# Chapter 21

# On the Laws of Motion of Liquid Matter.

*155. The motion of liquid matter is completely determined, if one is able to indicate for every point in time the velocity and direction with which each particle moves; to which end it is best to resolve the movement into three given, and mutually orthogonal directions.*

We intend to consider straight away the most general case of the movement of liquid bodies, since the steps required here are not much more difficult than those required for dealing with one or the other of special cases, and also to convince the reader that principles given above are not only general, but also sufficient to deal with all possible cases that could arise. Therefore to make the treatment quite general, and as was done in the preceding Chapter, we want to regard the liquid matter in relation to three mutually perpendicular planes *AOB, AOC, BOC* and three axes *OA,OB,OC*. We therefore consider a point *M* within the liquid body, the position of which is specified by its distances from the above three planes: $MX = x$, $MY = y$ and $MZ = z$. After a time $t$, reckoned from a certain time origin, has elapsed, the particle of liquid matter at *M* will have a certain movement, and whatever this movement is, it can always be resolved into three directions *MP, MQ* and *MR*, which are parallel to the assumed three axes *OC, OB,* and *OA*. We will therefore put: the movement along $MP = u$, along $MQ = v$, along $MR = w$, and on the determination of these three movements rests the entire description of the movement of the liquid matter. But these three speeds $u, v, w$ can undergo two different types of change, one depending on the location of the point *M*, i.e. depending on the three magnitudes $x, y, z,$ the other depending on the time $t$, where we want to retain the manner used above of expressing the increase in each of them if only one of the variables increases by an infinitesimal amount.

*156. These three movements must have a certain relation to the density of the liquid matter and to its change, both with respect to location and to time, leading to an equation that distinguishes possible movements from impossible ones. This equation is based on the principle of continuity.*

Let the density at $M = q$, which varies with time as well as with location. Let the point *M*, in the course of its movement, arrive after time $dt$ at $M'$ (Fig. 21, see Ch. 20), the location of which is given by the following three coordinates:

$$x + udt,\ \ y + vdt,\ \ z+wdt\ ;$$

consequently the density at $M'$ will be

$$q + dt\left(\frac{dq}{dt}\right) + udt\left(\frac{dq}{dx}\right) + vdt\left(\frac{dq}{dy}\right) + wdt\left(\frac{dq}{dz}\right).$$

We now assign to the liquid particle at *M* the shape of a cube *MPQRmpqr*, of volume *dxdydz* and mass *qdxdydz*, which through movement in time *dt* is shifted to *M'P'Q'R'm'p'q'r'*. If the movement of point *P* were the same as that of point *M*, then we would have *M'P'=MP*; however the three components of movement of point *P* are

$$u+dx\left(\frac{du}{dx}\right),$$

$$v+dx\left(\frac{dv}{dx}\right),$$

$$w+dx\left(\frac{dw}{dx}\right),$$

the last two of which do not change length *M'P'*, since they are perpendicular to it. But since due to the first the point *P* goes faster than point M by the amount $dx\left(\frac{du}{dx}\right)$, *M'P'* will be greater that *MP* by $dxdt\left(\frac{du}{dx}\right)$, so that

$$M'P'=dx+dxdt\left(\frac{du}{dx}\right)=dx\left(1+dt\left(\frac{du}{dx}\right)\right).$$

Similarly we have

$$M'Q'=dy\left(1+dt\left(\frac{dv}{dy}\right)\right)$$

and

$$M'R'=dz\left(1+dt\left(\frac{dw}{dz}\right)\right).$$

Although in this altered figure the angles are no longer right angles, the difference is infinitesimally small, and the volume is still given correctly by $M'P'\cdot M'Q'\cdot M'R'$. This volume is therefore

$$dxdydz\left(1+dt\left(\frac{du}{dx}\right)+dt\left(\frac{dv}{dy}\right)+dt\left(\frac{dw}{dz}\right)\right),$$

which,when multiplied by the density stated above, gives the mass as

$$qdxdydz\left(1+dt\left(\frac{du}{dx}\right)+dt\left(\frac{dv}{dy}\right)+dt\left(\frac{dw}{dz}\right)\right)+dxdydz\left(dt\left(\frac{dq}{dt}\right)+udt\left(\frac{dq}{dx}\right)+vdt\left(\frac{dq}{dy}\right)+wdt\left(\frac{dq}{dz}\right)\right),$$

which from the principle of continuity must be equal to the previously given mass *qdxdydz.* Dividing by *dtdxdydz* we obtain the equation

$$q\left(\frac{du}{dx}\right)+q\left(\frac{dv}{dy}\right)+q\left(\frac{dw}{dz}\right)+u\left(\frac{dq}{dx}\right)+v\left(\frac{dq}{dy}\right)+w\left(\frac{dq}{dz}\right)+\left(\frac{dq}{dt}\right)=0,$$

which can be contracted into

$$\left(\frac{dq}{dt}\right)+\left(\frac{d\cdot qu}{dx}\right)+\left(\frac{d\cdot qv}{dy}\right)+\left(\frac{d\cdot qw}{dz}\right)=0,$$

indicating a certain connection that must exist between the three movements and the density.

*157. With regard to the pressure, this must be such that from it, together with the forces acting on each particle, the movement of the latter results. The equation obtained from this serves to determine the pressure of the liquid matter everywhere and at all times.*

As in the previous Chapter, we want to assume that the particle is driven, proportionally to its mass *M*, by three forces *P,Q,R* in directions *MP,*MQ,MR. Since the mass is *qdxdydz*, the three forces are

towards $MP = Pqdxdydz$
towards $MQ = Qqdxdydz$
towards $MR = Rqdxdydz$.

Let us furthermore represent the pressure at *M* and at time *t* by the height *p*, changes in which also depend on the location and on time. The effect of this pressure on the particle *MPQRmpqr* has already be shown above (153); the particle is driven by the following forces:

$$\text{towards } PM = dxdydz\left(\frac{dp}{dx}\right),$$

$$\text{towards } QM = dxdydz\left(\frac{dp}{dy}\right),$$

$$\text{towards } RM = dxdydz\left(\frac{dp}{dz}\right).$$

The differences between these forces and the above must determine the increases in the speeds *u,v,w,* in time *dt*, during which point *M* moves to *M'*. During this change the quantities *t,x,y,z,* on which the three movements depend, experience the following increases: *dt*, *udt,vdt* and *wdt* ; therefore the increase in the speeds will be:

$$\text{for } u \quad dt\left(\frac{du}{dt}\right)+udt\left(\frac{du}{dx}\right)+vdt\left(\frac{du}{dy}\right)+wdt\left(\frac{du}{dz}\right)=Xdt$$

$$\text{for } v \quad dt\left(\frac{dv}{dt}\right)+udt\left(\frac{dv}{dx}\right)+vdt\left(\frac{dv}{dy}\right)+wdt\left(\frac{dv}{dz}\right)=Ydt,$$

$$\text{for } w \quad dt\left(\frac{dw}{dt}\right)+udt\left(\frac{dw}{dx}\right)+vdt\left(\frac{dw}{dy}\right)+wdt\left(\frac{dw}{dz}\right)=Zdt,$$

from which we see what is meant by the abbreviations *X,Y* and *Z*. But the increase of every velocity component multiplied by the mass *qdxdydz* is equal to the whole force acting in its direction multiplied by the time *dt*, as was shown above. Therefore, after division by *dtdxdydz* we obtain the following three equations:

$$Xq = Pq - \left(\frac{dp}{dx}\right), Yq = Qq - \left(\frac{dp}{dy}\right), Zq = Rq - \left(\frac{dp}{dz}\right).$$

Since

$$dx\left(\frac{dp}{dx}\right) + dy\left(\frac{dp}{dy}\right) + dz\left(\frac{dp}{dz}\right)$$

is the differential of *p*, where all three coordinates *x, y, z* are assumed variable, and the time *t* is assumed invariable, we will, as is customary, write this differential simply as *dp* so that these three equations are united into the following single one:

$$\frac{dp}{q} = Pdx + Qdy + Rdz - Xdx - Ydy - Zdz,$$

a differential equation in which the time *t* must be regarded as not changing. *P,Q,R* are the forces, and the values of *X,Y,Z* have been indicated above.

*158. In these two equations are contained all possible motions that can occur in liquid matter, irrespective of the nature of the forces acting on it, and of whether it is compressible or not.*

Summarizing what has so far been said, the whole matter depends on the following points. Firstly the state of the liquid matter must be described in terms of the three orthogonal planes *AOB,AOC,BOC,* and the three axes *OA,OB,OC,* that may be arbitrarily chosen. Secondly one considers, at time *t* after a certain starting time, a particle of liquid matter at point *M*, whose location is given by the coordinates *XM* = *x, YM* = *y* and *ZM* = *z*. Thirdly the forces, acting on it in proportion to the masses, are resolved in the directions *MP, MQ, MR* and are denoted by the letters *P,Q,* and *R* such that when multiplied with the mass they represent the forces themselves. Fourthly one puts the density of the liquid matter at point *M* and time t as *q* and expresses the pressure there by the height *p*. Fifthly the movement of the particle itself in the directions *MP,MQ,MR* is given by the speeds *u, v, w*; then in the quantities *p, q, u, v, w* a twofold variability must be taken into consideration, one of which depends only on time, but the other on the location, i.e. on the three coordinates *x, y* and *z*. In sixth place, to simplify the calculation, one puts

$$\left(\frac{du}{dt}\right) + u\left(\frac{du}{dx}\right) + v\left(\frac{du}{dy}\right) + w\left(\frac{du}{dz}\right) = X$$
$$\left(\frac{dv}{dt}\right) + u\left(\frac{dv}{dx}\right) + v\left(\frac{dv}{dy}\right) + w\left(\frac{dv}{dz}\right) = Y$$

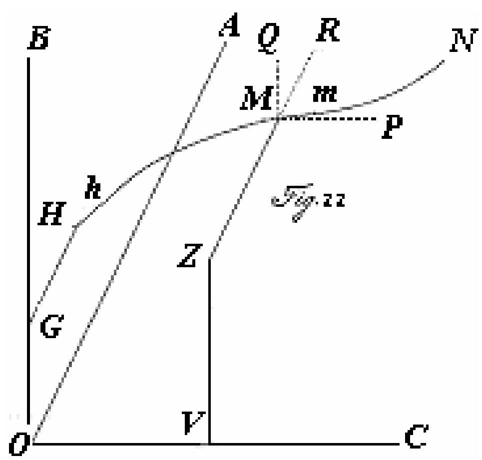
$$\left(\frac{dw}{dt}\right)+u\left(\frac{dw}{dx}\right)+v\left(\frac{dw}{dy}\right)+w\left(\frac{dw}{dz}\right)=Z\ .$$

Following this, the movement, whatever its character might be, is expressed by the following two equations:

$$\text{I. } \left(\frac{dq}{dt}\right)+\left(\frac{d\cdot qu}{dx}\right)+\left(\frac{d\cdot qv}{dy}\right)+\left(\frac{d\cdot qw}{dz}\right)=0$$

$$\text{II. } \frac{dp}{q}=Pdx+Qdy+Rdz-Xdx-Ydy-Zdz,$$

where it must be noted that in the second differential equation the time t must be taken as constant and the three coordinates x, y, z as varying. This must be taken into account in the integration, since the constant that is added here can involve the time. And in this manner all external circumstances are taken into account, such as when the liquid matter is pressed from the outside. What matters in all cases is that one find out how the three speeds *u* ,*v* ,*w* depend on the quantities *x, y, z,* and the time *t* , so that in the first place the first equation is satisfied, and subsequently the second equation can be integrated. But here the art of solving the latter is still deficient, since it has not yet been sufficiently developed for it to be done in the general case.




*159. Nearly all movements of liquid bodies that one can completely determine, are restricted to the case where the liquid matter moves through a tube, and if the solutions obtained are to be strictly accurate, then the tube must be effectively infinitely long. If this is not the case, the conclusions are not completely correct.*

Let us specify the tube *HMN* (Figure 22) with reference to our three planes *AOB, AOC* and *BOC* and consider its cross section *H* in the plane *AOB*. Let the width of the tube at *H* be *cc,* and for the location of point *H,* let *OG=g* and *GH=h*. After time *t* let the speed of the liquid matter at *H* be ω ,its density be Θ, and the pressure be π, then the quantities ω, Θ and π are only dependent on time t. For any other point *M* of the tube, determined by the coordinates *OV=x, VZ=y*, *ZM =z*, let the true speed at *M* in the direction of the tube *Mm* be *s*, the density *q,* the pressure *p*, the with of the tube *rr*, and, as before, the forces *MP=P, MQ=Q, MR=R,* which act proportionally to the masses. Since the shape of the tube of length *HM = s*, is regarded as known, *x, y, z* and *rr* are determined by *s*. From the true speed *s* and the direction *Mm=ds* one obtains the three speeds

$$u=\frac{sdx}{ds},\quad v=\frac{sdy}{ds,}\quad w=\frac{sdz}{ds,},$$

and therefore

$$udy = vdx, \;\; udz = wdx, \;\; vdz = wdy \quad \text{and} \;\; uu + vv + ww = ss.$$

Consequently one obtains

$$\begin{aligned} Xdx + Ydy + Zdz = &+dx\left(\frac{du}{dt}\right) + udx\left(\frac{du}{dx}\right) + udy\left(\frac{du}{dy}\right) + udz\left(\frac{du}{dz}\right) \\ &+dy\left(\frac{dv}{dt}\right) + vdx\left(\frac{dv}{dx}\right) + vdy\left(\frac{dv}{dy}\right) + vdz\left(\frac{dv}{dz}\right) \\ &+dz\left(\frac{dw}{dt}\right) + wdx\left(\frac{dw}{dx}\right) + wdy\left(\frac{dw}{dy}\right) + wdz\left(\frac{dw}{dz}\right). \end{aligned}$$

Since $\frac{dx}{ds}, \frac{dy}{ds}, \frac{dz}{ds}$ are not dependent on the time, we have

$$\left(\frac{du}{dt}\right) = \frac{dx}{ds}\left(\frac{ds}{dt}\right) \;\; \text{etc.},$$

and if in the other terms t is taken as constant, since

$$\frac{dx^2 + dy^2 + dz^2}{ds} = ds,$$

one obtains

$$Xdx + Ydy + Zdz = ds\left(\frac{ds}{dt}\right) + udu + vdv + wdw = ds\left(\frac{ds}{dt}\right) + sds.$$

Our differential equation will therefore assume the following form:

$$\frac{dp}{q} = Pdx + Qdy + Rdz - ds\left(\frac{ds}{dt}\right) - sds.$$

Putting the effectiveness of the forces = $V$, one obtains

$$Pdx + Qdy + Rdz = dV,$$

leading to the following equation:

$$\frac{dp}{q} = dV - ds\left(\frac{ds}{dt}\right) - sds,$$

where time $t$ is taken as constant. Consequently we have

$$\int \frac{dp}{q} = V - \frac{1}{2}ss - \int ds\left(\frac{ds}{dt}\right) + const.,$$

where the constant may include time $t$.
[ This result is due to Daniel Bernoulli, *Hydrodynamica*, Strassbourg 1738.]

*160. But apart from this equation, through which the pressure of liquid matter in the tube is determined, the principle of continuity must be taken into account, from which one obtains a new equation, which together with the previous one contains the entire motion.*

We find this new equation most readily by noting that the liquid matter, which now occupies the part *HM* of the tube, but after time *dt* occupies the part *hm*, must have the same mass in both cases. Since the width of the tube at $M = rr$, the mass of the liquid matter in the tube $HM = \int qrrds$, but after time *dt* the mass of the liquid matter contained within it will be

$$\int qrrds + dt\int rrds\left(\frac{dq}{dt}\right).$$

But in time $dt$ this mass flows from $H$ to $h$ and $M$ to $m$, so that $Hh = \omega dt$ and $Mm = sdt;$ therefore the mass contained in $Hh$ will be $\theta ccdt$ and that in $Mm = qrrsdt$, the former of which must be subtracted from the above, but the latter added. Therefore the mass contained in the part *hm* of the tube after time *dt* will be

$$\int qrrds + dt\int rrds\left(\frac{dq}{dt}\right) - \theta cc\omega dt + qrrsdt,$$

which must be equal to the former, that previously occupied part *HM*. If we now divide by *dt*, we obtain the following equation:

$$\theta cc\omega = qrrs + \int rrds\left(\frac{dq}{dt}\right),$$

where the integral $\int rrds\left(\frac{dq}{dt}\right)$ must be taken such that it vanishes if one takes the point $M$ to be at $H$, or $x=0$, $y=g$ and $z = h$, in which case $rr = cc$, $q = \Theta$ and $s = \omega$. But it must be noted that there are only two variables, namely $s$ and $t$ ; for $rr$ depends only on the shape of the tube, or on $s$, $\Theta$ and $\omega$ depend only on time $t$, and $q$ as well as $s$ are dependent on both.

If the density is the same at all times, one has that $\left(\frac{dq}{dt}\right) = 0$ and $q = \theta$, so that $cc\omega = rrs$, i.e. the speeds are inversely proportional to the widths; since

$$s = \frac{cc\omega}{rr} \text{ and } \left(\frac{ds}{dt}\right) = \frac{ccd\omega}{rrdt},$$

one finds from the first equation that

$$\frac{p}{\theta} = V - \frac{c^4\omega\omega}{2r^4} - \frac{d\omega}{dt}\int\frac{ccds}{rr} + C,$$

which contains nearly everything that has so far been said about the motion of liquid matter.

*161. If liquid matter moves in a vessel, the latter experiences a counter-pressure, which can be found if one deducts from all the forces, that act on the liquid matter ,those forces that are required for the changes that take place in all the particles.*

If liquid matter flows through a tube, or otherwise is ejected from a vessel, its effect on the vessel is different from that occurring if it were at rest, and this effect is called the reaction or counter-pressure, the determination of which is of the utmost importance in the theory of the movement of liquid matter. Although this point is usually treated in a manner associated with the greatest difficulties, it can be easily dealt with using the given formula. Let us designate all forces that act on the liquid matter by the letter $F$, but those that are needed to produce the changes that take place in the movement of all particles, by the letter $G$, and let finally $H$ represent the counter-pressure we are looking for. Since here $F$ must be regarded as the cause, but the quantities $G$ and $H$ as the total effect, the force $F$ must be equal to the forces $G$ and $H$, i.e. $F = G + H$, from which follows at once that $H = F - G$, as indicated by the given formula. The subtraction of the forces $G$ is done best by reversing for all particles of the liquid matter the forces necessary for the changes that take place in their movement, i.e. by imagining them to act in the opposite direction, and these reversed forces, together with the actual force $F$, will represent the counter-pressure. From what was said above in § 157, if we put the mass of the particle $MPQRmpqr = dM$ , then the reversed forces will be

$$\text{towards } PM = XdM,$$
$$\text{towards } QM = YdM,$$
$$\text{towards } RM = ZdM.$$

On account of every particle $dM$ of the liquid matter the following three forces are acting on the vessel:

$$\text{I.} \quad \text{towards } MP = (P - X)dM,$$
$$\text{II.} \quad \text{towards } MQ = (Q - Y)dM,$$
$$\text{III.} \quad \text{towards } MR = (R - Z)dM.$$

These forces that, originating from the particles $dM$, are acting on the vessel, can also be determined from the pressure acting there, according to the formulae given in § 157:

$$\text{I.} \quad \text{Force towards } MP = \frac{dM}{q}\left(\frac{dp}{dx}\right),$$
$$\text{II.} \quad \text{Force towards } MQ = \frac{dM}{q}\left(\frac{dp}{dy}\right),$$
$$\text{III.} \quad \text{Force towards } MR = \frac{dM}{q}\left(\frac{dp}{dz}\right).$$

All these forces taken together represent the total counter-pressure which the vessel must sustain because of the liquid matter it contains, provided one adds to these furthermore those that immediately press on the liquid matter from the outside.