Principia Motus Fluidorum
L. Euler *E*258 :Translated & Annotated by Ian Bruce. (Sept., 2020)

PRINCIPLES OF THE MOTION OF FLUIDS

E 258

Novi commentarii academiae scientiarum Petropolitanae 6 (1756/7), 1761, p. 271-311

FIRST PART

1. Since fluid bodies shall differ from solids chiefly by this means, in so far as their particles shall be free from each other generally, also they can receive the most diverse kinds of motion, yet the motion, by which each and every particle of the fluid is carried along, thus may be determined from the motion of the remaining particles, so that it shall not be possible for them to be progressing by another motion. Moreover the matter may be had by far otherwise in solid bodies, which if they were inflexible and in no way would they have been allowed to change their shapes, in whatever manner they may be moved, the individual particles of these always maintain the same position and distance between themselves ; from which it happens, so that, from the known motion of only two or three particles, at once the motion of any particle may be able to be defined ; nor also can the motion of two or three particles of a body of this kind be devised as it pleases, for thus it must be established, so that these particles may always maintain the same relative position between themselves.

2. But if the solid bodies were flexible, the motion of the individual particles is less well determined: since on account of being flexible both the separation as well as the relative position of the different particles may permit changes. Yet meanwhile the account of these flexures establishes a certain law, which the diverse particles of this kind must follow in their motion: certainly for which it will be required to take precautions, lest the parts, which may be allowed to be bent about each other, in turn either may be torn apart completely or forced into each other; from which indeed the hardness of all the latter common bodies is required.

3. But for fluid bodies, of which the particles are not joined together by any connection, also the motion of the different particles is much less restricted : nor from the motion of any given particle may the motion of the rest be determined. Indeed even if the motion of as many as a hundred particles may be assumed as known, it is evident the motion which the remaining particles may be going to take, at this stage is able to be varied indefinitely. From which it may be considered to be concluded the motion of each particle of the fluid plainly does not depend on the motion of the remaining particles, unless perhaps there were a dependence on these, so that by necessity it may be forced to follow these others.

4. Yet meanwhile it cannot be the case, that the motion of all the fluid particles may be entirely free from being combined together by some laws ; nor thus will it be permitted to fashion some desired motion, which may be considered for the particles present. Indeed since the particles are impenetrable, it is apparent at once a motion of this kind cannot be

sustained, where other particles may be able to pass through others, and thus may penetrate into each other: and for this reason, indeed such a motion cannot be considered to be present in fluids. Therefore since it will be required to exclude infinite motions, the remaining motions of which shall be prepared with this agreed on, and may be distinguished from these by which property, it will be seen to be worth the effort to define more accurately.

5. Indeed before the motion, by which some fluid actually may be disturbed, may be able to be designated, it is considered necessary that all motions may be discerned, which indeed may be able to be sustained in this fluid : which motions I call here possible, so that I may distinguish these from the impossible motions, which indeed do not have to be considered in the discussion. To this end a characteristic will be required to be established by us, agreeing with all possible motions, and removing these from the impossible ones ; with which done it will be necessary for some possible case to be determined from the possible motions, which actually will be required to be present. Then evidently it will be required for the forces to be considered, by which the water may be acted on, so that the motion, which shall be in agreement with these, may be possible to be defined from the principles of mechanics.

6. Therefore in accordance with any possible characteristic of the motion, evidently I will search for a safe impenetrable fluid to be established here. Moreover I assume a fluid of this nature, so that neither shall it be allowed to be forced into a smaller space, nor may it be possible for its continuity to be broken: without doubt I establish no space devoid of fluid to be left within a fluid with an enduring motion, but in that continuity to be conserved. Indeed the theory will be applied to fluids of this kind, thus without difficulty that will be extended also to fluids, the density of which is variable, and which indeed will not be required by necessity to be continuous.

7. Therefore if some portion of a fluid of this kind may be considered, the motions, by which its individual particles are carried along, must be prepared thus, so that for all equal times equal volumes may be filled. Indeed if this may happen in the individual parts, all changes may be constrained from being either an expansion into a greater volume or a contraction into a smaller volume ; and motion of this kind generally will be required to be had for every motion possible, if we may consider it only with regard to this characteristic, by which a fluid capable neither of expansion or contraction may be put in place. So that moreover it has been said regarding any part of the fluid, it is required to understand concerning its individual elements, that the volume of each element always must remain of the same amount.

8. Therefore so that this condition may be satisfied, any motion of any individual points of the fluid may be considered to be present; then for any element of the fluid taken the momentary translation of its individual boundaries may be investigated, and thus the element of the volume will become known, into which this element will be contained in the smallest increment of the time taken. Then this increment of the volume may be put equal to that increment, which it occupies before, and this equation will provide an

account of the motion as far as possible. So that indeed if the individual elements may occupy equal elements of volume in equal times, neither any compression or expansion of the fluid will arise ; and the motion will be prepared thus as required as far as possible.

9. But since here not only the speed of the motion must be considered, which may be considered for the individual points of the fluid present, but also its direction, this may be established for each with the most convenient consideration, if the motion of each point may be resolved along fixed directions. Moreover this resolution is accustomed to be resolved along either two or three directions : indeed with the first resolution allowed to be used if the motion of the points may be resolved in the same plane ; but if the motion of these may not be contained in the same plane, then it will be required to resolve the second motion along three fixed axes. Therefore since this latter case will have more difficulty than the former, it will be agreed the investigation of the possible motion to begin from the former case, from which the derivation of the latter case may be derived more easily.

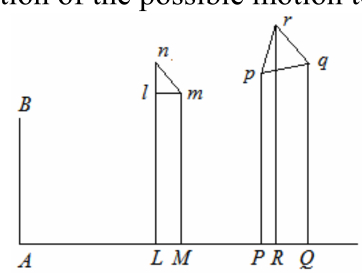

Fig. 1

10. At first therefore, I will attribute only two dimensions to the fluid, thus so that indeed not only now may its individual particles be found in the same plane, but also the motion of these may be resolved in the same plane. And thus this plane will be represented by the plane of the table (Fig.1) and some point l of the fluid may be considered, the position of which may be referred to by the coordinates $AL = x$ and $Ll = y$, then truly so that now its motion may be performed, may be shown resolved along the same directions where now indeed it is carried, following the same resolved directions the speed may be provided following the axis AL or following $lm = u$, or following the second axis AB, or following $ln = v$; thus so that the true speed of this point shall become $= \sqrt{(uu+vv)}$ and its direction inclined to the axis AL shall be by the angle, of which the tangent $= \frac{v}{u}$.

11. Since only the present motion stands, which shall agree with the individual points of the fluid, the proposition shall be established, the speeds u and v will depend especially on the position of the point l and therefore are required to be considered as if functions of the coordinates x and y. Therefore we may establish by differentiation :

$$du = Ldx + ldy \text{ and } dv = Mdx + mdy,$$

which differential formulas since they shall be complete, there shall be agreed to become

$$\frac{dL}{dy} = \frac{dl}{dx} \text{ and } \frac{dM}{dy} = \frac{dm}{dx},$$

where it is required to observe the differential $\frac{dL}{dy}$ of L or dL in an expression of this kind, only to be taken from the variation of y, and in a similar manner in the expression $\frac{dl}{dx}$ for dl this differential of l itself must be taken, which arises only if x may be taken for the variable.

12. Therefore it is required to take the proper precautions, lest in fractional expressions of this kind

$$\frac{dL}{dy}, \frac{dl}{dx}, \frac{dM}{dy} \text{ and } \frac{dm}{dx}$$

the numerators dL, dl, dM and dm may be thought to designate complete differentials of the functions L, l, M and m ; for these only denote the differentials of these always, which arise from the variation of a single coordinate, evidently of that, whose differential is shown in the denominator; and thus expressions of this kind will always represent finite and determined functions. [*i.e.* the relevant partial derivatives, for which at the time there was no special sign ∂.] Moreover in a similar manner there is understood to become:

$$L = \frac{du}{dx}, \quad l = \frac{du}{dy}, \quad M = \frac{dv}{dx} \text{ and } m = \frac{dv}{dy},$$

with which account requiring to be observed, first was used by the most illustrious Fontaine, and provides a worthy contribution to the abbreviation of the calculus, that I will use here also.

13. Therefore since there shall be

$$du = Ldx + ldy \text{ and } dv = Mdx + mdy,$$

hence it will be allowed to designate the twofold speeds of each other point, which indeed stands apart from the point l by an infinitely small distance ; indeed if the distance of such a point along the axis AL shall be $= dx$ from the point l, and along the axis $AB = dy$, then the speed of this point along the axis AL will become

$$= u + Ldx + ldy;$$

moreover the speed along the other axis AB

$$= v + Mdx + mdy.$$

Therefore in the infinitely small period of time dt this point will be moved along the direction of the axis AL by the infinitely small distance

$$= dt(u + Ldx + ldy)$$

and along the second direction of the other axis *AB* by an increment of distance

$$=dt\left(v+Mdx+mdy\right).$$

14. From these observations we will consider a triangular element of water *lmn* and we will seek the position to which it may be moved in the increment of time *dt*, on account of the motion which we consider to be present. Moreover the side *lm* of the element of the triangle *lmn* shall be parallel to the axis *AL,* truly the side *ln* shall be parallel to the side *AB* and there may be put $lm=dx$ and $ln=dy$; or the coordinates for the point *m* shall be $x+dx$ and *y,* and moreover for the point *n* the coordinates shall be x and $y+dy$. But it is apparent, because we have not defined the relation between the differentials dx and *dy* and each can be taken both negative as well as positive, the whole mass of the fluid is able to be divided into known elements of this kind, thus so that, what concerned with one element we will define in general, and that can be applied equally to all the elements.

15. Therefore so that it may become apparent on account of the intrinsic motion, we may seek the points *p, q* and *r* , into which the angles of this element *lmn* or the points *l, m, n,* of which may be transferred in the time *dt*. Therefore since there shall become

	point *l*	point *m*	point *n*
Speed along *AL*=	u	$=u+Ldx$	$=u+ldy$
Speed along *AB*=	v	$=v+Mdy$	$=v+mdy$,

the point *l* will arrive at *p* in the time increment *dt* , so that there shall become :

$$AP-AL=udt \text{ and } Pp-Ll=vdt.$$

Moreover the point *m* will arrive at *q,* so that there shall become :

$$AQ-AM=(u+Ldx)dt \text{ and } Qq-Mm=\left(v+Mdx\right)dt.$$

But the point *n* will be carried to *r,* so that there shall become :

$$AR-AL=(u+ldy)dt \text{ and } Rr-Ln=\left(v+mdy\right)dt.$$

16. Therefore since the points *l, m* and *n* may be transferred to the points *p, q* and *r* in the time *dt*, with the incremental lines *pq, pr* and *qr* joined, the situated triangle *lmn* , to which the triangle *pqr* corresponds, is considered to have arrived. Indeed since the triangle *lmn* is established very small, its sides are unable to receive any curvature by the motion, and thus the element of water *lmn* after the translation in the increment of time *dt* even now will retain the triangular figure *pqr,* and indeed be rectilinear. Therefore

since this element *lmn* must be unable either to be extended into a greater space, or compressed into a smaller space, the motion will be required to be prepared thus, so that the area of the triangle *pqr* may be expressed equal to the area of the triangle *lmn*.

17. Moreover the triangle *lmn,* since there shall be a right angle at l , the area is $=\frac{1}{2}dxdy$, therefore the area of the triangle *pqr* is required to be placed equal to this. But for this area requiring to be found the two coordinates of the points *p, q, r* are required to be found, which are :

$$AP = x + udt,\;\; AQ = x + dx + \left(u+Ldx\right)dt,\;\; AR = x + \left(u+ldy\right)dt,$$
$$Pp = y + vdt,\;\; Qq = y+(v + Mdx)dt,\; Rr = y + dy+\left(v+mdy\right)dt.$$

Then truly the area of the triangle *pqr* thus is found from the areas of the trapeziums, so that there shall become:

$$pqr = PprR + RrqQ - PpqQ.$$

But since these trapeziums shall have parallel sides and be perpendicular to the base *AQ*, the areas of these may be easily assigned.

18. Indeed there will be, as agreed from the elements:

$$PprR = \tfrac{1}{2}PR\left(Pp+Rr\right),$$
$$RrqQ = \tfrac{1}{2}RQ(Rr + Qq),$$
$$PpqQ = \tfrac{1}{2}PQ(Pp + Qq).$$

Therefore deduced from these there will be found:

$$\Delta pqr = \tfrac{1}{2}PQ\cdot Rr - \tfrac{1}{2}RQ\cdot Pp - \tfrac{1}{2}PR\cdot Qq.$$

For the sake of brevity there may be put

$$AQ = AP + Q,\; AR = AP + R,\; Qq = Pp + q \;\text{ and }\; Rr = Pp + r,$$

so that there shall become

$$PQ = Q,\;\; PR = R \;\text{ and }\; RQ = Q - R,$$

and there will become

$$\Delta pqr = \tfrac{1}{2}Q(Pp + r) - \tfrac{1}{2}\left(Q - R\right)Pp - \tfrac{1}{2}R\cdot(Pp+q)$$

or

$$\Delta pqr = \tfrac{1}{2}Q\cdot r - \tfrac{1}{2}R\cdot q.$$

19. Indeed from the values of the coordinates shown before

$$Q = dx + Ldxdt, \quad q = Mdxdt,$$
$$R = ldydt, \qquad r = dy + mdydt;$$

with which values substituted the area of the triangle will arise

$$\Delta\ pqr = \tfrac{1}{2}dxdy(1+Ldt)(1+mdt) - \tfrac{1}{2}Mldxdydt^2$$

or

$$\Delta\ pqr = \tfrac{1}{2}dxdy(1+Ldt+mdt+Lmdt^2-Mldt^2),$$

which since it must be equal to the area of the triangle *lmn,* which is $=\tfrac{1}{2}dxdy$, this equation will arise :

$$Ldt+mdt+Lmdt^2-Mldt^2=0$$

or

$$L+m+Lmdt-Mldt=0.$$

20. Therefore since the terms *Lmdt* and *Mldt* shall vanish before the finite terms *Lm* and *Ml* shall vanish, this equation will be had:

$$L+m=0.$$

On this account, so that the motion shall be possible, the speeds *u* and *v* of any point *l* thus must be prepared, so that with the differentials of these put in place

$$du = Ldx+ldy \ \text{ and } \ dv = Mdx+mdy$$

there shall be $L+m=0.$ Or since there shall be

$$L=\tfrac{du}{dx} \ \text{ and } \ m=\tfrac{dv}{dy},$$

the speeds *u* and *v,* which are taken to be present along the directions of the axes *AL* and *AB* from the point *l* , have to be functions of the coordinates *x* and *y* of this kind, so that there shall be

$$\tfrac{du}{dx}+\tfrac{dv}{dy}=0,$$

and thus a possible criterion of the motion consists of this equation, so that there shall be

$$\tfrac{du}{dx}+\tfrac{dv}{dy}=0,$$

unless this condition may be observed, the motion of the fluid is unable to exist.

21. By proceeding in the same manner, if the motion of the fluid may not be resolved in the same plane. Therefore, so that we may expedite the question to be taken in the widest sense, we may put in place the individual particles of the fluid to be interacting with each other by some kind of motion,

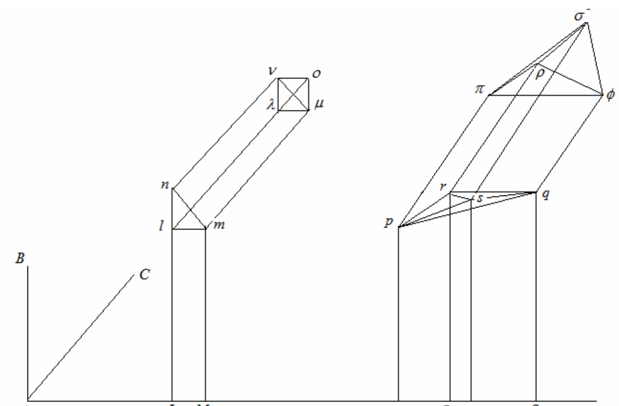

Fig. 2

only with this law being observed, so that neither condensation or expansion of the parts may eventuate anywhere: hence therefore similar manner is sought, which indeed may be approached for the determination for the speeds, which we consider for the individual points present, so that a possible motion may be returned: or, what amounts to the same thing, all the motions, which are opposed to these conditions, will require to be removed from the possible motions, from which the criterion of the possible motion will be established.

22. Therefore we will consider some point of the fluid λ, for the position of which we are required to use three fixed axes (Fig. 2) *AL, AB* and *AC* with these normal to each other. Therefore the three coordinates of the point λ parallel to these axes will be $AL = x$, $Ll = y$ and $l\lambda = z$; which will be obtained, if first the perpendicular λ*l* may be sent from the point λ to the plane determined by the two axes *AL* and *AB,* then truly the perpendicular *lL* may be acting from the point *l* to the axis *AL*. And thus in this manner the position of the point λ may be expressed most generally and can be applied to all the points of the fluid.

23. Again, whatever the motion of the point λ shall be*,* this will be able to be resolved along the three directions $\lambda\mu$, λv, and λo parallel to the axes *AL, AB* and *AC*. Therefore the speed of the point λ will be:

along the direction $\lambda\mu = u$,
along the direction $\lambda v = v$,
along the direction $\lambda o = w$,

and since these speeds shall be able to be varied for the various positions of the point λ, just as these functions of the three coordinates *x, y* and *z* are required to be considered. Therefore from these differentials, we may put to produce :

$$du = Ldx + ldy + \lambda dz,$$
$$dv = Mdx + mdy + \mu dz,$$
$$dw = Ndx + ndy + \nu dz,$$

and again where the quantities $L,\ l,\ \lambda,\ M,\ m,\ \mu,\ N,\ n,\ \nu$ will be functions of the coordinates *x, y* and *z.*

24. Because these differential formulas are complete, it follows in the same manner as above, to become :

$$\frac{dL}{dy} = \frac{dl}{dx},\quad \frac{dL}{dz} = \frac{d\lambda}{dx},\quad \frac{dl}{dz} = \frac{d\lambda}{dy},$$
$$\frac{dM}{dy} = \frac{dm}{dx},\quad \frac{dM}{dz} = \frac{d\mu}{dx},\quad \frac{dm}{dz} = \frac{d\mu}{dy},$$
$$\frac{dN}{dy} = \frac{dn}{dx},\quad \frac{dN}{dz} = \frac{d\nu}{dx},\quad \frac{dn}{dz} = \frac{d\nu}{dy},$$

[Thus, e.g. $\frac{dL}{dy} = \frac{dl}{dx}$, since $L = \frac{du}{dx}$ and $l = \frac{du}{dy}$, there becomes $\frac{dL}{dy} = \frac{dl}{dx} = \frac{ddu}{dxdy}$, etc.]

if indeed in the numerators only that one of the coordinates, of which the differential is shown in the denominator, is assumed to be the variable.

25. Therefore in this threefold motion, which we may consider to be present in the point X, this point λ thus will be moved in the time *dt*, so that it may advanced
along the direction of the *AL* axis through the distance increment $= udt$,
along the direction of the *AB* axis through the distance increment $= vdt$,
along the direction of the *AC* axis through the distance increment $= wdt$.

But if the true speed of the point λ , evidently which arises from the composition of this triple motion, may be called $= V$, it will be on account of the normality of the normality of the third direction

$$V = \sqrt{(uu + vv + ww)}$$

and the small distance, which truly it may perform in the time *dt* by its motion, will be $= Vdt$.

26. Now we will consider some whole element of the fluid, as we may see, in what direction that may be advanced in the incremental time *dt* ; and because it is a figure of this kind we may attribute to this element, provided that thus it may be defined generally, the whole mass of the fluid may be able to be taken divided into elements of this kind; in

order to be considered in the calculation, its figure shall be formed from a right angled triangular prism, terminating with a pyramid with the four solid angles $\lambda,\ \mu,\ v$ and o, thus so that for the individual points, the three coordinates shall be:

	point λ	point μ	point v	point o
along *AL*	x	$x+dx$	x	x
along *AB*	y	y	$y+dy$	y
along *AC*	z	z	z	$z+dz$

and since the base of this pyramid shall be

$$\lambda\mu v = lmn = \tfrac{1}{2}dxdy,$$

truly the height $\lambda o = dz$, and its volume will be

$$= \tfrac{1}{6}dxdydz.$$

27. Now we will investigate, in what direction the individual angles of this pyramid $\lambda,\ \mu,\ v$ and o may be transferred in the time dt : for which of these three speeds it will be required to consider along the directions of the three axis, which will be from the differential values of the speeds *u, v, w* :

Speed along	of point λ	of point μ	of point v	of point o
direction *AL*	u	$u+Ldx$	$u+ldy$	$u+\lambda dz$
direction *AB*	v	$v+Mdx$	$v+mdy$	$v+\mu dz$
direction *AC*	w	$w+Ndx$	$w+ndy$	$w+vdz$

28. So that if therefore we may have transferred the points $\lambda,\ \mu,\ v$ and o, to the points $\pi,\ \varphi,\ \rho$ and σ in the time dt, we may put the coordinates of these points to be parallel to the axes, the translations of the moments [*i.e.* products] along these axes will be:

$AP-AL=udt$	$Pp-Ll=vdt$	$p\pi-l\lambda=wdt$
$AQ-AM=(u+Ldx)dt$	$Qq-Mm=(v+Mdx)dt$	$q\varphi-m\mu=(w+Ndx)dt$
$AR-AL=(u+ldy)dt$	$Rr-Ln=(v+mdy)dt$	$r\rho-nv=(w+ndy)dt$
$AS-AL=(u+\lambda dz)dt$	$Ss-Ll=(v+\mu dz)dt$	$s\sigma-lo=(w+vdz)dt$

Hence the three coordinates of these four points $\pi,\ \varphi,\ \rho$ et σ therefore will be :

$AP=x+udt,$	$Pp=y+vdt,$	$p\pi=z+wdt.$
$AQ=x+dx+(u+Ldx)dt,$	$Qq=y+(v+Mdx)dt,$	$q\varphi=z+(w+Ndx)dt,$
$AR=x+(u+ldy)dt,$	$Rr=y+dy+(v+mdy)dt,$	$r\rho=z+(w+ndy)dt,$
$AS=x+(u+\lambda dz)dt,$	$Ss=y+(v+\mu dz)dt,$	$s\sigma=z+dz+(w+vdz)dt.$

29. Therefore since in the elapsed time dt the angles of the pyramid λ, μ, ν and o, shall be transposed to the points π, φ, ρ and σ, now equally the triangular pyramid $\pi\varphi\rho\sigma$ may be put in place with that pyramid ; and thus on account of the nature of the fluid it can be effected, so that the volume of the pyramid $\pi\varphi\rho\sigma$ shall be equal to the volume of the proposed pyramid $\lambda\mu\nu o$, or

$$=\tfrac{1}{6}dxdydz.$$

Therefore the whole procedure is reduced to this, so that the volume of the pyramid $\pi\varphi\rho\sigma$ may be determined. Moreover it is observed this pyramid to be left, if the volume $pq\pi\varphi\rho$ may be taken from the volume $prq\pi\varphi\rho\sigma$; so that the latter volume is a prism resting normally on the triangular base pqr and with the oblique section $\pi\rho\varphi$ being cut above.

30. But in a truncated prism of this kind also the volume $prq\pi\varphi\rho\sigma$ can be resolved into three other volumes, which are:

I. $pqs\pi\varphi\sigma$, II. $prs\pi\rho\sigma$, III. $qrs\varphi\rho\sigma$

and thus it must be effected, so that

$$\tfrac{1}{6}dxdydz = pqs\pi\varphi\sigma + prs\pi\rho\sigma + qrs\varphi\rho\sigma - pqr\pi\varphi\rho.$$

But since a prism of this kind may stand normally on its lower base, moreover it may have three unequal altitudes, and its volume will be found, if the base may be multiplied by the third part of the sum of these three altitudes.

31. Hence therefore the volumes of these truncated prisms will be :

$$pqs\pi\varphi\sigma = \tfrac{1}{3}pqs(p\pi + q\varphi + s\sigma),$$
$$prs\pi\rho\sigma = \tfrac{1}{3}prs(p\pi + r\rho + s\sigma),$$
$$qrs\varphi\rho\sigma = \tfrac{1}{3}qrs(q\varphi + r\rho + s\sigma),$$
$$pqr\pi\varphi\rho = \tfrac{1}{3}pqr(p\pi + q\varphi + r\rho).$$

But since there shall be

$$pqr = pqs + prs + qrs,$$

the sum of the three first prisms minus the latter prism will be either

$$\tfrac{1}{6}dxdydz = -\tfrac{1}{3}p\pi\cdot qrs - \tfrac{1}{3}q\varphi\cdot prs - \tfrac{1}{3}r\varphi\cdot pqs + \tfrac{1}{3}s\sigma\cdot pqr$$

or

$$dxdydz = 2pqr\cdot s\sigma - 2pqs\cdot r\rho - 2prs\cdot q\varphi - 2qrs\cdot p\pi.$$

32. Therefore it remains, so that the bases of these prisms may be defined; truly before we may do this, we may put together the following calculation:

$$AQ = AP + Q,\quad Qq = Pp + q,\quad q\varphi = p\pi + \varphi,$$
$$AR = AP + R,\quad Rr = Pp + r,\quad r\rho = p\pi + \rho,$$
$$AS = AP + S,\quad Ss = Pp + s,\quad s\sigma = p\pi + \sigma,$$

and with these latter values substituted, the terms containing $p\pi$ will cancel each other out and there will become

$$dxdydz = 2pqr\cdot\sigma - 2pqs\cdot\rho - 2prs\cdot\varphi$$

and thus the number of the bases requiring to be found is diminished by one.

33. Now the triangle pqr is found, if from the figure $PprqQ$ or from the sum of the trapeziums $PprR+RrqQ$ the trapezium $PpqQ$ may be taken away ; from which there will become

$$\Delta pqr = \tfrac{1}{2}PR\left(Pp + Rr\right) + \tfrac{1}{2}RQ\left(Rr + Qq\right) - \tfrac{1}{2}PQ\left(Pp + Qq\right);$$

or if on account of $PR = R$, $RQ = Q - R$ and $PQ = Q$ there will become

$$\Delta pqr = \tfrac{1}{2}R\left(Pp - Qq\right) + \tfrac{1}{2}Q\left(Rr - Pp\right) = \tfrac{1}{2}Qr - \tfrac{1}{2}Rq.$$

In a similar manner there will become :

$$\Delta pqs = \tfrac{1}{2}PS\left(Pp + Ss\right) + \tfrac{1}{2}SQ\left(Ss + Qq\right) - \tfrac{1}{2}PQ\left(Pp + Qq\right),$$

or

$$\Delta pqs = \tfrac{1}{2}S\left(Pp + Ss\right) + \tfrac{1}{2}\left(Q - S\right)\left(Ss + Qq\right) - \tfrac{1}{2}Q\left(Pp + Qq\right),$$

from which there will become:

$$\Delta pqs = \tfrac{1}{2}S\left(Pp - Qq\right) + \tfrac{1}{2}Q\left(Ss - Pp\right) = \tfrac{1}{2}Qs - \tfrac{1}{2}Sq.$$

And finally

$$\Delta prs = \tfrac{1}{2}PR(Pp + Rr) + \tfrac{1}{2}RS(Rr + Ss) - \tfrac{1}{2}PS(Pp + Ss)$$

or

$$\Delta prs = \tfrac{1}{2}R\left(Pp + Rr\right) + \tfrac{1}{2}\left(S - R\right)\left(Rr + Ss\right) - \tfrac{1}{2}S\left(Pp + Ss\right),$$

from which there becomes :

$$\Delta prs = \tfrac{1}{2}R\left(Pp - Ss\right) + \tfrac{1}{2}S\left(Rr - Pp\right) = \tfrac{1}{2}Sr - \tfrac{1}{2}Rs.$$

34. Therefore with these values substituted we will obtain

$$dxdydz = \left(Qr - Rq\right)\sigma + \left(Sq - Qs\right)\rho + \left(Rs - Sr\right)\varphi$$

or the volume of the pyramid $\pi\varphi\rho\sigma$ will become

$$\tfrac{1}{6}\left(Qr - Rq\right)\sigma + \tfrac{1}{6}\left(Sq - Qs\right)\rho + \tfrac{1}{6}\left(Rs - Sr\right)\varphi.$$

But it is shown from the values of the coordinates above in paragraph 28:

$$\begin{aligned} &Q = dx + Ldxdt, && q = Mdxdt, && \varphi = Ndxdt,\\ &R = ldydt, && r = dy + mdydt, && \rho = ndydt,\\ &S = \lambda dzdt, && s = \mu dzdt, && \sigma = dz + \nu dzdt. \end{aligned}$$

35. Therefore since there may become

$$\begin{aligned} &Qr - Rq = dxdy(1 + Ldt + mdt + Lmdt^2 - Mldt^2),\\ &Sq - Qs = dxdz(-\mu dt - L\mu dt^2 + M\lambda dt^2),\\ &Rs - Sr = dydz(-\lambda dt - m\lambda dt^2 + l\mu dt^2), \end{aligned}$$

the volume of the pyramid found $\pi\varphi\rho\sigma$ will be expressed thus:

$$\tfrac{1}{6}dxdydz\left\{\begin{matrix} 1 + Ldt + Lmdt^2 + Lm\nu dt^2\\ + mdt - Mldt^2 - Ml\nu dt^3\\ + \nu dt + L\nu dt^2 \;\; - Ln\mu dt^3\\ + m\nu dt^2 + Mn\lambda dt^3\\ - n\mu dt^2 - Nm\lambda dt^3\\ - N\lambda dt^2 + Nl\mu dt^3 \end{matrix}\right\}$$

which since it must be equal to the pyramid $\lambda\mu\nu o = \tfrac{1}{6}dxdydz$, and this equation will be had, with the division by dt put in place :

$$\begin{aligned} 0 = L + m \; + \nu + dt\left(Lm + L\nu + m\nu - Ml - N\lambda - n\mu\right)\\ + dt^2(Lm\nu + Mn\lambda + Nl\mu - Ln\mu - Ml\nu - Nm\lambda). \end{aligned}$$

36. Therefore with the infinitely small terms rejected this equation will be had :

$$L+m+v=0,$$

from which an account of the speeds *u, v, w* may be determined, so that the motion of the fluid may be made possible. Therefore since there shall be

$$L=\tfrac{du}{dx},\quad m=\tfrac{dv}{dy}\quad \text{and}\quad v=\tfrac{dw}{dz},$$

the criterion of the possible motion, if for any point of the fluid λ, of which the position of the three coordinates *x, y* and *z* is defined, speeds of this kind *u, v* and *w* may be given directed along the same coordinates, so that there shall be

$$\tfrac{du}{dx}+\tfrac{dv}{dy}+\tfrac{dw}{dz}=0.$$

Clearly it will prevail with this condition, so that no part of the fluid in motion may be transferred into a larger or smaller volume, and always both with the continuation of the fluid, as well as the same density may be maintained.

37. Moreover, this property is required to be explained thus, so that it may be extended to all the points of the fluid : evidently for the same moment the three speeds *u, v, w* of all the points must be such functions of the three coordinates *x, y* and *z,* in order that there shall become

$$\tfrac{du}{dx}+\tfrac{dv}{dy}+\tfrac{dw}{dz}=0.$$

and thus the nature of these functions will define the motion of the individual fluid points at the proposed instant. But at another time the motion of the same points will be able to be quite different, provided that for any point of time the property found may remain in place for the whole fluid, evidently just as long as we have considered the time as a constant quantity.

38. But if we may wish to regard the time also to be variable, thus so that the motion of the point *λ,* the position of which is indicated by the three coordinates $AL=x,\ Ll=y$ and $l\lambda=z$, must be able to be defined after the elapsed time *t*, it is evident the three speeds *u, v* and *w* to depend not only on the coordinates *x, y* and *z,* but in addition also to depend on the time *t*, or the functions to become of these four quantities *x, y, z* and *t* ; thus so that the differential forms shall be required to be had of this kind :

$$du=Ldx+ldy+\lambda dz+\mathfrak{L}dt,\quad dv=Mdx+mdy+\mu dz+\mathfrak{M}dt,$$
$$dw=Ndx+ndy+vdz+\mathfrak{N}dt.$$

Yet meanwhile there will be always $L+m+v=0$, therefore, since whatever time t may be had for the constant instant, or there shall become $dt=0$. Therefore in whatever manner the functions u, v and w may be changed with the time t, it is necessary, so that for any instant of time there shall be :

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0.$$

Since indeed it must be effected from this condition, that whatever the portion of the fluid may itself be transferred in the time dt into an equal volume, the same also by the same condition in the following element of time, therefore the same must happen in all the following elements of time.

SECOND PART

39. Therefore with these established, which so far pertain to possible motion, now we will investigate the nature of that motion, which actually can exist in the fluid. Here therefore, besides the continuity of the fluid and the permanence of the same density, an account will be required to be had of the forces, by which the individual elements of the fluid may be disturbed by the action. Indeed when any motion of the element either is not uniform or is not extended in a straight line, the changed motion must be fashioned by the forces acting on this element. Whereby since the changed motion may become known from these known forces, moreover the preceding formulas also shall be contained in this changed motion, hence new determinations will be deduced, by which the only motion possible at this stage is restrained to the actual motion.

40. Also we may establish this investigation in two parts; and at first we may consider the whole motion of the fluid to happen in the same plane. Therefore, as before (Fig. 1), the position of the coordinates defining any point l shall be $AL=x$, $Ll=y$; and now indeed with the time elapsed t, the two speeds of the point l following directions parallel to the axes AL and AB shall be u and v : u and v will be functions of x, y and t themselves, since now an account of the variation with time must be had, so that concerning which there may be put

$$du=Ludt+lvdt+\mathfrak{L}dt, \quad \text{and} \quad dv=Mudt+mvdt+\mathfrak{M}dt.$$

and on account of the first condition we have found above, there must become now $L+m=0$.

41. Therefore with the elapsed increment of time $=dt$ the point l shall be transferred to p, with the increment along the AL axis $=udt$, but the increment along the other axis AB will be $=vdt$: so that we may obtain the increments of the speeds u and v at the point l, which are themselves introduced in the very short time dt, for dx and dy it will be

required to write the very short distances *udt* and *vdt,* from which truly the increments of the speeds will be produced:

$$du = Ludt + lvdt + \mathfrak{L}dt, \quad \text{and} \quad dv = Mudt + mvdt + \mathfrak{M}dt.$$

From which the accelerating forces, which prevail to produce these accelerations, will be

accelerating force along $AL = 2(Lu + lv + \mathfrak{L})$,
accelerating force along $AB = 2(Mu + mv + \mathfrak{M})$,

which therefore, must be equal to the actual forces acting on the particle of water *l*.

42. But among these forces, which actually disturb the water particles, gravity becomes the first required to be considered ; but its effect, if the water is level, in which case the motion shall be horizontal, amounts to nothing. But if there were a slope and a declivity of the axis *AL* may follow, with the other *AB* proving to be horizontal, a constant accelerating force arises from gravity along *AL*, which shall be equal to $= \alpha$. Then not allowing for friction, by which the motion of the water often is greatly impeded ; moreover, though its laws have not yet been investigated well enough, perhaps we will not have differed much from the goal following the rules for the friction of solid bodies, if we may put in place the friction to be proportional to the pressure everywhere, by which the particles of water themselves are acted on.

43. But initially the pressure is required to be deduced in the computation, by which the particles of water everywhere may act on each other, from which it arises, so that any particle may be compressed by the neighbouring particles on all sides, and since this pressure may not have been equal everywhere, to this extend the motion of the particles may be affected. Evidently the water will be moving around everywhere in a certain compressed state, which will be similar to that present in still water at a certain depth. Therefore this depth, which may be found for that in still water in an equal state of compression, will be used most conveniently for the pressure being expressed by the force of the fluid at the point *l*. Therefore *p* shall be that height of depth, expressing the state of the compression at *l*, and *p* will be some function of the coordinates *x* and *y,* and if the pressure at *l* also may be varied with the time, also the time *t* will enter into the function *p*.

44. Therefore we may put

$$dp = Rdx + rdy + \mathfrak{R}dt$$

and we will consider the four angled rectangular element of water (Fig. 3) *lmno,* the sides of which shall be

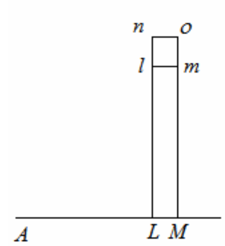

Fig. 3

$$lm = no = dx \quad \text{and} \quad ln = mo = dy$$

and the area $=dxdy$. Now since the pressure at *l* shall become $=p$, the pressure at *m* will be $=p+Rdx$, and at o shall be $=p+Rdx+rdy$. Hence the side *lm* is pressed on by a force $=dx\left(p+\frac{1}{2}Rdx\right)$; truly the side *no* is pressed on by an opposite force

$$=dx(p+\tfrac{1}{2}Rdx+rdy);$$

therefore from these two forces the element *lmno* will be pushed along the direction *ln* by the force $=-rdxdy$.
Moreover in a similar manner from the forces

$$dy(p+\tfrac{1}{2}rdy) \text{ and } dy(p+Rdx+\tfrac{1}{2}rdy),$$

which act on the sides *ln* and *mo,* the resultant force will be pushing the element along the direction $lm=-Rdxdy$.

45. Hence therefore the accelerating force will arise along lm, $=-R$ and the accelerating force along ln, $=-r$, of which that with the force arising from gravity α gives $\alpha-R$. Therefore with friction still separated out , we will have these equations :

$$\alpha-R=2Lu+2lv+2\mathfrak{L} \text{ or } R=\alpha-2Lu-2lv-2\mathfrak{L},$$
$$-r=2Mu+2mv+2\mathfrak{M} \text{ and } r=-2Mu-2mv-2\mathfrak{M},$$

from which we deduce to become

$$dp=\alpha dx-2(Lu+lv+\mathfrak{L})dx-2(Mu+mv+\mathfrak{M})dy+\mathfrak{R}dt,$$

which differential is required to be complete or integrable.

46. Since the term αdx by itself is integrable and for $\mathfrak{R}$ nothing is defined, by necessity it is required from the nature of the complete differential, so that it may be designated in the manner used above :

$$\frac{d.(Lu+lv+\mathfrak{L})}{dy}=\frac{d.(Mu+mv+\mathfrak{M})}{dx},$$

from which on account of

$$\tfrac{du}{dx}=L,\quad \tfrac{du}{dy}=l,\quad \tfrac{dv}{dx}=M \text{ and } \tfrac{dv}{dy}=m,$$

there may arise

$$Ll+\tfrac{udL}{dy}+lm+\tfrac{vdl}{dy}+\tfrac{d\mathfrak{L}}{dy}=Ml+\tfrac{udM}{dx}+mM+\tfrac{vdm}{dx}+\tfrac{d\mathfrak{M}}{dx},$$

which is reduced to this form :

$$(L+m)(l-M)+u\left(\frac{dL}{dy}-\frac{dM}{dx}\right)+v\left(\frac{dl}{dy}-\frac{dm}{dx}\right)+\frac{d\mathfrak{L}}{dy}-\frac{d\mathfrak{M}}{dx}=0.$$

47. Truly on account of the complete differentials

$$Ldx+ldy+\mathfrak{L}dt \text{ and } Mdx+mdy+\mathfrak{M}dt$$

we know to become

$$\frac{dL}{dy}=\frac{dl}{dx},\quad \frac{dm}{dx}=\frac{dM}{dy},\quad \frac{d\mathfrak{L}}{dy}=\frac{dl}{dt} \text{ and } \frac{d\mathfrak{M}}{dx}=\frac{dM}{dt},$$

from which we will have this same equation, with the values substituted :

$$(L+m)(l-M)+u\left(\frac{dl-dM}{dx}\right)+v\left(\frac{dl-dM}{dy}\right)+\frac{dl-dM}{dt}=0,$$

of which openly $l=M$ satisfies the equation: thus so that there shall become:

$$\frac{du}{dy}=\frac{dv}{dx}.$$

Therefore since this condition will be required, so that there shall be $\frac{du}{dy}=\frac{dv}{dx}$, in turn it will be apparent this differential formula $udx+vdy$ must become a complete differential, in which therefore the criterion of the actual motion is present.

48. This criterion depends on the preceding one, which furnished continuity of the fluid and constant uniform density. Whereby even if the fluid may change its density in moving, as in the motion of elastic fluids, just as is accustomed to happen with air, this property nevertheless must be accommodated, so that $udx+vdy$ shall be a complete differential. Or if the speeds *u* and *v* always must be functions of the coordinates *x* and *y* of this kind, besides the time *t,* so that with the time made constant, the formula $udx+vdy$ may be allowed to be integrated.

49. Hence moreover again we will be able to define that pressure *p* itself, which is completely necessary for determining the motion of a perfect fluid. Indeed since we shall find $M=l$, there will become

$$dp=\alpha dx-2u(Ldx+ldy)-2v(ldx+mdy)-2\mathfrak{L}dx-2\mathfrak{M}dy+\mathfrak{R}dt.$$

But there is

$$Ldx+ldy=du-\mathfrak{L}dt,\quad ldx+mdy=dv-\mathfrak{M}dt,$$

from which there becomes:

$$dp=\alpha dx-2udu-2vdv+2\mathfrak{L}udt+2\mathfrak{M}vdt-2\mathfrak{L}dx-2\mathfrak{M}dy+\mathfrak{R}dt.$$

So that therefore if we may wish to define the pressure at individual points of the fluid with the time present, by having that without respect to its change with time, it will be required for us to be considering this equation:

$$dp = \alpha dx - 2udu - 2vdv - 2\mathfrak{L}dx - 2\mathfrak{M}dy$$

and there is required to be designated accordingly in our mode :

$$\mathfrak{L} = \tfrac{du}{dt} \quad \text{and} \quad \mathfrak{M} = \tfrac{dv}{dt}$$

and hence

$$dp = \alpha dx - 2udu - 2vdv - 2\tfrac{du}{dt}dx - 2\tfrac{dv}{dt}dy,$$

in the integration of this equation the time t is required to be considered constant.

50. But this equation is integrable according to the hypothesis, and actually the integration of such may be performed, if we may attend to the criterion of its motion, where we have seen $udx+vdy$ must become a complete differential, if indeed the time t may be considered constant. Therefore S shall be its integral, so that therefore it will become a function of x, y and $t,$ so that on putting $dt = 0$ there may be produced

$$dS = udx + vdy;$$

but with some variable of the time taken t we may put in place to have

$$dS = udx + vdy + Udt$$

and therefore there will become

$$\tfrac{du}{dt} = \tfrac{dU}{dx} \quad \text{and} \quad \tfrac{dv}{dt} = \tfrac{dU}{dy}.$$

Then truly there will become

$$U = \tfrac{dS}{dt}.$$

51. With these values introduced there will be obtained: :

$$\tfrac{du}{dt}\cdot dx + \tfrac{dv}{dt}\cdot dy = \tfrac{dU}{dx}\cdot dx + \tfrac{dU}{dy}\cdot dy$$

and the integral of this formula, with the time t taken constant, evidently will be $= U$. So that this may appear more clearly, we may put

$$dU = Kdx + kdy,$$

there will become

$$\tfrac{dU}{dx} = K \quad \text{and} \quad \tfrac{dU}{dy} = k,$$

from which

$$\frac{dU}{dx}\cdot dx+\frac{dU}{dy}\cdot dy=Kdx+kdy=dU.$$

Therefore since the integral of this equation shall be

$$=U=\frac{dS}{dt},$$

there will become

$$dp=\alpha dx-2udu-2vdv-2dU,$$

from which on integrating there becomes :

$$p=\text{ Const.}+\alpha x-uu-vv-\frac{2dS}{dt},$$

with the function S arising of these variables *x, y* and *t,* of which the differential is $udx+vdy$, on putting $dt=0$.

52. So that we may understand the nature of this formula better, we will consider the true speed of the point l, which shall be

$$=V=\sqrt{(uu+vv)}.$$

And the pressure will become :

$$p=\text{Const.}+\alpha x-VV-\frac{2dS}{dt},$$

in which in the final term dS denotes the differential of

$$S=\int(udx+vdy),$$

so that only the time t may be considered as a variable.

53. If now we may wish to have some account of friction and we may establish that proportional to the pressure p, while the point l traverses the element ds, the retarding force arising from the friction will be $=\frac{p}{f}$; from which on putting $\frac{dS}{dt}=U$ our differential equation, on putting t constant, will be :

$$dp=\alpha dx-\frac{p}{f}ds-2VdV-2dU,$$

from which there arises by integration, on taking e for the number, whose hyperbolic logarithm is $=1$,

$$p=e^{-\frac{s}{f}}\int e^{\frac{s}{f}}\left(\alpha dx-2VdV-2dU\right)$$

or,

$$p=\alpha x-VV-2U-\frac{1}{f}e^{-\frac{s}{f}}\int e^{\frac{s}{f}}\left(\alpha dx-VdV-2dU\right)ds\,.$$

54. Therefore since the criterion of the motion, by which the fluid element truly is moved, is consistent with this, so that with the constant time differential in place, the differential $udx+vdy$ shall be completed, moreover with a continuous motion and constant uniform density shall be required, so that there shall be

$$\frac{du}{dx}+\frac{dv}{dy}=0,$$

hence it follows also for this differential $udy-vdx$ to be complete. Whereby each side joining the speeds u and v must be functions of the coordinates x and y with the time t of this kind , so that both these formulas

$$udx+vdy \text{ and } udy-vdx$$

shall be complete differentials.

55. Now we may establish the same investigation in general, and with the position of the point λ with the three speeds $u, v, w,$ directed along the axes AL, AB, AC these shall be functions both of the three coordinates x, y, z as well as of the time $t,$ so that by differentiation there may be established :

$$du=Ldx+ldy+\lambda dz+\mathfrak{L}dt,$$
$$dv=Mdx+mdy+\mu dz+\mathfrak{M}dt,$$
$$dw=Ndx+ndy+vdz+\mathfrak{N}dt,$$

and here likewise some time variable t has been assumed, yet so that the motion shall be possible, by the preceding condition there will be required to be either

$$L+m+v=0$$

which is returned to the same :

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0,$$

on which property indeed the present examination does not depend.

56. But in the elapsed time increment $dt,$ the point λ is transferred into π and following the axis AL it runs through a space increment $=udt$, along the AB axis the increment $=vdt$ and along the axis AC the increment wdt. Whereby now the three speeds of the point λ present at π will be :

$$\text{along } AL=u+Ludt+lvdt+\lambda wdt+\mathfrak{L}dt,$$
$$\text{along } AB=v+Mudt+mvdt+\mu wdt+\mathfrak{M}dt,$$
$$\text{along } AC=w+Nudt+nvdt+vwdt+\mathfrak{N}dt$$

and hence the accelerations along the same directions will be:

along $AL = 2(Lu + lv + \lambda w + \mathfrak{L})$,
along $AB = 2(Mu + mv + \mu w + \mathfrak{M})$,
along $AC = 2(Nu + nv + \nu w + \mathfrak{N})$.

57. If we may place the axis *AC* vertical, thus so that the two remaining axes *AL* and *AB* shall be horizontal, on account of the force of gravity the acceleration along the axis $AC = -1$. Then truly with the pressure put at $\lambda, = p$ and its differential, with the time taken constant,

$$dp = Rdx + rdy + \rho dz,$$

hence the three accelerating forces will become

along $AL = -R$, along $AB = -r$ and along $AC = -\rho$,

certainly which will be deduced in a similar manner, as we have used before in paragraphs 44 and 45, thus so that it would be superfluous to repeat the same reasoning here. On account of which, we will have these equations:

$$R = -2(Lu + lv + \lambda w + \mathfrak{L}),$$
$$r = -2\left(Mu + mv + \mu w + \mathfrak{M}\right),$$
$$\rho = -1 - 2\left(Nu + nv + \nu w + \mathfrak{N}\right).$$

58. Moreover since the formula $dp = Rdx + rdy + \rho dz$ must be a complete differential, there will become :

$$\tfrac{dR}{dy} = \tfrac{dr}{dx}, \quad \tfrac{dR}{dz} = \tfrac{d\rho}{dx}, \quad \tfrac{dr}{dz} = \tfrac{d\rho}{dy},$$

but with the differentiation performed, on being divided by -2, we will obtain the three following equations:

$$\text{I.}\quad \frac{udL}{dy}+\frac{vdl}{dy}+\frac{wd\lambda}{dy}+\frac{d\mathfrak{L}}{dy}+Ll+lm+\lambda n = \frac{udM}{dx}+\frac{vdm}{dx}+\frac{wd\mu}{dx}+\frac{d\mathfrak{M}}{dx}+ML+mM+\mu N$$

$$\text{II.}\quad \frac{udL}{dz}+\frac{vdl}{dz}+\frac{wd\lambda}{dz}+\frac{d\mathfrak{L}}{dz}+L\lambda+l\mu+\lambda v = \frac{udN}{dx}+\frac{vdn}{dx}+\frac{wdv}{dx}+\frac{d\mathfrak{N}}{dx}+NL+nM+vN$$

$$\text{III.}\quad \frac{udM}{dz}+\frac{vdm}{dz}+\frac{wd\mu}{dz}+\frac{d\mathfrak{M}}{dz}+M\lambda+m\mu+\mu v = \frac{udN}{dy}+\frac{vdn}{dy}+\frac{wdv}{dy}+\frac{d\mathfrak{N}}{dy}+Nl+nm+vn.$$

59. But from the nature of the complete differential :

$$\frac{dL}{dy}=\frac{dl}{dx},\ \frac{dm}{dx}=\frac{dM}{dy},\ \frac{d\lambda}{dy}=\frac{dl}{dz},\ \frac{d\mu}{dx}=\frac{dM}{dz},\ \frac{d\mathfrak{L}}{dy}=\frac{dl}{dt},\ \frac{d\mathfrak{M}}{dx}=\frac{dM}{dt},$$
$$\frac{dL}{dz}=\frac{d\lambda}{dx},\ \frac{dl}{dz}=\frac{d\lambda}{dy},\ \frac{dn}{dx}=\frac{dN}{dy},\ \frac{dv}{dx}=\frac{dN}{dz},\ \frac{d\mathfrak{L}}{dz}=\frac{d\lambda}{dt},\ \frac{d\mathfrak{N}}{dx}=\frac{dN}{dt},$$
$$\frac{dM}{dz}=\frac{d\mu}{dx},\ \frac{dN}{dy}=\frac{dn}{dx},\ \frac{dm}{dz}=\frac{d\mu}{dy},\ \frac{dv}{dy}=\frac{dn}{dz},\ \frac{d\mathfrak{M}}{dz}=\frac{d\mu}{dt},\ \frac{d\mathfrak{N}}{dy}=\frac{dn}{dt},$$

from which with the values substituted these three equations will be transformed into these [*i.e.* the vorticity equations]:

$$\left(\frac{dl-dM}{dt}\right)+u\left(\frac{dl-dM}{dx}\right)+v\left(\frac{dl-dM}{dy}\right)+w\left(\frac{dl-dM}{dz}\right)+\left(l-M\right)\left(L+m\right)+\lambda n-\mu N=0,$$
$$\left(\frac{d\lambda-dN}{dt}\right)+u\left(\frac{d\lambda-dN}{dx}\right)+v\left(\frac{d\lambda-dN}{dy}\right)+w\left(\frac{d\lambda-dN}{dz}\right)+(\lambda-N)(L+v)+l\mu-nM=0,$$
$$\left(\frac{d\mu-dn}{dt}\right)+u\left(\frac{d\mu-dn}{dx}\right)+v\left(\frac{d\mu-dn}{dy}\right)+w\left(\frac{d\mu-dn}{dz}\right)+\left(\mu-n\right)(m+v)+M\lambda-Nl=0.$$

60. Now it is evident for these three equations to be satisfied by the three following values [*i.e.* certainly true for irrotational motion where the curl of the velocity vector is zero]:

$$l=M,\ \ \lambda=N,\ \ \mu=n,$$

by which the criterion may be contained, which consideration may satisfy the requirements of the disturbing forces. Hence it follows on being designated in the following manner :

$$\frac{du}{dy}=\frac{dv}{dx},\quad \frac{du}{dz}=\frac{dw}{dx},\quad \frac{dv}{dz}=\frac{dw}{dy}.$$

But these are the very conditions which are required, so that the formula

$$udx+vdy+wdz$$

shall be a complete differential. From which this criterion stands therein so that the three velocities *u, v* and *w* must be functions of *x, y* and *z* together with *t,* so that with the time being put constant in the formula $udx+vdy+wdz$, it may allow integration.

61. Therefore since by putting the time *t* constant or $dt=0$ there shall become

$$du=Ldx+Mdy+Ndz,$$
$$dv=Mdx+mdy+ndz,$$
$$dw=Ndx+ndy+vdz,$$

moreover the values for *R, r* and ρ shall become :

$$R=-2(Lu+Mv+Nw+\mathfrak{L}),$$
$$r=-2(Mu+mv+nw+\mathfrak{M}),$$
$$\rho=-1-2(Nu+nv+vw+\mathfrak{N}),$$

for the state of the pressure p this equation will be had:

$$\begin{aligned} dp=dz&-2u(Ldx+Mdy+Ndz)\\ &-2v(Mdx+mdy+ndz)\\ &-2w(Ndx+ndy+vdz)\\ &-2\mathfrak{L}dx-2\mathfrak{M}dy-2\mathfrak{N}dz\\ =-dz&-2udu-2vdv-2wdw\\ &-2\mathfrak{L}dx-2\mathfrak{M}dy-2\mathfrak{N}dz. \end{aligned}$$

62. Truly since there shall be

$$\mathfrak{L}=\frac{du}{dt},\quad \mathfrak{M}=\frac{dv}{dt},\quad \mathfrak{N}=\frac{dw}{dt},$$

on integrating there shall become:

$$p=C-z-uu-vv-ww-2\int\left(\frac{du}{dt}\cdot dx+\frac{dv}{dt}\cdot dy+\frac{dw}{dt}\cdot dz\right).$$

But since by the condition, there shall be found $udx+vdy+wdz$ integrable, the integral of this may be put $=S$, so that, since also the time *t* can be involved, there shall become with *t* also to be variable:

$$dS=udx+vdy+wdz+Udt,$$

and there will become:

$$\tfrac{du}{dt}=\tfrac{dU}{dx},\quad \tfrac{dv}{dt}=\tfrac{dU}{dy},\quad \tfrac{dw}{dt}=\tfrac{dU}{dz}.$$

Whereby since in general with the time *t* taken constant, as that indeed may be assumed in the above integral,

$$\tfrac{dU}{dx}\cdot dx+\tfrac{dU}{dy}\cdot dy+\tfrac{dU}{dz}\cdot dz=dU,$$

we will have :

$$p=C-z-uu-vv-ww-2U$$

or

$$p=C-z-uu-vv-ww-2\cdot\tfrac{dS}{dt}.$$

63. Here it is evident for $uu+vv+ww$ to express the square of the true speed of the point λ thus, so that, if the true speed of this point may be called *V,* this equation will be had for the pressure :

$$p=C-z-VV-\tfrac{2dS}{dt},$$

for which therefore the first of the formulas requiring to be found $udx+vdy+wdz$, as it will be required to be complete, it will be sought from the integral *S,* and this may be differentiated again, with only the time *t* put variable, so that with the differential divided by *dt* the value of the formula $\tfrac{dS}{dt}$ will be given , which must be inserted into the expression for the pressure *p*.

64. But if now we may add here the prior criterion, by which at least a possible motion is held, the three speeds *u, v, w* must be functions of this kind of the three coordinates *x, y* and *z* together with the time *t* , so that in the first place

$$udx+vdy+wdz$$

shall be a complete differential; then truly, so that there shall become

$$\tfrac{du}{dx}+\tfrac{dv}{dy}+\tfrac{dw}{dz}=0$$

And from these two conditions the motion of all fluids may be set out, if indeed they may be provided with a constant density. Truly besides, if with the time *t* taken variable, this formula

$$udx+vdy+wdz+Udt$$

will be a complete differential, the status of the pressure at some point λ is expressed by the altitude *p,* so that there shall become :

$$p = C - z - uu - vv - ww - 2U,$$

if indeed the fluid may be blessed with natural weight and the plane *BAL* were horizontal.

65. If we may attribute another direction to gravity or we may assume also the forces to be variable in some manner, by which the individual particles of the fluid may be disturbed, thence only a distinction in the value of the pressure *p* may be introduced, nor thence the law, that the three speeds of each fluid point must follow, any change must be allowed. Therefore always, whatever were the disturbing forces, the three speeds *u, v* and *w* must be allowed to be prepared thus, if that formula $udx+vdy+wdz$ may become a complete differential and so that in addition there shall be

$$\frac{du}{dx} + \frac{dv}{dy} + \frac{dw}{dz} = 0$$

Therefore the three speed coordinates *u, v* and *w* will be able to be put in place in an infinite number of ways, in order that it may be satisfied by these two conditions, and then the pressure of the fluid will be able to be assigned at the individual points.

66. But a much more difficult question may arise, if, with the forces acting as well as with the pressure given at certain places, the very motion of the fluid at individual points will be required to be determined. Then indeed some equations of the form

$$p = C - z - uu - vv - ww - 2U,$$

will be required to be had, from which indeed since the constant *C* as well as the reckoning of the functions *u, v* and *w* must be defined, so that not only will it be required for these equations to be satisfied for these same cases, but also the rules advanced before will need to be observed, and certainly the work will require the maximum strength of calculation. Therefore it may be agreed in general to enquire into the nature of the functions, which shall be going to conform to each criterion.

67. Therefore we will begin from that convenient property of the integral, of which the differential will be required to be of the form $udx + vdy + wdz$ with the time put constant. Therefore for this integral *S*, which will be a function of *x, y* and *z,* with the time *t* involved with some constant quantities ; and if this quantity *S* is differentiated, the coefficients of the differential *dx, dy* and *dz* at once will produce the speeds *u, v* and *w,* which indeed agree for the present time with the point of the fluid *A,* of which the coordinates are *x, y* and *z.* Moreover the question is returned to this : so that there may be defined, what kind of functions of *x, y* and *z* must be assumed for *S*, so that also there may become

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0\,;$$

or since there shall become

$$u=\frac{dS}{dx},\quad v=\frac{dS}{dy},\quad w=\frac{dS}{dz},$$

so that there shall become

$$\frac{ddS}{dx^2}+\frac{ddS}{dy^2}+\frac{ddS}{dz^2}=0.$$

68. Since it may not be apparent, how this generally may be able to be set out generally, I will consider some more general cases. Therefore there shall be

$$S=\left(Ax+By+Cz\right)^n$$

and there will become

$$\frac{dS}{dx}=nA\left(Ax+By+Cz\right)^{n-1}$$

and

$$\frac{ddS}{dx^2}=n\left(n-1\right)AA\left(Ax+By+Cz\right)^{n-2}$$

and there will be similar forms for $\frac{ddS}{dy^2}$ and $\frac{ddS}{dz^2}$, from which it must be effected, so that there shall become

$$n\left(n-1\right)\left(Ax+By+Cz\right)^{n-2}\left(AA+BB+CC\right)=0,$$

to which at first it is satisfied, either if $n=0$ or $n=1$; from which now two suitable values will be found, evidently

$$S=\text{Const. and } S=Ax+By+Cz,$$

where the constants *A, B, C* and also the time can be included in these in some manner.

69. But if *n* can neither be $=0$ nor $=1$, it is necessary that there shall become:

$$AA+BB+CC=0,$$

and then a suitable value for the value *S* will be

$$S=\left(Ax+By+Cz\right)^n,$$

whatever number may be taken for the exponent n , also there is no reason why the time *t* may not be present in *n* [*i.e.* in the mathematical sense; however perhaps not in the physical sense for specific situations]. Also it is apparent the sum of any number of formulas of this kind may present a suitable value for *S*, thus, so that there may become :

$$S=\alpha+\beta x+\gamma y+\delta z+\varepsilon\left(Ax+By+Cz\right)^{n}+\zeta\left(A'x+B'y+C'z\right)^{n'}+$$
$$+\eta\left(A''x+B''y+C''z\right)^{n''}+\theta\left(A'''x+B'''y+C'''z\right)^{n'''}+\text{etc.}$$

provided there were :

$$AA+BB+CC=0,\ \ A'A'+B'B'+C'C'=0,$$
$$A''A''+B'B''+C''C''=0,\ \text{etc.}$$

70. Hence the following will be suitable values for *S* from the lower orders, where the coordinates *x, y, z* have either one, two, three or four dimensions,

I. $S=A$

II. $S=Ax+By+Cz$

III. $S=Axx+Byy+Czz+2Dxy+2Exz+2Fyz$
with $A+B+C=0$

IV. $S=AX^3+By^3+Cz^3+3Dxxy+3Fxxz+3Hyyz+6Kxyz$
$+3Exyy+3Gxzz+3Iyzz$
with $A+E+G=0,\ \ B+D+I=0,\ \ C+F+H=0$

$$V.\ S=\begin{cases}+Ax^4+6Dxxyy+4Gx^3\ y+4Hxy^3+12N\ xxyz\\ +By^4+6Exxzz\ +4IX^3z\ +4Kxz^3+12Oxyyz\\ +Cz^4+6Fyyzz+4Ly^3z+4Myz^3+12Pxyzz\end{cases}$$

with
$$A+D+E=0,\ \ G+H+P=0,$$
$$B+D+F=0,\ \ I+K+O=0,$$
$$C+E+F=0,\ \ L+M+N=0.$$

71. Hence it is evident, how these formulas shall be produced for any order : clearly at first the numerical coefficients will be given from the same individual terms, which will agree with the same terms permuted from the same law, or , which arise, if the trinomial $x+y+z$ will be raised to a power of the same order. But the indefinite letters *A, B, C* etc. shall be added to the numerical coefficients. Then with the numbers rejected it may be seen clearly, how often terms of this third kind occur

$$LZxx+MZyy+NZzz,$$

which clearly shall have the form of a common factor *Z* from the variables, as often as the sum of the coefficients of the letters $L+M+N$ may be put equal to zero. Thus so that for the fifth power there may be had:

$$S=\begin{cases}+Ax^5+5Dx^4y+5\mathfrak{D}x^4z+10Gx^3yy+10\mathfrak{C}x^3zz\ +20Kx^3yz+30Nxyyzz\\ +By^5+5Exy^4+5\mathfrak{E}y^4z+10Hx^2y^3\ \ +10\mathfrak{h}y^3zz\ +20Lxy^3z+30Oxxyzz\\ +Cz^5+5Fxz^4+5\mathfrak{F}yz^4+10Ixxz^3\ \ \ +10\mathfrak{J}yyz^3+20Mxyz^3+30Pxxyyz\end{cases}$$

the following determinations of the coefficients of the letters will be obtained :

$$A+G+\mathfrak{G}=0,\ D+H+O=0,\ \mathfrak{D}+I+P=0,$$
$$B+H+\mathfrak{h}=0,\ \ E+G+N=0,\ \mathfrak{E}\ +\mathfrak{J}+P=0,\ K+L+M=0,$$
$$C+I+\mathfrak{J}=0,\ \ \ F+\mathfrak{G}+N=0,\ \mathfrak{F}\ +\mathfrak{h}+O=0.$$

In a similar manner 15 determinations of this kind will be produced for the sixth order, for the seventh 21, for eighth 28, and thus so forth.

72. Now the first formula $S=A$, since the coordinates x, y and z plainly have no terms in common, the three speeds u, v and w will be placed equal to zero, and thus the state of the fluid will be shown to be at rest. Yet the pressure at any point will be able to be variable at whatever times. Indeed since A shall be some function of the time, for the given time t the pressure at the point A will be

$$p=C-\tfrac{2dA}{dt}-z,$$

from which formula the state of a fluid of this kind may be indicated, where the fluid may be disturbed by some moment arising from the forces acting, which yet itself always may be maintained in equilibrium, so that from these no motion of the fluid may be able to arise : where it may arise, if the fluid were enclosed in a vessel, from which at no time were it able to escape, and may be compressed in that by forces of some kind.

73. But the second formula differentiated $S=Ax+By+Cz$ will present these three velocities to the point λ :

$$u=A,\ \ v=B\ \text{ and }\ w=C.$$

Therefore in the same time all the points of the fluid will be carried equally along the same direction. From which the whole fluid will be moved likewise as a solid body, since it is carried by a single progressive motion. But at a different time both the speed as well as the direction of this motion will be able to be changed, just as the external forces acting will demand. Therefore the pressure at the point λ for the time t, of which A, B, C are functions, will become

$$p=C-z-AA-BB-CC-2x\cdot\tfrac{dA}{dt}-2y\cdot\tfrac{dB}{dt}-2z\cdot\tfrac{dC}{dt}$$

74. The third formula

$S = Axx + Byy + Czz + 2Dxy + 2Exz + 2Fyz$, where there is $A + B + C = 0$, will provide these three speeds to the point λ :

$$u = 2Ax + 2Dy + 2Ez,$$
$$v = 2By + 2Dx + 2Fz,$$
$$w = 2Cz + 2Ex + 2Fy$$
or
$$w = 2Ex + 2Fy - 2\left(A+B\right)z.$$

Therefore in this case also at the same moment of time different points of the fluid will be carried by a different motion; but in a succession of times also the motion of this same point can experience some variation in motion to emerge, since for the functions *A, B, D, E, F* it will be allowed to assume whatever value of the time *t*. But a much greater variation of the place will be had, if greater composite functions may be attributed to the point *S*.

75. Since in the second case the progression in the motion of the fluid will agree with the with the motion of a solid body, where clearly at each moment the individual parts are carried forwards by the equal motion: it may be allowed to suspect in other cases of fluid motion, also it may able to be in agreement with the motion of the solid body, either by rotation or by any other irregularity. Therefore besides the second case to be established well enough, an agreement of this kind will not be able to be shown . Indeed so that this may happen, it shall be necessary, that the pyramid (Fig. 2) $\pi\varphi\rho\sigma$ not only be equal, but also will become similar to the pyramid $\lambda\mu\nu o$, or so that there may become:

$$\pi\varphi = \lambda\mu = dx = \sqrt{(QQ + qq + \varphi\varphi)}$$
$$\pi\rho = \lambda\nu = dy = \sqrt{(RR + rr + \rho\rho)}$$
$$\pi\sigma = \lambda o = dz = \sqrt{(SS + ss + \sigma\sigma)}$$
$$\varphi\rho = \mu\nu = \sqrt{(dx^2 + dy^2)} = \sqrt{\left(\left(Q-R\right)^2 + \left(q-r\right)^2 + \left(\varphi-\rho\right)^2\right)}$$
$$\varphi\sigma = \mu o = \sqrt{(dx^2 + dz^2)} = \sqrt{\left(\left(Q-S\right)^2 + \left(q-s\right)^2 + \left(\varphi-\sigma\right)^2\right)}$$
$$\rho\sigma = \nu o = \sqrt{(dy^2 + dz^2)} = \sqrt{\left(\left(R-S\right)^2 + \left(r-s\right)^2 + \left(\rho-\sigma\right)^2\right)}$$

with the values used in paragraph 32.

76. Moreover, these latter equations taken together with the former, will be reduced to these :

$$QR+qr+\varphi\rho=0,\;\; QS+qs+\varphi\sigma=0 \text{ and } RS+rs+\rho\sigma=0,$$

but the three prior equations, if the values assigned in paragraph 34 may be substituted for the letters Q, R, S, q, r, s, φ, ρ, σ and the terms before the remaining terms may be allowed to vanish, will give these equations :

$$1=1+2Ldt,\quad l+M=0,$$
$$1=1+2mdt,\quad \lambda+N=0,$$
$$1=1+2\nu dt,\quad \mu+n=0,$$

from which there becomes $L=0$, $m=0$; $\nu=0$, $M=-l$, $N=-\lambda$, and $n=-\mu$.

77. Truly the three speeds of each point λ must be prepared thus, so that there may become

$$du=\quad +ldy+\lambda dz,$$
$$dv=\; -ldx+\mu dz,$$
$$dw=-\lambda dx-\mu dy.$$

Indeed the second condition demands a motion of the fluid, so that there shall be

$$l=M,\quad \lambda=N \text{ and } n=\mu\;;$$

so that all the coefficients l, λ and μ will vanish and the same or constant speeds *u, v* and *w* will be produced in all the particles of the fluid at the same time. Therefore it is apparent only in this case the motion of the solid body to be able to agree with the motion of the fluid.

78. But so that the effect of the forces, which act externally on the fluid, may be able to be defined, at first these forces must be defined which we assume to be present, which are required to bring about the motion: indeed for these forces, which actually disturb the fluid, equivalent forces must be established, moreover we have seen above in paragraph 56, at the point λ three accelerating forces to be required, which are related here. Whereby if the element of the fluid may be considered there, of which the volume or mass shall be

$$=dxdydz,$$

the motive forces for the motion required will be :

$$\text{along } AL = 2dxdydz(Lu + lv + \lambda w + \mathfrak{L})$$
$$= 2dxdydz(u \cdot \tfrac{du}{dx} + v \cdot \tfrac{du}{dy} + w \cdot \tfrac{du}{dz} + \tfrac{du}{dt}),$$
$$\text{along } AB = 2dxdydz(Mu + mv + \mu w + \mathfrak{M})$$
$$= 2dxdydz(u \cdot \tfrac{dv}{dx} + v \cdot \tfrac{dv}{dy} + w \cdot \tfrac{dv}{dz} + \tfrac{dv}{dt}),$$
$$\text{along } AC = 2dxdydz(Nu + nv + \nu w + \mathfrak{N})$$
$$= 2dxdydz(u \cdot \tfrac{dw}{dx} + v \cdot \tfrac{dw}{dy} + w \cdot \tfrac{dw}{dz} + \tfrac{dw}{dt}),$$

from which by a threefold integration the total forces, which must act on the mass of the fluid along these same directions, may be deduced.

79. Moreover since the second condition postulates, that $udx + vdy + wdz$ shall be a complete differential, of which the integral shall be $= S$, also with the time put to be variable, as before :

$$dS = udx + vdy + wdz + Udt,$$

from which, on account of

$$\tfrac{du}{dy} = \tfrac{dv}{dx}, \quad \tfrac{du}{dz} = \tfrac{dw}{dx}, \quad \tfrac{du}{dt} = \tfrac{dU}{dx} \text{ etc.}$$

these three motive forces will arise :

$$\text{along } AL = 2dxdydz\left(\tfrac{udu+vdvd+wdw+dU}{dx}\right),$$
$$\text{along } AB = 2dxdydz\left(\tfrac{udu+vdvd+wdw+dU}{dy}\right),$$
$$\text{along } AC = 2dxdydz\left(\tfrac{udu+vdvd+wdw+dU}{dz}\right).$$

80. Now there may be put

$$uu + vv + ww + 2U = T,$$

and T will become a function of the coordinates *x, y, z;* therefore now with the time constant, there may be put

$$dT = Kdx + kdy + \kappa dz,$$

and the three motive forces of the element *dxdydz*

$$\text{along } AL = Kdxdydz,$$
$$\text{along } AB = kdxdydz,$$
$$\text{along } AC = \chi dxdydz,$$

therefore with a threefold integration these formulas are required to be extended to the total mass of the fluid, so that thence the equivalent forces and the mean directions will be obtained for all of those. Truly this discussion is worthy of a higher investigation, to which I shall not tarry here.

81. Moreover this quantity

$$T = uu + vv + ww + 2U,$$

of which an account is required to be had in this calculation, also supplies a simpler formula for the height expressed by the pressure *p*; indeed there becomes

$$p = C - z - T,$$

if indeed the individual particles of the fluid may be acted on by gravity alone. But if any particle λ may be acted on by the three accelerating forces, which shall be Q, q and φ, acting along the three directions of the axes *AF, AB* and *AC* respectively, and from the calculation deduced as above, the pressure will be found:

$$p = C + \int (Qdx + qdy + \varphi dz) - T,$$

from which it will be apparent from the differential

$$Qdx + qdy + \varphi dz$$

must be complete, otherwise the state of equilibrium, or at least the possibility, will not be given. But this condition is required to be in agreement with the forces acting *Q, q* and φ, now demonstrated most clearly by the most celebrated Mr.Clairaut [1743].

82. Behold therefore the universal principles concerning the instruction of the motion of fluids, even if at the first glance they may not be seen to be exceedingly productive, yet almost everything has been included, which at this point is treated both for hydrostatics as well as in hydraulics, thus so that these principles shall be considered to be the widest to be allowed. So that which may appear clearer, it will be worth the effort to show, how from the hydrostatic and hydraulic precepts known so far, the principles shall follow treated clearly and distinctly.

83. Therefore we will consider first the fluid to be in a state of rest, thus so that there shall be $u = 0$, $v = 0$ and $w = 0$, and the pressure at some point λ in the fluid, on account of $T = 2U$, will become

$$p = C + \int (Qdx + qdy + \varphi dz) - 2U,$$

where, since *U* shall be a function of the time *t,* which we assume constant, since we want to investigate the pressure for a given time, this quantity *U* will be taken in this constant *C*, thus so that there shall be :

$$p = C + \int (Qdx + qdy + \varphi dz),$$

where *Q, q* and *p* are forces acting on the particle of water λ along the axes *AL, AB* and *AC*.

84. Since the pressure *p* can depend only on the point λ, that is on the coordinates *x, y* and *z,* it is necessary that

$$\int (Qdx + qdy + \varphi dz)$$

shall be a function requiring to be determined from these, which therefore permits integration. From which it is apparent at first, as I have just agreed, a fluid cannot exist in equilibrium, unless the forces disturbing the individual elements of the fluid, were prepared thus , so that the formula

$$Qdx + qdy + \varphi dz$$

shall be a complete differential. Therefore the complete differential of this if it may be put $= P$, the pressure at λ

$$p = C + P .$$

Thus so that only gravity shall be present acting along the direction *CA*, there will become $p = C - z$, from which if the pressure at one point λ may be agreed, from which the constant C may be agreed to be deduced, thence at the same time the pressure will defined at all points generally.

85. Yet meanwhile the fluid pressure at the same point will be able to be varied, which clearly will happen, if the external forces acting on the water were variable, of which an account is not yet had in terms of these forces, which are assumed to act on the individual elements one by one, thus so that they may in turn preserve equilibrium and will produce no motion. But if these forces preserve equilibrium, the letter *C* will actually denote a constant not depending on the time t ; and at the same place λ the same pressure always will be found : $p = C + P$.

86. Therefore in a fluid of this kind its permanent state will be able to be determined by an external figure, which is established with no forces present. For it may be contained at this extremity, where the fluid is itself remaining and not contained by the walls of the vessel, by which perhaps it is enclosed, it is necessary that the pressure shall be zero. [There will still be surface tension forces, and the air pressure: otherwise the liquid would evaporate.] Therefore this equation will be had : $P = \text{const.}$, by which the external figure of the surface of the fluid will be expressed by a relation between the three coordinates x, y and z . And if for the outer surface there were $P = E$, on account of $C = -E$, in any other place λ the internal pressure $p = P - E$. Thus so that if the particles of the fluid

may be acted on by gravity alone, on account of $p = C - z$, for the extremity of the surface $z = C$, by which it is understood the free surface to be horizontal.

[There will still be surface tension forces, and the air pressure: otherwise the liquid would evaporate. Nevertheless, we can understand what Euler is saying at this stage; in fact, his theory of fluid flow can be extended to the flow of heat and electric currents (not known to exist in Euler's day!) treated as fluids in mediums, where the function *S* at (*x, y, z*) can be treated as the temperature *T* or electrical potential *V* at this same point, satisfying the same kind of conditions, as occurs in the ubiquitous Laplacian $\frac{ddS}{dx^2} + \frac{ddS}{dy^2} + \frac{ddS}{dz^2} = 0.$]

87. Thence also everything, which at this point have been deduced concerned with the motion of fluids through pipes, are deduced easily from these principles. But pipes are accustomed to be considered to be the most narrow, or such are assumed, so that a fluid may flow through any section normal to the pipe with an equal motion : from which this rule arises, so that the speed of the fluid in place in some pipe shall be inversely proportional to its area of cross-section. Therefore (Fig. 2) λ shall be some point of this kind of pipe, of which the figure will be expressed by twin equations between the coordinates *x, y* and *z*, thus so that thence for any abscissa *x* both the remaining *y* and *z* may be able to be found.

88. In addition the cross-section of this pipe shall be at $\lambda, = rr$, but at another fixed place, where the cross-section $= ff$, at the present time the speed of the fluid shall be Γ, from hence in the minute time elapsed *dt* that may become $= \Gamma + d\Gamma$, and therefore Γ will be a function of the time *t* only, and equally $\frac{d\Gamma}{dt}$. Hence therefore truly the speed of the fluid at λ will be given at the time

$$V = \frac{ff\Gamma}{rr}$$

Now, since from the shape of the pipe, *y* and *z* will be given by *x,* there shall become

$$dy = \eta dx \text{ and } dz = \theta dx;$$

from which the following three speeds of the point at λ will be along the directions *AL, AB* and *AC* :

$$u = \frac{ff\Gamma}{rr} \cdot \frac{1}{\sqrt{(1+\eta\eta+\theta\theta)}}, \quad v = \frac{ff\Gamma}{rr} \cdot \frac{\eta}{\sqrt{(1+\eta\eta+\theta\theta)}}, \quad w = \frac{ff\Gamma}{rr} \cdot \frac{\theta}{\sqrt{(1+\eta\eta+\theta\theta)}}$$

and hence there will become

$$uu+vv+ww = VV = \frac{f^4\Gamma\Gamma}{r^4},$$

and *rr* is a function of *x,* and thence the dependence of *y* and *z.*

89. Now since $udx+vdy+wdz$ must be a complete differential, we may put its integral $=S$, and there will become :

$$dS=\tfrac{ff\Gamma}{rr}\cdot\tfrac{dx(1+\eta\eta+\theta\theta)}{\sqrt{(1+\eta\eta+\theta\theta)}}=\tfrac{ff\mho}{rr}\cdot dx\sqrt{(1+\eta\eta+\theta\theta)}.$$

But $dx\sqrt{(1+\eta\eta+\theta\theta)}$ expresses an element of this pipe, which if we may put $=ds$, there will become

$$dS=\tfrac{ff\Gamma ds}{rr},$$

from which since here the time t shall be assumed constant, of which Γ is a function, moreover the quantities s and rr shall not depend on the time t, but only on the shape of the cross-section of the pipe, there will become

$$S=\Gamma\int\tfrac{ffds}{rr}.$$

90. Now for finding the pressure p, which now which now is located in the pipe at the point λ, the quantity U must be considered, which arises from the differentiation of the quantity S, if only the time t may be treated as a variable, thus so that there shall become $U=\frac{dS}{dt}$.

Therefore since the formula of the integral $\int\frac{ffds}{rr}$ may not involve the time t, there will be generally

$$\tfrac{dS}{dt}=U=\tfrac{d\Gamma}{dt}\int\tfrac{ffds}{rr};$$

and thus there will become from paragraph 80:

$$T=\tfrac{f^4\Gamma\Gamma}{r^4}+\tfrac{2d\Gamma}{dt}\int\tfrac{ff}{rr}ds.$$

Whereby for whatever disturbing forces put in place Q, q and φ, the pressure at λ will become :

$$p=C+\int\left(Qdx+qdy+\varphi dz\right)-\tfrac{f^4\Gamma\Gamma}{r^4}-\tfrac{2d\Gamma}{dt}\int\tfrac{ffds}{rr},$$

which is that same formula, which is accustomed commonly to be deduced for the motion of a fluid through pipes, and thus appears to be much more general, since any fluid forces have been considered acting here, while commonly this formula is restricted to gravity alone. Meanwhile here it is required to be considering by necessity three forces Q, q and φ thus required to be prepared, so that $Qdx+qdy+\varphi dz$ shall be a complete differential or it may be allowed to be integrated.

PRINCIPIA MOTUS FLUIDORUM
Commentatio 258 indicis ENESTROEMIANI

Novi commentarii academiae scientiarum Petropolitanae 6 (1756/7), 1761, p. 271-311

PARS PRIOR

1. Cum corpora fluida a solidis hoc potissimum differant, quod eorum particulae a se invicem omnino sint dissolutae, hae etiam diversissimos motus recipere possunt, neque motus, quo unaquaeque fluidi particula fertur, a motu reliquarum particularum ita determinatur, ut alio motu progredi non possit. Longe aliter autem res se habet in corporibus solidis, quae si fuerint inflexibilla nullamque figurae sua mutationem patiantur, utcunque moveantur, singulae eorum particulae perpetuo eundem inter se situm ac distantiam servant ; unde fit, ut, cognito motu duarum triumve tantum particularum, statim alius cuiuscunque particulae motus definiri queat ; neque etiam duarum triumve huiusmodi corporum particularum motus ad lubitum fingi potest, sed is ita comparatus esse debet, ut hae particulae eundem perpetuo situm relativum inter se obtineant.

2. Quodsi autem corpora solida fuerint flexibilla, singulaeum particularum motus minus determinatur: cum ob flexuras tam distantia quam situs relativus diversarum particularum mutationes admittat. Interim tamen ipsa flexurae ratio legem quandam, quam diversae huiusmodi corporum particulae in motu suo sequi debent, constituit: quippe qua caveri oportet, ne partes, quae circa se invicem tantum inflecti se patiuntur, vel a se penitus divellantur vel in se invicem intrudantur; quod quidem posterius impenetrabilitas omnibus corporibus communis exigit.

3. In corporibus autem fluidis, quorum particulae nullo nexu inter se uniuntur, motus quoque diversarum particularum multo minus restringitur: neque ex motu quotcunque particularum motus reliquarum determinatur. Si enim vel centum particularum motus tanquam cognitus assumatur, manifestum est motus, quorum reliquae particulae capaces sint futurae, adhuc in infinitum variari posse. Ex quo concludendum videtur motum cuiusque particulae fluidi plane non a motu reliquarum pendere, nisi forte his ita fuerit interclusa, ut eas necessario sequi cogatur.

4. Interim tamen fieri non potest, ut motus omnium fluidi particularum nullis omnino legibus adstringatur; neque adeo pro lubitu motum, qui singulis particulis inesse concipitur, fingere licet. Cum enim particulae sint impenetrabiles, statim patet eiusmodi motum subsistere non posse, quo aliae particulae per alias transirent, sicque se mutuo penetrarent: atque, ob hanc causam, talis motus ne cogitatione quidem in fluido inesse concipi potest. Quoniam igitur infinitos motus excludi oportet, quorum pacto reliqui sint comparati, et quanam proprietate ab illis distinguantur, operae pretium videtur accuratius definire.

5. Antequam enim motus, quo fluidum quodpiam actu agitatur, assignari queat, necessarium videtur, ut omnes motus, qui quidem in hoc fluido subsistere possent, dignoscantur: quos motus hic possibiles vocabo, ut a motibus impossibilibus, qui ne locum quidem habere possunt, distinguam. In hunc finem nobis constituendus erit character, motibus possibilibus conveniens, eosque ab impossibilibus segregans ; quo facto ex motibus possibilibus quovis casu eum determinari oportebit, qui actu inesse debebit. Tum scilicet ad vires, quibus aqua sollicitatur, erit respiciendum, ut motus, qui illis sit conformis, ex mechanicae principiis definiri possit.

6. In characterem igitur motuum possibilium, quicunque scilicet salva impenetrabilitate in fluido inesse possunt, inquirere hic constitui. Fluidum autem eius indolis assumo, ut neque in arctius spatium compelli se patiatur, neque eius continuitas interrumpi possit: statuo nimirum in medio fluidi durante motu nullum spatium a fluido vacuum relinqui, sed continuitatem in eo iugiter conservari. Theoria enim ad fluida huius naturae accommodata, non adeo difficile erit eam ad fluida quoque, quorum densitas est variabilis et quae ne continuitatem quidem necessario requirunt, extendere.

7. Si igitur in huiusmodi fluido consideretur portio quaecunque, motus, quo singulae eius particulae feruntur, ita debet esse comparatus, ut omni tempore aequale spatium adimpleant. Hoc enim si in singulis portionibus eveniat, omnis vel expansio in maius spatium, vel coarctatio in minus spatium praepedietur ; atque huiusmodi motus, si ad hanc solam indolem respiciamus, qua fluidum neque expansionis neque condensationis capax statuitur, omnino pro possibili erit habendus. Quod autem hic de qualibet fluidi portione dictum est, de singulis eius elementis est intelligendum; ita ut cuiusque elementi volumen perpetuo eiusdem quantitatis manere debeat.

8. Quo ergo huic conditioni satisfiat, in singulis fluidi punctis motus quicunque inesse concipiatur; tum sumto quocunque fluidi elemento investigetur translatio momentanea singulorum eius terminorum, sicque innotescet spatiolum, in quo hoc elementum elapso tempusculo minimo continebitur. Deinde hoc spatiolum illi, quod ante occupaverat, aequale statuatur, haecque aequatio rationem motus, quatenus erit possibilis, indicabit. Quodsi enim singulae elementa singulis tempusculis aequalia spatiola occupent, neque ulla fluidi compressio neque expansio orietur; motusque ita erit comparatus, ut pro possibili sit habendus.

9. Cum autem hic non solum celeritas motus, qui singulis fluidi punctis inesse concipitur, spectari debeat, sed etiam eius directio, haec utraque consideratio commodissime instituetur, si motus cuiusque puncti secundum directiones fixas resolvatur. Haec autem resolutio vel secundum binas vel ternas directiones fieri solet : priori enim resolutione uti licet, si singulorum punctarum motus in eodem plano absolvatur; sin autem eorum motus non in eodem plano contineatur, tum motum secundum ternos axes fixos resolvi oportet. Quoniam igitur hic posterior

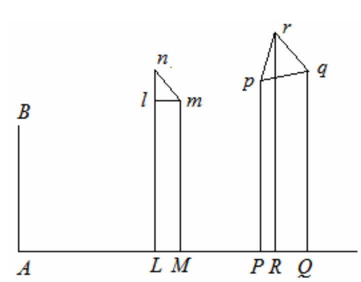

Fig. 1

casus
plus difficultatis habet quam prior, investigationem motuum possibilium a casu priori incipi conveniet, qua expedita casus posterior facilius expedietur.

10. Primum igitur fluido duas tantum dimensiones tribuam, ita ut singulae eius particulae non solum nunc quidem in eodem plano reperiantur, sed etiam earum motus in eodem plano absolvatur. Hoc itaque planum plano tabulae (Fig.1) representetur et consideretur fluidi quodcunque punctum *l,* cuius situs per coordinatas orthogonales $AL = x$ et $Ll = y$ referatur, tum vero eius motus, quo nunc quidem fertur, secundum easdem directiones resolutus praebeat celeritatem secundum axem *AL* vel secundum $lm = u$, et secundum alterum axem *AB* vel secundum $ln = v$; ita ut vera huius puncti celeritas futura sit $= \sqrt{(uu + vv)}$ eiusque directio ad axem *AL* inclinata sit angulo, cuius tangens $= \frac{v}{u}$.

11. Cum statum motus praesentem tantum, qui singulis fluidi punctis conveniat, evolvere sit propositum, celeritates *u* et *v* a situ puncti *l* unice pendebunt eruntque idcirco tanquam functiones coordinatarum *x* et *y* spectandae. Ponamus igitur esse differentiatione instituta:

$$du = Ldx + ldy \quad \text{et} \quad dv = Mdx + mdy,$$

quae formulae differentiales cum sint completae, constat fore

$$\frac{dL}{dy} = \frac{dl}{dx} \quad \text{et} \quad \frac{dM}{dy} = \frac{dm}{dx},$$

ubi notandum est in huiusmodi expressione $\frac{dL}{dy}$ differentiale ipsius *L* seu *dL,* tantum ex variabilitate ipsius *y* capiendum esse, similique modo in expressione $\frac{dl}{dx}$ pro *dl* id differentiale ipsius *l* sumi debet, quod oritur, si tantum *x* pro variabili habeatur.

12. Probe ergo cavendum est, ne in huiusmodi expressionibus fractis

$$\frac{dL}{dy}, \frac{dl}{dx}, \frac{dM}{dy} \text{ et } \frac{dm}{dx}$$

numeratores *dL, dl, dM* et *dm* differentialia completa functionum *L, l, M* et *m* designare putentur ; sed perpetuo ea tantum earum differentialia denotant, quae ex variabilitate unicae coordinatae, eius scilicet, cuius differentiale in denominatore exhibetur, oriuntur; sicque huiusmodi expressiones semper quantitates finitas ac determinatas repraesentabunt. Simili autem modo intelligitur fore

$$L = \frac{du}{dx}, \quad l = \frac{du}{dy}, \quad M = \frac{dv}{dx} \quad \text{et} \quad m = \frac{dv}{dy},$$

qua notandi ratione primum Clarissimus Fontaine usus est, et quia non contemnendum calculi compendium largitur, eam hic quoque adhibebo.

13. Cum igitur sit

$$du = Ldx + ldy \quad \text{et} \quad dv = Mdx + mdy,$$

hinc geminas celeritates cuiusque alius puncti, quod quidem infinite parum a puncto l distat, assignare licebit ; si enim talis puncti a puncto l distantia secundum axem AL sit $= dx$, et secundum axem $AB = dy$, tum huius puncti celeritas secundum axem AL erit

$$= u + Ldx + ldy;$$

celeritas autem secundum alterum axem AB

$$= v + Mdx + mdy.$$

Tempusculo ergo infinite parvo dt hoc punctum proferetur secundum directionem axis AL per spatiolum

$$= dt(u + Ldx + ldy)$$

et secundum directionem alterius axis AB per spatiolum

$$= dt\left(v + Mdx + mdy\right).$$

14. His notatis consideremus elementum aquae triangulare lmn et quaeramus situm, in quem id ob motum, quem ipsi insitum concipimus, tempusculo dt transferatur. Sit autem huius elementi triangularis lmn latus lm axi AL, latus vero ln axi AB parallelum ac ponatur $lm = dx$ et $ln = dy$; seu sint pro puncto m coordinatae $x + dx$ et y, pro puncto n autem sint coordinatae x et $y + dy$. Patet autem, quoniam relationem inter differentialia dx et dy non definimus eaque tam negative quam affirmative accipi possunt, totam fluidi massam in huiusmodi elementa cogitatione dividi posse, ita ut, quod de uno in genere definiemus, id aeque ad omnia pateat.

15. Ut igitur pateat, quorum elementum hoc lmn, ob motum insitum, tempusculo dt transferatur, quaeramus puncta p, q et r, in quae eius anguli seu puncta l, m et n tempusculo dt transferentur. Cum igitur sit

	puncti l	puncti m	puncti n
Celeritas secundum AL=	u	$= u + Ldx$	$= u + ldy$
Celeritas secundum AB=	v	$= v + Mdy$	$= v + mdy$,

punctum l perveniet tempusculo dt in p, ut sit :

$$AP - AL = udt \text{ et } Pp - Ll = vdt.$$

Punctum autem m perveniet in q, ut sit :

$$AQ - AM = (u + Ldx)dt \text{ et } Qq - Mm = \left(v + Mdx\right)dt.$$

At punctum n feretur in r, ut sit:

$$AR - AL = (u + ldy)dt \text{ et } Rr - Ln = \left(v + mdy\right)dt.$$

16. Cum igitur puncta *l, m* et *n* tempusculo *dt* in puncta *p, q* et *r* transferantur, iunctis lineolis rectis *pq, pr* et *qr* triangulum *lmn* in situm, quem triangulum *pqr* refert, pervenire censendum est. Quoniam enim triangulum *lmn* statuitur infinite parvum, eius latera per motum curvaturam recipere nequeunt, ideoque elementum aquae *lmn* post translationem tempusculo *dt* factam etiamnum figuram triangularem *pqr,* et quidem rectilineam, retinebit. Cum igitur hoc elementum *lmn* per motum neque in maius spatium extendi neque in minus compingi debeat, motum ita comparatum esse oportet, ut area trianguli *pqr* aequalis areae trianguli *lmn* reddatur.

17. Trianguli autem *lmn,* cum sit ad *l* rectangulum, area est $=\frac{1}{2}dxdy$, cui propterea area trianguli *pqr* aequalis est statuenda. Ad hanc autem aream inveniendam considerandae sunt punctorum *p, q, r* binae coordinatae, quae sunt:

$$AP = x+udt,\;\; AQ = x+dx+\left(u+Ldx\right)dt,\;\; AR = x+\left(u+ldy\right)dt,$$
$$Pp = y+vdt,\;\; Qq = y+(v+Mdx)dt,\;\; Rr = y+dy+\left(v+mdy\right)dt.$$

Tum vero area trianguli *pqr* ex arcis sequentium trapeziorum ita reperitur, ut sit:

$$pqr = PprR + RrqQ - PpqQ.$$

Cum autem haec trapezia bina latera parallela basique *AQ* perpendicularia habeant, eorum areae facile assignantur.

18. Erit enim, uti ex elementis constat :

$$PprR = \tfrac{1}{2}PR\left(Pp+Rr\right),$$
$$RrqQ = \tfrac{1}{2}RQ(Rr+Qq),$$
$$PpqQ = \tfrac{1}{2}PQ(Pp+Qq).$$

His igitur colligendis reperietur:

$$\Delta pqr = \tfrac{1}{2}PQ\cdot Rr - \tfrac{1}{2}RQ\cdot Pp - \tfrac{1}{2}PR\cdot Qq.$$

Ponatur brevitatis gratia

$$AQ = AP+Q,\; AR = AP+R,\; Qq = Pp+q \;\text{ et }\; Rr = Pp+r,$$

ut sit

$$PQ = Q,\;\; PR = R \;\text{ et }\; RQ = Q-R,$$

eritque

$$\Delta pqr = \tfrac{1}{2}Q(Pp+r) - \tfrac{1}{2}\left(Q-R\right)Pp - \tfrac{1}{2}R\cdot(Pp+q)$$

sive

$$\Delta pqr = \tfrac{1}{2}Q\cdot r - \tfrac{1}{2}R\cdot q.$$

19. Est vero ex valoribus coordinatarum ante exhibitis

$$Q = dx + Ldxdt, \quad q = Mdxdt,$$
$$R = ldydt, \qquad r = dy + mdydt;$$

quibus valoribus substitutis orietur area trianguli

$$\Delta\ pqr = \tfrac{1}{2}dxdy(1+Ldt)(1+mdt) - \tfrac{1}{2}Mldxdydt^2$$

sive

$$\Delta\ pqr = \tfrac{1}{2}dxdy(1+Ldt+mdt+Lmdt^2-Mldt^2),$$

quae cum aequalis esse debeat areae trianguli *lmn,* quae est $=\frac{1}{2}dxdy$, haec nascetur aequatio :

$$Ldt+mdt+Lmdt^2-Mldt^2=0$$

sive

$$L+m+Lmdt-Mldt=0.$$

20. Cum igitur termini *Lmdt* et *Mldt* prae finitis *Lm* et *Ml* evanescant, habebitur haec aequatio

$$L+m=0.$$

Quam ob rem, ut motus sit possibilis, celeritates *u* et *v* puncti cuiuscunque *l* ita debent esse comparatae, ut positis earum differentialibus

$$du = Ldx+ldy \ \text{ et } \ dv = Mdx+mdy$$

sit $L+m=0.$ Seu cum sit

$$L=\tfrac{du}{dx} \ \text{ et } \ m=\tfrac{dv}{dy},$$

celeritates *u* et *v,* quae puncto *l* secundum directiones axium *AL* et *AB* inesse concipiuntur, eiusmodi functiones coordinatarum *x* et *y* esse debent, ut sit

$$\tfrac{du}{dx}+\tfrac{dv}{dy}=0,$$

sicque motuum possibilium criterium in hoc consistit, ut sit

$$\tfrac{du}{dx}+\tfrac{dv}{dy}=0,$$

nisi enim haec conditio locum habeat, motus fluidi subsistere nequit.

21. Eodem modo erit procedendum, si motus fluidi non absolvatur in eodem plano. Ponamus igitur, ut quaestionem latissimo sensu acceptam expediamus, singulas fluidi particulas motu quocunque inter se agitari, hac solum lege observata, ut neque condensatio neque expansio partium usquam eveniat:
quaeritur igitur simili modo, quaenam hinc determinatio ad celeritates, quas singulis punctis inesse concipimus, accedat, ut motus possibilis reddatur : seu, quod eodem redit, omnes motus, qui hisce conditionibus adversantur, a possibilibus removere oportet, quo ipso criterium motuum possibilium constituetur.

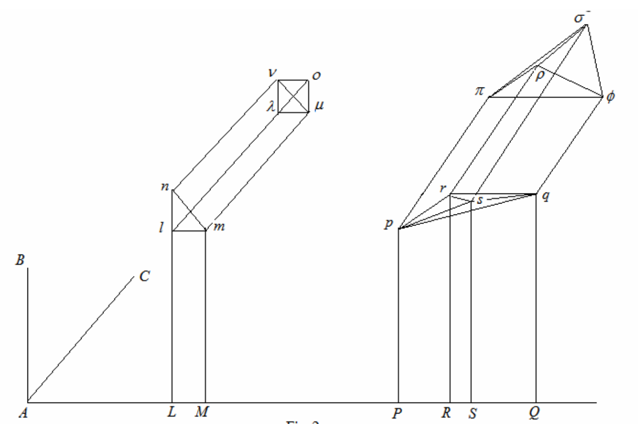

Fig. 2

22. Consideremus igitur punctum fluidi quodcunque λ, ad cuius situm repraesentandum utamur tribus axibus fixis (Fig. 2) *AL, AB* et *AC* inter se normalibus. Sint igitur ternae puncti λ coordinatae his axibus parallelae $AL = x$, $Ll = y$ et $l\lambda = z$; quae obtinentur, si primum a puncto λ ad planum duobus axibus *AL* et *AB* determinatum demittatur perpendiculum λ*l,* tum vero ex puncto *l* ad axem *AL* perpendicularis *lL* agatur. Hoc itaque modo situs puncti λ per ternas istas coordinatas generalissime exprimitur atque ad omnia fluidi puncta accommodari potest.

23. Quicunque porro sit motus puncti λ*,* is secundum ternas directiones $\lambda\mu$, λv, et λo axibus *AL, AB* et *AC* parallelas resolvi poterit. Sit igitur puncti λ

celeritas secundum directionem $\lambda\mu = u$,
celeritas secundum directionem $\lambda v = v$,
celeritas secundum directionem $\lambda o = w$,

et cum hae celeritates pro vario puncti λ situ utcunque variare possint, erunt eae tanquam functiones ternarum coordinatarum *x, y* et *z* considerandae. Iis igitur differentiatis, ponamus prodire:

$$du = Ldx + ldy + \lambda dz,$$
$$dv = Mdx + mdy + \mu dz,$$
$$dw = Ndx + ndy + vdz,$$

eruntque porro quantitates $L,\ l,\ \lambda,\ M,\ m,\ \mu,\ N,\ n,\ \nu$ functiones coordinatarum *x, y* et *z*.

24. Quoniam hae formulae differentiales sunt completae, sequitur simili modo, ut supra, fore :

$$\frac{dL}{dy}=\frac{dl}{dx},\quad \frac{dL}{dz}=\frac{d\lambda}{dx},\quad \frac{dl}{dz}=\frac{d\lambda}{dy},$$
$$\frac{dM}{dy}=\frac{dm}{dx},\quad \frac{dM}{dz}=\frac{d\mu}{dx},\quad \frac{dm}{dz}=\frac{d\mu}{dy},$$
$$\frac{dN}{dy}=\frac{dn}{dx},\quad \frac{dN}{dz}=\frac{d\nu}{dx},\quad \frac{dn}{dz}=\frac{d\nu}{dy},$$

siquidem in numeratoribus ea tantum coordinatarum, cuius differentiale in denominatore exhibetur, pro variabili assumatur.

25. Triplici ergo motu hoc, quem in puncto *X* inesse concipimus, hoc punctum λ tempusculo *dt* ita movebitur, ut

secundum directionem axis *AL* per spatiolum $=udt$,
secundum directionem axis *AB* per spatiolum $=vdt$,
secundum directionem axis *AC* per spatiolum $=wdt$

promoveatur. Sin autem puncti λ vera celeritas, quae scilicet ex compositione huius triplicis motus oritur, dicatur $=V$, erit ob normalitatem trium directionum

$$V=\sqrt{(uu+vv+ww)}$$

et spatiolum, quod tempusculo *dt* motu suo vero conficit, erit $=Vdt$.

26. Consideremus iam fluidi elementum quodpiam solidum, ut videamus, quorsum id tempusculo *dt* proferatur; et quoniam perinde est, cuiusmodi figuram isti elemento tribuamus, dummodo ita generatim definiatur, tota fluidi massa in eiusmodi elementa divisa concipi queat; sit, ut calculo consulatur, eius figura pyramis triangularis rectangula, terminata quatuor angulis solidis $\lambda,\ \mu,\ \nu$ et *o*, ita ut pro singulis sint ternae coordinatae:

	puncti λ	puncti μ	puncti ν	puncti o
secundum *AL*	x	$x+dx$	x	x
secundum *AB*	y	y	$y+dy$	y
secundum *AC*	z	z	z	$z+dz$

et cum basis huius pyramidis sit

$$\lambda\mu\nu=lmn=\tfrac{1}{2}dxdy,$$

altitudo vero $\lambda o=dz$, erit eius soliditas

$$=\tfrac{1}{6}dxdydz.$$

27. Investigemus iam, quorsum singuli isti pyramidis anguli $\lambda,\ \mu,\ \nu$ et o, tempusculo dt transferantur: ad quod eorum ternas celeritates secundum directiones ternorum axium contemplari oportet, quae ex celeritatum u, v, w valoribus differentialibus erunt :

Celeritas secundum	puncti λ	puncti μ	puncti ν	puncti o
directionem AL	u	$u+Ldx$	$u+ldy$	$u+\lambda dz$
directionem AB	v	$v+Mdx$	$v+mdy$	$v+\mu dz$
directionem AC	w	$w+Ndx$	$w+ndy$	$w+\nu dz$

28. Quodsi ergo puncta $\lambda,\ \mu,\ \nu$ et o, tempusculo dt in puncta $\pi,\ \varphi,\ \rho$ et σ transferri ponamus horumque punctorum ternas coordinatas axibus parallelas constituamus, erunt translationes momentaneae secundum hos axes

$AP-AL=udt$	$Pp-Ll=vdt$	$p\pi-l\lambda=wdt$
$AQ-AM=(u+Ldx)dt$	$Qq-Mm=(v+Mdx)dt$	$q\varphi-m\mu=(w+Ndx)dt$
$AR-AL=(u+ldy)dt$	$Rr-Ln=(v+mdy)dt$	$r\rho-n\nu=(w+ndy)dt$
$AS-AL=(u+\lambda dz)dt$	$Ss-Ll=(v+\mu dz)dt$	$s\sigma-lo=(w+\nu dz)dt$

Hinc ergo ternae coordinatae pro his quatuor punctis $\pi,\ \varphi,\ \rho$ et σ erunt:

$AP=x+udt,$	$Pp=y+vdt,$	$p\pi=z+wdt.$
$AQ=x+dx+(u+Ldx)dt,$	$Qq=y+(v+Mdx)dt,$	$q\varphi=z+(w+Ndx)dt,$
$AR=x+(u+ldy)dt,$	$Rr=y+dy+(v+mdy)dt,$	$r\rho=z+(w+ndy)dt,$
$AS=x+(u+\lambda dz)dt,$	$Ss=y+(v+\mu dz)dt,$	$s\sigma=z+dz+(w+\nu dz)dt.$

29. Cum igitur elapso tempusculo dt anguli pyramidis $\lambda,\ \mu,\ \nu$ et o, in puncta $\pi,\ \varphi,\ \rho$ et σ sint translati, ipsa pyramis nunc pyramidem pariter triangularem $\pi\varphi\rho\sigma$ constituet ; ideoque ob indolem fluidi efficiendum est, ut soliditas pyramidis $\pi\varphi\rho\sigma$ aequalis sit soliditati pyramidis propositae $\lambda\mu\nu o$ seu

$$=\tfrac{1}{6}dxdydz.$$

Totum ergo negotium huc redit, ut soliditas pyramidis $\pi\varphi\rho\sigma$ determinetur. Perspicuum autem est hanc pyramidem relinqui, si a solido $prq\pi\varphi\rho\sigma$ auferatur solidum $pq\pi\varphi\rho$ *;* quod posterius solidum est prisma basi triangulari pqr normaliter insistens et superne sectione obliqua $\pi\rho\varphi$ truncatum.

30. In huiusmodi autem prismata truncata tria quoque alterum solidum $prq\pi\varphi\rho\sigma$

resolvi potest, quae sunt:

$$\text{I. } pqs\pi\varphi\sigma, \quad \text{II. } prs\pi\rho\sigma, \quad \text{III. } qrs\varphi\rho\sigma$$

sicque effici debet, ut sit

$$\tfrac{1}{6}dxdydz = pqs\pi\varphi\sigma + prs\pi\rho\sigma + qrs\varphi\rho\sigma - pqr\pi\varphi\rho.$$

Cum autem huiusmodi prisma basi suae inferiori normaliter insistat, tres autem altitudines habeat inaequales, eius soliditas reperietur, si basis multiplicetur per trientem summae trium istarum altitudinum.

31. Hinc ergo soliditates horum prismatum truncatorum erunt:

$$pqs\pi\varphi\sigma = \tfrac{1}{3}pqs(p\pi + q\varphi + s\sigma),$$
$$prs\pi\rho\sigma = \tfrac{1}{3}prs(p\pi + r\rho + s\sigma),$$
$$qrs\varphi\rho\sigma = \tfrac{1}{3}qrs(q\varphi + r\rho + s\sigma),$$
$$pqr\pi\varphi\rho = \tfrac{1}{3}pqr(p\pi + q\varphi + r\rho).$$

Cum autem sit

$$pqr = pqs + prs + qrs,$$

erit summa trium priorum prismatum postremo minuta sive

$$\tfrac{1}{6}dxdydz = -\tfrac{1}{3}p\pi \cdot qrs - \tfrac{1}{3}q\varphi \cdot prs - \tfrac{1}{3}r\varphi \cdot pqs + \tfrac{1}{3}s\sigma \cdot pqr$$

sive

$$dxdydz = 2pqr \cdot s\sigma - 2pqs \cdot r\rho - 2prs \cdot q\varphi - 2qrs \cdot p\pi.$$

32. Superest igitur, ut horum prismatum bases definiantur; verum antequam hoc faciamus, ponamus ad sequentem calculum contrahendum :

$$AQ = AP + Q, \quad Qq = Pp + q, \quad q\varphi = p\pi + \varphi,$$
$$AR = AP + R, \quad Rr = Pp + r, \quad r\rho = p\pi + \rho,$$
$$AS = AP + S, \quad Ss = Pp + s, \quad s\sigma = p\pi + \sigma,$$

atque his postremis valoribus substitutis, termini $p\pi$ continentes se mutuo destruent eritque

$$dxdydz = 2pqr \cdot \sigma - 2pqs \cdot \rho - 2prs \cdot \varphi$$

sicque numerus basium investigandarum unitate est imminutus.

33. Iam triangulum *pqr* reperitur, si a figura *PprqQ* seu a summa trapeziorum *PprR*+*RrqQ* auferatur trapezium *PpqQ;* unde erit

$$\Delta pqr = \tfrac{1}{2}PR(Pp+Rr)+\tfrac{1}{2}RQ(Rr+Qq)-\tfrac{1}{2}PQ(Pp+Qq);$$

sive ob $PR = R,\ RQ = Q - R$ et $PQ = Q$ erit

$$\Delta pqr = \tfrac{1}{2}R(Pp-Qq)+\tfrac{1}{2}Q(Rr-Pp) = \tfrac{1}{2}Qr-\tfrac{1}{2}Rq.$$

Simili modo erit :

$$\Delta pqs = \tfrac{1}{2}PS(Pp+Ss)+\tfrac{1}{2}SQ(Ss+Qq)-\tfrac{1}{2}PQ(Pp+Qq),$$

seu

$$\Delta pqs = \tfrac{1}{2}S(Pp+Ss)+\tfrac{1}{2}(Q-S)(Ss+Qq)-\tfrac{1}{2}Q(Pp+Qq),$$

unde fit

$$\Delta pqs = \tfrac{1}{2}S(Pp-Qq)+\tfrac{1}{2}Q(Ss-Pp) = \tfrac{1}{2}Qs-\tfrac{1}{2}Sq.$$

Ac denique

$$\Delta prs = \tfrac{1}{2}PR(Pp+Rr)+\tfrac{1}{2}RS(Rr+Ss)-\tfrac{1}{2}PS(Pp+Ss)$$

seu

$$\Delta prs = \tfrac{1}{2}R(Pp+Rr)+\tfrac{1}{2}(S-R)(Rr+Ss)-\tfrac{1}{2}S(Pp+Ss),$$

unde fit:

$$\Delta prs = \tfrac{1}{2}R(Pp-Ss)+\tfrac{1}{2}S(Rr-Pp) = \tfrac{1}{2}Sr-\tfrac{1}{2}Rs.$$

34. His igitur valoribus substitutis obtinebimus

$$dxdydz = (Qr-Rq)\sigma+(Sq-Qs)\rho+(Rs-Sr)\varphi$$

sive pyramidis $\pi\varphi\rho\sigma$ soliditas erit

$$\tfrac{1}{6}(Qr-Rq)\sigma+\tfrac{1}{6}(Sq-Qs)\rho+\tfrac{1}{6}(Rs-Sr)\varphi.$$

Est autem ex coordinatarum valoribus supra paragrapho 28 exhibitis

$$\begin{aligned}
&Q = dx+Ldxdt, && q = Mdxdt, && \varphi = Ndxdt,\\
&R = ldydt, && r = dy+mdydt, && \rho = ndydt,\\
&S = \lambda dzdt, && s = \mu dzdt, && \sigma = dz+\nu dzdt.
\end{aligned}$$

35. Cum igitur hinc fiat

$$\begin{aligned}
&Qr-Rq = dxdy(1+Ldt+mdt+Lmdt^2-Mldt^2),\\
&Sq-Qs = dxdz(-\mu dt-L\mu dt^2+M\lambda dt^2),\\
&Rs-Sr = dydz(-\lambda dt-m\lambda dt^2+l\mu dt^2),
\end{aligned}$$

reperietur soliditas pyramidis $\pi\varphi\rho\sigma$ ita expressa

$$\tfrac{1}{6}dxdydz\left\{\begin{matrix} 1+Ldt+Lmdt^2+Lmvdt^2 \\ +mdt-Mldt^2-Mlvdt^3 \\ +vdt+Lvdt^2\ \ -Ln\mu dt^3 \\ +mvdt^2+Mn\lambda dt^3 \\ -n\mu dt^2-Nm\lambda dt^3 \\ -N\lambda dt^2+Nl\mu dt^3 \end{matrix}\right\}$$

quae cum debeat esse aequalis pyramidi $\lambda\mu vo=\frac{1}{6}dxdydz$, habebitur, divisione per *dt* instituta, haec aequatio :

$$0=L+m\ +v+dt\left(Lm+Lv+mv-Ml-N\lambda-n\mu\right)$$
$$+dt^2(Lmv+Mn\lambda+Nl\mu-Ln\mu-Mlv-Nm\lambda).$$

36. Reiectis igitur terminis infinite parvis habebitur haec aequatio:

$$L+m+v=0,$$

qua ratio celeritatum *u, v, w* determinatur, ut motus fluidi fiat possibilis. Cum igitur sit

$$L=\tfrac{du}{dx},\quad m=\tfrac{dv}{dy}\quad \text{et}\quad v=\tfrac{dw}{dz},$$

criterium motus possibilis, si puncto fluidi cuicunque λ, cuius situs ternis coordinatis *x, y* et *z* definitur, eiusmodi celeritates *u, v* et *w* secundum easdem coordinatas directae tribuantur, ut sit

$$\tfrac{du}{dx}+\tfrac{dv}{dy}+\tfrac{dw}{dz}=0.$$

Hac scilicet conditione id obtinetur, ut nulla fluidi pars in motu neque in magis neque in minus spatium transferatur ac perpetuo cum fluidi continuitas tum eadem densitas conservetur.

37. Haec autem proprietas ita est interpretanda, ut pro eodem temporis momento ad omnia fluidi puncta extendatur : eodem scilicet momento omnium punctorum ternae celeritates *u, v, w* tales esse debent functiones ternarum coordinatarum *x, y* et *z,* ut sit

$$\tfrac{du}{dx}+\tfrac{dv}{dy}+\tfrac{dw}{dz}=0.$$

sicque natura istarum functionum motum singulorum fluidi punctorum ad instans propositum definiet. Alio autem tempore eorundem punctorum motus utcunque diversus esse poterit, dummodo pro quovis temporis puncto inventa proprietas per totum fluidum locum habeat. Tempus scilicet hactenus tanquam quantitatem constantem sum contemplatus.

38. Sin autem tempus quoque variabile spectare velimus, ita ut motus puncti *λ,* cuius situs ternis coordinatis $AL = x,\ Ll = y$ et $l\lambda = z$ indicatur, elapso tempore *t* definiri debeat, manifestum est ternas celeritates *u, v* et *w* non solum a coordinatis *x,* y et z, sed insuper etiam a tempore *t* pendere, seu functiones fore quatuor harum quantitatum *x, y, z* et *t* ; ita ut earum differentialia huiusmodi formas sint habitura :

$$du = Ldx + ldy + \lambda dz + \mathfrak{L}dt,\quad dv = Mdx + mdy + \mu dz + \mathfrak{M}dt,$$
$$dw = Ndx + ndy + \nu dz + \mathfrak{N}dt.$$

Interim tamen semper erit $L + m + v = 0$, propterea, quod quovis instanti tempus *t* pro constanti habetur seu sit $dt = 0$. Utcunque igitur functiones *u, v* et *w* cum tempore *t* mutantur, necesse est, ut pro omni temporis momento sit:

$$\tfrac{du}{dx} + \tfrac{dv}{dy} + \tfrac{dw}{dz} = 0.$$

Cum enim hac conditione efficiatur, ut quaevis fluidi portio tempusculo *dt* in spatium sibi aequale transferatur, idem etiam per eandem conditionem in elemento temporis sequenti, omnibus ergo sequentibus temporis elementis evenire debebit.

PARS ALTERA

39. Expositis ergo iis, quae ad motum tantum possibilem attinent, investigemus nunc etiam indolem eius motus, qui re vera in fluido subsistere potest. Hic igitur, praeter fluidi continuitatem eiusdemque densitatis permanentiam, ratio quoque erit habenda virium, quibus singula fluidi elementa actu sollicitantur. Quando enim cuiusvis elementi motus vel non est uniformis vel non in directum porrigitur, motus immutatio viribus hoc elementum sollicitantibus conformis esse debet. Quare cum ex cognitis his viribus motus mutatio innotescat, praecedentes autem formulae etiam hanc motus mutationem contineant, hinc novae deducentur determinationes, quibus motus hactenus tantum possibilis ad motum actualem restringitur.

40. Instituamus quoque hanc investigationem bipartito ; ac primo concipiamus totum fluidi motum in eodem plano fieri. Sint ergo (Fig. 1), ut ante, coordinatae situm puncti cuiusvis *l* definientes $AL = x,\ Ll = y$*;* ac nunc quidem elapso tempore *t* sint puncti *l* binae celeritates secundum directiones axibus *AL* et *AB* parallelas *u* et *v:* erunt *u* et *v,* quoniam nunc variabilitatis temporis ratio haberi debet, functiones ipsarum *x, y* et *t,* quo circa ponatur

$$du = Ludt + lvdt + \mathfrak{L}dt,\quad \text{et}\quad dv = Mudt + mvdt + \mathfrak{M}dt.$$

atque ob priorem conditionem iam supra invenimus esse debere *L* +*m*= 0.

41. Cum igitur elapso tempusculo $=dt$ punctum *l* transferatur in *p,* absoluto secundum axem *AL* spatiolo $=udt$, secundum alterum axem *AB* autem spatiolo $=vdt$: ut incrementa celeritatum *u* et *v* puncti *l,* quae tempusculo *dt* ipsi inducuntur, obtineamus, pro *dx* et *dy* scribi oportet spatiola *udt* et *vdt,* unde haec vera celeritatum incrementa prodibunt:

$$du = Ludt + lvdt + \mathfrak{L}dt, \quad \text{et} \quad dv = Mudt + mvdt + \mathfrak{M}dt.$$

Ex quo vires acceleratrices, quae has accelerationes producere valent, erunt:

$$\text{Vis acceleratrix secundum } AL = 2(Lu + lv + \mathfrak{L}),$$
$$\text{Vis acceleratrix secundum } AB = 2(Mu + mv + \mathfrak{M}),$$

quibus ergo vires actu in aquae particulam *l* agentis aequales esse debebunt.

42. Inter vires autem, quae aquae particulas actu sollicitant, primum consideranda venit gravitas; cuius autem effectus, si planum, in quo fit motus, est horizontale, pro nihilo erit habendus. Sin autem fuerit declive axisque *AL* declivitatem sequatur, altero *AB* existente horizontali, a gravitate orietur vis acceleratrix secundum *AL* constans, quae sit $=\alpha$. Deinde non praetermittenda est frictio, qua saepe motus aquae non mediocriter impeditur; quanquam autem eius leges nondum sunt satis exploratae, tamen frictionem corporum solidorum sequentes non multum fortasse a scopo aberrabimus, si frictionem ubique pressioni, qua aquae particulae se invicem premunt, proportionalem statuerimus.

43. Inprimis autem in computum est ducenda pressio, qua particulae aquae ubique in se mutuo agunt, qua fit, ut quaelibet particula undique ab adiacentibus comprimatur, et quatenus haec pressio undequaque non fuerit aequalis, eatenus particulae motus afficiatur. Ubique scilicet aqua in certo quodam statu compressionis versabitur, qui similis erit ei, in quo aqua stagnans ad certam profunditatem existit. Haec ergo profunditas, ad quam in aqua stagnante aqua in pari compressionis statu reperitur, commodissime adhibebitur ad pressionem in quo vis fluidi puncto *l* exprimendam. Sit igitur *p* ista altitudo seu profunditas, statum compressionis in *l* exprimens, eritque *p* functio quaedam coordinatarum *x* et *y,* ac si pressio cum tempore in *l* quoque varietur, tempus quoque *t* in functionem *p* ingredietur.

44. Ponamus ergo

$$dp = Rdx + rdy + \mathfrak{R}dt$$

et consideremus (Fig. 3) elementum aquae quadrangulare rectangulum *lmno,* cuius latera sint

$$lm = no = dx \quad \text{et} \quad ln = mo = dy$$

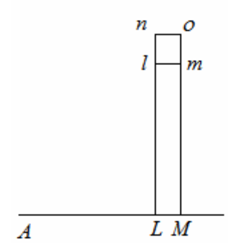

Fig. 3

areaque $=dxdy$. Cum iam pressio in *l* fit $=p$, pressio in *m* erit $=p+Rdx$, et in $o = p+Rdx+rdy$. Hinc latus *lm* premitur vi

=dx(p+J:Rdx), latus vero *no* contra premetur vi

$$= dx(p+\tfrac{1}{2}Rdx+rdy);$$

ab his ergo duabus viribus elementum *lmno* secundum directionem *ln* urgebitur vi

$$=-rdxdy.$$

Simili autem modo ex viribus

$$dy(p+\tfrac{1}{2}rdy) \text{ et } dy(p+Rdx+\tfrac{1}{2}rdy),$$

quae agunt in latera *ln* et *mo,* resultabit vis elementum urgens secundum directionem $lm=-Rdxdy$.

45. Hinc igitur orietur vis acceleratrix secundum $lm=-R$ et vis acceleratrix secundum $ln=\ -r$, quarum illa cum vi a gravitate orta α praebet $\alpha-R$. Frictione ergo adhuc semota, has habebimus aequationes :

$$\alpha-R=2Lu+2lv+2\mathfrak{L} \text{ seu } R=\alpha-2Lu-2lv-2\mathfrak{L},$$
$$-r\ =2Mu+2mv+2\mathfrak{M} \text{ et } r=-2Mu-2mv-2\mathfrak{M},$$

unde colligimus fore
$dp=\alpha dx-2(Lu+lv+\mathfrak{L})dx-2(Mu+mv+\mathfrak{M})dy+\mathfrak{R}dt$, quod differentiale oportet esse completum seu integrabile.

46. Quia terminus αdx per se est integrabilis et pro $\mathfrak{R}$ nihil est definitum ex natura differentialium completorum necesse est, ut sit signandi modo iam supra adhibito :

$$\frac{d.(Lu+lv+\mathfrak{L})}{dy}=\frac{d.(Mu+mv+\mathfrak{M})}{dx},$$

unde ob

$$\frac{du}{dx}=L,\quad \frac{du}{dy}=l,\quad \frac{dv}{dx}=M \text{ et } \frac{dv}{dy}=m,$$

orietur

$$Ll+\frac{udL}{dy}+lm+\frac{vdl}{dy}+\frac{d\mathfrak{L}}{dy}=Ml+\frac{udM}{dx}+mM+\frac{vdm}{dx}+\frac{d\mathfrak{M}}{dx},$$

quae reducitur ad hanc formam :

$$\left(L+m\right)\left(l-M\right)+u\left(\frac{dl-dM}{dx}\right)+v\left(\frac{dl-dM}{dy}\right)+\frac{dl-dM}{dt}=0,$$

47. Verum ob differentialia

$$Ldx+ldy+\mathfrak{L}dt \text{ et } Mdx+mdy+\mathfrak{M}dt$$

completa novimus esse

$$\frac{dL}{dy}=\frac{dl}{dx},\quad \frac{dm}{dx}=\frac{dM}{dy},\quad \frac{d\mathfrak{L}}{dy}=\frac{dl}{dt}\quad \text{et}\quad \frac{d\mathfrak{M}}{dx}=\frac{dM}{dt},$$

quibus valoribus substitutis habebimus istam aequationem:

$$(L+m)(l-M)+u\left(\frac{dL}{dy}-\frac{dM}{dx}\right)+v\left(\frac{dl}{dy}-\frac{dm}{dx}\right)+\frac{d\mathfrak{L}}{dy}-\frac{d\mathfrak{M}}{dx}=0.$$

cui aperte satisfacit $l=M$: ita ut sit

$$\frac{du}{dy}=\frac{dv}{dx}.$$

Cum igitur haec conditio requirat, ut sit $\frac{du}{dy}=\frac{dv}{dx}$, vicissim apparet formulam differentialem hanc $udx+vdy$ esse debere completam, in quo ergo criterium motus actualis consistit.

48. Criterium hoc independens est a praecedente, quod continuitas fluidi eiusque constans densitas uniformis suppeditavit. Quare etiamsi fluidum in motu densitatem suam mutaret, uti in motu fluidorum elasticorum, veluti aeris evenire solet, haec proprietas nihilominus locum habere debet, ut sit $udx + vdy$ differentiale completum. Sive celeritates u et v semper eiusmodi debent esse functiones coordinatarum x et y, praeter tempus t, ut posito tempore constante formula $udx + vdy$ integrationem admittat.

49. Hinc autem porro ipsam pressionem p definire poterimus, id quod absolute est necessarium ad motum fluidi perfecte determinandum. Cum enim invenerimus $M=l$, erit

$$dp=\alpha dx-2u(Ldx+ldy)-2v(ldx+mdy)-2\mathfrak{L}dx-2\mathfrak{M}dy+\mathfrak{R}dt.$$

At est

$$Ldx+ldy=du-\mathfrak{L}dt,\quad ldx+mdy=dv-\mathfrak{M}dt,$$

unde fit:

$$dp=\alpha dx-2udu-2vdv+2\mathfrak{L}udt+2\mathfrak{M}vdt-2\mathfrak{L}dx-2\mathfrak{M}dy+\mathfrak{R}dt.$$

Quodsi ergo pressionem in singulis fluidi punctis pro tempore praesente definire velimus, nullo respectu ad eius mutationem cum tempore oriundam habito, ista nobis consideranda erit aequatio:

$$dp=\alpha dx-2udu-2vdv-2\mathfrak{L}dx-2\mathfrak{M}dy$$

estque nostro designandi modo

$$\mathfrak{L}=\frac{du}{dt}\quad \text{et}\quad \mathfrak{M}=\frac{dv}{dt}$$

hincque

$$dp=\alpha dx-2udu-2vdv-2\frac{du}{dt}dx-2\frac{dv}{dt}dy,$$

in cuius aequationis integratione tempus t pro constanti est habendum.

50. Haec autem aequatio per hypothesin est integrabilis, atque revera talis deprehenditur, si ad criterium huius motus attendamus, quo vidimus esse debere $udx+vdy$ differentiale completum, si quidem tempus t constans assumamus. Sit igitur S eius integrale, quod ergo eiusmodi erit functio ipsarum x, y et t, ut posito $dt=0$ prodeat

$$dS = udx + vdy;$$

sumto autem quoque tempore t variabili ponamus haberi

$$dS = udx + vdy + Udt$$

eritque propterea

$$\tfrac{du}{dt} = \tfrac{dU}{dx} \text{ et } \tfrac{dv}{dt} = \tfrac{dU}{dy}.$$

Tum vero est

$$U = \tfrac{dS}{dt}.$$

51. His valoribus introductis habebitur:

$$\tfrac{du}{dt}\cdot dx + \tfrac{dv}{dt}\cdot dy = \tfrac{dU}{dx}\cdot dx + \tfrac{dU}{dy}\cdot dy$$

huiusque formulae, cum tempus t constans sumatur, integrale manifesto est $=U$. Quod quo clarius appareat, ponamus

$$dU = Kdx + kdy,$$

erit

$$\tfrac{dU}{dx} = K \text{ et } \tfrac{dU}{dy} = k,$$

unde

$$\tfrac{dU}{dx}\cdot dx + \tfrac{dU}{dy}\cdot dy = Kdx + kdy = dU.$$

Cum igitur huius integrale sit u

$$= U = \tfrac{dS}{dt},$$

erit

$$dp = \alpha dx - 2udu - 2vdv - 2dU,$$

unde integrando prodit :

$$p = \text{Const.} + \alpha x - uu - vv - \tfrac{2dS}{dt},$$

existente S functione ipsarum x, y et t, cuius differentiale posito $dt=0$ est $udx+vdy$.

52. Quo indoles huius formulae melius intelligatur, consideremus puncti l celeritatem veram, quae sit

$$= V = \sqrt{(uu+vv)}.$$

Atque erit pressio :

$$p = \text{Const.} + \alpha x - VV - \tfrac{2dS}{dt},$$

in quo postremo termino dS denotat differentiale ipsius

$$S = \int (udx + vdy),$$

si tantum tempus t ut variabile spectetur.

53. Si iam frictionis quoque rationem habere velimus eamque pressioni p proportionalem statuamus, dum punctum l elementum ds percurrit, erit vis retardatrix a frictione oriunda $= \frac{p}{f}$; unde posito $\frac{dS}{dt} = U$ aequatio nostra differentialis, posito t constante, erit:

$$dp = \alpha dx - \tfrac{p}{f} ds - 2VdV - 2dU,$$

unde integrando oritur, sumto e pro numero, cuius logarithmus hyperbolicus est $= 1$,

$$p = e^{-\frac{s}{f}} \int e^{\frac{s}{f}} \left(\alpha dx - 2VdV - 2dU \right)$$

sive

$$p = \alpha x - VV - 2U - \tfrac{1}{f} e^{-\frac{s}{f}} \int e^{\frac{s}{f}} \left(\alpha dx - VdV - 2dU \right) ds$$

54. Cum igitur criterium motus, quo fluidum re vera movetur, in hoc consistat, ut posito tempore constante differentiale $udx + vdy$ sit completum, continuitas autem et constans uniformis densitas exigat, ut sit

$$\tfrac{du}{dx} + \tfrac{dv}{dy} = 0,$$

hinc sequitur quoque hoc differentiale $udy - vdx$ fore completum. Quare utrinque coniunctim celeritates u et v eiusmodi debent esse functiones coordinatarum x et y cum tempore t, ut hae ambae formulae

$$udx + vdy \quad \text{et} \quad udy - vdx$$

sint differentialia completa.

55. Instituamus iam eandem investigationem in genere, positisque puncti λ ternis celeritatibus secundum axes *AL, AB, AC* directis $u, v, w,$ sint eae eiusmodi functiones cum coordinatarum x, y, z tum temporis t, ut differentiatione instituta fiat:

$$\begin{aligned} du &= Ldx + ldy + \lambda dz + \mathfrak{L}dt, \\ dv &= Mdx + mdy + \mu dz + \mathfrak{M}dt, \\ dw &= Ndx + ndy + \nu dz + \mathfrak{N}dt, \end{aligned}$$

et quanquam hic quoque tempus t variabile est assumtum, tamen, ut motus sit possibilis, per conditionem praecedentem oportet esse sive

$$L+m+\nu=0$$

quod eodem redit :

$$\tfrac{du}{dx}+\tfrac{dv}{dy}+\tfrac{dw}{dz}=0,$$

a qua proprietate quidem praesens examen non pendet.

56. Elapso autem tempusculo *dt,* punctum λ transfertur in π et secundum axem *AL* percurrit spatiolum $=udt$, secundum axem *AB* spatiolum $=vdt$ et secundum axem *AC* spatiolum *wdt.* Quare nunc puncti λ in π existentis ternae celeritates erunt:

$$\begin{aligned}&\text{secundum } AL=u+Ludt+lvdt+\lambda wdt+\mathfrak{L}dt,\\ &\text{secundum } AB=v+Mudt+mvdt+\mu wdt+\mathfrak{M}dt,\\ &\text{secundum } AC=w+Nudt+nvdt+\nu wdt+\mathfrak{N}dt\end{aligned}$$

hincque accelerationes secundum easdem directiones erunt:

$$\begin{aligned}&\text{secundum } AL=2(Lu+lv+\lambda w+\mathfrak{L}),\\ &\text{secundum } AB=2(Mu+mv+\mu w+\mathfrak{M}),\\ &\text{secundum } AC=2(Nu+nv+\nu w+\mathfrak{N})\ .\end{aligned}$$

57. Si axem *AC* verticalem statuamus, ita ut reliqui bini *AL* et *AB* sint horizontales, ob gravitatem vis acceleratrix oritur secundum axem $AC=-1$. Tum vero posita pressione in $\lambda=p$ eiusque differentiali, sumto tempore constante,

$$dp=Rdx+rdy+\rho dz,$$

hinc orientur ternae vires acceleratrices

$$\text{secundum } AL=-R,\ \text{secundum } AB=-r \text{ et secundum } AC=-\rho,$$

quippe quae facile simili modo colliguntur, quo ante paragraphia 44 et 45 sumus usi, ita ut superfluum foret idem ratiocinium repetere. Quam ob rem habebimus has aequationes:

$$\begin{aligned}&R=-2(Lu+lv+\lambda w+\mathfrak{L}),\\ &r=-2\left(Mu+mv+\mu w+\mathfrak{M}\right),\\ &\rho=-1-2\left(Nu+nv+\nu w+\mathfrak{N}\right).\end{aligned}$$

58. Cum autem formula $dp=Rdx+rdy+\rho dz$ debeat esse differentiale completum, erit

$$\tfrac{dR}{dy}=\tfrac{dr}{dx},\ \ \tfrac{dR}{dz}=\tfrac{d\rho}{dx},\ \ \tfrac{dr}{dz}=\tfrac{d\rho}{dy},$$

at differentiatione peracta obtinebuntur per -2 dividendo tres sequentes aequationes

$$\text{I.}\quad \frac{udL}{dy}+\frac{vdl}{dy}+\frac{wd\lambda}{dy}+\frac{d\mathfrak{L}}{dy}+Ll+lm+\lambda n = \frac{udM}{dx}+\frac{vdm}{dx}+\frac{wd\mu}{dx}+\frac{d\mathfrak{M}}{dx}+ML+mM+\mu N$$

$$\text{II.}\quad \frac{udL}{dz}+\frac{vdl}{dz}+\frac{wd\lambda}{dz}+\frac{d\mathfrak{L}}{dz}+L\lambda+l\mu+\lambda\nu = \frac{udN}{dx}+\frac{vdn}{dx}+\frac{wd\nu}{dx}+\frac{d\mathfrak{N}}{dx}+NL+nM+\nu N$$

$$\text{III.}\quad \frac{udM}{dz}+\frac{vdm}{dz}+\frac{wd\mu}{dz}+\frac{d\mathfrak{M}}{dz}+M\lambda+m\mu+\mu\nu = \frac{udN}{dy}+\frac{vdn}{dy}+\frac{wd\nu}{dy}+\frac{d\mathfrak{N}}{dy}+Nl+nm+\nu n.$$

59. Est autem ex natura differentialium completorum

$$\frac{dL}{dy}=\frac{dl}{dx},\ \frac{dm}{dx}=\frac{dM}{dy},\ \frac{d\lambda}{dy}=\frac{dl}{dz},\ \frac{d\mu}{dx}=\frac{dM}{dz},\ \frac{d\mathfrak{L}}{dy}=\frac{dl}{dt},\ \frac{d\mathfrak{M}}{dx}=\frac{dM}{dt},$$
$$\frac{dL}{dz}=\frac{d\lambda}{dx},\ \frac{dl}{dz}=\frac{d\lambda}{dy},\ \frac{dn}{dx}=\frac{dN}{dy},\ \frac{d\nu}{dx}=\frac{dN}{dz},\ \frac{d\mathfrak{L}}{dz}=\frac{d\lambda}{dt},\ \frac{d\mathfrak{N}}{dx}=\frac{dN}{dt},$$
$$\frac{dM}{dz}=\frac{d\mu}{dx},\ \frac{dN}{dy}=\frac{dn}{dx},\ \frac{dm}{dz}=\frac{d\mu}{dy},\ \frac{d\nu}{dy}=\frac{dn}{dz},\ \frac{d\mathfrak{M}}{dz}=\frac{d\mu}{dt},\ \frac{d\mathfrak{N}}{dy}=\frac{dn}{dt},$$

quibus valoribus substitutis tres illae aequationes abibunt in has:

$$\left(\frac{dl-dM}{dt}\right)+u\left(\frac{dl-dM}{dx}\right)+v\left(\frac{dl-dM}{dy}\right)+w\left(\frac{dl-dM}{dz}\right)+(l-M)(L+m)+\lambda n-\mu N=0,$$
$$\left(\frac{d\lambda-dN}{dt}\right)+u\left(\frac{d\lambda-dN}{dx}\right)+v\left(\frac{d\lambda-dN}{dy}\right)+w\left(\frac{d\lambda-dN}{dz}\right)+(\lambda-N)(L+\nu)+l\mu-nM=0,$$
$$\left(\frac{d\mu-dn}{dt}\right)+u\left(\frac{d\mu-dn}{dx}\right)+v\left(\frac{d\mu-dn}{dy}\right)+w\left(\frac{d\mu-dn}{dz}\right)+(\mu-n)(m+\nu)+M\lambda-Nl=0.$$

60. Manifestum iam est his tribus aequationibus satisfieri sequentibus tribus valoribus :

$$l=M,\quad \lambda=N,\quad \mu=n,$$

quibus continetur criterium, quod consideratio sollicitationum suppeditat. Hinc ergo sequitur fore recepto designandi modo

$$\frac{du}{dy}=\frac{dv}{dx},\quad \frac{du}{dz}=\frac{dw}{dx},\quad \frac{dv}{dz}=\frac{dw}{dy}.$$

Hae autem ipsae sunt illae conditiones, quae requiruntur, ut haec formula

$$udx+vdy+wdz$$

sit differentiale completum. Ex quo hoc criterium in eo consistit, ut ternae celeritates *u, v* et *w* eiusmodi esse debeant functiones ipsarum *x, y* et *z* una cum *t,* ut posito tempore constante formula $udx+vdy+wdz$ integrationem admittat.

61. Cum ergo posito tempore *t* constante seu $dt=0$ sit

$$du = Ldx + Mdy + Ndz,$$
$$dv = Mdx + mdy + ndz,$$
$$dw = Ndx + ndy + vdz,$$

valores autem pro *R, r* et ρ fiant:

$$R = -2(Lu+Mv+Nw+\mathfrak{L}),$$
$$r = -2(Mu+mv+nw+\mathfrak{M}),$$
$$\rho = -1-2(Nu+nv+vw+\mathfrak{N}) ,$$

pro statu pressionis *p* haec habebitur aequatio

$$\begin{aligned} dp = dz &- 2u(Ldx+Mdy+Ndz) \\ &- 2v(Mdx+mdy+ndz) \\ &- 2w(Ndx+ndy+vdz) \\ &- 2\mathfrak{L}dx - 2\mathfrak{M}dy - 2\mathfrak{N}dz \end{aligned}$$

62. Quia vero est

$$\mathfrak{L} = \tfrac{du}{dt},\ \ \mathfrak{M} = \tfrac{dv}{dt},\ \ \mathfrak{N} = \tfrac{dw}{dt},$$

erit integrando:

$$p = C - z - uu - vv - ww - 2\int\left(\tfrac{du}{dt}\cdot dx + \tfrac{dv}{dt}\cdot dy + \tfrac{dw}{dt}\cdot dz\right).$$

Cum autem per conditionem, inventam sit $udx+vdy+wdz$ integrabile, ponatur eius integrale $=S$, quod, quoniam etiam tempus *t* involvere potest, sit sumto quoque *t* variabili:

$$dS = udx + vdy + wdz + Udt,$$

eritque

$$\tfrac{du}{dt} = \tfrac{dU}{dx},\quad \tfrac{dv}{dt} = \tfrac{dU}{dy},\quad \tfrac{dw}{dt} = \tfrac{dU}{dz}.$$

Quare cum sit in genere sumto tempore *t* constante, uti id quidem in superiori integrali assumitur,

$$\frac{dU}{dx}\cdot dx+\frac{dU}{dy}\cdot dy+\frac{dU}{dz}\cdot dz=dU,$$

habebimus:

$$p=C-z-uu-vv-ww-2U$$

sive

$$p=C-z-uu-vv-ww-2\cdot\frac{dS}{dt}.$$

63. Perspicuum hic est $uu+vv+ww$ exprimere quadratum verae puncti λ celeritatis ita, ut, si celeritas huius puncti vera dicatur V, habeatur pro pressione ista aequatio :

$$p=C-z-VV-\frac{2dS}{dt},$$

ad quam ergo inveniendam primum formulae $udx+vdy+wdz$, quam completam esse oportet, quaeratur integrale S, hocque denuo differentietur, posito solo tempore t variabili, quod differentiale per dt divisum dabit valorem formulae $\frac{dS}{dt}$, quae in expressionem pro statu pressionis p inventam ingreditur.

64. Quodsi iam prius criterium, quo motus saltem possibilis continetur, hic adiungamus, ternae celeritates u, v, w eiusmodi functiones ternarum coordinatarum x, y et z una cum tempore t esse debent, ut primo sit

$$udx+vdy+wdz$$

differentiale completum ; deinde vero, ut sit

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0$$

Hisque duabus conditionibus omnis fluidorum motus, siquidem densitate invariabili sint praeditae, subiicitur. Praeterea vero, si sumto etiam tempore t variabili, haec formula

$$udx+vdy+wdz+Udt$$

fuerit differentiale completum, status pressionis in puncto quocunque λ exprimitur altitudine p, ut sit :

$$p=C-z-uu-vv-ww-2U,$$

siquidem fluidum gravitate naturali gaudeat et planum BAL fuerit horizontale.

65. Si gravitati aliam directionem tribuissemus sive etiam vires utcunque variabiles assumsissemus, quibus singulae fluidi particulae sollicitarentur, inde tantum discrimen in valorem pressionis p esset ingressum, neque inde lex, quam ternae cuiusque puncti fluidi celeritates sequi debent, ullam mutationem esset passa. Semper ergo, quaecunque fuerint vires sollicitantes, ternae celeritates u, v et w ita debent esse comparatae, ut formula $udx+vdy+wdz$ fiat differentiale completum atque ut insuper sit

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0$$

Infinitis igitur modis constitui poterunt ternae celeritates *u, v* et *w,* ut his duabus conditionibus satisfiat, atque tum pressio fluidi in singulis punctis poterit assignari.

66. Multo difficilior autem foret quaestio, si, datis viribus sollicitantibus una cum pressione in quibusdam locis, ipse motus fluidi in singulis punctis determinari deberet. Tum enim haberentur aliquot aequationes formae

$$p=C-z-uu-vv-ww-2U,$$

ex quibus cum constans *C* tum vero ratio functionum *u, v* et *w* ita definiri deberet, ut non solum his casibus istis aequationibus satisfieret, sed etiam ante allatae regulae observarentur, quod opus utique maximam calculi vim requireret. Conveniet igitur in genere in naturam functionum idonearum inquiri, quae utrique criterio futurae sint conformes.

67. Commodissime igitur incipiemus ab ipsa quantitate integrali, cuius differentiale esse oportet formulam $udx+vdy+wdz$ posito tempore constante. Sit ergo *S* hoc integrale, quod erit functio ipsarum *x, y* et *z,* tempore *t* in quantitatibus constantibus involuto; atque si haec quantitas *S* differentietur, coefficientes differentialium *dx, dy* et *dz* statim praebebunt celeritates *u,v* et *w,* quae quidem praesenti tempore conveniant puncto fluidi *A,* cuius coordinatae sunt *x, y* et *z.* Quaestio autem huc redit: ut definiatur, quales functiones ipsarum *x, y* et *z* pro *S* assumi debeant, ut etiam fiat

$$\frac{du}{dx}+\frac{dv}{dy}+\frac{dw}{dz}=0 \quad ;$$

seu cum sit

$$u=\frac{dS}{dx},\quad v=\frac{dS}{dy},\quad w=\frac{dS}{dz},$$

ut sit

$$\frac{ddS}{dx^2}+\frac{ddS}{dy^2}+\frac{ddS}{dz^2}=0.$$

68. Quoniam non patet, quomodo hoc generaliter praestari possit, casus quosdam generaliores contemplabor. Sit igitur

$$S=\left(Ax+By+Cz\right)^n$$

eritque

$$\frac{dS}{dx}=nA\left(Ax+By+Cz\right)^{n-1}$$

et

$$\frac{ddS}{dx^2}=n\left(n-1\right)AA\left(Ax+By+Cz\right)^{n-2}$$

similesque erunt formae pro $\frac{ddS}{dy^2}$ et $\frac{ddS}{dz^2}$, unde effici debet, ut sit

$$n(n-1)(Ax+By+Cz)^{n-2}(AA+BB+CC)=0,$$

cui primo satisfit, si vel $n=0$ vel $n=1$; ex quo iam duo valores idonei obtinentur, scilicet

$$S=\text{Const. et } S=Ax+By+Cz,$$

ubi constantes *A, B, C* etiam tempus utcunque in se complecti possunt.

69. Sin autem *n* neque $=0$ neque $=1$, necesse est, ut sit:

$$AA+BB+CC=0,$$

tumque pro *S* valor idoneus erit

$$S=(Ax+By+Cz)^n,$$

quicunque numerus pro exponente *n* sumatur, quin etiam ipsum tempus *t* in *n* poterit ingredi. Patet etiam aggregatum quotcunque huiusmodi formularum idoneum valorem pro *S* praebere, ita, ut sit :

$$S=\alpha+\beta x+\gamma y+\delta z+\varepsilon(Ax+By+Cz)^n+\zeta(A'x+B'y+C'z)^{n'}+$$
$$+\eta(A''x+B''y+C''z)^{n''}+\theta(A'''x+B'''y+C'''z)^{n''}+\text{etc.}$$

dummodo fuerit :

$$AA+BB+CC=0,\ \ A'A'+B'B'+C'C'=0,$$
$$A''A''+B'B''+C''C''=0,\ \text{etc.}$$

70. Hinc valores idonei pro *S* ex inferioribus ordinibus, ubi coordinatae *x, y, z* vel unam vel duas vel tres vel quatuor habent dimensiones, erunt sequentes

I. $S = A$

II. $S = Ax + By + Cz$

III. $S = Axx + Byy + Czz + 2Dxy + 2Exz + 2Fyz$

existente $A + B + C = 0$

IV. $S = AX^3 + By^3 + Cz^3 + 3Dxxy + 3Fxxz + 3Hyyz + 6Kxyz$
$+ 3Exyy + 3Gxzz + 3Iyzz$

existente $A + E + G = 0,\ \ B + D + I = 0,\ \ C + F + H = 0$

$$V.\ S = \begin{cases} +Ax^4 + 6Dxxyy + 4Gx^3\, y + 4Hxy^3 + 12N\, xxyz \\ +By^4 + 6Exxzz\ + 4IX^3z\ + 4Kxz^3 + 12Oxyyz \\ +Cz^4 + 6Fyyzz + 4Ly^3z + 4Myz^3 + 12Pxyzz \end{cases}$$

existente $A + D + E = 0,\ \ G + H + P = 0,$
$B + D + F = 0,\ \ I + K + O = 0,$
$C + E + F = 0,\ \ L + M + N = 0.$

71. Hinc perspicuum est, quomodo hae formulae pro quolibet ordine se sint habiturae : singulis scilicet terminis primo iidem dentur coefficientes numerici, qui iisdem terminis ex lege permutationum conveniunt, seu, qui oriuntur, si trinomium $x + y + z$ ad potestatem eiusdem ordinis elevetur. Numericis autem coefficientibus adiungantur litterales indefiniti *A, B, C* etc. Tum reiectis numericis dispiciatur, quoties eiusmodi terni termini occurrunt

$$LZxx + MZyy + NZzz,$$

qui scilicet factorem communem *Z* ex variabilibus formatum habeant, totiesque summa coefficientium litteralium *LMN* statuatur nihilo aequalis.Ita cum pro potestate quinta habeatur:

$$S = \begin{cases} +Ax^5 + 5Dx^4y + 5\mathfrak{D}x^4z + 10Gx^3yy + 10\mathfrak{C}x^3zz\ + 20Kx^3yz + 30Nxyyzz \\ +By^5 + 5Exy^4 + 5\mathfrak{E}y^4z + 10Hx^2y^3\ \ + 10\mathfrak{h}y^3zz\ + 20Lxy^3z + 30Oxxyzz \\ +Cz^5 + 5Fxz^4 + 5\mathfrak{F}yz^4 + 10Ixxz^3\ \ \ \ + 10\mathfrak{J}yyz^3 + 20Mxyz^3 + 30Pxxyyz \end{cases}$$

sequentes habebuntur coefficientium litteralium determinationes :

$A + G + \mathfrak{G} = 0,\ D + H + O = 0,\ \mathfrak{D} + I + P = 0,$
$B + H + \mathfrak{h} = 0,\ \ E + G + N = 0,\ \mathfrak{E}\ + \mathfrak{J} + P = 0,\ K + L + M\ = 0,$
$C + I + \mathfrak{J} = 0,\ \ \ F + \mathfrak{G} + N = 0,\ \ \mathfrak{F}\ + \mathfrak{h} + O = 0.$

Simili modo pro ordine sexto huiusmodi determinationes prodibunt 15, pro septimo 21, pro octavo 28 et ita porro.

72. Iam formula prima $S = A$, quoniam coordinatas x, y et z plane non in se complectitur, ternas celeritates u, v et w nihilo aequales praebebit, sicque statum fluidi quietum exhibebit. Pressio tamen in quovis puncto pro variis temporibus utcunque poterit esse variabilis. Cum enim A sit functio quaecunque temporis, ad datum tempus t pressio in puncto A erit

$$p = C - \tfrac{2dA}{dt} - z,$$

qua formula eiusmodi fluidi status indicatur, ubi fluidum quovis momento a viribus quibuscunque sollicitatur, quae tamen se semper in aequilibrio teneant, ut ab illis nullus motus in fluido oriri queat: ubi evenit, si fluidum vasi fuerit inclusum, ex quo nusquam erumpere queat, atque in eo a viribus quibuscunque comprimatur.

73. Formula autem secunda $S = Ax + By + Cz$ differentiata has praebebit puncti λ ternas celeritates:

$$u = A, \quad v = B \quad \text{et} \quad w = C.$$

Eodem ergo tempore omnia fluidi puncta pari motu feruntur secundum eandem directionem. Ex quo totum fluidum, perinde ac corpus solidum, movebitur, quod solo motu progressivo fertur. Diverso autem tempore huius motus tam celeritas quam directio utcunque variari poterit, prout vires extrinsecus urgentes exegerint. Pressio ergo in puncto λ ad tempus t, cuius A, B, C sunt functiones, erit

$$p = C - z - AA - BB - CC - 2x \cdot \tfrac{dA}{dt} - 2y \cdot \tfrac{dB}{dt} - 2z \cdot \tfrac{dC}{dt}$$

74. Formula tertia

$S = Axx + Byy + Czz + 2Dxy + 2Exz + 2Fyz$, ubi est $A + B + C = 0$, has praebebit ternas puncti λ celeritates:

$$u = 2Ax + 2Dy + 2Ez,$$
$$v = 2By + 2Dx + 2Fz,$$
$$w = 2Cz + 2Ex + 2Fy$$
seu
$$w = 2Ex + 2Fy - 2(A+B)z.$$

Hoc ergo casu etiam eodem temporis momento diversa fluidi puncta diverso motu feruntur; successu autem temporis etiam eiusdem puncti motus quomodocunque variabilis existere potest, quia pro A, B, D, E, F functiones quascunque temporis t assumere licet. Multo maior autem varietas locum habebit, si functioni S valores magis compositi tribuantur.

75. Quia casu secundo motus fluidi conveniebat cum motu corporis solidi progressivo, quo scilicet unoquoque momento singulae partes motu aequali sibique parallelo feruntur : suspicari liceat in aliis casibus motum fluidi, quoque cum motu corporis solidi, sive rotatorio sive utcunque anomalo convenire posse. Satis igitur erit ostendisse huiusmodi convenientiam, praeter casum secundum, nunquam locum habere posse. Ut enim hoc eveniret, necesse esset, ut (Fig. 2) pyramis $\pi\varphi\rho\sigma$ non solum aequalis, sed etiam similis fieret pyramidi $\lambda\mu\nu o$, seu ut foret:

$$\pi\varphi=\lambda\mu=dx=\sqrt{(QQ+qq+\varphi\varphi)}$$
$$\pi\rho=\lambda\nu=dy=\sqrt{(RR+rr+\rho\rho)}$$
$$\pi\sigma=\lambda o=dz=\sqrt{(SS+ss+\sigma\sigma)}$$
$$\varphi\rho=\mu\nu=\sqrt{(dx^2+dy^2)}=\sqrt{\left((Q-R)^2+(q-r)^2+(\varphi-\rho)^2\right)}$$
$$\varphi\sigma=\mu o=\sqrt{(dx^2+dz^2)}=\sqrt{\left((Q-S)^2+(q-s)^2+(\varphi-\sigma)^2\right)}$$
$$\rho\sigma=\nu o=\sqrt{(dy^2+dz^2)}=\sqrt{\left((R-S)^2+(r-s)^2+(\rho-\sigma)^2\right)}$$

adhibitis valoribus in paragrapho 32 usurpatis.

76. Ternae autem posteriores aequationes, cum prioribus coniunctae, reducentur ad has :

$$QR+qr+\varphi\rho=0,\quad QS+qs+\varphi\sigma=0 \ \text{ et } \ RS+rs+\rho\sigma=0,$$

ternae autem priores, si pro litteris Q, R, S, q, r, s, φ, ρ, σ valores in paragrapho 34 assignati substituantur terminique prae reliquis evanescentes praetermittantur, dabunt has aequationes :

$$1=1+2Ldt,\quad l+M=0,$$
$$1=1+2mdt,\quad \lambda+N=0,$$
$$1=1+2vdt,\quad \mu+n=0,$$

unde fit $L=0$, $m=0$ et $v=0$, $M=-l$, $N=-\lambda$ et $n=-\mu$.

77. Celeritates ergo ternae cuiusque puncti A esse deberent ita comparatae, ut foret

$$du=\quad +ldy+\lambda dz,$$
$$dv=\ -ldx+\mu dz,$$
$$dw=-\lambda dx-\mu dy.$$

Verum secunda conditio motus fluidorum postulat, ut sit

$$l = M, \quad \lambda = N \text{ et } n = \mu ;$$

unde omnes coefficientes l, λ et μ evanescent celeritatesque *u, v* et *w* pro eodem tempore in omnibus fluidi punctis eaedem, seu constantes, prodibunt. Patet igitur non nisi hoc casu fluidi motum cum motu corporis solidi convenire posse.

78. Ut autem effectus virium, quae extrinsecus in fluidum agunt, definiri possit, primum eae vires determinari debent, quae ad motum, quem fluido inesse assumimus, efficiendum requiruntur : his enim viribus eae, quae actu fluidum sollicitant, aequivalentes statui debent, supra autem paragrapho 56 vidimus, in puncto λ ternas vires acceleratrices requiri, quae ibi sunt relatae. Quare si fluidi elementum ibi concipiatur, cuius volumen seu massa sit

$$= dxdydz,$$

vires motrices ad motum requisitae erunt :

$$\begin{aligned}
\text{secundum } AL &= 2dxdydz(Lu + lv + \lambda w + \mathfrak{L}) \\
&= 2dxdydz(u \cdot \tfrac{du}{dx} + v \cdot \tfrac{du}{dy} + w \cdot \tfrac{du}{dz} + \tfrac{du}{dt}), \\
\text{secundum } AB &= 2dxdydz(Mu + mv + \mu w + \mathfrak{M}) \\
&= 2dxdydz(u \cdot \tfrac{dv}{dx} + v \cdot \tfrac{dv}{dy} + w \cdot \tfrac{dv}{dz} + \tfrac{dv}{dt}), \\
\text{secundum } AC &= 2dxdydz(Nu + nv + vw + \mathfrak{N}) \\
&= 2dxdydz(u \cdot \tfrac{dw}{dx} + v \cdot \tfrac{dw}{dy} + w \cdot \tfrac{dw}{dz} + \tfrac{dw}{dt}),
\end{aligned}$$

unde per triplicem integrationem vires totales, quae totam fluidi massam secundum easdem directiones sollicitare debent, colligentur.

79. Cum autem secunda conditio postulet, ut sit $udx + vdy + wdz$ differentiale completum, cuius integrale sit $= S$, ponatur posito quoque tempore variabili, ut ante :

$$dS = udx + vdy + wdz + Udt,$$

unde ob

$$\tfrac{du}{dy} = \tfrac{dv}{dx}, \quad \tfrac{du}{dz} = \tfrac{dw}{dx}, \quad \tfrac{du}{dt} = \tfrac{dU}{dx} \text{ etc.}$$

tres illae vires motrices evadent :

$$\begin{aligned}
\text{secundum } AL &= 2dxdydz\left(\tfrac{udu+vdvd+wdw+dU}{dx}\right), \\
\text{secundum } AB &= 2dxdydz\left(\tfrac{udu+vdvd+wdw+dU}{dy}\right), \\
\text{secundum } AC &= 2dxdydz\left(\tfrac{udu+vdvd+wdw+dU}{dz}\right).
\end{aligned}$$

80. Ponatur nunc

$$uu + vv + ww + 2U = T,$$

eritque T functio coordinatarum *x, y, z;* ponatur ergo posito tempore constante

$$dT = Kdx + kdy + \kappa dz,$$

eruntque tres illae vires motrices elementi $dxdydz$

$$\text{secundum } AL = Kdxdydz,$$
$$\text{secundum } AB = kdxdydz,$$
$$\text{secundum } AC = \kappa dxdydz,$$

triplici ergo integratione hae formulae per totam fluidi massam sunt extendendae, ut inde vires omnibus aequivalentes earumque mediae directiones obtineantur. Verum haec discussio est altioris indaginis, cui hic non immoror.

81. Quantitas autem haec

$$T = uu + vv + ww + 2U,$$

cuius in hoc calculo ratio est habenda, etiam simpliciorem formulam pro altitudine p pressionem exprimente suppeditat; est enim

$$p = C - z - T,$$

siquidem singulae fluidi particulae a sola gravitate urgeantur. Sin autem quaelibet particula λ a ternis viribus acceleratricibus sollicitetur, quae sint *Q, q* et φ, secundum directiones axium *AF, AB* et *AC* respective agentes, et calculo, ut supra, subducto reperietur pressio :

$$p = C + \int (Qdx + qdy + \varphi dz) - T,$$

unde patet differentiale

$$Qdx + qdy + \varphi dz$$

completum esse debere, alioquin status aequilibrii, vel saltem possibilis, non daretur. Hanc autem conditionem in vires sollicitantes *Q, q* et φ competere oportere, a Celeberrimo Domino Clairaut iam praeclare est demonstratum.

82. En ergo principia universae doctrinae de motu fluidorum, quae etsi primo intuitu non admodum foecunda videantur, tamen fere omnia, quae adhuc tam in hydrostatica quam in hydraulica sunt tradita, in se complectuntur, ita ut haec principia latissime patere sint censenda. Quod quo clarius appareat, operae pretium erit ostendere, quomodo cognita hydrostaticae et hydraulicae praecepta ex hactenus traditis principiis plane ac dilucide consequantur.

83. Consideremus igitur primo fluidum in statu quietis, ita ut sit $u = 0$, $v = 0$ et $w = 0$, eritque pressio in quovis fluido puncto *λ,* ob $T = 2U$,

$$p = C + \int (Qdx + qdy + \varphi dz) - 2U,$$

ubi, cum *U* sit functio ipsius temporis *t,* quod constans assumimus, quia pressionem ad datum tempus investigamus, haec quantitas *U* in ipsa constante *C* comprehendi poterit, ita ut sit :

$$p = C + \int (Qdx + qdy + \varphi dz),$$

ubi *Q, q* et *p* sunt vires particulam aquae *λ* secundum axes *AL, AB* et *AC* sollicitantes.

84. Quoniam pressio *p* non nisi a situ puncti *λ,* hoc est a coordinatis *x, y* et *z,* pendere potest, necesse est, ut

$$\int (Qdx + qdy + \varphi dz),$$

sit earum functio determinata, quae ergo integrationem admittat. Unde primo patet, quod modo innui, fluidum in aequilibrio subsistere non posse, nisi vires, singula fluidi elementa sollicitantes, ita fuerint comparatae, ut formula

$$Qdx + qdy + \varphi dz$$

sit differentiate completum. Cuius ergo integrale si ponatur in $= P$, erit pressio in λ

$$p = C+P .$$

Ita si sola adsit gravitas secundum directionem *CA* urgens, erit $p = C - z$, unde si pressio in uno puncta *λ* constet, unde constans *C* colligi queat, pro eodem tempore inde pressio in omnibus omnino punctis definietur.

85. Interim tamen tempore fluente pressio in eo~em loco variari poterit, id quod scilicet eveniet, si vires aquam extrinsecus urgentes, quarum ratio nondum est habita in iis viribus, quae in singula elementa singulatim agere assumuntur, fuerint variabiles, ita tamen, ut se mutuo in aequilibrio servent nullumque motum producant. Quodsi autem hae vires nulli mutationi sint obnoxiae, littera *C* denotabit quantitatem revera constantem neque a tempore *t* pendentem ; eodemque in loco *λ* perpetuo eadem pressio $p = C+P$ reperietur.

86. In huiusmodi ergo fluidi statu permanente eius extrema figura, quae nullis viribus est exposita, determinari poterit. In hac enim extremitate, qua fluidum sibi est relictum neque a parietibus vasis, cui forte est inclusum, continetur, necesse est, ut pressio sit nulla. Habebitur ergo haec aequatio : $P = \text{const.}$, qua figura extremae superficiei fluidi

per relationem inter ternas coordinatas *x, y* et *z* exprimetur. Atque si pro extremitate fuerit $P = E$, ob $C = -E$, in quovis alio loco λ interno erit pressio $p = P - E$. Ita si particulae fluidi a sola gravitate urgeantur, ob $p = C - z$, pro extremitate superficiei habebitur $z = C$, qua intelligitur extremam superficiem liberam esse horizontalem.

87. Deinde etiam omnia, quae adhuc de motu fluidi per tubos sunt eruta, ex his principiis facile deducuntur. Tubi autem vel angustissimi considerari solent, vel tales assumuntur, ut per quamlibet sectionem ad tubum normalem fluidum aequali motu transfluat : unde haec regula nascitur, ut celeritas fluidi in quovis tubi loco sit eius amplitudini reciproce proportionalis. Sit igitur (Fig. 2) λ punctum quodcunque huiusmodi tubi, cuius figura per geminam aequationem inter ternas coordinatas *x, y* et *z* exprimetur, ita ut inde pro quavis abscissa *x* ambae reliquae *y* et *z* definiri queant.

88. Sit praeterea huius tubi amplitudo in $\lambda = rr$, in alio autem tubi loco fixo, ubi amplitudo sit $= ff$, sit tempore praesente fluidi celeritas $= \Gamma$, de hinc autem elapso tempusculo *dt* evadat ea $= \Gamma + d\Gamma$, eritque ergo functio tempus *t* tantum, pariter ac $\frac{d\Gamma}{dt}$. Hinc ergo vera fluidi celeritas in λ erit tempore praesenti

$$V = \frac{ff\Gamma}{rr}$$

Cum nunc ex figura tubi dentur *y* et *z* per *x,* sit

$$dy = \eta dx \text{ et } dz = \theta dx;$$

unde ternae puncti fluidi in λ celeritates erunt secundum directiones *AL, AB* et *AC* sequentes:

$$u = \frac{ff\Gamma}{rr} \cdot \frac{1}{\sqrt{(1+\eta\eta+\theta\theta)}},\ \ v = \frac{ff\Gamma}{rr} \cdot \frac{\eta}{\sqrt{(1+\eta\eta+\theta\theta)}},\ \ w = \frac{ff\Gamma}{rr} \cdot \frac{\theta}{\sqrt{(1+\eta\eta+\theta\theta)}}$$

hincque fit

$$uu+vv+ww = VV = \frac{f^4\Gamma\Gamma}{r^4},$$

estque *rr* functio ipsius *x,* indeque pendentium y et z.

89. Cum nunc $udx + vdy + wdz$ debeat esse differentiale completum, cuius integrale posuimus $= S$, erit :

$$dS = \frac{ff\Gamma}{rr} \cdot \frac{dx(1+\eta\eta+\theta\theta)}{\sqrt{(1+\eta\eta+\theta\theta)}} = \frac{ff\Gamma}{rr} \cdot dx\sqrt{(1+\eta\eta+\theta\theta)}.$$

At $dx\sqrt{(1+\eta\eta+\theta\theta)}$ exprimit elementum ipsius tubi, quod si ponamus $= ds$, erit

$$dS = \frac{ff\Gamma ds}{rr},$$

unde cum hic tempus t constans sit assumtum, cuius functio est Γ, quantitates autem s et rr non a tempore t, sed tantum a figura tubi pendeant, erit

$$S = \Gamma \int \frac{ffds}{rr}.$$

90. Ad pressionem iam p, quae nunc in tubi puncto λ locum habet, inveniendam, considerari debet quantitas U, quae ex differentiatione quantitatis S oritur, si solum tempus t ut variabile tractetur, ita ut sit $U = \frac{dS}{dt}$.
Cum igitur formula integralis $\int \frac{ffds}{rr}$ tempus t non involvat, erit utique

$$\frac{dS}{dt} = U = \frac{d\Gamma}{dt} \int \frac{ffds}{rr};$$

sicque erit ex paragrapho 80:

$$T = \frac{f^4\Gamma\Gamma}{r^4} + \frac{2d\Gamma}{dt} \int \frac{ff}{rr} ds.$$

Quare positis quibuscunque viribus sollicitantibus Q, q et φ, erit pressio in λ :

$$p = C + \int (Qdx + qdy + \varphi dz) - \frac{f^4\Gamma\Gamma}{r^4} - \frac{2d\Gamma}{dt} \int \frac{ffds}{rr},$$

quae est ea ipsa formula, quae vulgo pro motu fluidi per tubos erui solet, atque adeo multo latius patens, quia vires quaecunque fluidum sollicitantes hic sunt assumtae, dum vulgo haec formula ad solam gravitatem adstringitur. Interim hic probe est recordandum ternas vires Q, q et φ necessario ita comparatas esse oportere, ut formula $Qdx + qdy + \varphi dz$ sit differentiale completum seu integrationem admittat.