FORMVLAE GENERALES PRO TRANSLATIONE QVACVNQVE CORPORVM RIGIDORVM

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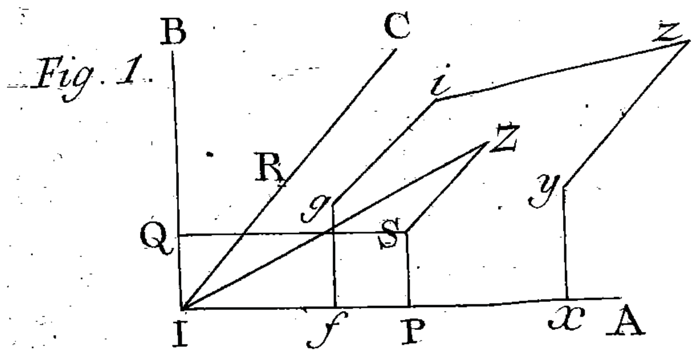
Novi Commentarii Academiae Scientiarum Imperialis Petropolitanae
pro Anno MDCCLXXV, Tom. XX, pp. 189-207

§1.

Quando corporis cuiusque rigidi motum determinari oportet, tota inuestigatio commode in duas partes distinguitur, alteram geometricam, alteram mechanicam. In priore enim parte sola translatio corporis ex dato situ in alium quemcunque sine vllo respectu habito ad motus principia per formulas analyticas repraesentari debet, quarum ope positio singulorum punctorum post translationem ex earum positione initiali definiri queat; quae ergo inuestigatio vnice ad Geometricam vel potius ad Stereometricam est referenda. Facile autem intelligitur, si ista inuestigatio ab altera, quae proprie ad Mechanicam pertinet, separetur, tum ipsam motus determinationem ex principiis motus multo facilus expediri posse, quam si vtraque inuestigatio coniunctim suscipiatur. Cum igitur in tractatu meo de motu corporum rigidorum hanc vtramque inuestigationem simul suscepissem, vnde tota tractatio non parum molesta et intricata est reddita: hoc loco solam partem geometricam accuratius euoluere constitui, quo deinceps pars mechanica faciliori negotio expediri possit.

§2. Vt igitur primo situm initialem corporis rigidi accurate definiam, positionem singulorum eius punctorum more solito per ternas coordinaatas inter se normales repraesentari conueniet. Hunc in finem (vide Tab. II, Fig. 1) constituo ternos axes fixos IA, IB et IC se inuicem in puncto I normaliter secantes, quorum bini IA et IB in ipso plano tabulae sint siti, tertius vero IC hinc plano perpendiulariter insistat. Nunc considero punctum corporis quodcunque Z, ex quo ad planum AIB demittatur perpendiculum ZS; tum vero ex puncto S ad axes IA et IB ducantur normales SP et SQ, ac vocemus coordinatas IP=QS=p, PS=IQ=q et ipsum perpendiculum SZ=r, cui in axe IC aequalis capiatur portio IR=r; ita vt punctum Z reperiatur in diagonali IZ parallelepipedi rectanguli, quod ex lateribus IP, IQ et IR formatur. Hoc igitur modo positio singulorum corporis punctorum commodissime per ternas coordinatas p, q, r, determinabitur.

§3. Quo autem deinceps facilus ista repraesentatio ad inuestigationem mechanicam accommodari possit, punctum I aptissime accipitur in ipso centro grauitatis seu potius inertiae



corporis rigidi propositi; sic enim istud insigne commodum impetramus, vt posita massula corporis in Z existentis $=d$M, per totam corporis extensionem fiat

$$1^\circ.\int p\,d\mathrm{M}=0. \qquad 2^\circ.\int q\,d\mathrm{M}=0. \qquad 3^\circ.\int r\,d\mathrm{M}=0. \tag{1}$$

siquidem haec integralia per totum corpus extendantur. Praeterea vero maximam vtilitatem afferet, si terni axes IA, IB, IC in ipsis axibus corporis principalibus constituantur; tum enim etiam valores trium sequentium formularum integralium pariter per totum corpus extensi nihilo aequales reddentur, quippe quae sunt

$$4^\circ.\int pq\,d\mathrm{M}=0. \qquad 5^\circ.\int pr\,d\mathrm{M}=0. \qquad 6^\circ.\int qr\,d\mathrm{M}=0. \tag{2}$$

haecque tantum hic in transitu notasse iuuabit, quandoquidem pars geometrica ab istis aequationibus neutiquam pendet.

§4. Iam facta quacunque corporis translatione consideremus primo locum i, in quem punctum corporis I fuerit translatum, pro quo vocemus coordinatas I$f=f$, $fg=g$ et $gi=h$; tum vero punctum Z est situ initiali translatum sit in z, pro quo statuamus coordinatas I$x=x$, $xy=y$ et $yz=z$, ac primo quidem statim manifestum est, distantiam $iz=$ etiamnunc aequalem esse debere distantiae IZ, qua, cum esset $\sqrt{pp+qq+rr}$, nunc vero sit

$$iz=\sqrt{(x-f)^2+(y-g)^2+(z-h)^2} \tag{3}$$

habebimus hanc aequationem:

$$pp+qq+rr=(x-f)^2+(y-g)^2+(z-h)^2. \tag{4}$$

Praeterea vero necesse est, vt distantiae inter bina corporis puncta quaecunque in situ translato etiamnunc aequales sint distantiis eorumdem punctorum in situ initiali, cui conditioni sequenti modo satisfaciemus.

§5. Sumamus punctum z in eo loco, in quem punctum P ex statu initiali fuerit translatum: hic enim non consultum videtur figuram nostram tot nouis lineis ducendis onerare. Deinde etiam in ipso statu initiali punctum Z vbicunque libuerit accipi potest; vnde si punctum Z in puncto P accipiatur, etiam punctum z in situ translato locum ipsi P respondentem exhibebit.

§6. Cum igitur punctum Z in punctum P incidat, si fiat $q = 0$ et $r = 0$, quoniam in genere ternas coordinatas x, y, z tanquam certas functiones ipsarum p, q et r considerare licet, quomodocunque hae functiones fuerint comparate, si in iis faciamus $q = 0$ et $r = 0$ hae coordinatae necessario tales formas accipere debebunt

$$x = f + \mathrm{F}p, \qquad y = g + \mathrm{G}p, \qquad z = h + \mathrm{H}p. \tag{5}$$

Quia enim ponimus $q = 0$ et $r = 0$ spectata p vt variabili, coordinatae x, y, z ostendere debent situm in quem linea recta IP fuerit translata; quae cum sit recta, ea in situ translato erit linea recta ipsi aequalis, ideoque coordinatae x, y, z positionem huius lineae rectae iz exprimere debent, vnde, cum sumto $p = 0$ etiam punctum z in i incidere debeat, euidens est, quantitates x, y, z ita per variabilem p definiri debere, vt posito $p = 0$ fiat $x = f$, $y = g$ et $z = b$. Tum vero quia aequatio debet esse pro linea recta, aliae formae locum habere nequeunt, nisi quas statuimus: scilicet

$$x = f + \mathrm{F}p, \qquad y = g + \mathrm{G}p, \qquad z = h + \mathrm{H}p. \tag{6}$$

vbi litterae F, G, H certas designant constantes ab indole translationis pendentes.

§7. Statim autem manifestum est, istas constantes ita comparatas esse debere, vt interuallum iz aequale sit interuallo IP$= p$, vnde sequitur ista determinatio:

$$iz^2 = \mathrm{F}^2p^2 + \mathrm{G}^2p^2 + \mathrm{H}^2p^2 = pp \tag{7}$$

quam ob rem necesse est vt sit $\mathrm{F}^2 + \mathrm{G}^2 + \mathrm{H}^2 = 1$. Quod si ergo sumamus F$= \sin\zeta$, fieri debet $\mathrm{G}^2 + \mathrm{H}^2 = \cos^2\zeta$; hanc ob rem statuamus

$$\mathrm{G} = \cos\zeta\sin\eta \qquad \text{et} \qquad \mathrm{H} = \cos\zeta\cos\eta, \tag{8}$$

ita vt sit

$$\mathrm{F} = \sin\zeta, \qquad \mathrm{G} = \cos\zeta\sin\eta \qquad \text{et} \qquad \mathrm{H} = \cos\zeta\cos\eta. \tag{9}$$

Hoc ergo modo tres litterae illae F, G, H ad duos tantum angulos ζ et η sunt reductae.

§8. Simili modo sumamus nunc punctum z in eo loco, in quem punctum Q ex situ initiali fuerit translatum; at vero punctum Z in punctum Q cadit sumendo $p = 0$ et $r = 0$. Hoc ergo casu ternae coordinatae x, y, z ita pendebunt a sola variabili q, vt facto $q = 0$ iterum fiat $x = f$, $y = g$ et $z = h$; quamobrem, cum aequatio etiam debeat esse pro linea recta, coordinatae talem formam habebunt:

$$x = f + \mathrm{F}'q, \qquad y = g + \mathrm{G}'q, \qquad z = h + \mathrm{H}'q \tag{10}$$

vbi ergo ob $iz = q$ etiam esse oportet $\mathrm{F}'\mathrm{F}' + \mathrm{G}'\mathrm{G}' + \mathrm{H}'\mathrm{H}' = 1$ cui conditione commode per binos angulos ζ' et η' ita satisfiet, vt sit

$$\mathrm{F}' = \sin\zeta', \qquad \mathrm{G}' = \cos\zeta'\sin\eta', \qquad \mathrm{H}' = \cos\zeta'\cos\eta'. \tag{11}$$

§9. Sumamus nunc punctum Z in R, quod euenit statuendo $p = 0$ et $q = 0$, vnde si iam punctum z exhibeat locum, in quem punctum R erit translatum, pro coordinatis eodem modo quo ante adipiscemur tales formas:

$$x = f + \mathrm{F}''r, \qquad y = g + \mathrm{G}''r, \qquad z = h + \mathrm{H}''r. \tag{12}$$

Et quia esse oportet $\mathrm{F}''\mathrm{F}'' + \mathrm{G}''\mathrm{G}'' + \mathrm{H}''\mathrm{H}'' = 1$ per binos nouos angulos ζ'' et η'' statuere poterimus

$$\mathrm{F}'' = \sin\zeta'', \qquad \mathrm{G}'' = \cos\zeta''\sin\eta'' \qquad \text{et} \qquad \mathrm{H}'' = \cos\zeta''\cos\eta''. \tag{13}$$

§10. Quoniam igitur coordinatarum x, y et z valores nacti sumus, quos inducere debent tribus casibus euolutis, vbi trium quantitatum p, q, r duae euanescebant, perspicuum hinc est, quomodo coordinatae x, y, z a singulis quantitatibus p, q, r pendent. Quamobrem, si omnes istae litterae simul in computum ingrediantur, ita vt iis punctum corporis quodcunque Z indicetur, cui in situ translato respondeat punctum z, coordinatae x, y, z sequentes habere debebunt valores:

$$x = f + \mathrm{F}p + \mathrm{F}'q + \mathrm{F}''r \tag{14}$$
$$y = g + \mathrm{G}p + \mathrm{G}'q + \mathrm{G}''r \tag{15}$$
$$z = h + \mathrm{H}p + \mathrm{H}'q + \mathrm{H}''r \tag{16}$$

has autem nouem litteras vidimus reduci ad sex angulos ζ, η, ζ', η', ζ'', η''.

§11. Sumamus nunc punctum Z in ipso puncto s ita vt sit $r = 0$, ac si in situ translato isti puncto respondeat punctum z, posito $r = 0$ ternae coordinatae ita se habebunt:

$$x = f + \mathrm{F}p + \mathrm{F}'q \qquad y = g + \mathrm{G}p + \mathrm{G}'q \qquad z = h + \mathrm{H}p + \mathrm{H}'q. \tag{17}$$

Vbi necesse est, vt fiat distantia iz distantiae IS aequalis, quae cum sit $\sqrt{pp + qq}$, hinc nascetur ista aequatio:

$$pp + qq = (\mathrm{F}p + \mathrm{F}'q)^2 + (\mathrm{G}p + \mathrm{G}'q)^2 + (\mathrm{H}p + \mathrm{H}'q)^2 \tag{18}$$

et facta euolutione fiet

$$pp + qq = pp(\mathrm{FF} + \mathrm{GG} + \mathrm{HH}) + qq(\mathrm{F}'\mathrm{F}' + \mathrm{G}'\mathrm{G}' + \mathrm{H}'\mathrm{H}') + 2pq(\mathrm{FF}' + \mathrm{GG}' + \mathrm{HH}') \tag{19}$$

Cum igitur sit $\mathrm{FF} + \mathrm{GG} + \mathrm{HH} = 1$ et $\mathrm{F}'\mathrm{F}' + \mathrm{G}'\mathrm{G}' + \mathrm{H}'\mathrm{H}' = 1$ superest, vt euadat $\mathrm{FF}' + \mathrm{GG}' + \mathrm{HH}' = 0$.

§12. Eodem modo patebit, si sumamus $q = 0$, tum istam proprietatem locum habere debere, vt sit $\mathrm{FF}'' + \mathrm{GG}'' + \mathrm{HH}'' = 0$: at si statuamus $p = 0$, inde resultabit ista aequatio $\mathrm{F}'\mathrm{F}'' + \mathrm{G}'\mathrm{G}'' + \mathrm{H}'\mathrm{H}'' = 0$. Quibus tribus conditionibus cum fuerit satisfactum, tota translatio erit determinata; ac nostrae formulae pro omnibus corporis punctis easdem exhibebunt distantias in situ translato, quas tennerunt in situ initiali.

§13. Substituamus nunc in his aequationibus valores ante inuentos, ac prima $\mathrm{FF}' + \mathrm{GG}' + \mathrm{HH}' = 0$ dabit

$$\sin\zeta\sin\zeta' + \cos\zeta\cos\zeta'\sin\eta\sin\eta' + \cos\zeta\cos\zeta'\cos\eta\cos\eta' = 0 \tag{20}$$

siue

$$\sin\zeta\sin\zeta' + \cos\zeta\cos\zeta'\cos(\eta-\eta') = 0 \quad \text{ob} \quad \cos\eta\cos\eta' + \sin\eta\sin\eta' = \cos(\eta-\eta') \tag{21}$$

quae aequatio per $\cos\zeta\cos\zeta'$ diuisa praebet $\tan\zeta\tan\zeta' = -\cos(\eta-\eta')$. Eodem modo binae reliquae aequationes dabunt

$$\tan\zeta'\tan\zeta'' = -\cos(\eta'-\eta'') \qquad \text{et} \qquad \tan\zeta''\tan\zeta = -\cos(\eta''-\eta). \tag{22}$$

Ex his igitur tribus aequationibus ternos angulos ζ, ζ' et ζ'' determinare licebit, ita vt omnia per ternos angulos η, η' et η'' definiri queant.

§14. Quod quo facilus fieri possit, multiplicemus has tres aequationes in se inuicem, vt fiat

$$\tan^2\zeta\tan^2\zeta'\tan^2\zeta'' = -\cos(\eta-\eta')\cos(\eta'-\eta'')\cos(\eta''-\eta). \tag{23}$$

Vnde statim patet, nisi productum horum trium cosinuum fuerit negatiuum, casum esse impossibilem; quocirca ante omnia necesse est, vt horum cosinuum vel vnus vel omnes tres sint negatiuum. Statuamus igitur breuitatis gr.

$$\cos(\eta-\eta')\cos(\eta'-\eta'')\cos(\eta''-\eta) = -\Delta\Delta, \tag{24}$$

vt nanciscamur $\tan\zeta\tan\zeta'\tan\zeta'' = \Delta$, quae aequatio per singulas praecedentes diuisa nobis suppeditat hos valores

$$\tan\zeta'' = \frac{-\Delta}{\cos(\eta-\eta')}; \quad \tan\zeta = \frac{-\Delta}{\cos(\eta'-\eta'')}; \quad \tan\zeta' = \frac{-\Delta}{\cos(\eta''-\eta)} \tag{25}$$

hoc igitur modo omnes nouem coefficientes initio assumti F, G, H, F′, G′, H′, F″, G″, H″, per solos ternos angulos η, η', η'' determinatur hoc modo:

$$\mathrm{F} = \sin\zeta, \quad \mathrm{G} = \cos\zeta\sin\eta, \quad \mathrm{H} = \cos\zeta\cos\eta \tag{26}$$
$$\mathrm{F}' = \sin\zeta', \quad \mathrm{G}' = \cos\zeta'\sin\eta', \quad \mathrm{H}' = \cos\zeta'\cos\eta' \tag{27}$$
$$\mathrm{F}'' = \sin\zeta'', \quad \mathrm{G}'' = \cos\zeta''\sin\eta'', \quad \mathrm{H}'' = \cos\zeta''\cos\eta'' \tag{28}$$

§15. Omnes igitur translationes, quibus situs corporis rigidi mutari potest, per sex elementa determinari possunt. Primo enim ternae coordinatae f, g, h determinant translationem puncti I in i, quae ergo penitus a nostro arbitrio pendet. Deinde, quomodocunque corpus circa hoc punctum i fuerit interea conuersum, eius situs per ternos angulos η, η', η'' penitus determinatur; sumto enim in situ initiali elemento corporis quocunque Z, cuius positio per ternas coordinatas p, q, r, definitur, id in situ translato reperietur in puncto z, cuius positio per istas ternas coordinatas definietur:

$$x = f + \mathrm{F}p + \mathrm{F}'q + \mathrm{F}''r \tag{29}$$
$$y = g + \mathrm{G}p + \mathrm{G}'q + \mathrm{G}''r \tag{30}$$
$$z = h + \mathrm{H}p + \mathrm{H}'q + \mathrm{H}''r. \tag{31}$$

§16. Quo autem magis conuincamur, per has formulas omnia quae ad translationem pertinent perfecte determinari, totum negotium etiam sequenti modo absolui potest. Concipiamus in statu initiali praeter punctum Z aliud quodcunque Z′, in figura quidem non expressum, cuius locus his coordinatis definiatur p', q', r'. Hoc autem punctum translatum sit in z', cui respondeant coordinatae x', y', z', quarum ergo valores ita exprimentur

$$x' = f + \mathrm{F}p' + \mathrm{F}'q' + \mathrm{F}''r' \quad (32)$$

$$y' = g + \mathrm{G}p' + \mathrm{G}'q' + \mathrm{G}''r' \quad (33)$$

$$z' = h + \mathrm{H}p' + \mathrm{H}'q' + \mathrm{H}''r'. \quad (34)$$

Quibus positis natura corporum rigidorum postulat, vt interuallum in situ translato zz' aequale sit interuallo ZZ′ in situ initiali, quandoquidem in his corporibus omnia interualla inter bina eorum puncta quaecunque perpetuo eandem quantitatem seruare debent.

§17. Iam vero distantiae punctorum Z et Z′ in statu initiali quadratum est

$$(p'-p)^2 + (q'-q)^2 + (r'-r)^2 \quad (35)$$

in statu autem translato quadratum distantiae inter puncta z et z' est

$$(x'-x)^2 + (y'-y)^2 + (z'-z)^2, \quad (36)$$

quod ergo ex tribus sequentibus quadratis componitur

$$\begin{aligned} &(\mathrm{F}(p'-p) + \mathrm{F}'(q'-q) + \mathrm{F}''(r'-r))^2 \\ +&(\mathrm{G}(p'-p) + \mathrm{G}'(q'-q) + \mathrm{G}''(r'-r))^2 + (\mathrm{H}(p'-p) + \mathrm{H}'(q'-q) + \mathrm{H}''(r'-r))^2 \end{aligned} \quad (37)$$

quorum ergo summa aequalis esse debet illi formulae $(p'-p)^2 + (q'-q)^2 + (r'-r)^2$.

§18. Euolutis autem ternis illis quadratis sequens expressio resultabit

$$\begin{aligned} &(p'-p)^2(\mathrm{FF} + \mathrm{GG} + \mathrm{HH}) \\ +\ &(q'-q)^2(\mathrm{F}'\mathrm{F}' + \mathrm{G}'\mathrm{G}' + \mathrm{H}'\mathrm{H}') \\ +\ &(r'-r)^2(\mathrm{F}''\mathrm{F}'' + \mathrm{G}''\mathrm{G}'' + \mathrm{H}''\mathrm{H}'') \\ +\ &2(p'-p)(q'-q)(\mathrm{FF}' + \mathrm{GG}' + \mathrm{HH}') \\ +\ &2(p'-p)(r'-r)(\mathrm{FF}'' + \mathrm{GG}'' + \mathrm{HH}'') \\ +\ &2(q'-q)(r'-r)(\mathrm{F}'\mathrm{F}'' + \mathrm{G}'\mathrm{G}'' + \mathrm{H}'\mathrm{H}'') \end{aligned}$$

quamobrem, vt ista expressio priori $(p'-p)^2 + (q'-q)^2 + (r'-r)^2$ reddatur aequalis, quomodocunque coordinatae p, q, r, p', q', r' fuerint assumtae sex sequentibus conditionibus satisfieri oportet

$$\text{I.}\quad \mathrm{FF} + \mathrm{GG} + \mathrm{HH} = 1 \quad (38)$$

$$\text{II.}\quad \mathrm{F}'\mathrm{F}' + \mathrm{G}'\mathrm{G}' + \mathrm{H}'\mathrm{H}' = 1 \quad (39)$$

$$\text{III.}\quad \mathrm{F}''\mathrm{F}'' + \mathrm{G}''\mathrm{G}'' + \mathrm{H}''\mathrm{H}'' = 1 \quad (40)$$

$$\text{IV.}\quad \mathrm{FF}' + \mathrm{GG}' + \mathrm{HH}' = 0 \quad (41)$$

$$\text{V.}\quad \mathrm{FF}'' + \mathrm{GG}'' + \mathrm{HH}'' = 0 \quad (42)$$

$$\text{VI.}\quad \mathrm{F}'\mathrm{F}'' + \mathrm{G}'\mathrm{G}'' + \mathrm{H}'\mathrm{H}'' = 0. \quad (43)$$

§19. At vero omnes istas sex conditiones iam in superioribus adimpleuimus, vbi ostendimus, quemadmodum omnes his nouem coefficientes per ternos angulos η, η', η'' determinari queant. Ex quo eo clarius intelligitur, solutionem nostram quaestionis circa translationem quamcunque corporum rigidorum penitus esse determinatam et adaequatam, ita vt in parte geometrica, quam motus talium corporum determinatio postulat, nihil amplius desiderari possit.

§20. Quomodocunque autem translatio corporis fuerit facta, qua punctum corporis I in punctum i est translatum; notum est si translatio fuerit infinite parua, tum semper in situ translato dari quaedam recta iz, cuius situs parallelus erit ei, quem eadem recta in statu initiali habuit, ita vt, si punctum I quieuisset, ista recta penitus immota mansisset. Euidens autem est, istam rectam repraesentare axem corporis circa quem gyratio fuerit facta, dum corpus in situm translatum peruenit. Quamobrem maximi momenti erit inuestigare, vtrum, si translatio fuerit finita, etiam detur talis axis.

§21. Manifestum autem est, vt recta iz etiam nunc parallela sit rectae IZ, ad hoc requiri tres istas conditiones:

$$1^\circ . x - f = p, \qquad 2^\circ . y - g = q, \qquad 3^\circ . z - h = r \tag{44}$$

vnde nascuntur hae aequationes:

$$p = \mathrm{F}p + \mathrm{F}'q + \mathrm{F}''r \tag{45}$$
$$q = \mathrm{G}p + \mathrm{G}'q + \mathrm{G}''r \tag{46}$$
$$r = \mathrm{H}p + \mathrm{H}'q + \mathrm{H}''r \tag{47}$$

ex quibus aequationibus litteras p, q, r eliminari oportet. Valores autem ipsius p hinc deducti erunt

$$\frac{\mathrm{F}'q + \mathrm{F}''r}{1 - \mathrm{F}}, \quad \frac{(1 - \mathrm{G}')q - \mathrm{G}''r}{\mathrm{G}}, \quad \frac{(1 - \mathrm{H}'')r - \mathrm{H}'q}{\mathrm{H}} \tag{48}$$

Horum valorum primus secundo aequatus istam dabit rationem inter q et r, scilicet

$$\frac{q}{r} = \frac{\mathrm{G}''(\mathrm{F} - 1) - \mathrm{F}''\mathrm{G}}{\mathrm{G}\mathrm{F}' - (1 - \mathrm{F})(1 - \mathrm{G}')} \tag{49}$$

ac primus valor tertio aequatus perducit ad hanc relationem:

$$\frac{q}{r} = \frac{(1 - \mathrm{F})(1 - \mathrm{H}'') - \mathrm{F}''\mathrm{H}}{\mathrm{F}'\mathrm{H} + \mathrm{H}'(1 - \mathrm{F})}. \tag{50}$$

Hos igitur duos valores reuera inter se aequari necesse est, siquidem talis axis gyrationis datur.

§22. Quod si autem hos duos valores inter se aequales ponamus, perueniemus ad istam aequationem:

$$\begin{aligned} & (1 - \mathrm{F})\mathrm{F}''\mathrm{G}\mathrm{H}' + (1 - \mathrm{F})\mathrm{F}'\mathrm{G}''\mathrm{H} + (1 - \mathrm{F})(1 - \mathrm{G}')\mathrm{F}''\mathrm{H} + (1 - \mathrm{F})(1 - \mathrm{H}'')\mathrm{F}'\mathrm{G} \\ + \;& (1 - \mathrm{F})^2\mathrm{G}''\mathrm{H}' - (1 - \mathrm{F})^2(1 - \mathrm{G}')(1 - \mathrm{H}'') = 0 \end{aligned} \tag{51}$$

cuius aequationis omnia membra factore communi gaudent $(1-\mathrm{F})$; hoc ergo per diuisionem sublato remanebit ista aequatio:

$$\mathrm{F}''\mathrm{G}\mathrm{H}' + \mathrm{F}'\mathrm{G}''\mathrm{H} + (1 - \mathrm{G}')\mathrm{F}''\mathrm{H} + (1 - \mathrm{H}'')\mathrm{F}'\mathrm{G} + (1 - \mathrm{F})(\mathrm{G}''\mathrm{H}' - (1 - \mathrm{G}')(1 - \mathrm{H}'')) = 0 \tag{52}$$

quae singulis membris euolutis dat hanc aequationem

$$\begin{aligned}0 = -1 + \mathrm{F} &- \mathrm{FG}' + \mathrm{FG}'\mathrm{H}''\\ +\mathrm{G}' &- \mathrm{FH}'' - \mathrm{FG}''\mathrm{H}'\\ +\mathrm{H}'' &+ \mathrm{F}'\mathrm{G} - \mathrm{F}'\mathrm{GH}''\\ &- \mathrm{G}'\mathrm{H}'' - \mathrm{F}''\mathrm{G}'\mathrm{H}\\ &+ \mathrm{G}''\mathrm{H}' + \mathrm{F}'\mathrm{G}''\mathrm{H}\\ &+ \mathrm{F}''\mathrm{H} + \mathrm{F}''\mathrm{GH}'. \qquad (53)\end{aligned}$$

Hic autem non liquet, quomodo ista expressio ad nihilum redigatur; ac nimis taediosum foret loco litterarum F, G, H, eorum valores penitus euolutos substituere.

§23. Missa igitur hac inuestigatione, quoniam pro translatione quacunque formulas dedimus, quarum ope ex data cuiusque puncti positione in statu initiali eiusdem positio in statu translato assignari potest, scopo quem nobis proposuemarus plene satisfecimus, ita vt in hac parte nihil amplius desiderari queat, cum tota haec inuestigatio in determinatione 9 coefficientium F, G, H, F′, G′, H′, F″, G″, H″ contineatur.

Additamentum.

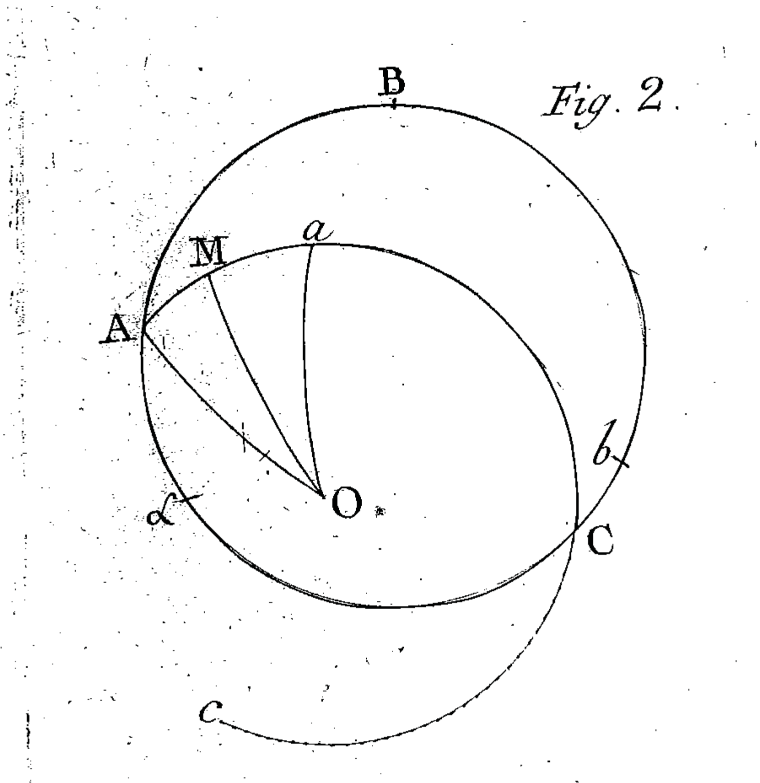
§24. Cum formulae, quas supra pro quouis situ translato dedimus, maxime sint generales, et omnes translationes in se complectantur, mirum videri debet, quod ex illis haud pateat, vtrum in omnia situ translato talis detur recta iz, quae eandem directionem teneat, quam in situ initiali habuit. Aequatio enim §22. inuenta tantopere est implicata, vt nimis molestum foret, loco singularum litterarum valores quos ipsis assignauimus substituere. Interim tamen aliunde certum est, quomodocunque corpus rigidum ex vno situ in alium transferatur, semper dari eiusmodi rectam iz, cuius directio nullam mutationem patiatur. Ad hoc enim demonstrandum concipiamus corpori rigido, cuiscunque fuerit figurae, sphaeram circumscribi cum ipso connexam simulque mobilem, quae centrum habeat in puncto I, quo facilius istam inuestigationem ad doctrinam sphaericam traducere liceat.

Theorema.

Quomodocunque sphaera circa centrum suum conuertatur, semper assignari potest diameter, cuius directio in situ translato conueniat cum situ initiali

Demonstratio.

§25. (Vide Tab II. Fig. 2) Referat circulus A, B, C circulum sphaerae maximum quemcunque, in statu initiali, qui facta translatione peruenerit in situm a, b, c, ita vt puncta A, B, C translata sint in puncta a, b, c; puntum autem A sit simul intersectio horum duorum circulorum. Quo posito demonstrandum est, semper dari punctum O, quod pari modo referetur tam ad circulum A, B, C, quam ad circulum a, b, c. Ad hoc igitur necesse est, vt primo distantiae OA et Oa sint inter se aequales; deinde vero, vt etiam arcus OA et Oa ad illos duos circulos aequaliter sint inclinati, siue vt sit angulus Oab=angulo OAB: erunt ergo etiam complementa ad duos rectos, hoc est anguli OaA et OAα inter se aequales. Quoniam autem arcus Oa et OA sunt aequales, erit quoque angulus OaA=angulo OAa, ideoque OAa=OAα; vnde patet, si angulus aAα bisecetur arcu OA, tum punctum quaesitum O alicubi in isto arcu AO fore situm; quod igitur reperietur si arcus aO ita ducatur, vt angulus AaO aequalis euadat angulo OAa. Intersectio enim horum arcuum dabit punctum O, per quod si ducatur diameter Sphaerae, eius positio in situ translato etiamnunc eadem erit, quae fuerat in situ initiali.



§26. Ad hoc punctum O facilius definiendum, bisecari potest arcus Aa in puncto M, vbi constituatur arcus MO ad Aa normalis; tum vero ducatur arcus AO, ita vt angulum aAα bisecet; atque intersectio horum arcuum O monstrabit punctum quaesitum. Hic obseruatur, si arcus $a\alpha$ aequalis capiatur arcui aA, fore α punctum Sphaerae, quod facta translatione peruenerit in punctum A, quamobrem iste angulus aAα bisecari debet, non vero eius deinceps positus αAB.

§27. Vulgo quidem punctum I (Vide Tab. II, Fig. 1) ad quod positio corporis initialis refertur, sumi solet in eius centro grauitatis. Verum ex demonstratione data apparet, veritatem theorematis etiam subsistere, quodcunque aliud punctum pro centro Sphaerae fuerit assumtum. Quamobrem, si in corpore rigido loco I accipiatur punctum quodcunque, per id semper duci poterit linea recta, cuius positio in situ translato non erit immutata; quin etiam nihil impedit, quo minus istud punctum I adeo extra corpus accipiatur. Quamobrem cauendum est, ne ista insignis proprietas tanquam centro grauitatis propria spectetur: ideo enim tantum punctum illud I in ipso centro grauitatis corporis constitui solet, quo formulae analyticae, quibus motus talium corporum definitur, fiunt simpliciores.

§28. Cum igitur solidissimis rationibus sit euictum, in omni situ translato semper dari eiusmodi lineam rectam iz, cuius directio non discrepet a directione, quam eadem recta IZ in situ initiali tenuit, etiam certi esse possumus, aequationem §22. datam semper locum esse habituram, postquam scilicet loco omnium litterarum valores assignati fuerint substituti; hoc enim facto necessario euenire debet, vt omnes plane termini sponte se mutuo tollant, etiamsi hoc ex sex illis conditionibus principalibus, quibus satisfieri oportuit neutiquam appareat. Quamobrem ista eximia proprietas, cuius veritas geometrice tam facile est ostensa ratione formularum analyticarum pro maxime abscondita est habenda; atque ob hanc ipsam rationem ex ea pulcherrima incrementa per totam mechanicam merito expectare possumus.

§29. Interim tamen formulas, quas pro illis litteris maiusculis supra inuenimus, diligentius euoluamus, quo inde forsitan facilius perspici queat, quemadmodum aequatio illa §22. data adimpleatur. Introductis autem sex angulis ζ, ζ', ζ''; η, η per quos illas §14. expressimus, ternos priores per posteriores ita determinauimus, vt posito

$$-\cos(\eta-\eta')\cos(\eta'-\eta'')\cos(\eta''-\eta)=\Delta\Delta \qquad \text{esset} \tag{54}$$

$$\tan\zeta''=\frac{-\Delta}{\cos(\eta-\eta')};\quad \tan\zeta=\frac{-\Delta}{\cos(\eta'-\eta'')};\quad \tan\zeta'=\frac{-\Delta}{\cos(\eta''-\eta)}; \tag{55}$$

vnde ergo tam sinus quam cosinus illorum angulorum deduci oportet.

§30. Quod quo facilius fieri possit, loco angulorum η, η' et η'' introducamus alios angulos θ, θ', θ'', ita vt sit $\eta-\eta'=\theta''$; $\eta'-\eta''=\theta$ et $\eta''-\eta=\theta'$, vnde patet fore $\theta+\theta'+\theta''=0$, ita vt hi tres noui anguli tantum duobus aequiualeant; ideoque hinc ex angulis η, η' et η'' vnus manebit indefinitus, qui si fuerit η erit $\eta'=\eta-\theta''$ et $\eta''=\eta+\theta'$. His igitur angulis introductis erit $\Delta\Delta=-\cos\theta\cos\theta'\cos\theta''$, hocque valore adhibito habebimus

$$\tan\zeta''=-\sqrt{-\frac{\cos\theta\cos\theta'}{\cos\theta''}},\quad \tan\zeta=-\sqrt{-\frac{\cos\theta'\cos\theta''}{\cos\theta}},\quad \tan\zeta'=-\sqrt{-\frac{\cos\theta''\cos\theta}{\cos\theta'}}. \tag{56}$$

§31. Ex his formulis pro tangentibus inuentis colligamus formulas pro sinibus et cosinibus, ac pro primo quidem erit

$$\sin\zeta''=-\frac{\sqrt{-\cos\theta\cos\theta'}}{\sqrt{-\cos\theta''-\cos\theta\cos\theta'}} \quad \text{et} \quad \cos\zeta''=-\frac{\sqrt{\cos\theta''}}{\sqrt{\cos\theta''-\cos\theta\cos\theta'}}. \tag{57}$$

Cum autem sit $\theta''=-\theta-\theta'$ erit $\cos\theta''=\cos(\theta+\theta')=\cos\theta\cos\theta'-\sin\theta\sin\theta'$, quo valore substituo fiet

$$\sin\zeta''=-\frac{\sqrt{-\cos\theta\cos\theta'}}{\sqrt{-\sin\theta\sin\theta'}}=-\sqrt{\frac{\cos\theta\cos\theta'}{\sin\theta\sin\theta'}}=-\sqrt{\cot\theta\cot\theta'} \tag{58}$$

similique modo

$$\cos\zeta''=\frac{\sqrt{\cos\theta\cos\theta'-\sin\theta\sin\theta'}}{\sqrt{-\sin\theta\sin\theta'}}=\sqrt{\frac{-\cos\theta\cos\theta'}{\sin\theta\sin\theta'}+1}=\sqrt{1-\cot\theta\cot\theta'}. \tag{59}$$

Atque hinc iam perspicuum est, pro binis reliquis angulis fore

$$\sin\zeta = -\sqrt{\cot\theta'\cot\theta''} \quad \text{et} \quad \cos\zeta = \sqrt{1-\cot\theta'\cot\theta''} \tag{60}$$
$$\sin\zeta' = -\sqrt{\cot\theta''\cot\theta} \quad \text{et} \quad \cos\zeta' = \sqrt{1-\cot\theta''\cot\theta} \tag{61}$$
$$\sin\zeta'' = -\sqrt{\cot\theta'\cot\theta'} \quad \text{et} \quad \cos\zeta'' = \sqrt{1-\cot\theta\cot\theta''}. \tag{62}$$

§32. Quod si nunc isti valores euoluti substituantur loco angulorum ζ, ζ' et ζ'', formulae pro nouem litteris F, G, H, F$'$, G$'$, H$'$ etc. supra inuentae sequenti modo exprimentur, postquam scilicet breuitatis gratia posuerimus

$$\cot\theta = t; \quad \cot\theta' = t' \quad \text{et} \quad \cot\theta'' = t'' \tag{63}$$

$$\mathrm{F} = -\sqrt{t't''}; \quad \mathrm{G} = \sin\eta\sqrt{1-t't''}; \quad \mathrm{H} = \cos\eta\sqrt{1-t't''}; \tag{64}$$
$$\mathrm{F}' = -\sqrt{t''t}; \quad \mathrm{G}' = \sin\eta'\sqrt{1-t''t}; \quad \mathrm{H}' = \cos\eta'\sqrt{1-t''t}; \tag{65}$$
$$\mathrm{F}'' = -\sqrt{tt'}; \quad \mathrm{G}'' = \sin\eta''\sqrt{1-tt'}; \quad \mathrm{H}'' = \cos\eta''\sqrt{1-tt'}. \tag{66}$$

§33. Verum etiamsi hos valores in aequatione §22. substituamus, nullo tamen modo perspicitur, quomodo singula eius membra se mutuo destruere queant. Quamobrem necesse erit, insuper eius conditionis rationem habere, quod sit $\theta+\theta'+\theta''=0$; vnde inter litteras t, t', t'' ista relatio nascitur, vt sit $tt'+t't''+tt''=1$, siue vt summa productorum ex binis vnitate aequatur. Praeterea vero etiam ad eam conditionem est attendenda, qua erat $\eta'=\eta-\theta''$ et $\eta''=\eta+\theta'$, atque his conditionibus rite obseruatis et per calculum euolutis nullum dubium superesse potest, quin ista aequatio adimpleatur. At vero nemo facile stupendum hunc laborem in se suscipere volet; quamobrem egregia ista proprietas omnium corporum rigidorum multo magis ardua est censenda, et Geometris pulcherrimam occasionem praebere potest, vires suas in ista proprietate penitus enucleanda exercendi.

General formulas for any translation[1] of rigid bodies
by Leonhard Euler (translated by Johan Sten)

§1.

When it is required to determine the motion of any rigid body, it is convenient to separate the inquiry into two parts, the one part being geometrical and the other mechanical. For in the first part only the translation of the body from a given situation to some other is present, without the body having any other motion that must be represented by analytical formulas with regard to the principles of the motion, and with the help of this, the position of these individual points can be defined, after the translation from their initial position: which investigation hence should be referred only to geometry or rather to stereometry. Moreover it is easily understood that if this investigation is separated from the other, which properly relates to Mechanics, then the determination of the motion itself is much easier to set out from the principles of motion, than if each investigation is undertaken jointly. Hence as I undertook each investigation at the same time in my [previous] discussion on the motion of rigid bodies, it followed that the whole discussion was rendered rather intricate and troublesome : in this place it has been agreed to set out the geometrical part with more care, from which the part concerned with the mechanical part can be more easily explained.

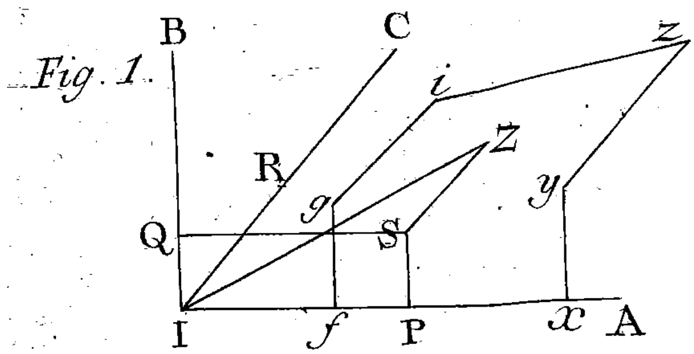
§2. First, in order to define the initial situation of the rigid body, it is customary to represent the position of a single point of it by three mutually orthogonal coordinates. To this end (see Fig. 1), I set up three fixed axes IA, IB and IC, intersecting each other normally in the point I, two of which, IA and IB, are situated in the very plane of the board, the third IC standing perpendicularly to this plane. Now consider an arbitrary point of a body Z, from which a perpendicular ZS is emitted towards the plane AIB; then from the point S to the axes IA and IB are emitted the normals SP and SQ, and so we say that the coordinates IP=QS=p, PS=IQ=q and the perpendicular SZ=r, which occupies the portion IR=r; so that the point Z is found on the diagonal IZ of a rectangular parallelepiped formed out of the three sides IA, IB, and IC. In this way, then, the position of a single point of a body is determined by means of the three coordinates p, q, r.

§3. However, to make this representation even more easily adapted to mechanical investigations, the point I is most suitably taken at the centre of gravity or rather [centre of] inertia of the rigid body in question; for in this way we obtain this desirable property, that given the mass of a body at Z equal to dM, when taken over the whole extension of the body gives

$$1^\circ. \int p\, d\mathrm{M} = 0. \qquad 2^\circ. \int q\, d\mathrm{M} = 0. \qquad 3^\circ. \int r\, d\mathrm{M} = 0. \tag{1}$$

if indeed these integrals are extended over the whole body. In addition, maximum utility is achieved if the three axes IA, IB, IC are attached to the principal axes of the body; for then

[1]A caveat must be given concerning terminology: For Euler, the word 'translatione' can mean both a linear and a rotational movement or change of a body in a broad sense. In the present translation the same word has been retained, since its significance should be clear from the context.



also the values of the three following integral formulas taken over the entire extension of the body render zero, as is clear when

$$4^\circ.\int pq\,d\mathrm{M}=0. \qquad 5^\circ.\int pr\,d\mathrm{M}=0. \qquad 6^\circ.\int qr\,d\mathrm{M}=0. \tag{2}$$

and these are noted here only in passing, since the geometrical part of the investigation by no means depends on these equations.

§4. Having already performed the translation of a body, let us first consider the location i, into which the point I of the body has been translated, and whose coordinates we say to be $\mathrm{I}f=f$, $fg=g$ and $gi=h$; then it is true that the point Z is translated from its initial location to z, for which we assign the coordinates $\mathrm{I}x=x$, $xy=y$ and $yz=z$, and it is immediately clear that the distance iz must still be equal to the distance IZ, being $\sqrt{pp+qq+rr}$ and since

$$iz=\sqrt{(x-f)^2+(y-g)^2+(z-h)^2} \tag{3}$$

we have this equation:

$$pp+qq+rr=(x-f)^2+(y-g)^2+(z-h)^2. \tag{4}$$

Moreover it is necessary that the distance between two points of any body in translation is all the time the same as in the initial state, which condition we can satisfy in the following way.

§5. Let us take a point z in the location, into which the point P has been translated from its initial state: for thus the figure we are consulting does not seem too burdened with a great number of new lines. Then also in the initial state itself the point Z can admit any place whatever; whence if the point Z would be admitted into the point P, the point z in the translated state would also furnish the corresponding location of P itself.

§6. When the point Z falls into the point P, if $q=0$ and $r=0$, because in general the tree coordinates x, y, z can be considered as certain functions of p, q and r, however these functions were to be compared, if we make $q=0$ and $r=0$ in them, these coordinates must necessarily take the following form

$$x=f+\mathrm{F}p, \qquad y=g+\mathrm{G}p, \qquad z=h+\mathrm{H}p. \tag{5}$$

For indeed, setting $q = 0$ and $r = 0$, considering p as a variable, the coordinates x, y, z should point out the location into which the straight line IP has been translated; which being straight, will be equal to the straight line in its translated position, and so the coordinates x, y, z must express the position of the straight line, whence, assuming $p = 0$ also the point z should fall into i, it is evident that the quantities x, y, z should be defined by the variable p in such a way, that the position $p = 0$ makes $x = f$, $y = g$ and $z = b$. Then truly because the equation should be for a straight line, the location cannot be anything else but as stated, that is

$$x = f + \mathrm{F}p, \qquad y = g + \mathrm{G}p, \qquad z = h + \mathrm{H}p. \tag{6}$$

where the letters F, G, H denote certain constants depending inherently of the translation.

§7. However, it is immediately clear that these constants should be in such a way compared, that the distance iz is equal to the distance IP$=p$, wherupon the following conclusion follows:

$$iz^2 = \mathrm{F}^2p^2 + \mathrm{G}^2p^2 + \mathrm{H}^2p^2 = pp \tag{7}$$

owing to which it is necessary that $\mathrm{F}^2 + \mathrm{G}^2 + \mathrm{H}^2 = 1$. But if we then take F$= \sin\zeta$, we must have $\mathrm{G}^2 + \mathrm{H}^2 = \cos^2\zeta$; from this fact we get

$$\mathrm{G} = \cos\zeta\sin\eta \qquad \text{and} \qquad \mathrm{H} = \cos\zeta\cos\eta, \tag{8}$$

so that

$$\mathrm{F} = \sin\zeta, \qquad \mathrm{G} = \cos\zeta\sin\eta \qquad \text{and} \qquad \mathrm{H} = \cos\zeta\cos\eta. \tag{9}$$

In this way, then, are these three letters F, G, H reduced to just these two angles ζ and η.

§8. In the same fashion let us assume the point z at the place where the point Q has been translated from its original position; but for the point Z to fall into the point Q it will be assumed that $p = 0$ and $r = 0$. Thus, in this case the three coordinates x, y, z depend on the sole variable q, so that having set $q = 0$ for the second time, $x = f$, $y = g$ and $z = h$; for this reason, as the equation still has to describe a straight line, the coordinates will have the following form:

$$x = f + \mathrm{F}'q, \qquad y = g + \mathrm{G}'q, \qquad z = h + \mathrm{H}'q \tag{10}$$

where thanks to $iz = q$ it is required also that $\mathrm{F}'\mathrm{F}' + \mathrm{G}'\mathrm{G}' + \mathrm{H}'\mathrm{H}' = 1$ which condition is conveniently satisfied by the two angles ζ' and η', so that

$$\mathrm{F}' = \sin\zeta', \qquad \mathrm{G}' = \cos\zeta'\sin\eta', \qquad \mathrm{H}' = \cos\zeta'\cos\eta'. \tag{11}$$

§9. Let us now assume the point Z in R, which happens by setting $p = 0$ and $q = 0$, whence if the point z gives the location in which the point R is to be translated, one obtains in the same manner as before the following forms for the coordinates:

$$x = f + \mathrm{F}''r, \qquad y = g + \mathrm{G}''r, \qquad z = h + \mathrm{H}''r. \tag{12}$$

And because it is required $\mathrm{F}''\mathrm{F}'' + \mathrm{G}''\mathrm{G}'' + \mathrm{H}''\mathrm{H}'' = 1$ we can state them by means of two new angles ζ'' and η'' as

$$\mathrm{F}'' = \sin\zeta'', \qquad \mathrm{G}'' = \cos\zeta''\sin\eta'' \qquad \text{and} \qquad \mathrm{H}'' = \cos\zeta''\cos\eta''. \tag{13}$$

§10. Now since we have obtained the coordinates x, y and z, which should provide us with the three cases explained, where of the three quantities p, q, r two vanish, it is clear how the coordinates x, y, z, depend on each of the quantities p, q, r. Therefore, if all these letters enter simultaneously into the calculation, so that they indicate some point Z of a body, which in its translated state corresponds to the point z, then the coordinates x, y, z will have to have the following values:

$$x = f + \mathrm{F}p + \mathrm{F}'q + \mathrm{F}''r \tag{14}$$
$$y = g + \mathrm{G}p + \mathrm{G}'q + \mathrm{G}''r \tag{15}$$
$$z = h + \mathrm{H}p + \mathrm{H}'q + \mathrm{H}''r \tag{16}$$

but these nine letters have been seen to reduce to only six angles ζ, η, ζ', η', ζ'', η''.

§11. Now let us assume the point Z to be in the point s itself, so that $r = 0$, and if in the translated state this point corresponds to the point z, by putting $r = 0$ the three coordinates are as follows:

$$x = f + \mathrm{F}p + \mathrm{F}'q \qquad y = g + \mathrm{G}p + \mathrm{G}'q \qquad z = h + \mathrm{H}p + \mathrm{H}'q. \tag{17}$$

Here it is necessary to put the distance iz equal to the distance IS, which being $\sqrt{pp+qq}$ gives rise to this equation:

$$pp + qq = (\mathrm{F}p + \mathrm{F}'q)^2 + (\mathrm{G}p + \mathrm{G}'q)^2 + (\mathrm{H}p + \mathrm{H}'q)^2 \tag{18}$$

and having accomplished the expansion

$$pp + qq = pp(\mathrm{FF} + \mathrm{GG} + \mathrm{HH}) + qq(\mathrm{F'F'} + \mathrm{G'G'} + \mathrm{H'H'}) + 2pq(\mathrm{FF'} + \mathrm{GG'} + \mathrm{HH'}) \tag{19}$$

Then, since $\mathrm{FF}+\mathrm{GG}+\mathrm{HH} = 1$ and $\mathrm{F'F'}+\mathrm{G'G'}+\mathrm{H'H'} = 1$ we are left with $\mathrm{FF'}+\mathrm{GG'}+\mathrm{HH'} = 0$.

§12. In the same way it is clear that if we assume $q = 0$, then the location must have this property that $\mathrm{FF''}+\mathrm{GG''}+\mathrm{HH''} = 0$: but if we set $p = 0$, this equation $\mathrm{F'F''}+\mathrm{G'G''}+\mathrm{H'H''} = 0$ will result. These three conditions being satisfied, the entire translation is determined; And our formulas give the same distances between the points of any body in translation as they had in their initial position.

§13. Let us now substitute in these equations the values invented above, first $\mathrm{FF'}+\mathrm{GG'}+\mathrm{HH'} = 0$ will give

$$\sin\zeta\sin\zeta' + \cos\zeta\cos\zeta'\sin\eta\sin\eta' + \cos\zeta\cos\zeta'\cos\eta\cos\eta' = 0 \tag{20}$$

or

$$\sin\zeta\sin\zeta' + \cos\zeta\cos\zeta'\cos(\eta - \eta') = 0 \quad \text{for} \quad \cos\eta\cos\eta' + \sin\eta\sin\eta' = \cos(\eta - \eta') \tag{21}$$

which equation divided by $\cos\zeta\cos\zeta'$ render $\tan\zeta\tan\zeta' = -\cos(\eta - \eta')$. In the same way the remaining equations give

$$\tan\zeta'\tan\zeta'' = -\cos(\eta' - \eta'') \qquad \text{and} \qquad \tan\zeta''\tan\zeta = -\cos(\eta'' - \eta). \tag{22}$$

From these three equations, then, the three angles ζ, ζ' and ζ'' may be determined, so that everything can be defined using three angles η, η' and η''.

§14. But as can be easily done, let us multiply these three equations with each other, which gives

$$\tan^2\zeta\tan^2\zeta'\tan^2\zeta'' = -\cos(\eta-\eta')\cos(\eta'-\eta'')\cos(\eta''-\eta). \tag{23}$$

From this it is immediately clear that if not the product of these three cosines be negative, the case is impossible; on this account it is first and foremost necessary, that of these cosines either one or all three are negative. For example, let us put

$$\cos(\eta-\eta')\cos(\eta'-\eta'')\cos(\eta''-\eta) = -\Delta\Delta, \tag{24}$$

so that $\tan\zeta\tan\zeta'\tan\zeta'' = \Delta$ is obtained, which equation divided by each of the precending ones supplies us with the values

$$\tan\zeta'' = \frac{-\Delta}{\cos(\eta-\eta')};\quad \tan\zeta = \frac{-\Delta}{\cos(\eta'-\eta'')};\quad \tan\zeta' = \frac{-\Delta}{\cos(\eta''-\eta)} \tag{25}$$

in this way, then, all nine coefficients initially assumed F, G, H, F′, G′, H′, F″, G″, H″, can be determined by only three angles η, η', η'' in such a way:

$$\mathrm{F} = \sin\zeta,\quad \mathrm{G} = \cos\zeta\sin\eta,\quad \mathrm{H} = \cos\zeta\cos\eta \tag{26}$$

$$\mathrm{F}' = \sin\zeta',\quad \mathrm{G}' = \cos\zeta'\sin\eta',\quad \mathrm{H}' = \cos\zeta'\cos\eta' \tag{27}$$

$$\mathrm{F}'' = \sin\zeta'',\quad \mathrm{G}'' = \cos\zeta''\sin\eta'',\quad \mathrm{H}'' = \cos\zeta''\cos\eta'' \tag{28}$$

§15. Every translation, then, which can change the situation of a rigid body, can be determined by means of six elements. Indeed, first the three coordinates f, g, h determine the translation of the point I into i, which therefore is ultimately dependent on our choice. Then, in whatever way the body in the meantime revolves, its situation is determined by basically three angles η, η', η''; Indeed, assuming the element Z of an arbitrary body in its initial state, whose position is defined by the three coordinates p, q, r, in its translated state it is discovered in the point z , whose position is defined by these three coordinates:

$$x = f + \mathrm{F}p + \mathrm{F}'q + \mathrm{F}''r \tag{29}$$

$$y = g + \mathrm{G}p + \mathrm{G}'q + \mathrm{G}''r \tag{30}$$

$$z = h + \mathrm{H}p + \mathrm{H}'q + \mathrm{H}''r. \tag{31}$$

§16. But to be more convinced that by means of these three formulas, all that pertains to translation can be determined completely, the whole work can be acquitted in the following manner. Let us conceive in its initial state in addition to the point Z another one Z′, though not expressed in the figure, whose location is defined by these coordinates p', q', r'. But this point translated is in z', corresponding to the coordinates x', y', z', its values are thus expressed

$$x' = f + \mathrm{F}p' + \mathrm{F}'q' + \mathrm{F}''r' \tag{32}$$

$$y' = g + \mathrm{G}p' + \mathrm{G}'q' + \mathrm{G}''r' \tag{33}$$

$$z' = h + \mathrm{H}p' + \mathrm{H}'q' + \mathrm{H}''r'. \tag{34}$$

This is then required by the nature of rigid bodies, that the distance in the translated state zz' must be equal to the distance ZZ$'$ in the initial state, seeing that in this body all distances between any two of its points must be kept forever the same quantity.

§17. Now the distance squared between the points Z and Z$'$ in the initial state is

$$(p'-p)^2+(q'-q)^2+(r'-r)^2 \tag{35}$$

while the distance squared between the points z and z' in the translated state is

$$(x'-x)^2+(y'-y)^2+(z'-z)^2, \tag{36}$$

which therefore is composed of three following squares

$$\begin{aligned}(\mathrm{F}(p'-p) &+ \mathrm{F}'(q'-q)+\mathrm{F}''(r'-r))^2\\ +(\mathrm{G}(p'-p) &+ \mathrm{G}'(q'-q)+\mathrm{G}''(r'-r))^2+(\mathrm{H}(p'-p)+\mathrm{H}'(q'-q)+\mathrm{H}''(r'-r))^2\end{aligned} \tag{37}$$

the sum of which shall be equal to this formula $(p'-p)^2+(q'-q)^2+(r'-r)^2$.

§18. By expanding these three squares the following expression results

$$\begin{aligned}&(p'-p)^2(\mathrm{FF}+\mathrm{GG}+\mathrm{HH})\\ +\;&(q'-q)^2(\mathrm{F'F'}+\mathrm{G'G'}+\mathrm{H'H'})\\ +\;&(r'-r)^2(\mathrm{F''F''}+\mathrm{G''G''}+\mathrm{H''H''})\\ +\;&2(p'-p)(q'-q)(\mathrm{FF'}+\mathrm{GG'}+\mathrm{HH'})\\ +\;&2(p'-p)(r'-r)(\mathrm{FF''}+\mathrm{GG''}+\mathrm{HH''})\\ +\;&2(q'-q)(r'-r)(\mathrm{F'F''}+\mathrm{G'G''}+\mathrm{H'H''})\end{aligned}$$

on account of which, to render equal to this expression $(p'-p)^2+(q'-q)^2+(r'-r)^2$, whatever coordinates p, q, r, p', q', r' being assumed, the following six conditions must be satisfied

$$\text{I.}\quad \mathrm{FF}+\mathrm{GG}+\mathrm{HH}=1 \tag{38}$$
$$\text{II.}\quad \mathrm{F'F'}+\mathrm{G'G'}+\mathrm{H'H'}=1 \tag{39}$$
$$\text{III.}\quad \mathrm{F''F''}+\mathrm{G''G''}+\mathrm{H''H''}=1 \tag{40}$$
$$\text{IV.}\quad \mathrm{FF'}+\mathrm{GG'}+\mathrm{HH'}=0 \tag{41}$$
$$\text{V.}\quad \mathrm{FF''}+\mathrm{GG''}+\mathrm{HH''}=0 \tag{42}$$
$$\text{VI.}\quad \mathrm{F'F''}+\mathrm{G'G''}+\mathrm{H'H''}=0. \tag{43}$$

§19. But truly we have already fulfilled these six conditions above, where we showed how all these nine coefficients could be determined via the three angles η, η', η''. From which it is clearly understood that our solution to the question of any translation of a rigid body is thoroughly determined and adequate, so that what the geometrical part requires for determining the change of such a body, nothing further can be asked for.

§20. However the body is moved, through which the point I of a body is translated into i, it is known that if the translation be infinitely small, then it is always given in the translated state some straight line iz, whose location is parallel with that which the same line had in the initial state, so that if the point I was at rest, this line ultimately stays unchanged. But it is evident that this line represents the axis around which a rotation has been made, while the body reaches its translated state. On this account it is of greatest importance to investigate, whether if the translation was finished, such an axis is also given.

§21. It is clear, however, that for the line iz to be parallel with the line IZ, the following three conditions are required:

$$1^\circ . x - f = p, \qquad 2^\circ . y - g = q, \qquad 3^\circ . z - h = r \tag{44}$$

giving rise to these equations:

$$\begin{aligned} p &= \mathrm{F}p + \mathrm{F}'q + \mathrm{F}''r & (45)\\ q &= \mathrm{G}p + \mathrm{G}'q + \mathrm{G}''r & (46)\\ r &= \mathrm{H}p + \mathrm{H}'q + \mathrm{H}''r & (47) \end{aligned}$$

from which equations the letters p, q, r ought to be eliminated. But the values of p are deduces as

$$\frac{\mathrm{F}'q + \mathrm{F}''r}{1 - \mathrm{F}}, \quad \frac{(1 - \mathrm{G}')q - \mathrm{G}''r}{\mathrm{G}}, \quad \frac{(1 - \mathrm{H}'')r - \mathrm{H}'q}{\mathrm{H}} \tag{48}$$

Equating the first of these values with the second one gives the following ratio between q and r, namely

$$\frac{q}{r} = \frac{\mathrm{G}''(\mathrm{F} - 1) - \mathrm{F}''\mathrm{G}}{\mathrm{G}\mathrm{F}' - (1 - \mathrm{F})(1 - \mathrm{G}')} \tag{49}$$

and equating the first with the third leads to this relation:

$$\frac{q}{r} = \frac{(1 - \mathrm{F})(1 - \mathrm{H}'') - \mathrm{F}''\mathrm{H}}{\mathrm{F}'\mathrm{H} + \mathrm{H}'(1 - \mathrm{F})}. \tag{50}$$

These two values should therefore truly be equal to each other, if indeed such an axis of rotation is given.

§22. But if we put these two values equal to each other, we arrive at the following equation:

$$\begin{aligned} & (1-\mathrm{F})\mathrm{F}''\mathrm{G}\mathrm{H}' + (1-\mathrm{F})\mathrm{F}'\mathrm{G}''\mathrm{H} + (1-\mathrm{F})(1-\mathrm{G}')\mathrm{F}''\mathrm{H} + (1-\mathrm{F})(1-\mathrm{H}'')\mathrm{F}'\mathrm{G} \\ + \;& (1-\mathrm{F})^2\mathrm{G}''\mathrm{H}' - (1-\mathrm{F})^2(1-\mathrm{G}')(1-\mathrm{H}'') = 0 \end{aligned} \tag{51}$$

whose every member enjoy the common factor (1−F); this is now done away with by division to leave this equation:

$$\mathrm{F}''\mathrm{G}\mathrm{H}' + \mathrm{F}'\mathrm{G}''\mathrm{H} + (1-\mathrm{G}')\mathrm{F}''\mathrm{H} + (1-\mathrm{H}'')\mathrm{F}'\mathrm{G} + (1-\mathrm{F})(\mathrm{G}''\mathrm{H}' - (1-\mathrm{G}')(1-\mathrm{H}'')) = 0 \tag{52}$$

whose every member expanded gives the equation[2]

$$0 = -1 + \mathrm{F} \quad - \quad \mathrm{F}\mathrm{G}' + \mathrm{F}\mathrm{G}'\mathrm{H}''$$

[2] In the original manuscript, the few last terms of this long expression were in error and are corrected in this translation (J.S.)

$$
\begin{aligned}
+\mathrm{G}' \quad &- \quad \mathrm{FH}'' - \mathrm{FG}''\mathrm{H}' \\
+\mathrm{H}'' \quad &+ \quad \mathrm{F}'\mathrm{G} - \mathrm{F}'\mathrm{GH}'' \\
&- \quad \mathrm{G}'\mathrm{H}'' - \mathrm{F}''\mathrm{G}'\mathrm{H} \\
&+ \quad \mathrm{G}''\mathrm{H}' + \mathrm{F}'\mathrm{G}''\mathrm{H} \\
&+ \quad \mathrm{F}''\mathrm{H} + \mathrm{F}''\mathrm{GH}'. \qquad (53)
\end{aligned}
$$

It is not clear, however, how this expression is rendered zero; besides it would have been exceedingly tedious to substitute in place of the letters of F, G, H their final values.

§23. Leaving now this investigation, seeing that for any translation we give formulas, by means of which from the position of a given point in its initial state the position of the same point in its translated state can be assigned, the scope which we put forward being thus entirely satisfied, so that in this respect nothing more can be desired, the whole investigation being based on the determination of the 9 coefficients F, G, H, F′, G′, H′, F″, G″, H″.

Addition (supplement).

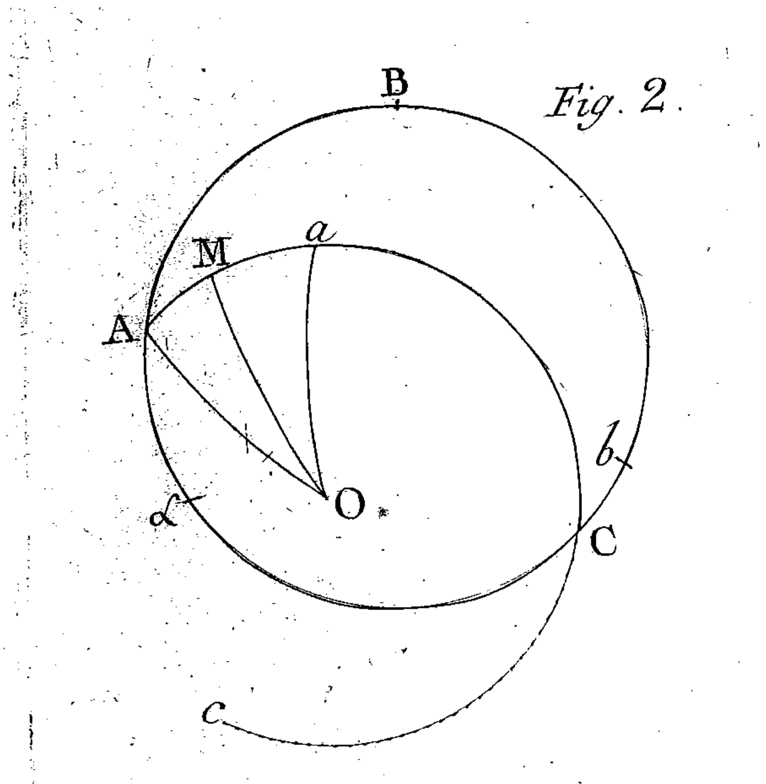
§24. Given that the formulas we gave for the translated situation are the most general and embrace all translations, it would appear remarkable, that from them it is not evident, whether in all translated states there is given such a line iz, which holds the same direction as it had initially. For the equation found in §22. is so involved, that it would be too tiresome to substitute for each of the letters the values we assigned. Yet from another point it is at the same time certain, that however a rigid body is transferred from one point to another, such a line iz is always given, whose direction does not suffer any change. Indeed, to show this let us conceive a rigid body, whatever be its figure, connected with a circumscribed sphere moving about with it, centred at the point I, which easily lets us bring this investigation into the theory of the sphere [=spherical trigonometry].

Theorem.

In whatever way a sphere is turned about its centre, it is always possible to assign a diameter, whose direction in the translated state agrees with that of the initial state.

Demonstration.

§25. (See Fig. 2) The circle A, B, C refers to a great circle in its initial state, which after the translation arrive at a, b, c, so that the points A, B, C are translated to a, b, c; but the point A is at the same time the intersection between these two circles. What is to be shown is that there is always given a point O, which equally refers to the circle A, B, C as to the circle a, b, c. Now for this to hold it is necessary, that first of all the distances OA and Oa are equal to each other; second, also that the arcs OA and Oa are similarly inclined towards the two circles, that is, so that the angle Oab=angle OAB: now since they are the complementary angles of two right angles, that is the angles OaA and OAα are equal to each other. But since the arcs Oa and OA are equal, likewise the angle OaA=angle OAa, whence OAa=OAα; from this it clear that, if the angle aAα bisects the arc OA, then the point O searched for must be situated somewhere on this arc AB; which, then, is found if the arc aO is drawn so that the angle AaO be equal to the angle OAa. In fact, the intersection these two arcs gives the point O, through which if a diameter of the sphere be drawn, its situation in the translated state would be the same as it had in the initial state.



§26. To determine this point O more easily, the arc Aa can be bisected in the point M, where the arc MO is placed normal to Aa; then draw the arc AO so that the angle aAα is bisected; and the intersection O of these arcs gives the point searched for. Here it is observed that if the arc $a\alpha$ is understood as being equal to the arc aA, then α would be a point on the sphere, which after the translation would arrive at A, on account of which it is the angle aAα that ought to be bisected, not truly its future position αAB.

§27. Indeed, generally the point I (See Fig. 1) to which the initial position of the body is referred, is taken to be at the centre of gravity of the body. However, from the given proof it is apparent, that the truth of the theorem still subsists, however many other points were taken as the centre of the sphere. Owing to this fact, if whatever point I be taken in a rigid body, a straight line can always be drawn through it, whose position in its translated state will not be changed; why not even, as nothing prevents it, that this point I would be taken from outside the body. From this one must be careful not to regard this outstanding property as just a property of the centre of gravity: therefore, indeed, only when the point I is placed in the centre of gravity, the analytical formulas which define the motion of such a body happen to become simpler.

§28. As it thus has been seen by the most solid reasoning, that in every translated situation there is always given such a straight line iz, whose direction does not differ from that which the same line IZ held in its initial situation, also we can always have the place as would be given by the equations of §22, namely after that the assigned value of every letter has been substituted; indeed this turns out to be necessary in order that all the terms will destroy each other mutually by themselves, even if this by no means appears from the six principal conditions, that ought to be satisfied. On this account this excellent property, the truth of which is so easily shown geometrically will be most hidden by rules of analytical formulas; and owing to this we can anticipate the rules themselves to be the most important advancement for the whole science of mechanics.

§29. Yet at the same time we develop more carefully the formulas for the capital letters which we found above, from which perhaps it can be seen more easily, how the equation given in §22 is fulfilled. By introducing then the six angles ζ, ζ', ζ''; η, η through which we express those of §14, the three former by the latter, so setting

$$-\cos(\eta-\eta')\cos(\eta'-\eta'')\cos(\eta''-\eta)=\Delta\Delta \qquad \text{would give} \tag{54}$$

$$\tan\zeta''=\frac{-\Delta}{\cos(\eta-\eta')};\quad \tan\zeta=\frac{-\Delta}{\cos(\eta'-\eta'')};\quad \tan\zeta'=\frac{-\Delta}{\cos(\eta''-\eta)}; \tag{55}$$

from which the sine as well as the cosine of these angles must now be deduced.

§30. To make this more easy, let us introduce instead of the angles η, η' and η'' other angles θ, θ', θ'', such that $\eta-\eta'=\theta''$; $\eta'-\eta''=\theta$ and $\eta''-\eta=\theta'$, from which it clearly follows that $\theta+\theta'+\theta''=0$, so that these three new angles will be equivalent to only two; and hence one of the angles η, η' and η'' stays indefinite, which if it was η would give $\eta'=\eta-\theta''$ and $\eta''=\eta+\theta'$. Introducing now these values gives $\Delta\Delta=-\cos\theta\cos\theta'\cos\theta''$, and using this value we have

$$\tan\zeta''=-\sqrt{-\frac{\cos\theta\cos\theta'}{\cos\theta''}},\quad \tan\zeta=-\sqrt{-\frac{\cos\theta'\cos\theta''}{\cos\theta}},\quad \tan\zeta'=-\sqrt{-\frac{\cos\theta''\cos\theta}{\cos\theta'}}. \tag{56}$$

§31. From these formulas discovered for the tangents we assemble the formulas for the sines and cosines, and so for the first one they are

$$\sin\zeta''=-\frac{\sqrt{-\cos\theta\cos\theta'}}{\sqrt{-\cos\theta''-\cos\theta\cos\theta'}}\quad\text{and}\quad \cos\zeta''=-\frac{\sqrt{\cos\theta''}}{\sqrt{\cos\theta''-\cos\theta\cos\theta'}}. \tag{57}$$

But as $\theta''=-\theta-\theta'$, then $\cos\theta''=\cos(\theta+\theta')=\cos\theta\cos\theta'-\sin\theta\sin\theta'$, which value subsituted gives

$$\sin\zeta''=-\frac{\sqrt{-\cos\theta\cos\theta'}}{\sqrt{-\sin\theta\sin\theta'}}=-\sqrt{\frac{\cos\theta\cos\theta'}{\sin\theta\sin\theta'}}=-\sqrt{\cot\theta\cot\theta'} \tag{58}$$

and in the same way

$$\cos\zeta''=\frac{\sqrt{\cos\theta\cos\theta'-\sin\theta\sin\theta'}}{\sqrt{-\sin\theta\sin\theta'}}=\sqrt{\frac{-\cos\theta\cos\theta'}{\sin\theta\sin\theta'}+1}=\sqrt{1-\cot\theta\cot\theta'}. \tag{59}$$

And hence it is very clear, that for the two remaining angles

$$\sin\zeta = -\sqrt{\cot\theta'\cot\theta''} \quad \text{and} \quad \cos\zeta = \sqrt{1-\cot\theta'\cot\theta''} \tag{60}$$
$$\sin\zeta' = -\sqrt{\cot\theta''\cot\theta} \quad \text{and} \quad \cos\zeta' = \sqrt{1-\cot\theta''\cot\theta} \tag{61}$$
$$\sin\zeta'' = -\sqrt{\cot\theta'\cot\theta'} \quad \text{and} \quad \cos\zeta'' = \sqrt{1-\cot\theta\cot\theta''}. \tag{62}$$

§32. Now if these values expanded in the place of the angles ζ, ζ' and ζ'', formulas for the nine letters F, G, H, F′, G′, H′ etc. derived above are expressed in the following way, namely after putting for brevity

$$\cot\theta = t; \quad \cot\theta' = t' \quad \text{and} \quad \cot\theta'' = t'' \tag{63}$$

$$\mathrm{F} = -\sqrt{t't''}; \quad \mathrm{G} = \sin\eta\sqrt{1-t't''}; \quad \mathrm{H} = \cos\eta\sqrt{1-t't''}; \tag{64}$$
$$\mathrm{F}' = -\sqrt{t''t}; \quad \mathrm{G}' = \sin\eta'\sqrt{1-t''t}; \quad \mathrm{H}' = \cos\eta'\sqrt{1-t''t}; \tag{65}$$
$$\mathrm{F}'' = -\sqrt{tt'}; \quad \mathrm{G}'' = \sin\eta''\sqrt{1-tt'}; \quad \mathrm{H}'' = \cos\eta''\sqrt{1-tt'}. \tag{66}$$

§33. Even if we truly were to substitute these values into the equation §22 it still is not clear how each of its members can destroy each other. For this reason it becomes necessary to have in addition a condition, which is $\theta + \theta' + \theta'' = 0$; from which there arises between the letters t, t', t'' the relation that $tt' + t't'' + tt'' = 1$, or that the sum of the products of two of them is unity. In addition to this condition there also applies that $\eta' = \eta - \theta''$ and $\eta'' = \eta + \theta'$, and by duly observing these conditions while developing them by calculus there cannot possibly be left any doubt, that this equation will be fulfilled. But truly nobody who is easily stunned will undertake this work; on this account this extraordinary property of all rigid bodies is judged as much more difficult, and can provide the best opportunity for the geometers to exercise their powers by thoroughly explaining this property.