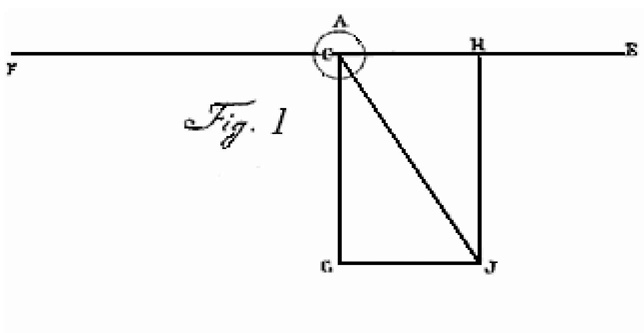
Chapter 7.

On the Effect of Forces on the Speed of Material Bodies.

51. To alter the speed of a body, a force is required, which acts on the body in its particular direction and pushes it either forward or backwards; in the former case its speed is increased, in the latter it is decreased.

We have seen that all forces that originate from impenetrability are of the nature of a pressure, through which the bodies interact at their point of contact, so each one tries to push the other away from itself. In the case of every force two things need to be considered: firstly its magnitude, and secondly its direction, since every force pushes with a certain effort in a certain direction. We consider here firstly the direction by noting the direction of the moving body on which it acts. If we suppose the body A (Fig.1) [Note : The figure numbers 1 to 22 of this paper correspond to numbers 220 to 241 of the *opera postuma*] moves in direction CE with a certain speed, and is pushed in this direction CE by some force, then it is clear that its speed must be increased without a change in direction, and in this case one says that the body is pushed forward. But if the force were to push the body backwards along CF, then there would also be no change in direction, but the body's speed would be reduced. From this one understands that if the force pushes the body A sideward along CG, where CG is perpendicular to CE, then at least at the first moment the speed of the body undergoes no change, but only its direction is shifted from AE to the side towards CG.



But if the force pushes the body in an inclined direction CJ, then, as is known from the theory of equilibrium, such an inclined force CJ has the same effect as two other forces CG and CH, that form a rectangular quadrangle, of which CJ is the diagonal. The force CH will only change the speed, but the force CG will only change the direction of the body.

52. If a moving body is pushed forwards by a force, then the increase in speed is the greater, the longer the force acts on the body, and the same applies to the decrease in speed if the force acts on the body in a backward direction.

Here the time during which the body is pushed by the force must of necessity be taken into consideration; for if a pressure is to produce an effect, it must be of some duration, however short this might be. Thus the longer the force acts on the body, the greater will be the change that is produced in its state; in double the time it will be twice, and in three times the time it will be three times as large, and so on. If we now assume that he body is pushed forward by the force, then the change in its state is an increase in its speed, so that in twice the time the increase in speed is twice as large, in three times the time three times as large, etc; that means the increase in speed produced in a body by the same force is proportional to the time. Therefore, if we call v the speed that the body now has, and dv the increase produced in time dt, then dv is proportional to dt; in another time interval ndt the increase in velocity will be ndv, and this is true irrespective of whether we regard the time dt and the corresponding increase in speed dv as infinitesimally small, or as finite quantities, provided only the magnitude of the force remains constant over the whole time. But since the increase in speed that is produced in a finite time can only be finite, it follows that that the increase in speed dv, produced in an infinitely small time interval dt, must be infinitely small. The same applies to the decrease in speed, if the force pushes the body backward; the decrease in velocity is then called $-dv$, and $-dv$ is proportional to dt.

53. If a body is pushed forwards, then the increase in speed that is produced in a certain time is the greater or smaller, the greater or smaller the force is that acts on the body; the same applies to the decrease in velocity if the body is pushed backward by the force.

A force twice as big must in the same time produce twice as big an effect, because just in view of that do we consider it twice as big. But when a force, whose magnitude is called p, pushes a body, and if, as before, the time of the interaction is dt, then the increase in speed dv is proportional to p, provided the time interval dt remains the same. But we have seen that when the force is constant, but time dt is considered variable, then the speed increase dv is proportional to dt ; from this follows that dv must be proportional to pdt , i.e. the increase in speed is proportional to the force p multiplied by the time dt. From this we conclude that even the smallest force is able to change the state of a body; for whilst it is certain that a large force can produce a certain change in a body, a force that is a thousand times smaller will in the same time produce a change a thousand times smaller, and were it to act on the body for a time a thousand time longer, it would produce in it the same change as the large force. It is therefore without foundation that, as some maintain, a force must have a certain magnitude before it is able to change the state of a body.

Finally it is evident that if the same force p pushes the body backwards, the loss in speed $-dv$ is proportional to pdt.

54. If a moving body is pushed forward by a certain force, then the increase in speed produced in a certain time is the larger, the smaller the persistence of the body is; that means the increase is inversely proportional to the persistence.

Since it is because of its persistence that a body attempts to remain in its state, the persistence opposes all change, and because of this forces are required to produce a change. The greater the persistence, the greater the force required to effect the same change in the same time, and from this follows that persistence falls into the category of magnitudes, and as such is accessible to measurement. Since from the above a doubling of the persistence requires a doubling of the force, if the effect is to remain the same, halving the force will only produce half the effect; that means the effect of the same force in the same time will be so much smaller as the persistence is larger. In our case the effect is the increase in speed; therefore if the force is represented by p, the time by $d\,t$, the increase in speed by dv, and the persistence by M, then dv is inversely proportional to M, or dv varies as $1/M$ when the force p and the time dt remain the same. If we now summarize what has been shown regarding the relation between the speed increase dv, the time dt and the force p, we find that the speed increase dv is given by pdt/M, that means it is given by the force p multiplied by the time dt and divided by the persistence M. If the force were to push the body backward, then the loss of speed $-dv$ would likewise be given by pdt/M.

55. The magnitude of the persistence of a body is called its mass or the quantity of matter of which it consists; accordingly one must take into consideration the mass of a body, if one wants to determine the change which a given force will produce in its state.

In this manner we arrive at a clear concept of what is called the mass or quantity of matter of a body. One differentiates thus between the mass of a body and the magnitude of its extent, because frequently a small body requires an equally great force as a large one to cause a certain change in its state, and from this one has concluded that one cannot judge the quantity of matter in a body from the magnitude of its extent. Some judge the mass from the weight; but since the weight arises from an external cause, and can be different in different locations and different circumstances, it cannot properly be used to measure an essential property of all bodies. Only if the bodies are weighed in vacuum on the same spot on earth can one say with confidence that their masses are proportional to their weights, otherwise one can not do this. Others estimate the mass of a body from the so called force of inertia, which is in complete agreement with the present concept, where we replace this inconvenient nomenclature by the term persistence. Here we find at once an incorrect consequence of this nomenclature, since some maintain that no force is able to put a body in motion unless it is greater than its force of inertia. But apart from the fact that it has been shown that this property cannot in any way be regarded as a force, and thus cannot be drawn into comparison with other forces, we recognize from the preceding argument that even the smallest force is able to put the largest body into motion or otherwise to change its state. For since *dv* varies as *pdt/M* , one sees that the persistence or mass *M* is not related to the force *p* in such a manner that *p* would have to be greater than *M* , but that however small *p* and however large *M* , there would always be an effect.

56. Thus when a moving body is driven forward by a force, the increase in speed produced in a given time varies as the force multiplied by the time and divided by the mass of the body.

This has already been shown above, if only one replaces the magnitude of the persistence by that of the mass, and this is the entire basis on which the whole theory of motion rests. Since this theory has been extremely well verified by experience, one could not doubt the truth of this basis, even if it were not buttressed by such uncontradictable reasons. Whoever is unable to see the strength of these reasons we refer to the indisputable truth of the whole theory of motion, which has in its totality been derived from this single basis. If one now puts the repulsive force = *p,* the time during which it acts on a body = d*t,* the resulting increase in speed = *dv* and the mass of the body = *M,* then, as already shown, *dv* varies as *pdt/M.* Therefore if one knows in a single case how large an increase in speed a given force has produced in a given time on a given body, one can by means of this relation determine the effect in all other cases. It is only necessary to express the various quantities involved, namely force, time, speed and mass in an arbitrary manner, depending on ones taste, in terms of numbers, and if on maintains this scheme of expression in all other cases, one can find the effect with the help of this relation. Thus if after adopting a certain system of units one finds in a particular case that *dv=npdt/M,* then in all other cases we must also have that *dv=npdt/M.*

57. When a body A (Fig.2), that so far has been at rest at point C, is acted on by a steady force in the direction CS, then movement in that direction will be imparted to it and after some time its velocity will vary as the force multiplied by the time, and divided by its mass.

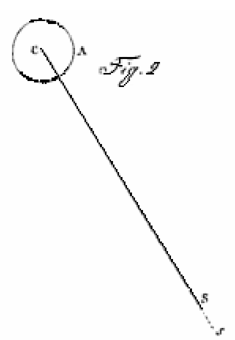


Fig. 2

Because the body was initially at rest, the force acting on it in the direction CS must at once impart on it movement in that direction, and since the force steadily acts in that direction, the movement of the body will maintain this same direction, but its speed will continually increase.

Also, since the magnitude of the force remains constant, the increase in speed will vary as the time; but since initially the body had no speed, after some time its total speed will be equal to the increase sustained during that time. If therefore as before we designate the force by *p,* the mass of the body by *M* and the speed attained in time *t* by *v,* then this speed *v* will vary as *pt/M*. We obtain the same result by integration if we only consider the increase in speed in every infinitely small time interval. For let *v* be the speed attained during time t, and *dv* its increase in time dt, then from the above we must have $dv= npdt/M$, if the relevant quantities are expressed in certain units, and the value of n is known from a particular case. But n,p and M are here constants; therefore one obtains by integration the total speed increase in time t, i.e. the total speed of the body $v = npt/M$. From this last calculation one sees at once how one would have to proceed if the force were of variable magnitude, although of constant direction; it would then be necessary when integrating the formula $npdt/M$ to take into account also the variability of the force, and one would obtain $v= n/M \int pdt$. In the following chapter we shall discuss the change in speed as well as direction produced by a force acting at an angle to the direction of the body.

58. Under these circumstances the path CS traversed by the body in a certain time will vary as the force multiplied by the square of the time and divided by the mass of the body.

Since the body moves in the straight line CS, let us call CS= s the path that it traverses in time t, and S at its end,with speed $=v$. If the force is p and the mass M, then from the preceding $v=npt/M$; and with this velocity it would in time dt uniformly traverse the infinitely small path $Ss=ds$, because the meanwhile produced increase in speed is infinitely small and can be neglected. But we have seen above that in the case of uniform motion the speed is obtained by dividing the path by the time; that means in this case

$$v = ds/dt$$

and therefore

$$ds/dt = npt/M$$

or

$$ds = nptdt/M$$

so that by integration one obtains

$$s = nptt/2M$$

since np/M is a constant; therefore the path s varies as ptt/M, i.e. as the force multiplied by the square of the time tt and divided by the mass of the body. A complete description of the motion is therefore contained in the two equations

$$v = npt/M \qquad \text{and} \qquad s = nptt/2M,$$

from which one can derive for any time the speed of the body and the path traversed in that time.

If one divides the second equation by the first one obtains

$$s/v = t/2 \qquad or \qquad s = tv/2.$$

But tv expresses the path traversed in time t with uniform velocity, which consequently is twice as long as $CS = s$, the path travelled through in the present case.

Furthermore, since $t = 2s/v,$ if this value of t is introduced into the first equation one obtains

$$v = 2nps/Mv \qquad \text{or} \qquad vv = 2nps/M.$$

Therefore in this movement the square of the speed varies with the force p multiplied by the distance traversed s, and divided by the mass M of the body. This can however immediately be derived from the following proposition.

59. If a body is moved forward by a force, then the increase in the square of its speed varies with the force multiplied by the distance traversed by the body in the meantime, and divided by its mass.

Let M be the mass of a body and v its present speed, with which in time dt it traverses the infinitesimally small distance $ds,$ whilst at the same time being pushed forward by force p; we therefore have

$$dv = npdt/M.$$

But for the movement through ds, which can be regarded as uniform, we have

$$v = ds/dt$$

so that

$$ds = vdt;$$

multiplying the above equation by $2v$ and writing in the last term ds for vdt one obtains

$$2vdv = 2npvdt/M = 2npds/M$$

But here $2vdv$ expresses the increase in the square of the speed, since it is the derivative of vv; therefore the increase in the square of the speed varies as pds/M, i.e. as the force p multiplied by the path ds and divided by the mass of the body. If the force p remains constant in magnitude and direction, one obtains from integration as before

$$vv = 2nps/M$$

provided the body was initially at rest. But if the force is variable, one can for any time express the change, produced in the state of the body, by the differential equation

$$2vdv = 2npds/M\ ;$$

In this case we therefore have two formulae from which one can determine the change, either from the time dt or from the traversed path ds. We have

$$dv = npdt/M \qquad \text{and} \qquad 2vdv = 2npds/M,$$

which are basically the same, their difference being that the first yields the increase in the speed itself, whilst the second yield the increase in the square of the speed. If however the force were to push the body back, one would have the following formulae:

$$-dv = npdt/M \quad \text{and} \quad -2vdv = 2npds/M$$

60. If a body at rest is set into motion by a constant force,then firstly the mass multiplied by the speed varies as the force multiplied by the time; secondly the mass multiplied by the square of the speed varies as the force multiplied by the path traversed.

The truth of these relations follows immediately from our two equations

$$v = npt/M \quad \text{and} \quad vv = 2nps/M\ ,$$

which, when multiplied with M yield

$$Mv = npt \quad \text{and} \quad Mvv = 2nps\ ;$$

The first of these determines the product of mass and speed, but the other determines the product of the mass and the square of the speed. Because of the importance of these products it is customary to give them special names. The former is called the *magnitude of motion* and the latter the *living force*. Although such names are arbitrary, the latter is not appropriate here, since we have determined a certain concept for the word force. The product Mvv can with as little justification be regarded as a force as the other product Mv, and since it equals the product $2nps$, where p represents a true force, it can not simply be compared with a force, but must be compared with the product of a force and a path, that is to say with line, just as the magnitude of motion, Mv, is related to the product of force and time. If one wanted to chose for this product an appropriate name, this could also well be applied to the product Mvv; but in doing so it is necessary to remember that this is strictly only permissible if one regards this product Mvv as being produced in a body at rest by a force. In that case one sees that the force multiplied by the time represents the magnitude of the motion, but that the force multiplied by the path indicates the so called living force. However if one holds fast to the definite concept of a force, as we have given it here, all difficulties that arise in the dispute concerning living forces are automatically removed, and the two formulae that we have found must in all cases indicate the truth.

61. A body in motion is brought back to rest, if a backward directed force acts on it for the same time as was required for a forward directed force of equal magnitude to act in order to produce its initial motion.

Let M be the mass of the body, v its speed, and p the backward force acting on it and gradually reducing its speed, t the time in which the body is brought to complete rest, and s the path that it traverses until then. Then, because a backwards acting force reduces the speed by the same amount as it would increase it, if it acted on the body in a forward direction, the following two equations would hold:

$$Mv = npt$$

and

$$Mvv = 2nps.$$

Therefore if two diffrerent moving bodies are to be brought to rest in the same time, the forces required must vary as Mv, i.e. as the magnitudes of motion of the bodies. But if the same bodies were to be brought to rest not in the same time, but whilst traversing the same path, the required forces would have to vary as Mvv, that is as the masses multiplied with the square of the speed, or as their so called living forces. This is the cause of the dispute arising from the fact that some maintain that the force of a moving body must be estimated from the product of its mass with its speed, whilst others maintain that it must be estimated from the product of the mass and the square of the speed. The misunderstanding arises manifestly from the fact that one wants to associate an actual force with a moving body, although neither Mv nor Mvv can be compared with a force, since in one case the force must be connected with time, and in the other with the path. One can also not in a general way say how large a force is required to bring a moving body to rest, since any force is able to do this; but if it is to occur in a certain and definite time, then those are correct who maintain that the force varies as the magnitude of movement; but if it is to occur over a certain path, then the others are correct, and seen in this light the whole matter dissolves into a mere quarrel about words. If however the force p is assumed to be known, then the time in which the body is brought to rest varies as the magnitude of motion; however the path, that the body must traverse before coming to rest, varies as the so called living force, or the mass multiplied with the square of the speed.

[See Joh. Bernoulli (1667-1748), Selected Theorems for the Conservation of Living Forces to be demonstrated, and to be confirmed by experiments. Also excerpts from letters to his son Daniel, dated 11. Oct. and 20. Dec. 1727, Comment. Acad. Sc. Petrop. **2** (1727), 1729, p. 200; Opera omnia, Lausannae et Genevae 1742, t. III, p.239 (here the title is De vera notione virium vivarum, or : Concerning the true notion of living forces].

See further I. Kant (1724-1804), Thoughts on the correct estimate of the living forces, Koenigsberg 1746; Kants Collected Writings, published by the Prussian Academy of Sciences, Berlin 1902,Vol.1, p.7.