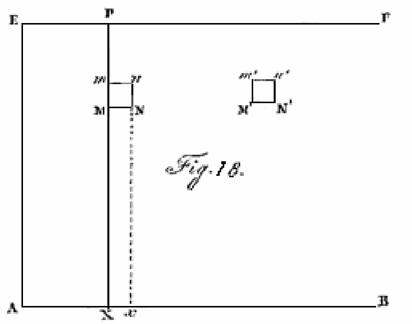
On the laws of equilibrium in liquid substances.

147. A liquid substance, the particles of which are not subject to any forces other than the pressure due to adjacent particles, can, whatever it density might be, not be in equilibrium or at rest, if the pressure is not the same at all points.

If a liquid substance is to be at rest, then all its particles must also be at rest, and therefore the forces acting on each of them must compensate each other, i.e. maintain equilibrium. Since the particles are not subject to any forces other than the pressure due to the adjacent ones, the pressure from all directions must be of equal strength, which is the case when the height, that represents the pressure, is everywhere the same. Here differences in density of the liquid substance produce no change, other than to the extent that the density depends on the magnitude of the pressure. When therefore the liquid substance is such that wherever the pressure is equally strong, the density must also be the same, as is the case for liquid substances of the kind, that can be compressed, and that so much more the greater the compressing forces are, then also in this case the density must everywhere be equally great. But since the aether is such a liquid substance, the particles of which are not acted upon by any foreign forces, and the density of which is solely determined by the pressure, or the elastic force, the aether can not be other than in equilibrium, when its elastic force, and consequently also its density, has everywhere the same magnitude. Since, whilst explaining gravity, we have seen that the elastic force of the aether must be very different at different locations, there must be a similar difference in its density, and there must be a strong internal movement between its parts. But if we consider the case where different liquid substances are mixed together, each having a particular density that does not depend on the pressure, then for a state of rest and equilibrium it would be necessary that the magnitude of the pressure would have to be the same everywhere, although the density would be very different. But if such a substance has weight, then we must consider it separately.

148. A liquid substance that has weight can only be in equilibrium if at equal heights the pressure as well as the density are the same. But at different heights the pressure will be different, decreasing with increasing height, until at the surface it vanishes entirely.

Imagine a horizontal plane *AB* (Fig.18) , above which is situated the liquid substance, of which we want to consider a particle *MNmn* in the shape of a cube. Its height above the



horizontal plane *A* is *XM = z*, so that it is pushed downwards by gravity in the direction *MX*. Let the density of the particle = *q*, the height *Mm =dz* ,the length *MN = dx* (putting *AX*=*x*) and let the width, not shown in the figure, be *dy*, then the volume of the particle = *dxdydz,* which, when multiplied by the density *q* yields its mass = *qdxdydz,* which at the same time expresses its weight. Therefore the force of gravity that pushes the particle downwards along *MX* equal to *qdxdydz*, and its effect must be counterbalanced by the pressure of the adjacent liquid matter. Let therefore the pressure at *M* be given by the height *p*, which must be dependent on the coordinates *x,y* and *z*, so that it has for every point *M* a certain magnitude. Because the particle *MNmn* is to remain at rest, the pressure from all directions must be of equal magnitude, and therefore the height *p* must undergo no change, although *x* or *y* is changed, i.e. *p* must be solely dependent on the height *XM = z.* Let therefore the pressure at *m* and *n* be *p+dp*, then the force pressing on the upper side *mn* will be (*p+dp*)*dxdy*; on the lower side *MN* the particle is pushed upwards by the force *pdxdy*. From both therefore results a force –*dpdxdy* that pushes the particle upwards, and that must be equal to the downwards pushing force *qdxdydz* due to gravity. For equilibrium it is therefore necessary that –*dp = qdz* or *dp* = - *qdz*. Because, as we have seen, *p* is only dependent on *z*, this equation is impossible, unless *q* also is solely dependent on *z*, therefore we have

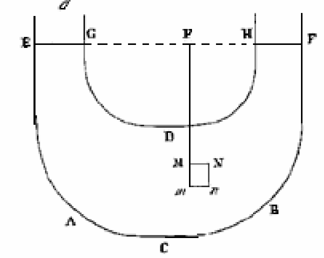
$$p = c - \int qdz$$

Therefore at equal heights not only the pressure, but also the density must have the same magnitude, and the greater the height, the smaller will the pressure become, and if we take the magnitude of the height *AE* such that $\int qdz = c$, then the pressure at *E* and the whole horizontal plane *EF* passing through *E* will totally vanish. From this follows that at equilibrium the surface of a heavy liquid body must always be horizontal. From this it is also clear that if an other particle *M'N'm'n'* at the same height were denser or less dense, i.e. heavier or lighter than *MNmn*, then since it is pressed upward by the pressure with equal strength, it would either have to sink down or rise up, and therefore could not maintain equilibrium.

149. In stagnant water the pressure is always proportional to the depth under its surface and a body immersed in it is driven upwards by a force that is equal to the weight of a quantity of water of equal size. If therefore the body is either heavier or lighter, it will either sink down or rise.

Water represents for us here liquid matter, the density of which is unchangeable and not dependent on pressure, so that q is a constant quantity. According to the foregoing calculation the pressure $p = c - qz$, and if we denote the height to the uppermost water surface *EF* (Fig.19) by $AE = a$, then, since at *E* the pressure vanishes, we have $0=c-qa$, or $c = qa$. Therefore $p = qa - qz = q(a - z)$ and $a - z$ represents the depth of point *M* below the surface *EF*, so that $p = q \times PM$. But since the surface of free standing water is where there is no pressure, it is always horizontal, and whatever shape one chooses for the vessel *ACBD*, the surface *EG*…….*HF* must be horizontal. Below this surface the pressure begins, and it is exactly proportional to the depth below this surface; thus at *M* the pressure is determined by the height $p = q \times PM$, or if this height refers to the water itself, as we have determined it above for uniform matter, then $p = PM$; therefore the height that determines the pressure of the water at every point *M*, is equal to the depth of this point below the surface. For here an infinitely small surface of area ds^2 is pressed such that the force equals the weigh of a column of water with base area ds^2 and height *PM*. The deeper a point *M* is assumed to be below the surface of the water, the greater will the pressure of the water be there; from which follows how with very little water a very large pressure can be produced, namely if the top of the vessel terminates in a narrow tube, that can be filled to a great height by a small quantity of water. Let us now consider a body *MNmn* of volume e^3, immersed in the water, then it must experience the same pressure as if water were in its place; but this water would be in equilibrium , and therefore be equally strongly pushed upwards, as it is pushed downwards by its weight. Therefore the body will be pushed upwards by the pressure of the water with a force that equals the weight of a quantity of water that occupies the space e^3. If the bodies' own weight is greater or smaller, it will be pushed by the excess downwards or upwards.

Fig.18.



150. If the air is to be in equilibrium, then at the same height above the earth not only the pressure and the density, but also the temperature must everywhere be the same. If these conditions are not met, then the air can not be at rest, but a wind must develop.

The elastic force of the air is not only dependent on its density, but the temperature contributes also very much to its increase. Therefore if the elastic force at a location is p and the density is q, but if z, as above, indicates the height of this location above the earth, or above an arbitrary horizontal plane *AB* (Fig.18), then p can not be determined solely from q, but one must also take into account the temperature = r. Although the way to make this determination is not exactly known, we will not make a great error if we assume p to be proportional to qr , i.e. $q = \beta p/r$. But because at equal height above the earth, or above the horizontal plane *AB,* both the pressure p and the density q must everywhere be the same, it is clear that the temperature there must also be the same, irrespective of how p is determined by q and r. But if we assume the above formula $q=\beta p/r$, then we have

$$dp = \frac{-\beta pdz}{r}$$

or

$$\frac{dp}{p} = \frac{-\beta dz}{r} .$$

To make the meaning of this equation clearer, we want to express the pressure of the air at any height above the earth by the height of a column of water, and set the density of water equal to 1. Furthermore let the pressure on the plane $AB = h$, the density = g and the temperature = f. Since

$$g = \frac{\beta h}{f} ,$$

we have that

$$\beta = \frac{fg}{h}$$

and therefore

$$q = \frac{fgp}{hr} ,$$

from which we obtain

$$\frac{dp}{p} = \frac{-fgdz}{hr} .$$

If the temperature r were the same at all heights, or $r = f$, then we would have

$$lp = C - \frac{gz}{h}$$

and since for $z = 0$ we must have $p = h$, we obtain

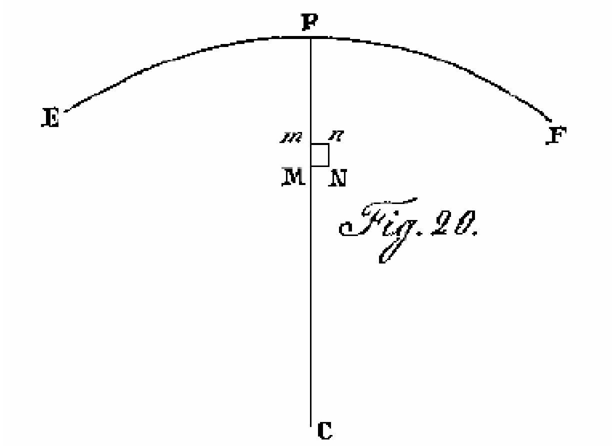
$$lp = lh - \frac{gz}{h}$$

or

$$\frac{gz}{h} = l\frac{h}{p} .$$

If the temperature were to decrease steadily with height, then from the law governing this decrease the pressure at all heights could be determined. In addition this theorem provides us with a rich source for wind, for since at every height the temperature changes all the time, the air must for this reason alone be in continuous motion, thus causing winds.

151. When all parts of a liquid matter are driven towards one point in proportion to their mass, and when the forces depend on the distances according to an arbitrary law, then equilibrium demands that at equal distances from the imagined point the pressure as well as the density of the liquid matter must have the same magnitude.



Let C (Fig.20) be the point towards which all bodies are being driven, such that, when at distance $CM = z$ there is a body of mass $= M$ that is being driven towards C by a force $= MZ$, where Z is determined through z in an arbitrary manner. Consider now at M a cubic shaped body $MNmn$, of height $Mm = dz$, length $= dx$ and width $= dy$, and density $= q$, then its mass will be $qdxdydz$, and its weight directed towards C will be $qZdxdydz$. Let furthermore the pressure at M be expressed by the height p, then it is clear that the pressure from the sides must be equally large and that therefore p does not change as long as the distance z remains the same, i.e. p must be solely dependent on z. Because at m the pressure is given by the height $p+dp$, and both the upper and lower base plane are given by $dxdy$, the particle $MNmn$ must be pushed upwards by the pressure from the adjacent liquid matter with a force $pdxdy$ and downwards with a force $(p + dp)dxdy$. Therefore, including gravity, the total downward force will be

$$\left(p + dp\right) dxdy + qZdxdydz$$

which must equal the upward force $pdxdy$, from which follows the equation

$$dp = -qZdz.$$

Since p is only dependent on the distance $CM = z$, it follows that q is also solely dependent on it, and that therefore at the same distance both the pressure and the density are the same. From this we can determine the top surface of the liquid matter, for since there the pressure must vanish and $p=0$, it is clear that all points on it must be equally distant from the centre C. This surface EPF is therefore given as the surface of a sphere, the centre of which is C. If the density is everywhere the same, and the force Z is inversely proportional to the square of the distance $CM = z$ or $Z = aa/zz$, then one has

$$dp = \frac{-aaqdz}{zz}$$

and therefore

$$p = C + \frac{aaq}{z}.$$

If one puts $CP = c$, where the pressure vanishes, then one obtains

$$0 = C + \frac{aaq}{c}$$

and therefore

$$p = \frac{aaq}{z} - \frac{aaq}{c}$$

or

$$p = aaq\left(\frac{1}{CM} - \frac{1}{CP}\right) = \frac{aaq \cdot PM}{CM \cdot CP}$$

152. Whatever the forces might be that act on the particles of liquid matter in proportion to their mass, they can be reduced to three, whose directions are perpendicular to each other, and are parallel to three arbitrarily chosen lines.

We now come to the determination of equilibrium of liquid matter in the broadest context, and since we have so far have only considered gravity and such forces that push towards a fixed point, the forces shall now be however one may imagine them. But it is known that these can always be reduced to three, whose directions are parallel to three arbitrarily chosen mutually orthogonal lines. Let therefore *OA,OB,OC* (Fig.21) be these three given lines, that determine three planes *AOB,AOC,BOC,* that are also mutually orthogonal. Now consider at *M* an infinitesimally small particle of liquid matter, whose mass = *M*, and to determine its location we use its distance from the three imagined planes and put $MX = x$, $MY = y$, $MZ = z$,

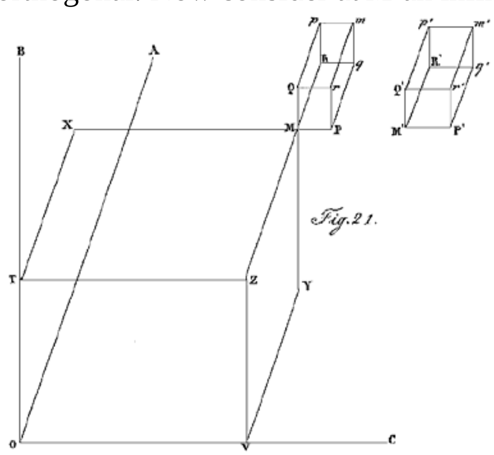


Fig.21.

then we have

$$OV = TZ = x,\ VZ = TO = y,\ VY = TX = z.$$

The particle at point *M* with mass *M* [1] is driven in the directions of the three axes *OA, OB, OC* by the following three forces, namely by the force along $MP = M \times P$, along $MQ = M \times Q$ and along $MR = M \times R$. Now allot to this particle the shape of a cube *MPQRmpqr* of length $MP = dx$, width $MQ = dy$ and height $MR = dz$, so that its volume is *dxdydz.* If one furthermore puts the density of the liquid matter at *M* as *q*, then the mass of the particle is *qdxdydz*, and consequently the forces acting on the particle will be as follows: along $MP = Pqdxdydz$, along $MQ = Qqdxdydz$, along $MR = Rqdxdydz$. On this cubic particle we have six surfaces, to which we give the following names to distinguish better between them:

The front surface $MPQr = dxdy$, the lower surface $MPRq = dxdz,$ the back surface $mpqR = dxdy$, the upper surface $mprQ = dxdz,$ the left surface $MQRp = dydz,$ the right surface $mqrP = dydz.$

[1] In the original both location and mass are allotted the symbol *M* (translator)

153. To determine the equilibrium of such liquid matter it is necessary to know the required pressure at all points. But this depends solely on the effectiveness of the forces by which each movable particle of the liquid matter is driven.

Assuming everything that was put forward in the preceding section, let p be the height that represents the pressure at M, changes in which depend on all three of the variables x,y,z. To take these into account in a more convenient way, we want to designate the increment in p, when only x increases by dx whilst y and z remain unaltered, by $dx\left(\frac{dp}{dx}\right)$, the increment when only y changes by dy by $dy\left(\frac{dp}{dy}\right)$ and the increment when only z changes by dz by $dz\left(\frac{dp}{dz}\right)$. Therefore when all three variables x,y,z change by dx ,dy, dz, then the increment in p will be $dx\left(\frac{dp}{dx}\right)+dy\left(\frac{dp}{dy}\right)+dz\left(\frac{dp}{dz}\right)$, which, as usual, will be the true value of dp. One sees here at once that one finds the value of $dx\left(\frac{dp}{dx}\right)$, if one differentiates p and assumes only x to be variable, from which one sees the meaning of the not yet very common notation $\left(\frac{dp}{dx}\right)$. Because the pressure on the plane $Pqrm$ exceeds that on the plane $MQRp$ by $dx\left(\frac{dp}{dx}\right)$, the particle is driven to the left towards PM by the force

$$x\left(\frac{dp}{dx}\right)\cdot dydz = dxdydz\left(\frac{dp}{dx}\right).$$

Furthermore, since the pressure on the upper plane $mprQ$ is greater by $dy\left(\frac{dp}{dy}\right)$ than that on the lower one, the particle will be pushed downwards towards QM by the force

$$dy\left(\frac{dp}{dy}\right)dxdz = dxdydz\left(\frac{dp}{dy.}\right)$$

And since finally the pressure on the plane at the back $mpqR$ is greater than that on the front plane $MPQr$ by $dz\left(\frac{dp}{dz}\right)$, the particle is pushed forward towards RM by the force

$$dz\left(\frac{dp}{dz}\right)\cdot dxdy = dxdydz\left(\frac{dp}{dz}\right).$$

These three forces must therefore be equal and opposite to the above three forces that act on the particle, since otherwise there could not be equilibrium. We therefore obtain the following three equations:

$$Pq = \left(\frac{dp}{dx}\right), \quad Qq = \left(\frac{dp}{dy}\right), \quad Rq = \left(\frac{dp}{dz}\right),$$

from which, introducing the total differential of *p* one obtains

$$q(Pdx + Qdy + Rdz) = dp.$$

But $\int(Pdx + Qdy + Rdz)$ expresses what in the above was called the effectiveness of the forces. Therefore at points where the effectiveness is the same, there must also the pressure and the density of liquid matter be the same.

154. Liquid matter that either is equally dense everywhere, or the density of which is solely dependent on the pressure, can never be in equilibrium, unless the forces acting on it are such that their effectiveness can be calculated.

When the density *q* is either constant or only dependent on pressure *p*, then $\frac{dp}{q}$ can be integrated and the integral $\int\frac{dp}{q}$ has a certain value. Since we have found that

$$\frac{dp}{q} = Pdx + Qdy + Rdz\ ,$$

one must be able to determine by integration the value of $\int(Pdx + Qdy + Rdz)$ such that one can find it at every point, whatever values might be chosen for *x,y* and *z*, that means the expression *Pdx + Qdy +Rdz* must be integrable; not necessarily algebraic, but such that it arises from the differentiation of an expression depending on *x,y* and *z*. This means that the forces *P,Q, R* are such that their effectiveness, represented by $\int(Pdx + Qdy + Rdz)$ can be calculated; in which case also the density and the pressure of the liquid matter are determined at all points. All real forces that we know are indeed such that their effectiveness, or the magnitude of the integral $\int(Pdx + Qdy + Rdz)$ can be calculated. But if we imagined forces such that integration is impossible, for example if we were to put *P=x, Q=y* and *R=x*, then the integral $\int(xdx + ydy + xdz)$ would not be integrable and therefore liquid matter driven by such forces would never be in equilibrium, an absurdity that must be ascribed to the invented forces. In addition to this, if the effectiveness of the forces can be calculated and one puts $\int(Pdx + Qdy + Rdz) = V$, then one has $\int\frac{dp}{q} = C + V$, and if *p=0*, then one has for the shape of the surface of the liquid matter *C+V=0* or alternatively the differential equation *Pdx+Qdy+Rdz=0*, by which the shape of any liquid matter in equilibrium is always determined, and the connection of which with the effectiveness it is well worth noting.