Chapter 9.

Determination of the motion of a body that is acted upon by forces.

69. As long as a body moves uniformly in some direction, it recedes from, or approaches uniformly an arbitrarily selected plane. From this it is clear what we mean by the movement of a body from a plane, and that in the cited case the movement is uniform.

If in connection with the movement of a body we are only concerned with its distance from a certain plane, then we call the change in this distance the movement of the body from this plane. We can regard this movement as a speed, and we can determine it by dividing the increase in distance by the time. Calling the distance of the body from the plane x, and $x+dx$ the distance after an infinitesimally small time interval dt, so that in dt the distance has increased by dx, then we say that at this instant its movement from the plane is dx/dt, just as the true speed of a body is obtained by dividing the path by the time. This concept of the movement of a body from a plane will lead us to a simpler and more general way to visualize, from the acting forces, any motion, and the changes it undergoes. Here it is before all else necessary to note that from geometry the distance of a body from a plane is given by the straight line that is drawn perpendicularly from the body to the plane.

If therefore a body moves at constant speed in a straight line, then this line either will, or will not maintain the same distance from the assumed plane. In the first case the body always remains at the same distance from the plane, and its movement from the latter will be expressed by 0. In the latter case, since the body will on its line traverse equal path in equal times, its distance from the plane will in equal times increase or decrease by the same amount, and its movement from the plane will be at constant speed. If the distance increases we will denote it by +, if it decreases by -.

70. If a body is repelled from a plane by a force, its movement from the plane will increase such that its increase varies as the force multiplied by the time and divided by the mass of the body.

Let the mass of the body be M and its movement from the assumed plane be u, through which actually the speed of this movement from the plane is indicated; but for the sake of brevity we will call u the movement from the plane, since there is no fear of ambiguity. With this understood, as we have just seen, u would always remain the same, if no force acted on the body, which would always have to move uniformly along a straight line. The effect of the force, that we want to designate P, must therefore be that the movement u increases, since we assume that the body is pushed away from the plane by this force. Thus what we have shown above regarding the increase in speed, when the force pushes the body away from the plane, applies here too, and therefore the increase in movement produced in the infinitely small time interval dt is $du = nPdt/M$, or one finds $Mdu =$

nPdt, where *n* is the number determined by the way in which the quantities involved are expressed by numbers. If one wants to bring into the calculation the distance *x* of the body from the plane, one need merely write *dx/dt* instead of *u*, and if one assumes constant the increase in time *dt*, the increase in movement is $du = ddx/dt$. From this one gets

$$Mddx/dt = nPdt$$

or

$$Mddx = nPdt^2 ,$$

which is a differential equation of the second degree.

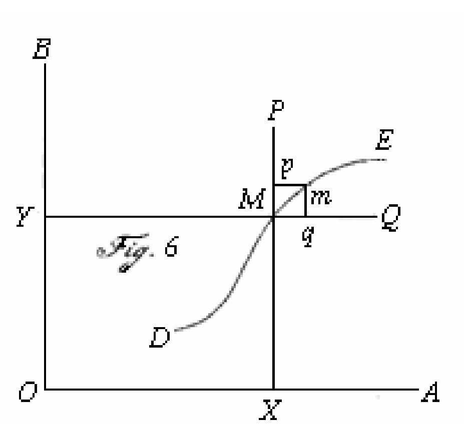
If apart from force *P* there were another force *Q*, which however acted on the body in a direction parallel to the plane, then there would be no change in movement from the plane due to the latter, and one would also have

$$Mdu = nPdt \quad \text{or} \quad Mddx = nPdt^2.$$

For if the force *P* were not present, the body would, irrespective of the force Q, move with constant speed from the plane, since this force, being parallel to the plane, would cause no change in its distance from the plane, a circumstance to be borne in mind in order to understand the following statements.

71. When the entire movement takes place in a plane, and therefore the forces are also acting in that plane, then this movement is described if one determines the movement of the body from two lines, drawn on the plane at right angles to each other.

Let us assume that the body of mass *M* (Fig. 6) moves on a plane, represented by a copper plate, and that it describes on it a line *DME*, then the forces acting on it must have their directions in that plane. Imagine now an other plane, that stands, perpendicularly to the plane of the paper, on the line *OA* , and consider the movement of the body from this plane. When it is at *M*, its distance will be represented by the line *MX*, that is perpendicular to *OA*; therefore it is only necessary to examine how the distance from the line *OA* varies from time to time, that means one only needs to examine the movement of the body from the line *OA*. Imagine now another plane that stands, perpendicular to the plane of the paper, on the line *OB*, where *OB* is at right angles to *OA*, and similarly examine the movement of the body from the line *OB*. It is then clear that if one knows at any time how far the body is from both lines *OA* and *OB*, then the true location of the body is known. Whatever the force acting on the body at *M* may be, it can always be decomposed into two forces *MP* and *MQ*, one of which, *MP*, pushes the body away from the line *OA,* whilst the other, *MQ*, pushes it away from the



line *OB*; and since *MQ* is parallel to *OA*, and *MP* is parallel to *OB*, the effect of one force is not affected by the other, so that one can determine their effects separately.

Let the distance *MX* of the body *M* from *OA* i.e. *MX* be designated by *x*, the movement from the same line *OA* by *u*, and the force *MP*, that drives it from this line by *P*. Similarly let the distance *MY* of the body from *OB* be called *y*, its movement from the same line *OB* be *v*, and the force *MQ*, pushing it away from *OB* be called *Q*. In the infinitesimally small time interval *dt* the action of these two forces will produce the following changes:

$$Mdu = nPdt \quad \text{or} \quad Mddx = nPdt^2,$$
$$Mdv = nQdt \quad \text{or} \quad Mddy = nQdt^2,$$

provided the instant in time *dt* is assumed constant. But from the foregoing it is clear that

$$u = dx/dt \quad \text{and} \quad v = dy/dt.$$

72. If one has determined the movement of a body in a plane from two lines that are in this plane at right angles to each other, then one obtains from this its true movement, i.e. its speed and its direction for every instant of time.

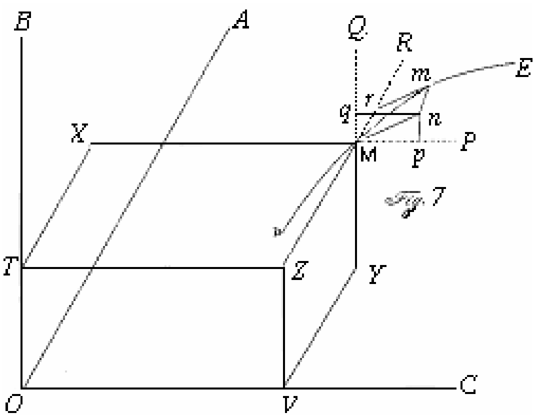
In the above we have given two particular rules, one to determine the change in speed that takes place, and the other to determine the change in direction that occurs. But with the present method we only need a single rule, that teaches us how the movement of a body from any plane is changed, whatever the acting force may be; this is so because the force can always be decomposed in two other forces, one of which repels from the plane, whilst the other acts parallel to the plane, and therefore does not interfere with the action of the former. In the previous section we have on this basis determined what change in the movement of the body from the two lines *OA* and *OB* (Fig.6) must occur, and that this must be contained in the two equations

$$Mdu = nPdt \quad \text{and} \quad Mdv = nQdt.$$

If one has determined from this for every moment of time the two movements *u* and *v*, one of which is directed towards *Mp* whilst the other is directed towards *Mq*, then one sees, if *Mp* and *Mq* are expressed by *u* and *v*, that the body must move along *Mm*, where *Mm* is the diagonal of the rectangular quadrangle *Mpmq*. Therefore the true speed which the body at *M* has in direction *Mm* is given by $\sqrt{uu+vv}$. In addition one also derives from this the direction, or the position of the line *Mm* with respect to the lines *OA* and *OB*. For since $Mp = u$ and $Mq = v$, one finds in the rectangular triangle *Mqm* the angle *QME*, whose tangent is *u/v*, which gives the direction of the movement *Mm* with respect to the line *MQ*; similarly the tangent of the angle *PME* will be *v/u*. If further one can for every time indicate the distances $MX = x$ and $MY = y$, then the true location *M* of the body at the time in question is determined. But if one knows the location of the body for every moment of time, then that is the most complete knowledge one could ever desire of the movement of a body.

73. If the movement of a body does not take place in a plane, then one can represent it in the clearest possible way, if one arbitrarily assumes three planes that are perpendicular to each other, and considers the movement of the body from each of these three planes.

Let one of these planes be represented by a copper plate, on which one arbitrarily draws two lines *OB* and *OC* (Fig.7), and at point *O* also erects the line *OA* perpendicular to the plate, so that it is also perpendicular to the lines *OB* and *OC*. The three lines *OA,OB* and *OC* define for us three planes *AOB, AOC* and *BOC* that are perpendicular to each other, and of which *BOC* may be regarded as the base plane, and the other two, *AOB* and *AOC*, as two walls, erected perpendicularly on it. Wherever the body may be, such as for example at *M*, one can determine how far it is from the three planes; to do this, if one draws from *M* firstly the line *MX*, perpendicular to the plane AO*B*, then the line *MY* perpendicular to the plane *AOC*, and finally the line *MZ* perpendicular to the plane *BOC*, then these lines will indicate the desired distances. Call these distances *MX* = *x*, *MY* = *y* and *MZ* = *z*; also from *X* on *OB* draw perpendicularly the line *XT*, and from *Z* on *OC* the line *ZV,* then *OV = x, VZ = OT =y* and *ZM = z*. When one knows these three lines, one can from them determine the points *V, Z* and *M*, and thus determine the true location of the body at *M*. If the body now moves in any arbitrary fashion, then one considers how much the distances *x, y, z,* change in the time interval *dt*, and denotes these changes by *dx, dy* and *dz*. Then the ratios *dx/dt, dy/dt, dz/dt* will represent the movements of the body from these three planes. Calling now the movement [*i. e.* speed] from the plane *AOB = u*, from plane *AOC = v* and from plane *BOC = w*, then we have that



$$u = dx/dt, \quad v = dy/dt, \quad w = dz/dt.$$

Finally, if we take these in the directions *Mp=u, Mq=v* and *Mr=w,* or *Mp=u, pv=v* and *vm=w*, then the line *Mm* will indicate the direction of motion of the body, and its true speed will be $\sqrt{uu+vv+ww}$. Consequently in this way the true movement of the body is visualized in the clearest manner.

74. Whatever the forces acting on a body might be, they can always be decomposed into three forces, each of which repels the body from one of the assumed planes, and the action of each is not disturbed by the other two forces.

Let the body be at point M, and let all nomenclature be as in the preceding section. It is known that whatever the forces acting on a body might be, they can always be replaced by three forces in the directions MP, MQ, MR, which are equivalent to them. Let us therefore call the force $MP = P$, the force $MQ = Q$, the force $MR = R$, the first of which, P, pushes the body from the plane OAB, the other, Q, from the plane AOC, and the third from the plane BOC, and should these forces push the body toward the planes, then as already said, they should carry the sign -. It is however clear that the action of the force $MP = P$ is not affected by the other two, Q and R, because their directions MQ and MR are parallel to the plane AOB, and do not attempt to either increase or decrease the distance of the body from this plane; the same applies to the remaining two forces. Therefore the action of each force can be separately determined without considering the others, so that we obtain for the changes produced at the point in time dt the following three equations:

$$Mdu = nPt, \quad Mdv = nQdt \quad Mdw = nRdt.$$

Alternatively we can also use the following, where the differential of time dt is assumed constant:

$$Mddx = nPdt^2, \quad Mddy = nQdt^2, \quad Mddz = nRdt^2.$$

We can also eliminate consideration of time, since

$$dx = udt, \quad dy = vdt, \quad dz = wdt,$$

which yields

$$Mudu = nPdx, \quad Mvdv = nQdy, \quad Mwdw = nRdz,$$

where we must remember that $dx/u = dy/v = dz/w$. If the forces are known, these equations are sufficient to determine the true location of the body at any time, and thus to indicate its true movement.

75. We understand the effectiveness of a force to be the integral quantity that is found if one multiplies the force with the differential of the distance from the plane, from which it pushes the body, and then integrates.

The last of the equations given above lead us to this new concept, to which we give the name *Effectiveness,* since we find by integration that

$$Muu = 2n\int Pdx, \quad Mvv = 2n\int Qdy, \quad Mww = 2n\int Rdz$$

from which the movement of a body from any plane can be derived. [We note that these are the equations for the kinetic energy along the three axes on dividing each by 2; thus Euler's *Effectiveness* is double the kinetic energy or the *Vis Viva,* or *living force* along each axis. IB.] This concept is the more of the utmost importance, because the sum of the effectiveness of all forces

$$\int Pdx + \int Qdy + \int Rdz$$

always has the same magnitude, even if we had assumed three different planes; and if the three forces have arisen from the decomposition of a single force, then the sum of their effectiveness is equal to the effectiveness of the single force, from which they were derived. Thus the arbitrariness of the assumed plane has no influence on the total effectiveness, the value of which remains always constant. Such a remarkable relation is not obtained from the other formula, and the magnitude of

$$\int Pdt + \int Qdt + \int Rdt$$

derived from it would always be different, according to how we change the three planes, apart from the fact that one can not ascribe to the forces a connection with *dt*, such as one can with the differentials *dx, dy,dz*. But what makes this concept of effectiveness in itself most remarkable is the fact that the whole theory of equilibrium is based on it. For it can be shown that equilibrium can not occur, if the sum of the effectivenesses is not a minimum, or occasionally a maximum. This marvelous theorem was first derived by the world renowned President de Maupertuis[1] and is closely connected with the other general principle of frugality. From this we see at the very least that the effectiveness has a major influence on all movements that can be produced by forces, and that it deserves to be given a special name.

76. In whatever way the movement of a body may be changed by forces, the so called living force, i.e. the mass of the body multiplied by the square of the speed, varies as the effectiveness of all forces acting on the body.

Combining the above three integral equations, we obtain

$$Muu + Mvv + Mww = 2n\int Pdx + 2n\int Qdy + 2n\int Rdz$$

or

$$M(uu + vv + ww) = 2n\int(Pdx + Qdy + Rdz).$$

Since we have shown above that the true speed of the body is given by

$$\sqrt{uu + vv + ww}$$

Note from the *Opera Omnia* edition : P.L. Moreau de Maupertuis (1698-1759) had in 1741 been called to Berlin by Frederick the Great, and been made President of the Academy, of which Euler was the Director of the mathematical class from 1744. Since the first of the papers by Maupertuis, to which Euler refers, dates from the year 1744, it is clear from this part of the text, that Euler's *Introduction to Natural Science* was at the earliest written in 1745, as rightly pointed out by Eneströem in his *Index*. Later on Euler devoted himself to the " Principle of least Action" the ground breaking papers 145, 146, 176, 181, 182, 186, 197, 198, 199, 200 (of Eneströem's *Index*); *Leonardo Euleri Opera omnia,* series II, vol.4. (F.R.) Indeed, one of the reasons why Euler returned to St. Petersburg was the strained relations he had with Frederick.

$uu + vv + ww$ is the square of the speed, and consequently $M(uu + vv + ww)$ is the so called living force of the body. Furthermore we have seen that

$$\int Pdx + \int Qdy + \int Rdz$$

is the effectiveness of the forces acting on the body, from which it is clear that the living force is equal to the effectiveness multiplied by the number 2n. However it must be remembered that the effectiveness, being a quantity obtained by integration, is not determined to the extent that an arbitrary constant may be added to it. This is also demanded by the nature of the matter, since the present movement of the body is dependent on the arbitrary initial movement. But if one has initially added the constant quantity, by which the effectiveness must be augmented to obtain the living force, then this will subsequently apply to the entire movement, and can be used to determine the body's living force correctly for any time and any location . This is a considerable advantage that the product of the mass of a body and the square of its speed has over the product of the mass and the speed itself, and that raises the concept of the living force far above that of the quantity of movement, since from our equations, that apply to all possible movements, it does not appear that the quantity of movement, given by

$$M\sqrt{uu + vv + ww}$$

is of particular significance.