SPHERICAL TRIGONOMETRY

ALL DERIVED BRIEFLY AND CLEARLY FROM FIRST PRINCIPLES

[E524]

1. Some spherical triangle shall be proposed (Fig. 1), the angles of which may be designated by the capital letters *A*, *B*, *C*, and the sides by the small letters *a*, *b*, *c* described in the figure, thus so that the same small letters may be placed opposite the same capital letters. Now from the centre of the sphere, to which we may attribute the letter *O*, the right lines *OC*, *Oa*, and *Ob* may be drawn through the individual angles, which establish a solid angle from the centre *O*, the plane angles of which will measure the sides of the triangle, and moreover the mutual inclinations of these planes measure the angles of the triangle.

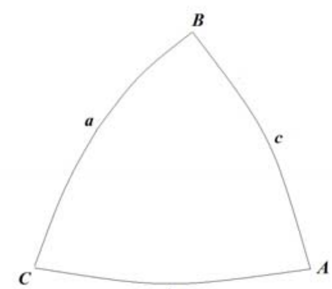

Fig. 1

2. With these in place (Fig. 2) *OC* shall itself be taken for the radius of the sphere equal to 1, so that the right lines *Ca* and *Cb* may be put in place normally to *OC* in each plane *COa* and *COb ;* while indeed the perpendicular *bp* may be sent from *b* to *Ca*, which likewise will be normal to the plane *COa* ; truly in addition the normal *pq* may be drawn from *p* to *Oa*, with *bq* drawn, that also will be normal to *Oa*. In this manner the whole figure needed will be constructed.

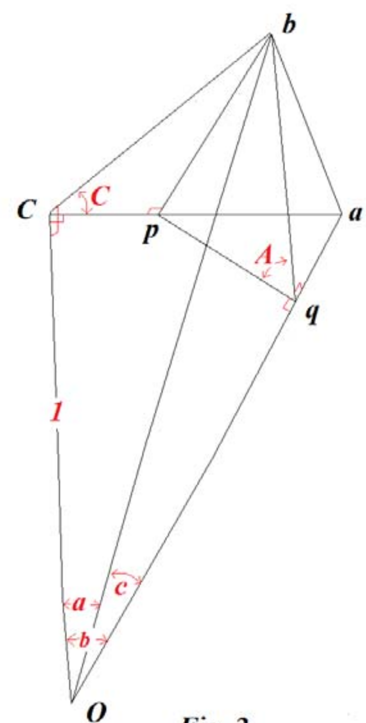

Fig. 2

3. Now since the angle *COa* shall be equal to the side *b*, there will be

$$Ca = \text{tang}\, b \text{ and } Oa = \sec b = \frac{1}{\cos b}.$$

In a similar manner, on account of the angle $COb = a$, there will be

$$Cb = \text{tang}\, a \text{ and } Ob = \sec a = \frac{1}{\cos a}.$$

Moreover again, since there shall be the angle

$$aOb = c \text{ and } Ob = \frac{1}{\cos a},$$

there will be

$$bq = \frac{\sin c}{\cos a} \text{ and } Oq = \frac{\cos c}{\cos a}.$$

Hence for the remaining lines of the figure being expressed, on account of the angle $aCb = C$, there will be

$$bp = Cb \sin C = \operatorname{tang} a \sin C$$

and

$$Cp = Cb \cos C = \operatorname{tang} a \cos C,$$

from which again it is gathered,

$$ap = Ca - Cp = \operatorname{tang} b - \operatorname{tang} a \cos C,$$

and because the angle $CaO = 90° - b$, there will be had

$$pq = ap \cos b = \sin b - \operatorname{tang} a \cos b \cos C$$

and

$$aq = ap \sin b = \frac{\sin b^2}{\cos b} - b \operatorname{tang} a \sin b \cos C.$$

Whereby, since we have found $Oq = \frac{\cos c}{\cos a}$, there becomes

$$Oa = \frac{1}{\cos b} = \frac{\cos c}{\cos a} + \frac{\sin b^2}{\cos b} - b \operatorname{tang} a \sin b \cos C.$$

and thus there will be

$$\frac{\cos c}{\cos a} = \cos b + \operatorname{tang} a \sin b \cos C.$$

or

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$

4. Now since the angle *bqp* provides the inclination of the plane *aOb* to the plane *aOC*, this same angle will be $bqp = A$, so that from the triangle *bpq* there will be had in the first place :

$$\sin A = \frac{bp}{bq} = \frac{\sin a \sin C}{\operatorname{Sin} c} \quad \text{or} \quad \frac{\sin C}{\sin c} = \frac{\sin A}{\sin a};$$

from which now it follows that the sines of the angles of our triangle are to be proportional to the sines of the opposite sides. Then the equation

$$\cos A = \frac{pq}{bq} = \frac{\cos a \sin b - \sin a \cos b \cos C}{\sin c},$$

together with the two preceding adjoining equations, enfolds the whole science of spherical triangles, but which requires a richer exposition, so that we may elaborate more on these three equations.

EXTENDING THE FIRST FORMULA

$$\frac{\sin C}{\sin c} = \frac{\sin A}{\sin a}.$$

5. Since both the capital letters *A*, *B*, *C* as well as the small letters *a*, *b*, *c* can be interchanged amongst themselves, provided they may be left with the same small letter opposite to the large letter, there will be also $\frac{\sin C}{\sin c} = \frac{\sin B}{\sin b}$, and thus the three fold equation will be produced :

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c}.$$

Then also it will help to have noted the following equalities:

$$\sin A \sin b = \sin B \sin a,$$
$$\sin A \sin c = \sin C \sin a,$$
$$\sin B \sin c = \sin C \sin b.$$

EVOLVING THE FORMULA

$$\cos A \sin c = \cos a \sin b - \sin a \cos b \cos C.$$

6. Because $\sin A \sin c = \sin C \sin a$, the first member of this equation first may be divided by $\sin A \sin c$, truly the latter by $\sin C \sin a$,and there will be obtained

$$\cot A = \frac{\cos a \sin b - \sin a \cos b \cos C}{\sin a \sin C}$$

so that now the angle *A* is able to be found from the two sides *a* and *b*, with the intercepted angle *C* ; and in a similar manner I may derive by this formula from that, with the same given the angle *B*, and with the letters *A*, *B*, *a*, *b* interchanged :

$$\cot B = \frac{\cos b \sin a - \cos a \sin b \cos C}{\sin b \sin C}.$$

7. If again we may multiply the first term of the same as we have set down here, by $\frac{\sin C}{\sin c}$, the second by $\frac{\sin B}{\sin b}$, and truly the third by $\frac{\sin A}{\sin a}$, then this memorable equation will arise :

$$\cos A\sin C = \cos a\sin B - \cos b\sin A\cos C\ ,$$

$$\cos a = \frac{\cos A\sin C + \sin A\cos C\cos b}{\sin B}$$

and with the letters *B* and *C*, likewise *b* and *c* interchanged between each other, there will be

$$\cos a = \frac{\cos A\sin B + \sin A\cos B\cos c}{\sin C}$$

or

$$\cos a\sin C = \cos A\sin B + \sin A\cos B\cos c\ ,$$

which does not differ at all from the proposed equation, except that the large and small letters will be interchanged among themselves, and truly in addition all the cosines may be taken negative.

8. But if now we may divide the first member of this latter equation by $\sin a\sin C$, and the latter by $\sin A\sin c$, this equation will arise :

$$\cot a = \frac{\cos A\sin B + \sin A\cos B\cos c}{\sin A\sin c}$$

which will serve for finding the side *a* from the two given angles *A* and *B* with the intercepted side *c;* then truly the side *b* will be defined by this equation from the same given equation :

$$\cot b = \frac{\cos B\sin A + \sin B\cos A\cos c}{\sin B\sin c}\ .$$

9. Truly besides from the same formula proposed another more difficult case will be elicited, by which the sides may be postulated from the three given angles. For since there shall be

$$\cos A \sin c = \cos a \sin b - \sin a \cos b \cos C ,$$

in a similar manner there will be, with the letters *A* and *B* interchanged,

$$\cos B \sin c = \cos b \sin a - \sin b \cos a \cos C .$$

If the latter, multiplied by $\cos C$, may be added to the first, this equation will be produced :

$$\sin c \left(\cos A + \cos B \cos \mathrm{C} \right) = \cos a \sin b \sin \mathrm{C}^2 ;$$

but truly on account of $\sin b \sin C = \sin B \sin c$ that equation may adopt this form :

$$\cos A + \cos B \cos C = \cos a \sin B \sin C$$

or

$$\cos A = -\cos B \cos C + \sin C \cos a \sin B .$$

Therefore with the letters *A* and *C* interchanged, with *B* remaining, it becomes

$$\cos C = -\cos B \cos A + \sin B \sin A \cos c ,$$

which from our third formula:

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

arises, if the greater and lesser letters may be interchanged between themselves, and moreover all the cosines may be taken negative.

THE EVOLUTION OF THE FORMULA

$$\cos c = \cos a \cos b + \sin a \sin b \cos C .$$

10. Here it is apparent at once that this formula has two outstanding uses, the one , by which from the given sides *a, b, c* the angles are to be defined, which happens with the aid of this formula :

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b} ;$$

and truly the other, when from two sides *a* and *b* with the angle *C* of the third side intercepted, the side *c* is sought, which comes about with the aid of this formula:

$$\cos c = \cos a \cos b + \sin a \sin b \cos C .$$

11. Now therefore we will be able to transfer this use to angles, because as we have just found,

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c .$$

Hence also immediately, if the two angles *A* and *B* may be given with the side intercepted *c*, the third angle *C* may be determined. Then truly, if all three spherical angles may be given, whatever side, such as *c*, is defined in this way :

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B} .$$

12. Therefore since all spherical trigonometry shall depend on the three equations found above, generally the interchange of all the angles and sides has a place, provided all the cosines may be taken negatively. For in the first formula :

$$\frac{\sin C}{\sin c} = \frac{\sin B}{\sin b} = \frac{\sin A}{\sin a}$$

this interchange is evident at once, because no cosines occur, then that interchange for both the remaining formulas now has been brought about, so that the following conspicuous Theorem arises.

THEOREM

For any spherical triangle proposed, the angles and sides of which shall be A, B, C and a, b, c, another analogous triangle can be shown always, the angles of which shall be the complements of these sides to two right angles, and the sides truly the complements of the angles to two right angles.

For in this manner all the sines remain the same, truly all the cosines become negative and thus also the tangents and cotangents. Moreover such a triangle is agreed to be formed from the poles of the three sides of the proposed triangle.

13. Therefore for a practical use all the precepts can be represented under four forms, of which two thus indeed may be closely joined together, so that the one may be formed from the other, while the capital and ordinary letters may be interchanged between themselves, with the cosines taken negatively, thus so that only two forms may suffice to be ordered to memory. Therefore we may put before us these four forms with all the variations, which they are able to receive by the transposition of the letters.

THE FIRST FORM

14. This form involves two cases, of which on the one hand a certain angle may be found from three given sides, and truly on the other hand, from two given sides and the angle intercepted, the third side can be found.

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad \Big| \quad \cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c} \quad \Big| \quad \cos b = \cos a \cos c + \sin a \sin c \cos B,$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b} \quad \Big| \quad \cos c = \cos a \cos b + \sin a \sin b \cos C.$$

SECOND FORM

15. This form also contains two cases, of which the one from the three given angles some side is found, and truly the other from two given angles with the intercepted side, the third angle is found:

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} \quad \Big| \quad \cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C} \quad \Big| \quad \cos B = -\cos A \cos C + \sin A \sin C \cos b,$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B} \quad \Big| \quad \cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

THIRD FORM

16. This form includes that case, in which from two sides with the intercepted angle, the remaining two angles may be determined, which formulas with their variations thus may be had :

$$\cot A = \frac{\cos a \sin b - \sin a \cos b \cos C}{\sin a \sin C} \quad \Big| \quad \cot B = \frac{\sin a \cos b - \cos a \sin b \cos C}{\sin b \sin C},$$

$$\cot B = \frac{\cos b \sin c - \sin b \cos c \cos A}{\sin b \sin A} \quad \Big| \quad \cot C = \frac{\sin b \cos c - \cos b \sin c \cos A}{\sin c \sin A},$$

$$\cot C = \frac{\cos c \sin a - \sin c \cos a \cos B}{\sin c \sin B} \quad \Big| \quad \cot A = \frac{\sin c \cos a - \cos c \sin a \cos B}{\sin a \sin B}.$$

FOURTH FORM

17. This form considers the case, in which from two angles with the intercepted side, the two remaining sides may be defined, which formulas with variations thus themselves may be had :

$$\cot a = \frac{\cos A \sin B + \sin A \cos B \cos c}{\sin A \sin c} \quad \Big| \quad \cot b = \frac{\sin A \cos B + \cos A \sin B \cos c}{\sin B \sin c},$$

$$\cot b = \frac{\cos B \sin C + \sin B \cos C \cos a}{\sin B \sin a} \quad \Big| \quad \cot c = \frac{\sin B \cos C + \cos B \sin C \cos a}{\sin C \sin a},$$

$$\cot c = \frac{\cos C \sin A + \sin C \cos A \cos b}{\sin C \sin b} \quad \Big| \quad \cot a = \frac{\sin C \cos A + \cos C \sin A \cos b}{\sin A \sin b}.$$

18. This simplicity is more noteworthy on that account, because the resolution of right angles triangles thus requires six formulas clearly different in turn amongst themselves. But if the angle C were right and thus c were the hypotenuse and both a and b were perpendicular, the six formulas required are the following :

$$\cos c = \cos a \cos b$$

$$\cos c = \cot A \cot B$$

$$\sin a = \sin c \sin A \quad \text{or} \quad \sin b = \sin c \sin B$$

$$\operatorname{tang} b = \operatorname{tang} c \cos A \quad \text{or} \quad \operatorname{tang} a = \operatorname{tang} c \cos B$$

$$\operatorname{tang} a = \operatorname{tang} A \sin b \quad \text{or} \quad \operatorname{tang} b = \operatorname{tang} B \sin a$$

$$\cos A = \cos a \sin B \quad \text{or} \quad \cos B = \cos b \sin A,$$

which formulas may be derived from the above on putting

$$\cos C = 0 \text{ and } \sin C = 1.$$

[Thus Euler demonstrates the proof of Napier's Circular Parts in the solution of right angled triangles, introduced in his seminal work on logarithms, his *Descriptio*…. of 1614.]

19. But so that logarithms may be called into use, others of a different nature are required to be derived from the above forms, which depend on factors ; that it is possible to obtain by certain transformations, from which we deduce half the angles as well as half the sides. Moreover it is possible to put these transformations in place succinctly in the following ways.

THE FIRST TRANSFORMATION

20. This transformation may be derived most conveniently from the first formula

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Hence indeed in the first place it follows :

$$1 - \cos A = \frac{\cos(b-c) - \cos a}{\sin b \sin c},$$

$$1 + \cos A = \frac{\cos a - \cos(b+c)}{\sin b \sin c}.$$

Hence since there shall be

$$\frac{1-\cos A}{1+\cos A} = \operatorname{tang} \tfrac{1}{2} A^2, \quad \text{there will be} \quad \operatorname{tang} \tfrac{1}{2} A^2 = \frac{\cos(b-c) - \cos a}{\cos a - \cos(b+c)};$$

but it is agreed that

$$\cos p - \cos q = 2 \sin \frac{q-p}{2} \sin \frac{p+q}{2},$$

so that we will have

$$\operatorname{tang} \tfrac{1}{2} A = \sqrt{\frac{\sin \frac{a-b+c}{2} \cdot \sin \frac{a+b-c}{2}}{\sin \frac{b+c-a}{2} \cdot \sin \frac{a+b+c}{2}}}.$$

SECOND TRANSFORMATION

21. This is sought from the formula of the first form

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C}$$

so that there is deduced

$$1 - \cos a = -\frac{\cos(B+C) + \cos A}{\sin B \sin C},$$

$$1 + \cos a = \frac{\cos A + \cos(B-C)}{\sin B \sin C}$$

and thus there will be

$$\operatorname{tang}\tfrac{1}{2}a^2 = -\frac{\cos(B+C)+\cos A}{\cos(B-C)+\cos A}$$

Since now there shall be

$$\cos p + \cos q = 2\cos\frac{p-q}{2}\cos\frac{p+q}{2},$$

the formula becomes :

$$\operatorname{tang}\tfrac{1}{2}a = \sqrt{-\frac{\cos\frac{B+C-A}{2}\cdot\cos\frac{B+C+A}{2}}{\cos\frac{B+A-C}{2}\cdot\cos\frac{A+C-B}{2}}}.$$

THE THIRD TRANSFORMATION.

22. This transformation also can be set out from the first form, by combining these two formulas :

$$\cos a - \cos b\cos c = \sin b\sin c\cos A,$$
$$\cos b - \cos a\cos c = \sin a\sin c\cos B\ ;$$

of which the former divided by the latter provides :

$$\frac{\cos a - \cos b\cos c}{\cos b - \cos a\cos c} = \frac{\sin b\cos A}{\sin a\cos B} = \frac{\sin B\cos A}{\sin A\cos B}.$$

One may be added to each side and the formula becomes

$$\frac{(\cos a + \cos b)(1-\cos c)}{\cos b - \cos a\cos c} = \frac{\sin(A+B)}{\sin A\cos B}$$

and one may be taken from both sides, producing

$$\frac{(\cos a - \cos b)(1+\cos c)}{\cos b - \cos a\cos c} = \frac{\sin(A-B)}{\sin A\cos B}$$

which equation divided by the first gives

$$\frac{\cos a-\cos b}{\cos a+\cos b}\cdot\cot\tfrac{1}{2}c^2=\frac{\sin(B-A)}{\sin(A+B)}.$$

But it is agreed that

$$\frac{\cos p-\cos q}{\cos p+\cos q}=\operatorname{tang}\frac{q+p}{2}\operatorname{tang}\frac{q-p}{2},$$

from which it is deduced :

$$\operatorname{tang}\frac{b-a}{2}\operatorname{tang}\frac{b+a}{2}\cdot\cot\tfrac{1}{2}c^2=\frac{\sin(B-A)}{\sin(B+A)}$$

23. Now we may call into help this formula from the first property :

$$\frac{\sin b}{\sin a}=\frac{\sin B}{\sin A}$$

from which we deduce

$$\frac{\sin b-\sin a}{\sin b+\sin a}=\frac{\sin B-\sin A}{\sin B+\sin A}$$

which is reduced to this form:

$$\operatorname{tang}\frac{b-a}{2}\cot\frac{b+a}{2}=\operatorname{tang}\frac{B-A}{2}\cot\frac{B+A}{2}.$$

But if now we may multiply the equation found before by this, this itself will be produced :

$$\left(\operatorname{tang}\frac{b-a}{2}\right)^2\cdot\cot\tfrac{1}{2}c^2=\frac{\left(\sin\frac{B-A}{2}\right)^2}{\left(\sin\frac{B+A}{2}\right)^2};$$

or with the root extracted :

$$\operatorname{tang}\frac{b-a}{2}\cdot\cot\tfrac{1}{2}c=\frac{\sin\frac{B-A}{2}}{\sin\frac{B+A}{2}}.$$

But truly the first formula divided by the latter gives

$$\operatorname{tang}\frac{b+a}{2}\cdot\cot\tfrac{1}{2}c=\frac{\cos\frac{B-A}{2}}{\cos\frac{B+A}{2}}.$$

Therefore from these formulas the case is resolved, in which two angles *A* and *B* with the side intercepted *c* are given and both the sides *a* and *b* are sought, which comes about with the aid of the formulas:

$$\operatorname{tang}\frac{b-a}{2}=\operatorname{tang}\tfrac{1}{2}c\frac{\sin\frac{B-A}{2}}{\sin\frac{B+A}{2}}$$

$$\operatorname{tang}\frac{b+a}{2}=\operatorname{tang}\tfrac{1}{2}c\frac{\cos\frac{B-A}{2}}{\cos\frac{B+A}{2}}$$

TRANSFORMATION FOUR

24. This is deduced in a similar manner from these formulas :

$$\cos A+\cos B\cos C=\sin B\sin C\cos a$$
$$\cos B+\cos A\cos C=\sin A\sin C\cos b,$$

of which the former divided by the latter provides

$$\frac{\cos A+\cos B\cos C}{\cos B+\cos A\cos C}=\frac{\sin B\cos a}{\sin A\cos b}=\frac{\sin b\cos a}{\sin a\cos b}.$$

So that on both adding and subtracting one the following new equations may be derived:

$$\frac{(\cos A+\cos B)(1+\cos C)}{\cos B+\cos A\cos C}=\frac{\sin(a+b)}{\sin a\cos b}$$

$$\frac{(\cos A-\cos B)(1-\cos C)}{\cos B+\cos A\cos C}=\frac{\sin(b-a)}{\sin a\cos b},$$

on dividing the first by the second we find :

$$\frac{\cos A+\cos B}{\cos A-\cos B}\cdot\cot\tfrac{1}{2}C^{2}=\frac{\sin(a+b)}{\sin(b-a)}$$

or

$$\operatorname{tang}\frac{B-A}{2}\cdot\operatorname{tang}\frac{B+A}{2}=\cot\tfrac{1}{2}C^{2}\cdot\frac{\sin(a-b)}{\sin(b+a)},$$

which equation multiplied and divided by this one :

$$\operatorname{tang}\frac{B-A}{2}\cdot\cot\frac{B+A}{2}=\operatorname{tang}\frac{b-a}{2}\cot\frac{b+a}{2},$$

produces

$$\operatorname{tang}\frac{B-A}{2}=\cot\tfrac{1}{2}C\cdot\frac{\sin\frac{b-a}{2}}{\sin\frac{b+a}{2}}$$

$$\operatorname{tang}\frac{B+A}{2}=\cot\tfrac{1}{2}C\cdot\frac{\cos\frac{b-a}{2}}{\cos\frac{b+a}{2}},$$

which formulas prevail for the case, in which two sides with the intercepted angle are given.

25. Because we have put in place all of the preceding forms, we may set out also these four cases with all the variations evident.

$$\operatorname{tang}\tfrac{1}{2}A=\sqrt{\frac{\sin\frac{a+b-c}{2}\sin\frac{a+c-b}{2}}{\sin\frac{b+c-a}{2}\sin\frac{a+b+c}{2}}}$$

$$\operatorname{tang}\tfrac{1}{2}B=\sqrt{\frac{\sin\frac{b+c-a}{2}\sin\frac{a+b-c}{2}}{\sin\frac{a+c-b}{2}\sin\frac{a+b+c}{2}}}$$

$$\operatorname{tang}\tfrac{1}{2}C=\sqrt{\frac{\sin\frac{a+c-b}{2}\sin\frac{b+c-a}{2}}{\sin\frac{a+b-c}{2}\sin\frac{a+b+c}{2}}}$$

$$\operatorname{tang}\tfrac{1}{2}a=\sqrt{-\frac{\cos\frac{B+C-A}{2}\cos\frac{A+B+C}{2}}{\cos\frac{A+B-C}{2}\cos\frac{A+C-B}{2}}}$$

$$\operatorname{tang}\tfrac{1}{2}b=\sqrt{-\frac{\cos\frac{A+C-B}{2}\cos\frac{A+B+C}{2}}{\cos\frac{B+C-A}{2}\cos\frac{A+B-C}{2}}}$$

$$\operatorname{tang}\tfrac{1}{2}c=\sqrt{-\frac{\cos\frac{A+B-C}{2}\cos\frac{A+B+C}{2}}{\cos\frac{A+C-B}{2}\cos\frac{B+C-A}{2}}}$$

$\operatorname{tang}\frac{b-a}{2}=\operatorname{tang}\tfrac{1}{2}c\cdot\frac{\sin\frac{B-A}{2}}{\sin\frac{B+A}{2}}$	$\operatorname{tang}\frac{b+a}{2}=\operatorname{tang}\tfrac{1}{2}c\cdot\frac{\cos\frac{B-A}{2}}{\cos\frac{B+A}{2}}$
$\operatorname{tang}\frac{c-b}{2}=\operatorname{tang}\tfrac{1}{2}a\cdot\frac{\sin\frac{C-B}{2}}{\sin\frac{C+B}{2}}$	$\operatorname{tang}\frac{c+b}{2}=\operatorname{tang}\tfrac{1}{2}a\cdot\frac{\cos\frac{C-B}{2}}{\cos\frac{C+B}{2}}$
$\operatorname{tang}\frac{a-c}{2}=\operatorname{tang}\tfrac{1}{2}b\cdot\frac{\sin\frac{A-C}{2}}{\sin\frac{A+C}{2}}$	$\operatorname{tang}\frac{a+c}{2}=\operatorname{tang}\tfrac{1}{2}b\cdot\frac{\cos\frac{A-C}{2}}{\cos\frac{A+C}{2}}$
$\operatorname{tang}\frac{B-A}{2}=\cot\tfrac{1}{2}C\cdot\frac{\sin\frac{b-a}{2}}{\sin\frac{b+a}{2}}$	$\operatorname{tang}\frac{B+A}{2}=\cot\tfrac{1}{2}C\cdot\frac{\cos\frac{b-a}{2}}{\cos\frac{b+a}{2}}$
$\operatorname{tang}\frac{C-B}{2}=\cot\tfrac{1}{2}A\cdot\frac{\sin\frac{c-b}{2}}{\sin\frac{c+b}{2}}$	$\operatorname{tang}\frac{C+B}{2}=\cot\tfrac{1}{2}A\cdot\frac{\cos\frac{c-b}{2}}{\cos\frac{c+b}{2}}$
$\operatorname{tang}\frac{A-C}{2}=\cot\tfrac{1}{2}B\cdot\frac{\sin\frac{a-c}{2}}{\sin\frac{a+c}{2}}$	$\operatorname{tang}\frac{A+C}{2}=\cot\tfrac{1}{2}B\cdot\frac{\cos\frac{a-c}{2}}{\cos\frac{a+c}{2}}$

26. Now from these final formulas the case will be arranged easily, that we have not yet examined, in which two sides with the opposite angle are given, and either the third side or the third angle is sought, each of which can be done in two ways. Therefore we may put in place these formulas with the variations :

$\operatorname{tang}\tfrac{1}{2}c = \operatorname{tang}\dfrac{b-a}{2}\cdot\dfrac{\sin\dfrac{B+A}{2}}{\sin\dfrac{B-A}{2}}$	$\operatorname{tang}\tfrac{1}{2}c = \operatorname{tang}\dfrac{b+a}{2}\cdot\dfrac{\cos\dfrac{B+A}{2}}{\cos\dfrac{B-A}{2}}$
$\operatorname{tang}\tfrac{1}{2}a = \operatorname{tang}\dfrac{c-b}{2}\cdot\dfrac{\sin\dfrac{C+B}{2}}{\sin\dfrac{C-B}{2}}$	$\operatorname{tang}\tfrac{1}{2}a = \operatorname{tang}\dfrac{c+b}{2}\cdot\dfrac{\cos\dfrac{C+B}{2}}{\cos\dfrac{C-B}{2}}$
$\operatorname{tang}\tfrac{1}{2}b = \operatorname{tang}\dfrac{a-c}{2}\cdot\dfrac{\sin\dfrac{A+C}{2}}{\sin\dfrac{A-C}{2}}$	$\operatorname{tang}\tfrac{1}{2}b = \operatorname{tang}\dfrac{a+c}{2}\cdot\dfrac{\cos\dfrac{A+C}{2}}{\cos\dfrac{A-C}{2}}$
$\cot\tfrac{1}{2}C = \operatorname{tang}\dfrac{B-A}{2}\cdot\dfrac{\sin\dfrac{b+a}{2}}{\sin\dfrac{b-a}{2}}$	$\cot\tfrac{1}{2}C = \operatorname{tang}\dfrac{B+A}{2}\cdot\dfrac{\cos\dfrac{b+a}{2}}{\cos\dfrac{b-a}{2}}$
$\cot\tfrac{1}{2}A = \operatorname{tang}\dfrac{C-B}{2}\cdot\dfrac{\sin\dfrac{c+b}{2}}{\sin\dfrac{c-b}{2}}$	$\cot\tfrac{1}{2}A = \operatorname{tang}\dfrac{C+B}{2}\cdot\dfrac{\cos\dfrac{c+b}{2}}{\cos\dfrac{c-b}{2}}$
$\cot\tfrac{1}{2}B = \operatorname{tang}\dfrac{A-C}{2}\cdot\dfrac{\sin\dfrac{a+c}{2}}{\sin\dfrac{a-c}{2}}$	$\cot\tfrac{1}{2}B = \operatorname{tang}\dfrac{A+C}{2}\cdot\dfrac{\cos\dfrac{a+c}{2}}{\cos\dfrac{a-c}{2}}$

So that in this manner the present discussion is able to consider the complete system of the whole of spherical trigonometry.

TRIGONOMETRIA SPHAERICA UNIVERSA

EX PRIMIS PRINCIPIIS BREVITER ET DILUCIDE DERIVATA

[E524]

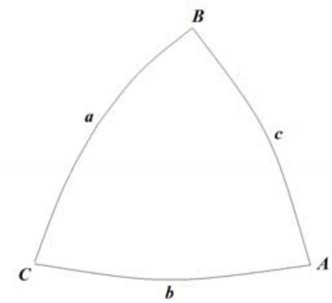

Fig. 1

1. Propositum sit triangulum sphaericum (Fig. 1) quodcunque, cuius anguli litteris maiusculis *A, B, C*, latera autem minusculis *a, b, c* in figura adscriptis designentur, ita ut iisdem litteris maiusculis eaedem minusculae opponantur. Iam ex centro Sphaerae, cui literam *O* tribuamus, per singulos angulos educantur rectae *OC*, *Oa*, *Ob*, quae in centro *O* angulum solidum constituent, cuius anguli plani metientur latera trianguli, eorum autem inclinationes mutuae angulos trianguli.

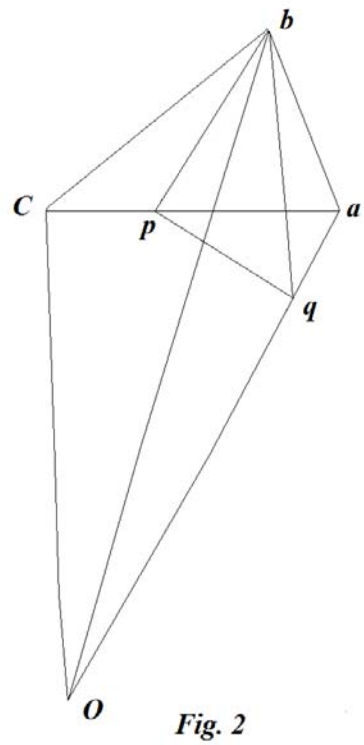

Fig. 2

2. His praemissis capiatur (Fig. 2) *OC* ipsi radio Sphaerae aequalis = 1, unde ad *OC* in utroque plano *COa* et *COb* normaliter statuantur rectae *Ca* et *Cb;* tum vero ex *b* ad *Ca* demittatur perpendiculum *bp*, quod simul ad planum *COa* erit normale; praeterea vero ex *p* ad *Oa* normalis ducatur *pq* sicque, ducta *bq*, ea etiam ad *Oa* erit normalis. Hoc modo tota figura, qua indigemus, erit constructa.

3. Cum iam sit angulus *COa* lateri *b* aequalis, erit

$$Ca = \operatorname{tang} b \text{ et } Oa = \sec b = \frac{1}{\cos b}.$$

Simili modo, ob angulum $COb = a$, erit

$$Cb = \operatorname{tang} a \text{ et } Ob = \sec a = \frac{1}{\cos a}.$$

Porro autem, cum sit angulus $aOb = c$ et $Ob = \frac{1}{\cos a}$,

erit

$$bq = \frac{\sin c}{\cos a} \text{ et } Oq = \frac{\cos c}{\cos a}.$$

Hinc pro reliquis figurae lineis exprimendis, ob angulum *aCb* = *C*, erit

$$bp = Cb\sin C = \text{tang}\, a \sin C$$

et

$$Cp = Cb\cos C = \text{tang}\, a \cos C,$$

unde porro colligitur

$$ap = Ca - Cp = \text{tang}\, b - \text{tang}\, a \cos C\ ,$$

et quia angulus $CaO = 90° - b$, habebitur

$$pq = ap\ \cos\ b = \sin b - \text{tang}\, a \cos b \cos C$$

et

$$aq = ap \sin b = \frac{\sin b^2}{\cos b} - b\,\text{tang}\, a \sin b \cos C.$$

Quare, cum invenerimus $Oq = \dfrac{\cos c}{\cos a}$, fiet

$$Oa = \frac{1}{\cos b} = \frac{\cos c}{\cos a} + \frac{\sin b^2}{\cos b} - b\,\text{tang}\, a \sin b \cos C.$$

sicque erit

$$\frac{\cos c}{\cos a} = \cos b + \text{tang}\, a \sin b \cos C.$$

sive

$$\cos c = \cos a \cos b + \sin a \sin b \cos C\ .$$

4. Cum iam angulus *bqp* praebeat inclinationem plani *aOb* ad *aOC*, erit iste angulus $bqp = A$, unde ex triangulo *bpq* primo habebitur

$$\sin\ A = \frac{bp}{bq} = \frac{\sin a \sin C}{\text{Sin}\ c} \quad \text{sive} \quad \frac{\sin C}{\sin\ c} = \frac{\sin\ A}{\sin a}\ ;$$

unde iam sequitur sinus angulorum nostri trianguli proportionales esse sinubus laterum oppositorum. Deinde aequatio

$$\cos\ A = \frac{pq}{bq} = \frac{\cos a \sin b - \sin a \cos b \cos C}{\sin c}$$

cum binis praecedentibus coniuncta totam Doctrinam sphaericam complectitur, quod autem uberiorem explicationem postulat, unde singulas has tres aequationes magis evolvamus.

EVOLUTIO PRIMAE FORMULAE

$$\frac{\sin C}{\sin c}=\frac{\sin A}{\sin a}.$$

5. Cum tam litteras maiusculas *A, B, C* quam minusculas *a, b, c* inter se permutare liceat, si modo iisdem litteris maiusculis eaedem minusculae oppositae relinquantur, erit etiam $\frac{\sin C}{\sin c}=\frac{\sin B}{\sin b}$, sicque prodibit tergemina aequatio :

$$\frac{\sin A}{\sin a}=\frac{\sin B}{\sin b}=\frac{\sin C}{\sin c}$$

Deinde etiam notasse iuvabit sequentes aequalitates:

$$\sin A\sin b=\sin B\sin a,$$
$$\sin A\sin c=\sin C\sin a,$$
$$\sin B\sin c=\sin C\sin b.$$

EVOLUTIO FORMULAE

$$\cos A\sin c=\cos a\sin b-\sin a\cos b\cos C$$

6. Quia $\sin A\sin c=\sin C\sin a$, dividatur huius aequationis membrum prius per $\sin A\sin c$, posterius vero per $\sin C\sin a$, atque obtinebitur

$$\cot A=\frac{\cos a\sin b-\sin a\cos b\cos C}{\sin a\sin C}$$

unde iam ex datis binis lateribus *a* et *b*, cum angulo intercepto *C*, angulus *A* reperiri potest; similique modo ex iisdem datis colligetur angulus *B* per hanc formuIam ex illa, literas *A*, *B*, *a*, *b* permutando, derivatam:

$$\cot B=\frac{\cos b\sin a-\sin a\cos b\cos C}{\sin b\sin C}.$$

7. Si porro eiusdem, quam hic consideramus, formulae primum terminum per $\frac{\sin C}{\sin c}$, secundum per $\frac{\sin B}{\sin b}$, tertium vero per $\frac{\sin A}{\sin a}$ multiplicemus, orietur ista aequatio memorabilis:

$$\cos A \sin C = \cos a \sin B - \cos b \sin A \cos C ,$$

$$\cos a = \frac{\cos A \sin C + \sin A \cos C \cos b}{\sin B}$$

et litteris *B* et *C*, item *b* et *c* inter se permutandis erit

$$\cos a = \frac{\cos A \sin B + \sin A \cos B \cos c}{\sin C}$$

sive

$$\cos a \sin C = \cos A \sin B + \sin A \cos B \cos c ,$$

quae a proposita aliter non discrepat, nisi quod literae maiusculae et minusculae inter se permutentur, insuper vero omnes cosinus negative accipiantur.

8. Quodsi iam huius postremae aequationis primum membrum per $\sin a \sin C$, posterius per $\sin A \sin c$ dividamus, orietur haec aequatio:

$$\cot a = \frac{\cos A \sin B + \sin A \cos B \cos c}{\sin A \sin c}$$

quae inservit lateri *a* inveniendo ex datis duobus angulis *A*, *B* cum latere intercepto *c;* tum vero ex iisdem datis etiam latus *b* definietur hac aequatione:

$$\cot b = \frac{\cos B \sin A + \sin B \cos A \cos c}{\sin B \sin c}$$

9. Praeterea vero ex eadem formula proposita casus alias difficillimus, quo ex datis tribus angulis latera postulantur, eruitur. Cum enim sit

$$\cos A \sin c = \cos a \sin b - \sin a \cos b \cos C ,$$

erit simili modo, literis *A* et *B* permutatis,

$$\cos B \sin c = \cos b \sin a - \sin b \cos a \cos C .$$

Si posterior, ducta in cos *C*, ad priorem addatur, prodibit ista aequatio:

$$\sin c \left(\cos A + \cos B \cos \mathrm{C} \right) = \cos a \sin b \sin \mathrm{C}^2 ;$$

at vero ob $\sin b \sin C = \sin B \sin c$ aequatio illa induet hanc formam:

$$\cos A + \cos B \cos C = \cos a \sin B \sin C$$

sive

$$\cos A = -\cos B \cos C + \sin C \cos a \sin B \, .$$

Permutatis igitur literis *A* et *C*, manente *B*, fiet

$$\cos C = -\cos B \cos A + \sin B \sin A \cos c \, ,$$

quae ex nostra tertia formula:

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

nascitur, si litterae maiusculae et minusculae inter se permutentur, omnes autem cosinus negative accipiantur.

EVOLUTIO FORMULAE

$$\cos c = \cos a \cos b + \sin a \sin b \cos C$$

10. Hic statim evidens est hanc formuIam duplicem usum praestare, alterum, quo ex datis lateribus *a, b, c* anguli sunt definiendi, quod fit ope huius formulae

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b} \;;$$

alterum vero, quando ex binis lateribus *a* et *b* cum angulo intercepto *C* tertium latus *c* quaeritur, quod fit ope huius formulae:

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \, .$$

11. Nunc igitur hunc usum etiam ad angulos transferre poterimus, quoniam modo invenimus

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \, .$$

Hinc enim statim, si dentur duo anguli *A, B* cum latere intercepto *c*, determinatur tertius angulus *C.* Deinde vero, si dentur omnes tres anguli trianguli sphaerici, quodvis latus, veluti *c*, hoc modo definitur:

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B} \, .$$

12. Cum igitur tota Trigonometria Sphaerica tribus aequationibus supra inventis innitatur, permutatio angulorum et laterum generaliter locum habet, si modo omnes cosinus negative accipiantur. In prima enim formula:

$$\frac{\sin C}{\sin c} = \frac{\sin B}{\sin b} = \frac{\sin A}{\sin a}$$

haec permutabilitas per se est manifesta, quia nulli cosinus occurrunt, deinde ista permutabilitas pro ambabus reliquis formulis iam est evicta, unde sequens Theorema insigne nascitur.

THEOREMA

Proposito quocunque triangulo sphaerico, cuius anguli sint A, B, C et latera a, b, c, semper aliud triangulum analogum exhiberi potest, cuius anguli sint complementa laterum illius ad duos rectos, latera vero complementa angulorum ad duos rectos.

Hoc enim modo omnes sinus manent iidem, omnes vero cosinus evadunt negativi ideoque etiam tangentes et cotangentes. Constat autem tale triangulum formari ex Polis trium laterum trianguli propositi.

13. Ad usum ergo practicum omnia praecepta sub quatuor formis repraesentari possunt, quarum binae adeo ita arcte colligantur, ut altera ex altera formetur, dum litterae maiusculae et minusculae inter se permutantur, cosinibus negative sumtis, ita ut sufficiat duas tantum formas memoriae mandasse.Has igitur quatuor formas cum omnibus variationibus, quas transpositione litterarum recipere possunt, ante oculos exponamus.

FORMA PRIMA

14. Haec forma duos involvit casus, quorum altero ex datis tribus lateribus quidam angulus, altero vero ex datis duobus lateribus, cum angulo intercepto, tertium latus invenitur.

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c} \quad \Big| \quad \cos a = \cos b \cos c + \sin b \sin c \cos A,$$

$$\cos B = \frac{\cos b - \cos a \cos c}{\sin a \sin c} \quad \Big| \quad \cos b = \cos a \cos c + \sin a \sin c \cos B,$$

$$\cos C = \frac{\cos c - \cos a \cos b}{\sin a \sin b} \quad \Big| \quad \cos c = \cos a \cos b + \sin a \sin b \cos C.$$

FORMA SECUNDA

15. Haec forma etiam duos casus continet, quorum altero ex datis tribus angulis aliquod latus, altero vero ex datis duobus angulis cum latere intercepto tertius angulus quaeritur:

$$\cos a = \frac{\cos A + \cos B \cos C}{\sin B \sin C} \quad \Big| \quad \cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

$$\cos b = \frac{\cos B + \cos A \cos C}{\sin A \sin C} \quad \Big| \quad \cos B = -\cos A \cos C + \sin A \sin C \cos b,$$

$$\cos c = \frac{\cos C + \cos A \cos B}{\sin A \sin B} \quad \Big| \quad \cos C = -\cos A \cos B + \sin A \sin B \cos c.$$

FORMA TERTIA

16. Haec forma eum casum complectitur, quo ex duobus lateribus cum angulo intercepto duo reliqui anguli determinantur, quae formulae cum suis variationibus ita se habebunt:

$$\cot A = \frac{\cos a \sin b - \sin a \cos b \cos C}{\sin a \sin C} \quad \Big| \quad \cot B = \frac{\sin a \cos b - \cos a \sin b \cos C}{\sin b \sin C},$$

$$\cot B = \frac{\cos b \sin c - \sin b \cos c \cos A}{\sin b \sin A} \quad \Big| \quad \cot C = \frac{\sin b \cos c - \cos b \sin c \cos A}{\sin c \sin A},$$

$$\cot C = \frac{\cos c \sin a - \sin c \cos a \cos B}{\sin c \sin B} \quad \Big| \quad \cot A = \frac{\sin c \cos a - \cos c \sin a \cos B}{\sin a \sin B}.$$

FORMA QUARTA

17. Haec forma respicit casum, quo ex duobus angulis cum latere intercepto bina reliqua latera definiuntur, quae formulae cum variationibus ita se habent:

$$\cot a = \frac{\cos A \sin B + \sin A \cos B \cos c}{\sin A \sin c} \quad \Big| \quad \cot b = \frac{\sin A \cos B + \cos A \sin B \cos c}{\sin B \sin c},$$

$$\cot b = \frac{\cos B \sin C + \sin B \cos C \cos a}{\sin B \sin a} \quad \Big| \quad \cot c = \frac{\sin B \cos C + \cos B \sin C \cos a}{\sin C \sin a},$$

$$\cot c = \frac{\cos C \sin A + \sin C \cos A \cos b}{\sin C \sin b} \quad \Big| \quad \cot a = \frac{\sin C \cos A + \cos C \sin A \cos b}{\sin A \sin b}.$$

18. Haec simplicitas eo magis est notatu digna, quod resolutio triangulorum rectangulorum adeo sex formulas a se invicem prorsus diversas requirat. Quodsi enim angulus C fuerit rectus ideoque c hypothenusa et a et b ambo catheti, sex formulae requisitae sunt sequentes:

$$\cos c = \cos a \cos b$$
$$\cos c = \cot A \cot B$$
$$\sin a = \sin c \sin A \quad \text{sive} \quad \sin b = \sin c \sin B$$
$$\operatorname{tang} b = \operatorname{tang} c \cos A \quad \text{sive} \quad \operatorname{tang} a = \operatorname{tang} c \cos B$$
$$\operatorname{tang} a = \operatorname{tang} A \sin b \quad \text{sive} \quad \operatorname{tang} b = \operatorname{tang} B \sin a$$
$$\cos A = \cos a \sin B \quad \text{sive} \quad \cos B = \cos b \sin A,$$

quae formulae ex superioribus sponte derivantur posito

$$\cos C = 0 \text{ et } \sin C = 1.$$

19. Quo autem logarithmi in usum vocari queant, ex formis superioribus aliae eius indolis sunt derivandae, quae ex factoribus constent; id quod per certas transformationes obtineri potest, quibus ad semisses tam angulorum quam laterum deducimur. Has autem transformationes sequentibus modis succincte instituere licet.

TRANSFORMATIO PRIMA

20. Haec transformatio ex primae formae hac formula

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}$$

commodissime derivatur. Hinc enim primo sequitur :

$$1 - \cos A = \frac{\cos(b-c) - \cos a}{\sin b \sin c},$$
$$1 + \cos A = \frac{\cos a - \cos(b+c)}{\sin b \sin c}.$$

Hinc cum sit

$$\frac{1-\cos A}{1+\cos A} = \operatorname{tang} \tfrac{1}{2} A^2, \quad \text{erit} \quad \operatorname{tang} \tfrac{1}{2} A^2 = \frac{\cos(b-c) - \cos a}{\cos a - \cos(b+c)} \;;$$

constat autem esse

$$\cos p - \cos q = 2 \sin \frac{q-p}{2} \sin \frac{p+q}{2},$$

unde habebimus

$$\operatorname{tang}\tfrac{1}{2}A=\sqrt{\frac{\sin\frac{a-b+c}{2}\cdot\sin\frac{a+b-c}{2}}{\sin\frac{b+c-a}{2}\cdot\sin\frac{a+b+c}{2}}}.$$

TRANSFORMATIO SECUNDA

21. Haec petitur ex formae prioris formula

$$\cos a=\frac{\cos A+\cos B\cos C}{\sin B\sin C}$$

unde deducitur

$$1-\cos a=-\frac{\cos(B+C)+\cos A}{\sin B\sin C},$$

$$1+\cos a=\frac{\cos A+\cos(B-C)}{\sin B\sin C}$$

sicque erit

$$\operatorname{tang}\tfrac{1}{2}a^2=-\frac{\cos(B+C)+\cos A}{\cos(B-C)+\cos A}$$

Cum iam sit

$$\cos p+\cos q=2\cos\frac{p-q}{2}\cos\frac{p+q}{2},$$

erit

$$\operatorname{tang}\tfrac{1}{2}a=\sqrt{-\frac{\cos\frac{B+C-A}{2}\cdot\cos\frac{B+C+A}{2}}{\cos\frac{B+A-C}{2}\cdot\cos\frac{A+C-B}{2}}}.$$

TRANSFORMATIO TERTIA

22. Hanc transformationem etiam ex prima forma expedire licet, combinandis his duabus formulis:

$$\cos a-\cos b\cos c=\sin b\sin c\cos A,$$
$$\cos b-\cos a\cos c=\sin a\sin c\cos B;$$

quarum illa per hanc divisa praebet

$$\frac{\cos a-\cos b\cos c}{\cos b-\cos a\cos c}=\frac{\sin b\cos A}{\sin a\cos B}=\frac{\sin B\cos A}{\sin A\cos B}.$$

Addatur utrinque unitas fietque

$$\frac{(\cos a+\cos b)(1-\cos c)}{\cos b-\ \cos a\cos c}=\frac{\sin(A+B)}{\sin A\cos B}$$

subtrahatur utrinque unitas, prodibit

$$\frac{(\cos a-\cos b)(1+\cos c)}{\cos b-\ \cos a\cos c}=\frac{\sin(A-B)}{\sin A\cos B}$$

quae aequatio per priorem divisa dat

$$\frac{\cos a-\cos b}{\cos a+\cos b}\cdot\cot\tfrac{1}{2}c^{2}=\frac{\sin(B-A)}{\sin(A+B)}.$$

Constat autem esse

$$\frac{\cos p-\cos q}{\cos p+\cos q}=\operatorname{tang}\frac{q+p}{2}\operatorname{tang}\frac{q-p}{2},$$

unde colligitur:

$$\operatorname{tang}\frac{b-a}{2}\operatorname{tang}\frac{b+a}{2}\cdot\cot\tfrac{1}{2}c^{2}=\frac{\sin(B-A)}{\sin(B+A)}$$

23. Iam in subsidium vocemus ex proprietate primaria hanc formulam :

$$\frac{\sin b}{\sin a}=\frac{\sin B}{\sin A}$$

unde deducimus

$$\frac{\sin b-\sin a}{\sin b+\sin a}=\frac{\sin B-\sin A}{\sin B+\sin A}$$

quae reducitur ad hanc formam:

$$\operatorname{tang}\frac{b-a}{2}\cot\frac{b+a}{2}=\operatorname{tang}\frac{B-A}{2}\cot\frac{B+A}{2}.$$

Quodsi iam aequationem ante inventam per hanc multiplicemus, prodibit ista:

$$\left(\operatorname{tang}\frac{b-a}{2}\right)^2\cdot\cot\tfrac{1}{2}c^2=\frac{\left(\sin\frac{B-A}{2}\right)^2}{\left(\sin\frac{B+A}{2}\right)^2}$$

sive extracta radice

$$\operatorname{tang}\frac{b-a}{2}\cdot\cot\tfrac{1}{2}c=\frac{\sin\frac{B-A}{2}}{\sin\frac{B+A}{2}}.$$

At vero prior formula per posteriorem divisa dat

$$\operatorname{tang}\frac{b+a}{2}\cdot\cot\tfrac{1}{2}c=\frac{\cos\frac{B-A}{2}}{\cos\frac{B+A}{2}}.$$

His igitur formulis resolvitur casus, quo dantur duo anguli A et B cum latere intercepto c et quaeruntur ambo latera a et b, quod fit ope harum formularum:

$$\operatorname{tang}\frac{b-a}{2}=\operatorname{tang}\tfrac{1}{2}c\frac{\sin\frac{B-A}{2}}{\sin\frac{B+A}{2}}$$

$$\operatorname{tang}\frac{b+a}{2}=\operatorname{tang}\tfrac{1}{2}c\frac{\cos\frac{B-A}{2}}{\cos\frac{B+A}{2}}$$

TRANSFORMATIO QUARTA

24. Haec simili modo deducitur ex his formulis:

$$\cos A+\cos B\cos C=\sin B\sin C\cos a$$
$$\cos B+\cos A\cos C=\sin A\sin C\cos b,$$

quarum illa per hanc divisa praebet

$$\frac{\cos A+\cos B\cos C}{\cos B+\cos A\cos C}=\frac{\sin B\cos a}{\sin A\cos b}=\frac{\sin b\cos a}{\sin a\cos b}.$$

Unde unitatem tam addendo quam subtrahendo sequentes novae derivantur aequationes:

$$\frac{(\cos A+\cos B)(1+\cos C)}{\cos B+\cos A\cos C}=\frac{\sin(a+b)}{\sin a\cos b}$$

$$\frac{(\cos A-\cos B)(1-\cos C)}{\cos B+\cos A\cos C}=\frac{\sin(a-b)}{\sin a\cos b},$$

dividendo illam per hanc nanciscimur:

$$\frac{\cos A+\cos B}{\cos A-\cos B}\cdot\cot\tfrac{1}{2}C^2=\frac{\sin(a+b)}{\sin(b-a)}$$

sive

$$\operatorname{tang}\frac{B-A}{2}\cdot\operatorname{tang}\frac{B+A}{2}=\cot\tfrac{1}{2}C^2\cdot\frac{\sin(a-b)}{\sin(b+a)},$$

quae aequatio multiplicata et divisa per istam:

$$\operatorname{tang}\frac{B-A}{2}\cdot\cot\frac{B+A}{2}=\operatorname{tang}\frac{b-a}{2}\cot\frac{b+a}{2},$$

producit

$$\operatorname{tang}\frac{B-A}{2}=\cot\tfrac{1}{2}C\cdot\frac{\sin\frac{b-a}{2}}{\sin\frac{b+a}{2}}$$

$$\operatorname{tang}\frac{B+A}{2}=\cot\tfrac{1}{2}C\cdot\frac{\cos\frac{b-a}{2}}{\cos\frac{b+a}{2}},$$

quae formulae valent pro casu, quo dantur duo latera cum angulo intercepto.

25. Quoniam praecedentium formarum omnes variationes apposuimus, etiam hos quatuor casus cum omnibus variationibus conspectui exponamus.

$$\operatorname{tang}\tfrac{1}{2}A=\sqrt{\frac{\sin\frac{a+b-c}{2}\sin\frac{a+c-b}{2}}{\sin\frac{b+c-a}{2}\sin\frac{a+b+c}{2}}}$$

$$\operatorname{tang}\tfrac{1}{2}B=\sqrt{\frac{\sin\frac{b+c-a}{2}\sin\frac{a+b-c}{2}}{\sin\frac{a+c-b}{2}\sin\frac{a+b+c}{2}}}$$

$$\operatorname{tang}\tfrac{1}{2}C=\sqrt{\frac{\sin\frac{a+c-b}{2}\sin\frac{b+c-a}{2}}{\sin\frac{a+b-c}{2}\sin\frac{a+b+c}{2}}}$$

$$\operatorname{tang}\tfrac{1}{2}a=\sqrt{-\frac{\cos\frac{B+C-A}{2}\cos\frac{A+B+C}{2}}{\cos\frac{A+B-C}{2}\cos\frac{A+C-B}{2}}}$$

$$\operatorname{tang}\tfrac{1}{2}b=\sqrt{-\frac{\cos\frac{A+C-B}{2}\cos\frac{A+B+C}{2}}{\cos\frac{B+C-A}{2}\cos\frac{A+B-C}{2}}}$$

$$\operatorname{tang}\tfrac{1}{2}c=\sqrt{-\frac{\cos\frac{A+B-C}{2}\cos\frac{A+B+C}{2}}{\cos\frac{A+C-B}{2}\cos\frac{B+C-A}{2}}}$$

$\operatorname{tang}\frac{b-a}{2}=\operatorname{tang}\frac{1}{2}c\cdot\frac{\sin\frac{B-A}{2}}{\sin\frac{B+A}{2}}$	$\operatorname{tang}\frac{b+a}{2}=\operatorname{tang}\frac{1}{2}c\cdot\frac{\cos\frac{B-A}{2}}{\cos\frac{B+A}{2}}$
$\operatorname{tang}\frac{c-b}{2}=\operatorname{tang}\frac{1}{2}a\cdot\frac{\sin\frac{C-B}{2}}{\sin\frac{C+B}{2}}$	$\operatorname{tang}\frac{c+b}{2}=\operatorname{tang}\frac{1}{2}a\cdot\frac{\cos\frac{C-B}{2}}{\cos\frac{C+B}{2}}$
$\operatorname{tang}\frac{a-c}{2}=\operatorname{tang}\frac{1}{2}b\cdot\frac{\sin\frac{A-C}{2}}{\sin\frac{A+C}{2}}$	$\operatorname{tang}\frac{a+c}{2}=\operatorname{tang}\frac{1}{2}b\cdot\frac{\cos\frac{A-C}{2}}{\cos\frac{A+C}{2}}$
$\operatorname{tang}\frac{B-A}{2}=\cot\frac{1}{2}C\cdot\frac{\sin\frac{b-a}{2}}{\sin\frac{b+a}{2}}$	$\operatorname{tang}\frac{B+A}{2}=\cot\frac{1}{2}C\cdot\frac{\cos\frac{b-a}{2}}{\cos\frac{b+a}{2}}$
$\operatorname{tang}\frac{C-B}{2}=\cot\frac{1}{2}A\cdot\frac{\sin\frac{c-b}{2}}{\sin\frac{c+b}{2}}$	$\operatorname{tang}\frac{C+B}{2}=\cot\frac{1}{2}A\cdot\frac{\cos\frac{c-b}{2}}{\cos\frac{c+b}{2}}$
$\operatorname{tang}\frac{A-C}{2}=\cot\frac{1}{2}B\cdot\frac{\sin\frac{a-c}{2}}{\sin\frac{a+c}{2}}$	$\operatorname{tang}\frac{A+C}{2}=\cot\frac{1}{2}B\cdot\frac{\cos\frac{a-c}{2}}{\cos\frac{a+c}{2}}$

26. Ex his postremis formulis iam facile expeditur casus, quem nondum attigimus, quo dantur duo latera cum angulis oppositis, et vel tertium latus vel tertius angulus quaeritur, quorum utrumque duplici modo fieri potest. Has ergo formulas cum variationibus apponamus

$\operatorname{tang}\frac{1}{2}c = \operatorname{tang}\frac{b-a}{2}\cdot\frac{\sin\frac{B+A}{2}}{\sin\frac{B-A}{2}}$	$\operatorname{tang}\frac{1}{2}c = \operatorname{tang}\frac{b+a}{2}\cdot\frac{\cos\frac{B+A}{2}}{\cos\frac{B-A}{2}}$
$\operatorname{tang}\frac{1}{2}a = \operatorname{tang}\frac{c-b}{2}\cdot\frac{\sin\frac{C+B}{2}}{\sin\frac{C-B}{2}}$	$\operatorname{tang}\frac{1}{2}a = \operatorname{tang}\frac{c+b}{2}\cdot\frac{\cos\frac{C+B}{2}}{\cos\frac{C-B}{2}}$
$\operatorname{tang}\frac{1}{2}b = \operatorname{tang}\frac{a-c}{2}\cdot\frac{\sin\frac{A+C}{2}}{\sin\frac{A-C}{2}}$	$\operatorname{tang}\frac{1}{2}b = \operatorname{tang}\frac{a+c}{2}\cdot\frac{\cos\frac{A+C}{2}}{\cos\frac{A-C}{2}}$
$\cot\frac{1}{2}C = \operatorname{tang}\frac{B-A}{2}\cdot\frac{\sin\frac{b+a}{2}}{\sin\frac{b-a}{2}}$	$\cot\frac{1}{2}C = \operatorname{tang}\frac{B+A}{2}\cdot\frac{\cos\frac{b+a}{2}}{\cos\frac{b-a}{2}}$
$\cot\frac{1}{2}A = \operatorname{tang}\frac{C-B}{2}\cdot\frac{\sin\frac{c+b}{2}}{\sin\frac{c-b}{2}}$	$\cot\frac{1}{2}A = \operatorname{tang}\frac{C+B}{2}\cdot\frac{\cos\frac{c+b}{2}}{\cos\frac{c-b}{2}}$
$\cot\frac{1}{2}B = \operatorname{tang}\frac{A-C}{2}\cdot\frac{\sin\frac{a+c}{2}}{\sin\frac{a-c}{2}}$	$\cot\frac{1}{2}B = \operatorname{tang}\frac{A+C}{2}\cdot\frac{\cos\frac{a+c}{2}}{\cos\frac{a-c}{2}}$

Hoc igitur modo praesens tractatio tanquam systema completum totius Trigonometriae sphaericae spectari potest.