# THE THIRD PART:

# CONCERNING THE LINEAR FLOW OF FLUIDS, MAINLY WATER.

E409

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## CHAPTER I

## THE PRINCIPLES OF FLUID FLOW ALONG LINES

### DEFINITION

1. A fluid is said to be carried in a linear manner, when it is moved thus in its course along a certain line *DE* (Fig. 32), which may be called the direction of the motion, so that for an individual point *Z*, the motion may become along the direction of that line, and the speed shall be the same at all points through the whole section *UV* made at right angles to the direction of the speed.

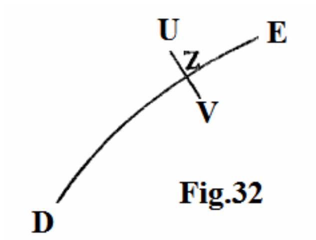

Fig.32

### COROLLARY 1

2. Therefore two things are required for linear motion, in the first place so that everywhere the motion may follow the direction of a certain determined line *DE*, which may be called its directrix [*i.e.* directed line], then truly, so that at the individual sections, all the elements of the fluid shall be advanced normally to the direction *UV* along the same direction with equal speed.

### COROLLARY 2

3. Therefore with the directrix line known, if the speed of the fluid were given at any point *Z* of this line, the same also is common of the whole section *UV*, and since the direction agrees with the tangent of the directrix at the point *Z*, the whole motion of the section *UV* will be determined.

### SCHOLIUM 1

4. When a fluid (Fig. 33) is forced to pass through the narrowest tube *DE*, its motion rightly can be had as linear, as we have described here, for on account of the narrowness of the tube at the individual points *Z*, the direction of the motion is unable to be any way other than the course of the tube allows, and if we may wish to consider the matter with greater care, it may be agreed to consider the line *DzE*

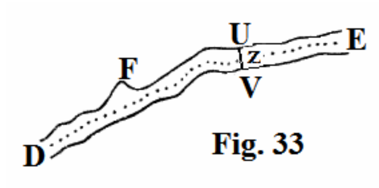

Fig. 33

produced through the middle of the cavity of the tube, which will represent the directrix of the motion, and from the direction of which, the direction of the motion itself will become known at the individual points $Z$. Then truly, because the tube is narrowest, at some section of that $UV$, both the speed as well as the direction of the fluid will be normal to the same everywhere, in order that all inequality may be excluded completely, where certainly it may be able to happen, so that the speed through the part $ZU$ may be made faster or slower than through the part $ZV$, but here we exclude such motion, while we may examine a linear motion, only those will be going to be examined, which shall be convenient to be defined. But truly if such an inequality shall be present, it is evident the area of the tube on being required to be continually contracted, all the unevenness of this kind ought to cease finally, on account of which, if we may put in place an infinitely narrow tube, lest indeed no place may be left for this to be the exception. And in this case also the directrix line likewise does not differ on account of the tube, since the thickness of the tube may be taken for the directrix; yet meanwhile it is not necessary, that the same area may be attributed everywhere to the tube, since it will be able to be allowed further significant differences, provided that nowhere a large jump may occur, just as will eventuate, if the continuity of the tube may be interrupted somewhere at $F$ by a swelling, which even if it may be infinitely small, yet the motion may no longer be able to follow the prescribed law, while the fluid in the swelling may be almost at rest, and the rest thence will flow past, as if that swelling were not present. Therefore irregularities of this kind are required to be excluded, disturbing the continuity of the motion in the tube generally .

## SCHOLIUM 2

5. Because proper linear motion demands an infinitely narrow tube, lest irregularities of this kind may not be present, which may act against this kind of motion, still also it can happen in wide enough tubes that the motion of the fluid may follow this same law, and in this case the motion also will be of this kind, however great were the tubes, rightly are referred to as linear motion. So that also, even if the motion may depart from this law briefly, in practice this distinction is scarcely accustomed to be considered, and the conclusions drawn from calculation are had as approximately true, the situation being under control. Thus also the outflow of water from the largest vessels made through an opening is usually defined from these principles, so that hardly any disagreement may be observed from these trials, and thus may stray less from the truth in that case, where the opening were smaller, since however in this case with the whole vessel level to the normal direction, the flow certainly may not be carried by a common motion. Truly here it usually comes about, that the shape of the vessel arising from the calculation again may be extended out finally, and the efflux may be taken to occur in the same manner, as if the vessel actually will have a figure adapted for the manner of lines. But even if this consideration of linear motion has the greatest use, yet, since in wider tubes the motion may follow another law, from which aberrations must arise, although they may be scarcely perceived, our investigations, it will be agreed, to refer only to the narrowest of tubes, for this reason also the motion, which I am going to consider here, I have called the motion of lines of flow.

## SCHOLIUM 3

6. This treatment includes a huge variation arising from the shape of the tubes, therefore in the first place, I will consider the tubes to be straight, or rather the directrices of which shall be along straight lines, to which indeed any variable amount can be attributed to the area. Then I am going to investigate the motion of a fluid through a tube, the directrices of which are curved lines; which just as if they will have been produced either in the same plane or otherwise in some distinct calculation, while in the first case the figure is required to be defined by two coordinates only, truly in the latter by three coordinates. Then for the most part it is of concern, whether the fluid may flow in tubes of this kind always, or it may flow out elsewhere; then truly the nature of the fluid itself and the forces acting are required to be led into the calculation. But generally the determination of the motion must be demanded from the principles established before, and since these principle motions have been set out in two ways, in whatever case it will be appropriate to use that, which will be seen to be the most suitable. Indeed at first I will consider the flow in tubes placed at rest, then we will be going to look at motion in moving tubes.

## PROBLEM 42

7. *To describe the calculation for the linear motion of a fluid through a rectilinear tube, by using the method established above in the first part.*

## SOLUTION

*OIAa* (Fig. 34) shall be the tube proposed and its directrix line shall be *OA*, the area of which shall be variable in any manner. In the elapsed time $=t$, a particle of the fluid occupying the volume *XVxv* will be considered, and the distance from a fixed end *O* may be put $OX=x$, and the area of the tube at *X* or the section made normally $XV=\omega$, thus so that $\omega$ shall be a given function of *x*. Now the speed of the fluid passing through the section *XV* shall be the speed of the fluid along the direction $XA=u$, the density $=q$ and the pressure $=p$; and *u*, *q* and *p* will be functions of the two variables *x* and *t;* then truly from the nature of the fluid the relation between *p*, *q* and the heat [*i.e.* temperature] *r* is given, if perhaps an account of this were available. $Xx=dx$ may be attributed to this particular thickness, and its volume will be $=\omega dx$ and the mass $=q\omega dx$. Now in the increment of the time *dt* this particle may progress to *X' V' x' v'*; and since the speed at *X* shall be $=u$, the length increment $XX'=udt$; at *x* truly the speed $=u+dx\left(\frac{du}{dx}\right)$ which will give the distance increment $xx'=udt+dtdx\left(\frac{du}{dx}\right)$, thus so that there shall become:

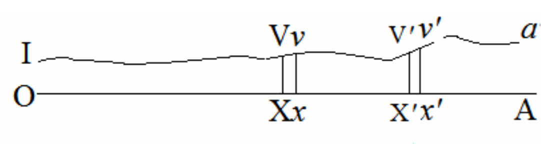

Fig.34

$$X'x'=dx+dtdx\left(\tfrac{du}{dx}\right).$$

But since the area at $X'$ shall be $=\omega+udt\cdot\frac{d\omega}{dx}$, the volume of this same particle will be

$$=\omega dx+\omega dtdx\left(\frac{du}{dx}\right)+udtdx\cdot\frac{d\omega}{dx}.$$

But the density of our particle thence must be deduced at $X'$, so that, since $q$ shall be a function of $x$ and $t$, the initial increment of $x$ may be taken $XX'=udt$, then truly the increment $dt$ of $t$, from which the density at $X'$ will become

$$=q+udt\left(\frac{dq}{dx}\right)+dt\left(\frac{dq}{dt}\right),$$

by which if the volume may be multiplied in the manner found, the mass of our translated particle will be found :

$$X'V'x'v'=q\omega dx+dxdt\left(q\omega\left(\frac{du}{dx}\right)+qu\frac{d\omega}{dx}+\omega u\left(\frac{dq}{dx}\right)+\omega\left(\frac{dq}{dt}\right)\right),$$

which since it must be equal to the initial mass $q\omega dx$ ,this consideration will provide this first equation for the motion:

$$q\omega\left(\frac{du}{dx}\right)+qu\frac{d\omega}{dx}+\omega u\left(\frac{dq}{dx}\right)+\omega\left(\frac{dq}{dt}\right)=0$$

or

$$qu\frac{d\omega}{\omega dx}+\left(\frac{d\cdot qu}{dx}\right)+\left(\frac{dq}{dt}\right)=0.$$

Again since the speed $u$ after the time $dt$ shall be changed into

$$u+udt\left(\frac{du}{dx}\right)+dt\left(\frac{dq}{dt}\right),$$

since not only the increase $dt$ of the time $t$, but also the increase $XX'=udt$ of the distance $OX=x$ must be attributed, the acceleration of the particle $XVxv=u\left(\frac{du}{dx}\right)+\left(\frac{du}{dt}\right)$, which must agree with the accelerative forces. For these requiring to be found in the first place it will depend on the pressure, which since it shall be $p$ on the face $XV$, on the face $xv$

$$=p+dx\left(\frac{dp}{dx}\right)\text{on account of } Xx=dx.$$

Hence the accelerative force acting backwards on the particle $=\frac{1}{q}\left(\frac{dp}{dx}\right)$, if indeed the motive force thence arising $=\omega\cdot dx\left(\frac{dp}{dx}\right)$, which divided by the mass $q\omega dx$ gives the accelerative force. Again from the forces acting directly on the particles of the fluid, the accelerative force shall arise along the direction $XA$ shall be $=P$, thus so than now the total force along the same direction shall become $=P-\frac{1}{q}\left(\frac{dp}{dx}\right)$ which multiplied by $2g$ is required to be equal to the acceleration, from which the other equation thus is itself had for the motion of the fluid:

$$2gP-\tfrac{2g}{q}\left(\tfrac{dp}{dx}\right)=u\left(\tfrac{du}{dx}\right)+\left(\tfrac{du}{dt}\right),$$

or, if the time *t* may be taken constant :

$$\tfrac{2gdp}{q}=2gPdx-udx\left(\tfrac{du}{dx}\right)-dx\left(\tfrac{du}{dt}\right)=2gPdx-udu-dx\left(\tfrac{du}{dt}\right).$$

Therefore this equation together with the one found before:

$$qu\,\tfrac{d\omega}{\omega dx}+\left(\tfrac{d\cdot qu}{dx}\right)+\left(\tfrac{dq}{dt}\right)=0\,,$$

if the relation between $p$ and $q$ may be called in to help from the nature of the fluid, will determine the whole motion.

COROLLARY 1

8. Also the first equation may be multiplied by *dx* and in its integration the time *t* will be considered constant ; and there is going to be had:

$$\tfrac{qud\omega}{\omega}+d\cdot qu+dx\left(\tfrac{dq}{dt}\right)=0\ \text{ seu }\ \omega dx\left(\tfrac{dq}{dt}\right)+d\cdot qu\omega=0,$$

from which, if the density of the fluid were the constant $q=b$, it is deduced that $u\omega=\text{Const.}$ which constant can involve the time *t* in some manner, from which it is apparent the speeds located in diverse places of the tube for whatever time to be inversely proportional with the areas, which is the most noteworthy property of the motion of fluids through tubes.

COROLLARY 2

9. But if the density of the fluid *q* were not constant, but may depend also on the pressure *p*, by necessity both the equations must be joined together, so that from these thence for any location and any time *t* both the speed *u* as well as the pressure may be defined ; which investigation therefore will emerge often with the greatest difficulty.

SCHOLION

10. Since this motion shall be an extra special case of the most general problem, of which we have shown above the twofold solution, and here I may have made use of the first method, it is necessary, that the solution found here may be contained in that most general solution, so that indeed it is evident that the pressure *p* by itself may be found from the other equation ; while indeed here with the single variable *x* present besides the

time *t*, also these terms, which will include the differentials *dy* and *dz*, are missing, where initially it is required to be observed both the speeds *v* and *w* vanish, therefore since for the whole section *XV* the motion may be considered to be made along the same direction *OA*. But the other equation is seen to be completely adverse on account of the term $qu\frac{d\omega}{\omega dx}$; moreover with the same cross-section of the tube present everywhere this matter is agreed on especially well. Yet truly with the matter thought out well this equation can be deduced at once from the general form. Indeed if the tube may diverge from *V* to *V'*, the direction of the motion around *V* must be deflected a little from the direction *OA*. Therefore on putting $XV = y$, the speed $v = \alpha y$ may be put in place, in order that at *X* there shall be no deflection ; moreover in order that the third speed $\omega$ may vanish, there may be take $z = \text{const.} = \gamma$, so that there shall become $\omega = \gamma y$. But since the density *q* clearly will not depend on *y*, there will become $\left(\frac{d\cdot qv}{dy}\right) = \alpha q;$ but since *y* is a function of *x*, there will become $X'V' = y + udt\frac{dy}{dx}$, then truly from the speed *v* there will become also

$$X'V' = y + vdt = y + \alpha ydt,$$

from which there becomes

$$\alpha = u\frac{dy}{ydx} = u\frac{d\omega}{\omega dx} \text{ and thus } \left(\frac{d\cdot qv}{dy}\right) = qu\frac{d\omega}{\omega dx}.$$

Hence therefore it is clearly understood, how the special solution given here may agree most neatly with the general equation and may arise from that.

## PROBLEM 43

11. *To resolve the preceding problem, where the motion of the fluid is sought in a rectilinear tube, by the latter method had with respect to the initial state.*

## SOLUTION

The particle of the fluid (Fig. 35), of which we wish to investigate the motion, initially $t = 0$ will occupy the element of the tube *XVX'V'*, for which we may put $OX = X$, $XX' = dX$, the cross-section area of the tube at $X = \boldsymbol{\Omega}$, so that the volume of the particles shall be $= \Omega dX$.

Again the density at $x$ shall be $= Q$, the pressure $= P$ and the speed along the $XA = U$, which is common for all the points of the particle ; from which its mass $= Q\Omega dX$. Now in the elapsed time *t*

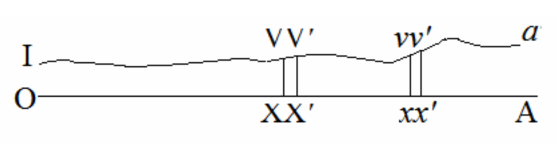

Fig.35

the same particle will be found at *xvx'v'*, for which we may put $Ox = x$, the area $xv = \omega$, then truly the density at $x = q$, the pressure $p$ and the speed along $xA = u$, and the letters *x*, *q*, *p*, *u* will be functions of the two variables *X* and *t*, but the area ω is certainly a function of *x* requiring to be defined from the figure of the tube. Now since the section *X'V'* will arrive at *x'v'* in the same time *t*, there will be

$$xx' = dX\left(\tfrac{dx}{dX}\right);$$

from which the volume of the particle $xvx'v'$ will become $\omega dX\left(\tfrac{dx}{dX}\right)$ and the mass $= q\omega dX\left(\tfrac{dx}{dX}\right)$, which since it may remain the same always, we will have this first equation determined for the motion :

$$q\omega\left(\tfrac{dx}{dX}\right) = Q\Omega.$$

Thence since the speed at $x$ shall be $u = \left(\tfrac{dx}{dt}\right)$, the acceleration of the particle $xvx'v'$ $= \left(\tfrac{ddx}{dt^2}\right)$. From the forces acting the accelerative force arises $= \mathfrak{P}$ for this particle along the direction $xA$ ; then truly, since the pressure at $x' = p + dX\left(\tfrac{dp}{dX}\right)$, hence the motive force acting backwards $= \omega dX\left(\tfrac{dp}{dX}\right)$, which divided by the mass $q\omega dX\left(\tfrac{dx}{dX}\right)$ gives the accelerative force :

$$= \frac{1}{q\left(\frac{dx}{dX}\right)}\left(\frac{dp}{dX}\right)$$

with that being subtracted from $\mathfrak{P}$, from which the other equation is deduced containing a determination of the motion:

$$\left(\frac{ddx}{dt^2}\right) = 2g\mathfrak{P} - \frac{2g}{q\left(\frac{dx}{dX}\right)}\left(\frac{dp}{dX}\right),$$

which with the time $t$ taken constant will be changed into this equation :

$$\frac{2g\,dp}{q} = 2g\mathfrak{P}dx - dx\left(\frac{ddx}{dt^2}\right).$$

## COROLLARY 1

12. It is evident the quantities $x$, $q$, $p$ and $u$ must be functions of the two variables $X$ and $t$ of this kind, so that on putting $t = 0$ there shall become $x = X$, $q = Q$, $p = P$ and $u = U$ ;it is required for which conditions to be satisfied by the integration.

## COROLLARY 2

13. The magnitude $\omega$ is considered as a given function of the quantity $Ox = x$, which since that shall be a function of $X$ and $t$, until now the magnitude $\omega$ has been considered to depend on the time.

## SCHOLIUM

14. Because this problem contains especially the special case of the most general problem 41, also the solution given here will be required to be contained in that most generally, since indeed it is apparent from the latter equation, certainly which arises from the most general form, if the terms may be removed, which here involve the two variables $Y$ and $Z$. Moreover, just as it is less clear how the first equation may agree with the most general solution. But again it can be shown to be agreed on, only if the variation of the magnitude of the tube may be considered correctly. Finally we may consider these three dimensional tubes, and if both $Y$ and $Z$ are infinitely small besides $OX = X$ and indeed $Z$ may be considered as constant through the whole length of the tube, so that there shall become $z = Z$ and thus $\left(\frac{dz}{dZ}\right) = 1$, then truly there will be $\Omega = YZ$. There may be put $y = LY$, with the function $L$ being some function of the time $t$, and there will become $\left(\frac{dy}{dY}\right) = L$ and $\left(\frac{dy}{dX}\right)$ will vanish with the remaining differential formulas, and the area at $x$ will be $\omega = LYZ = L\Omega$. Now from the general solution the value $K = \left(\frac{dx}{dX}\right) L \cdot 1$ is deduced, which on account of $L = \frac{\omega}{\Omega}$ will become $K = \frac{\omega}{\Omega}\left(\frac{dx}{dX}\right)$: from which evidently there is found $Kq = Q$, so that the general solution is had; and thus this whole solution will be contained in general, and from that special cases can be derived.

## PROBLEM 44

15. *If* (Fig. 36) *the directrix IYK of the tube shall be some curved line placed in the same plane and its cross-section YV shall be variable in some manner along the length of the tube, to investigate the motion of any fluid treated above by the first method.*

## SOLUTION

In the elapsed time $= t$ a particle of the fluid in the tube will be considered occupying the volume *YVyv*, and the two coordinates $OX = x$, $XY = y$ may be put in place for the point *Y*, the mutual relation of which with the directrix from the figure of the tube may be given, $y$ will be a function of $x$; truly the minimum cross-section of the tube at $Y$ may be put $YV = \omega$, which also is required to be seen as a given function of $x$. Again the density of the fluid at $Y = q$ and the pressure $= p$, which two quantities will be functions of the two variables $x$ and $t$, and thence the speed of the fluid in the tube, which shall be $= \mathfrak{T}$, since its direction shall be *Yy*, if for the sake of brevity we may call this element

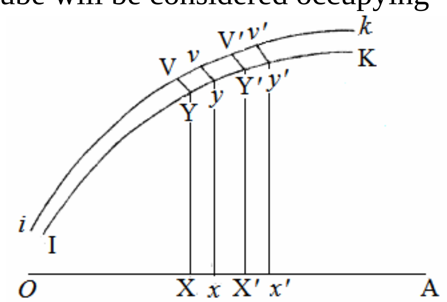

Fig. 36

$$Yy = \sqrt{\left(dx^2 + dy^2\right)} = ds,$$

its speed along $OX = \frac{\mathfrak{T}dx}{ds} = u$ and along $XY = \frac{\mathfrak{T}dy}{ds} = v$. Now since the section $YV = \omega$ as normal to the directrix, the volume of our particle $YVyv$ will become $= \omega ds$ and the mass $= q\omega ds$. Now in the time *dt* this particle may progress to *Y'V'y'v'*, thus so that the element *Y* will arrive at *Y'*, and the increment of distance

$$XX' = udt = \frac{\mathfrak{T}dx}{ds}\cdot dt \text{ and } X'Y' - XY = vdt = \frac{\mathfrak{T}dy}{ds}\cdot dt.$$

But since the speeds at *y* shall be $u + dx\left(\frac{du}{dx}\right)$ and $v + dx\left(\frac{dv}{dx}\right)$, there will be with the progress of the time :

$$xx' = udt + dtdx\left(\frac{du}{dx}\right) \text{ and } x'y' - xy = vdt + dtdx\left(\frac{dv}{dx}\right),$$

and hence

$$X'x' = dx + dtdx\left(\frac{du}{dx}\right) \text{ and } x'y' - X'Y' = dy + dtdx\left(\frac{dv}{dx}\right).$$

On account of which we will have:

$$Y'y' = ds + \frac{dtdx^2}{ds}\left(\frac{du}{dx}\right) + \frac{dtdxdy}{ds}\left(\frac{dv}{dx}\right).$$

Thence the cross-section at *Y'* is $Y'y' = \omega + udt\cdot\frac{d\omega}{dx}$ and the density

$$= q + udt\left(\frac{dq}{dx}\right) + dt\left(\frac{dq}{dt}\right);$$

from which it is concluded the volume of the particle *Y'V'y'v'*

$$= \omega ds + \frac{\omega dtdx^2}{ds}\left(\frac{du}{dx}\right) + \frac{\omega dtdxdy}{ds}\left(\frac{dv}{dx}\right) + udtds\frac{d\omega}{dx}$$

and the mass

$$= q\omega ds + \frac{q\omega dtdx^2}{ds}\left(\frac{du}{dx}\right) + \frac{q\omega dtdxdy}{ds}\left(\frac{dv}{dx}\right) + qudtds\frac{d\omega}{dx} + u\omega dtds\left(\frac{dq}{dx}\right) + \omega dtds\left(\frac{dq}{dt}\right),$$

which since it must be equal to the preceding $q\omega ds$, on dividing by $\omega dtds$, we will arrive at this equation:

$$\frac{qdx^2}{ds^2}\left(\frac{du}{dx}\right) + \frac{qdxdy}{ds^2}\left(\frac{dv}{dx}\right) + qu\frac{d\omega}{\omega dx} + u\left(\frac{dq}{dx}\right) + \left(\frac{dq}{dt}\right) = 0;$$

but these values may be substituted $u = \frac{\mathfrak{T}dx}{ds}$, $v = \frac{\mathfrak{T}dy}{ds}$ and, since $\mathfrak{T}$ is a function of *t* and *x*, truly the fractions $\frac{dx}{ds}$ and $\frac{dy}{ds}$ will depend on *x* alone, this equation will be come upon:

$$q\mathfrak{T}\frac{d\omega}{\omega ds}+\frac{dx}{ds}\left(\frac{d\cdot q\mathfrak{T}}{dx}\right)+\left(\frac{dq}{dt}\right)=0\,.$$

Thereupon since he speeds at $Y'$ along the directions $OX$ and $XY$ shall be

$$u+udt\left(\frac{du}{dx}\right)+dt\left(\frac{du}{dt}\right)\ \text{ and }\ v+udt\left(\frac{dv}{dx}\right)+dt\left(\frac{dv}{dt}\right),$$

the accelerations will become :

$$u\left(\frac{du}{dx}\right)+\left(\frac{du}{dt}\right)\ \text{ and }\ u\left(\frac{dv}{dx}\right)+\left(\frac{dv}{dt}\right).$$

Now from the forces acting we may put these along the same directions resulting from the accelerative forces $P$ and $Q$. Truly at last on account of the pressure at $y = p+dx\left(\frac{dp}{dx}\right)$, the element $YVyv$ will be acted on backwards by the accelerative force $=\frac{dx}{qds}\left(\frac{dp}{dx}\right)$ in the direction $yY$ , from which these forces arise along the directions $OX$ and $XY$ :

$$P-\frac{dx^2}{qds^2}\left(\frac{dp}{dx}\right)\ \text{ and }\ Q-\frac{dxdy}{qds^2}\left(\frac{dp}{dx}\right);$$

and hence these equations again:

$$2gP-\frac{2gdx^2}{qds^2}\left(\frac{dp}{dx}\right)=u\left(\frac{du}{dx}\right)+\left(\frac{du}{dt}\right)=\frac{\mathfrak{T}\mathfrak{T}}{ds}d\cdot\frac{dx}{ds}+\frac{\mathfrak{T}dx^2}{ds^2}\left(\frac{d\mathfrak{T}}{dx}\right)+\frac{dx}{ds}\left(\frac{d\mathfrak{T}}{dt}\right),$$
$$2gQ-\frac{2gdxdy}{qds^2}\left(\frac{dp}{dx}\right)=u\left(\frac{dv}{dx}\right)+\left(\frac{dv}{dt}\right)=\frac{\mathfrak{T}\mathfrak{T}}{ds}d\cdot\frac{dy}{ds}+\frac{\mathfrak{T}dxdy}{ds^2}\left(\frac{d\mathfrak{T}}{dx}\right)+\frac{dy}{ds}\left(\frac{d\mathfrak{T}}{dt}\right),$$

from which the first multiplied by $dy$ if it may be subtracted from the other multiplied by $dx$, there is left :

$$2g\left(Pdy-Qdx\right)=\frac{\mathfrak{T}\mathfrak{T}}{ds}\left(\frac{dyddx-dxddy}{ds}\right)=\mathfrak{T}\mathfrak{T}d\cdot\text{Ang tang}\frac{dx}{dy}.$$

Truly hence besides there is deduced :

$$2g(Pdx+Qdy)-\frac{2gdx}{q}\left(\frac{dp}{dx}\right)=\mathfrak{T}dx\left(\frac{d\mathfrak{T}}{dx}\right)+\ ds\left(\frac{d\mathfrak{T}}{dt}\right).$$

But the first equation is required to be excluded completely, because we have calculated the acceleration only from the pressure acting along the direction of the motion, while thence also any other pressure arising from the sides of the tube to be removed from the equation. For the forces $P$ and $Q$ do not otherwise affect the motion, unless perhaps they may act along the direction of the motion $Yy$ , from which the second equation alone is left in the calculation, which, if the time $t$ may be placed constant, the pressure $p$ is defined thus, so that there shall become:

$$\frac{2gdp}{q} = 2g(Pdx + Qdy) - \mathfrak{T}d\mathfrak{T}\left(\frac{dp}{dx}\right) - ds\left(\frac{d\mathfrak{T}}{dt}\right).$$

COROLLARY 1

16. Therefore it is evident the curvature of the directrix line to disturb nothing in the flow of the fluid, while the differentials of the formulas $\frac{dx}{ds}$ and $\frac{dy}{ds}$, by which the curvature may be defined, will have vanished completely from the calculation.

COROLLARY 2

17. Also the first equation found can be reduced to a more convenient form. Indeed since *s* shall be a function of *x* only, it is likewise if $\mathfrak{T}$ may be considered as a function of *x* and *t,* or of *s* and *t:* from which there becomes :

$$dx\left(\frac{d\mathfrak{T}}{dx}\right) = ds\left(\frac{d\mathfrak{T}}{ds}\right)$$

and by the same reason

$$dx\left(\frac{d\cdot q\mathfrak{T}}{dx}\right) = ds\left(\frac{d\cdot q\mathfrak{T}}{ds}\right)$$

from which the first equation will be changed into this form:

$$q\mathfrak{T}\frac{d\omega}{\omega ds} + \left(\frac{d\cdot q\mathfrak{T}}{ds}\right) + \left(\frac{dq}{dt}\right) = 0$$

and again, into this form : $\left(\frac{d\cdot q\mathfrak{T}\omega}{ds}\right) + \omega\left(\frac{dq}{dt}\right) = 0$.

SCHOLIUM 1

18. Therefore I have deduced the solution of this problem by so many indirect unnecessary ways, so that it may clearly become more evident the curvature of the directrix line plainly to disturb nothing in the motion of the fluid; which principle, if I may have wished to establish at once, deservedly that would have been allowed to be called into doubt. But now at last we know with certainty, whatever is spent concerned with the forces changing the direction of the motion of the fluid, that is to be absorbed as if by the sides of the tube, thus so that they may produce the same determination of the motion, as if the tube were rectilinear. Thus it is required to observe concerned with so many forces acting for any fluid element thence that single accelerative force must be deduced, which shall act along the same direction as the line of the motion, clearly with all the other forces ignored, evidently which on the whole are expended on the sides of the tube. With this observed it is evident the solution of this problem plainly does not differ from problem 42 from each it shall be able to derive at once, only by writing *s* and

$\mathfrak{T}$ in place of $x$ and $u$, and in place of the force $P$ this force $P\frac{dx}{ds}+Q\frac{dy}{ds}$. Therefore from this compendium of the laws involved, we will be able now to resolve the following problems much more easily.

## SCHOLIUM 2

19. Lest a place may remain for any doubt, it is seen by declaring more clearly, why in the estimation of the pressure, by which the particle of the fluid *YVyv* is pushed forwards, no account of the inequality of the bases *yv* and *YV* will have been given, yet while in the remaining investigations for that so much care may be required to be considered, since indeed for the section $YV=\omega$, the section *yv* certainly shall be made $=\omega+dx\left(\frac{d\omega}{dx}\right)$, and just as the pressure acting on that may be considered $=p+dx\left(\frac{dp}{dx}\right)$, the total pressure the base *yv* may sustain as well, becomes

$$p\omega+pdx\left(\frac{d\omega}{dx}\right)+\omega dx\left(\frac{dp}{dx}\right),$$

while that, which the base *YV* sustains, is only $=p\omega$ and thus the total repelling force may become greater than I have assumed in the solution, which same difficulty is observed to be present in the preceding problem. Truly this doubt may be reduced, if from these expressions, which have been treated above concerning the nature of the pressure, we may record all the pressures which are represented by equal heights, to maintain each other in mutual equilibrium, even if the sections may be acted on especially unequal at the bases ; hence the strength of this $p+dx\left(\frac{dp}{dx}\right)$, which the base *xy* sustains, the part *p* plainly is in equilibrium with the force $p$ acting on the base *YV*, even if the base *xy* may become especially uneven to this one *YV*, whereby of that pressure, which we have seen the base *yv* to be impressed, not only the part $p\omega$ is cancelled, but this is to be removed by an opposing pressure $p\omega+pdx\left(\frac{d\omega}{dx}\right)$, thus so that an excess pressure has to be estimated from the single term $\omega dx\left(\frac{d\omega}{dx}\right)$, just as I have done at once in the solutions of these problems. Then if there may be still be seen to be some doubt about this, in what manner the setting out of the equation is to found, a differential equations of the second order vanish from the calculation, thus so that there shall become

$$\frac{\mathfrak{T}\mathfrak{T}}{ds}dxd\frac{dx}{ds}+\frac{\mathfrak{T}\mathfrak{T}}{ds}dyd\frac{dy}{ds}=0,$$

that may involve only these formulas, and there will be found

$$\frac{\mathfrak{T}\mathfrak{T}}{ds}\left(dx\frac{ddx}{ds}-\frac{dx^2dds}{ds^2}+dy\frac{ddy}{ds}-\frac{dy^2dds}{ds^2}\right),$$

which on account of the form

$$dx^2 + dy^2 = ds^2 \text{ and } dxddx + dyddy = dsdds,$$

will become

$$\tfrac{\mathfrak{T}\mathfrak{T}}{ds}\left(\tfrac{dsdds}{ds} - \tfrac{ds^2dds}{ds^2}\right) = 0.$$

## PROBLEM 45

20. *If the directrix of the tube shall be shall be the curved line IYK placed in the same plane* (Fig. 37), *of which the section shall be variable in some manner along the length of tubes, by the latter method treated above with respect to the initial state had, to define the motion of the fluid of whatever kind in this tube.*

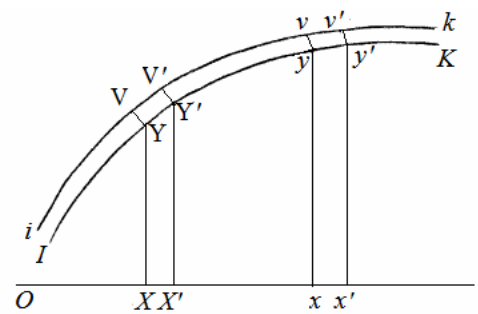

Fig. 37

## SOLUTION

In the initial state we will consider a particle of the fluid *YVY'V'* and for the point *Y* with the coordinates $OX = X$, $XY = Y$ the arc itself shall be $IY = S$ and at *Y* the section of the tube $YV = \Omega$, which, either so that a function of $X$ may be considered or which likewise is of *S*. Then truly the density at $Y = Q$, the pressure $= P$ and the speed along the direction of the tube $YY' = \Upsilon$, if now the element of the tube may be taken $YY' = dS$, our particular volume will be $= \Omega dS$ and the mass $= Q\Omega dS$. From these, which pertain to the initial state, with the time put $= t$, our particle shall be transferred into $yvy'v'$ and for the point *y* there may be called $Ox = x$, $xy = y$ and the arc of the directricix $Iy = s$, then truly the density $= q$ and the pressure $= p$, which quantities are all functions of the two variables *S* and *t* and these such that, so that with the time put $t = 0$ there may become $x = X$, $y = Y$, $s = S$, $q = Q$ and $p = P$, truly the cross-section of the tube at *y* shall be $yv = \omega$ of the function $s$ itself. But if now the theory may be put in place as in problem 43, since the curvature of the tube disturbs nothing in the motion of the fluid, in place of the quantity $X$ denoting there the length of the tube, here it will be required to write the letter *S*. Moreover, from the forces acting, if these accelerative forces, which act along the directions *Ox* and *xy* of the coordinates, shall be $\mathfrak{P}$ and $\mathfrak{Q}$, this force, which will act along the direction of the tube at *y*, will be $\frac{\mathfrak{P}dx+\mathfrak{Q}dy}{ds}$ and hence the motion of the fluid will be expressed by the two following equations

$$q\omega\left(\tfrac{ds}{dS}\right) = Q\Omega \text{ and } \tfrac{2gdp}{q} = 2g\left(\mathfrak{P}dx+\mathfrak{Q}dy\right) - ds\left(\tfrac{dds}{dt^2}\right),$$

in which for the latter the time *t* is assumed to be constant.

## COROLLARY 1

21. Therefore $\left(\frac{ds}{dt}\right)$ denotes the speed of the fluid in the tube at the point *y* with the elapsed time $=t$, which therefore will be required to be prepared thus, so that on making $t=0$ there may become $\left(\frac{ds}{dt}\right)=\Upsilon$, which of course is to be regarded as the speed in the initial state.

## COROLLARY 2

22. Therefore the curvature of the tube only produces in effect a variation of the forces acting, since for any particle of the fluid the accelerative force must be deduced there only from the disturbing forces, which act along the direction of the tube.

## SCHOLIUM

23. It is appropriate to pay attention here in the initial state neither the speed $\Upsilon$ nor the pressure *P* can be formed as it pleases, since indeed these quantities may not be present in the equations found for the motion, thus it will be required to prepare these, so that, after the equations found were integrated, with the values of $\mathfrak{T}$ and $p$ established for the time $t=0$, these quantities may arise. But whatever the maximum section the integration may bring in with it, yet it cannot be effected, that in the initial state for the individual elements both these quantities $\Upsilon$ and *P* may be able to obtain given values. But if indeed the fluid shall be capable of no compression, once the speed may be given in a single section of the tube, likewise it may be determined in all the remaining sections, then truly also for the pressure at a single location of the tube the pressures are determined at all the remaining locations. From which it is understood well enough, even if the density were variable, since it is connected always with the speed and pressure in a certain way, that it cannot be put in place at the individual places as it pleases for the initial state, as can be done for the initial speed and pressure.

## PROBLEM 46

24. *If* (Fig. 38) *the directrix of the tube IZzK were some curved line not situated in the same plane and its amplitude were variable in some manner, to define the motion of the fluid in a tube of this kind following the first method I set out above.*

## SOLUTION

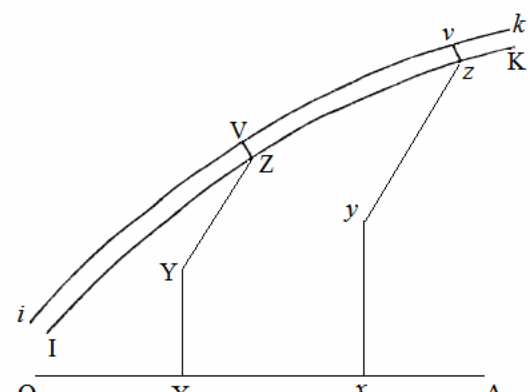

Fig. 38

By using this method we consider at once the state of the fluid at some indefinite time $=t$ elapsed from a certain initial time. Therefore with the directrix defined by the three coordinates $Ox = x,\ xy = y,\ yz = z,$ which thus in turn will depend on each other, so that they may be had from a single variable, moreover the arc of the directrix $Iz = s$, and at $z$ the section of the tube shall be $zv = \omega$, which also is required to be considered as function of $s$. Now we may consider a particle of the fluid at this location $z$ for the time $=t$, the density of which shall be $=q$, the pressure $=p$, and the speed along the direction of the tube $zK = \mathfrak{T}$ ; which quantities are functions of the two variables evidently of $s$ and of the time $t$. Finally from the remaining forces acting at the point $z$ these three accelerative forces arise $P$, $Q$, $R$ along the directions of the coordinates $Ox$, $xy$ et $yz$; from which again the accelerating force along the direction of the tube $zK$ is deduced $\frac{Pdx+Qdy+Rdz}{ds}$.
With these in place the curvature of the tube will change nothing in the motion, the motion sought will be expressed by the two following equations, of which the first defines the relation between the density, speed and cross-section of the tube, and thus itself becomes:

$$q\mathfrak{T}\frac{d\omega}{\omega ds}+\left(\frac{d\cdot q\mathfrak{T}}{ds}\right)+\left(\frac{dq}{dt}\right)=0 \text{ or } \left(\frac{d\cdot q\mathfrak{T}\omega}{ds}\right)+\omega\left(\frac{dq}{dt}\right)=0,$$

truly the other in addition involves the pressure, and there the time $t$ may be considered as constant:

$$\frac{2gdp}{q}=2g(Pdx+Qdy+Rdz)-\mathfrak{T}d\mathfrak{T}-ds\left(\frac{d\mathfrak{T}}{dt}\right).$$

PROBLEM 47

25. *If* (Fig. 38) *the directrix of the tube shall be some curved line IZzK not placed in the same plane and its cross-section variable in some manner, to define the motion of the fluid in a tube of this kind following the second method, with respect to the initial state considered.*

SOLUTION

While the individual points of the directrix *Z* are defined by the three coordinates $OX = X$, $XY = Y$ and $YZ = Z$, in addition there may be put the length of the arc $IZ = S$ and the cross-section of the tube at $Z = \Omega$. Now some particle of the fluid may be considered, which in the initial state, where the time was $t = 0$, will be at *Z*, and its density $= Q$, then truly the pressure $= P$ and the speed along the direction of the tube $ZK = \Upsilon$; which quantities therefore can be regarded as given and functions of the single variable $S$. Now in the time elapsed $= t$ the same particle will arrive at the point *z* of the tube, with the coordinates defined $Ox = x$, $xy = y$ and $yz = z$, where the arc shall become $Iz = s$ and the cross-section of the tube $z = \omega$, which is a function of *s*, then truly the density of the fluid at $z = q$, the pressure $= p$ and the speed along $zK = \Upsilon$, as by these new denominations to become $\mathfrak{T} = \left(\frac{ds}{dt}\right)$, and all these quantities are required to be considered as functions of the two variables *S* and *t*. Finally from the resolution of the force acting on the particle at *z* these three accelerations $\mathfrak{P}$, $\mathfrak{Q}$ and $\mathfrak{R}$ will be derived along the directions *Ox*, *xy* and *yz* . From which in place since the curvature of the tube will disturb nothing in our investigation, we will have these two equations used in problem 45 for the motion of the fluid in this tube :

$$q\omega\left(\frac{ds}{dS}\right) = Q\Omega$$

and

$$\frac{2gdp}{q} = 2g\left(\mathfrak{P}dx + \mathfrak{Q}dy + \mathfrak{R}dz\right) - ds\left(\frac{dds}{dt^2}\right)$$

in which the time *t* of the latter is assumed constant.

SCHOLIUM

26. We have given all these problems a twofold solution, as we have used each method in the preceding section. Indeed these twin forms generally disagree in the reckoning, but yet, there is no reason why they may not agree exceptionally well amongst themselves. But just as questions were to be asked, as to whether it would be more expedite to use the first or the second solution; but always each on being used, not only the agreement of these will confirm the truth there, but also will provide significant explanations, from which we may understand thus the true nature of the motion more accurately. But the first

part of this treatment produces a diversity of fluids, whereby the density of these is either constant or variable, just as a mixture of fluids can be added, as if the continuity of the fluid in the tube were interrupted by bubbles. The reckoning of the forces acting will affect the treatment to a great extent, so that scarcely will there be a need to consider other forces besides gravity, nor here also does it hinder in any way, if perhaps the action of the forces $Pdx+Qdy+Rdz$ may not be allowed to be integrated, since here $x$, $y$ and $z$ among themselves in turn are agreed to depend on, and to constitute a single variable, on account of that difficulty the integral cannot be found, [unless perhaps numerically]. But truly the cross-section of the tube enters into the calculation to such an extent, that hence it may be especially conducive to demand a separation into parts; according to which initially we are going to consider tubes of the same cross-section everywhere. Finally any diversity of the fluids will be reviewed for two kinds of fluids, air and water.

# SECTIO TERTIA

# DE MOTU FLUIDORUM LINEARI POTISSIMUM AQUAE

Commentatio 409 indicis ENESTROEMIANI

Novi commentarii academiae scientiarum Petropolitanae 15 (1770), 1771, p. 219-360

## CAPUT I

## DE PRINCIPIIS MOTUS LINEARIS FLUIDORUM

### DEFINITIO

1. Fluidum motu lineari ferri dicitur, quando (Fig. 32) eius vena ita secundum lineam quandam *DE*, quam motus directricem vocare licet, movetur, ut in singulis punctis *Z* motus fiat secundum directionem eius lineae et per totam sectionem *UV* ad directricem normaliter factam celeritas in omnibus punctis sit eadem.

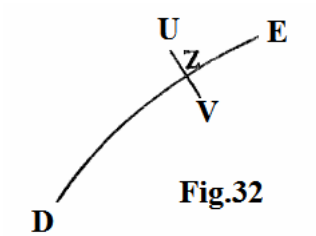

Fig.32

### COROLLARIUM 1

2. Ad motum ergo linearem duo requiruntur, primo ut motus ubique sequatur directionem certae cuiusdam lineae *DE*, quae eius directrix vocatur, tum vero, ut in singulis sectionibus *UV* ad directricem normalibus omnia fluidi elementa pari celeritate secundum eandem directionem proferantur.

### COROLLARIUM 2

3. Cognita ergo linea directrice, si in quovis eius puncto *Z* fluidi celeritas fuerit data, eadem quoque toti sectioni *UV* est communis, et quia directio convenit cum directricis tangente in puncto *Z*, totus motus sectionis *UV* erit determinatus.

### SCHOLION 1

4. Quando fluidum (Fig. 33) per tubum angustissimum *DE* transire cogitur, eius motus recte pro lineari, quem hic descripsimus, haberi potest, ob angustiam enim tubi in singulis punctis *Z* alia motus directio esse nequit, nisi quam tractus tubi permittit, ac si rem accuratius cognoscere velimus, per mediam tubi cavitatem lineam productam *DZE* concipere licet, quae motus directricem repraesentabit

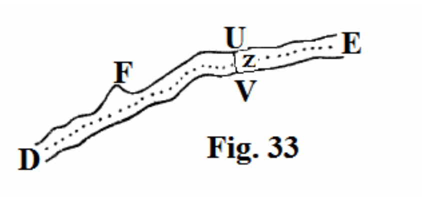

Fig. 33

et ex cuius directione in singulis punctis *Z* ipsa motus directio innotescet. Tum vero, quia tubus est angustissimus, in qualibet eius sectione *UV* ad directricem normali tam fluidi celeritas quam directio ubique erit eadem, non quod omnis inaequalitas absolute excludatur, quum utique fieri posset, ut per partem *ZU* celerius vellentius feratur quam per partem *ZV*, sed tales motus hic excludimus, dum in motum linearem inquirimus, tantum eos, qui sint definitioni consentanei, consideraturi. Quodsi vero talis inaequalitas adsit, perspicuum est tubi amplitudinem continuo magis coarctando, tandem omnem huiusmodi inaequalitatem cessare debere, quocirca, si tubos infinite angustos statuamus, huic exceptioni ne locus quidem relinquitur. Atque hoc casu etiam linea directrix non discrepat a ductu ipsius tubi perindeque erit, quodnam tubi latus pro directrice accipiatur; interim tamen non est necesse, ut tubo ubique eadem amplitudo tribuatur, quin potius insignis diversitas admitti poterit, dummodo nusquam enormis saltus occurrat, veluti eveniret, si uspiam in *F* tubi continuitas tumore interrumperetur, qui etiamsi esset infinite parvus, tamen motus non amplius legem praescriptam sequi posset, dum fluidum in tumore fere stagnaret, et reliquum perinde praeterflueret, ac si tumor ille abesset. Huiusmodi ergo irregularitates in tubo motus continuitatem perturbantes omnino sunt excludendae.

## SCHOLION 2

5. Quanquam motus linearis proprie tubos infinite angustos postulat, ne eiusmodi inaequalitates, quae huius motus indoli adversarentur, locum habere queant, tamen etiam in tubis satis amplis fieri potest, ut motus fluidi istam legem sequatur, hocque casu istiusmodi etiam motus, quantumvis tubi fuerint ampli, recte ad motum linearem referuntur. Quin etiam, etsi motus ab hac lege parumper recedat, in praxi hoc discrimen vix spectari solet, et conclusiones ex calculo deductae pro veris proxime habentur, experientia non admodum reclamante. Ita effluxus aquae ex vasis etiam amplissimis perforamen factus ex his principiis ita definiri solet, ut vix ullus dissensus ab experientia percipiatur, atque adeo a veritate eo minus aberratur, quo minus fuerit foramen, cum tamen hoc casu tota vasis strata ad directricem normalia certe non communi motu ferantur. Verum hic commode usu venit, ut vasis figura ex calculo ad finem perducto iterum egrediatur et effluxus eodem modo fieri deprehendatur, ac si vas revera haberet figuram ad modum linearem accommodatam. Etsi autem haec motus linearis consideratio amplissimum habet usum, tamen, quia in tubis amplioribus motus aliam legem sequitur, unde aberrationes, quamvis vix sentiantur, nasci debent, nostras investigationes tantum ad tubos angustissimos referri conveniet, quam ob causam etiam motus, quos hic sum consideraturus, lineares vocavi.

## SCHOLION 3

6. Tractatio haec ingentem includit varietatem ex tuborum figura oriundam, primum ergo tubos considerabo rectos, seu potius quorum directrices sint lineae rectae, quibus quidem amplitudines utcunque variabiles tribuere licet. Deinde fluidi motus sum investigaturus per tubos, quorum directrices sunt lineae curvae; quae prout fuerint velin eodem plano velsecus? in calculo aliquod discrimen pariunt, dum priori casu figura per duas tantum

coordinatas, posteriori vero per tres est definienda. Plurimum deinde interest, utrum fluidum perpetuo in huiusmodi tubis fluat, an alicubi effluat; tum vero ipsa fluidi natura et vires sollicitantes in computum sunt ducendae. Omnino autem motus determinatio ex principiis ante stabilitis peti debet, et quoniam duplici modo haec motus principia sunt evoluta, quovis casu uti conveniet eo, qui maxime accommodatus videbitur. Primum quidem tubos in quiete positos spectabo, deinceps seorsim in motum per tubos mobiles inquisiturus.

## PROBLEMA 42

7. *Motum linearem fluidi per tubum rectilineum ad calculum revocare, methodum adhibendo supra priori loco expositam.*

## SOLUTIO

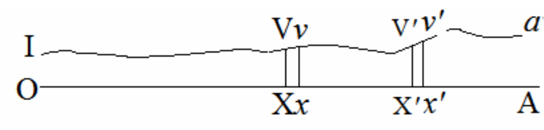

Fig.34

Sit (Fig. 34) *OIAa* tubus propositus eiusque directrix linea recta *OA*, cuius amplitudo sit utcunque variabilis. Elapso tempore $=t$ consideretur fluidi particula tubi spatiolum *XVxv* occupans, atque a termino fixo *O* ponatur distantia $OX=x$, tubique amplitudo in *X* seu sectio normaliter facta $XV=\omega$, ita ut $\omega$ sit functio data ipsius *x*. Iam per sectionem *XV* sit celeritas fluidi secundum directionem $XA=u$, densitas $=q$ et pressio $=p$; eruntque *u, q* et *p* functiones duarum variabilium *x* et *t;* tum vero ex fluidi natura datur relatio inter *p* et *q* et calorem *r,* si forte eius ratio fuerit habenda. Tribuatur huic particulae crassities $Xx=dx$ eritque eius volumen $=\omega dx$ et massa $=q\omega dx$. Iam tempusculo *dt* progrediatur haec particula in *X′ V′ x′ v′*; et cum celeritas in *X* sit $=u$, erit spatiolum $XX'=udt$; in *x* vero celeritas $=u+dx\left(\frac{du}{dx}\right)$ dabit spatiolum $xx'=udt+dtdx\left(\frac{du}{dx}\right)$, ita ut sit

$$X'x'=dx+dtdx\left(\frac{du}{dx}\right).$$

Cum autem amplitudo in *X′* sit $=\omega+udt\cdot\frac{d\omega}{dx}$, erit istius particulae volumen

$$=\omega dx+\omega dtdx\left(\frac{du}{dx}\right)+udtdx\cdot\frac{d\omega}{dx}.$$

At densitas nostrae particulae in *X′* inde colligi debet, quod, cum *q* sit functio ipsarum *x* et *t*, prior *x* incrementum capiat $XX'=udt$, posterior vero *t* incrementum *dt*, ex quo densitas in *X′* erit

$$=q+udt\left(\frac{dq}{dx}\right)+dt\left(\frac{dq}{dt}\right),$$

per quam si volumen modo inventum multiplicetur, prodit massa nostrae particulae translatae:

$$X'V'x'v' = qwdx + dxdt\left(q\omega\left(\tfrac{du}{dx}\right)+qu\tfrac{d\omega}{dx}+\omega u\left(\tfrac{dq}{dx}\right)+\omega\left(\tfrac{dq}{dt}\right)\right),$$

quae quia aequalis esse debet massae priori $q\omega dx$, haec consideratio pro motu suppeditat hanc priorem aequationem

$$q\omega\left(\tfrac{du}{dx}\right)+qu\tfrac{d\omega}{dx}+\omega u\left(\tfrac{dq}{dx}\right)+\omega\left(\tfrac{dq}{dt}\right)=0$$

seu

$$qu\tfrac{d\omega}{\omega dx}+\left(\tfrac{d\cdot qu}{dx}\right)+\left(\tfrac{dq}{dt}\right)=0.$$

Cum porro celeritas *u* post tempusculum *dt* abeat in

$$u+udt\left(\tfrac{du}{dx}\right)+dt\left(\tfrac{dq}{dt}\right),$$

quia non solum tempori *t* augmentum *dt*, sed etiam distantiae $OX = x$ augmentum $XX' = udt$ tribui debet, erit acceleratio particulae $XVxv = u\left(\tfrac{du}{dx}\right)+\left(\tfrac{du}{dt}\right)$, quae cum viribus acceleratricibus convenire debet. Ad has inveniendas primo pressio perpendatur, quae cum in facie *XV* sit *p*, erit in facie *xv*

$$= p+dx\left(\tfrac{dp}{dx}\right)\text{ob } Xx = dx.$$

Hinc nascitur vis acceleratrix particulam retro urgens $=\tfrac{1}{q}\left(\tfrac{dp}{dx}\right)$, siquidem vis motrix inde nata est $=\omega\cdot dx\left(\tfrac{dp}{dx}\right)$, quae per massam $q\omega dx$ divisa illam dat vim acceleratricem. Ex viribus porro fluidi particulas singulas immediate sollicitantibus, nascatur vis acceleratrix secundum directionem $XA = P$, ita ut iam secundum eandem directionem tota sit vis acceleratrix $= P-\tfrac{1}{q}\left(\tfrac{dp}{dx}\right)$ quae per 2*g* multiplicata accelerationi est aequanda, unde pro motu fluidi altera aequatio ita se habet:

$$2gP-\tfrac{2g}{q}\left(\tfrac{dp}{dx}\right)=u\left(\tfrac{du}{dx}\right)+\left(\tfrac{du}{dt}\right),$$

seu si tempus *t* constans accipiatur:

$$\tfrac{2gdp}{q}=2gPdx-udx\left(\tfrac{du}{dx}\right)-dx\left(\tfrac{du}{dt}\right)=2gPdx-udu-dx\left(\tfrac{du}{dt}\right).$$

Haec ergo aequatio cum ante inventa:

$$qu\tfrac{d\omega}{\omega dx}+\left(\tfrac{d\cdot qu}{dx}\right)+\left(\tfrac{dq}{dt}\right)=0.$$

coniuncta, si relatio inter *p* et *q* ex natura fluidi in subsidium vocetur, totum motum determinabit.

## COROLLARIUM 1

8. Multiplicetur etiam prior aequatio per *dx* et in eius integratione tempus *t* constans spectetur; habebiturque

$$\frac{qud\omega}{\omega}+d\cdot qu+dx\left(\frac{dq}{dt}\right)=0 \text{ seu } \omega dx\left(\frac{dq}{dt}\right)+d\cdot qu\omega=0,$$

unde, si densitas fluidi fuerit constans $q=b$, colligitur $u\omega=\text{Const.}$ quae constans tempus *t* utcunque involvere potest, unde patet celeritates in diversis tubi locis quovis tempore eius amplitudinibus reciproce esse proportionales, quae est notissima motus fluidorum per tubos proprietas.

## COROLLARIUM 2

9. Sin autem densitas fluidi *q* non fuerit constans, sed etiam a pressione *p* pendeat, necessario ambae aequationes coniungi debent, ut ex iis deinceps pro quovis loco et ad quodvis tempus *t* tam celeritas *u* quam pressio definiatur; quae investigatio propterea saepe vehementer difficilis evadit.

## SCHOLION

10. Cum hic motus sit casus maxime specialis problematis generalissimi, cuius supra duplicem solutionem exhibuimus, hicque methodo priori sim usus, necesse est, ut solutio hic inventa in illa generalissima contineatur, quod quidem de altera aequatione pressionem *p* definiente per se est perspicuum; dum enim hic praeter tempus *t* unica variabilis *x* adest, ii etiam termini, qui differentialia *dy* et *dz* complectebantur, sunt praetermissi, ubi imprimis notandum est binas celeritates *v* et *w* evanescere, propterea quod per totam sectionem *XV* motus secundum eandem directionem *OA* fieri concipitur. Altera autem aequatio ob terminum $qu\frac{d\omega}{\omega dx}$ generali formae penitus adversari videtur; amplitudine autem tubi ubique eadem existente res egregie convenit. Verum tamen re bene perpensa et haec aequatio immediate ex forma generali deduci potest. Si enim tubus ab *V* ad *V'* divergat, motus directio circa *V* aliquantillum a directione *OA* deflectere debet. Posita ergo $XV=y$, statuatur celeritas $v=\alpha y$, ut in *X* nulla sit deflexio; ut tertia autem celeritas $\omega$ evanescat, sumatur $z=\text{const.}=\gamma$, ut sit $\omega=\gamma y$. Quia autem densitas *q* plane non ab *y* pendet, erit $\left(\frac{d\cdot qv}{dy}\right)=\alpha q;$ ; at quia y est functio ipsius *x*, erit $X'V'=y+udt\frac{dy}{dx}$, tum vero ex celeritate *v* fit etiam

$$X'V'=y+vdt=y+\alpha ydt,$$

unde fit

$$\alpha=u\frac{dy}{ydx}=u\frac{d\omega}{\omega dx} \text{ ideoque } \left(\frac{d\cdot qv}{dy}\right)=qu\frac{d\omega}{\omega dx}.$$

Hinc ergo clare intelligitur, quomodo solutio specialis hic data pulcerrime cum aequatione generali cohaereat atque ex ea nascatur.

## PROBLEMA 43

11. *Praecedens problema, quo motus fluidi in tubo rectilineo quaeritur, per methodum posteriorem respectu ad statum initialem habito resolvere.*

## SOLUTIO

Fluidi particula (Fig. 35), cuius motum investigamus, initio $t = 0$ occupaverit tubi elementum *XVX'V'*, pro quo ponamus $OX = X$, $XX' = dX$, amplitudinem tubi in $X = \Omega$, ut volumen particulae sit $= \Omega dX$. Sit porro densitas in $x = Q$, pressio $= P$ et celeritas secundum directionem $XA = U$, quae omnibus particulae punctis est communis; unde eius massa $= Q\Omega dX$. Iam elapso tempore *t* eadem particula reperiatur in *xvx'v'*, pro qua ponamus $Ox = x$, amplitudinem $xv = \omega$, tum vero densitatem in $x = q$, pressionem *p* et celeritatem secundum $xA = u$, eruntque litterae *x, q, p, u* functiones binarum variabilium *X* et *t*, at amplitudo west certa functio ipsius *x* ex tubi figura definienda. Cum iam sectio *X' V'* eodem tempore *t* pervenerit in *x'v'*, erit

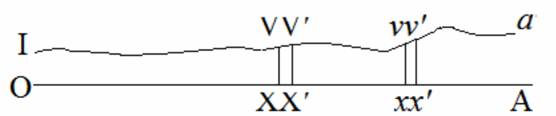

Fig.35

$$xx' = dX\left(\tfrac{dx}{dX}\right);$$

unde particulae *xvx'v'* volumen erit $\omega dX\left(\tfrac{dx}{dX}\right)$ et massa $= q\omega dX\left(\tfrac{dx}{dX}\right)$, quae cum semper maneat eadem, habebimus pro motus determinatione hanc primam aequationem:

$$q\omega\left(\tfrac{dx}{dX}\right) = Q\Omega.$$

Cum deinde celeritas in *x* sit $u = \left(\tfrac{dx}{dt}\right)$, erit particulae *xvx'v'* acceleratio $= \left(\tfrac{ddx}{dt^2}\right)$. Ex viribus sollicitantibus nascatur pro hac particula vis acceleratrix secundum directionem $xA = \mathfrak{P}$; tum vero, quia in *x'* pressio est $= p + dX\left(\tfrac{dp}{dX}\right)$, hinc oritur vis motrix retro urgens $= \omega dX\left(\tfrac{dp}{dX}\right)$, quae per massam $q\omega dX\left(\tfrac{dx}{dX}\right)$ divisa dat vim acceleratricem

$$= \frac{1}{q\left(\frac{dx}{dX}\right)}\left(\frac{dp}{dX}\right)$$

ab illa $\mathfrak{P}$ subtrahendam, unde altera aequatio motus determinationem continens colligitur:

$$\left(\frac{ddx}{dt^2}\right) = 2g\mathfrak{P} - \frac{2g}{q\left(\frac{dx}{dX}\right)}\left(\frac{dp}{dX}\right),$$

quae sumto tempore *t* constante abit in hanc:

$$\frac{2gdp}{q} = 2g\mathfrak{P}dx - dx\left(\frac{ddx}{dt^2}\right).$$

COROLLARIUM 1

12. Perspicuum est quantitates *x, q, p* et *u* eiusmodi functiones esse debere binarum variabilium *X* et *t,* ut facto $t = 0$ fiat $x = X$, $q = Q$, $p = P$ et $u = U$*;* quibus conditionibus per integrationem est satisfaciendum.

COROLLARIUM 2

13. Amplitudo $\omega$ ut functio data spectatur quantitatis $Ox = x$, quae cum ipsa sit functio ipsarum *X* et *t,* eatenus amplitudo $\omega$ a tempore pendere est censenda.

SCHOLION

14. Quia hoc problema continet casum maxime specialem problematis generalissimi 41, etiam solutionem hic datam in illa generalissima contentam esse oportet, quod quidem de aequatione posteriori statim patet, quippe qui ex forma generali nascitur, si termini, qui ibi binas variabiles *Y* et *Z* involvunt, expungantur. Quemadmodum autem aequatio prior cum solutione generali conveniat, minus clare perspicitur. Consensus autem iterum ostendi potest, si modo ad amplitudinis tubi variationem rite respiciatur. Hunc in finem spectentur ternae tubi dimensiones, et si binae *Y* et *Z* prae $OX = X$ sunt infinite parvae ac *Z* quidem ut constantis per totum tubum magnitudinis consideretur, ut sit $z = Z$ ideoque $\left(\frac{dz}{dZ}\right) = 1$, tum vero erit $\Omega = YZ$. Statuatur $y = LY$, existante *L* functione solius temporis *t,* fietque $\left(\frac{dy}{dY}\right) = L$ et $\left(\frac{dy}{dX}\right)$ cum reliquis formulis differentialibus evanescit, atque amplitudo in *x* erit $\omega = LYZ = L\Omega$. Nunc ex solutione generali colligitur valor $K = \left(\frac{dx}{dX}\right)L \cdot 1$, qui ob $L = \frac{\omega}{\Omega}$ abit in $K = \frac{\omega}{\Omega}\left(\frac{dx}{dX}\right)$: quo invento manifesto est $Kq = Q$, ut solutio habet generalis; sicque tota haec solutio in generali continetur ex eaque ut casus specialis derivari potest.

PROBLEMA 44

15. *Si* (Fig. 36) *tubi directrix IYK sit linea curva quaecunque in eodem plano posita eiusque amplitudo YV per tubi longitudinem utcunque sit variabilis, per methodum priorem supra traditam motum fluidi cuiuscunque in hoc tubo investigare.*

SOLUTIO

Elapso tempore $=t$ consideretur fluidi particula in tubo occupans spatiolum *YVyv*, ac pro puncto *Y* statuantur binae coordinatae $OX = x,\ \ XY = y$, quarum relatio mutua cum ex figura directricis tubi detur, erit *y* functio ipsius *x;* tubi vero amplitudo in *Y* minima ponatur $YV = \omega$, quae Fig. 36 etiam ut functio ipsius *x* data est spectanda. Sit porro densitas fluidi in et pressio $= p$, quae duae quantitates erunt functiones duarum variabilium *x* et *t*, perinde ac celeritas fluidi in tubo, quae sit $=\mathfrak{T}$, cuius directio cum sit *Yy*, si hoc elementum brevitatis gratia vocemus

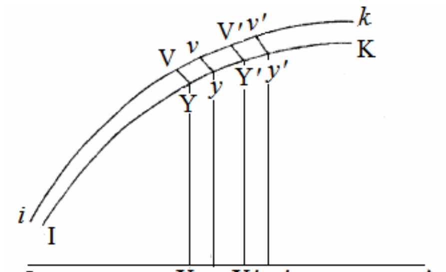

Fig. 36

$Yy = \sqrt{\left(dx^2 + dy^2\right)} = ds$, erit celeritas secundum $OX = \frac{\mathfrak{T}dx}{ds} = u$ et secundum $= XY = \frac{\mathfrak{T}dy}{ds} = v$. Quia nunc sectio $YV = \omega$ ad directricem est normalis, volumen nostrae particulae *YVyv* erit $=\omega ds$ et massa $=q\omega ds$. Iam tempusculo *dt* progrediatur haec particula in *Y′V′y′v′*, ita ut elementum *Y* pervenerit in *Y′*, eritque spatiolum

$$XX' = udt = \tfrac{\mathfrak{T}dx}{ds}\cdot dt \ \text{ et } \ X'Y' - XY = vdt = \tfrac{\mathfrak{T}dy}{ds}\cdot dt\,.$$

Cum autem in *y* essent celeritates $u + dx\left(\frac{du}{dx}\right)$ et $v + dx\left(\frac{dv}{dx}\right)$, erit progressu temporis

$$xx' = udt + dtdx\left(\tfrac{du}{dx}\right) \ \text{ et } \ x'y' - xy = vdt + dtdx\left(\tfrac{dv}{dx}\right),$$

hincque

$$X'x' = dx + dtdx\left(\tfrac{du}{dx}\right) \ \text{ et } \ x'y' - X'Y' = dy + dtdx\left(\tfrac{dv}{dx}\right).$$

Quamobrem habebimus:

$$Y'y' = ds + \tfrac{dtdx^2}{ds}\left(\tfrac{du}{dx}\right) + \tfrac{dtdxdy}{ds}\left(\tfrac{dv}{dx}\right).$$

Deinde in *Y′* est amplitudo $Y'y' = \omega + udt\cdot\frac{d\omega}{dx}$ et densitas

$$= q + udt\left(\tfrac{dq}{dx}\right) + dt\left(\tfrac{dq}{dt}\right);$$

ex quo concluditur particulae *Y′V′y′v′* volumen

$$= \omega ds + \frac{\omega dt dx^2}{ds}\left(\frac{du}{dx}\right) + \frac{\omega dt dx dy}{ds}\left(\frac{dv}{dx}\right) + udtds\,\frac{d\omega}{dx}$$

et massa

$$= q\omega ds + \frac{q\omega dt dx^2}{ds}\left(\frac{du}{dx}\right) + \frac{q\omega dt dx dy}{ds}\left(\frac{dv}{dx}\right) + qudtds\,\frac{d\omega}{dx}$$
$$+ u\omega dtds\left(\frac{dq}{dx}\right) + \omega dtds\left(\frac{dq}{dt}\right),$$

quae cum praecedenti $q\omega ds$ aequalis esse debeat, per $\omega dtds$ dividendo perveniemus ad hanc aequationem:

$$\frac{qdx^2}{ds^2}\left(\frac{du}{dx}\right) + \frac{qdxdy}{ds^2}\left(\frac{dv}{dx}\right) + qu\,\frac{d\omega}{\omega dx} + u\left(\frac{dq}{dx}\right) + \left(\frac{dq}{dt}\right) = 0\,;$$

substituantur autem hic valores $u = \frac{\mathfrak{T}dx}{ds}$, $v = \frac{\mathfrak{T}dy}{ds}$ et, quia $\mathfrak{T}$ est functio ipsarum $t$ et $x$, fractiones vero $\frac{dx}{ds}$ et $\frac{dy}{ds}$ a sola $x$ pendent, pervenietur ad hanc aequationem :

$$q\mathfrak{T}\,\frac{d\omega}{\omega ds} + \frac{dx}{ds}\left(\frac{d\cdot q\mathfrak{T}}{dx}\right) + \left(\frac{dq}{dt}\right) = 0\,.$$

Deinde cum celeritates in $Y'$ secundum directiones $OX$ et $XY$ sint

$$u + udt\left(\frac{du}{dx}\right) + dt\left(\frac{du}{dt}\right) \text{ et } v + udt\left(\frac{dv}{dx}\right) + dt\left(\frac{dv}{dt}\right),$$

erunt accelerationes:

$$u\left(\frac{du}{dx}\right) + \left(\frac{du}{dt}\right) \text{ et } u\left(\frac{dv}{dx}\right) + \left(\frac{dv}{dt}\right).$$

Ex viribus nunc sollicitantibus ponamus secundum easdem directiones resultare vires acceleratrices $P$ et $Q$. Postea vero ob pressionem in $y = p + dx\left(\frac{dp}{dx}\right)$, elementum $YVyv$ in directione $yY$ retro urgetur vi acceleratrice $= \frac{dx}{qds}\left(\frac{dp}{dx}\right)$, unde nascuntur vires secundum directiones $OX$ et $XY$ hae:

$$P - \frac{dx^2}{qds^2}\left(\frac{dp}{dx}\right) \text{ et } Q - \frac{dxdy}{qds^2}\left(\frac{dp}{dx}\right)$$

hincque porro istae aequationes:

$$2gP - \frac{2gdx^2}{qds^2}\left(\frac{dp}{dx}\right) = u\left(\frac{du}{dx}\right) + \left(\frac{du}{dt}\right) = \frac{\mathfrak{T}\mathfrak{T}}{ds}\,d\cdot\frac{dx}{ds} + \frac{\mathfrak{T}dx^2}{ds^2}\left(\frac{d\mathfrak{T}}{dx}\right) + \frac{dx}{ds}\left(\frac{d\mathfrak{T}}{dt}\right),$$
$$2gQ - \frac{2gdxdy}{qds^2}\left(\frac{dp}{dx}\right) = u\left(\frac{dv}{dx}\right) + \left(\frac{dv}{dt}\right) = \frac{\mathfrak{T}\mathfrak{T}}{ds}\,d\cdot\frac{dy}{ds} + \frac{\mathfrak{T}dxdy}{ds^2}\left(\frac{d\mathfrak{T}}{dx}\right) + \frac{dy}{ds}\left(\frac{d\mathfrak{T}}{dt}\right),$$

a quarum prima per $dy$ multiplicata si subtrahatur altera in $dx$ ducta, relinquitur:

$$2g\left(Pdy-Qdx\right)=\tfrac{\mathfrak{T}\mathfrak{T}}{ds}\left(\tfrac{dyddx-dxddy}{ds}\right)=\mathfrak{T}\mathfrak{T}d\cdot\text{Ang tang}\,\tfrac{dx}{dy}.$$

Praeterea vero hinc colligitur:

$$2g(Pdx+Qdy)-\tfrac{2gdx}{q}\left(\tfrac{dp}{dx}\right)=\mathfrak{T}dx\left(\tfrac{d\mathfrak{T}}{dx}\right)+\ ds\left(\tfrac{d\mathfrak{T}}{dt}\right).$$

Prior autem aequatio penitus hinc est excludenda, propterea quod ex pressione solam accelerationem secundum motus directionem computavimus, dum inde alia quoque nasceretur a lateribus tubi excepta et destructa. Vires enim *P* et *Q* aliter motum non afficiunt, nisi quatenus secundum directionem motus *Yy* agunt, ex quo sola aequatio posterior in calculo relinquitur, quae, si tempus *t* constans statuatur, pressionem *p* ita definit, ut sit

$$\tfrac{2gdp}{q}=2g(Pdx+Qdy)-\mathfrak{T}d\mathfrak{T}\left(\tfrac{dp}{dx}\right)-ds\left(\tfrac{d\mathfrak{T}}{dt}\right).$$

### COROLLARIUM 1

16. Patet ergo curvaturam lineae directricis nihil in motu fluidi turbare, dum formularum $\tfrac{dx}{ds}$ et $\tfrac{dy}{ds}$ differentialia, quibus curvatura definitur, penitus ex calculo evanuerunt.

### COROLLARIUM 2

17. Prior quoque aequatio inventa ad formam commodiorem reduci potest. Cum enim *s* sit functio ipsius *x* tantum, perinde est, sive $\mathfrak{T}$ consideretur ut functio ipsarum *x* et *t*, sive ipsarum *s* et *t:* ex quo erit

$$dx\left(\tfrac{d\mathfrak{T}}{dx}\right)=ds\left(\tfrac{d\mathfrak{T}}{ds}\right)$$

et ob eandem rationem

$$dx\left(\tfrac{d\cdot q\mathfrak{T}}{dx}\right)=ds\left(\tfrac{d\cdot q\mathfrak{T}}{ds}\right)$$

unde prior aequatio abit in hanc formam:

$$q\mathfrak{T}\tfrac{d\omega}{\omega ds}+\left(\tfrac{d\cdot q\mathfrak{T}}{ds}\right)+\left(\tfrac{dq}{dt}\right)=0$$

et porro in hanc: $\left(\tfrac{d\cdot q\mathfrak{T}\omega}{ds}\right)+\omega\left(\tfrac{dq}{dt}\right)=0$.

### SCHOLION 1

18. Solutionem huius problematis ideo per tantas ambages minus necessarias deduxi, quo clarius appareat curvaturam lineae directricis nihil plane in motu fluidi turbare; quod principium si statim stabilire voluissem, merita id in dubium vocare licuisset. Nunc autem demum certo agnoscimus, quidquid de viribus ad motum fluidi inflectendum insumitur,

id quasi a tubi lateribus absorberi, ita ut eaedem prodeant motus determinationes, ac si tubus esset rectilineus. De viribus tantum sollicitantibus observandum est pro quovis fluidi elemento inde eam solum vim acceleratricem elici debere, quae secundum ipsam motus directionem agat, reliquis viribus plane neglectis, quippe quae totae in latera tubi impenduntur. Hoc notato evidens est solutionem huius problematis plane non discrepare a problemate 42 ex eoque statim derivari potuisse, tantum loco $x$ et $u$ scribendo $s$ et $\mathfrak{T}$, et loco vis $P$ hanc $P\frac{dx}{ds}+Q\frac{dy}{ds}$. Hoc ergo compendio iam animadverso multo facilius· sequentia problemata resolvemus.

## SCHOLION 2

19. Ne ulli dubio locus relinquatur, clarius declarandum videtur, cur in aestimatione pressionum, quibus particula fluidi *YVyv* urgetur, nullam rationem inaequalitatis basium *yv* et *YV* habuerim, dum tamen in reliquis investigationibus ad eam tam sollicite respici oporteat cum enim posita amplitudine $YV=\omega$ amplitudo *yv* utique fiat $=\omega+dx\left(\frac{d\omega}{dx}\right)$ in eamque pressio agat $=p+dx\left(\frac{dp}{dx}\right)$, tota pressio, quam basis *yv* sustentat, fit

$$p\omega+pdx\left(\frac{d\omega}{dx}\right)+\omega dx\left(\frac{dp}{dx}\right),$$

dum ea, quam basis *YV* sustinet, tantum est $=p\omega$ sicque vis retro pellens maior foret, quam in solutione assumseram, quae eadem difficultas etiam praecedentia problemata premere videtur. Verum hoc dubium facile diluitur, si ex iis, quae supra de indole pressionum sunt tradita, recordemur omnes pressiones, quae per aequales altitudines repraesentantur, se mutuo in aequilibrio tenere, etiamsi in bases maxime inaequales agant; hinc illius vis $p+dx\left(\frac{dp}{dx}\right)$, quam basis *xy* sustinet, pars *p* plane est in aequilibrio cum vi *p* basin *YV* urgente, etiamsi basis *xy* maxime foret inaequalis huic *YV*, ex quo pressionis illius, qua basin *yv* impelli vidimus, non sola pars $p\omega$, sed haec $p\omega+pdx\left(\frac{d\omega}{dx}\right)$ a pressione opposita destruitur, ita ut excessus ex sola parte $\omega dx\left(\frac{d\omega}{dx}\right)$ aestimari debeat, prorsus uti in horum problematum solutionibus feci. Deinde si cui dubium adhuc videatur, quomodo in evolutione aequationum inventarum differentialia secundi gradus ex calculo evanescant, ita ut sit

$$\frac{\mathfrak{T}\mathfrak{T}}{ds}dxd\frac{dx}{ds}+\frac{\mathfrak{T}\mathfrak{T}}{ds}dyd\frac{dy}{ds}=0,$$

is has formulas tantum evolvat, ac reperiet

$$\frac{\mathfrak{T}\mathfrak{T}}{ds}\left(dx\frac{ddx}{ds}-\frac{dx^2dds}{ds^2}+dy\frac{ddy}{ds}-\frac{dy^2dds}{ds^2}\right),$$

quae forma ob

$$dx^2+dy^2=ds^2 \quad \text{et} \quad dxddx+dyddy=dsdds$$

abit in

$$\frac{\mathfrak{T}\mathfrak{T}}{ds}\left(\frac{dsdds}{ds}-\frac{ds^2dds}{ds^2}\right)=0.$$

PROBLEMA 45

20. *Si* (Fig. 37) *tubi directrix sit IYK linea curva in eodem plano posita, cuius amplitudo per longitudinem tubi utcunque sit variabilis, per methodum posteriorem supra traditam respectu ad statum initialem habito, motum fluidi cuiuscunque in hoc tubo definire.*

SOLUTIO

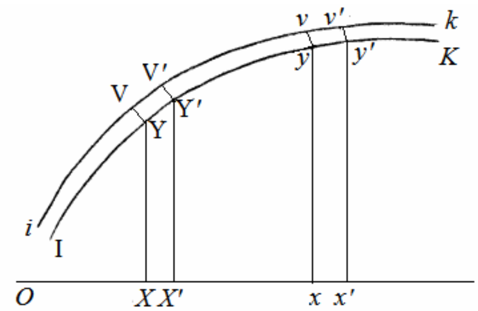

Fig. 37

In statu initiali consideremus fluidi particulam *YVY'V'* et pro puncto *Y* positis coordinatis $OX = X$, $XY = Y$ sit ipse arcus $IY = S$ et in *Y* amplitudo tubi $YV = \Omega$, quae, sive ut functio ipsius *X* spectetur sive ipsius *S*, perinde est. Tum vero sit densitas in $Y = Q$, pressio $= P$ et celeritas secundum tubi directionem $YY' = T$, si iam capiatur tubi elementum $YY' = dS$, erit particulae nostrae volumen $= \Omega dS$ et massa $= Q\Omega dS$. His, quae ad statum initialem pertinent, positis tempore $= t$ nostra particula transferatur in *yvy'v'* et pro puncto *y vocetur* $Ox = x$, $xy = y$ et arcus directricis $Iy = s$, tum vero densitas $= q$ et pressio $= p$, quae quantitates omnes sunt functiones duarum variabilium *S* et *t* eaeque tales, ut posito tempore $t = 0$ fiat $x = X$, $y = Y$, $s = S$, $q = Q$ et $p = P$, amplitudo vero tubi in *y* sit $yv = \omega$ functioni ipsius *8.* Quodsi iam ratiocinium instituatur ut in problemate 43, quoniam tubi curvatura nihil turbat in motu fluidi, loco quantitatis *X* ibi tubi IONgitudinem denotantis hic litteram *S* scribi oportet. Ex viribus autem sollicitantibus, si eae vires acceleratrices, quae secundum coordinatarum *Ox* et *xy* directiones agunt, sint $\mathfrak{P}$ et $\mathfrak{Q}$, ea, quae in *y* secundum tubi directionem urget, erit $\frac{\mathfrak{P}dx+\mathfrak{Q}dy}{ds}$ atque hinc motus fluidi sequentibus duabus aequationibus exprimetur

$$q\omega\left(\frac{ds}{dS}\right) = Q\Omega \text{ et } \frac{2gdp}{q} = 2g\left(\mathfrak{P}dx+\mathfrak{Q}dy\right) - ds\left(\frac{dds}{dt^2}\right),$$

in qua posteriori tempus *t* constans est assumtum.

COROLLARIUM 1

21. Denotat ergo $\left(\frac{ds}{dt}\right)$ celeritatem fluidi in tubi puncto *y* elapso tempore $= t$, quam ergo ita comparatum esse oportet, ut facto $t = 0$ fiat $\left(\frac{ds}{dt}\right) = \Upsilon$, quippe quae celeritas in statu initiali ut est spectanda.

COROLLARIUM 2

22. Curvamen ergo tubi tantum in effectu virium sollicitantium variationem parit, quoniam pro quavis fluidi particula ex viribus sollicitantibus ea tantum vis acceleratrix colligi debet, quae secundum tubi directionem agit.

SCHOLION

23. Animadverti hic convenit in statu initiali neque celeritatem $\Upsilon$ neque pressionem *P* pro lubitu fingi posse, cum enim hae quantitates in aequationes pro motu inventas non ingrediantur, eas ita comparatas esse oportet, ut, postquam aequationes inventae fuerint integratae, ex valoribus ipsarum $\mathfrak{T}$; et *p* posito tempore $t = 0$ illae quantitates oriantur. Quanquam autem integratio maximam amplitudinem secum importat, tamen effici nequit, ut in statu initiali pro singulis elementis illae ambae quantitates $\Upsilon$ et *P* datos valores obtineant. Quodsi enim fluidum nullius compressionis sit capax, statim atque in unica tubi sectione celeritas datur, simul in omnibus reliquis determinatur, tum vero etiam per pressionem in unico tubi loco pressiones in omnibus reliquis determinantur. Unde satis intelligitur, etiamsi densitas fuerit variabilis, quia semper cum celeritate et pressione certo modo cohaeret, tamen non in singulis locis pro statu initiali celeritatem et pressionem pro lubitu fingi posse.

PROBLEMA 46

24. *Si* (Fig. 38) *tubi directrix IZzK fuerit linea curva quaecunque non in eodem plano sita eiusque amplitudo utcunque variabilis, definire motum fluidi in huiusmodi tubo secundum methodum priorem supra expositam.*

SOLUTIO

Hac methodo utentes statim consideramus fluidi statum ad tempus quodcunque indefinitum $= t$ a certo initio elapsum. Definita igitur directrice per ternas coordinatas $Ox = x,\ xy = y,\ yz = z,$ quae ita a se invicem pendent, ut pro unica variabili haberi queant, statuamus praeterea arcum directricis $Iz = s$, et in *z* sit tubi amplitudo $zv = \omega$, quae etiam ut functio ipsius *s* est spectanda. In hoc iam loco *z* ad tempus $= t$ contemplamur fluidi particulam, cuius densitas sit $= q$, pressio $= p$ et celeritas

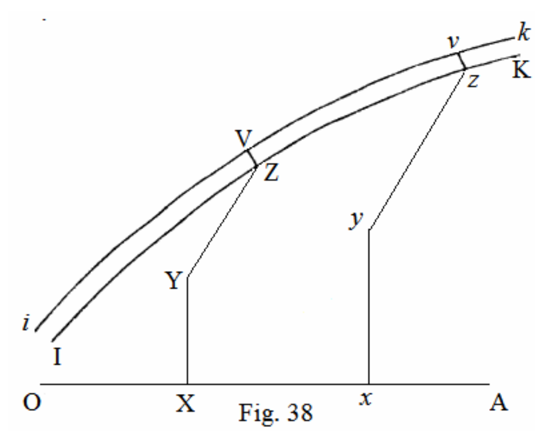

Fig. 38

secundum tubi directionem $zK = \mathfrak{T}$; quae quantitates sunt functiones duarum variabilium ipsius *s* scilicet et temporis *t.* Ex viribus denique sollicitantibus oriantur pro puncto *z* hae tres vires acceleratrices *P, Q, R* secundum directiones coordinatarum *Ox, xy* et *yz;* ex quibus porro vis secundum tubi directionem *zK* accelerans colligitur

$$\frac{Pdx+Qdy+Rdz}{ds}.$$

His positis cum tubi curvatura in motu fluidi nihil immutet, motus quaesitus sequentibus duabus aequationibus exprimetur, quarum prior relationem inter densitatem, celeritatem et amplitudinem tubi definit et ita se habet:

$$q\mathfrak{T}\frac{d\omega}{\omega ds}+\left(\frac{d\cdot q\mathfrak{T}}{ds}\right)+\left(\frac{dq}{dt}\right)=0 \text{ seu } \left(\frac{d\cdot q\mathfrak{T}\omega}{ds}\right)+\omega\left(\frac{dq}{dt}\right)=0,$$

altera vero praeterea pressionem involvit, in eaque tempus $t$ ut constans spectatur:

$$\frac{2gdp}{q}=2g(Pdx+Qdy+Rdz)-\mathfrak{T}d\mathfrak{T}-ds\left(\frac{d\mathfrak{T}}{dt}\right).$$

PROBLEMA 47

25. *Si* (Fig. 38) *tubi directrix IZzK sit linea curva quaecunque non in eodem plano posita eiusque amplitudo utcunque variabilis, definire motum fluidi in huiusmodi tubo secundum methodum posteriorem respectu ad statum initialem habito.*

SOLUTIO

Dum singula directricis puncta $Z$ per ternas coordinatas definiuntur $OX=X$, $XY=Y$ et $YZ=Z$, ponatur insuper IONgitudo arcus $IZ=S$ et tubi amplitudo in $Z=\Omega$. Consideretur iam fluidi particula quaecunque, quae in statu initiali, ubi erat tempus $t=0$, erat in $Z$, eiusque densitas $=Q$, tum vero pressio $=P$ et celeritas secundum tubi directionem $ZK=\Upsilon$; quae ergo quantitates ut datae et functiones unius variabilis $S$ spectari possunt. Nunc elapso tempore $=t$ eadem particula pervenerit in tubi punctum $z$, coordinatis $Ox=x$, $xy=y$ et $yz=z$ definitum, ubi sit arcus $Iz=s$ et tubi amplitudo $z=\omega$, quae est functio ipsius $s$, tum vero in $z$ sit fluidi densitas $=q$, pressio $=p$ et celeritas secundum $zK=\Upsilon$, quam per has denominationes novimus esse $\mathfrak{T}=\left(\frac{ds}{dt}\right)$, haeque quantitates omnes ut functiones duarum variabilium $S$ et $t$ sunt spectandae. Denique ex resolutione virium particulam in $z$ sollicitantium deriventur secundum directiones $Ox$, $xy$ et $yz$ hae tres vires acceleratrices $\mathfrak{P}$, $\mathfrak{Q}$ et $\mathfrak{R}$. Quibus positis cum tubi curvatura nihil in nostra investigatione perturbet, habebimus uti in problemate 45 pro motu fluidi in hoc tubo has duas aequationes:

$$q\omega\left(\frac{ds}{dS}\right)=Q\Omega$$

et

$$\frac{2gdp}{q}=2g\left(\mathfrak{P}dx+\mathfrak{Q}dy+\mathfrak{R}dz\right)-ds\left(\frac{dds}{dt^2}\right)$$

in qua posteriori tempus $t$ constans est assumtum.

SCHOLl ON

26. Omnia haec problemata duplici modo dedimus soluta, dum utramque methodum in praecedente sectione expositam adhibuimus. Solutiones quidem hae geminae ratione formae plurimum discrepant, verumtamen, quin semper egregie inter se conveniant, nullo modo dubitari potest. Prout autem quaestiones fuerint comparatae, modo magis expediet uti solutione priori, modo posteriori; semper autem utramque adhibendo non solum earum consensus veritatem eo magis confirmabit, sed etiam insignes dilucidationes suppeditabit, unde veram motus naturam eo accuratius cognoscemus. Huius autem tractationis primariam divisionem praebet fluidorum diversitas, quatenus eorum densitas vel est constans vel variabilis, quorsum addi potest fluidum mixtum, veluti si fluidi continuitas in tubo bullis aereis fuerit interrupta. Virium sollicitantium ratio tam parum afficit hanc tractationem, ut vix operae pretium sit alias vires praeter gravitatem considerari, neque hic etiam quicquam impedit, si forte actio virium $Pdx+Qdy+Rdz$ integrationem non admittat, hic enim $x$, $y$ et $z$ a se invicem pendent et unicam variabilem constituere sunt censendae, quamobrem illa difficultas locum habere nequit. At vero amplitudinis tubi variatio tantopere in calculum influit, ut maxime conducat hinc divisionem petere; ex quo primo tubos eiusdem ubique amplitudinis sum contemplaturus. Denique omnem fluidorum diversitatem ad duas species aquae et aëris revocare licet.