $ACc=\frac{1}{3}\alpha c^3$, which after the time t will occupy the volume of the tube *MmNn,* from which on account of $AM=m$ and $AN=n$ will be $n^3=c^3+m^3$. Then truly on putting the fixed altitude $AO=a$, at first the cross-sections will be

$$Mm=\mu=\alpha mm,\ Nn=\nu=\alpha nn\ \text{ and }\ z\upsilon=\omega=\alpha ss$$

on putting $Az=s$, thence the altitudes

$$OM=\mathfrak{m}=a-m,\ ON=\mathfrak{n}=a-n\ \text{ and }\ Oz=z=a-s.$$

Again on account of $\int\frac{ds}{\omega}=-\frac{1}{\alpha s}$, there becomes $\mathfrak{M}=-\frac{1}{\alpha m}$ and $\mathfrak{N}=-\frac{1}{\alpha n}$. Whereby if the pressures at M and N may be equal only to the atmospheric pressure k, which arises, if the opening at the apex A may be considered, there will become $M=N=k$. But if now it may be agreed the speed of the cross-section ff to be $=\upsilon$ acting downwards, our equation of the integral is deduced for this case:

$$f^4\ \upsilon\upsilon\left(\tfrac{1}{\alpha m}-\tfrac{1}{\alpha n}\right)=4g\int\alpha mmdm(n-m)=4\alpha g\left(\tfrac{1}{4}n^4-\tfrac{1}{4}m^4\right)+\text{Const.}$$

on account of $mmdm = nndn$. But since the descent may start from rest, on making $m=0$ and $n=c$ the speed must vanish, so that there will be had :

$$f^4\ \upsilon\upsilon\left(\tfrac{1}{m}-\tfrac{1}{n}\right)=\alpha\alpha g\left(n^4-m^4-c^4\right)$$

and hence

$$f\upsilon\upsilon=\alpha\sqrt{\frac{gmn\left(n^4-m^4-c^4\right)}{n-m}}$$

and thus the time is deduced:

$$t=\int\frac{\mu dm}{ff\upsilon}=\int\frac{mdm\sqrt{m(n-m)}}{\sqrt{gn\left(n^4-m^4-c^4\right)}},$$

the integral is required to be taken on account of this formula $n^3=c^3+m^3$, so that for the given time *t* the distance $AM=m$ may be able to be defined. Finally for the pressure at *z,* which is *p,* this equation will be obtained by finding the above on being multiplied by *z*

$$\left(4g\left(p+a-s\right)+\frac{f^4\upsilon\upsilon}{2\omega\omega s^4}\right)\left(\tfrac{1}{\alpha m}-\tfrac{1}{\alpha n}\right)=\begin{cases}+\left(4g\left(k+a-m\right)+\frac{f^4\upsilon\upsilon}{\alpha\alpha m^4}\right)\left(\tfrac{1}{\alpha s}-\tfrac{1}{\alpha n}\right)\\+\left(4g\left(k+a-n\right)+\frac{f^4\upsilon\upsilon}{\alpha\alpha n^4}\right)\left(\tfrac{1}{\alpha m}-\tfrac{1}{\alpha s}\right),\end{cases}$$

which on account of

$$\frac{f^4\upsilon\upsilon}{\alpha\alpha}=\frac{gmn\left(n^4-m^4-c^4\right)}{n-m}$$

will be changed into this :

$$\frac{4(p+a-s)(n-m)}{mn}=\frac{4(k+a-m)(n-s)}{ns}+\frac{4(k+a-n)(s-m)}{ms}$$
$$+\frac{(s-m)(n-s)\ (n^4-m^4-c^4)(mmnn+mn(m+n)s+(mm+mn+nn)ss)}{m^3n^3s^4}$$

from which we deduce :

$$p=k+\tfrac{mn}{s}-m-n+s+\frac{(s-m)(n-s)\left(n^4-m^4-c^4\right)\left(mmnn+mn(m+n)s+(mm+mn+nn)ss\right)}{4mmnn(n-m)s^4}.$$

EXAMPLE 2

71. *In the case of the preceding example, if the tube may be closed at A, so that the superior surface Mm may sustain no pressure, to determine the motion of the water.*

Since everything may remain as in the preceding example, except that here there shall be $M=0$ and $N=k$, the first equation will be changed into this form

$$f^4\ \upsilon\upsilon\left(\tfrac{1}{\alpha m}-\tfrac{1}{\alpha n}\right)=4g\int\alpha mmdm(n-m-k)$$

or

$$f^4\ \upsilon\upsilon\left(\tfrac{1}{m}-\tfrac{1}{n}\right)=\alpha\alpha g\left(n^4-m^4-c^4-\tfrac{4}{3}km^3\right).$$

But I observe here at first, where $m=0$ and $n=c$, the motion may not have been able to begin, unless there were $c>k$; indeed if there shall be $c<k$ or also $c=k$, the water will stick for ever to the top of the tube and no motion will follow. But if there shall be $c>k$, at first the motion indeed will be accelerated, until there may become

$$n=\sqrt[3]{\left(c^3+m^3\right)}=m+k,$$

that is:

$$c^3=3kmm+3kkm+k^3\ \text{ or }\ m=\sqrt{\left(\tfrac{c^3}{3k}-\tfrac{1}{12}kk\right)}-\tfrac{1}{2}k.$$

Then truly the speed will decrease, and thus will vanish, when there becomes :

$$n^4=\left(\mathrm{c}^3+m^3\right)^{\frac{4}{3}}=c^4+m^4+\tfrac{4}{3}km^3,$$

which expanded out becomes:

$$\begin{aligned}4km^8+\tfrac{16}{3}kkm^7+\tfrac{64}{27}k^3m^6+3c^4m^5+8kc^4m^4+\tfrac{16}{3}kkc^4m^3+3c^8m+4kc^8&\\ -4c^3m^6\qquad\qquad\qquad\qquad -6c^6m^3\qquad -4c^9&=0.\end{aligned}$$

So that hence we may be able to conclude somewhat easier, we may put c very small and to exceed k we may put $c=(1+\delta)k$, with δ denoting some minimum fraction, and since m also will be a very small interval, there will become $n=c+\frac{m^3}{3cc}$ and hence

$$f^4\upsilon\upsilon\left(\tfrac{1}{m}-\tfrac{1}{c}+\tfrac{m^3}{3c^4}\right)=\alpha\alpha g\left(\tfrac{4\delta cm^3}{3(1+\delta)}-m^4+\tfrac{2m^6}{3cc}\right)$$

and the maximum speed in place will correspond to

$$m=\tfrac{\delta c}{1+\delta}+\tfrac{m^3}{3cc}=\tfrac{\delta c}{1+\delta}+\tfrac{\delta^3c}{3(1+\delta)^2}$$

and again will vanish , when $m=\frac{4\delta c}{3(1+\delta)}+\frac{2m^3}{3cc}$

or closely enough $m=\frac{4\delta c}{3(1+\delta)}$. Then truly there is deduced :

$$ff\upsilon=\alpha mm\sqrt{g\left(\tfrac{4\delta c}{3(1+\delta)}-m\right)}=\alpha mm\sqrt{g\left(\tfrac{4}{3}\delta k-m\right)}$$

and hence the time

$$t=\int\frac{dm}{\sqrt{g\left(\frac{4}{3}\delta k-m\right)}}=2\sqrt{\frac{4\delta k}{3g}}-2\sqrt{\frac{4\delta k-3m}{3g}}$$

and the time of the total descent $=4\sqrt{\frac{\delta k}{3g}}$.

EXAMPLE 3

72. *If the tube* (Fig. 52) *may have two vertically erect arms AB and CO joined by the horizontal branch BC and any part shall be equally wide, but different from the others, to define the motion of the oscillating water in this tube.*

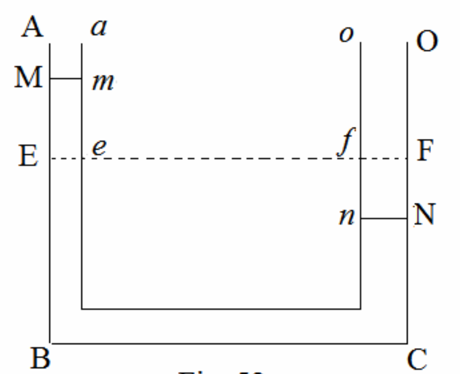


Fig. 52

The cross-section of the tube AB shall be $=\mu$ everywhere, in which Mm is found at the end of the tube, truly the cross-section of the other tube OC is found $=\nu$, and that of the horizontal $BC=\lambda$. Since the water on both sides is in equilibrium, it shall extend to the horizontal EF, and there may be put $BE=CF=a$, and $BC=b$, so that the whole volume of the water shall be $=a\mu+b\lambda+a\nu$. Now in a state of motion at the time $=t$, EM may be called $=\nu x$ and there will become $FN=\mu x$, and now the quantities μ and ν are constants. There may be put $AE=e$, there will be

$$m=e-\nu x,\ \mathfrak{m}=a+\nu x,\ \mathfrak{M}=\frac{e-\nu x}{\mu},$$

again

$$n=e+2a+b-\mu x,\ \mathfrak{n}=a-\mu x \ \text{ and } \ \mathfrak{N}=\frac{e+a}{\mu}+\frac{b}{\lambda}+\frac{a-\mu x}{\nu},$$

so that

$$\mathfrak{N}-\mathfrak{M}=\frac{b}{\lambda}+\frac{a+\nu x}{\mu}+\frac{a-\mu x}{\nu}.$$

Then on account of the atmospheric pressure $M=N=k$ and $\mathfrak{m}-\mathfrak{n}=(\mu t+\nu)x$, there will be

$$f^4\upsilon\upsilon\left(\frac{b}{\lambda}+\frac{a+\nu x}{\mu}+\frac{a-\mu x}{\nu}\right)=-4g\int\mu\nu dx\left(\mu+\nu\right)x$$
$$=-2g\mu\nu\left(\mu+\nu\right)xx+C.$$

On making $x=0$ we may put the speed of the agreed cross-section ff to become $\upsilon=2\sqrt{gc}$, hence so that the constant may be determined thus :

$$4gcf^4\left(\tfrac{b}{\lambda}+\tfrac{a(\mu+\nu)}{\mu\nu}\right)=C,$$

and there will be had:

$$ff\upsilon\sqrt{\left(\tfrac{b}{\lambda}+\tfrac{a+\nu x}{\mu}+\tfrac{a-\mu x}{\nu}\right)}=\sqrt{2g\mu\nu(\mu+\nu)\left(\tfrac{2cf^4}{\mu\nu(\mu+\nu)}\left(\tfrac{b}{\lambda}+\tfrac{a(\mu+\nu)}{\mu\nu}\right)-xx\right)},$$

therefore there will become for the maximum excursions:

$$x=\pm\sqrt{\left(\tfrac{2cf^4}{\mu\nu(\mu+\nu)}\left(\tfrac{b}{\lambda}+\tfrac{a(\mu+\nu)}{\mu\nu}\right)\right)}.$$

Moreover for the time, this equation will be required to be integrated:

$$t=-\mu\nu\int\tfrac{dx}{ff\upsilon},$$

but which formula is exceedingly complex, just as may be suspected from its derivation, except in the case, so that c and hence also x is taken as a minimum quantity. For in general the time is defined with such a form:

$$t=\int\frac{dx\sqrt{(A+Bx)}}{\sqrt{(hh-xx)}},$$

the integration of which is evident with the term Bx rejected. But with the term Bx allowed certainly the whole oscillations will be isochronous, but the times, for which the end *Mm* either rises above or falls below the upper level EF, will not be equal to the times, with which it may turn about the lower level.

PROBLEM 54

73. *If water may flow out through the orifice Oo* (Fig. 50), *from a tube with a cross-section uneven in some manner, and its directrix is some curved line, so that its amount in the tube may be diminished continually, to determine its motion.*

SOLUTION

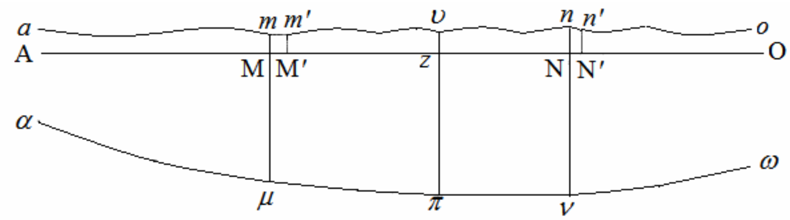


Fig. 50

Everything may remain as in the solution of the preceding problem, except that the constant cross-section *ff* may be attributed to the opening *Oo*, through which now after the time $=t$ the water may be flowing out with the speed $=\upsilon$, truly the other end may remain at *Mm,* where the cross-section shall be $=\mu$, with the height above the horizontal $M\mu=\mathfrak{m}$ and the pressure $=M$, which indeed, if it may stand open to the air, will be equal to the pressure of the atmosphere *k,* and likewise at the opening *Oo* itself. But in the place of the tube the distance shall be $AM=m$ along its directrix from the given *A* and the total length $AO=a$, then truly the speed, with which the upper surface of the water *Mm* is moving through the tube, will be $=\frac{ff\upsilon}{\mu}$. Now we may put *z* for some place of the tube, the length $Az=s$, the cross-section $zv=\omega$, the height $z\pi=z$ and the pressure $=p$, and the principles of motion will provide this equation to us:

$$2gp=\varDelta:t-2gz-\frac{f^4\upsilon\upsilon}{2\omega\omega}-\frac{ffd\upsilon}{dt}\int\frac{ds}{\omega},$$

which initially from the end *Mm,* then truly it will be required to be transferred to the opening *Oo*, since the pressure has been given in these two places. But for that one *Mm* there becomes $p=M$, $z=\mathfrak{m}$, $\omega=\mu$, truly the value of the integral $\int\frac{ds}{\omega}$ here may become $=\mathfrak{M}$, so that there becomes:

$$2gM=\varDelta:t-2g\mathfrak{m}-\frac{f^4vv}{2\omega\omega}-\frac{ffdv}{dt}\mathfrak{M}.$$

Truly for the opening *Oo,* if indeed the water may flow out into air, the pressure will be had $p=k$, the cross-section $\omega=ff$, truly the height $O\omega$ becomes zero, since it may be allowed to flow through this opening *Oo* in the horizontal plane, but the value of the integral formula $\int\frac{ds}{\omega}$ moved to this point may become $=\mathfrak{A}$, certainly which will be constant, from which our equation will become

$$2gk=\varDelta:t-\tfrac{1}{2}\upsilon\upsilon-\frac{ffdv}{dt}\cdot\mathfrak{A},$$

which taken away from that other one leaves

$$2g(M-k)=-2g\mathfrak{m}-\frac{f^4\upsilon\upsilon}{2\mu\mu}+\tfrac{1}{2}\upsilon\upsilon+\frac{ffd\upsilon}{dt}(\mathfrak{A}-\mathfrak{M}),$$

which equation, in which only the time *t* remains variable, will be required to be integrated, by calling in as an aid $dm = \frac{ff\upsilon dt}{\mu}$: from which on account of $dt = \frac{\mu dm}{ff\upsilon}$ there will be had :

$$2g\left(M-k\right)\mu dm = -2g\mathfrak{m}\mu dm - \tfrac{1}{2}\upsilon\upsilon\mu dm\left(1-\tfrac{f^4}{\mu\mu}\right) + \tfrac{ffd\upsilon}{dt}f^4\upsilon d\upsilon\left(\mathfrak{A}-\mathfrak{M}\right),$$

where $\mathfrak{M} = \int\frac{dm}{\mu}$, from which the value of the quantity $\mathfrak{A}$ arises, if there may be become $m = a$. Moreover μ and $\mathfrak{m}$ are given functions of *m,* so that this equation involves only the two variables *m* and *v*, from which the value of $\upsilon\upsilon$ will be allowed to be deduced easily, with which found with the aid of the formula $dt = \frac{\mu dm}{ff\upsilon}$ for any time t both with the length $AM = m$ as well as the speed υ, from which it will be able for the water to be assigned flowing out through the opening *Oo*. Then truly also for the pressure p at some place z there will be had :

$$2g(p-k) = -2gz + \tfrac{1}{2}\upsilon\upsilon\left(1-\tfrac{f^4}{\omega\omega}\right) + \tfrac{ffd\upsilon}{dt}\left(\mathfrak{A}-\int\tfrac{ds}{\omega}\right),$$

whereby, if the term $\frac{ffdv}{dt}$ may be removed, there will be deduced:

$$\begin{aligned}2g(M-k)\left(\mathfrak{A}-\int\tfrac{ds}{\omega}\right)+2g\left(k-p\right)\left(\mathfrak{A}-\mathfrak{M}\right) &= 2gz\left(\mathfrak{A}-\mathfrak{M}\right)-2g\mathfrak{m}\left(\mathfrak{A}-\int\tfrac{ds}{\omega}\right)\\ &+\tfrac{1}{2}\upsilon\upsilon\left(\mathfrak{M}-\int\tfrac{ds}{\omega}\right)-\tfrac{f^4\upsilon\upsilon}{2\mu\mu}\left(\mathfrak{A}-\int\tfrac{ds}{\omega}\right)+\tfrac{f^4\upsilon\upsilon}{2\omega\omega}\left(\mathfrak{A}-\mathfrak{M}\right).\end{aligned}$$

COROLLARY 1

74. We may suppose the fixed end of the tube *A* to be at the opening *O* itself*,* so that there shall be $a = 0$, and we may call $OM = m$ and $Oz = s$, thus so that in the formulas found these two quantities *m* and *s* must be taken negative; then truly there will become $\mathfrak{A} = 0$, and in place of $\mathfrak{M}$ and $\int\frac{ds}{\omega}$ it will be required to write

$$-\int\tfrac{dm}{\mu} \text{ and } -\int\tfrac{ds}{\omega},$$

from which we will have for the pressure:

$$\begin{aligned}&2g\left(M-k\right)\int\tfrac{ds}{\omega}+2g\left(k-p\right)\int\tfrac{dm}{\mu}\\ &= 2gz\int\tfrac{dm}{\mu}-\tfrac{1}{2}\upsilon\upsilon\left(1-\tfrac{f^4}{\omega\omega}\right)\int\tfrac{dm}{\mu}+\tfrac{1}{2}\upsilon\upsilon\left(1-\tfrac{f^4}{\mu\mu}\right)\int\tfrac{ds}{\omega}-g\mathfrak{m}\int\tfrac{ds}{\omega}.\end{aligned}$$

COROLLARY *2*

75. Moreover with the remaining $OM = m$ and $Oz = s$, at first for the time there will be had $dt = -\frac{\mu dm}{ffv}$, since with the time *t* passing the interval $OM = m$ is diminished, but the speed υ of the outflow must be defined from this equation

$$2g\left(M-k+\mathfrak{m}\right)\mu dm = \tfrac{1}{2}\upsilon\upsilon\mu dm\left(1-\tfrac{f^4}{\mu\mu}\right)-f^4\upsilon d\upsilon\int\tfrac{dm}{\mu},$$

which may be represented more conveniently thus:

$$2f^4\upsilon d\upsilon+\tfrac{f^4\upsilon\upsilon dm}{\mu}-\upsilon\upsilon\mu dm+4g\left(M-k+\mathfrak{m}\right)\mu dm = 0.$$

COROLLARY 3

76. Here there may be put $f^4\upsilon\upsilon\int\frac{dm}{\mu} = u$, so that there shall be had:

$$du-\tfrac{\mu u dm}{f^4\int\frac{dm}{\mu}}+4g\left(M-k+\mathfrak{m}\right)\mu dm = 0,$$

which, so that it may be rendered integrable, must be multiplied by e^o with there being $o = -\frac{1}{f^4}\int\frac{\mu dm}{\int\frac{dm}{\mu}}$ and then there will become $e^o u+4g\int e^o\left(M-k+\mathfrak{m}\right)\mu dm = \text{Const.}$

SCHOLIUM

77. Everything is contained in this solution, which commonly are accustomed to be used for the outflow of water from vessels of whatever kind of figure, but which can only be allowed to the extent, as far as these vessels either are very narrow or the motion through these shall become thus, so that the individual layers may be carried along the direction of the common motion taken ; indeed unless this condition may be fulfilled, the speed of the outflow defined here may no longer be true, even if on most occasions the difference of the experiments put in place is scarcely perceived. So that if in the formulas found there may be put in place $M = k$, the case will be had, where the upper surface of the water open and the outflow shall be into the air, moreover if the water may flow into a space devoid of air, there must be taken $k = 0$, but if the opening of the tube *Oo* may be immersed in still water, the letter *k* must adopt the pressure of this water at the opening. But the whole matter always is reduced to this differential equation found, from the integration of which for whatever in place, where the upper surface of the water may adhere, the speed of the outflow may be known, then truly by introducing the formula $dt = -\frac{\mu dx}{ffv}$ (§ 75) to help find the time, where the water may settle at a given location *Mm*

in the tube ; and at last, since in the elapsed time t water may flow out through Oo, the cross-section of which is $= ff$, with a speed $= v$, which will have flowed out in the time t , will be

$$= ff\int vdt = -\int \mu dm.$$

So that it may become clearer, we may set out some cases.

EXAMPLE 1

78. *If the directrix of the tube AO (Fig. 53) shall be a vertical right line, while the tube initially will be full of water from Aa, which may then begin to flow out through the opening $Oo = ff$, to determine the speed of the outflow and the pressure for some given time at some cross-section $z\upsilon$.*

Therefore putting the interval $OM = m$ and with the cross-section $Mm = \mu$, which we assume to have settled down as the place of the water from Aa in the elapsed time $= t$, also there will become with the height $OM = \mathfrak{m} = m$ and the pressure $M = k$. Now with the speed of the outflow through the opening put $= \upsilon$, that will be required to be defined from the equation :

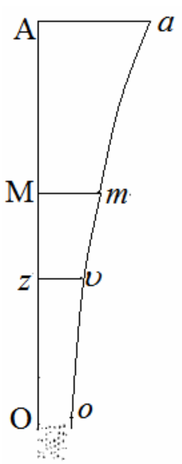


Fig. 53

$$2f^4\upsilon d\upsilon\int\tfrac{dm}{\mu}+\tfrac{f^4\upsilon\upsilon dm}{\mu}-\upsilon\upsilon\mu dm+4g\mu mdm=0,$$

which on putting $f^4\upsilon\upsilon\int\tfrac{dm}{\mu}=u$ will be changed into this form

$$du-\frac{u\mu dm}{f^4\int\frac{dm}{\mu}}+4g\mu mdm=0.$$

Then for the pressure at some cross-section $z\upsilon$ there shall be $Oz = s$ and the amplitude $z\upsilon = \omega$, also the height will be $z = s$ and thus

$$2g(k-p)\int\tfrac{dm}{\mu}=2gs\int\tfrac{dm}{\mu}-2gm\int\tfrac{ds}{\omega}-\tfrac{1}{2}\upsilon\upsilon\left(1-\tfrac{f^4}{\omega\omega}\right)\int\tfrac{dm}{\mu}+\tfrac{1}{2}\upsilon\upsilon\left(1-\tfrac{f^4}{\mu\mu}\right)\int\tfrac{ds}{\omega}.$$

But that differential equation thus must be integrated, so that with the heights made $OM = m = OA = a$ the speed υ may vanish, then truly with the speed υ found , the calculation for the time will be adapted with the aid of this formula $t=-\int\frac{\mu dm}{ff\upsilon}$, which must vanish on putting $m = a$. So that if subsequently there may be put $m = 0$, the time of the whole efflux will become known.

Truly besides with the outpouring enduring, since the speed υ increases continually from the start, the maximum speed thus is defined when $d\upsilon = 0$, so that there shall become

$$\upsilon\upsilon = \frac{4gm\mu\mu}{\mu\mu - f^4} \quad \text{and thus} \quad \upsilon = \frac{2\mu\sqrt{gm}}{\sqrt{\mu\mu - f^4}}.$$

Therefore since there shall be $\upsilon > \sqrt{2gm}$, that maximum speed will be greater, than a weight falling from the height *m* will acquire.

COROLLARY 1

79. If the vessel (Fig. 54) shall be equal in cross-section everywhere or $\mu = \omega = cc$, its bottom OC is bored through with the opening $Oo = ff$, we will have :

$$du - \frac{c^4}{f^4} \cdot \frac{udm}{m} + 4gccmdm = 0.$$

Let there be $\frac{c^4}{f^4} = \lambda$, there will become

$$m^{-\lambda}u + \frac{4ggc}{2-\lambda} m^{2-\lambda} = C$$

and hence

$$u = Cm^{\lambda} + \frac{4ggc}{\lambda - 2} mm = \frac{f^4 mvv}{cc}$$

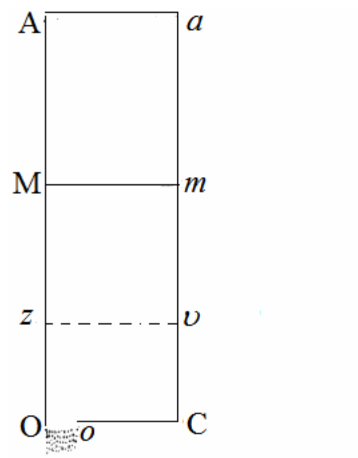


FIG. 54

and with the constant duly defined

$$f^4 mvv = \frac{4gc^4 m}{\lambda - 2} mm\left(1 - a^{2-\lambda} m^{\lambda - 2}\right)$$

or

$$v = \sqrt{\frac{4\lambda gm}{\lambda - 2}\left(1 - \frac{m^{\lambda - 2}}{a^{2-\lambda}}\right)},$$

from which there is deduced the pressure

$$p = k + \frac{\lambda - 1}{\lambda - 2}(m - s)\left(1 - \frac{m^{\lambda - 2}}{a^{\lambda - 2}}\right)$$

for the cross-section $z\upsilon$ for the height $Oz = s$, finally for the time there will become :

$$t = -\int \frac{dm\sqrt{(\lambda - 2)}}{2\sqrt{gm\left(1 - a^{2-\lambda} m^{\lambda - 2}\right)}}$$

moreover the maximum speed becomes

$$=\frac{2cc\sqrt{gm}}{\sqrt{c^4-f^4}}=\frac{2\sqrt{\lambda gm}}{\sqrt{\lambda-1}},$$

which agrees with the height m hence on defining

$$\frac{\lambda-2}{\lambda-1}=1-\frac{m^{\lambda-2}}{a^{\lambda-2}},$$

thus so that there shall become :

$$m=\frac{a}{\sqrt[\lambda-2]{\lambda-1}}.$$

COROLLARY 2

80. The case, in which $\lambda=2$ or $c^4=2f^4$, demands a singular expansion ; since the equation

$$du-\frac{2udm}{m}+4gccmdm=0$$

gives on integration:

$$u=4gccmml\frac{a}{m}=\frac{ccm\upsilon\upsilon}{2},$$

hence

$$\upsilon=\sqrt{8gml\frac{a}{m}} \text{ and } t=-\int\frac{dm}{\sqrt{4gml\frac{a}{m}}},$$

truly there will be for the pressure: $p=k+(m-s)l\frac{a}{m}$.

COROLLARY 3

81. A conical tube shall be truncated for the opening, and $\mu=(f+\alpha m)^2$, and $\omega=(f+\alpha s)^2$, hence there becomes :

$$\int\frac{dm}{\mu}=\frac{1}{\alpha f}-\frac{1}{\alpha(f+\alpha m)}=\frac{m}{f(f+\alpha m)},$$

and in a similar manner

$$\int\frac{ds}{\omega}=\frac{s}{f(f+\alpha s)}.$$

Therefore there will be had for the motion:

$$du - \tfrac{udm}{m}\left(1+\tfrac{\alpha m}{f}\right)^3 + 4gmdm\left(f+\alpha m\right)^2 = 0,$$

from which u will be found:

$$f^4 \upsilon\upsilon = \tfrac{f(f+m)u}{m}.$$

COROLLARY 4

82. The tube may be expanded indefinitely above according to this equation $\omega\omega = \tfrac{af^4}{a-s}$, thus so that initially the upper surface *Aa* will have become infinite ; and that even now will have been reduced to zero, so that in the elapsed time t the height will become $m = a$ and $\mu\mu = \tfrac{af^4}{a-m} = \infty$. For this reason the differential equation produces at once $\upsilon\upsilon = 4gm = 4ga$, thus so that the water may flow out with the same speed. But since here the efflux motion is uniform on account of $\tfrac{d\upsilon}{dt} = 0$, the pressure at $z\upsilon$ thus is defined found from the first equation:

$$2g\left(p-k\right) = -2gs + 2ga\left(1-\tfrac{a-s}{a}\right) = 0,$$

and evidently everywhere the pressure is equal to the atmospheric pressure or the sides of the tube to be equally pressed outwards and will sustain no force, and thus the flow will behave likewise for these remote parts.

EXAMPLE 2

83. *The upper part of the tube (Fig. 55) AaBb shall be vertical and of equal cross-section, truly the lower part BbOo shall be curved in some manner and of unequal cross-section, to define the motion of the outflow of the water from that, as long as the surface of the water Mm may be moving in the upper part.*

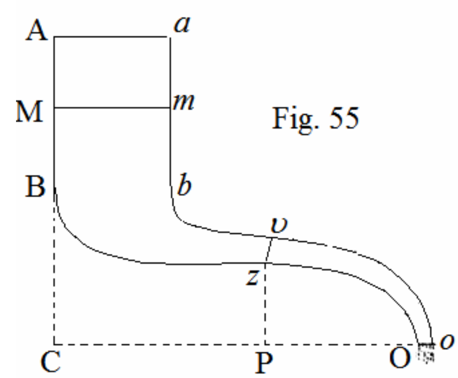


Fig. 55

The cross-section of the upper part shall be $Mm = \mu = cc$, the length of the lower tube $BzO = a$, the height $BC = b$ and $BM = x$; therefore there will become $m = a+x$ and $m = b+x$; then with the length taken $Oz = s$, to which there may correspond the cross-section $z\upsilon = \omega$ and the height $Pz = z$, the value of the integral $\int \tfrac{ds}{\omega}$ shall be through the whole extended lower part $= B$, since this value will be constant; therefore then the same integral extended to the highest surface *Mm* will become

$$= B + \tfrac{x}{cc} = \int \tfrac{dm}{\mu} :$$

from which we will have this equation on account of $M = k$:

$$2gcc(b+x)dx = \tfrac{1}{2}cc\upsilon\upsilon dx\left(1-\tfrac{f^4}{c^4}\right) - f^4\upsilon d\upsilon\left(B+\tfrac{x}{cc}\right),$$

which on putting $f^4\upsilon\upsilon\left(B+\tfrac{x}{cc}\right) = u$ will become :

$$du - \tfrac{c^4 udx}{f^4(Bcc+x)} - 4gcc(b+x)dx = 0.$$

There may be put $\tfrac{c^4}{f^4} = \lambda$ and on multiplying by $(Bcc+x)^{-\lambda}$ the integral will become:

$$\upsilon\upsilon = C(Bcc+x)^{\lambda-1} - \tfrac{4\lambda g}{(1-\lambda)(2-\lambda)}\big((2-\lambda) - Bcc + (1-\lambda)x\big)b.$$

If now we may assume the descent to have begun from *Aa* with there being $AB = e$, there will become

$$C = \tfrac{4\lambda g}{(1-\lambda)(2-\lambda)} \cdot \tfrac{(2-\lambda)b - Bcc + (1-\lambda)e}{(Bcc+x)^{\lambda-1}}$$

and thus

$$\upsilon\upsilon = \tfrac{4\lambda g\left((2-\lambda)b - Bcc + (1-\lambda)e\right)}{(1-\lambda)(2-\lambda)} \cdot \left(\frac{Bcc+x}{Bcc+e}\right)^{\lambda-1} - \tfrac{4\lambda g\left((2-\lambda)b - Bcc + (1-\lambda)x\right)}{(1-\lambda)(2-\lambda)}$$

or

$$\upsilon\upsilon = \tfrac{4\lambda g\left((2-\lambda)b - Bcc\right)}{(1-\lambda)(2-\lambda)} \cdot \left(\left(\tfrac{Bcc+x}{Bcc+e}\right)^{\lambda-1} - 1\right) + \tfrac{4\lambda g}{2-\lambda}\left(e\left(\tfrac{Bcc+x}{Bcc+e}\right)^{\lambda-1} - x\right)$$

or

$$\upsilon\upsilon = \tfrac{4\lambda g\left(Bcc + (\lambda-2)b\right)}{(\lambda-1)(\lambda-2)} \cdot \left(1 - \left(\tfrac{Bcc+x}{Bcc+e}\right)^{\lambda-1}\right) + \tfrac{4\lambda g}{\lambda-2}\left(x - e\left(\tfrac{Bcc+x}{Bcc+e}\right)^{\lambda-1}\right).$$

And if in the time $= t$ the water will have settled from *Aa* to *Mm*, there will be $dt = \tfrac{-dx\sqrt{\lambda}}{\upsilon}$: but when the water will flow out with the maximum speed, there will become

$$\upsilon\upsilon = \tfrac{4\lambda g(b+x)}{\lambda-1},$$

which therefore happens, when there will be

$$x=-Bcc+\frac{(Bcc+e)^{\frac{\lambda-1}{\lambda-2}}}{(Bcc+(\lambda-2)b+(\lambda-1)e)^{\frac{1}{\lambda-2}}}.$$

Finally, for the pressure p, by which the lower part of the tube may be forced into the section $z\upsilon$, the above equation found will adopt this form:

$$2g(k-p)\left(B+\tfrac{x}{cc}\right)=\left(2gz-\tfrac{1}{2}\upsilon\upsilon\left(1-\tfrac{f^4}{\omega\omega}\right)\right)\left(B+\tfrac{x}{cc}\right)$$
$$-\left(2g(b+x)-\tfrac{1}{2}\upsilon\upsilon\left(1-\tfrac{1}{\lambda}\right)\right)\int\tfrac{ds}{\omega},$$

from which there becomes:

$$p=k+\frac{cc\left(b+x-\frac{(\lambda-1)\upsilon\upsilon}{4\lambda g}\right)\int\frac{ds}{\omega}}{Bcc+x}-z+\frac{\upsilon\upsilon}{4g}\left(1-\frac{f^4}{\omega\omega}\right).$$

This case is especially noteworthy, where $\lambda=\frac{c^4}{f^4}$ is a very large number, in which case it follows from the differential equation:

$$4\lambda g(b+x)dx=(\lambda-1)\upsilon\upsilon dx-2(Bcc+x)\upsilon d\upsilon,$$

$\upsilon\upsilon=4g(b+x)$ is deduced at once, evidently because the opening Oo is a minimum, just as from the start the speed shall be a maximum at once, and the pressure produced in the section $z\upsilon$:

$$p=k-z+(b+x)\left(1-\frac{f^4}{\omega\omega}+\frac{cc}{\lambda(Bcc+x)}\right)\int\frac{ds}{\omega},$$

and because the final term divided by λ will be allowed to be omitted, there will become

$$p=k-z+(b+x)\left(1-\frac{f^4}{\omega\omega}\right).$$

COROLLARY 1

84. This same case, where $\lambda=\frac{c^4}{f^4}$ is a very large number, is especially noteworthy, because the experiments may be adapted to that easily; from which also the speed of the outflow prevails to scarcely differ from the value found.

COROLLARY 2

85. But concerning the pressures in the lower part of the tube BO, in this case it will be agreed to be observed mainly that these not only to be diminished below k, but also can become negative. Indeed if the cross-section $z\upsilon = \omega$ shall be equal to the opening ff, there will become

$$p = k - z = k - zP,$$

but if this cross-section is smaller than the opening ff, the pressure is diminished much more.

COROLLARY 3

86. But when the pressure p actually shall become negative, the continuity of the fluid is removed, and because the sides of the tube on the tube being abandoned itself contracts into a smaller volume, neither does a further stability rule follow. But as long as the pressure indeed is positive, but less than k, then, since the outer pressure meanwhile exceeds the inner pressure, if the tube may be punctured there by a small hole, the air or another fluid placed outside will enter, thus so that there the tube may be considered to be endowed with an attractive force.

[It is evident from Euler's descriptions above and in the following, that he conducted experiments either personally, or with the aid of assistants, whereby the results he had found by calculation were in fact in agreement with experiment, or otherwise: the last case treated presumably may have involved the stomach and associated tubes of some unfortunate animal, now diseased.]

SCHOLIUM 1

87. At this point almost all these matters correspond, which are accustomed to be treated from the discharge of water from tubes or vessels of whatever form, which now since they have been examined extensively and with great care, I am unwilling to establish further: and that thus on account of this main reason, because in most cases, in which this theory is accustomed to be applied, the calculation may be taken to differ greatly from the truth. Indeed it is evident at once, and the vessel used in figure 54 has the whole cross-section $z\upsilon$ the level not to subside equally before the opening Oo, but the parts closer to the opening are forced to descend more. Then truly, where the tube is narrowed suddenly at the opening, certainly there by no means has the motion of the fluid been prepared as we have assumed in this section. The true motion will disagree so much from this hypothesis, so that it shall be wondered at that the experiments do not differ more from the calculation. Yet meanwhile a significant difference arises, when the bottom of the vessel OC has been perforated by the most tenuous opening Oo, in which case a huge contraction in the outflow from the vessel thence arises, so that the water erupting from the sides flows out obliquely; with which it happens, so that a smaller amount of water is

being ejected through the opening than in accordance with its size. To this inconvenience those, who desire the experiments to be in agreement with the calculation, thus supply the remedy, so that they shall be accustomed to insert a cylindrical tube into the opening, as in this way the oblique motion shall be avoided.

SCHOLIUM 2

88. The case, where $\lambda = \frac{c^4}{f^4}$ is an exceedingly great number, demands a singular exposition, by which it may be explained more clearly, how water, while its motion begins from rest, suddenly may gain the maximum speed. Towards this end the formula $\left(\frac{Bcc+x}{Bcc+e}\right)^{\lambda-1}$ duly will require to be established, so that the motion may be shown generally from the beginning. Hence as the motion begins at once, the height x shall be less than e; therefore we may put

$$\frac{Bcc+x}{Bcc+e} = 1 - \frac{y}{\lambda-1},$$

so that there shall become $x = e - \frac{y(Bcc+e)}{\lambda-1}$, and with ε denoting the number, of which the hyperbolic logarithm is unity, there will become approximately:

$$\left(1-\frac{y}{\lambda-1}\right)^{\lambda-1} = \varepsilon^{-y}.$$

Therefore we will have:

$$\upsilon\upsilon = \frac{4\lambda g(Bcc+\lambda b)}{\lambda\lambda}\left(1-\varepsilon^{-y}\right) + \frac{4\lambda g}{\lambda}\left(1-e\varepsilon^{-y}\right),$$

or

$$\upsilon\upsilon = 4gb\left(1-\varepsilon^{-y}\right) + 4g\left(x-e\varepsilon^{-y}\right) = 4g\left(b+x\right) - 4g\left(b+e\right)\varepsilon^{-y},$$

from which it is apparent in the very first place, where $x = e$ and $y = 0$, on account of $\varepsilon^{-y} = 1$, actually to become $\upsilon = 0$, but very soon and the water will have settled through a minimum interval $\frac{y(Bcc+e)}{\lambda-1}$, since y is allocated a measurable value, the quantity ε^{-y} is to vanish and thus the expression becomes $\upsilon\upsilon = 4g\left(b+x\right)$. Then truly from the equation for the time $dt = \frac{-dx\sqrt{\lambda}}{\upsilon}$, since it will be allowed to write e in place of x in the value of $\upsilon\upsilon$, so that there shall become $\upsilon\upsilon = 4g\left(b+e\right)\left(1-\varepsilon^{-y}\right)$, there will become

$$dt = \frac{-dx\sqrt{\lambda}}{2\sqrt{g(b+e)(1-\varepsilon^{-y})}} = \frac{dy(Bcc+e)}{2\sqrt{\lambda g(b+e)(1-\varepsilon^{-y})}},$$

from which there is deduced on integrating

$$t = \frac{Bcc+e}{2\sqrt{\lambda g(b+e)}}\, l\, \frac{1+\sqrt{(1-\varepsilon^{-y})}}{1-\sqrt{(1-\varepsilon^{-y})}}.$$

And likewise therefore on account of ε^{-y} taken as a minimal fraction,

$$l\frac{1+\sqrt{(1-\varepsilon^{-y})}}{1-\sqrt{(1-\varepsilon^{-y})}}=l\left(4\varepsilon^{y}-1\right)=l4\varepsilon^{y}=y+l4$$

there will become

$$t=\frac{Bcc+e}{2\sqrt{\lambda g(b+e)}}\left(y+l4\right)=\frac{Bcc+e}{2\sqrt{\lambda g(b+e)}}\left(\frac{\lambda(e-x)}{Bcc+e}+l4\right).$$

Again, since the speed may emerge a maximum, when

$$x=-Bcc+\left(Bcc+e\right)\left(\frac{Bcc+e}{Bcc+\lambda(b+e)}\right)^{\frac{1}{\lambda}}=e-\frac{(Bcc+e)l\lambda}{\lambda},$$

this will arise, when $y=l\lambda$, and thus after the time

$$t=\frac{(Bcc+e)l4\lambda}{2\sqrt{\lambda g(b+e)}},$$

from the initial outflow, which since $\frac{l\lambda}{\sqrt{\lambda}}$ shall vanish, if $\lambda=\infty$, will be minimal, thus so that the water may arrive at the maximum speed as if from the first instant. Hence likewise it is understood, so that the part of the tube below *BO* were longer and likewise narrower, there it is going to arrive at the maximum speed more slowly.

SCHOLIUM 3

89. With the case established, where $\lambda=\frac{e^4}{f^4}$ is as if an infinite number, that case also, where λ is a moderately great, is seen to be worthy of being established more accurately. Therefore since we will find:

$$\frac{(1-\lambda)(2-\lambda)\upsilon\upsilon}{4\lambda g}=Bcc+\left(\lambda-2\right)b+\left(\lambda-1\right)x$$
$$-B\left(cc+\left(\lambda-2\right)b+\left(\lambda-1\right)e\right)\left(\frac{Bcc+x}{Bcc+e}\right)^{\lambda-1},$$

we may put as before $\frac{Bcc+x}{Bcc+e}=1-\frac{y}{\lambda-1}$, so that there shall become

$$y=\frac{(\lambda-1)(e-x)}{Bcc+e};$$

and since the whole concern for convenience is reduced to the establishment of the formula $\left(1-\frac{y}{\lambda-1}\right)^{\lambda-1}$, with that put $=Y$ there becomes

$$lY=\left(\lambda-1\right)l\left(1-\frac{y}{\lambda-1}\right),$$

and since there is always $y<\lambda-1$, on account of $\frac{y}{\lambda-1}=\frac{e-x}{Bcc+e}$ there will become:

$$lY=-y-\frac{yy}{2(\lambda-1)}-\frac{y^3}{3(\lambda-1)^2}-\frac{y^4}{4(\lambda-1)^3}-\text{etc.},$$

which series certainly converges strongly. Hence therefore the value found for *Y* will be:

$$\frac{(1-\lambda)(2-\lambda)\upsilon\upsilon}{4\lambda g}=\left(Bcc+(\lambda-2)b+(\lambda-1)e\right)(1-Y)-B(cc+e)y,$$

where there is

$$y=(\lambda-1)\left(1-Y^{\frac{1}{\lambda-1}}\right)\ \text{ and }\ x=(Bcc+e)Y^{\frac{1}{\lambda-1}}-Bcc.$$

Now since the maximum speed shall be:

$$\frac{(\lambda-1)(\lambda-2)\upsilon\upsilon}{4\lambda g}=(\lambda-2)(b+x),$$

here the place is defined from this equation

$$Y^{\frac{\lambda-2}{\lambda-1}}=\frac{(Bcc+e)}{\left(Bcc+(\lambda-2)b+(\lambda-1)e\right)},$$

and on putting

$$\frac{Bcc+(\lambda-2)b+(\lambda-1)e}{Bcc+e}=E$$

there becomes

$$Y^{\frac{1}{\lambda-1}}=E^{-\frac{1}{\lambda-2}}=\varepsilon^{-\frac{1}{\lambda-2}lE}=1-\frac{lE}{\lambda-2}+\frac{(lE)^2}{2(\lambda-2)^2}-\frac{(lE)^3}{6(\lambda-2)^3}+\text{etc.},$$

from which the speed will be a maximum, when

$$x=e-(Bcc+e)\left(\frac{lE}{\lambda-2}-\frac{(lE)^2}{2(\lambda-2)^2}+\frac{(lE)^3}{6(\lambda-2)^3}-\text{etc.}\right)$$

Now since again there shall be

$$\frac{(\lambda-1)(\lambda-2)\upsilon\upsilon}{4\lambda g}=(Bcc+e)\left(E(1-Y)-y\right),$$

for the time there will become

$$dt=\frac{-dx\sqrt{\lambda}}{\upsilon}=\frac{-Y^{\frac{2-\lambda}{\lambda-1}}dY\sqrt{(\lambda-2)(Bcc+e)}}{2\sqrt{(\lambda-1)g\left(E(1-Y)-(\lambda-1)\left(1-Y^{\frac{1}{\lambda-1}}\right)\right)}}$$

and on making $Y=u^{k-1}$ there becomes

$$dt = \frac{-dx\sqrt{\lambda}}{\upsilon} = \frac{-du\sqrt{(\lambda-1)(\lambda-2)(Bcc+e)}}{2\sqrt{g\left(E-\lambda+1+(\lambda-1)u-Eu^{\lambda-1}\right)}}.$$

Indeed a more convenient calculation will be established by retaining only the quantity y and by putting :

$$Y = \varepsilon^{-y}\left(1-\frac{yy}{2(\lambda-1)}-\frac{y^3}{3(\lambda-1)^2}+\frac{y^4}{8(\lambda-1)^2}\right),$$

from which the solution of the preceding paragraph will be produced closer to the truth, while also the terms divisible by λ are introduced. But since these are purely analytical, they will not be treated further here.

PROBLEM 55

90. *If the tube* (Fig. 56)*, while water flows out through the opening Oo, and it may accept a new supply of water continually into the other end Aa, to define its motion, so that it may be kept full as far as Aa, and there the water may be thrust forwards by some force.*

SOLUTION

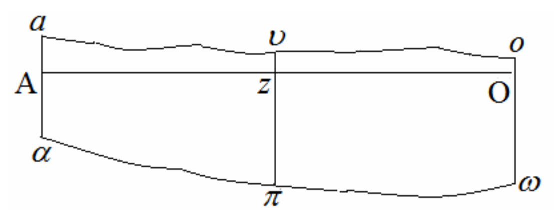


Fig. 56

With the cross-section of the orifice $Oo = ff$, the speed shall be υ, with which now in the elapsed time $= t$ water will flow out there, in another place truly some *z*, of which the distance from the start *A* shall be $Az = s$, and the cross-section of the tube $z\upsilon = \omega$ and the height above a fixed horizontal plane $z\pi = z$, if indeed the applied lines of the curve $\alpha\pi\omega$ are assumed to show the heights of the individual points of the tube above the same plane. With these in place if in the cross-section $z\upsilon$ the pressure may be put $= p$, the principle of motion provide this equation:

$$2gp = \varDelta : t - 2gz - \frac{f^4\upsilon\upsilon}{2\omega\omega} - \frac{ffd\upsilon}{dt}\int\frac{ds}{\omega}.$$

If now the pressure at $Aa = L$, with the cross-section $Aa = cc$ and the height $A\alpha = \mathfrak{a}$, and since here $s = 0$ and likewise the integral $\int\frac{ds}{\omega}$ vanishes, on account of $p = L,\ z = \mathfrak{a}$ and $\omega = cc$ there will become :

$$2gL = \varDelta : t - 2g\mathfrak{a} - \frac{f^4\upsilon\upsilon}{2\omega\omega}.$$

Then for the opening *Oo*, the pressure here shall be $=k$, referring to the weight of the atmosphere, and the value of the integral $\int\frac{ds}{\omega}$ along the extent of the whole tube *AO* may become $=\mathfrak{O}$, truly the height $O\omega=\mathfrak{o}$. Wherefore on account of $p=k,\ z=\mathfrak{o}$ and $\omega=ff$ there will be had

$$2gk=\Delta:t-2g\mathfrak{o}-\tfrac{1}{2}\upsilon\upsilon-\tfrac{f^4d\upsilon}{dt}\cdot\mathfrak{O}.$$

Now this equation taken from that equation above leaves

$$2g\left(L-k\right)=2g\left(\mathfrak{o}-\mathfrak{a}\right)+\tfrac{1}{2}\upsilon\upsilon\left(1-\tfrac{f^4}{c^4}\right)-\tfrac{f^4d\upsilon}{dt}\cdot\mathfrak{O}$$

or

$$2g\left(L-k+\mathfrak{o}-\mathfrak{a}\right)dt-\upsilon\upsilon dt\left(1-\tfrac{f^4}{c^4}\right)=2\mathfrak{O}ffd\upsilon,$$

from which, since $\mathfrak{a},\ \mathfrak{o}$ and $\mathfrak{O}$ shall be constant quantities, truly the pressure *L* shall be able to denote a function of the time, if indeed that may be varied with the time, the speed υ must be defined for some time Moreover with the pressure *L* constant an equation of this kind will be able to be resolved:

$$dt=\frac{Ad\upsilon}{B\pm C\upsilon\upsilon},$$

with there being $A=2\mathfrak{O}ff,\ B=4g(L-k+\mathfrak{a}-\mathfrak{o})$ and $\pm C=\frac{f^4}{c^4}-1;$ therefore three cases are required to be established.

I. If $cc=ff$ or the cross-section *Aa* is equal to the opening $Oo=ff$, there will become

$$C=0\ \text{ and }\ t=\tfrac{Av}{B}\ \text{ or }\ v=\tfrac{B}{A}t+\text{Const.}$$

from which, if $B>0$, the speed will be able to increase continually.

II. If $cc>ff$ or the cross-section *Aa* may be greater than *Oo*, on putting $C=1-\frac{f^4}{c^4}$ the equation $dt=\frac{Ad\upsilon}{B-C\upsilon\upsilon}$ integrated gives

$$t=\tfrac{A}{2\sqrt{BC}}\,l\frac{\sqrt{B}+\upsilon\sqrt{C}}{\sqrt{B}-\upsilon\sqrt{C}}+\text{Const.},$$

which constant vanishes, if the motion may have began from rest ; and in this case the speed certainly increases, or also in an infinite passage of the time it will not increase beyond $\upsilon=\frac{\sqrt{B}}{\sqrt{C}}$.

III. If $cc<ff$ or the cross-section *Aa* shall be smaller than the opening *Oo*, on putting $C=1-\frac{f^4}{c^4}$ the integrated equation $dt=\frac{Ad\upsilon}{B+C\upsilon\upsilon}$ gives:

$$t = \frac{A}{\sqrt{BC}}\,\text{arc.tan}\,\frac{\upsilon\sqrt{C}}{\sqrt{B}} \text{ or } \upsilon = \frac{\sqrt{B}}{\sqrt{C}}\,\text{arc.tan}\,\frac{t\sqrt{BC}}{A},$$

where this outcome is noteworthy, so that in the elapsed time $t = \frac{A}{\sqrt{BC}} \cdot \frac{\pi}{2}$ the speed may emerge infinitely great.

An outflow found with the speed υ for any time *t*, at some middle location $z\upsilon$, with the pressure *p*, is expressed thus:

$$2g\left(p-k\right) = 2g\left(\mathfrak{o}-\mathfrak{a}\right) + \tfrac{1}{2}\upsilon\upsilon\left(1-\frac{f^4}{\omega\omega}\right) + \frac{ffd\upsilon}{dt}\left(\mathfrak{D} - \int\frac{ds}{\omega}\right),$$

which the derived differential formula provides, with $\frac{ffd\upsilon}{dt}$ replaced:

$$4g\left(p-L+z-\mathfrak{a}\right)\mathfrak{D} = \upsilon\upsilon\left(\frac{f^4}{c^4}-\frac{f^4}{\omega\omega}\right)\mathfrak{D} + 2g\left(\mathfrak{o}-\mathfrak{a}\right) + \upsilon\upsilon\left(1-\frac{f^4}{c^4}\right)\int\frac{ds}{\omega}$$
$$-4g\left(L-k+\mathfrak{a}-\mathfrak{o}\right)\int\frac{ds}{\omega},$$

and thus everything pertaining to the motion has been determined.

COROLLARY 1

91. If uniform motion may arise, thus so that now water may be expelled constantly with the same speed through the opening *Oo* , on account of $d\upsilon = 0$ this equation will be had:

$$4g\left(L-k+\mathfrak{a}-\mathfrak{o}\right) = \upsilon\upsilon\left(1-\frac{f^4}{c^4}\right),$$

from which, if the cross-section at *Aa* shall be equal to the orifice Oo, as in the preceding chapter, the pressure at *Aa* must become $L = k+\mathfrak{o}-\mathfrak{a}$, hence the speed υ cannot be determined.

COROLLARY 2

92. But if the cross-section $Aa = cc$ were greater than the opening $Oo = ff$, the speed of the outflow υ is defined thus for motion with a constant speed, so that there shall become:

$$\upsilon\upsilon = \frac{4gc^4\left(L-k+\mathfrak{a}-\mathfrak{o}\right)}{c^4-f^4}.$$

Therefore in this case it is necessary, so that there shall be $L > k+\mathfrak{o}-\mathfrak{a}$, and from this excess the speed of the outflow is determined.

COROLLARY 3

93. But if the cross-section $Aa = cc$ were smaller than the opening $Oo = ff$, uniform motion provides this equation

$$\upsilon\upsilon = \frac{4gc^4(k+\mathfrak{o}-\mathfrak{a}-L)}{f^4-c^4},$$

from which it is apparent uniform motion cannot be obtained, unless there shall be $L < k+\mathfrak{o}-\mathfrak{a}$; and the speed of the outflow is determined from this difference.

SCHOLIUM 1

94. All these certainly are especially puzzling, since from the same pressure *L*, by which the water may be forced forwards at the cross-section *Aa*, a speed of any magnitude shall be able to be found to arise; and this will be seen to be especially absurd, since in the third case thus an infinite speed may be able to be impressed by a finite force *L* in a finite time. But this absurdity will vanish at once, only if we may consider more carefully the hypothesis on which this problem depends; for we suppose, while the water is being driven forwards through the section *Aa*, continually a new supply of water to flow in there from elsewhere with the same speed, neither here do we look after, from where this water may have come and from which force this motion itself may be introduced ; evidently this force is far different from the force *L*, which acts on nothing else, except so that the water with that entering speed now may be driven further through the tube. Therefore while this force *L* may prevail to accelerate the motion of the water entering through *Aa*, the speed of the outflow will increase and thus by the hypothesis new water is assumed always to be carried in with a greater speed by that external force. Therefore when the calculation shows the speed to become greater and indeed soon to become infinite, here the minimal effect must be attributed to our finite force *L* requiring to drive the water forwards through the tube, but evidently must be attributed to that external force, which in this case certainly becomes infinite ; indeed which forces new water into the tube at an infinite speed. And for the same reason it is required to consider that paradox also, as the outflow speed itself will not be determined in the problem; where indeed the greater speed and abundance of the new water must be supplied by that external force, whatever that shall be, therefore a greater speed also the same pressure *L* at *Aa* will prevail to propel that through the tube; therefore since no account of that external force is found in our calculations, it is no wonder, that the calculation implies so great a paradox in itself, but which with the matter better expended are refuted at once.

SCHOLION 2

95. But with the introduction of a power *L* of this kind, which may press continually with an equal force, the water will be moved through the tube either more quickly or more slowly, according to the nature of the forces, which are used to propel the water,

with the maximum repulsion, since all these forces shall be prepared thus, so that, quo where the water may be moved through the tube quicker, thus they shall be weakened more. On account of which, if we may wish to adapt this problem to real cases, in which water must be raised to a certain level, the nature of these forces will have to be considered properly, which it is required to use to this end. Now since I shall have set these out clearly in the preceding chapter, then for the present situation I repeat only the procedure requiring to be carried out: a certain force to be used *F*, which shall act with a certain velocity $= e$, thus so that now the whole question may correspond to this: how the machine may be agreed to be constructed, so that by this same force acting the water may be able to be propelled through the tube uniformly.

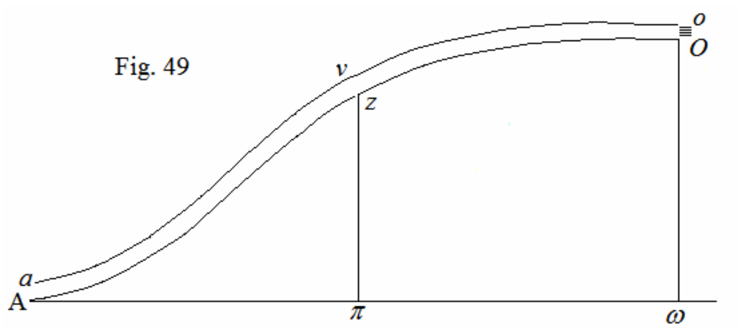
PROBLEM 56

96. *If* (Fig. 49) *water must be raised by a given force F to a given height* $O\omega = a$*, in a uniform motion through a tube of some kind unequal cross-section Aa Oo, which shall work at the given speed* $= e$*, to invent a machine, with the aid of which this effect may be able to be obtained, and likewise to define the copious supply of water requiring to be raised in a given time.*

SOLUTION

Because the nature of all machines consists in this, so that the force by which the water

Fig. 49



may be moved, may be transferred to another place and that likewise may be increased or diminished in a given ratio, we may put that machine sought to be outstanding, so that the force propelling the water through the lower opening *Aa* shall become $= nF$; and from the nature of the machine this force will act with the speed $= \frac{e}{n}$, thus so that the construction of the machine may depend on the number *n* only, which therefore will require *n* to be defined. Now by calling the preceding problem in to help, which with the cross-section *Aa* has been put $= cc$, at which the force *nF*, the pressure exerted there will be $= \frac{nF}{cc}$, which since it may be helped by the pressure of the atmosphere *k*, we will have the pressure put there to be $L = \frac{nF}{cc} + k$; and water will be propelled through the lower opening *Aa* with the speed $=\frac{e}{n}$. Whereby since the cross-section of the upper opening *Oo* shall be $=ff$, water will be expelled there with the speed $=\frac{cce}{nff}$, thus so that there shall become $\upsilon=\frac{cce}{nff}$ and $d\upsilon=0$. Now again with the height of the opening *Oo* put before $=\mathrm{o}$ here there becomes $O\omega=a$, truly the height of the lower *Aa* is zero or $\mathfrak{a}=0$, and from which the equation for the motion found there will adopt this form:

$$4g\left(\tfrac{nF}{cc}-a\right)-\tfrac{c^4ee}{nnf^4}\left(1-\tfrac{f^4}{c^4}\right)=0 \text{ or } 4gnn\left(\tfrac{nF}{cc}-a\right)=ee\left(1-\tfrac{c^4}{f^4}\right),$$

and thus from which the machine will be allowed to define the number n . But then it is necessary, that the water brought in continuously from elsewhere with the speed $=\frac{e}{n}$ at the opening *Aa* and the volume pouring in per second shall be $=\frac{cce}{n}$, but just as much will be discharged per second through the above opening *Oo*.

Then if at some place z of the tube the cross-section may be put $z\upsilon=\omega$, the height $z\pi=z$ and with the length $Az=s$, therefore the at that place $=p$, from the preceding problem there will become :

$$2g\left(p-k\right)=2g\left(a-z\right)+\tfrac{c^4ee}{2nnf^4}\left(1-\tfrac{f^4}{\omega\omega}\right),$$

from which, if by the preceding equation we may eliminate $\frac{c^4ee}{nnf^4}$, there will become

$$p=k-z+\left(a\left(\tfrac{f^4}{\omega\omega}-\tfrac{f^4}{c^4}\right)+\tfrac{nF}{cc}\left(1-\tfrac{f^4}{cc}\right)\right):\left(1-\tfrac{f^4}{c^4}\right).$$

COROLLARY 1

97. If both the orifices *Aa* and *Oo* are equal or $cc=ff$, there becomes $\frac{nF}{cc}-a=0$ and thus $n=\frac{acc}{F}$; then for the construction of the machine the volume of water ejected per second $=\frac{Fe}{a}$; truly just as great an amount meanwhile must be supplied continually with the speed $=\frac{Fe}{acc}$ into the orifice *Aa* ; for which there is need of a particular force, concerning which we do not consider here.

COROLLARY 2

98. If there shall be $cc>ff$ and thus $\frac{c^4}{f^4}-1>0$, there becomes $\frac{n}{cc}>\frac{a}{F}$ hence the volume of water ejected per second will be $<\frac{Fe}{a}$ and the same amount of water must be carried

into the opening *Aa* with a speed smaller than $\frac{Fe}{acc}$, for which a smaller force than in the preceding case is required.

COROLLARY 3

99. But if the upper orifice *ff* shall be smaller than the lower one *cc*, there becomes $\frac{n}{cc} < \frac{a}{F}$, and the volume of water ejected per second becomes $> \frac{Fe}{a}$, thus so that in this way more water is raised than in the case $ff = cc$; indeed also with so much more water introduced by that external force at the opening *Aa*, and that must be raised with a speed greater than $\frac{Fe}{acc}$.

SCHOLIUM

100. Therefore it must not be seen as a wonder, so that by the same force a machine by moving either a larger or smaller amount of water to the same height, provided the above orifice *Oo* were larger or smaller than the orifice *Aa*. But if we may wish to consider the whole effect, the whole cause also is required to be observed, which will be had, if to that force by which we may suppose the machine to be made to move, and in addition that force may be added, which is required to bring the water continually to the orifice *Aa*, but these two forces taken jointly for that case, where a smaller supply of water may be raised, certainly gives rise to a smaller sum than the other case, where a greater supply may be raised, thus so that here nothing may happen, so that it may oppose the equality between the cause and the effect. Truly because in practice the same force, by which water is propelled through a tube, also water must be supplied continually into the tube, in as much as this twofold effect is produced from the same cause, the use in practice is required to be investigated more carefully. Therefore since with the aid of pumps water may be drawn into a tube, as well as may be accustomed to be propelled through that, for this to be investigated, we put in place a special chapter, which in practice has the greatest use.

CAPUT III

DE MOTU AQUAE IN TUBIS INAEQUALITER AMPLIS

PROBLEMA 53

64. *Si data aquae quantitas in tubo, cuius amplitudo utcunque est variabilis, moveatur et utrinque a viribus quibuscunque prematur, eius motum et pressionem in singulis punctis determinare.*

SOLUTIO

Quamcunque directrix tubi habuerit figuram (Fig. 50), ea ut linea recta *AO* consideretur, cui autem adiungatur linea curva $\alpha\omega$, cuius applicatae $z\pi$

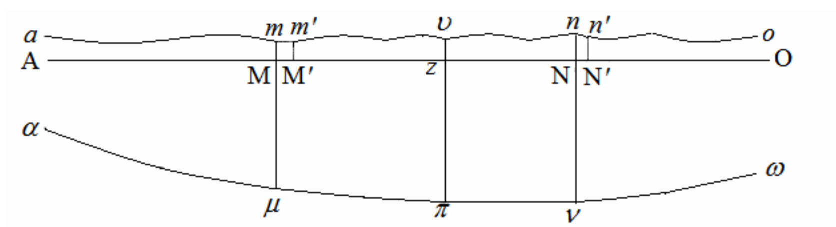


Fig. 50

singulorum punctorum *z* altitudines super plano horizontali fixo denotent. Iam elapso tempore *t* consideretur aquae particula quaecunque, quae versetur circa tubi punctum *z*, existente directricis longitudine $Az = s$ a puncto fixo *A* computata, ibique sit tubi amplitudo $z\upsilon = \omega$ et altitudo $z\pi = z$, quae per *s* datae assumuntur. Densitas aquae unitate exprimatur, ut sit $q = 1$, in *z* vero vocetur pressio $= p$ pariter ad aquam relata, et celeritas huius particulae in tubo versus *O* sit $= \mathfrak{T}$, quae sunt functiones duarum variabilium *s* et *t*.

Quibus positis problema 46 primo nobis suppeditat hanc aequationem $\left(\frac{d\cdot\mathfrak{T}\omega}{ds}\right) = 0$, unde $\mathfrak{T}\omega$ functioni solius temporis aequetur necesse est, cum ergo hoc tempore ubique celeritas sit reciproce ut amplitudo, concipiamus alicubi amplitudinem datam $= ff$, in qua sit celeritas $= v$, functio ipsius tempore *t*, eritque

$$\mathfrak{T}\omega = ff\upsilon \quad \text{et} \quad \mathfrak{T} = \frac{ff\upsilon}{\omega};$$

ita ut, si definita fuerit celeritas *v* amplitudini datae ff conveniens pro hoc tempore, ex ea celeritas in quacunque alia amplitudine ω innotescat ad idem tempus; hacque formula $\mathfrak{T} = \frac{ff\upsilon}{\omega}$ iam prima determinatio contineatur, ubi probe notetur celeritatem υ esse functionem solius temporis *t*, amplitudinem vero ω spatii *s* tantum. Nunc ad alteram aequationem progrediamur, qua pressio *p* definitur, et quia aquam a sola gravitate animari ponimus, erit $P = 0$, $Q = 0$ et $R = -1$, tum vero, quia in hac aequatione tempus *t* constans accipitur, erit

$$d\mathfrak{T} = \frac{-ff\upsilon d\omega}{\omega\omega} \text{ et } \left(\frac{d\mathfrak{T}}{dt}\right) = \frac{ffd\upsilon}{\omega dt},$$

sicque aequatio posterior abit in hanc formam:

$$2gdp = -2gdz + \frac{f^4\upsilon\upsilon d\omega}{\omega^3} - \frac{ffds}{\omega} \cdot \frac{d\upsilon}{dt},$$

quae, quia υ et $\frac{d\upsilon}{dt}$ ut constantes spectantur, per integrationem dat:

$$2gp = \Delta : t - 2gz + \frac{f^4\upsilon\upsilon}{2\omega\omega} - \frac{ffd\upsilon}{dt}\int\frac{ds}{\omega},$$

ubi eum ω sit functio solius *s,* integrale $\int\frac{ds}{\omega}$ ut quantitas cognita spectari potest. Nunc ad ambos terminos nostrae massae aqueae respiciamus qui sint in *M* et *N* existente $AM = m,\ AN = n$, amplitudine in $M = \mu$, in $N = \nu$, altitudine $M\mu = \mathfrak{m}$, in $N\nu = \mathfrak{n}$; integralis $\int\frac{ds}{\omega}$ valore in $M = \mathfrak{M}$, in $N = \mathfrak{N}$; tum vero pressione in $M = M$ et in $N = N$. Cum igitur celeritas in M sit $= \frac{ff\upsilon}{\mu}$, in $N = \frac{ff\upsilon}{\nu}$, tempusculo *dt* ambo termini *M* et *N* promovebuntur in *M', N',* ut sit

$$MM' = \frac{ff\upsilon}{\mu}dt \text{ et } NN' = \frac{ff\upsilon dt}{\nu},$$

unde, quia *m* et *n* sunt functiones solius temporis *t,* erit

$$dm = \frac{ff\upsilon dt}{\mu},\ dn = \frac{ff\upsilon dt}{\nu}$$

hincque $\mu dm = \nu dn$. Ex cognitis autem pressionibus in *M* et *N* has duas obtinemus aequationes:

$$2gM = \Delta : t - 2g\mathfrak{m} - \frac{f^4\upsilon\upsilon}{2\mu\mu} - \frac{ffd\upsilon}{dt}\cdot\mathfrak{M},$$

$$2gN = \Delta : t - 2g\mathfrak{n} - \frac{f^4\upsilon\upsilon}{2\nu\nu} - \frac{ffd\upsilon}{dt}\cdot\mathfrak{N},$$

unde colligimus:

$$2g(M-N) = 2g(\mathfrak{n}-\mathfrak{m}) - \frac{f^4\upsilon\upsilon}{2}\left(\frac{1}{\nu\nu} - \frac{1}{\mu\mu}\right) + \frac{ffd\upsilon}{dt}(\mathfrak{N}-\mathfrak{M}),$$

quae aequatio tantum functiones ipsius temporis *t* involvit, indeque propterea celeritas υ definiri poterit. Tum vero pro pressione invenitur:

$$2g(M-p) = 2g(z-\mathfrak{m}) - \frac{f^4\upsilon\upsilon}{2}\left(\frac{1}{\omega\omega} - \frac{1}{\mu\mu}\right) + \frac{ffd\upsilon}{dt}\left(\int\frac{ds}{\omega} - \mathfrak{M}\right),$$

quae elisa formula $\frac{ffd\upsilon}{dt}$ praebet hanc aequationem:

$$2g(p+z)+\frac{f^4\upsilon\upsilon}{2\omega\omega}(\mathfrak{N}-\mathfrak{M})=\begin{cases}+2g(M+\mathfrak{m})+\frac{f^4\upsilon\upsilon}{2\mu\mu}\left(\mathfrak{N}-\int\frac{ds}{\omega}\right)\\+2g(N+\mathfrak{n})+\frac{f^4\upsilon\upsilon}{2\nu\nu}\left(\int\frac{ds}{\omega}-\mathfrak{M}\right).\end{cases}$$

COROLLARIUM 1

65. Cum detur massa fluidi in tubo contenta, ex dato spatio $AM=m$, quo simul quantitates μ, $\mathfrak{m}$ et $\mathfrak{M}=\int\frac{dm}{\mu}$ determinantur, definitur spatium $AN=n$, cum $\int vdn-\int\mu dm$ praebeat illam massam, sicque etiam *n* cum $v,\mathfrak{n}$ et $\mathfrak{N}=\int\frac{dn}{v}$ ut functiones solius quantitatis *m* spectari poterunt.

COROLLARIUM 2

66. Quoniam totum negotium a resolutione aequationis differentialis inventae pendet et est $dt=\frac{\mu dm}{ff\upsilon}$, si ea aequatio per $\mu dm=ff\upsilon dt$ multiplicetur, habebitur:

$$2g(M-N+\mathfrak{m}-\mathfrak{n})\mu dm=\tfrac{1}{2}f^4\upsilon\upsilon\mu dm\left(\frac{1}{vv}-\frac{1}{\mu\mu}\right)+f^4\upsilon d\upsilon(\mathfrak{N}-\mathfrak{M}),$$

quae posito $f^4\upsilon\upsilon=V$ abit in hanc

$$2g(M-N+\mathfrak{m}-\mathfrak{n})\frac{\mu dm}{\mathfrak{N}-\mathfrak{M}}=dV+\frac{V\mu dm}{\mathfrak{N}-\mathfrak{M}}\left(\frac{1}{vv}-\frac{1}{\mu\mu}\right),$$

ex quo quantitatem *V* elici oportet, qua inventa primo reperitur celeritas $\upsilon=\frac{\sqrt{V}}{ff}$, indeque porro tempus $t=\int\frac{\mu dm}{\sqrt{V}}$.

COROLLARIUM 3

67. Si enim pressiones *M* et *N* vel sint constantes, vel a spatiis *m* et *n* pendeant, quia *n* per *m* determinatur, aequatio illa duas tantum variabiles *m* et *V* continere est censenda et integrabilis redditur, si multiplicetur per e^Q existente

$$Q=\int\frac{\mu dm}{\mathfrak{N}-\mathfrak{M}}\left(\frac{1}{vv}-\frac{1}{\mu\mu}\right).$$

Quia vero est $\mu dm=vdn$ et $\frac{dn}{v}=d\mathfrak{N}$ item $\frac{dm}{\mu}=d\mathfrak{M}$, fit

$$Q=\int\frac{d\mathfrak{N}-d\mathfrak{M}}{\mathfrak{N}-\mathfrak{M}}$$

hincque multiplicator $e^{Q} = \mathfrak{N} - \mathfrak{M}$.

COROLLARIUM 4

68. Quamobrem illius aequationis integrale est:

$$\left(\mathfrak{N} - \mathfrak{M}\right)V = f^4 \upsilon\upsilon\left(\mathfrak{N} - \mathfrak{M}\right) = 4g\int \mu dm(M - N + \mathfrak{m} - \mathfrak{n}),$$

ubi notandum est, cum sit celeritas aquae in $M = \frac{ff\upsilon}{\mu}$ et in $N = \frac{ff\upsilon}{\nu}$, expressionem

$$f^4\upsilon\upsilon\left(\mathfrak{N} - \mathfrak{M}\right) = \int \frac{f^4\upsilon\upsilon}{\nu\nu}\cdot \nu dn - \int \frac{f^4\upsilon\upsilon}{\mu\mu}\cdot \mu dm$$

designare vim vivam massae aquae *MmNn,* quandoquidem νdn est eius elementum *NnN'n',* idque in celeritatis quadratum ducitur.

SCHOLIUM

69. Omni attentione dignum est, quod aequatio differentialis inventa tam commode integrari potuerit eiusque integrale ad vim vivam aquae in tubo contentae sit perductum, unde summus usus principii conservationis virium vivarum, quo iam olim Celeberrimus BERNOULLI in Hydrodynamica felicissimo successu est usus, clarissime perspicitur. Hinc scilicet intelligimus, si vires utrinque prementes *M* et *N* fuerint aequales et tubi directrix horizontalis, ut nullae adsint vires motum fluidi vel accelerantes vel retardantes, tum fluidi massam eandem perpetuo vim vivam esse conservaturam, posito enim $M = N$ et $\mathfrak{m} = 0$ et $\mathfrak{n} = 0$ seu in genere $z = 0$, prodit vis viva

$$f^4\upsilon\upsilon\left(\mathfrak{N} - \mathfrak{M}\right) = \text{Const.},$$

sin autem altitudines $\mathfrak{m}$ et $\mathfrak{n}$ non evanescant, aequatio inventa ob $\mu dm = \nu dn$ ita repraesentari potest:

$$f^4\upsilon\upsilon\left(\mathfrak{N} - \mathfrak{M}\right) = 4g\int (M + \mathfrak{m})\mu dm - 4g\int (N + \mathfrak{n})\nu dm,$$

unde manifestum est, quantum incrementum vis viva capiat a vi accelerante; quandoquidem pressio *M* motum accelerat, pressio vero *N* retardat, ac praeterea ex altitudinibus $\mathfrak{m}$ et $\mathfrak{n}$ singulorum elementorum vel ascensus vel descensus definitur. Ceterum hic imprimis notari meretur, quod aequatio differentialis inventa sola multiplicatione per

$$2\mu dm = 2\nu dn = 2ff\upsilon dt$$

statim integrabilis reddatur, dum prodit

$$4g(M-N+\mathfrak{m}-\mathfrak{n})\mu dm = f^4\upsilon\upsilon\left(\tfrac{dn}{\nu}-\tfrac{dm}{\mu}\right)+2f^4\upsilon d\upsilon\left(\mathfrak{N}-\mathfrak{M}\right),$$

cuius integrabilitas ob $\frac{dn}{\nu}=d\mathfrak{N}$ et $\frac{dm}{\mu}=d\mathfrak{M}$ statim in oculos incurrit; ita ut iam totum negotium ad integrationem primae partis reducatur. Ad maiorem ergo dilucidationem sufficit, ut nonnulla exempla proferamus.

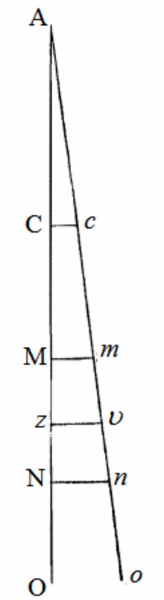


Fig. 51

EXEMPLUM 1

70. *Si tubus* (Fig. 51) *sit conicus eiusque directrix AO verticalis, in quo massa aquea ACc libere descendat, eius motum definire.*

Sit $AC=c$ et amplitudo tubi in *C* nempe $Cc=\alpha cc$, ut fiat tota massa aquae $ACc=\frac{1}{3}\alpha c^3$, quae post tempus *t* occupet tubi spatium *MmNn,* unde ob $AM=m$ et $AN=n$ erit $n^3=c^3+m^3$. Tum vero posita altitudine fixa $AO=a$, erunt primo amplitudines

$$Mm=\mu=\alpha mm,\ Nn=\nu=\alpha nn \text{ et } z\upsilon=\omega=\alpha ss$$

posito $Az=s$, deinde altitudines

$$OM=\mathfrak{m}=a-m,\ ON=\mathfrak{n}=a-n \text{ et } Oz=z=a-s.$$

Porro ob $\int\frac{ds}{\omega}=-\frac{1}{\alpha s}$, fit $\mathfrak{M}=-\frac{1}{\alpha m}$ et $\mathfrak{N}=-\frac{1}{\alpha n}$. Quare si pressiones in *M* et *N* aequentur soli pressioni atmosphaerae *k,* quod evenit, si tubus in apice *A* apertus concipiatur, erit $M=N=k$. Quodsi iam amplitudini *ff* conveniat celeritas $=v$ deorsum tendens, aequatio nostra integralis pro hoc casu colligitur:

$$f^4\ \upsilon\upsilon\left(\tfrac{1}{\alpha m}-\tfrac{1}{\alpha n}\right)=4g\int\alpha mmdm(n-m)=4\alpha g\left(\tfrac{1}{4}n^4-\tfrac{1}{4}m^4\right)+\text{Const.}$$

ob $mmdm=nndn$. Cum autem descensus ex quiete incipiat, facto $m=0$ et $n=c$ celeritas evanescere debeat, unde habebitur:

$$f^4\ \upsilon\upsilon\left(\tfrac{1}{m}-\tfrac{1}{n}\right)=\alpha\alpha g\left(n^4-m^4-c^4\right)$$

hincque

$$f\upsilon\upsilon=\alpha\sqrt{\frac{gmn\left(n^4-m^4-c^4\right)}{n-m}}$$

sicque colligitur tempus

$$t=\int\frac{\mu dm}{ff\upsilon}=\int\frac{mdm\sqrt{m(n-m)}}{\sqrt{gn\left(n^4-m^4-c^4\right)}},$$

cuius formulae ob $n^3 = c^3 + m^3$ integrale est capiendum, ut ad datum tempus *t* spatium $AM = m$ definiri possit. Denique pro pressione in *z,* quae est *p,* invenienda habetur haec aequatio superiorem per *z* multiplicando

$$\left(4g\left(p+a-s\right)+\tfrac{f^4\upsilon\upsilon}{2\omega\omega s^4}\right)\left(\tfrac{1}{\alpha m}-\tfrac{1}{\alpha n}\right)=\begin{cases}+\left(4g\left(k+a-m\right)+\tfrac{f^4\upsilon\upsilon}{\alpha\alpha m^4}\right)\left(\tfrac{1}{\alpha s}-\tfrac{1}{\alpha n}\right)\\+\left(4g\left(k+a-n\right)+\tfrac{f^4\upsilon\upsilon}{\alpha\alpha n^4}\right)\left(\tfrac{1}{\alpha m}-\tfrac{1}{\alpha s}\right),\end{cases}$$

quae ob

$$\frac{f^4 vv}{\alpha\alpha}=\frac{gmn\left(n^4-m^4-c^4\right)}{n-m}$$

abit in hanc:

$$\frac{4(p+a-s)(n-m)}{mn}=\frac{4(k+a-m)(n-s)}{ns}+\frac{4(k+a-n)(s-m)}{ms}$$
$$+\frac{(s-m)(n-s)\ (n^4-m^4-c^4)(mmnn+mn(m+n)s+(mm+mn+nn)ss)}{m^3n^3s^4}$$

unde deducimus:

$$p=k+\tfrac{mn}{s}-m-n+s+\frac{(s-m)(n-s)\left(n^4-m^4-c^4\right)\left(mmnn+mn\left(m+n\right)s+\left(mm+mn+nn\right)ss\right)}{4mmnn\left(n-m\right)s^4}.$$

EXEMPLUM 2

71. *In casu praecedentis exempli si tubus sit in A clausus, ut superior superficies Mm nullam pressionem sustineat, motum aquae determinare.*

Cum omnia maneant ut in praecedente exemplo, nisi quod hic sit $M = 0$ et $N = k$, aequatio prior abit in hanc formam

$$f^4\ \upsilon\upsilon\left(\tfrac{1}{\alpha m}-\tfrac{1}{\alpha n}\right)=4g\int\alpha mmdm(n-m-k)$$

seu

$$f^4\ \upsilon\upsilon\left(\tfrac{1}{m}-\tfrac{1}{n}\right)=\alpha\alpha g\left(n^4-m^4-c^4-\tfrac{4}{3}km^3\right).$$

Hic autem primum observo initio, ubi $m = 0$ et $n = c$, motum incipere non potuisse, nisi fuerit $c > k$; si enim sit $c < k$ vel etiam $c = k$, aqua perpetuo in summitate tubi haerebit nullusque motus sequetur. Sin autem sit $c > k$, motus primo quidem accelerabitur, donec fiat

$$n=\sqrt[3]{\left(c^3+m^3\right)}=m+k,$$

hoc est

$$c^3 = 3kmm + 3kkm + k3 \ \text{ seu } m=\sqrt{\left(\tfrac{c^3}{3k}-\tfrac{1}{12}kk\right)}-\tfrac{1}{2}k.$$

Inde vero celeritas decrescet, atque adeo evanescet, quando fiet

$$n^4=\left(c^3+m^3\right)^{\frac{4}{3}}=c^4+m^4+\tfrac{4}{3}km^3,$$

quae evoluta dat

$$\begin{aligned}4km^8+\tfrac{16}{3}kkm^7+\tfrac{64}{27}k^3m^6+3c^4m^5+8kc^4m^4+\tfrac{16}{3}kkc^4m^3+3c^8m+4kc^8\\ -4c^3m^6 \qquad\qquad\qquad -6c^6m^3 \qquad -4c^9=0.\end{aligned}$$

Quo hinc aliquid facilius concludere queamus, ponamus c valde parum excedere k statuamusque $c=(1+\delta)k$, denotante δ fractionem minimam, et quia m quoque erit spatium valde parvum, fiet $n=c+\frac{m^3}{3cc}$ hincque

$$f^4\upsilon\upsilon\left(\tfrac{1}{m}-\tfrac{1}{c}+\tfrac{m^3}{3c^4}\right)=\alpha\alpha g\left(\tfrac{4\delta cm^3}{3(1+\delta)}-m^4+\tfrac{2m^6}{3cc}\right)$$

ac celeritas maxima respondebit loco

$$m=\tfrac{\delta c}{1+\delta}+\tfrac{m^3}{3cc}=\tfrac{\delta c}{1+\delta}+\tfrac{\delta^3c}{3(1+\delta)^2}$$

rursusque evanescet, ubi

$$m=\tfrac{4\delta c}{3(1+\delta)}+\tfrac{2m^3}{3cc}$$

seu satis exacte $m=\frac{4\delta c}{3(1+\delta)}$. Deinde vero colligitur

$$ff\upsilon=\alpha mm\sqrt{g\left(\tfrac{4\delta c}{3(1+\delta)}-m\right)}=\alpha mm\sqrt{g\left(\tfrac{4}{3}\delta k-m\right)}$$

hincque tempus

$$t=\int\frac{dm}{\sqrt{g\left(\frac{4}{3}\delta k-m\right)}}=2\sqrt{\frac{4\delta k}{3g}}-2\sqrt{\frac{4\delta k-3m}{3g}}$$

et tempus totius descensus $=4\sqrt{\frac{\delta k}{3g}}$.

EXEMPLUM 3

72. *Si* (Fig. 52) *tubus habeat duo brachia verticaliter erecta AB et CO iuncta ramo horizontali BC et quaelibet pars sit aequaliter ampla, sed a reliquis diversa, definire motum oscillatorium aquae in hoc tubo.*

Tubi *AB,* in quo alter venae terminus *Mm* reperitur, amplitudo sit ubique $=\mu$, tubi vero $OC=\nu$, horizontalis vero $BC=\lambda$.

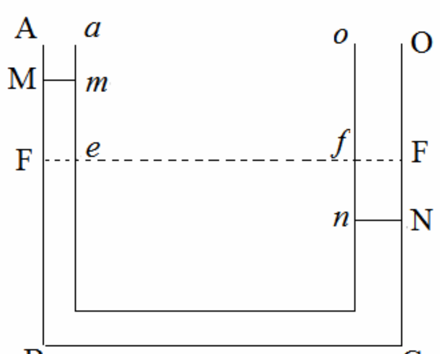


Fig. 52

Cum aqua utrinque est in aequilibrio, pertingat ad horizontalem *EF,* ponaturque $BE = CF = a$, et $BC = b$, ut totum aquae volumen sit $= a\mu + b\lambda + av$. Iam in statu motus ad tempus $= t$ vocetur $EM = vx$ eritque $FN = \mu x$, et iam quantitates μ et v sunt constantes. Statuatur
$AE = e$, erit

$$m = e - vx,\ \mathfrak{m} = a + vx,\ \mathfrak{M} = \tfrac{e-vx}{\mu},$$

porro

$$n = e + 2a + b - \mu x,\ \mathfrak{n} = a - \mu x \text{ et } \mathfrak{N} = \tfrac{e+a}{\mu} + \tfrac{b}{\lambda} + \tfrac{a-\mu x}{v},$$

unde

$$\mathfrak{N} - \mathfrak{M} = \tfrac{b}{\lambda} + \tfrac{a+vx}{\mu} + \tfrac{a-\mu x}{v}.$$

Deinde ob $M = N = k$ pressioni atmosphaerae et $\mathfrak{m} - \mathfrak{n} = (\mu t + v)x$ erit

$$f^4\upsilon\upsilon\left(\tfrac{b}{\lambda} + \tfrac{a+vx}{\mu} + \tfrac{a-\mu x}{v}\right) = -4g\int \mu v dx\left(\mu + v\right)x$$
$$= -2g\mu v\left(\mu + v\right)xx + C.$$

Ponamus facto $x = 0$ celeritatem amplitudini *ff* convenientem fieri $\upsilon = 2\sqrt{gc}$, ut hinc constans ita determinetur

$$4gcf^4\left(\tfrac{b}{\lambda} + \tfrac{a(\mu+v)}{\mu v}\right) = C,$$

atque habebitur

$$ff\upsilon\sqrt{\left(\tfrac{b}{\lambda} + \tfrac{a+vx}{\mu} + \tfrac{a-\mu x}{v}\right)} = \sqrt{2g\mu v\left(\mu + v\right)\left(\tfrac{2cf^4}{\mu v(\mu+v)}\left(\tfrac{b}{\lambda} + \tfrac{a(\mu+v)}{\mu v}\right) - xx\right)},$$

pro excursionibus maximis ergo erit

$$x = \pm\sqrt{\left(\tfrac{2cf^4}{\mu v(\mu+v)}\left(\tfrac{b}{\lambda} + \tfrac{a(\mu+v)}{\mu v}\right)\right)}.$$

Pro tempore autem hanc aequationem integrari oportet

$$t = -\mu v\int \tfrac{dx}{ff\upsilon},$$

quae formula autem nimis est perplexa, quam ut eius evolutio suscipi queat, nisi casu, quo c ac proinde etiam *x* est quantitas quam minima. In genere enim tempus tali forma definitur

$$t = \int \tfrac{dx\sqrt{(A+Bx)}}{\sqrt{(hh-xx)}},$$

cuius integratio reiecto termino *Bx* est manifesta. Admisso autem termino *Bx* totae quidem oscillationes erunt isochronae, sed tempora, quibus terminus *Mm* supra libellam *EF* vel ascendit vel descendit, non erunt aequalia temporibus, quibus infra libellam versatur.

PROBLEMA 54

73. *Si* (Fig. 50) *aqua ex tubo utcunque inaequaliter amplo et cuius directrix est linea curva quaecunque, per orificium Oo effluat, ut eius quantitas in tubo continuo minuatur, eius motum determinare.*

SOLUTIO

Maneant omnia uti in solutione problematis praecedentis, nisi quod amplitudo illa constans *ff* iam ipsi orificio *Oo* tribuatur, per quod nunc aqua elapso tempore $=t$ effluat celeritate $=\upsilon$, alter vero aquae terminus haereat in *Mm,* ubi amplitudo sit $=\mu$, altitudo supra horizontem $M\mu=\mathfrak{m}$ et pressio $=M$, quae quidem, si aëri pateat, erit aequalis pressioni atmosphaerae *k,* perinde ac in ipso orificio *Oo.* A loco autem tubi dato *A* secundum eius directricem sit distantia $AM=m$ et tota longitudo $AO=a$, tum vero celeritas, qua aquae suprema superficies *Mm* per tubum promovetur, erit $=\frac{ff\upsilon}{\mu}$.
Statuamus nunc pro loco tubi quocunque *z,* longitudinem $Az=s$, amplitudinem $zv=\omega$, altitudinem $z\pi=z$ et pressionem $=p$, ac principia motus hanc nobis suppeditant aequationem:

$$2gp=\Delta:t-2gz-\frac{f^4\upsilon\upsilon}{2\omega\omega}-\frac{ffd\upsilon}{dt}\int\frac{ds}{\omega},$$

quam primo ad extremitatem *Mm,* tum vero ad orificium *Oo* transferri oportet, quandoquidem in his duobus locis pressio est data. Pro illa autem *Mm* fit $p=M$, $z=\mathfrak{m}$, $\omega=\mu$, integralis vero $\int\frac{ds}{\omega}$ valor hic fiat $=\mathfrak{M}$, unde fit

$$2gM=\Delta:t-2g\mathfrak{m}-\frac{f^4\upsilon\upsilon}{2\omega\omega}-\frac{ffd\upsilon}{dt}\mathfrak{M}.$$

Pro orificio vero *Oo,* siquidem aqua in aërem effluat, habetur pressio $p=k$, amplitudo $\omega=ff$, altitudo vero $O\omega$ fit nulla, quoniam planum horizontale per ipsum orificium *Oo* ducere licet, valor autem formulae integralis $\int\frac{ds}{\omega}$ ad hunc locum translatus fiat $=\mathfrak{A}$, quippe qui erit constans, ex quo nostra aequatio fiet

$$2gk=\Delta:t-\tfrac{1}{2}\upsilon\upsilon-\frac{ffd\upsilon}{dt}\cdot\mathfrak{A},$$

quae ab illa subtracta relinquit

$$2g\left(M-k\right)=-2g\mathfrak{m}-\frac{f^4\upsilon\upsilon}{2\mu\mu}+\tfrac{1}{2}\upsilon\upsilon+\frac{ffd\upsilon}{dt}\left(\mathfrak{A}-\mathfrak{M}\right)\mathfrak{A},$$

quam aequationem, in qua solum tempus t variabile inest, integrari oportet, hanc formulam $dm = \frac{ff\upsilon dt}{\mu}$ in subsidium vocando: unde ob $dt = \frac{\mu dm}{ff\upsilon}$ habetur

$$2g\left(M-k\right)\mu dm = -2g\mathfrak{m}\mu dm - \tfrac{1}{2}\upsilon\upsilon\mu dm\left(1-\tfrac{f^4}{\mu\mu}\right) + \tfrac{ffdv}{dt}f^4\upsilon d\upsilon\left(\mathfrak{A}-\mathfrak{M}\right)\mathfrak{A},$$

ubi est $\mathfrak{M} = \int\frac{dm}{\mu}$, ex quo valore nascitur quantitas $\mathfrak{A}$, si fiat $m = a$. Sunt autem μ et $\mathfrak{m}$ functiones datae ipsius *m,* unde haec aequatio duas tantum variabiles *m* et *v* involvit, ex qua valorem ipsius $\upsilon\upsilon$ facile elicere licet, quo invento ope formulae $dt = \frac{\mu dm}{ff\upsilon}$ ad quodvis tempus *t* cum longitudo $AM = m$ tum celeritas υ, qua aqua per orificium *Oo* effluit, assignari poterit. Deinde vero etiam pro pressione p in loco quocunque *z* habebitur:

$$2g(p-k) = -2gz + \tfrac{1}{2}\upsilon\upsilon\left(1-\tfrac{f^4}{\omega\omega}\right) + \tfrac{ffd\upsilon}{dt}\left(\mathfrak{A}-\int\tfrac{ds}{\omega}\right),$$

quare, si terminus $\frac{ffd\upsilon}{dt}$ elidatur, colligitur:

$$2g(M-k)\left(\mathfrak{A}-\int\tfrac{ds}{\omega}\right)+2g\left(k-p\right)\left(\mathfrak{A}-\mathfrak{M}\right) = 2gz\left(\mathfrak{A}-\mathfrak{M}\right)-2g\mathfrak{m}\left(\mathfrak{A}-\int\tfrac{ds}{\omega}\right)$$
$$+\tfrac{1}{2}\upsilon\upsilon\left(\mathfrak{M}-\int\tfrac{ds}{\omega}\right)-\tfrac{f^4\upsilon\upsilon}{2\mu\mu}\left(\mathfrak{A}-\int\tfrac{ds}{\omega}\right)+\tfrac{f^4\upsilon\upsilon}{2\omega\omega}\left(\mathfrak{A}-\mathfrak{M}\right).$$

COROLLARIUM 1

74. Sumamus tubi terminum fixum *A* in ipso orificio *O,* ut sit $a = 0$, et vocemus $OM = m$ atque $Oz = s$, ita ut iam in formulis inventis hae duae quantitates *m* et *s* negative capi debeant; tum vero erit $\mathfrak{A} = 0$, et loco $\mathfrak{M}$ et $\int\frac{ds}{\omega}$ scribi oportebit

$$-\int\tfrac{dm}{\mu} \text{ et } -\int\tfrac{ds}{\omega},$$

unde pro pressione habebimus:

$$2g\left(M-k\right)\int\tfrac{ds}{\omega}+2g\left(k-p\right)\int\tfrac{dm}{\mu}$$
$$= 2gz\int\tfrac{dm}{\mu}-\tfrac{1}{2}\upsilon\upsilon\left(1-\tfrac{f^4}{\omega\omega}\right)\int\tfrac{dm}{\mu}+\tfrac{1}{2}\upsilon\upsilon\left(1-\tfrac{f^4}{\mu\mu}\right)\int\tfrac{ds}{\omega}-g\mathfrak{m}\int\tfrac{ds}{\omega}.$$

COROLLARIUM *2*

75. Manentibus autem $OM = m$ et $Oz = s$, primum pro tempore habebitur $dt = -\frac{\mu dm}{ff\upsilon}$, quoniam labente tempore *t* intervallum $OM = m$ minuitur, celeritas autem effluxus υ ex hac aequatione debet definiri

$$2g(M-k+\mathfrak{m})\mu dm = \tfrac{1}{2}\upsilon\upsilon\mu dm\left(1-\tfrac{f^4}{\mu\mu}\right) - f^4\upsilon d\upsilon\int\tfrac{dm}{\mu},$$

quae commodius ita repraesentatur:

$$2f^4\upsilon d\upsilon + \tfrac{f^4\upsilon\upsilon dm}{\mu} - \upsilon\upsilon\mu dm + 4g(M-k+\mathfrak{m})\mu dm = 0.$$

COROLLARIUM 3

76. Ponatur hic $f^4\upsilon\upsilon\int\frac{dm}{\mu} = u$, ut habeatur:

$$du - \tfrac{\mu u dm}{f^4\int\frac{dm}{\mu}} + 4g(M-k+\mathfrak{m})\mu dm = 0,$$

quae ut integrabilis reddatur, multiplicari debet per e^o existente $o = -\frac{1}{f^4}\int\frac{\mu dm}{\int\frac{dm}{\mu}}$ eritque tum

$$e^o u + 4g\int e^o(M-k+\mathfrak{m})\mu dm = \text{Const.}$$

SCHOLION

77. In hac solutione omnia continentur, quae vulgo de effluxu aquae ex vasis cuiuscunque figurae tradi solent, quae autem eatenus tantum admitti possunt, quatenus ea vasa vel sunt angustissima vel motus per ea ita fiat, ut singula strata ad directricem normaliter sumta communi motu ferantur; nisi enim haec conditio locum habeat, celeritas effluxus hic definita a veritate recedet, etsi saepenumero discrimen experimentis instituendis vix percipitur. Quodsi in formulis inventis statuatur $M = k$, habebitur casus, quo suprema aquae superficies est aperta et effluxus fit in aërem, sin autem aqua in spatium aëre vacuum efflueret, sumi deberet $k = 0$, at si tubi orificium *Oo* aquae stagnanti esset immersum, littera *k* pressionem huius aquae in orificium exprimere deberet. Totum autem negotium semper reducitur ad aequationem differentialem inventam, ex cuius integratione pro quovis loco, ubi superficies aquae suprema haeret, celeritas effluxus cognoscetur, tum vero formulam $dt = -\frac{\mu dx}{ffv}$ (§ 75) in subsidium vocando tempus innotescet, quo aqua in tubo ad datum locum *Mm* subsidit; ac denique, cum elapso tempore *t* aqua per orificium *Oo*, cuius amplitudo est $= ff$, celeritate $= v$ effluat, omnis aqua, quae tempore *t* effluxerit, erit

$$= ff\int vdt = -\int\mu dm.$$

Quod quo clarius appareat, aliquot casus evolvamus.

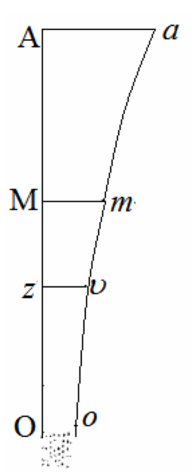


Fig. 53

EXEMPLUM 1

78. *Si (Fig. 53) tubi directrix AO sit recta verticalis, at tubus initio ad Aa fuerit aqua plenus, quae tum per orificium $Oo = ff$ effluere inceperit, ad datum quodvis tempus celeritatem effluxus et pressionem in quavis sectione $z\upsilon$ determinare.*

Posito ergo intervallo $OM = m$ et amplitudine $Mm = \mu$, quem in locum aquam ex Aa elapso tempore $= t$ subsedisse assumimus, erit etiam altitudo $OM = \mathfrak{m} = m$ et pressio $M = k$. Posita nunc celeritate effluxus per orificium $= \upsilon$, eam ex aequatione definiri oportet:

$$2f^4\upsilon d\upsilon\int\tfrac{dm}{\mu}+\frac{f^4\upsilon\upsilon dm}{\mu}-\upsilon\upsilon\mu dm+4g\mu m dm=0,$$

quae posito $f^4\upsilon\upsilon\int\frac{dm}{\mu}=u$ abit in hanc formam

$$du-\frac{u\mu dm}{f^4\int\frac{dm}{\mu}}+4g\mu m dm=0.$$

Deinde pro pressione in sectione quacunque $z\upsilon$ sit $Oz = s$ et amplitudo $z\upsilon = \omega$, erit quoque altitudo $z = s$ ideoque

$$2g(k-p)\int\frac{dm}{\mu}=2gs\int\frac{dm}{\mu}-2gm\int\frac{ds}{\omega}-\frac{1}{2}\upsilon\upsilon\left(1-\frac{f^4}{\omega\omega}\right)\int\frac{dm}{\mu}+\frac{1}{2}\upsilon\upsilon\left(1-\frac{f^4}{\mu\mu}\right)\int\frac{ds}{\omega}.$$

Aequatio autem illa differentialis ita integrari debet, ut facta altitudine $OM = m = OA = a$ celeritas υ evanescat, tum vero inventa celeritate υ calculus ad tempus accommodabitur ope huius formulae $t=-\int\frac{\mu dm}{ff\upsilon}$, quae posito $m = a$ evanescere debet. Quodsi deinceps ponatur $m = 0$, tempus totius effluxus innotescet.

Praeterea vero durante effluxu, quoniam ab initio celeritas υ continuo crescit, celeritas maxima, ubi $d\upsilon = 0$, ita definitur, ut sit

$$\upsilon\upsilon=\frac{4gm\mu\mu}{\mu\mu-f^4}\quad\text{ideoque}\quad\upsilon=\frac{2\mu\sqrt{gm}}{\sqrt{\mu\mu-f^4}}.$$

Cum ergo sit $\upsilon > \sqrt{2gm}$, celeritas maxima maior erit ea, quam grave delabens ex altitudine *m* acquirit.

COROLLARIUM 1

79. Si (Fig. 54) vas ubique sit aeque amplum seu $\mu = \omega = cc$, cuius fundum *OC* foramine $Oo = ff$ est pertusum, habebimus:
Si (Fig. 54) vas ubique sit aeque amplum seu $\mu = \omega = cc$, cuius fundum *OC* foramine $Oo = ff$ est pertusum, habebimus:

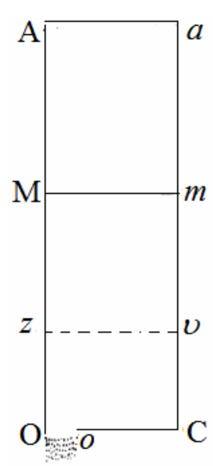


FIG. 54

$$du - \tfrac{c^4}{f^4}\cdot\tfrac{udm}{m} + 4gccmdm = 0.$$

Sit $\frac{c^4}{f^4} = \lambda$, erit

$$m^{-\lambda}u + \tfrac{4ggc}{2-\lambda}m^{2-\lambda} = C$$

hincque

$$u = Cm^{\lambda} + \tfrac{4ggc}{\lambda-2}mm = \tfrac{f^4mvv}{cc}$$

et constante rite definita

$$f^4mvv = \tfrac{4gc^4m}{\lambda-2}mm\left(1 - a^{2-\lambda}m^{\lambda-2}\right)$$

seu

$$v = \sqrt{\tfrac{4\lambda gm}{\lambda-2}\left(1 - \tfrac{m^{\lambda-2}}{a^{2-\lambda}}\right)},$$

unde colligitur pressio

$$p = k + \tfrac{\lambda-1}{\lambda-2}\left(m-s\right)\left(1 - \tfrac{m^{\lambda-2}}{a^{\lambda-2}}\right)$$

pro sectione $z\upsilon$ ad altitudinem $Oz = s$, denique pro tempore erit

$$t = -\int \frac{dm\sqrt{(\lambda-2)}}{2\sqrt{gm\left(1-a^{2-\lambda}m^{\lambda-2}\right)}}$$

celeritas autem maxima fit

$$= \frac{2cc\sqrt{gm}}{\sqrt{c^4-f^4}} = \frac{2\sqrt{\lambda gm}}{\sqrt{\lambda-1}},$$

quae convenit altitudini *m* hinc definiendae

$$\tfrac{\lambda-2}{\lambda-1} = 1 - \tfrac{m^{\lambda-2}}{a^{\lambda-2}},$$

ita ut sit

$$m = \frac{a}{\sqrt[\lambda-2]{\lambda-1}}.$$

COROLLARIUM 2

80. Casus, quo $\lambda = 2$ seu $c^4 = 2f^4$, singularem postulat evolutionem; quia aequatio

$$du - \tfrac{2udm}{m} + 4gccmdm = 0$$

integrata dat

$$u = 4gccmml\tfrac{a}{m} = \tfrac{ccm\upsilon\upsilon}{2},$$

hinc

$$\upsilon = \sqrt{8gml\frac{a}{m}} \quad \text{et} \quad t = -\int \frac{dm}{\sqrt{4gml\frac{a}{m}}},$$

pro pressione vero $p = k + (m-s)l\frac{a}{m}$.

COROLLARIUM 3

81. Sit tubus conus ad orificium truncatus et $\mu = (f+\alpha m)^2$ atque $\omega = (f+\alpha s)^2$, hinc fit

$$\int \frac{dm}{\mu} = \frac{1}{\alpha f} - \frac{1}{\alpha(f+\alpha m)} = \frac{m}{f(f+\alpha m)},$$

similique modo

$$\int \frac{ds}{\omega} = \frac{s}{f(f+\alpha s)}.$$

Pro motu ergo habetur:

$$du - \frac{udm}{m}\left(1+\frac{\alpha m}{f}\right)^3 + 4gmdm(f+\alpha m)^2 = 0,$$

unde invento u erit

$$f^4\upsilon\upsilon = \frac{f(f+m)u}{m}.$$

COROLLARIUM 4

82. Expandatur tubus superne in infinitum secundum hanc aequationem $\omega\omega = \frac{af^4}{a-s}$, ita ut initio suprema superficies *Aa* fuerit infinita; eaque etiamnunc nihil subsederit, ut sit elapso tempore *t* altitudo $m = a$ et $\mu\mu = \frac{af^4}{a-m} = \infty$. Quam ob causam aequatio differentialis statim praebet $\upsilon\upsilon = 4gm = 4ga$, ita ut aqua constanter eadem celeritate effluat. Quia autem hic motus effluxus est uniformis ob $\frac{d\upsilon}{dt} = 0$, pressio ad $z\upsilon$ ex aequatione primum inventa ita definitur:

$$2g(p-k) = -2gs + 2ga\left(1 - \frac{a-s}{a}\right) = 0,$$

ubique scilicet pressio aequalis erit pressioni atmosphaerae seu latera tubi extrinsecus aequaliter pressa nullam vim sustinent, iisque adeo remotis fluxus perinde fieret.

EXEMPLUM 2

83. *Sit (Fig. 55) superior tubi pars AaBb verticalis et aequaliter ampla, inferior vero pars BbOo utcunque curva et inaequaliter ampla, definire aquae ex eo effluentis motum, quamdiu suprema aquae superficies Mm in parte superiori versatur.*

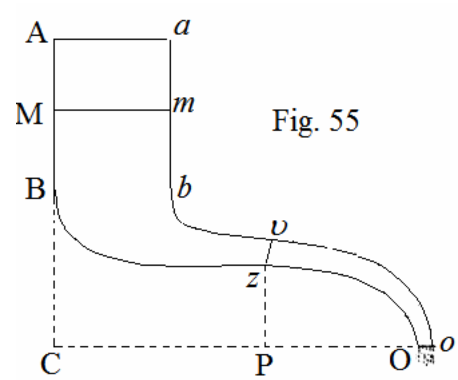


Fig. 55

Sit amplitudo partis superioris $Mm = \mu = cc$, longitudo tubi inferioris $BzO = a$, altitudo $BC = b$ et $BM = x$; erit ergo $m = a + x$ et $m = b + x$; tum sumta longitudine $Oz = s$, cui respondeat amplitudo $z\upsilon = \omega$ et altitudo $Pz = z$, sit valor integralis $\int \frac{ds}{\omega}$ per totam partem inferiorem extensi $= B$, quandoquidem hic valor erit constans; tum igitur idem integrale ad superficiem supremam *Mm* extensum erit

$$= B + \tfrac{x}{cc} = \int \tfrac{dm}{\mu} ::$$

unde hanc habebimus aequationem ob $M = k$:

$$2gcc(b+x)dx = \tfrac{1}{2}cc\upsilon\upsilon dx\left(1 - \tfrac{f^4}{c^4}\right) - f^4\upsilon d\upsilon\left(B + \tfrac{x}{cc}\right),$$

quae posito $f^4\upsilon\upsilon\left(B + \tfrac{x}{cc}\right) = u$ abit in hanc:

$$du - \frac{c^4 udx}{f^4(Bcc+x)} - 4gcc(b+x)dx = 0.$$

Ponatur $\frac{c^4}{f^4} = \lambda$ et multiplicando per $(Bcc+x)^{-\lambda}$ erit integrale:

$$\upsilon\upsilon = C(Bcc+x)^{\lambda-1} - \frac{4\lambda g}{(1-\lambda)(2-\lambda)}\big((2-\lambda) - Bcc + (1-\lambda)x\big)b.$$

Si iam descensum ex *Aa* incepisse assumamus existente $AB = e$, fiet

$$C = \frac{4\lambda g}{(1-\lambda)(2-\lambda)} \cdot \frac{(2-\lambda)b - Bcc + (1-\lambda)e}{(Bcc+x)^{\lambda-1}}$$

ideoque

$$\upsilon\upsilon = \frac{4\lambda g\big((2-\lambda)b - Bcc + (1-\lambda)e\big)}{(1-\lambda)(2-\lambda)} \cdot \left(\frac{Bcc+x}{Bcc+e}\right)^{\lambda-1}$$
$$- \frac{4\lambda g\big((2-\lambda)b - Bcc + (1-\lambda)x\big)}{(1-\lambda)(2-\lambda)}$$

vel

$$\upsilon\upsilon = \frac{4\lambda g\left((2-\lambda)b-Bcc\right)}{(1-\lambda)(2-\lambda)}\cdot\left(\left(\frac{Bcc+x}{Bcc+e}\right)^{\lambda-1}-1\right)$$
$$+\frac{4\lambda g}{2-\lambda}\left(e\left(\frac{Bcc+x}{Bcc+e}\right)^{\lambda-1}-x\right)$$

seu

$$\upsilon\upsilon = \frac{4\lambda g\left(Bcc+(2-\lambda)b\right)}{(\lambda-1)(\lambda-2)}\cdot\left(1-\left(\frac{Bcc+x}{Bcc+e}\right)^{\lambda-1}\right)$$
$$+\frac{4\lambda g}{2-\lambda}\left(x-e\left(\frac{Bcc+x}{Bcc+e}\right)^{\lambda-1}\right).$$

Ac si tempore $=t$ aqua ab *Aa* ad *Mm* subsederit, erit $dt=\frac{-dx\sqrt{\lambda}}{\upsilon}$: cum autem aqua maxima celeritate effluit, fiet

$$\upsilon\upsilon=\frac{4\lambda g(b+x)}{\lambda-1},$$

quod ergo evenit, ubi erit

$$x=-Bcc+\frac{(Bcc+e)^{\frac{\lambda-1}{\lambda-2}}}{\left(Bcc+(\lambda-2)b+(\lambda-1)e\right)^{\frac{1}{\lambda-2}}}.$$

Denique pro pressione *p*, qua tubi pars inferior in sectione $z\upsilon$ urgetur, aequatio supra inventa hanc induet formam:

$$2g(k-p)\left(B+\frac{x}{cc}\right)=\left(2\,gz-\tfrac{1}{2}\upsilon\upsilon\left(1-\frac{f^4}{\omega\omega}\right)\right)\left(B+\frac{x}{cc}\right)$$
$$-\left(2g(b+x)-\tfrac{1}{2}\upsilon\upsilon\left(1-\tfrac{1}{\lambda}\right)\right)\int\frac{ds}{\omega},$$

unde fit

$$p=k+\frac{cc\left(b+x-\frac{(\lambda-1)\upsilon\upsilon}{4\lambda g}\right)\int\frac{ds}{\omega}}{Bcc+x}-z+\frac{\upsilon\upsilon}{4g}\left(1-\frac{f^4}{\omega\omega}\right).$$

Casus hic imprimis notari meretur, quo $\lambda=\frac{c^4}{f^4}$ est numerus valde magnus, quo casu ex aequatione differentiali

$$4\lambda g(b+x)dx=(\lambda-1)\upsilon\upsilon dx-2(Bcc+x)\upsilon d\upsilon$$

statim colligitur $\upsilon\upsilon=4g(b+x)$, scilicet quia orificium *Oo* est minimum, quasi a primo statim initio celeritas fit maxima, et pressio in sectione $z\upsilon$ prodit

$$p=k-z+(b+x)\left(1-\frac{f^4}{\omega\omega}+\frac{cc}{\lambda(Bcc+x)}\right)\int\frac{ds}{\omega},$$

et quia ultimum terminum per λ divisum omittere licet, erit

$$p = k - z + (b + x)\left(1 - \frac{f^4}{\omega\omega}\right).$$

COROLLARIUM 1

84. Casus iste, quo $\lambda = \frac{c^4}{f^4}$ est numerus valde magnus, imprimis notari meretur, quia experimenta facillime ad eum accommodantur; quibus etiam evincitur celeritatem effluxus vix discrepare a valore invento.

COROLLARIUM 2

85. Circa pressiones autem in tubi parte inferiori *BO*, hoc casu potissimum observari convenit eas non solum ultra *k* diminui, sed etiam negativas fieri posse. Si enim sectio $z\upsilon = \omega$ aequalis sit orificio *ff*, erit

$$p = k - z = k - zP,$$

at si haec sectio minor est foramine *ff*, pressio multo magis diminuitur.

COROLLARIUM 3

86. Quando autem pressio *p* revera fit negativa, fluidi continuitas tollitur, et quia latera tubi deserendo se in arctius spatium contrahit, neque amplius legem stabilitam sequitur. Quamdiu autem pressio est positiva quidem, sed minor quam *k*, tum, quia pressio externa superat internam, si tubus ibi foraminulo perforetur, aër aliudve fluidum extra positum intrudetur, ita ut tubus ibi vi attractrice praeditus videatur.

SCHOLION 1

87. Huc fere redeunt, quae de effluxu aquae ex tubis vel vasis cuiuscunque formae tradi solent, quae quia iam copiose ac diligenter sunt pertractata, hic fusius evolvere nolo: idque adeo ob hanc potissimum rationem, quod in plerisque casibus, ad quos haec Theoria applicari solet, calculus non mediocriter a veritate aberrare deprehendatur. Statim enim, ac vas, uti figura 54, notabilem prae foramine *Oo* habet amplitudinem, manifestum

est tota strata $z\upsilon$ non aequaliter subsidere, sed partes foramini imminentes magis ad descensum impelli. Tum vero, ubi tubus subito in foramen coarctatur, ibi certe neutiquam aquae motus ita est comparatus, uti in hac sectione assumimus. Tantopere potius verus motus ab hac hypothesi discrepabit, ut mirandum sit experimenta non multo magis a calculo discrepare. Interim tamen dissensus insignis se prodit, quando fundus vasis *OC* tenuissimo foramine *Oo* est pertusus, quo casu in vena effluente ingens contractio animadvertitur inde oriunda, quod aqua a lateribus erumpens oblique effluit; quo fit, ut per foramen minor aquae copia quam pro eius amplitudine eiiciatur. Cui incommodo ii, qui experimenta calculo consentanea reddere volunt, ita medentur, ut foramini tubulum cylindricum inserere soleant, ut hoc modo obliquitas motus evitetur.

SCHOLION 2

88. Casus, quo $\lambda = \frac{c^4}{f^4}$; est numerus vehementer magnus, evolutionem singularem postulat, qua dilucide explicetur, quomodo aqua, dum eius motus a quiete incipit, subito maximam celeritatem adipiscatur. Hunc in finem formula $\left(\frac{Bcc+x}{Bcc+e}\right)^{\lambda-1}$ rite evolvi oportet, ut motum ab initio genitum exhibeat. Statim ergo, ac motus incipit, altitudo x fit minor quam e; ponamus igitur

$$\frac{Bcc+x}{Bcc+e} = 1 - \frac{y}{\lambda-1},$$

ut sit $x = e - \frac{y(Bcc+e)}{\lambda-1}$, ac denotante ε numerum, cuius logarithmus hyperbolicus est unitas, erit proxime

$$\left(1 - \frac{y}{\lambda-1}\right)^{\lambda-1} = \varepsilon^{-y}.$$

Habebimus ergo:

$$\upsilon\upsilon = \frac{4\lambda g(Bcc+\lambda b)}{\lambda\lambda}\left(1-\varepsilon^{-y}\right) + \frac{4\lambda g}{\lambda}\left(1 - e\varepsilon^{-y}\right),$$

seu

$$\upsilon\upsilon = 4gb\left(1-\varepsilon^{-y}\right) + 4g\left(x - e\varepsilon^{-y}\right) = 4g\left(b+x\right) - 4g\left(b+e\right)\varepsilon^{-y},$$

unde patet in ipso initio, ubi $x = e$ et $y = 0$, ob $\varepsilon^{-y} = 1$ revera fieri $\upsilon = 0$, statim autem, atque aqua subsederit per intervallum minimum $\frac{y(Bcc+e)}{\lambda-1}$, quoniam y valorem notabilem sortitur, quantitatem ε^{-y} evanescere ideoque fieri $\upsilon\upsilon = 4g\left(b+x\right)$. Deinde vero ex aequatione pro tempore $dt = \frac{-dx\sqrt{\lambda}}{\upsilon}$, quoniam in valore ipsius $\upsilon\upsilon$ loco x scribere licet e, ut sit $\upsilon\upsilon = 4g\left(b+e\right)\left(1-\varepsilon^{-y}\right)$, erit

$$dt = \frac{-dx\sqrt{\lambda}}{2\sqrt{g\left(b+e\right)\left(1-\varepsilon^{-y}\right)}} = \frac{dy(Bcc+e)}{2\sqrt{\lambda g\left(b+e\right)\left(1-\varepsilon^{-y}\right)}},$$

unde colligitur integrando

$$t=\frac{Bcc+e}{2\sqrt{\lambda g(b+e)}}l\frac{1+\sqrt{(1-\varepsilon^{-y})}}{1-\sqrt{(1-\varepsilon^{-y})}}.$$

Simul igitur atque ε^{-y} fit fractio quam minima, ob

$$l\frac{1+\sqrt{(1-\varepsilon^{-y})}}{1-\sqrt{(1-\varepsilon^{-y})}}=l\left(4\varepsilon^{y}-1\right)=l4\varepsilon^{y}=y+l4$$

erit

$$t=\frac{Bcc+e}{2\sqrt{\lambda g(b+e)}}\left(y+l4\right)=\frac{Bcc+e}{2\sqrt{\lambda g(b+e)}}\left(\frac{\lambda(e-x)}{Bcc+e}+l4\right).$$

Cum porro celeritas evadat maxima, ubi

$$x=-Bcc+\left(Bcc+e\right)\left(\frac{Bcc+e}{Bcc+\lambda(b+e)}\right)^{\frac{1}{\lambda}}=e-\frac{(Bcc+e)l\lambda}{\lambda},$$

eveniet hoc, ubi $y=l\lambda$, ideoque postquam ab initio effluxerit tempus

$$t=\frac{(Bcc+e)l4\lambda}{2\sqrt{\lambda g(b+e)}},$$

quod cum $\frac{l\lambda}{\sqrt{\lambda}}$ evanescat, si $\lambda=\infty$, erit quam mininum, ita ut aqua primo quasi instanti maximam celeritatem adipiscatur. Interim hinc intelligitur, quo longior simulque angustior fuerit tubi pars inferior *BO*, eo tardius ad celeritatem maximam perventum iri.

SCHOLION 3

89. Evoluto casu, quo $\lambda=\frac{e^4}{f^4}$ est quasi numerus infinitus, etiam is, quo λ est numerus mediocriter magnus, accuratiori evolutione dignus videtur. Cum igitur invenerimus:

$$\frac{(1-\lambda)(2-\lambda)\upsilon\upsilon}{4\lambda g}=Bcc+\left(\lambda-2\right)b+\left(\lambda-1\right)x$$
$$-B\left(cc+\left(\lambda-2\right)b+\left(\lambda-1\right)e\right)\left(\frac{Bcc+x}{Bcc+e}\right)^{\lambda-1},$$

ponamus ut ante $\frac{Bcc+x}{Bcc+e}=1-\frac{y}{\lambda-1}$, ut sit

$$y=\frac{(\lambda-1)(e-x)}{Bcc+e};$$

et quia totum negotium ad commodam evolutionem formulae $\left(1-\frac{y}{\lambda-1}\right)^{\lambda-1}$ reducitur, posita ea $=Y$ fit

$$lY=\left(\lambda-1\right)l\left(1-\frac{y}{\lambda-1}\right),$$

ac quia semper est $y<\lambda-1$, ob $\frac{y}{\lambda-1}=\frac{e-x}{Bcc+e}$ erit

$$lY=-y-\frac{yy}{2(\lambda-1)}-\frac{y^3}{3(\lambda-1)^2}-\frac{y^4}{4(\lambda-1)^3}-\text{etc.},$$

quae series utique valde convergit. Hinc ergo invento valore Y erit:

$$\frac{(1-\lambda)(2-\lambda)\upsilon\upsilon}{4\lambda g}=\left(Bcc+(\lambda-2)b+(\lambda-1)e\right)(1-Y)-B(cc+e)y,$$

ubi est

$$y=(\lambda-1)\left(1-Y^{\frac{1}{\lambda-1}}\right) \text{ et } x=(Bcc+e)Y^{\frac{1}{\lambda-1}}-Bcc.$$

Cum iam celeritas maxima sit:

$$\frac{(\lambda-1)(\lambda-2)\upsilon\upsilon}{4\lambda g}=(\lambda-2)(b+x),$$

hic locus definitur hac aequatione

$$Y^{\frac{\lambda-2}{\lambda-1}}=\frac{(Bcc+e)}{(Bcc+(\lambda-2)b+(\lambda-1)e)},$$

et posito

$$\frac{Bcc+(\lambda-2)b+(\lambda-1)e}{Bcc+e}=E$$

fit

$$Y^{\frac{1}{\lambda-1}}=E^{-\frac{1}{\lambda-2}}=\varepsilon^{-\frac{1}{\lambda-2}lE}=1-\frac{lE}{\lambda-2}+\frac{(lE)^2}{2(\lambda-2)^2}-\frac{(lE)^3}{6(\lambda-2)^3}+\text{etc.},$$

unde celeritas erit maxima, ubi

$$x=e-(Bcc+e)\left(\frac{lE}{\lambda-2}-\frac{(lE)^2}{2(\lambda-2)^2}+\frac{(lE)^3}{6(\lambda-2)^3}-\text{etc.}\right)$$

Cum nunc porro sit

$$\frac{(\lambda-1)(\lambda-2)\upsilon\upsilon}{4\lambda g}=(Bcc+e)\left(E(1-Y)-y\right),$$

erit pro tempore

$$dt=\frac{-dx\sqrt{\lambda}}{\upsilon}=\frac{-Y^{\frac{2-\lambda}{\lambda-1}}dY\sqrt{(\lambda-2)(Bcc+e)}}{2\sqrt{(\lambda-1)g\left(E(1-Y)-(\lambda-1)\left(1-Y^{\frac{1}{\lambda-1}}\right)\right)}}$$

et facto $Y=u^{k-1}$ fit

$$dt=\frac{-dx\sqrt{\lambda}}{\upsilon}=\frac{-du\sqrt{(\lambda-1)(\lambda-2)(Bcc+e)}}{2\sqrt{g\left(E-\lambda+1+(\lambda-1)u-Eu^{\lambda-1}\right)}}.$$

Verum calculus commodius instituetur solam quantitatem y retinendo et ponendo:

$$Y = \varepsilon^{-y}\left(1 - \frac{yy}{2(\lambda-1)} - \frac{y^3}{3(\lambda-1)^2} + \frac{y^4}{8(\lambda-1)^2}\right),$$

unde solutio paragraphi praecedentis propius ad veritatem perducetur, dum etiam termini per λ divisi introducuntur. Sed quia haec mere sunt analytica, ea hic uberius non petracto.

PROBLEMA 55

90. *Si tubus* (Fig. 56), *dum aqua per orificium Oo effluit, in altero termino Aa continuo novum aquae supplementum accipiat, ut perpetua ad Aa usque plenus conservetur, ibique aqua a vi quacunque iugiter protrudatur, eius motum definire.*

SOLUTIO

Posita amplitudine orificii $Oo = ff$ sit υ celeritas, qua iam elapso tempore $= t$ aqua ibi effluit, in alio vero loco quocunque z, cuius distantia ab initio A sit $Az = s$, tubique amplitudo $z\upsilon = \omega$ et altitudo super plano horizontali fixo $z\pi = z$, siquidem curvae $\alpha\pi\omega$ applicatae singulorum tubi punctorum altitudines super eodem plano exhibere assumuntur. His positis si in sectione $z\upsilon$ statuatur pressio $= p$, principia motus hanc suppeditant aequationem:

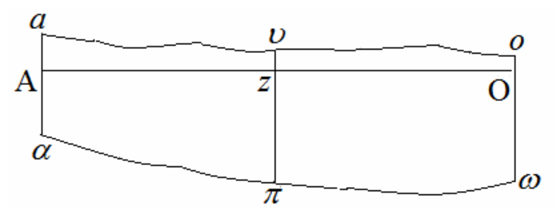


Fig. 56

$$2gp = \Delta : t - 2gz - \frac{f^4\upsilon\upsilon}{2\omega\omega} - \frac{ffd\upsilon}{dt}\int\frac{ds}{\omega}.$$

Sit nunc pressio in $Aa = L$, amplitudo $Aa = cc$ et altitudo $A\alpha = \mathfrak{a}$, et quia hic $s = 0$ simulque integrale $\int\frac{ds}{\omega}$ evanescit, ob $p = L,\ z = \mathfrak{a}$ et $\omega = cc$ erit:

$$2gL = \Delta : t - 2g\mathfrak{a} - \frac{f^4\upsilon\upsilon}{2\omega\omega}.$$

Deinde pro orificio *Oo*, sit ibi pressio $= k$, pondus atmosphaerae referens, et valor integralis $\int\frac{ds}{\omega}$ per totum tubum *AO* extensi fiat $= \mathfrak{O}$, altitudo vero $O\omega = \mathfrak{o}$. Quocirca ob $p = k,\ z = \mathfrak{o}$ et $\omega = ff$ habebitur

$$2gk = \Delta : t - 2g\mathfrak{o} - \tfrac{1}{2}\upsilon\upsilon - \frac{f^4d\upsilon}{dt}\cdot\mathfrak{O}.$$

Nunc haec aequatio ab illa subtracta relinquit

$$2g(L-k) = 2g(\mathfrak{o}-\mathfrak{a}) + \tfrac{1}{2}\upsilon\upsilon\left(1 - \frac{f^4}{c^4}\right) - \frac{f^4d\upsilon}{dt}\cdot\mathfrak{O}$$

seu

$$2g\left(L-k+\mathfrak{o}-\mathfrak{a}\right)dt-\upsilon\upsilon dt\left(1-\tfrac{f^4}{c^4}\right)=2\mathfrak{O}ff d\upsilon,$$

unde, cum $\mathfrak{a}$, $\mathfrak{o}$ et $\mathfrak{O}$ sint quantitates constantes, pressio vero *L* functionem temporis denotare possit, siquidem ea cum tempore varietur, celeritas υ ad quodvis tempus definiri debet. Posita autem pressione *L* constante huiusmodi aequatio erit resolvenda:

$$dt=\tfrac{Ad\upsilon}{B\,\pm\,C\upsilon\upsilon},$$

existante $A=2\mathfrak{O}ff$, $B=4g(L-k+\mathfrak{a}-\mathfrak{o})$ et $\pm C=\tfrac{f^4}{c^4}-1$; tres ergo casus sunt evolvendi.

I. Si $cc>ff$ seu amplitudo *Aa* orificio $Oo=ff$ aequalis, erit

$$C=0 \text{ et } t=\tfrac{Av}{B} \text{ seu } v=\tfrac{B}{A}t+\text{Const.}$$

unde, si $B>0$, celeritas continuo crescere posset.

II. Si $cc>ff$ seu amplitudo *Aa* orificium *Oo* superet, posito $C=1-\tfrac{f^4}{c^4}$ aequatio $dt=\tfrac{Ad\upsilon}{B-C\upsilon\upsilon}$ integrata dat

$$t=\tfrac{A}{2\sqrt{BC}}\,l\,\frac{\sqrt{B}+\upsilon\sqrt{C}}{\sqrt{B}-\upsilon\sqrt{C}}+\text{Const.},$$

quae constans, si motus a quiete inceperit, evanescit; hocque casu celeritas quidem crescit, sed elapso etiam tempore infinito non ultra $\upsilon=\tfrac{\sqrt{B}}{\sqrt{C}}$ augetur.

III. Si $cc<ff$ seu amplitudo *Aa* minor sit orificio *Oo*, posito $C=1-\tfrac{f^4}{c^4}$ aequatio $dt=\tfrac{Ad\upsilon}{B+C\upsilon\upsilon}$ integrata dat:

$$t=\tfrac{A}{\sqrt{BC}}\text{Ang. tang}\tfrac{\upsilon\sqrt{C}}{\sqrt{B}} \text{ seu } \upsilon=\tfrac{\sqrt{B}}{\sqrt{C}}\text{Ang. tang}\tfrac{t\sqrt{BC}}{A},$$

ubi hoc memoratu dignum evenit, ut elapso tempore $t=\tfrac{A}{\sqrt{BC}}\cdot\tfrac{\pi}{2}$ celeritas iam infinita evadat.

Inventa celeritate effluxus υ ad quodvis tempus *t*, in quovis loco medio $z\upsilon$ pressio *p* ita exprimitur:

$$2g\left(p-k\right)=2g\left(\mathfrak{o}-\mathfrak{a}\right)+\tfrac{1}{2}\upsilon\upsilon\left(1-\tfrac{f^4}{\omega\omega}\right)+\tfrac{ffd\upsilon}{dt}\left(\mathfrak{O}-\int\tfrac{ds}{\omega}\right),$$

quae elisa formula differentiali $\tfrac{ffd\upsilon}{dt}$ praebet:

$$4g\left(p-L+z-\mathfrak{a}\right)\mathfrak{O}=\upsilon\upsilon\left(\frac{f^4}{c^4}-\frac{f^4}{\omega\omega}\right)\mathfrak{O}+2g\left(\mathfrak{o}-\mathfrak{a}\right)+\upsilon\upsilon\left(1-\frac{f^4}{c^4}\right)\int\frac{ds}{\omega}$$
$$-4g\left(L-k+\mathfrak{a}-\mathfrak{o}\right)\int\frac{ds}{\omega},$$

sicque omnia, quae ad motum pertinent, sunt determinata.

COROLLARIUM 1

91. Si motus ad uniformitatem pervenerit, ita ut iam aqua constanter eadem celeritate per orificium *Oo* expellatur, ob $\mathrm{d}\upsilon=0$ habebitur haec aequatio:

$$4g\left(L-k+\mathfrak{a}-\mathfrak{o}\right)=\upsilon\upsilon\left(1-\frac{f^4}{c^4}\right),$$

unde, si amplitudo in *Aa* aequalis sit orificio *Oo*, uti in capite praecedente pressio in *Aa* debet esse $L=k+\mathfrak{o}-\mathfrak{a}$, neque hinc celeritas υ ipsa determinatur.

COROLLARIUM 2

92. At si amplitudo $Aa=cc$ maior fuerit quam orificium $Oo=ff$,
pro motus uniformitate celeritas effluxus υ ita definitur, ut sit:

$$\upsilon\upsilon=\frac{4gc^4\left(L-k+\mathfrak{a}-\mathfrak{o}\right)}{c^4-f^4}.$$

Hoc ergo casu necesse est, ut sit $L>k+\mathfrak{o}-\mathfrak{a}$, atque ex hoc excessu celeritas effluxus determinatur.

COROLLARIUM 3

93. Sin autem amplitudo $Aa=cc$ minor sit orificio $Oo=ff$, motus
uniformitas hanc praebet aequationem

$$\upsilon\upsilon=\frac{4gc^4\left(k+\mathfrak{o}-\mathfrak{a}-L\right)}{f^4-c^4},$$

unde patet motum aequabilem obtineri non posse, nisi sit $L<k+\mathfrak{o}-\mathfrak{a}$; atque ex hoc defectu celeritas effluxus determinatur.

SCHOLION 1

94. Omnia haec certe sunt maxime paradoxa, cum ex eadem pressione *L*, qua aqua in sectione *Aa* urgetur, celeritas quantumvis magna oriri posse sit inventa; atque hoc imprimis videbitur absurdum, quod in casu tertio a vi finita *L* tempore finito aquae celeritas adeo infinita imprimi possit. Haec autem absurditas statim evanescet, si modo hypothesin, cui totum problema innititur, attentius perpendamus; assumimus enim, dum

aqua per sectionem *Aa* propellitur, continuo aliunde novam aquae copiam eadem celeritate eo influere, neque hic curamus, unde haec aqua adveniat et a quanam vi ipsi hic motus inducatur; longe diversa scilicet haec vis est a vi *L*, quae nihil aliud agit, nisi ut aquam iam illa celeritate intrusam ulterius per tubum propellat. Dum ergo haec vis *L* valeat aquae per *Aa* ingressae motum accelerare, celeritas effluxus increscet ideoque per hypothesin aqua nova etiam continuo maiori celeritate ab illa vi peregrina ingeri assumitur. Quando igitur calculus ostendit celeritatem mox fieri adeo infinitam, hic effectus minime vi nostrae finitae *L*, aquam per tubum propellenti, sed manifesta vi illi peregrinae, quae hoc casu utique fit infinita, tribui debet; quippe quae aquam novam celeritate infinita in tubum compellit. Atque eidem causae est etiam illud paradoxon adscribendum, quod celeritas effluxus ipsa in problemate non determinetur; quo celerius enim et copiosius aqua nova a vi illa peregrina, quaecunque ea sit, subministratur, eo celerius etiam eadem pressio *L* in *Aa* eam per tubum propellere valebit; quoniam igitur illius vis peregrinae nulla ratio in nostro calculo habetur, mirum non est, quod calculus tam immania paradoxa in se implicet, quae autem re bene expensa sponte diluuntur.

SCHOLION 2

95. Introductio autem eiusmodi potentiae L, quae iugiter pari vi premat, sive aqua per tubum celerius promoveatur sive tardius, a natura virium, quae ad aquam propellendam usurpantur, maxime abhorret, cum omnes istae vires ita sint comparatae, ut, quo celerius iam aqua per tubum promovetur, eo magis debilitentur. Quamobrem, si hoc problem *A* ad casus reales, quibus aqua ad certam altitudinem elevari debet, accommodare velimus, naturam earum virium, quibus ad hunc finem est utendum, probe considerari oportet. Quam indolem cum iam in praecedente capite dilucide exposuerim, inde ad praesens institutum id tantum repeto ad opus peragendum adhiberi certam vim *F*, quae certa velocitate *e* agat, ita ut iam tota quaestio eo redeat, quomodo machinam instrui conveniat, ut ab ista vi hac celeritate agente aqua uniformiter per tubum propelli possit.

PROBLEMA 56

96. *Si (Fig. 49) aqua per tubum utcunque inaequaliter amplum Aa Oo ad altitudinem datam* $O\omega = a$ *a motu uniformi elevari debeat a data vi F, quae data celeritate* $= e$ *operetur, machinam invenire, cuius ope hic effectus obtineri queat, simulque copiam aquae dato tempore elevandae definire.*

SOLUTIO

Quia omnis machinae indoles in hoc consistit, ut vim, qua agitatur, in alium locum transferat eamque simul in data ratione vel augeat vel minuat, ponamus machinam quaesitam id praestare, ut vis aquam per orificiuro inferius *Aa* propellens fiat $= nF$; atque ex natura machinarum ista vis hic aget celeritate $= \frac{e}{n}$, ita ut machinae constructio a solo numero *n* pendeat, quem ergo *n* definiri oportet. Nunc praecedens problema in subsidium vocando, qui amplitudo *Aa* posita est $= cc$, in quam vis *nF* agit, pressio ibi exerta erit $=\frac{nF}{cc}$, quae cum pressione atmosphaerae k adiuvetur, habebimus pressionem ibi positam $L=\frac{nF}{cc}+k$; atque aqua per orificium inferius Aa propelletur celeritate $=\frac{e}{n}$ Quare cum superioris orificii Oo amplitudo sit posita $=ff$, aqua ibi expelletur celeritate $=\frac{cce}{nff}$, ita ut sit $\upsilon=\frac{cce}{nff}$ et $d\upsilon=0$. Nune porro altitutudo orificii Oo ante posita $=\mathfrak{o}$ hic est $O\omega=a$, inferioris vero Aa nulla seu $\mathfrak{a}=0$, ex quo aequatio pro motu ibi inventa induet hanc formam:

$$4g\left(\tfrac{nF}{cc}-a\right)-\tfrac{c^4ee}{nnf^4}\left(1-\tfrac{f^4}{c^4}\right)=0 \text{ seu } 4gnn\left(\tfrac{nF}{cc}-a\right)=ee\left(1-\tfrac{c^4}{f^4}\right),$$

unde numerum n ideoque machinam definire licet. Tum autem necesse est, ut aqua aliunde iugiter celeritate $=\frac{e}{n}$ in orificium Aa advehatur singulisque minutis secundis aquae advectae volumen sit $=\frac{cce}{n}$, tantum autem singulis minutis secundis per orificium superius Oo exonerabitur.

Deinde si in loco tubi quovis z ponatur amplitudo $z\upsilon=\omega$, altitudo $z\pi=z$ et longitudo $Az=s$, pressio vero ibidem $=p$, erit ex problemate praecedente:

$$2g(p-k)=2g(a-z)+\tfrac{c^4ee}{2nnf^4}\left(1-\tfrac{f^4}{\omega\omega}\right),$$

unde, si per praecedentem aequationem $\frac{c^4ee}{nnf^4}$ eliminerous, fit

$$p=k-z+\left(a\left(\tfrac{f^4}{\omega\omega}-\tfrac{f^4}{c^4}\right)+\tfrac{nF}{cc}\left(1-\tfrac{f^4}{cc}\right)\right):\left(1-\tfrac{f^4}{c^4}\right).$$

COROLLARIUM 1

97. Si ambo orificia Aa et Oo sunt aequalia seu $cc=ff$, fit $\frac{nF}{cc}-a=0$ ideoque $n=\frac{acc}{F}$; pro machinae instructione tum singulis minutis secundis eiicitur aquae volumen $=\frac{Fe}{a}$; tanta vero copia interea continuo celeritate $=\frac{Fe}{acc}$ in orificium Aa suppeditari debet; ad quod peculiari opus est vi, ad quam hic non respicimus.

COROLLARIUM 2

98. Si sit $cc>ff$ ideoque $\frac{c^4}{f^4}-1>0$, fit $\frac{n}{cc}>\frac{a}{F}$ hinc aquae volumen uno minuto secundo eiectum erit $<\frac{Fe}{a}$ ac tantumdem aquae in orificium Aa advehi debet celeritate minore quam $\frac{Fe}{acc}$, ad quod minori vi peculiari opus est quam casu praecedente.

COROLLARIUM 3

99. Sin autem orificium superius ff minus sit quam inferius cc, prodit $\frac{n}{cc}<\frac{a}{F}$, et volumen aquae uno minuto secundo eiectae fit $>\frac{Fe}{a}$, ita ut hoc modo plus aquae elevetur quam casu primo $ff=cc$; verum etiam tanto plus aquae a vi illa peregrina in orificium *Aa*, idque maiori celeritate quam $\frac{Fe}{acc}$ advehi debet.

SCHOLION

100. Mirum igitur non debet videri, quod ab eadem vi machinam movente modo maior modo minor aquae copia ad eandem altitudinem elevetur, prout superius orificium *Oo* fuerit maius vel minus inferiore *Aa*. Si enim integrum effectum perpendere velimus, etiam integra causa est spectanda, quae habetur, si ad eam vim, qua machinam agitari assumimus, insuper adiungatur illa vis, quae ad aquam continuo in orificium *Aa* ingerendam requiritur, hae autem ambae vires iunctim sumtae eo casu, quo minor aquae copia elevatur, utique minorem praebent summam quam altero casu, quo maior copia elevatur, ita ut hic nihil occurrat, quod aequalitati inter causam et effectum adversetur. Quoniam vero in praxi eadem vis, qua aqua per tubum propellitur, etiam aquam continuo in tubum suppeditare debet, quatenus hic duplex effectus ab eadem causa producitur, in usum practicum accuratius est investigandum. Cum igitur antliarum ope aqua tam in tubum attrahi, quam per cum propelli soleat, huic investigationi, quae in praxi amplissimum habet usum, caput peculiare destinamus.