CONCERNING THE USE OF A NEW ALGORITHM IN SOLVING PELLIAN PROBLEMS

*E*323

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1. Whatever whole numbers may be assumed for the letters *l*, *m* and *n*, also innumerable whole numbers can be found for the letter x , by which this formula $lxx+mx+n$ is returned square, if indeed the following conditions may have a place :

I. so that l shall be a positive non-square number,

II. so that at least one known value shall be known for *x*.

For if l shall be either negative or square, it is evident infinitely many solutions in whole numbers cannot be shown, even if one may be known. Then truly also it can eventuate, that the formula $lxx+mx+n$ may be quite adverse to the nature of being square, as happens in the case $3xx+2$. But truly once a single solution may be found, it will be allowed always to find innumerable others.

2. Whereby if we may put

$$lxx+mx+nyy=yy$$

and one case will be put in place, so that this condition may be satisfied, thus so that on putting $x=a$ there may be produced:

$$laa+ma+n=bb$$

and thus by taking $x=a$ there may be obtained $y=b$, of which with the aid of several rules, itself thus can have an infinite number of solutions to be elicited:

In the first place from the given number l , two numbers of this kind *p* and *q* may be investigated, so that there shall be

$$pp=lqq+l \text{ or } p=\sqrt{(lqq+l)},$$

from which found from the first known solution this new solution is deduced at once [See *E*29 and *E*279, esp. § 13, in this series of translations]:

$$x= pa+qb+\tfrac{m}{2l}\left(p-1\right),$$

from which there becomes

$$y= pb+lqa+\tfrac{1}{2}mq ,$$

from which the others may be derived in a similar manner. For if we may substitute these values in place of a and *b*, this same solution arises,

$$x=(2pp-1)a+2pqb+mqq$$

and

$$y=(2pp-1)b+2lpqa+mpq,$$

which certainly is in whole numbers, if perhaps the preceding ones were fractions at this point.

3. Therefore since in this manner new solutions may be able to be found continually, it helps for a shortening of the calculation to be noting these values continually, both of x as well as of y themselves, following the recurring progression, of which the individual terms certainly will be determined by a sure and constant law from the two preceding terms. Evidently if the values of these two continually progressions were

$$\text{of } x: a, \ . \ . \ . \ P,\ Q,\ R,\ S \ \text{ etc.},$$
$$\text{of } y: b, \ . \ . \ . \ \mathfrak{P},\ \mathfrak{Q},\ \mathfrak{R},\ \mathfrak{S} \ \text{ etc.},$$

by the law of the recurrent series there will be

$$R=2pQ-P+\tfrac{m(p-1)}{l}, \quad \mathfrak{R}=2p\mathfrak{Q}-\mathfrak{P},$$
$$S=2pR-Q+\tfrac{m(p-1)}{l}, \quad \mathfrak{S}=2p\mathfrak{R}-\mathfrak{Q}.$$

And hence these values for the general expressions can be understood, which thus themselves will have :

$$x=\tfrac{2la+m+2b\sqrt{l}}{4l}\left(p+q\sqrt{l}\right)^{\mu}+\tfrac{2la+m-2b\sqrt{l}}{4l}\left(p-q\sqrt{l}\right)^{\mu}-\tfrac{m}{2l},$$
$$y=\tfrac{2la+m+2b\sqrt{l}}{4\sqrt{l}}\left(p+q\sqrt{l}\right)^{\mu}-\tfrac{2la+m-2b\sqrt{l}}{4\sqrt{l}}\left(p-q\sqrt{l}\right)^{\mu},$$

from which, whatever whole numbers may be attributed to the exponent μ, rational values will result always for x and y.

4. But this investigation thus can be extended much wider, so that with the proposed equation of this kind between the two numbers x and y :

$$Axx+2Bxy+Cyy+2Dx+2Ey+F=0,$$

all the solutions shall be elicited in terms of rational numbers and integers. Indeed here it is equally necessary that one solution be known, which shall be $x=a$ and $y=b$, thus so that there shall be :

$$Aaa+2Bab+Cbb+2Da+2Eb+F=0.$$

Then truly two numbers p and q may be sought, so that there shall be

$$pp=\left(BB-AC\right)qq+1,$$

which indeed cannot happen, unless there shall be $BB > AC$. And the new solution will be prepared thus :

$$x = a\left(p+Bq\right)+bCq+Eq+\tfrac{BE-CD}{BB-AC}\left(p-1\right),$$
$$y = b\left(p-Bq\right)-aAq-Dq+\tfrac{BD-AE}{BB-AC}\left(p-1\right),$$

from which more will be allowed to be produced continually by the same law.

5. This therefore is seen to become central, so that it may be understood for all the resolutions of this kind that are generally required, so that for any proposed non-square positive integer of this kind l , two equally whole numbers p and q may be able to be found, so that there shall be

$$pp = lqq+1 \text{ or } p = \sqrt{(lqq+1)}\,.$$

And this is that other former problem indeed especially celebrated called Pellian after the most ingenious solution by the author Pell, where for some number l of this kind the square number qq is required, which multiplied by l and with one added on may become another square number. Indeed with fractions this sought will be had with no difficulty, since on taking $q = \frac{2rs}{lss-rr}$ there shall become $p = \frac{lss+rr}{lss-rr}$; truly since whole numbers are desired, there again the matter may be resolved, so that the denominator $lss - rr$ may be changed into one.
[Note: Euler was in error in attributing this problem to Pell, which originally arose from Fermat's work; solutions originally were obtained by Brouncker and Wallis: see *Wallis*, *A Treatise of Algebra*, Ch. 98-99.]

6. Even if moreover the Pellian solution of this problem shall be the most elegant, yet most often the implementation of this solution is so very tedious, which are accustomed to be created not only with great tedium but also with great labor. Therefore since I may have considered that new algorithm, of which I have established the nature quite recently, these calculations for which there is a need here, taken together supply significant aid, certainly it will scarcely be allowed to show a clearer example, whereby the use of this algorithm may be illustrated and recommended. Where that happens to be especially noteworthy, so that the whole saving supplied to the calculation thence may be due mainly to the use of suitable signs.

7. Operations, in which the Pell method is used, indeed are known well enough from elsewhere and now I will take the opportunity to describe these further in another way ; from which the work is less, so that I may linger here with these requiring to be explained anew, since I shall soon be going to establish the whole analysis here in another long account. Evidently the principle of this I draw from this source, so that, since there shall be $pp = lqq+1$, there may become approximately $\frac{p}{q} = \sqrt{l}$, from which it is evident $\frac{p}{q}$ to be a fraction of this same kind, which irrational value of $\sqrt{l}$ may express so very closely or it may exceed that so little, so that it cannot be made more accurate except by using

larger numbers. The solution to which problem at one time was found happily by Wallis; indeed now I too have extracted the same by continued fractions much more conveniently a little while ago.

8. Therefore so that I may deal with this argument more clearly and in order, I will show how to set out the square root for some number as a continued fraction, and with the least trouble by that method. Then I will show, how thence the fractions $\frac{p}{q}$ must be formed expressing the irrational value to $\sqrt{l}$ approximately by calling in the aid from the new algorithm explained above. Then truly it will be readily apparent, how hence the numbers p and q may be required to be defined, so that there may become $pp = lqq + 1$. Finally I shall add a table, in which for all of the numbers l less than one hundred no greater values of the two numbers p and q are shown.

THE ESTABLISHMENT OF SQUARE ROOTS BY CONTINUED FRACTIONS

9. To this end, the operations required to be put in place will be explained most easily by an example. Therefore the square root proposed shall be from the number 13, and since the closest rational root shall be 3, there may be put

$$\sqrt{13} = 3 + \frac{1}{a}.$$

Hence there is deduced

$$a = \frac{1}{\sqrt{13}-3} = \frac{\sqrt{13}+3}{4},$$

the value of which in terms of the closest smaller whole number is 1, which thence is apparent, if 3 may be written in place $\sqrt{13}$. And thus I may put

$$a = \frac{\sqrt{13}+3}{4} = 1 + \frac{1}{b}$$

and hence

$$b = \frac{4}{\sqrt{13}-1} = \frac{4\left(\sqrt{13}+1\right)}{12} = \frac{\sqrt{13}+1}{3} = 1 + \frac{1}{c},$$

hence

$$c = \frac{3}{\sqrt{13}-2} = \frac{3\left(\sqrt{13}+2\right)}{9} = \frac{\sqrt{13}+2}{3} = 1 + \frac{1}{d},$$

hence

$$d = \frac{3}{\sqrt{13}-1} = \frac{3\left(\sqrt{13}+1\right)}{12} = \frac{\sqrt{13}+1}{} = 1 + \frac{1}{e},$$

hence

$$e = \frac{4}{\sqrt{13}-3} = \frac{4\left(\sqrt{13}+3\right)}{4} = \sqrt{13}+3 = 6 + \frac{1}{f},$$

hence

$$f = \frac{1}{\sqrt{13}-3} = \frac{\sqrt{13}+3}{4} = 1 + \frac{1}{g};$$

and here the operation is allowed to be interrupted, since the value f produced itself will be equal to *a* and thus the following are repeated in the same order. And thus we have found to be

$$\sqrt{13}=3+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\text{etc.}}}}}}}}}}}$$

10. So that the nature of these operations may be examined better, I shall add an example demanding a more extended calculation. Proposed as it were shall be $\sqrt{61}$; of which 7 shall be the nearby smaller value, I put

$$\sqrt{61}=7+\frac{1}{a}$$

and operations are required to be put in place in the following manner:

$$\text{I.}\quad a=\frac{1}{\sqrt{61}-7}=\frac{\sqrt{61}+7}{12}=1+\frac{1}{b},$$

$$\text{II.}\quad b=\frac{1}{\sqrt{61}-5}=\frac{12\left(\sqrt{61}+5\right)}{36}=\frac{\left(\sqrt{61}+5\right)}{3}=4+\frac{1}{c},$$

$$\text{III.}\quad c=\frac{3}{\sqrt{61}-7}=\frac{3\left(\sqrt{61}+7\right)}{12}=\frac{\sqrt{61}+7}{4}=3+\frac{1}{d},$$

$$\text{IV.}\quad d=\frac{4}{\sqrt{61}-5}=\frac{4\left(\sqrt{61}+5\right)}{36}=\frac{\sqrt{61}+5}{9}=1+\frac{1}{e},$$

$$\text{V.}\quad e=\frac{9}{\sqrt{61}-4}=\frac{9\left(\sqrt{61}+4\right)}{45}=\frac{\sqrt{61}+4}{5}=2+\frac{1}{f},$$

$$\text{VI.}\quad f=\frac{5}{\sqrt{61}-6}=\frac{5\left(\sqrt{61}+6\right)}{25}=\frac{\sqrt{61}+6}{5}=2+\frac{1}{g},$$

$$\text{VII.}\quad g=\frac{5}{\sqrt{61}-4}=\frac{5\left(\sqrt{61}+4\right)}{45}=\frac{\sqrt{61}+4}{9}=1+\frac{1}{h},$$

$$\text{VIII.}\quad h=\frac{9}{\sqrt{61}-5}=\frac{9\left(\sqrt{61}+5\right)}{36}=\frac{\sqrt{61}+5}{4}=3+\frac{1}{i},$$

$$\text{IX.}\quad i=\frac{4}{\sqrt{61}-7}=\frac{4\left(\sqrt{61}+7\right)}{12}=\frac{\sqrt{61}+7}{3}=4+\frac{1}{k},$$

$$\text{X.}\quad i=\frac{3}{\sqrt{61}-5}=\frac{12\left(\sqrt{61}+7\right)}{36}=\frac{\sqrt{61}+5}{12}=1+\frac{1}{l},$$

$$\text{XI.}\quad l=\frac{12}{\sqrt{61}-7}=\frac{12\left(\sqrt{61}+7\right)}{12}=\sqrt{61}+7=14+\frac{1}{m},$$

$$\text{XII.}\quad m=\frac{1}{\sqrt{61}-7},$$

therefore $m = a$ and hence again $n = b,\ o = c$ etc. From which the indices for the continued fraction will be:

7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2 etc.

nor is there a need to show this continued fraction here.

11. At this point it will help to be adding another example, where the number of indices shall be odd before the same repeat. Let this example be

$$\sqrt{67} = 8 + \tfrac{1}{a}$$

and it will be required to put in place the following operations:

$$\begin{aligned}
&\text{I.}\ a = \tfrac{1}{\sqrt{67}-8} = \tfrac{\sqrt{67}+8}{3} = 5 + \tfrac{1}{b},\\
&\text{II.}\ b = \tfrac{3}{\sqrt{67}-7} = \tfrac{3\left(\sqrt{67}+8\right)}{18} = \tfrac{\sqrt{67}+7}{6} = 2 + \tfrac{1}{c},\\
&\text{III.}\ c = \tfrac{6}{\sqrt{67}-5} = \tfrac{6\left(\sqrt{67}+5\right)}{42} = \tfrac{\sqrt{67}+5}{7} = 1 + \tfrac{1}{d},\\
&\text{IV.}\ d = \tfrac{7}{\sqrt{67}-2} = \tfrac{7\left(\sqrt{67}+2\right)}{63} = \tfrac{\sqrt{67}+2}{9} = 1 + \tfrac{1}{e},\\
&\text{V.}\ e = \tfrac{9}{\sqrt{67}-7} = \tfrac{9\left(\sqrt{67}+7\right)}{18} = \tfrac{\sqrt{67}+7}{2} = 7 + \tfrac{1}{f},\\
&\text{VI.}\ f = \tfrac{2}{\sqrt{67}-7} = \tfrac{2\left(\sqrt{67}+7\right)}{18} = \tfrac{\sqrt{67}+7}{9} = 1 + \tfrac{1}{g},\\
&\text{VII.}\ g = \tfrac{9}{\sqrt{67}-2} = \tfrac{9\left(\sqrt{67}+2\right)}{63} = \tfrac{\sqrt{67}+2}{7} = 1 + \tfrac{1}{h},\\
&\text{VIII.}\ h = \tfrac{7}{\sqrt{67}-5} = \tfrac{7\left(\sqrt{67}+5\right)}{42} = \tfrac{\sqrt{67}+5}{6} = 2 + \tfrac{1}{i},\\
&\text{IX.}\ i = \tfrac{6}{\sqrt{67}-7} = \tfrac{6\left(\sqrt{67}+7\right)}{18} = \tfrac{\sqrt{67}+7}{3} = 5 + \tfrac{1}{k},\\
&\text{X.}\ k = \tfrac{3}{\sqrt{67}-8} = \tfrac{3\left(\sqrt{67}+8\right)}{3} = \sqrt{67}+8 = 16 + \tfrac{1}{l},\\
&\text{XI.}\ k = \tfrac{1}{\sqrt{67}-8},
\end{aligned}$$

therefore $l = a$ and thence the indices b, c, d etc. recur in that order ; whereby the indices of the continued fraction sought shall be

8, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16 etc.

12. With these examples considered properly we will be able now to describe the operations in general, by which for the square root of any number a continued fraction equal to itself, or the indices constituting that, are found. Evidently the number proposed *z* and its approximate smaller integer root *v*, but truly this may be expressed by the continued fraction

$$\sqrt{z}=v+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{etc.}}}}}$$

of which the indices a, b, c, d etc., after the first *v* will have been found by the following operations:

There may be taken	then truly	and there becomes
I. $A=v,$	$\alpha=z-AA=z-vv$	$a<\frac{v+A}{\alpha},$
II. $B=\alpha a-A,$	$\beta=\frac{z-BB}{\alpha}=1+a(A-B)$	$b<\frac{v+B}{\beta},$
III. $C=\beta b-B,$	$\gamma=\frac{z-CC}{\beta}=\alpha+b(B-C)$	$c<\frac{v+C}{\gamma},$
IV. $D=\gamma c-C,$	$\delta=\frac{z-DD}{\gamma}=\beta+c(C-D)$	$d<\frac{v+D}{\delta},$
V. $E=\delta d-D,$	$\varepsilon=\frac{z-EE}{\delta}=\gamma+d(D-E)$	$e<\frac{v+E}{\varepsilon},$
	etc.	

where in the last column the sign < indices for the letters *a, b, c, d* etc. the nearest smaller whole numbers must be added to the fractions, unless these fractions themselves may become whole numbers, in which case these numbers themselves will become the indices.

13. Therefore for the indices a, b, c, d etc. required to be elicited it will be necessary to investigate two other series of numbers, of which the first to be designated by the capital letters

$$A, B, C, D \text{ etc.},$$

truly the latter by the greek letters

$$\alpha,\ \beta,\ \gamma,\ \delta\ \text{ etc.}$$

Concerning the first numbers I observe those numbers at no time can be greater than the number *v*. Indeed the first of these is $A=v$, but since there shall be $a<\frac{v+A}{\alpha}$, there will become $\alpha a-A<v$ and thus $B<v$ or at most $B=v$, in which case there becomes $\beta=1$ and $b=2v.$ Then on account of $b<\frac{v+B}{\beta}$ there becomes $\beta b-B=C<v$ and in a similar manner $D<v,\ E<v$ etc., thus so that none of these numbers may be produced greater than *v*. Then besides the case is clear, by which some of these greek letters shall be unity, all the indices *a, b, c* etc. cannot be greater than *v* itself, since the numerators in the fractions $\frac{v+B}{\beta},\ \frac{v+C}{\gamma}$ etc. cannot exceed 2*v*, truly the denominators for a minimum

shall be $=2$. Finally when it may arrive at the index $=2v$, the sequence will arise in turn *a, b, c, d* etc.

14. Also we may illustrate these operations with some examples.

I. Let $z=31$; there will be $v=5$.

$A=5$,	$\alpha=6$,	$a<\frac{10}{6}=1$,
$B=6-5=1$,	$\beta=1+1\cdot4=5$,	$b<\frac{6}{5}=1$,
$c=5-1=4$,	$\gamma=6-1\cdot3=3$,	$c<\frac{9}{3}=3$,
$D=9-4=5$,	$\delta=5-3\cdot1=2$,	$d<\frac{10}{2}=5$,
$E=10-5=5$,	$\varepsilon=3+5\cdot0=3$,	$e<\frac{10}{3}=3$,
$F=9-5=4$,	$\xi=2+3\cdot1=5$,	$f<\frac{9}{5}=1$,
$G=5-4=1$,	$\eta=3+1\cdot3=6$,	$g<\frac{6}{6}=1$,
$H=6-1=5$,	$\theta=5-1.4=1$,	$h<\frac{10}{1}=10$.

II. Let $z=46$; there will be $v=6$.

$A=6$,	$\alpha=10$,	$a<\frac{12}{10}=1$,
$B=10-6=4$,	$\beta=1+1\cdot2=3$,	$b<\frac{10}{3}=3$,
$C=9-4=5$,	$\gamma=10-3\cdot1=7$,	$c<\frac{11}{7}=1$,
$D=7-5=2$,	$\delta=3+1\cdot3=6$,	$d<\frac{8}{6}=1$,
$E=6-2=4$,	$\varepsilon=7-1\cdot2=5$,	$e<\frac{10}{5}=2$,
$F=10-4=6$,	$\xi=6-2\cdot2=2$,	$f<\frac{12}{2}=6$,
$G=12-6=6$	$\eta=5+6\cdot0=5$,	$g<\frac{12}{5}=2$,
$H=10-6=4$,	$\theta=2+2\cdot2=6$,	$h<\frac{10}{6}=1$,
$I=6-4=2$,	$\iota=5+1\cdot2=7$,	$i<\frac{8}{7}=1$,
$K=7-2=5$,	$\chi=6-1\cdot3=3$,	$k<\frac{11}{3}=3$,
$L=9-5=4$,	$\lambda=7+3\cdot1=10$,	$l<\frac{10}{10}=1$,
$M=10-4=6$,	$\mu=3-1\cdot2=1$,	$m<\frac{12}{1}=12$.

III. Let $z=54$; there will be $v=7$.

$$\begin{array}{lll}
A = 7, & \alpha = 5, & a < \tfrac{14}{5} = 2, \\
B = 10 - 7 = 3, & \beta = 1 + 2 \cdot 4 = 9, & b < \tfrac{10}{9} = 1, \\
C = 9 - 3 = 6, & \gamma = 5 - 1 \cdot 3 = 2, & c < \tfrac{13}{2} = 6, \\
D = 12 - 6 = 6, & \delta = 9 + 6 \cdot 0 = 9, & d < \tfrac{13}{9} = 1, \\
E = 9 - 6 = 3, & \varepsilon = 2 + 1 \cdot 3 = 5, & e < \tfrac{10}{5} = 2, \\
F = 10 - 3 = 7, & \zeta = 9 - 2 \cdot 4 = 1, & f < \tfrac{14}{1} = 14.
\end{array}$$

15. Therefore the following table for the square roots of the individual numbers contains the indices, from which the equal continued fractions themselves may be able to be formed. Likewise truly the subscripts are found to be the appropriate values of the individual greek letters.

Root of the number	Indices
$\sqrt{2}$	$1_1,\ 2_1,\ 2_1,\ 2_1$ etc.
$\sqrt{3}$	$1_1,\ 1_2,\ 2_1,\ 1_2,\ 2_1,\ 1_2,\ 2_1$ etc.
$\sqrt{5}$	$2_1,\ 4_1,\ 4_1,\ 4_1$ etc.
$\sqrt{6}$	$2_1, 2_2, 4_1, 2_2, 4_1, 2_2, 4_1$ etc.
$\sqrt{7}$	$2_1, 1_3, 1_2, 1_3, 4_1, 1_3, 1_2, 1_3, 4_1$ etc.
$\sqrt{8}$	$2_1, 1_4, 4_1, 1_4, 4_1, 1_4, 4_1$ etc.
$\sqrt{10}$	$3_1, 6_1, 6_1, 6_1$ etc.
$\sqrt{11}$	$3_1, 3_2, 6_1, 3_2, 6_1, 3_2, 6_1$ etc.
$\sqrt{12}$	$3_1, 2_3, 6_1, 2_3, 6_1, 2_3, 6_1$ etc.
$\sqrt{13}$	$3_1, 1_4, 1_3, 1_3, 1_4, 6_1, 1_4, 1_3, 1_3, 1_4, 6_1$ etc.
$\sqrt{14}$	$3_1, 1_5, 2_2, 1_5, 6_1, 1_5, 2_2, 1_5, 6_1$ etc.
$\sqrt{15}$	$3_1, 1_6, 6_1, 1_6, 6_1, 1_6, 6_1$ etc.
$\sqrt{17}$	$4_1, 8_1, 8_1, 8_1, 8_1$ etc.
$\sqrt{18}$	$4_1, 4_2, 8_1, 4_2, 8_1, 4_2, 8_1, 4_2, 8_1$ etc.

$\sqrt{19}$	4, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8 etc. 1 3 5 2 6 3 1 3 5 2 5 3 1
$\sqrt{20}$	4, 2, 8, 2, 8, 2, 8, 2, 8 etc. 1 4 1 4 1 4 1 4 1
$\sqrt{21}$	4, 1, 1, 2, 1, 1, 8, 1, 1, 2, 1, 1, 8 etc. 1 5 4 3 4 5 1 5 4 3 4 5 1
$\sqrt{22}$	4, 1, 2, 4, 2, 1, 8, 1, 2, 4, 2, 1, 8 etc. 1 6 3 2 3 6 1 6 3 2 3 6 1
$\sqrt{23}$	4, 1, 3, 1, 8, 1, 3, 1, 8 etc. 1 7 2 7 1 7 2 7 1
$\sqrt{24}$	4, 1, 8, 1, 8, 1, 8 etc. 1 8 1 8 1 8 1
$\sqrt{26}$	5, 10, 10, 10 etc. 1 1 1 1
$\sqrt{27}$	5, 5, 10, 5, 10, 5, 10 etc. 1 2 1 2 1 2 1
$\sqrt{28}$	5, 3, 2, 3, 10, 3, 2, 3, 10 etc. 1 3 4 3 1 3 4 3 1
$\sqrt{29}$	5, 2, 1, 1, 2, 10, 2, 1, 1, 2, 10 etc. 1 4 5 5 4 1 4 5 5 4 1
$\sqrt{30}$	5, 2, 10, 2, 10, 2, 10, 2, 10 etc. 1 5 1 5 1 5 1 5 1
$\sqrt{31}$	5, 1, 1, 3, 5, 3, 1, 1, 10 etc. 1 6 5 8 2 3 5 6 1
$\sqrt{32}$	5, 1, 1, 1, 10, 1, 1, 1, 10 etc. 1 7 4 7 1 7 4 7 1
$\sqrt{33}$	5, 1, 2, 1, 10, 1, 2, 1, 10 etc. 1 8 3 8 1 8 3 8 1
$\sqrt{34}$	5, 1, 4, 1, 10, 1, 4, 1, 10 etc. 1 9 2 9 1 9 2 9 1
$\sqrt{35}$	5, 1, 10, 1, 10, 1, 10 etc. 1 10 1 10 1 10 1
$\sqrt{37}$	6, 12, 12, 12 etc. 1 1 1 1
$\sqrt{38}$	6, 6, 12, 6, 12, 6, 12 etc. 1 2 1 2 1 2 1
$\sqrt{39}$	6, 4, 12, 4, 12, 4, 12 etc. 1 3 1 3 1 3 1
$\sqrt{40}$	6, 3, 12, 3, 12, 3, 12 etc. 1 4 1 4 1 4 1

$\sqrt{41}$	6, 2, 2, 12, 2, 2, 12 etc.	1 5 5 1 5 5 1
$\sqrt{42}$	6, 2, 12, 2, 12, 2, 12 etc.	1 6 1 6 1 6 1
$\sqrt{43}$	6, 1, 1, 3, 1, 5, 1, 3, 1, 1, 12 etc.	1 7 6 3 9 2 9 3 6 7 1
$\sqrt{44}$	6, 1, 1, 1, 2, 1, 1, 1, 12 etc.	1 8 5 7 4 7 5 8 1
$\sqrt{45}$	6, 1, 2, 2, 2, 1, 12, 1, 2, 2, 2, 1, 12 etc.	1 9 4 5 4 9 1 9 4 5 4 9 1
$\sqrt{46}$	6, 1, 3, 1, 1, 2, 6, *2*, 1, 1, 3, 1, 12 etc.	1 10 3 7 6 5 2 5 6 7 3 10 1
$\sqrt{47}$	6, 1, 5, 1, 12, 1, 5, 1, 12 etc.	1 11 2 11 1 11 2 11 1
$\sqrt{48}$	6, 1, 12, 1, 12, 1, 12 etc.	1 12 1 12 1 12 1
$\sqrt{50}$	7, 14, 14, 14 etc.	1 1 1 1
$\sqrt{51}$	7, 7, 14, 7, 14, 7, 14 etc.	1 2 1 2 1 2 1
$\sqrt{52}$	7, 4, 1, 2, 1, 4, 14, 4, 1, 2, 1, 4, 14 etc.	1 3 9 4 9 3 1 3 9 4 9 3 1
$\sqrt{53}$	7, 3, 1, 1, 3, 14, 3, 1, 1, 3, 14 etc.	1 4 7 7 4 1 4 7 7 4 1
$\sqrt{54}$	7, 2, 1, 6, 1, 2, 14, 2, 1, 6, 1, 2, 14 etc.	1 5 9 2 9 5 1 6 9 2 9 5 1
$\sqrt{55}$	7, 2, 2, 2, 14, 2, 2, 2, 14, 2, 2, 2, 14 etc.	1 6 5 6 1 6 5 6 1 6 5 6 1
$\sqrt{56}$	7, 2, 14, 2, 14, 2, 14 etc.	1 7 1 7 1 7 1
$\sqrt{57}$	7, 1, 1, 4, 1, 1, 14 etc.	1 8 7 3 7 8 1
$\sqrt{58}$	7, 1, 1, 1, 1, 1, 1, 14 etc.	1 9 6 7 7 6 9 1
$\sqrt{59}$	7, 1, 2, 7, 2, 1, 14 etc.	1 10 5 2 5 10 1

$\sqrt{60}$	7, 1, 2, 1, 14 etc.	1 11 4 11 1
$\sqrt{61}$	7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14 etc.	1 12 3 4 9 5 5 9 4 3 12 1
$\sqrt{62}$	7, 1, 6, 1, 14 etc.	1 13 2 13 1
$\sqrt{63}$	7, 1, 14, 1, 14 etc.	1 14 1 14 1
$\sqrt{65}$	8, 16, 16 etc.	1 1 1
$\sqrt{66}$	8, 8, 16, 8, 16 etc.	1 2 1 2 1
$\sqrt{67}$	8, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16 etc.	1 3 6 7 9 2 9 7 6 3 1
$\sqrt{68}$	8, 4, 16, 4, 16 etc.	1 4 1 4 1
$\sqrt{69}$	8, 3, 3, 1, 4, 1, 3, 3, 16 etc.	1 5 4 11 3 11 4 5 1
$\sqrt{70}$	8, 2, 1, 2, 1, 2, 16 etc.	1 6 9 5 9 6 1
$\sqrt{71}$	8, 2, 2, 1, 7, 1, 2, 2, 16 etc.	1 7 5 11 2 11 5 7 1
$\sqrt{72}$	8, 2, 16, 2, 16 etc.	1 8 1 8 1
$\sqrt{73}$	8, 1, 1, 5, 5, 1, 1, 16 etc.	1 9 8 3 3 8 9 1
$\sqrt{74}$	8, 1, 1, 1, 1, 16 etc.	1 10 7 7 10 1
$\sqrt{75}$	8, 1, 1, 1, 16 etc.	1 11 6 11 1
$\sqrt{76}$	8, 1, 2, 1, 1, 5, 4, 5, 1, 1, 2, 1, 16 etc.	1 12 5 8 9 3 4 3 9 8 5 12 1
$\sqrt{77}$	8, 1, 3, 2, 3, 1, 16 etc.	1 13 4 7 4 13 1
$\sqrt{78}$	8, 1, 4, 1, 16 etc.	1 14 3 14 1
$\sqrt{79}$	8, 1, 7, 1, 16 etc.	1 15 2 15 1
$\sqrt{80}$	8, 1, 16, 1, 16 etc.	1 16 1 16 1

$\sqrt{82}$	9, 18, 18, 18 etc. 1 1 1 1
$\sqrt{83}$	9, 9, 18, 9, 18 etc. 1 2 1 2 1
$\sqrt{84}$	9, 6, 18, 6, 18 etc. 1 3 1 3 1
$\sqrt{85}$	9, 4, 1, 1, 4, 18 etc. 1 4 9 9 4 1
$\sqrt{86}$	9, 3, 1, 1, 1, 8, 1, 1, 1, 3, 18 etc. 1 5 10 7 11 2 11 7 10 5 1
$\sqrt{87}$	9, 3, 18, 3, 18 etc. 1 6 1 6 1
$\sqrt{88}$	9, 2, 1, 1, 1, 2, 18 etc. 1 7 9 8 9 7 1
$\sqrt{89}$	9, 2, 3, 3, 2, 18 etc. 1 8 6 5 8 1
$\sqrt{90}$	9, 2, 18, 2, 18 etc. 1 9 1 9 1
$\sqrt{91}$	9, 1, 1, 5, 1, 5, 1, 1, 18 etc. 1 10 9 *8* 14 3 9 10 1
$\sqrt{92}$	9, 1, 1, 2, 4, 2, 1, 1, 18 etc. 1 11 8 7 4 7 8 11 1
$\sqrt{93}$	9, 1, 1, 1, 4, 6, 4, 1, 1, 1, 18 etc. 1 12 7 11 4 3 4 11 7 12 1
$\sqrt{94}$	9, 1, 2, 3, 1, 1, 5, 1, 8, 1, 5, 1, 1, 3, 2, 1, 18 etc. 1 13 6 6 9 10 3 15 2 15 8 10 9 6 6 13 1
$\sqrt{95}$	9, 1, 2, 1, 18 etc. 1 14 5 14 1
$\sqrt{96}$	9, 1, 3, 1, 18 etc. 1 15 4 15 1
$\sqrt{97}$	9, 1, 5, 1, 1, 1, 1, 1, 1, 5, 1, 18 etc. 1 16 3 11 8 9 9 8 11 3 16 1
$\sqrt{98}$	9, 1, 8, 1, 18 etc. 1 17 2 17 1
$\sqrt{99}$	9, 1, 18, 1, 18 etc. 1 18 1 18 1
$\sqrt{101}$	10, 20, 20 etc. 1 1 1
$\sqrt{102}$	10, 10, 20, 10, 20 etc. 1 2 1 2 1

$\sqrt{103}$	10, 6, 1, 2, 1, 1, 9, 1, 1, 2, 1, 6, 20 etc.	1 3 18 6 9 11 2 11 9 6 13 3 1
$\sqrt{104}$	10, 5, 20, 5, 20 etc.	1 4 1 4 1
$\sqrt{105}$	10, 4, 20, 4, 20 etc.	1 5 1 5 1
$\sqrt{106}$	10, 3, 2, 1, 1, 1, 1, 2, 3, 20 etc.	1 6 7 10 9 9 10 7 6 1
$\sqrt{107}$	10, 2, 1, 9, 1, 2, 20 etc.	1 7 13 2 13 7 1
$\sqrt{108}$	10, 2, 1, 1, 4, 1, 1, 2, 20 etc.	1 8 9 11 4 11 9 8 1
$\sqrt{109}$	10, 2, 3, 1, 2, 4, 1, 6, 6, 1, 4, 2, 1, 3, 2, 20 etc.	1 9 5 12 7 4 15 3 3 15 4 7 12 5 9 1
$\sqrt{110}$	10, 2, 20, 2, 20 etc.	1 10 1 10 1
$\sqrt{111}$	10, 1, 1, 6, 1, 1, 20 etc.	1 11 10 3 10 11 1
$\sqrt{112}$	10, 1, 1, 2, 1, 1, 20 etc.	1 12 9 7 9 12 1
$\sqrt{113}$	10, 1, 1, 1, 2, 2, 1, 1, 1, 20 etc.	1 13 8 11 7 7 11 8 13 1
$\sqrt{114}$	10, 1, 2, 10, 2, 1, 20 etc.	1 14 7 2 7 14 1
$\sqrt{115}$	10, 1, 2, 1, 1, 1, 1, 1, 2, 1, 20 etc.	1 15 6 11 9 10 9 11 6 15 1
$\sqrt{116}$	10, 1, 3, 2, 1, 4, 1, 2, 3, 1, 20 etc.	1 16 5 7 13 4 13 7 5 16 1
$\sqrt{117}$	10, 1, 4, 2, 4, 1, 20 etc.	1 17 4 9 4 17 1
$\sqrt{118}$	10, 1, 6, 3, 2, 10, 2, 3, 6, 1, 20 etc.	1 18 3 6 9 2 9 6 3 18 1
$\sqrt{119}$	10, 1, 9, 1, 20 etc.	1 19 2 19 1
$\sqrt{120}$	10, 1, 20, 1, 20 etc.	1 20 1 20 1

16. In all these series of indices the periods are taken either in a shorter or longer manner, by which these indices are included, which are twice as long in the first sequence, and these periods thus may be observed more clearly, if the first indices of each series may be doubled. Then the same order is observed in any index, either in the forwards or backwards direction ; from which in any period either one or two middle indices is given, just as the number of terms were either even or odd. Truly similar periods are observed in the greek letters also, where it is required to be noted especially for all the indices the greek letters 2ν to become one. This significant property shall be the most interesting to be observed properly in the following, which can be seen more easily with these operations than may be demonstrated by a long account in words.

17. From these examples certain general forms will be allowed to be deduced, which may be established thus:

I.	If $z = nn+1$,	the indices will be	$\underset{1}{n},\ \underset{1}{2n},\ \underset{1}{2n},\ \underset{1}{2n}$ etc.,
II.	" $z = nn+2$,	"	$\underset{1}{n}\ \underset{2}{n}\ \underset{1}{2n}\ \underset{2}{n}\ \underset{1}{2n}$ etc.,
III.	" $z = nn+n$	"	$\underset{1}{n},\ \underset{n}{2},\ \underset{1}{2n},\ \underset{n}{2},\ \underset{1}{2n}$ etc.,
IV.	" $z = nn+2n-1$	"	$\underset{1}{n},\ \underset{2n-1}{1},\ \underset{2}{n-1},\ \underset{2n-1}{1},\ \underset{1}{2n}$ etc.,
V.	if $z = nn+2n$	"	$\underset{1}{n},\ \underset{2n}{1},\ \underset{1}{2n},\ \underset{2n}{1},\ \underset{1}{2n}$ etc.

And indeed the value of the continued fractions in general is easily defined from the formation of these indices and likewise is taken, as we have designated here. Then truly also it is evident,

VI.	if there shall be	$z = 4nn+4$,	the indices to become	$\underset{1}{2n},\ \underset{4}{n},\ \underset{1}{4n},\ \underset{4}{n},\ \underset{1}{4n}$ etc.,
VII.	"	$z = 9nn+3$,	"	$\underset{1}{3n},\ \underset{3}{2n},\ \underset{1}{6n},\ \underset{3}{2n},\ \underset{1}{6n}$ etc.,
VIII.	"	$z = 9nn+6$	"	$\underset{1}{3n},\ \underset{6}{n},\ \underset{1}{6n},\ \underset{6}{n},\ \underset{1}{6n}$ etc.

CONCERNED WITH THE RESOLUTION OF THE FORMULA $p = \sqrt{(lqq+1)}$ IN TERMS OF INTEGERS

18. With the indices found for the square root of some number z, this may be expressed by a continued fraction in this way :

$$\sqrt{z} = v + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}}$$

and from these indices *v, a, b, c, d* etc. the fractions $\frac{x}{y}$ can be formed, which approach so near to $\sqrt{z}$, so that only with greater numbers required to be used its value may be shown more accurately. Moreover these fractions thus may be formed :

$$\begin{array}{lllllllll} \text{Indices} & v, & a, & b, & c, \cdot \;\cdot\;\cdot\;\cdot & m, & n, \\ \frac{x}{y} & \frac{1}{0}, & \frac{v}{1}, & \frac{av+1}{a}, & \frac{(ab+1)v+b}{ab+1}, \cdot\;\cdot\;\cdot\;\cdot & \frac{M}{P}, & \frac{N}{Q}, & \frac{nN+M}{nQ+P}, \end{array}$$

which continually express the irrational value $\sqrt{z}$ closer.

19. But the new algorithm supplies a succinct way being required these fractions conveniently be indices, which shall be had thus :

$$\frac{1}{0}, \ \frac{(v)}{1}, \ \frac{(v,\ a)}{(a)}, \ \frac{(v,\ a,\ b)}{(a,\ b)}, \ \frac{(v,\ a,\ b,\ c)}{(a,\ b,\ c)}, \ \frac{(v,\ a,\ b,\ c,\ d)}{(a,\ b,\ c,\ d)} \ \text{ etc.;}$$

where since from the nature of the progression there shall be

$$\begin{array}{lll} (v,\ a) = a(v)+1, & (v,\ a,\ b) = b(v,\ a)+(v), & (v,\ a,\ b,\ c) = c(v,\ a,\ b)+(v,\ a), \\ (a) = a1+0, & (a,\ b) = b(a)+1, & (a,\ b,\ c) = c(a,\ b)+(a), \end{array}$$

also from the nature of the formulas there will become

$$(v, a) = v(a) + 1,\ (v, a, b) = v(a, b) + b,\ (v, a, b, c) = v(a, b, c) + (b, c);$$

then also the following transformations to be shown:

$$\begin{aligned} (v,\ a,\ b,\ c,\ d,\ e) &= v(a,\ b,\ c,\ d,\ e)+(b,\ c,\ d,\ e), \\ (v,\ a,\ b,\ c,\ d,\ e) &= (v,\ a)(b,\ c,\ d,\ e)+v(c,\ d,\ e), \\ (v,\ a,\ b,\ c,\ d,\ e) &= (v,\ a,\ b)(c,\ d,\ e)+(v,a)(d,\ e), \\ (v,\ a,\ b,\ c,\ d,\ e) &= (v,\ a,\ b,\ c)(d,\ e)+(v,a,\ b)(e), \end{aligned}$$

which it will help to be noted properly generally in the following.

20. We may see now, as these individual fractions approach the proper value $\sqrt{z}$, as will be apparent from our most splendid principle, if we may deduce the value $xx - zyy$ from some fraction $\frac{x}{y}$; certainly as which will have been smaller besides the values from these numbers *x* and *y*, for that value of the fraction $\frac{x}{y}$ to become more exactly equal to $\sqrt{z}$. And indeed initially if there were $\frac{x}{y} = \frac{1}{0}$, there will become $xx - zyy = 1$. Then on taking $\frac{x}{y} = \frac{v}{1}$ there becomes $xx - zyy = vv - z,$ which differences put in place by the above operations (§12) may be designated by the first greek letter taken negative, $-\alpha$. Again on putting $\frac{x}{y} = \frac{(v,\, a)}{(a)} = \frac{va+1}{a}$, there is deduced

$$xx - zyy = (vv - z)aa + 2va + 1 = -\alpha aa + 2va + 1,$$

therefore

$$xx - zyy = 1 + a(2v - \alpha a) = 1 + a(A - B) = \beta$$

on account of which $v = A$ and $\alpha a = A + B$. Wherefore in this case there becomes $xx - zyy = \beta$.

21. Therefore since we shall have obtained

$$vv - z = -\alpha \text{ and } (v,a)^2 - z(a)^2 = \beta,$$

hence we will be able to progress further. Therefore there shall be

$$\frac{x}{y} = \frac{(v,\, a,\, b)}{(a,b)} = \frac{b(v,a)+v}{b(a)+1}$$

and with these reduced we will obtain

$$xx - zyy = \beta bb + 2vb(v,a) - 2zb(a) - \alpha ,$$

therefore on account of $(v,a) = v(a) + 1$ there will be

$$xx - zyy = \beta bb - 2\alpha ab + 2vb - \alpha = -\alpha - b(2\alpha a - \beta b - 2v):$$

but there is $v = A,\ \alpha a = A + B$ and $\beta b = B + C$ and thus

$$xx - zyy = -\alpha - b(B - C) = -\gamma,$$

so that there shall be

$$(v,\ a,\ b)^2 - z(a,\ b)^2 = -\gamma .$$

22. Now we will consider the following fraction

$$\frac{x}{y}=\frac{(v,\ a,\ b,\ c)}{(a,\ b,\ c)}=\frac{c(v,\ a,\ b)+(v,\ a)}{c(a,\ b)+a},$$

from which there is deduced

$$xx-zyy=-\gamma cc+2c(v,a,\ b)(v,\ a)+\beta-2zca(a,\ b),$$

the middle part of which is reduced to $2c(\beta b-\alpha a+v)$, from which on account of

$$v=A,\ \alpha a=A+B,\quad \beta b=B+C,\quad \gamma c=C+D$$

there results

$$xx-zyy=\beta+c(C-D)=\delta,$$

thus so that there shall be

$$(v,\ a,\ b,\ c)^2-z(a,\ b,\ c)^2=\delta,$$

from which by induction the following values are deduced readily.

23. But lest this proof by induction may appear excessive to be attributed, this investigation can be established in the following manner. There shall be

$$\begin{array}{lll}
(v)^2 & -z1^2 & =\mathfrak{A}\\
(v,\ a)^2 & -z(a)^2 & =\mathfrak{B}\\
(v,\ a,\ b)^2 & -z(a,\ b)^2 & =\mathfrak{C}\\
(v,\ a,\ b,\ c)^2 & -z(a,\ b,\ c)^2 & =\mathfrak{D}\\
(v,\ a,\ b,\ c,\ d)^2 & -z(a,\ b,\ c,\ d)^2 & =\mathfrak{E}\\
 & \text{etc.} &
\end{array}$$

where indeed now we have seen to be $\mathfrak{A}=-\alpha,\ \mathfrak{B}=\beta,\ \mathfrak{C}=-\gamma$ etc. There since there shall be

$$\begin{array}{llll}
(v,\ a) & =a(v)+1, & (a) & =a,\\
(v,\ a,\ b) & =b(v,\ a)+(v), & (a,\ b) & =b(a)+1,\\
(v,\ a,\ b,\ c) & =c(v,\ a,\ b)+(v,\ a), & (a,\ b,\ c) & =c(a,\ b)+(a),\\
(v,\ a,\ b,\ c,\ d) & =c(v,\ a,\ b,\ c)+(v,\ a,\ b), & (a,\ b,\ c,\ d) & =d(a,\ b,\ c)+(a,\ b)
\end{array}$$

we will have

$$\mathfrak{B} = \mathfrak{A}aa + 1 + 2a(v),$$
$$\mathfrak{C} = \mathfrak{B}bb + \mathfrak{A} + 2b\big((v,\ a)(v) - z(a)\big),$$
$$\mathfrak{D} = \mathfrak{C}cc + \mathfrak{B} + 2c\big((v,\ a,\ b)(v,a) - z(a,\ b)(a)\big),$$
$$\mathfrak{E} = \mathfrak{D}dd + \mathfrak{C} + 2d\big((v,\ a,\ b,\ c)(v,\ a,\ b) - z(a,\ b,\ c)(a,\ b)\big),$$
$$\mathfrak{F} = \mathfrak{E}ee + \mathfrak{D} + 2e\big((v, a,\ b,\ c,\ d)(v,\ a,\ b,\ c) - z(a,\ b,\ c,\ d)(a,\ b, c)\big)$$

etc.

24. For the sake of brevity we may take

$$\mathfrak{B} = 1 + \mathfrak{A}ab + 2a \cdot O,$$
$$\mathfrak{C} = \mathfrak{A} + \mathfrak{B}bb + 2b \cdot P,$$
$$\mathfrak{D} = \mathfrak{B} + \mathfrak{C}cc + 2c \cdot Q,$$
$$\mathfrak{E} = \mathfrak{C} + \mathfrak{D}dd + 2d \cdot R,$$
$$\mathfrak{F} = \mathfrak{D} + \mathfrak{E}ee + 2e \cdot S$$

etc.

and from the above reductions we will deduce

$$P - O = a(v)^2 - za = \mathfrak{A}a,$$
$$Q - P = b(v,a)^2 - zb(a)^2 = \mathfrak{B}b,$$
$$R - Q = c(v,\ a,\ b)^2 - cz(a,\ b)^2 = \mathfrak{C}c,$$
$$S - R = d(v,\ a,\ b,\ c)^2 - dz(a,\ b, c)^2 = \mathfrak{D}d$$

etc.

and thus there will become

$$O = v,$$
$$P = v + \mathfrak{A}a,$$
$$Q = v + \mathfrak{A}a + \mathfrak{B}b,$$
$$R = v + \mathfrak{A}a + \mathfrak{B}b + \mathfrak{C}c,$$
$$S = v + \mathfrak{A}a + \mathfrak{B}b + \mathfrak{C}c + \mathfrak{D}d$$

etc.

25. Moreover the formulas used above provide us with :

$$\begin{aligned}
A &= v,\\
B &= -v+\alpha a,\\
C &= v-\alpha a+\beta b,\\
D &= -v+\alpha a-\beta b+\gamma c,\\
E &= v-\alpha a+\beta b-\gamma c+\delta d
\end{aligned}$$

etc.,

from which it is apparent to be:

$$O = A \text{ and } P = -B \text{ on account of } \mathfrak{A} = -\alpha.$$

Now since there shall be $\mathfrak{B} = 1-\alpha aa+2av = 1+a(A-B)$, certainly there will be

$$\mathfrak{B} = \beta \text{ and hence } Q = C,$$

from which again there is deduced

$$\mathfrak{C} = -\alpha+\beta bb-2bB = -\alpha-b\left(2B-\beta b\right) = -\alpha-b\left(B-C\right),$$

and thus

$$\mathfrak{C} = -\gamma \text{ and } R = -D;$$

in a similar manner

$$\mathfrak{D} = \beta-\gamma cc+2cC = \beta+c\left(2C-\gamma c\right) = \beta+c(C-D)$$

and thus there is

$$\mathfrak{D} = \delta \text{ et } S = E.$$

Then again truly

$$\mathfrak{E} = -\gamma+\delta dd-2dD = -\gamma-d(2D-\delta d) = -\gamma-d(D-E)$$

and therefore

$$\mathfrak{E} = -\varepsilon,$$

from which the above induction is confirmed well enough.

26. Therefore for the fractions $\frac{x}{y}$, we may come upon the following relations of the formula approximately equal to the square root $\sqrt{z}$:

If there may be taken		there will be
$x = 1,$	$y = 0,$	$xx = zyy+1,$
$x = (v),$	$y = 1,$	$xx = zyy-\alpha,$
$x = (v,\ a),$	$y = (a),$	$xx = zyy+\beta,$
$x = (v,\ a,\ b),$	$y = (a,\ b),$	$xx = zyy-\gamma,$
$x = (v,\ a,\ b,\ c),$	$y = (a,\ b,\ c),$	$xx = zyy+\delta,$

$$x=(v,\ a,\ b,\ c,\ d),\qquad y=(a,\ b,\ c,\ d),\qquad xx=zyy-\varepsilon$$

etc.,

from which the Pellian problem will be solved, as often as certain jumps of the greek letters $\beta,\ \delta,\ \zeta$ etc. are selected, some of which will become unity.

27. But we have seen above, except for these indices which are $2v$, correspond to greek letters going into unity ; therefore since any of the periods, which we have observed in the order of the indices, will be advancing towards the index $2v$, it is evident, if we may define the numbers *x* and *y* by the indices of the first period, to become either

$$xx=zyy-1$$

or

$$xx=zyy+1;$$

and indeed the first eventuates, if the individual periods were to constitute an odd number, truly the latter, if it were even. Therefore in this case the solution of the Pellian problem will be had, were it is required, so that there shall be

$$pp=zqq+1,$$

whenever it will be required to take $p=x$ and $q=y$.

28. But if from the first period there may be produced $xx=zyy-1$, which eventuates, if the index number is odd, then the indices as far as to the start of the third period will be taken for defining the numbers *x* and *y* ; the number of which shall be even, in this way suitable numbers will be obtained for *p* and *q*. Truly in the case found, where there becomes fit $xx=zyy-1$, thence the numbers *p* and *q* can be found much easier, so that there shall be $pp=zqq+1$. Indeed there may be taken

$$p=2xx+1 \text{ and } q=2xy$$

and there will

$$pp-zqq=4x^4+4xx+1-4zxxyy=1+4xx(xx-zyy+1);$$

but $xx-zyy+1=0$ and thus

$$pp-zqq=1 \quad\text{or}\quad pp=zqq+1,$$

just as the Pellian problem demands.

Therefore we may see, how for any number z thence the numbers arising from the indices the numbers p and q shall be required to be defined, so that there may become $pp=zqq+1$, where indeed we will run through the following case periods.

I. THE CASE WHERE THE INDICES FOR THE NUMBER z ARE

$$v,\ 2v,\ 2v \text{ etc.}$$

29. Here the individual periods contain a single index; therefore on taking

$$x=(v) \text{ and } y=1$$

there will be

$$xx=zyy-1.$$

Whereby so that there may become $pp=zqq+1$, there may be taken

$$p=2xx+1=2vv+1 \text{ and } q=2xy=2v.$$

This case, as we have seen above, has a place, if there shall be

$$z=vv+1,$$

or where the number z exceeds the square by one; then therefore there must be taken

$$p=2vv+1 \text{ or } p=2z-1 \text{ and } q=2v,$$

with which agreed it satisfies the Pell problem, so that there shall be $p=\sqrt{(zqq+1)}$.

Thus if there shall be	there will be	and thus
$z=2,$	$p=3$ and $q=2$	$p=\sqrt{(2qq+1)},$
$z=5,$	$p=9$ and $q=4$	$p=\sqrt{(5qq+1)},$
$z=10,$	$p=19$ and $q=6$	$p=\sqrt{(10qq+1)},$
$z=17,$	$p=33$ and $q=8$	$p=\sqrt{(17qq+1)}$

etc.

II. THE CASE WHERE THE INDICES FOR THE NUMBER z ARE

$$v,\ a,\ 2v,\ a,\ 2v \text{ etc.}$$

30. The first period depends on the two numbers v, a, from which by taking

$$x=(v,a)=va+1 \text{ and } y=(a)=a$$

there will be had

$$xx = zyy + 1.$$

So that therefore for the Pellian problem there may become $pp = zqq + 1$, it will be required to take

$$p = va + 1 \text{ and } q = a.$$

Moreover it is evident from the indices this case is relevant, provided the number will have been

$$z = vv + \tfrac{2v}{a},$$

from which it is understood this case in integers, concerning which this is to be performed, cannot exist, unless a shall be a divisor of $2v$, where two cases are required to be considered:

1. If $a = 2n$, $\quad v = mn \quad$ and $\quad \tfrac{2v}{a} = m$;
2. If $a = 2n+1$, $\quad v = m(2n+1) \quad$ and $\quad \tfrac{2v}{a} = 2m$.

III. THE CASE WHERE THE INDICES FOR THE NUMBER z ARE

$$v,\ a,\ a,\ 2v,\ a,\ a,\ 2v \text{ etc.}$$

31. From the first period with the numbers x and y taken, thus so that there shall be

$$x = (v,\ a,\ a) \text{ and } y = (a,\ a),$$

there will be

$$xx = zyy - 1;$$

from which so that there may become $pp = zqq + 1$, there must be taken

$$p = 2xx + 1 \quad \text{and} \quad q = 2xy.$$

Therefore

$$y = aa + 1 \text{ et } x = vy + a,$$

from which the numbers p and q may be defined easily. Moreover from the indices the number z will have a form of this kind

$$z = vv + u$$

with there being present

$$u = \tfrac{2av+1}{aa+1},$$

from which it is apparent the number a must be even. Therefore if there may be put $a = 2n$, it is necessary there shall be

$$v = n + m(4nn+1),$$

and then there shall be

$$u = 1+4mn.$$

IV. THE CASE WHERE THE INDICES FOR THE NUMBER z ARE

$$v,\ a,\ b,\ a,\ 2v,\ a,\ b,\ a,\ 2v \text{ etc.}$$

32. Since the index number in each period is even, if there may be put

$$x = (v,\ a,\ b,\ a) \quad \text{et} \quad y = (a,\ b,\ a),$$

there will become

$$xx = zyy+1$$

and thus

$$p = x \text{ and } q = y.$$

But by the transformations we have shown above the twofold index can be shown in this manner

$$x = (a)(v,\ a,\ b)+(v,\ a) \text{ and } y = (a)(a,\ b)+(a).$$

Hence the following fractions may be formed from the indices v, a, b

indices		v,	a,	b
fractions	$\frac{1}{0}$,	$\frac{\mathfrak{A}}{\mathfrak{a}}$,	$\frac{\mathfrak{B}}{\mathfrak{b}}$,	$\frac{\mathfrak{C}}{\mathfrak{c}}$,

on account of

$$\mathfrak{A} = (v), \quad \mathfrak{B} = (v, a), \quad \mathfrak{C} = (v,\ a,\ b)$$

and

$$\mathfrak{a} = 1, \qquad \mathfrak{b} = (a), \qquad \mathfrak{c} = (a,\ b)$$

there will be

$$x = \mathfrak{b}\mathfrak{C} + \mathfrak{a}\mathfrak{B} \quad \text{and} \quad y = \mathfrak{b}\mathfrak{c}+\mathfrak{a}\mathfrak{b}.$$

But from the indices there becomes

$$z = vv+u$$

with there being

$$2v = m(a,\ b,\ a) - b(a,\ b)$$

and

$$u = m(a,\ b) - b(b).$$

V. THE CASE WHERE THE INDICES FOR THE NUMBER z ARE $v,\ a,\ b,\ b,\ a,\ 2v$ etc.

33. On account of the index of each period being an odd number, if we may take

$$x=(v,a,b,b,a) \text{ and } y=(a,b,b,a),$$

there will become

$$xx=zyy-1;$$

hence so that for the Pellian problem there may become $pp=zqq+1$, it will be required to put

$$p=2xx+1 \text{ and } q=2xy.$$

But so that the numbers x and y may be found more easily, the following transformations may be put in place

$$x=(a,\ b)(v,\ a,\ b)+(a)(v,\ a) \text{ and } y=(a,\ b)(a,\ b)+(a)(a),$$

which therefore will be defined by the single indices v, a, b from the fractions thence formed:

indices		v,	a,	b
fractions	$\frac{1}{0}$,	$\frac{\mathfrak{A}}{\mathfrak{a}}$,	$\frac{\mathfrak{B}}{\mathfrak{b}}$,	$\frac{\mathfrak{C}}{\mathfrak{c}}$,

where

$$\mathfrak{A}=v,\quad \mathfrak{B}=a\mathfrak{A}+1,\quad \mathfrak{C}=b\mathfrak{B}+\mathfrak{A}$$

and

$$\mathfrak{a}=1,\qquad \mathfrak{b}=a\mathfrak{a}+0,\qquad \mathfrak{c}=b\mathfrak{b}+\mathfrak{a};$$

then also it will be require to take

$$x=\mathfrak{c}\mathfrak{C}+\mathfrak{b}\mathfrak{B} \text{ et } y=\mathfrak{c}\mathfrak{c}+\mathfrak{b}\mathfrak{b}.$$

Moreover this case can be found, whenever on putting

$$z=vv+u$$

there will have become

$$2v=m(a,\ b,\ b,\ a)+(b,\ b)(a,\ b,\ b)$$

and

$$u=m(a,\ b,\ b)+(b,\ b)(b,\ b).$$

VI. THE CASE WHERE THE INDICES FOR THE NUMBER z ARE
$v,\ a,\ b,\ c,\ b,\ a,\ 2v$ etc.

34. Since here the index number in any interval is even, if we may take

$$x=(v,\ a,\ b,\ c,\ b,\ a)\ \text{ and }\ y=(a,\ b,\ c,\ b,\ a),$$

there will become

$$xx=zyy+1$$

and thus for the Pellian problem there is had at once

$$p=x\ \text{ and }\ q=y.$$

Moreover the numbers x and y will be found by using these transformations

$$x=(a,\ b)\ (v,\ a,\ b,\ c)(a)(v,\ a,\ b)\ \text{ and }\ y=\ (a,\ b)(a,\ b,\ c)\ +(a)(a,\ b);$$

from which if from the indices v, a, b, c the customary fractions formed may be set out

indices	$v,$	$a,$	$b,$	c	
fractions	$\frac{1}{0},$	$\frac{\mathfrak{A}}{\mathfrak{a}},$	$\frac{\mathfrak{B}}{\mathfrak{b}},$	$\frac{\mathfrak{C}}{\mathfrak{c}},$	$\frac{\mathfrak{D}}{\mathfrak{d}},$

it is required to take

$$x=\mathfrak{c}\mathfrak{D}+\mathfrak{b}\mathfrak{C}\ \text{ and }\ y=\mathfrak{c}\mathfrak{d}+\mathfrak{b}\mathfrak{c}.$$

Moreover, this case can be used, whenever on putting

$$z=vv+u$$

there were

$$2v=m(a,\ b,\ c,\ b,\ a)-\ (b,\ c,\ b)(a,\ b,\ c,\ b)$$

and

$$u=m(a,\ b,\ c,\ b)+(b,\ c,\ b)(b,\ c,\ b).$$

VII. THE CASE WHERE THE INDICES FOR THE NUMBER z ARE

$v,\ a,\ b,\ c,\ c,\ b,\ a,\ 2v$ etc.

35. Here again the index number in any period is odd ; and thus if we may put

$$x=(v,\ a,\ b,\ c,\ c,\ b,\ a)\ \text{ and }\ y=(a,\ b,\ c,\ c,\ b,\ a),$$

there will be

$$xx = zyy - 1;$$

from which so that there may be put $pp = zqq + 1$, it is required to take

$$p = 2xx + 1 \text{ and } q = 2xy.$$

But for finding the numbers x and y more easily from the indices v, a, b, c the fractions will be formed:

$$\begin{array}{llllll} \text{indices} & v, & a, & b, & c & \\ \text{fractions} & \frac{1}{0}, & \frac{\mathfrak{A}}{\mathfrak{a}}, & \frac{\mathfrak{B}}{\mathfrak{b}}, & \frac{\mathfrak{C}}{\mathfrak{c}}, & \frac{\mathfrak{D}}{\mathfrak{d}}, \end{array}$$

and hence there will be

$$x = \mathfrak{d}\mathfrak{D} + \mathfrak{c}\mathfrak{C} \text{ and } y = \mathfrak{d}\mathfrak{d} + \mathfrak{c}\mathfrak{c}.$$

But here a case may be had, whenever

$$z = vv + u$$

there will become

$$2v = m(a,\ b,\ c,\ c,\ b,\ a) + (b,\ c,\ c,\ b)(a,\ b,\ c,\ c,\ b)$$

and

$$u = m(a,\ b,\ c,\ c,\ b) + (b,\ c,\ c,\ b)(b,\ c,\ c,\ b).$$

VIII. THE CASE WHERE THE INDICES FOR THE NUMBER z ARE $v,\ a,\ b,\ c,\ d,\ c,\ b,\ a, 2v$ etc.

36. Here any period contains eight indices; and thus if we may put

$$x = (v,\ a,\ b,\ c,\ d,\ c,\ b,\ a) \text{ and } y = (a,\ b,\ c,\ d,\ c,\ b,\ a),$$

there will be

$$xx = zyy + 1$$

and for the PELLIAN problem it is required to take

$$p = x \text{ and } q = y,$$

so that there may become $pp = zqq + 1$. But for the transformations used it will be permitted to define the numbers x and y only by the indices v, a, b, c, d. Thence indeed with the fractions formed

Example of a special algorithm

*E*323 :Translated & Annotated by Ian Bruce. (Aug., 2020)

indices $v, \quad a, \quad b, \quad c, \quad d$

fractions $\frac{1}{0},\ \frac{\mathfrak{A}}{\mathfrak{a}},\ \frac{\mathfrak{B}}{\mathfrak{b}},\ \frac{\mathfrak{C}}{\mathfrak{c}},\ \frac{\mathfrak{D}}{\mathfrak{d}},\ \frac{\mathfrak{E}}{\mathfrak{e}},$

there will become

$$x = \mathfrak{d}\mathfrak{E} + \mathfrak{c}\mathfrak{D} \text{ and } y = \mathfrak{d}\mathfrak{e} + \mathfrak{c}\mathfrak{d}.$$

Here truly the case occurs, whenever there becomes

$$z = vv + u$$

there were be

$$2v = m(a,\ b,\ c,\ d,\ c,\ b,\ a) - \left(b,\ c,\ d,\ c,\ b\right)\left(a,\ b,\ c,\ d,\ c,\ b\right)$$

and

$$u = m\left(a,\ b,\ c,\ d,\ c,\ b\right) - (b,\ c,\ d,\ c,\ b)\left(b,\ c,\ d,\ c,\ b\right).$$

ESTABLISHING THE CALCULATION FOR ANY NUMBER z

SO THAT THERE MAY BECOME

$$pp = zqq + 1$$

37. Therefore it is required to investigate by the first method set out above the indices required for the square root of the number z ; but as there is no need for the operation to be continued further, as while the indices shall begin to be produced in the reverse order, with which agreed on we will be able to supersede establishing the above with half the labor. But since in the first period one or two indices may appear in the central position, these cases are required to be distinguished properly, since, if only a single middle may be present, the discovery of the numbers p and q in the cases II, IV, VI and VIII treated must be established, but if two middle indices were present, only that is described, which in the cases I, III, V and VII. Evidently if the first may eventuate, the numbers p and q may be taken equal for the numbers x et y, but if the latter as we have seen, it is necessary to put in place $p = 2xx + 1$ and $q = 2xy$, thus so that for these cases the numbers p and q may be found with other much greater equal parts.

38. See therefore examples of the first kind, where a single central index is given in any period .

I. If $z = 6$, the indices are 2, 2, 4; hence the operation:

$$2,\quad 2$$

$$\frac{1}{0},\quad \frac{2}{1},\quad \frac{5}{2};$$

hence

$$x = 1\cdot 5 + 0\cdot 2, \qquad p = 5,$$

therefore

$$y = 1\cdot 2 + 0\cdot 1, \qquad q = 2.$$

II. If $z = 14$, the indices are 3, 1, 2, 1, 6:

$$3,\quad 1,\quad 2$$

$$\frac{1}{0},\quad \frac{3}{1},\quad \frac{4}{1}\quad \frac{11}{3};$$

hence

$$x = 1\cdot 11 + 1\cdot 4, \qquad p = 15,$$

therefore

$$y = 1\cdot 3 + 1\cdot 1, \qquad q = 4.$$

III. If $z = 19$, the indices are 4, 2, 1, 3, 1, 2, 8:

$$4,\quad 2,\quad 1,\quad 3$$

$$\frac{1}{0},\quad \frac{4}{1},\quad \frac{9}{2},\quad \frac{13}{3},\quad \frac{48}{11};$$

hence

$$x = 3\cdot 48 + 2\cdot 13, \qquad p = 170,$$

therefore

$$y = 3\cdot 11 + 2\cdot 3, \qquad q = 39.$$

IV. If z = 31, the indices are 5, 1, 1, 3, 5, 3, 1, 1, 10:

$$5,\quad 1,\quad 1,\quad 3,\quad 5$$

$$\frac{1}{0},\quad \frac{5}{1},\quad \frac{6}{1},\quad \frac{11}{2},\quad \frac{39}{7},\quad \frac{206}{37};$$

hence

$$x = 7\cdot 206 + 2\cdot 39, \qquad p = 1520,$$

therefore

$$y = 7\cdot 37 + 2\cdot 7, \qquad q = 273.$$

V. If z = 44, the indices are 6, 1, 1, 1, 2, 1, 1, 1, 12:

$$6, \quad 1, \quad 1, \quad 1, \quad 2$$
$$\tfrac{1}{0}, \quad \tfrac{6}{1}, \quad \tfrac{7}{1}, \quad \tfrac{13}{2}, \quad \tfrac{20}{3}, \quad \tfrac{53}{8};$$
hence
$$x = 3\cdot 53 + 2\cdot 20, \qquad p = 199,$$
therefore
$$y = 3\cdot \ 8 + 2\cdot \ 3, \qquad q = \ 30.$$

VI. If $z = 55$, the indices are 7, 2, 2, 2, 14:

$$7, \quad 2, \quad 2,$$
$$\tfrac{1}{0}, \quad \tfrac{7}{1}, \quad \tfrac{15}{2}, \quad \tfrac{37}{5};$$
hence
$$x = 2\cdot 37 + 1\cdot 15, \qquad p = 89,$$
therefore
$$y = 2\cdot \ 5 + 1\cdot \ 2, \qquad q = 12.$$

39. Truly of the other kind, where two middle indices are given in any period, I add these examples.

I. If $z = 13$, the indices are 3, 1, 1, 1, 1, 6:

$$3, \quad 1, \quad 1,$$
$$\tfrac{1}{0}, \quad \tfrac{3}{1}, \quad \tfrac{4}{1}, \quad \tfrac{7}{2};$$
hence
$$x = 2\cdot 7 + 1\cdot 4 = 18,$$
$$y = 2\cdot 2 + 1\cdot 1 = \ 5.$$
therefore
$$p = 2xx + 1 = 649,$$
$$q = 2xy \quad = 180.$$

II. If $z = 29$, the indices are 5, 2, 1, 1, 2, 10:

$$5,\quad 2,\quad 1,$$

$$\tfrac{1}{0},\quad \tfrac{5}{1},\quad \tfrac{11}{2},\quad \tfrac{16}{3};$$

hence

$$x = 3\cdot 16 + 2\cdot 11 = 70,$$
$$y = 3\cdot\ \ 3 + 2\cdot\ \ 2 = 13.$$

therefore

$$p = 2xx+1 = 9801,$$
$$q = 2xy \quad = 1820.$$

III. If $z = 58$, the indices are 7, 1, 1, 1, 1, 1, 1, 14:

$$7,\quad 1,\quad 1,\quad 1$$

$$\tfrac{1}{0},\quad \tfrac{7}{1},\quad \tfrac{8}{1},\quad \tfrac{15}{2}\quad \tfrac{23}{3};$$

hence

$$x = 3\cdot 23 + 2\cdot 15 = 99,$$
$$y = 3\cdot\ \ 3 + 2\cdot\ \ 2 = 13.$$

therefore

$$p = 2xx+1 = 19603,$$
$$q = 2xy \quad = \ 2574.$$

IV. If $z = 61$, the indices are 7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14:

$$7,\quad 1,\quad 4,\quad 3,\quad 1,\quad 2$$

$$\tfrac{1}{0},\quad \tfrac{7}{1},\quad \tfrac{8}{1},\quad \tfrac{39}{5},\quad \tfrac{125}{16},\quad \tfrac{164}{21},\quad \tfrac{453}{58};$$

hence there becomes

$$x = 58\cdot 453 + 21\cdot 164 = 29718,$$
$$y = 58\cdot\ \ 58 + 21\cdot\ \ 21 = \quad 3805.$$

therefore

$$p = 2xx+1 = 1766319049,$$
$$q = 2xy \quad = \ 226153980.$$

40. But if for larger numbers z, than have be presented before, the numbers p and q must be sought, so that there shall be $pp = zqq+1$, initially by the method set out above (§ 12) it is necessary to find the indices *v*, *a*, *b*, *c*, *d* etc., but there is no need to continue these further, than at this point to the middle index or the two middle indices may be arrived at; then truly from these operations described here in the first place the numbers

x and *y*, then truly the numbers themselves sought *p* and *q* will be determined. It will be appropriate to illustrate that by some examples.

I. The numbers p and q are sought, so that there shall become $pp = 157qq + 1$

41. Since here there shall be $z = 157$, there will be $v = 12$ and $\alpha = 13$, from which the index found will be obtained thus :

$$\begin{array}{lll} A = 12, & \alpha = 13, & a = 1, \\ B = 1, & \beta = 12, & b = 1, \\ C = 11, & \gamma = 3, & c = 7, \\ D = 10, & \delta = 19, & d = 1, \\ E = 9, & \varepsilon = 4, & e = 5, \\ F = 11, & \zeta = 9, & f = 2, \\ G = 7, & \eta = 12, & g = 1, \\ H = 5, & \theta = 11, & \left.\begin{array}{l} h = 1 \\ i = 1 \end{array}\right\} \\ I = 6, & \iota = 11, & \end{array}$$

Hence on account of the two equal middle indices the example belongs to the second kind and the operations thus are required to be put in place:

$$12,\ 1,\ 1,\ 7,\ 1,\ 5,\ 2,\ 1,\ 1$$

$$\frac{1}{0},\ \frac{12}{1},\ \frac{13}{1},\ \frac{25}{2},\ \frac{188}{15},\ \frac{213}{17},\ \frac{1253}{100},\ \frac{2719}{217},\ \frac{3972}{317},\ \frac{6691}{534}.$$

Hence there will be

$$x = 534 \cdot 6691 + 317 \cdot 3972 = 4832118$$

and

$$y = 534 \cdot 534 + 317 \cdot 317 = 385645.$$

Regarding which

$$p = 2xx + 1 = 46698728731849$$

and

$$q = 2xy \quad = \quad 3726964292220$$

and thus these are the smallest whole numbers satisfying the formula $p = \sqrt{(157qq + 1)}$.

II. The numbers p and q are sought, so that there shall be $pp = 367qq + 1$

42. Here therefore there is $z = 367$, $v = 19$ and hence

$$\begin{array}{lll}
A=19, & \alpha=6, & a=6,\\
B=17, & \beta=13, & b=2,\\
C=\ 9, & \gamma=22, & c=1,\\
D=13, & \delta=9, & d=3,\\
E=14, & \varepsilon=19, & e=1,\\
F=5, & \zeta=18, & f=1,\\
G=13, & \eta=11, & g=2,\\
H=9, & \theta=26, & h=1,\\
I=17, & \iota=3, & \text{i} = 12,\\
K=19, & \chi=2, & k=\text{middle no. } 19,\\
L=19, & \lambda=3, & l=12.
\end{array}$$

Therefore this example pertains to the first kind.

$$\begin{array}{cccccccccccc}
 & 19, & 6, & 2, & 1, & 3, & 1, & 1, & 2, & 1, & 12, & 19\\
\frac{1}{0}, & \frac{19}{1}, & \frac{115}{6}, & \frac{249}{13}, & \frac{364}{19}, & \frac{1341}{70}, & \frac{1705}{89}, & \frac{3046}{159}, & \frac{7797}{407}, & \frac{10843}{566}, & \frac{137913}{7199}, & \frac{2631190}{137347}.
\end{array}$$

Hence there will be

$$x = 7199 \cdot 2631190 + 566 \cdot 137913$$

and

$$y = 7199 \cdot 137347 + 566 \cdot 7199,$$

from which the smallest satisfying numbers are

$$p = 19019995568,$$
$$q = \quad 992835687.$$

Table of the numbers p and q, for which there becomes
$pp = lqq + 1$ for all the values of the number l as far as to 100

l	q	p
2	3	3
3	1	2
5	4	9
6	2	5
7	3	8
8	1	3
10	6	19
11	3	10

12	2	7
13	180	649
14	4	15
15	1	4
17	8	33
18	4	17
19	39	170
20	2	9
21	12	55
22	42	197
23	5	24
24	1	5

26	10	51
27	5	26
28	24	127
29	1820	9801
30	2	11
31	273	1520
32	3	17
33	4	23
34	6	35
35	1	6
37	12	73
38	6	37
39	4	25
40	3	19
41	320	2049
42	2	13
43	531	3482
44	30	199
45	24	161
46	3588	24335
47	7	48
48	1	7
50	14	99
51	7	50
52	90	649
53	9100	66249
54	66	485
55	12	89
56	2	15
57	20	151
58	2574	19603

59	69	530
60	4	31
61	226153980	1766319049
62	8	63
63	1	8
65	16	129
66	8	65
67	5967	48842
68	4	33
69	936	7775
70	30	251
71	413	3480
72	2	17
73	267000	2281249
74	430	3699
75	3	26
76	6630	57799
77	40	351
78	6	53
79	9	80
80	1	9
82	18	163
83	9	82
84	6	55
85	30996	285769
86	1122	10405
87	3	28
88	21	197
89	53000	500001
90	2	19
91	165	1574
92	120	1151
93	1260	12151
94	221064	21432295
95	4	39
96	5	49
97	6377352	62809633
98	10	99
99	1	10

Finally I shall add some examples of numbers for greater values assumed for l :

If there shall be $l=103$, there will become $\begin{cases} p = 227528, \\ q = \quad 22419; \end{cases}$

if $l = 109$, there will become $\begin{cases} p = 158070671986249, \\ q = 15140424455100; \end{cases}$

if $l=113$, there will become $\begin{cases} p = 1204353, \\ q = \quad 113296; \end{cases}$

if $l=157$, there will become $\begin{cases} p = 46698728731849 \\ q = \quad 3726964292220; \end{cases}$

if $l=367$, there will become $\begin{cases} p = 19019995568, \\ q = \quad 992835687. \end{cases}$

DE USU NOVI ALGORITHMI
IN PROBLEMATE PELLIANO SOLVENDO

Commentatio 323 indicis ENESTROEMIANI
Novi commentarii academiae scientiarum Petropolitanae **11** (1765), 1767, p. 28-66

1. Quicunque numeri integri pro litteris *l*, *m* et *n* assumantur, innumerabiles quoque numeri integri pro *x* inveniri possunt, quibus haec formula $lxx+mx+n$ reddatur quadratum, siquidem sequentes conditiones habeant locum:

I. ut *l* sit numerus positivus non quadratus,

II. ut pro *x* unus saltem valor sit cognitus.

Nam si *l* est numerus vel negativus vel quadratus, manifestum est infinitas solutiones in numeris integris exhiberi non posse, etiamsi una innotuerit.Tum vero etiam evenire potest, ut formula $lxx+mx+n$ naturae quadrati prorsus adversetur, uti fit hoc casu $3xx+2$. Verum statim atque unica solutio habetur, semper innumerabiles invenire licet.

2. Quare si statuamus

$$lxx + mx + nyy = yy$$

unusque casus constet, quo huic conditioni satisfiat, ita ut posito $x = a$ prodeat

$$laa + ma + n = bb$$

sicque sumto $x = a$ obtineatur $y = b$, regula, cuius ope plures, imo infinitae solutiones elici possunt, ita se habet:

Primo ex dato numero l duo huiusmodi numeri p et q investigentur, ut sit

$$pp = lqq + l \text{ seu } p = \sqrt{(lqq + l)},$$

quibus inventis ex solutione prima cognita statim eruitur haec nova

$$x = pa + qb + \tfrac{m}{2l}(p-1),$$

unde fit

$$y = pb + lqa + \tfrac{1}{2}mq,$$

ex qua deinceps simili modo aliae derivantur. Si enim hos valores loco a et b substituamus, nascitur tertia solutio ista

$$x = (2pp-1)a + 2pqb + mqq$$

et

$$y = (2pp-1)b + 2lpqa + mpq,$$

quae certe est in numeris integris, si forte praecedentes adhuc fuerint fracti.

3. Cum igitur hoc modo continuo novae solutiones inveniri queant, ad calculi compendium plurimum iuvat notasse continuos istos valores, tam ipsius x quam ipsius y, secundum seriem recurrentem progredi, cuius singuli termini per binos praecedentes certa et constante lege determinentur. Scilicet si fuerint valores hi continuo progredientes

$$\text{ipsius } x \;\; a, \;\; \ldots \;\; P, \; Q, \; R, \; S \;\; \text{etc.},,$$
$$\text{ipsius } y \;\; b, \;\; \ldots \;\; \mathfrak{P}, \; \mathfrak{Q}, \; \mathfrak{R}, \; \mathfrak{S} \;\; \text{etc.},$$

erit per legem seriei recurrentis

$$R = 2pQ - P + \tfrac{m(p-1)}{l}, \quad \mathfrak{R} = 2p\mathfrak{Q} - \mathfrak{P},$$
$$A = 2pR - Q + \tfrac{m(p-1)}{l}, \quad \mathfrak{S} = 2p\mathfrak{R} - \mathfrak{Q}.$$

Atque hinc isti valores expressionibus generalibus comprehendi possunt, quae ita se habent

$$x=\tfrac{2la+m+2b\sqrt{l}}{4l}\left(p+q\sqrt{l}\right)^{\mu}+\tfrac{2la+m-2b\sqrt{l}}{4l}\left(p-q\sqrt{l}\right)^{\mu}-\tfrac{m}{2l},$$
$$y=\tfrac{2la+m+2b\sqrt{l}}{4\sqrt{l}}\left(p+q\sqrt{l}\right)^{\mu}-\tfrac{2la+m-2b\sqrt{l}}{4\sqrt{l}}\left(p-q\sqrt{l}\right)^{\mu},$$

unde, quicunque numeri integri exponenti μ tribuantur, semper valores rationales pro x et y resultant.

4. Haec autem investigatio multo latius ita potest extendi, ut proposita inter binos numeros x et y huiusmodi aequatione

$$Axx+2Bxy+Cyy+2Dx+2Ey+F=0$$

omnes solutiones in numeris rationalibus et integris sint eruendae. Hic quidem pariter necesse est unam solutionem esse cognitam, quae sit $x=a$ et $y=b$, ita ut sit

$$Aaa+2Bab+Cbb+2Da+2Eb+F=0\,.$$

Tum vero quaerantur bini numeri p et q, ut sit

$$pp=\left(BB-AC\right)qq+1,$$

quod quidem fieri nequit, nisi sit $BB>AC$. Atque nova solutio ita erit comparata

$$x=a\left(p+Bq\right)+bCq+Eq+\tfrac{BE-CD}{BB-AC}\left(p-1\right),$$
$$y=b\left(p-Bq\right)-aAq-Dq+\tfrac{BD-AE}{BB-AC}\left(p-1\right),$$

unde per eandem legem continuo plures elicere licet.

5. Haec ideo in medium afferre est visum, ut intelligatur ad omnes huius generis resolutiones id omnino requiri, ut proposito quocunque numero integro positivo non quadrato l eiusmodi binos numeros pariter integras p et q inveniri oporteat, ut sit

$$pp=\ lqq+1 \text{ seu } p=\sqrt{(lqq+1)}\,.$$

Atque hoc est illud problema olim quidem maxime celebratum a solutionis ingeniosissimae auctore Pellianum vocatum, quo pro quovis huiusmodi numero l numerus quadratus qq requiritur, qui per l multiplicatus adiuncta unitate fiat quadratus. In fractis quidem haec quaestio nullam haberet difficultatem, cum sumto $q=\frac{2rs}{lss-rr}$ fiat

$p=\frac{lss+rr}{lss-rr}$; verum quia numeni integri desiderantur, negotium iterum eo revocatur, ut denominator $lss-rr$ in unitatem abeat.

6. Etiamsi autem solutio Pelliana huius problematis sit elegantissima, tamen saepenumero tam operosis calculis implicatur, qui non minus taedii quam laboris creare solent. Cum igitur observassem Algorithmum illum novum, cuius nuper indolem exposui, ad hos calculos, quibus hic est opus, contrahendos insignia subsidia suppeditare, praeclarius certe specimen exhibere vix licebit, quo usus istius Algorithmi illustretur et commendetur. Ubi id imprimis notatu dignum occurrit, quod totum compendium inde subministratum potissimum idoneorum signorum usu contineatur.

7. Operationes, quibus Pellius est usus, aliunde quidem satis sunt notae egoque iam eas alia occasione fusius descripsi ; ex quo eo minus opus est, ut iis denuo explicandis hic immorer, cum totam Analysin hic longe alia ratione sim instituturus. Eius scilicet principium ex hoc fonte haurio, quod, cum sit $pp=lqq+1$, proxime fiat $\frac{p}{q}=\sqrt{l}$, ex quo manifestum $\frac{p}{q}$ eiusmodi esse fractionem, quae valorem irrationalem $\sqrt{l}$ tam prope exprimat seu eum tam parum excedat, ut id nisi maioribus numeris adhibendis accuratius fieri nequeat. Quod problema olim feliciter a Wallisio solutum equidem quoque iam dudum per fractiones continuas multo commodius expedivi.

8. Quo ergo hoc argumentum luculentius et ordine pertractem, primum radicem quadratam ex quovis numero in fractionem continuam evolvere docebo, idque methodo quam minime molesta. Deinde ostendam, quomodo inde fractiones $\frac{p}{q}$ valorem irrationalem $\sqrt{l}$ proxime exprimentes formari debeant in subsidium vocato Algorithmo novo supra explicato. Tum vero facile patebit, quomodo hinc numeros p et q definiri oporteat, ut fiat $pp=lqq+1$.
Denique tabulam subiungam, in qua pro omnibus numeris l centenarium non superantibus numeri bini p et q exhibentur.

DE EVOLUTIONE RADICUM QUADRATARUM PER FRACTIONES CONTINUAS

9. Operationes in hunc finem constituendae in exemplo facillime explicabuntur. Sit igitur proposita radix quadrata ex numero 13, et cum radix rationalis proxime minor sit 3, statuo

$$\sqrt{13}=3+\frac{1}{a}.$$

Hinc colligitur

$$a=\frac{1}{\sqrt{13}-3}=\frac{\sqrt{13}+3}{4},$$

cuius valor in integris proxime minor est 1, quod inde patet, si 3 loco $\sqrt{13}$ scribatur. Pono itaque

$$a = \tfrac{\sqrt{13}+3}{4} = 1 + \tfrac{1}{b}$$

hincque

$$b = \tfrac{4}{\sqrt{13}-1} = \tfrac{4\left(\sqrt{13}+1\right)}{12} = \tfrac{\sqrt{13}+1}{3} = 1 + \tfrac{1}{c},$$

ergo

$$c = \tfrac{3}{\sqrt{13}-2} = \tfrac{3\left(\sqrt{13}+2\right)}{9} = \tfrac{\sqrt{13}+2}{3} = 1 + \tfrac{1}{d},$$

ergo

$$d = \tfrac{3}{\sqrt{13}-1} = \tfrac{3\left(\sqrt{13}+1\right)}{12} = \tfrac{\sqrt{13}+1}{} = 1 + \tfrac{1}{e},$$

ergo

$$e = \tfrac{4}{\sqrt{13}-3} = \tfrac{4\left(\sqrt{13}+3\right)}{4} = \sqrt{13}+3 = 6 + \tfrac{1}{f},$$

ergo

$$f = \tfrac{1}{\sqrt{13}-3} = \tfrac{\sqrt{13}+3}{4} = 1 + \tfrac{1}{g};$$

atque hic operationem abrumpere licet, quia valor *f* ipsi *a* aequalis prodiit ideoque sequentes eodem ordine repetuntur. Sicque invenimus esse

$$\sqrt{13} = 3 + \cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\text{etc.}}}}}}}}}}}$$

10. Quo indoles harum operationum melius perspiciatur, aliud exemplum prolixiorem calculum postulans adiungam. Proposita scilicet sit $\sqrt{61}$; cuius valor proxime minor cum sit 7, pono

$$\sqrt{61} = 7 + \tfrac{1}{a}$$

et operationes sequenti modo erunt instituendae:

I. $a=\frac{1}{\sqrt{61}-7}=\frac{\sqrt{61}+7}{12}=1+\frac{1}{b}$,

II. $b=\frac{1}{\sqrt{61}-5}=\frac{12\left(\sqrt{61}+5\right)}{36}=\frac{\left(\sqrt{61}+5\right)}{3}=4+\frac{1}{c}$,

III. $c=\frac{3}{\sqrt{61}-7}=\frac{3\left(\sqrt{61}+7\right)}{12}=\frac{\sqrt{61}+7}{4}=3+\frac{1}{d}$,

IV. $d=\frac{4}{\sqrt{61}-5}=\frac{4\left(\sqrt{61}+5\right)}{36}=\frac{\sqrt{61}+5}{9}=1+\frac{1}{e}$,

V. $e=\frac{9}{\sqrt{61}-4}=\frac{9\left(\sqrt{61}+4\right)}{45}=\frac{\sqrt{61}+4}{5}=2+\frac{1}{f}$,

VI. $f=\frac{5}{\sqrt{61}-6}=\frac{5\left(\sqrt{61}+6\right)}{25}=\frac{\sqrt{61}+6}{5}=2+\frac{1}{g}$,

VII. $g=\frac{5}{\sqrt{61}-4}=\frac{5\left(\sqrt{61}+4\right)}{45}=\frac{\sqrt{61}+4}{9}=1+\frac{1}{h}$,

VIII. $h=\frac{9}{\sqrt{61}-5}=\frac{9\left(\sqrt{61}+5\right)}{36}=\frac{\sqrt{61}+5}{4}=3+\frac{1}{i}$,

IX. $i=\frac{4}{\sqrt{61}-7}=\frac{4\left(\sqrt{61}+7\right)}{12}=\frac{\sqrt{61}+7}{3}=4+\frac{1}{k}$,

X. $i=\frac{3}{\sqrt{61}-5}=\frac{12\left(\sqrt{61}+7\right)}{36}=\frac{\sqrt{61}+5}{12}=1+\frac{1}{l}$,

XI. $l=\frac{12}{\sqrt{61}-7}=\frac{12\left(\sqrt{61}+7\right)}{12}=\sqrt{61}+7=14+\frac{1}{m}$,

XII. $m=\frac{1}{\sqrt{61}-7}$,

ergo $m=a$ hincque porro $n=b,\ o=c$ etc. Ex quo indices pro fractione continua erunt

7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14, 1, 4, 3, 1, 2 etc.

neque opus est ipsam fractionem continuam hic exhibere.

11. Adhuc aliud exemplum adiecisse iuvabit, ubi indicum numerus, antequam iidem recurrunt, fit impar. Esto hoc exemplum

$$\sqrt{67}=8+\frac{1}{a}$$

et operationes sequentes institui oportebit:

$$\text{I.}\quad a=\tfrac{1}{\sqrt{67}-8}=\tfrac{\sqrt{67}+8}{3}=5+\tfrac{1}{b},$$

$$\text{II.}\quad b=\tfrac{3}{\sqrt{67}-7}=\tfrac{3\left(\sqrt{67}+8\right)}{18}=\tfrac{\sqrt{67}+7}{6}=2+\tfrac{1}{c},$$

$$\text{III.}\quad c=\tfrac{6}{\sqrt{67}-5}=\tfrac{6\left(\sqrt{67}+5\right)}{42}=\tfrac{\sqrt{67}+5}{7}=1+\tfrac{1}{d},$$

$$\text{IV.}\quad d=\tfrac{7}{\sqrt{67}-2}=\tfrac{7\left(\sqrt{67}+2\right)}{63}=\tfrac{\sqrt{67}+2}{9}=1+\tfrac{1}{e},$$

$$\text{V.}\quad e=\tfrac{9}{\sqrt{67}-7}=\tfrac{9\left(\sqrt{67}+7\right)}{18}=\tfrac{\sqrt{67}+7}{2}=7+\tfrac{1}{f},$$

$$\text{VI.}\quad f=\tfrac{2}{\sqrt{67}-7}=\tfrac{2\left(\sqrt{67}+7\right)}{18}=\tfrac{\sqrt{67}+7}{9}=1+\tfrac{1}{g},$$

$$\text{VII.}\quad g=\tfrac{9}{\sqrt{67}-2}=\tfrac{9\left(\sqrt{67}+2\right)}{63}=\tfrac{\sqrt{67}+2}{7}=1+\tfrac{1}{h},$$

$$\text{VIII.}\quad h=\tfrac{7}{\sqrt{67}-5}=\tfrac{7\left(\sqrt{67}+5\right)}{42}=\tfrac{\sqrt{67}+5}{6}=2+\tfrac{1}{i},$$

$$\text{IX.}\quad i=\tfrac{6}{\sqrt{67}-7}=\tfrac{6\left(\sqrt{67}+7\right)}{18}=\tfrac{\sqrt{67}+7}{3}=5+\tfrac{1}{k},$$

$$\text{X.}\quad k=\tfrac{3}{\sqrt{67}-8}=\tfrac{3\left(\sqrt{67}+8\right)}{3}=\sqrt{67}+8=16+\tfrac{1}{l},$$

$$\text{XI.}\quad k=\tfrac{1}{\sqrt{67}-8},$$

ergo $l=a$ indeque indices *b, c, d* etc. ordine recurrunt; quare indices fractionis continuae quaesitae sunt

8, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16 etc.

12. His exemplis probe perpensis poterimus iam in genere operationes describere, quibus pro cuiusvis numeri radice quadrata fractio continua ipsi aequalis seu indices eam constituentes inveniuntur. Sit scilicet numerus propositus *z* eiusque radix integra proxime minor *v*, vera autem hac fractione continua exprimatur

$$\sqrt{z}=v+\cfrac{1}{a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{etc.}}}}}$$

cuius indices *a, b, c, d* etc. post primum *v* per se cognitum sequentibus operationibus reperiuntur:

Capiatur	tum vero	eritque
I. $A=v,$	$\alpha = z - AA = z - vv$	$a < \frac{v+A}{\alpha},$
II. $B = \alpha a - A,$	$\beta = \frac{z-BB}{\alpha} = 1+a\left(A-B\right)$	$b < \frac{v+B}{\beta},$
III. $C = \beta b - B,$	$\gamma = \frac{z-CC}{\beta} = \alpha + b\left(B-C\right)$	$c < \frac{v+C}{\gamma},$
IV. $D = \gamma c - C,$	$\delta = \frac{z-DD}{\gamma} = \beta + c\left(C-D\right)$	$d < \frac{v+D}{\delta},$
V. $E = \delta d - D,$	$\varepsilon = \frac{z-EE}{\delta} = \gamma + d\left(D-E\right)$	$e < \frac{v+E}{\varepsilon},$
	etc.	

ubi in postrema columna signum < indicat pro litteris *a*, *b*, *c*, *d* etc. sumi debere numeros integros proxime minores fractionibus adiectis, nisi hae fractiones ipsae in numeros integros abeant, quo casu hi ipsi erunt indices.

13. Pro indicibus igitur *a, b, c, d* etc. eliciendis binas alias numerorum series investigari oportet, quarum priorem litteris maiusculis

$$A,\ B,\ C,\ D \text{ etc.},$$

posteriorem vero graecis

$$\alpha,\ \beta,\ \gamma,\ \delta \text{ etc.}$$

designavi. Circa priores numeros observo eos numerum *v* nunquam superare posse. Eorum quidem primus est $A = v$, at cum sit $a < \frac{v+A}{\alpha}$, erit $\alpha a - A < v$ ideoque $B < v$ vel ad summum $B = v$, quo casu fit $\beta = 1$ et $b = 2v$. Deinde ob $b < \frac{v+B}{\beta}$ est $\beta b - B = C < v$ similique modo $D < v,\ E < v$ etc., ita ut horum numerorum nullus ipso *v* maior prodire possit. Deinde patet praeter casus, quibus graecarum litterarum quaepiam fit unitas, indices *a, b, c* etc. omnes ipso *v* maiores esse non posse, quandoquidem in fractionibus $\frac{v+B}{\beta},\ \frac{v+C}{\gamma}$ etc. numeratores non excedere possunt 2*v*, denominatores vero ad minimum sint $= 2$. Denique cum fuerit perventum ad indicem $= 2v$, sequentes iterum prodeunt *a, b, c, d* etc.

14. Illustremus etiam has operationes nonnullis exemplis.

I. Sit $z = 31$; erit $v = 5$.

$$A = 5, \qquad \alpha = 6, \qquad a < \tfrac{10}{6} = 1,$$

$$B = 6 - 5 = 1, \qquad \beta = 1 + 1 \cdot 4 = 5, \qquad b < \tfrac{6}{5} = 1,$$

$$c = 5 - 1 = 4, \qquad \gamma = 6 - 1 \cdot 3 = 3, \qquad c < \tfrac{9}{3} = 3,$$

$$D = 9 - 4 = 5, \quad \delta = 5 - 3 \cdot 1 = 2, \quad d < \tfrac{10}{2} = 5,$$

$$E = 10 - 5 = 5, \quad \varepsilon = 3 + 5 \cdot 0 = 3, \quad e < \tfrac{10}{3} = 3,$$

$$F = 9 - 5 = 4, \quad \xi = 2 + 3 \cdot 1 = 5, \quad f < \tfrac{9}{5} = 1,$$

$$G = 5 - 4 = 1, \quad \eta = 3 + 1 \cdot 3 = 6, \quad g < \tfrac{6}{6} = 1,$$

$$H = 6 - 1 = 5, \quad \theta = 5 - 1.4 = 1, \quad h < \tfrac{10}{1} = 10 .$$

II. Sit $z = 46$; erit $v = 6$.

$$A = 6, \quad \alpha = 10, \quad a < \tfrac{12}{10} = 1,$$

$$B = 10 - 6 = 4, \quad \beta = 1 + 1 \cdot 2 = 3, \quad b < \tfrac{10}{3} = 3,$$

$$C = 9 - 4 = 5, \quad \gamma = 10 - 3 \cdot 1 = 7, \quad c < \tfrac{11}{7} = 1,$$

$$D = 7 - 5 = 2, \quad \delta = 3 + 1 \cdot 3 = 6, \quad d < \tfrac{8}{6} = 1,$$

$$E = 6 - 2 = 4, \quad \varepsilon = 7 - 1 \cdot 2 = 5, \quad e < \tfrac{10}{5} = 2,$$

$$F = 10 - 4 = 6, \quad \xi = 6 - 2 \cdot 2 = 2, \quad f < \tfrac{12}{2} = 6,$$

$$G = 12 - 6 = 6 \quad \eta = 5 + 6 \cdot 0 = 5, \quad g < \tfrac{12}{5} = 2,$$

$$H = 10 - 6 = 4, \quad \theta = 2 + 2 \cdot 2 = 6, \quad h < \tfrac{10}{6} = 1,$$

$$I = 6 - 4 = 2, \quad \iota = 5 + 1 \cdot 2 = 7, \quad i < \tfrac{8}{7} = 1,$$

$$K = 7 - 2 = 5, \quad \chi = 6 - 1 \cdot 3 = 3, \quad k < \tfrac{11}{3} = 3,$$

$$L = 9 - 5 = 4, \quad \lambda = 7 + 3 \cdot 1 = 10, \quad l < \tfrac{10}{10} = 1,$$

$$M = 10 - 4 = 6, \quad \mu = 3 - 1 \cdot 2 = 1, \quad m < \tfrac{12}{1} = 12.$$

III. Sit $z = 54$; erit $v = 7$.

$$A = 7, \quad \alpha = 5, \quad a < \tfrac{14}{5} = 2,$$

$$B = 10 - 7 = 3, \quad \beta = 1 + 2 \cdot 4 = 9, \quad b < \tfrac{10}{9} = 1,$$

$$C = 9 - 3 = 6, \quad \gamma = 5 - 1 \cdot 3 = 2, \quad c < \tfrac{13}{2} = 6,$$

$$D = 12 - 6 = 6, \quad \delta = 9 + 6 \cdot 0 = 9, \quad d < \tfrac{13}{9} = 1,$$

$$E = 9 - 6 = 3, \quad \varepsilon = 2 + 1 \cdot 3 = 5, \quad e < \tfrac{10}{5} = 2,$$

$$F = 10 - 3 = 7, \quad \zeta = 9 - 2 \cdot 4 = 1, \quad f < \tfrac{14}{1} = 14.$$

15. Tabulam ergo hic subiungam pro singulorum numerorum radicibus quadratis indices continentem, ex quibus fractiones continuae ipsis aequales formari queant. Simul vero litterarum graecarum singulis convenientium valores subscripti reperiuntur.

Numeri surdi	Indices
$\sqrt{2}$	1, 2, 2, 2, etc. 1, 1, 1, 1
$\sqrt{3}$	1, 1, 2, 1, 2, 1, 2 etc. 1 2 1 2 1 2 1
$\sqrt{5}$	2, 4, 4, 4 etc. 1, 1, 1, 1
$\sqrt{6}$	2, 2, 4, 2, 4, 2, 4 etc. 1 2 1 2 1 2 1
$\sqrt{7}$	2, 1, 1, 1, 4, 1, 1, 1, 4 etc. 1 3 2 3 1 3 2 3 1
$\sqrt{8}$	2, 1, 4, 1, 4, 1, 4 etc. 1 4 1 4 1 4 1
$\sqrt{10}$	3, 6, 6, 6 etc. 1 1 1 1
$\sqrt{11}$	3, 3, 6, 3, 6, 3, 6 etc. 1 2 1 2 1 2 1
$\sqrt{12}$	3, 2, 6, 2, 6, 2, 6 etc. 1 3 1 3 1 3 1
$\sqrt{13}$	3, 1, 1, 1, 1, 6, 1, 1, 1, 1, 6 etc. 1 4 3 3 4 1 4 3 3 4 1
$\sqrt{14}$	3, 1, 2, 1, 6, 1, 2, 1, 6 etc. 1 5 2 5 1 5 2 5 1
$\sqrt{15}$	3, 1, 6, 1, 6, 1, 6 etc. 1 6 1 6 1 6 1
$\sqrt{17}$	4, 8, 8, 8, 8 etc. 1 1 1 1 1
$\sqrt{18}$	4, 4, 8, 4, 8, 4, 8, 4, 8 etc. 1 2 1 2 1 2 1 2 1
$\sqrt{19}$	4, 2, 1, 3, 1, 2, 8, 2, 1, 3, 1, 2, 8 etc. 1 3 5 2 6 3 1 3 5 2 5 3 1
$\sqrt{20}$	4, 2, 8, 2, 8, 2, 8, 2, 8 etc. 1 4 1 4 1 4 1 4 1
$\sqrt{21}$	4, 1, 1, 2, 1, 1, 8, 1, 1, 2, 1, 1, 8 etc. 1 5 4 3 4 5 1 5 4 3 4 5 1

$\sqrt{22}$	4, 1, 2, 4, 2, 1, 8, 1, 2, 4, 2, 1, 8 etc. 1 6 3 2 3 6 1 6 3 2 3 6 1
$\sqrt{23}$	4, 1, 3, 1, 8, 1, 3, 1, 8 etc. 1 7 2 7 1 7 2 7 1
$\sqrt{24}$	4, 1, 8, 1, 8, 1, 8 etc. 1 8 1 8 1 8 1
$\sqrt{26}$	5, 10, 10, 10 etc. 1 1 1 1
$\sqrt{27}$	5, 5, 10, 5, 10, 5, 10 etc. 1 2 1 2 1 2 1
$\sqrt{28}$	5, 3, 2, 3, 10, 3, 2, 3, 10 etc. 1 3 4 3 1 3 4 3 1
$\sqrt{29}$	5, 2, 1, 1, 2, 10, 2, 1, 1, 2, 10 etc. 1 4 5 5 4 1 4 5 5 4 1
$\sqrt{30}$	5, 2, 10, 2, 10, 2, 10, 2, 10 etc. 1 5 1 5 1 5 1 5 1
$\sqrt{31}$	5, 1, 1, 3, 5, 3, 1, 1, 10 etc. 1 6 5 8 2 3 5 6 1
$\sqrt{32}$	5, 1, 1, 1, 10, 1, l, 1, 10 etc. 1 7 4 7 1 7 4 7 1
$\sqrt{33}$	5, 1, 2, 1, 10, 1, 2, 1, 10 etc. 1 8 3 8 1 8 3 8 1
$\sqrt{34}$	5, 1, 4, 1, 10, 1, 4, 1, 10 etc. 1 9 2 9 1 9 2 9 1
$\sqrt{35}$	5, 1, 10, 1, 10, 1, 10 etc. 1 10 1 10 1 10 1
$\sqrt{37}$	6, 12, 12, 12 etc. 1 1 1 1
$\sqrt{38}$	6, 6, 12, 6, 12, 6, 12 etc. 1 2 1 2 1 2 1
$\sqrt{39}$	6, 4, 12, 4, 12, 4, 12 etc. 1 3 1 3 1 3 1
$\sqrt{40}$	6, 3, 12, 3, 12, 3, 12 etc. 1 4 1 4 1 4 1
$\sqrt{41}$	6, 2, 2, 12, 2, 2, 12 etc. 1 5 5 1 5 5 1
$\sqrt{42}$	6, 2, 12, 2, 12, 2, 12 etc. 1 6 1 6 1 6 1
$\sqrt{43}$	6, 1, 1, 3, 1, 5, 1, 3, 1, 1, 12 etc. 1 7 6 3 9 2 9 3 6 7 1
$\sqrt{44}$	

6, 1, 1, 1, 2, 1, 1, 1, 12 etc.
1 8 5 7 4 7 5 8 1

$\sqrt{45}$

6, 1, 2, 2, 2, 1, 12, 1, 2, 2, 2, 1, 12 etc.
1 9 4 5 4 9 1 9 4 5 4 9 1

$\sqrt{46}$

6, 1, 3, 1, 1, 2, 6, *2*, 1, 1, 3, 1, 12 etc.
1 10 3 7 6 5 2 5 6 7 3 10 1

$\sqrt{47}$

6, 1, 5, 1, 12, 1, 5, 1, 12 etc.
1 11 2 11 1 11 2 11 1

$\sqrt{48}$

6, 1, 12, 1, 12, 1, 12 etc.
1 12 1 12 1 12 1

$\sqrt{50}$ 7, 14, 14, 14 etc.
1 1 1 1

$\sqrt{51}$ 7, 7, 14, 7, 14, 7, 14 etc.
1 2 1 2 1 2 1

$\sqrt{52}$ 7, 4, 1, 2, 1, 4, 14, 4, 1, 2, 1, 4, 14 etc.
1 3 9 4 9 3 1 3 9 4 9 3 1

$\sqrt{53}$ 7, 3, 1, 1, 3, 14, 3, 1, 1, 3, 14 etc.
1 4 7 7 4 1 4 7 7 4 1

$\sqrt{54}$ 7, 2, 1, 6, 1, 2, 14, 2, 1, 6, 1, 2, 14 etc.
1 5 9 2 9 5 1 6 9 2 9 5 1

$\sqrt{55}$ 7, 2, 2, 2, 14, 2, 2, 2, 14, 2, 2, 2, 14 etc.
1 6 5 6 1 6 5 6 1 6 5 6 1

$\sqrt{56}$ 7, 2, 14, 2, 14, 2, 14 etc.
1 7 1 7 1 7 1

$\sqrt{57}$ 7, 1, 1, 4, 1, 1, 14 etc.
1 8 7 3 7 8 1

$\sqrt{58}$ 7, 1, 1, 1, 1, 1, 1, 14 etc.
1 9 6 7 7 6 9 1

$\sqrt{59}$ 7, 1, 2, 7, 2, 1, 14 etc.
1 10 5 2 5 10 1

$\sqrt{60}$ 7, 1, 2, 1, 14 etc.
1 11 4 11 1

$\sqrt{61}$ 7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14 etc.
1 12 3 4 9 5 5 9 4 3 12 1

$\sqrt{62}$ 7, 1, 6, 1, 14 etc.
1 13 2 13 1

$\sqrt{63}$ 7, 1, 14, 1, 14 etc.

	1 14 1 14 1
$\sqrt{65}$	8, 16, 16 etc. 1 1 1
$\sqrt{66}$	8, 8, 16, 8, 16 etc. 1 2 1 2 1
$\sqrt{67}$	8, 5, 2, 1, 1, 7, 1, 1, 2, 5, 16 etc. 1 3 6 7 9 2 9 7 6 3 1
$\sqrt{68}$	8, 4, 16, 4, 16 etc. 1 4 1 4 1
$\sqrt{69}$	8, 3, 3, 1, 4, 1, 3, 3, 16 etc. 1 5 4 11 3 11 4 5 1
$\sqrt{70}$	8, 2, 1, 2, 1, 2, 16 etc. 1 6 9 5 9 6 1
$\sqrt{71}$	8, 2, 2, 1, 7, 1, 2, 2, 16 etc. 1 7 5 11 2 11 5 7 1
$\sqrt{72}$	8, 2, 16, 2, 16 etc. 1 8 1 8 1
$\sqrt{73}$	8, 1, 1, 5, 5, 1, 1, 16 etc. 1 9 8 3 3 8 9 1
$\sqrt{74}$	8, 1, 1, 1, 1, 16 etc. 1 10 7 7 10 1
$\sqrt{75}$	8, 1, 1, 1, 16 etc. 1 11 6 11 1
$\sqrt{76}$	8, 1, 2, 1, 1, 5, 4, 5, 1, 1, 2, 1, 16 etc. 1 12 5 8 9 3 4 3 9 8 5 12 1
$\sqrt{77}$	8, 1, 3, 2, 3, 1, 16 etc. 1 13 4 7 4 13 1
$\sqrt{78}$	8, 1, 4, 1, 16 etc. 1 14 3 14 1
$\sqrt{79}$	8, 1, 7, 1, 16 etc. 1 15 2 15 1
$\sqrt{80}$	8, 1, 16, 1, 16 etc. 1 16 1 16 1
$\sqrt{82}$	9, 18, 18, 18 etc. 1 1 1 1
$\sqrt{83}$	9, 9, 18, 9, 18 etc. 1 2 1 2 1
$\sqrt{84}$	9, 6, 18, 6, 18 etc. 1 3 1 3 1

$\sqrt{85}$	9, 4, 1, 1, 4, 18 etc.	1 4 9 9 4 1
$\sqrt{86}$	9, 3, 1, 1, 1, 8, 1, 1, 1, 3, 18 etc.	1 5 10 7 11 2 11 7 10 5 1
$\sqrt{87}$	9, 3, 18, 3, 18 etc.	1 6 1 6 1
$\sqrt{88}$	9, 2, 1, 1, 1, 2, 18 etc.	1 7 9 8 9 7 1
$\sqrt{89}$	9, 2, 3, 3, 2, 18 etc.	1 8 6 5 8 1
$\sqrt{90}$	9, 2, 18, 2, 18 etc.	1 9 1 9 1
$\sqrt{91}$	9, 1, 1, 5, 1, 5, 1, 1, 18 etc.	1 10 9 *8* 14 3 9 10 1
$\sqrt{92}$	9, 1, 1, 2, 4, 2, 1, 1, 18 etc.	1 11 8 7 4 7 8 11 1
$\sqrt{93}$	9, 1, 1, 1, 4, 6, 4, 1, 1, 1, 18 etc.	1 12 7 11 4 3 4 11 7 12 1
$\sqrt{94}$	9, 1, 2, 3, 1, 1, 5, 1, 8, 1, 5, 1, 1, 3, 2, 1, 18 etc.	1 13 6 6 9 10 3 15 2 15 8 10 9 6 6 13 1
$\sqrt{95}$	9, 1, 2, 1, 18 etc.	1 14 5 14 1
$\sqrt{96}$	9, 1, 3, 1, 18 etc.	1 15 4 15 1
$\sqrt{97}$	9, 1, 5, 1, 1, 1, 1, 1, 1, 5, 1, 18 etc.	1 16 3 11 8 9 9 8 11 3 16 1
$\sqrt{98}$	9, 1, 8, 1, 18 etc.	1 17 2 17 1
$\sqrt{99}$	9, 1, 18, 1, 18 etc.	1 18 1 18 1

Example of a special algorithm

$\sqrt{101}$	10, 20, 20 etc. 1 1 1
$\sqrt{102}$	10, 10, 20, 10, 20 etc. 1 2 1 2 1
$\sqrt{103}$	10, 6, 1, 2, 1, 1, 9, 1, 1, 2, 1, 6, 20 etc. 1 3 18 6 9 11 2 11 9 6 13 3 1
$\sqrt{104}$	10, 5, 20, 5, 20 etc. 1 4 1 4 1
$\sqrt{105}$	10, 4, 20, 4, 20 etc. 1 5 1 5 1
$\sqrt{106}$	10, 3, 2, 1, 1, 1, 1, 2, 3, 20 etc. 1 6 7 10 9 9 10 7 6 1
$\sqrt{107}$	10, 2, 1, 9, 1, 2, 20 etc. 1 7 13 2 13 7 1
$\sqrt{108}$	10, 2, 1, 1, 4, 1, 1, 2, 20 etc. 1 8 9 11 4 11 9 8 1
$\sqrt{109}$	10, 2, 3, 1, 2, 4, 1, 6, 6, 1, 4, 2, 1, 3, 2, 20 etc. 1 9 5 12 7 4 15 3 3 15 4 7 12 5 9 1
$\sqrt{110}$	10, 2, 20, 2, 20 etc. 1 10 1 10 1
$\sqrt{111}$	10, 1, 1, 6, 1, 1, 20 etc. 1 11 10 3 10 11 1
$\sqrt{112}$	10, 1, 1, 2, 1, 1, 20 etc. 1 12 9 7 9 12 1
$\sqrt{113}$	10, 1, 1, 1, 2, 2, 1, 1, 1, 20 etc. 1 13 8 11 7 7 11 8 13 1
$\sqrt{114}$	10, 1, 2, 10, 2, 1, 20 etc. 1 14 7 2 7 14 1
$\sqrt{115}$	10, 1, 2, 1, 1, 1, 1, 1, 2, 1, 20 etc. 1 15 6 11 9 10 9 11 6 15 1
$\sqrt{116}$	10, 1, 3, 2, 1, 4, 1, 2, 3, 1, 20 etc. 1 16 5 7 13 4 13 7 5 16 1
$\sqrt{117}$	10, 1, 4, 2, 4, 1, 20 etc. 1 17 4 9 4 17 1
$\sqrt{118}$	10, 1, 6, 3, 2, 10, 2, 3, 6, 1, 20 etc. 1 18 3 6 9 2 9 6 3 18 1
$\sqrt{119}$	10, 1, 9, 1, 20 etc. 1 19 2 19 1
	10, 1, 20, 1, 20 etc.

$\sqrt{120}$	1 20 1 20 1

16. In omnibus his indicum seriebus periodi deprehenduntur modo strictiores modo largiores, quae indicibus iis, qui primo duplo sunt maiores, includuntur, atque hae periodi eo clarius in oculos incidunt, si primi indices cuiusque seriei duplicantur. Deinde in qualibet periodo idem indicum ordo, sive antrorsum sive retrorsum, observatur; ex quo in qualibet periodo vel unus datur index medius vel duo, prout terminorum numerus fuerit par vel impar. In litteris vero etiam graecis similes periodi observantur, ubi imprimis animadvertendum pro omnibus indicibus 2*v* litteram graecam in unitatem abire. Hanc proprietatem insignem, quae in ipsis operationibus facilius perspicitur quam verborum ambage demonstratur, probe notasse in sequentibus plurimum intererit.

17. Ex his autem exemplis formas quasdam generales colligere licet, quae ita se habent:

I.	Si $z = nn + 1$,	erunt indices	n, $2n$, $2n$, $2n$ etc.,	1 1 1 1
II.	si $z = nn + 2$,	erunt indices	n n $2n$ n $2n$ etc.,	1 2 1 2 1
III.	si $z = nn + n$	erunt indices	n, 2, $2n$, 2, $2n$ etc.,	1 n 1 n 1
IV.	si $z = nn + 2n - 1$	erunt indices	n, 1, n-1, 1, $2n$ etc.,	1 $2n$-1 2 $2n$-1 1
V.	si $z = nn + 2n$	erunt indices	n, 1, $2n$, 1, $2n$ etc.	1 $2n$ 1 $2n$ 1

Ac fractionum quidem continuarum ex his indicibus formatarum valor in genere facile definitur idemque, quem hic assignavimus, deprehenditur. Tum vero etiam patet,

VI.	si sit $z = 4nn+4$,	erunt indices	$\overset{2n,}{\scriptstyle 1}\ \overset{n,}{\scriptstyle 4}\ \overset{4n,}{\scriptstyle 1}\ \overset{n,}{\scriptstyle 4}\ \overset{4n}{\scriptstyle 1}$ etc.,
VII.	si sit $z = 9nn+3$,	erunt indices	$\overset{3n,}{\scriptstyle 1}\ \overset{2n,}{\scriptstyle 3}\ \overset{6n,}{\scriptstyle 1}\ \overset{2n,}{\scriptstyle 3}\ \overset{6n}{\scriptstyle 1}$ etc.,
VIII.	si sit $z = 9nn+6$	erunt indices	$\overset{3n,}{\scriptstyle 1}\ \overset{n,}{\scriptstyle 6}\ \overset{6n,}{\scriptstyle 1}\ \overset{n,}{\scriptstyle 6}\ \overset{6n}{\scriptstyle 1}$ etc.

DE RESOLUTIONE FORMULAE $p = \sqrt{(lqq+1)}$ IN NUMERIS INTEGRIS

18. Inventis indicibus pro radice quadrata numeri cuiusvis z ea hoc modo per fractionem continuam exprimitur:

$$\sqrt{z} = v + \cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}}$$

atque ex his indicibus v, a, b, c, d etc. fractiones $\frac{x}{y}$ formari possunt, quae tam prope ad $\sqrt{z}$ accedunt, ut nonnisi maioribus numeris adhibendis eius valor accuratius exhiberi possit. Hae fractiones autem ita formantur:

$$\begin{array}{lccccccc} \text{Indices} & v, & a, & b, & c, \cdot\ \cdot\ \cdot\ \cdot & m, & n, & \\ \frac{x}{y} & \frac{1}{0}, & \frac{v}{1}, & \frac{av+1}{a}, & \frac{(ab+1)v+b}{ab+1}, \cdot\ \cdot\ \cdot\ \cdot & \frac{M}{P}, & \frac{N}{Q}, & \frac{nN+M}{nQ+P}, \end{array}$$

quae continuo propius valorem irrationalem $\sqrt{z}$ exprimunt.

19. Novus autem Algorithmus succinctum modum suppeditat has fractiones commode per indices repraesentandi, quae ita se habent:

$$\frac{1}{0},\ \frac{(v)}{1},\ \frac{(v,a)}{(a)},\ \frac{(v,a,b)}{(a,b)},\ \frac{(v,a,b,c)}{(a,b,c)},\ \frac{(v,a,b,c,d)}{(a,b,c,d)} \text{ etc.;}$$

ubi cum ex natura progressionis sit

$$(v,\ a) = a(v)+1,\quad (v,\ a,\ b) = b(v,\ a)+(v),\quad (v,\ a,\ b,\ c) = c(v,\ a,\ b)+(v,\ a),$$
$$(a) = a1+0,\qquad (a,\ b) = b(a)+1,\qquad (a,\ b,\ c) = c(a,\ b)+(a),$$

erit etiam ex natura harum formularum

$$(v,\ a) = v(a) + 1,\ (v,\ a,\ b) = v(a,\ b) + b,\ (v,\ a,\ b,\ c) = v(a,\ b,\ c) + (b,\ c);$$

deinde etiam sequentes transformationes demonstravi

$$(v,\ a,\ b,\ c,\ d,\ e) = v(a,\ b,\ c,\ d,\ e) + (b,\ c,\ d,\ e),$$
$$(v,\ a,\ b,\ c,\ d,\ e) = (v,\ a)(b,\ c,\ d,\ e) + v(c,\ d,\ e),$$
$$(v,\ a,\ b,\ c,\ d,\ e) = (v,\ a,\ b)(c,\ d,\ e) + (v,a)(d,\ e),$$
$$(v,\ a,\ b,\ c,\ d,\ e) = (v,\ a,\ b,\ c)(d,\ e) + (v,a,\ b)(e),$$

quas probe notasse in sequentibus plurimum iuvabit.

20. Videamus iam, quam prope singulae istae fractiones ad valorem $\sqrt{z}$ accedant, quod pro instituto nostro luculentissime inde patebit, si ex quaque fractione $\frac{x}{y}$ valorem $xx - zyy$ colligamus; quippe qui quo minor fuerit prae *y* ipsis numeris *x* et *y*, eo exactius fractio $\frac{x}{y}$ valori $\sqrt{z}$ aequabitur. Ac primo quidem si $\frac{x}{y} = \frac{1}{0}$, erit $xx - zyy = 1$. Deinde sumto $\frac{x}{y} = \frac{v}{1}$ fit

$$xx - zyy = vv - z,$$

quae differentia per operationes supra (§12) expositas prima littera graeca negative sumta $-\alpha$ designatur. Porro posito $\frac{x}{y} = \frac{(v,\ a)}{(a)} = \frac{va+1}{a}$ colligitur

$$xx - zyy = (vv - z)aa + 2va + 1 = -\alpha aa + 2va + 1,$$

ergo

$$xx - zyy = 1 + a(2v - aa) = 1 + a(A - B) = \beta$$

ob $v = A$ et $\alpha a = A + B$. Quocirca hoc casu fit $xx - zyy = \beta$.

21. Cum igitur nacti simus

$$vv - z = -\alpha \ \text{ et } \ (v,a)^2 - z(a)^2 = \beta,$$

hinc ulterius progredi poterimus. Sit igitur

$$\frac{x}{y} = \frac{(v,\ a,\ b)}{(a,b)} = \frac{b(v,a)+v}{b(a)+1}$$

atque adhibitis illis reductionibus obtinebimus

$$xx - zyy = \beta bb + 2vb(v,\ a) - 2zb(a) - \alpha,$$

ergo ob $(v,a) = v(a) + 1$ erit

$$xx - zyy = \beta bb - 2\alpha ab + 2vb - \alpha = -\alpha - b(2\alpha a - \beta b - 2v):$$

at est $v = A,\ \alpha a = A + B$ et $\beta b = B + C$ ideoque

$$xx - zyy = -\alpha - b\left(B - C\right) = -\gamma,$$

ita ut sit

$$\left(v,\ a,\ b\right)^2 - z\left(a,\ b\right)^2 = -\gamma\,.$$

22. Consideremus nunc fractionem sequentem

$$\frac{x}{y} = \frac{(v,\,a,\,b,\,c)}{\left(a,\,b,\,c\right)} = \frac{c\left(v,\,a,\,b\right)+(v,\,a)}{c\left(a,\,b\right)+a},$$

ex qua colligitur

$$xx - zyy = -\gamma cc + 2c(v, a,\ b)\left(v,\ a\right) + \beta - 2zca\left(a,\ b\right),$$

cuius pars media reducitur ad $2c\left(\beta b - \alpha a + v\right)$, unde ob

$$v = A,\ \alpha a = A + B,\quad \beta b = B + C,\quad \gamma c = C + D$$

resultat

$$xx - zyy = \beta + c\left(C - D\right) = \delta,$$

ita ut sit

$$(v,\ a,\ b,\ c)^2 - z\left(a,\ b,\ c\right)^2 = \delta,$$

unde per inductionem sequentes valores facile colliguntur.

23. Ne autem hic inductioni nimium videar tribuisse, sequenti modo haec investigatio instituti potest. Sit

$$\begin{array}{lll}
(v)^2 & -z1^2 & = \mathfrak{A} \\
(v,\ a)^2 & -z\left(a\right)^2 & = \mathfrak{B} \\
(v,\ a,\ b)^2 & -z\left(a,\ b\right)^2 & = \mathfrak{C} \\
\left(v,\ a,\ b,\ c\right)^2 & -z\left(a,\ b,\ c\right)^2 & = \mathfrak{D} \\
\left(v,\ a,\ b,\ c,\ d\right)^2 & -z\left(a,\ b,\ c,\ d\right)^2 & = \mathfrak{E}
\end{array}$$

etc.

ubi quidem iam vidimus esse $\mathfrak{A} = -\alpha,\ \ \mathfrak{B} = \beta,\ \ \mathfrak{C} = -\gamma$ etc. Cum vero sit

$$\begin{array}{llll}
\left(v,\ a\right) & = a(v) \quad +1, & \left(a\right) & = a, \\
\left(v,\ a,\ b\right) & = b(v,\ a) \quad +\left(v\right), & \left(a,\ b\right) & = b\left(a\right) \quad +1, \\
\left(v,\ a,\ b,\ c\right) & = c(v,\ a,\ b) \quad +\left(v,\ a\right), & (a,\ b,\ c) & = c\left(a,\ b\right) \quad +\left(a\right), \\
\left(v,\ a,\ b,\ c,\ d\right) & = c(v,\ a,\ b,\ c) + \left(v,\ a,\ b\right), & (a,\ b,\ c,\ d) & = d\left(a,\ b,\ c\right) + \left(a,\ b\right)
\end{array}$$

habebimus

$$\mathfrak{B} = \mathfrak{A}aa + 1 + 2a(v),$$
$$\mathfrak{C} = \mathfrak{B}bb + \mathfrak{A} + 2b\big((v,\ a)(v) - z(a)\big),$$
$$\mathfrak{D} = \mathfrak{C}cc + \mathfrak{B} + 2c\big((v,\ a,\ b)(v,a) - z(a,\ b)(a)\big),$$
$$\mathfrak{E} = \mathfrak{D}dd + \mathfrak{C} + 2d\big((v,\ a,\ b,\ c)(v,\ a,\ b) - z(a,\ b,\ c)(a,\ b)\big),$$
$$\mathfrak{F} = \mathfrak{E}ee + \mathfrak{D} + 2e\big((v,a,\ b,\ c,\ d)(v,\ a,\ b,\ c) - z(a,\ b,\ c,\ d)(a,\ b,c)\big)$$

etc.

24. Statuamus iam brevitatis gratia

$$\mathfrak{B} = 1 + \mathfrak{A}ab + 2a\cdot O,$$
$$\mathfrak{C} = \mathfrak{A} + \mathfrak{B}bb + 2b\cdot P,$$
$$\mathfrak{D} = \mathfrak{B} + \mathfrak{C}cc + 2c\cdot Q,$$
$$\mathfrak{E} = \mathfrak{C} + \mathfrak{D}dd + 2d\cdot R,$$
$$\mathfrak{F} = \mathfrak{D} + \mathfrak{E}ee + 2e\cdot S$$

etc.

et ex superioribus reductionibus colligemus

$$P - O = a(v)^2 - za = \mathfrak{A}a,$$
$$Q - P = b(v,a)^2 - zb(a)^2 = \mathfrak{B}b,$$
$$R - Q = c(v,\ a,\ b)^2 - cz(a,\ b)^2 = \mathfrak{C}c,$$
$$S - R = d(v,\ a,\ b,\ c)^2 - dz(a,\ b,c)^2 = \mathfrak{D}d$$

etc.

sicque fiet

$$O = v,$$
$$P = v + \mathfrak{A}a,$$
$$Q = v + \mathfrak{A}a + \mathfrak{B}b,$$
$$R = v + \mathfrak{A}a + \mathfrak{B}b + \mathfrak{C}c,$$
$$S = v + \mathfrak{A}a + \mathfrak{B}b + \mathfrak{C}c + \mathfrak{D}d$$

etc.

25. Formulae autem supra usurpatae praebent

$$\begin{aligned} A &= v, \\ B &= -v+\alpha a, \\ C &= v-\alpha a+\beta b, \\ D &= -v+\alpha a-\beta b+\gamma c, \\ E &= v-\alpha a+\beta b-\gamma c+\delta d \end{aligned}$$

etc.,

unde patet esse

$$O = A \text{ et } P=-B \text{ ob } \mathfrak{A}=-\alpha.$$

Cum iam sit $\mathfrak{B}=1-\alpha aa+2av=1+a(A-B)$, erit utique

$$\mathfrak{B}=\beta \text{ hincque } Q=C,,$$

ex quo porro colligitur

$$\mathfrak{C}=-\alpha+\beta bb-2bB=-\alpha-b\left(2B-\beta b\right)=-\alpha-b\left(B-C\right),$$

sicque est

$$\mathfrak{C}=-\gamma \text{ et } R=-D;$$

simili modo

$$\mathfrak{D}=\beta-\gamma cc+2cC=\beta+c\left(2C-\gamma c\right)=\beta+c(C-D)$$

ideoque est

$$\mathfrak{D}=\delta \text{ et } S=E.$$

Tum vero porro

$$\mathfrak{E}=-\gamma+\delta dd-2dD=-\gamma-d(2D-\delta d)=-\gamma-d(D-E)$$

ac propterea

$$\mathfrak{E}=-\varepsilon,$$

unde superior inductio satis confirmatur.

26. Pro fractionibus ergo $\frac{x}{y}$ formulae radicali $\sqrt{z}$ proxime aequalibus y sequentes adipiscimur relationes:

Si sumatur		erit
$x=1,$	$y=0,$	$xx=zyy+1,$
$x=\left(v\right),$	$y=1,$	$xx=zyy-\alpha,$
$x=\left(v,\ a\right),$	$y=\left(a\right),$	$xx=zyy+\beta,$
$x=\left(v,\ a,\ b\right),$	$y=\left(a,\ b\right),$	$xx=zyy-\gamma,$
$x=\left(v,\ a,\ b,\ c\right),$	$y=\left(a,\ b,\ c\right),$	$xx=zyy+\delta,$
$x=\left(v,\ a,\ b,\ c,\ d\right),$	$y=\left(a,\ b,\ c,\ d\right),$	$xx=zyy-\varepsilon$

etc.,

unde problema Pellianum solvetur, quoties litterarum graecarum per saltum excerptarum $\beta,\ \delta,\ \zeta$ etc. quaepiam in unitatem abit.

27. Vidimus autem supra nonnisi iis indicibus, qui sunt 2v, respondere litteram graecam in unitatem abeuntem; cum igitur quaelibet periodorum, quas in indicum ordine observavimus, indice *2v* inchoetur, perspicuum est, si numeros *x* et *y* per indices primae periodi definiamus, fore vel

$$xx = zyy - 1$$

vel

$$xx = zyy + 1;$$

ac prius quidem evenit, si indicum singulas periodos constituentium numerus fuerit impar, posterius vero, si is fuerit par. Hoc igitur casu statim habetur solutio problematis Pelliani, quo requiritur, ut sit

$$pp = zqq + 1,$$

quandoquidem capi oportet $p = x$ et $q = y$.

28. At si ex prima periodo prodeat $xx = zyy - 1$, quod evenit, si indicum numerus est impar, tum indices usque ad initium tertiae periodi ad definiendos numeros *x* et *y* capi possent; quorum numerus cum sit par, hoc modo idonei numeri pro *p* et *q* obtinerentur. Verum casu inventa, quo fit $xx = zyy - 1$, multo facilius inde numeri *p* et *q* reperiri possunt, ut sit $pp = zqq + 1$. Sumatur enim

$$p = 2xx + 1 \ \text{ et } \ q = 2xy$$

eritque

$$pp - zqq = 4x^4 + 4xx + 1 - 4zxxyy = 1 + 4xx(xx - zyy + 1);$$

at $xx - zyy + 1 = 0$ ideoque

$$pp - zqq = 1 \quad \text{seu} \quad pp = zqq + 1,$$

quemadmodum problema Pellanium postulat.

Videamus igitur, quomodo pro quovis numero *z* ex indicibus inde natis numeri *p* et *q* sint definiendi, ut fiat $pp = zqq + 1$, ubi quidem casus secundum periodos percurramus.

I. CASUS QUO PRO NUMERO *z* INDICES SUNT

$$v,\ 2v,\ 2v \ \text{ etc.}$$

29. Hic singulae periodi unicum indicem continent; sumto ergo

$$x=(v) \text{ et } y=1$$

erit

$$xx=zyy-1.$$

Quamobrem ut fiat $pp=zqq+1$, capiatur

$$p=2xx+1=2vv+1 \text{ et } q=2xy=2v.$$

Hic casus, ut supra vidimus, locum habet, si sit

$$z=vv+1,$$

seu quo numerus z unitate superat quadratum; tum igitur capi debet

$$p=2vv+1 \text{ seu } p=2z-1 \text{ et } q=2v,$$

quo pacto problemati Pelliano satisfit, ut sit $p=\sqrt{(zqq+1)}$.

Ita si sit	erit	sicque
$z=2,$	$p=3$ et $q=2$	$p=\sqrt{(2qq+1)},$
$z=5,$	$p=9$ et $q=4$	$p=\sqrt{(5qq+1)},$
$z=10,$	$p=19$ et $q=6$	$p=\sqrt{(10qq+1)},$
$z=17,$	$p=33$ et $q=8$	$p=\sqrt{(17qq+1)}$

etc.

II. CASUS QUO PRO NUMERO z INDICES SUNT

$$v,\ a,\ 2v,\ a,\ 2v \text{ etc.}$$

30. Prima periodus constat binis numeris v, a, unde sumtis

$$x=(v,a)=va+1 \text{ et } y=(a)=a$$

habebitur

$$xx=zyy+1.$$

Ut igitur pro problemate Pelliano fiat $pp=zqq+1$, capi oportet

$$p=va+1 \text{ et } q=a.$$

Ex indicibus autem patet hunc casum locum habere, quoties fuerit numerus

$$z = vv + \tfrac{2v}{a},$$

unde intelligitur hunc casum in integris, de quibus hic agitur, existere non posse, nisi sit a divisor ipsius $2v$, ubi duo casus sunt considerandi:

1. Si $a = 2n$, erit $v = mn$ et $\tfrac{2v}{a} = m$;
2. Si $a = 2n+1$, erit $v = m(2n+1)$ et $\tfrac{2v}{a} = 2m$.

III. CASUS QUO PRO NUMERO z INDICES SUNT

$$v,\ a,\ a,\ 2v,\ a,\ a,\ 2v \text{ etc.}$$

31. Ex prima periodo sumtis numeris x et y, ita ut sit

$$x = (v,\ a,\ a) \text{ et } y = (a,\ a),$$

erit

$$xx = zyy - 1;$$

unde ut fiat $pp = zqq + 1$, sumi debet

$$p = 2xx + 1 \quad \text{et} \quad q = 2xy.$$

Hic vero est

$$y = aa + 1 \text{ et } x = vy + a,$$

unde numeri p et q facillime definiuntur. Ex indicibus autem numerus z eiusmodi habebit formam

$$z = vv + u$$

existente

$$u = \tfrac{2av+1}{aa+1},$$

unde patet numerum a esse debere parem. Si ergo statuatur $a = 2n$, necesse est sit

$$v = n + m(4nn + 1),$$

tumque fit

$$u = 1 + 4mn.$$

IV. CASUS QUO PRO NUMERO z INDICES SUNT

$$v,\ a,\ b,\ a,\ 2v,\ a,\ b,\ a,\ 2v \text{ etc.}$$

32. Quia numerus indicum in quaque periodo est par, si sumatur

$$x=(v,\ a,\ b,\ a)\quad \text{et}\quad y=(a,\ b,\ a),$$

erit

$$xx=zyy+1$$

ideoque

$$p=x\ \text{ et }\ q=y.$$

Per transformationes autem supra ostensas duplicatio indicum tolli potest hoc modo

$$x=(a)(v,\ a,\ b)+(v,\ a)\ \text{ et }\ y=(a)(a,\ b)+(a).$$

Hinc si ex indicibus *v, a, b* sequentes fractiones formentur

$$\begin{array}{ll} \text{indices} & v,\quad a,\quad b \\ \text{fractiones} & \frac{1}{0},\ \frac{\mathfrak{A}}{\mathfrak{a}},\ \frac{\mathfrak{B}}{\mathfrak{b}},\ \frac{\mathfrak{C}}{\mathfrak{c}}, \end{array}$$

ob

$$\mathfrak{A}=(v),\quad \mathfrak{B}=(v,a),\quad \mathfrak{C}=(v,\ a,\ b)$$

et

$$\mathfrak{a}=1,\qquad \mathfrak{b}=(a),\qquad \mathfrak{c}=(a,\ b)$$

erit

$$x=\mathfrak{b}\mathfrak{C}+\mathfrak{a}\mathfrak{B}\quad \text{et}\quad y=\mathfrak{b}\mathfrak{c}+\mathfrak{a}\mathfrak{b}.$$

Ex indicibus autem fit

$$z=vv+u$$

existente

$$2v=m(a,\ b,\ a)-b(a,\ b)$$

et

$$u=m(a,\ b)-b(b).$$

V. CASUS QUO PRO NUMERO z INDICES SUNT $v,\ a,\ b,\ b,\ a,\ 2v$ etc.

33. Ob indicum cuiusque periodi numerum imparem, si capiamus

$$x=(v,a,b,b,a)\ \text{ et }\ y=(a,b,b,a),$$

erit

$$xx=zyy-1;$$

hinc pro problemate Pelliano ut fiat $pp=zqq+1$, statui oportet

$$p = 2xx + 1 \text{ et } q = 2xy.$$

Quo autem numeri x et y facilius inveniri queant, sequentes transformationes instituantur

$$x = (a,\ b)(v,\ a,\ b) + (a)(v,\ a) \ \text{ et } \ y = (a,\ b)(a,\ b) + (a)(a),$$

qui ergo per solos indices v, a, b fractionibus inde formandis definientur:

$$\begin{array}{llll} \text{indices} & v, & a, & b \\ \text{fractiones} & \frac{1}{0}, & \frac{\mathfrak{A}}{\mathfrak{a}}, & \frac{\mathfrak{B}}{\mathfrak{b}}, & \frac{\mathfrak{C}}{\mathfrak{c}}, \end{array}$$

ubi

$$\mathfrak{A} = v, \quad \mathfrak{B} = a\mathfrak{A} + 1, \quad \mathfrak{C} = b\mathfrak{B} + \mathfrak{A}$$

et

$$\mathfrak{a} = 1, \qquad \mathfrak{b} = a\mathfrak{a} + 0, \qquad \mathfrak{c} = b\mathfrak{b} + \mathfrak{a};$$

tum enim capi oportet

$$x = \mathfrak{c}\mathfrak{C} + \mathfrak{b}\mathfrak{B} \ \text{ et } \ y = \mathfrak{c}\mathfrak{c} + \mathfrak{b}\mathfrak{b}.$$

Hic autem casus locum habet, quoties posito

$$z = vv + u$$

fuerit

$$2v = m(a,\ b,\ b,\ a) + (b,\ b)(a,\ b,\ b)$$

et

$$u = m(a,\ b,\ b) + (b,\ b)(b,\ b).$$

VI. CASUS QUO PRO NUMERO z INDICES SUNT $v,\ a,\ b,\ c,\ b,\ a,\ 2v$ etc.

34. Quoniam hic numerus indicum in qualibet periodo est par, si sumamus

$$x = (v,\ a,\ b,\ c,\ b,\ a) \ \text{ et } \ y = (a,\ b,\ c,\ b,\ a),$$

erit

$$xx = zyy + 1$$

ideoque pro Pelliano problemate statim habetur

$$p = x \ \text{ et } \ q = y.$$

Facilius autem numeri x et y his transformationibus adhibitis invenientur

$$x = (a,\ b)\ (v,\ a,\ b,\ c)(a)(v,\ a,\ b) \ \text{ et } \ y = \ (a,\ b)(a,\ b,\ c)\ + (a)(a,\ b);$$

unde si ex indicibus *v, a, b, c* more exposito fractiones formentur

$$\begin{array}{llllll} \text{indices} & v, & a, & b, & c & \\ \text{fractiones} & \frac{1}{0}, & \frac{\mathfrak{A}}{\mathfrak{a}}, & \frac{\mathfrak{B}}{\mathfrak{b}}, & \frac{\mathfrak{C}}{\mathfrak{c}}, & \frac{\mathfrak{D}}{\mathfrak{d}}, \end{array}$$

sumi oportet

$$x = \mathfrak{c}\mathfrak{D} + \mathfrak{b}\mathfrak{C} \text{ et } y = \mathfrak{c}\mathfrak{d} + \mathfrak{b}\mathfrak{c}.$$

At hic casus locum habet, quoties posito

$$z = vv + u$$

fuerit

$$2v = m(a,\ b,\ c,\ b,\ a) - \ (b,\ c,\ b)(a,\ b,\ c,\ b)$$

et

$$u = m(a,\ b,\ c,\ b) + (b,\ c,\ b)(b,\ c,\ b).$$

VII. CASUS QUO PRO NUMERO *z* INDICES SUNT

$$v,\ a,\ b,\ c,\ c,\ b,\ a,\ 2v \text{ etc.}$$

35. Hic iterum indicum numerus in qualibet periodo est impar; ideoque si ponamus

$x = (v,\ a,\ b,\ c,\ c,\ b,\ a)$ et $y = (a,\ b,\ c,\ c,\ b,\ a)$, erit

$$xx = zyy - 1;$$

ex quo ut fiat $pp = zqq + 1$, sumi oportet

$$p = 2xx + 1 \text{ et } q = 2xy.$$

Pro faciliori autem numerorum *x* et *y* inventione ex indicibus *v, a, b, c* formentur fractiones

$$\begin{array}{llllll} \text{indices} & v, & a, & b, & c & \\ \text{fractiones} & \frac{1}{0}, & \frac{\mathfrak{A}}{\mathfrak{a}}, & \frac{\mathfrak{B}}{\mathfrak{b}}, & \frac{\mathfrak{C}}{\mathfrak{c}}, & \frac{\mathfrak{D}}{\mathfrak{d}}, \end{array}$$

hincque erit

$$x = \mathfrak{d}\mathfrak{D} + \mathfrak{c}\mathfrak{C} \text{ et } y = \mathfrak{d}\mathfrak{d} + \mathfrak{c}\mathfrak{c}.$$

At hic casus locum habebit, quoties posito

$$z = vv + u$$

fuerit

$$2v = m(a,\ b,\ c,\ c,\ b,\ a) + (b, c,\ c,\ b)(a,\ b,\ c,\ c,\ b)$$

et

$$u = m(a,\ b,\ c,\ c,\ b) + (b,\ c,\ c,\ b)(b,\ c,\ c,\ b).$$

VIII. CASUS QUO PRO NUMERO z INDICES SUNT $v,\ a,\ b,\ c,\ d,\ c,\ b,\ a, 2v$ etc.

36. Hic quaelibet periodus octo continet indices; ideoque si ponamus

$$x = (v,\ a,\ b,\ c,\ d,\ c,\ b,\ a) \text{ et } y = (a,\ b,\ c,\ d,\ c,\ b,\ a),$$

erit

$$xx = zyy + 1$$

et pro problemate PELLIANO capi oportet

$$p = x \text{ et } q = y,$$

ut fiat $pp = zqq + 1$. Transformationibus autem adhibitis numeros x et y per solos indices v, a, b, c, d definire licet. Formatis enim inde fractionibus

indices	$v,$	$a,$	$b,$	$c,$	d	
fractiones	$\frac{1}{0},$	$\frac{\mathfrak{A}}{\mathfrak{a}},$	$\frac{\mathfrak{B}}{\mathfrak{b}},$	$\frac{\mathfrak{C}}{\mathfrak{c}},$	$\frac{\mathfrak{D}}{\mathfrak{d}},$	$\frac{\mathfrak{E}}{\mathfrak{e}},$

fiet

$$x = \mathfrak{d}\mathfrak{E} + \mathfrak{c}\mathfrak{D} \text{ et } y = \mathfrak{d}\mathfrak{e} + \mathfrak{c}\mathfrak{d}.$$

Hic vero casus locum habet, quoties posito

$$z = vv + u$$

fuerit

$$2v = m(a,\ b,\ c,\ d,\ c,\ b,\ a) - (b,\ c,\ d,\ c,\ b)(a,\ b,\ c,\ d,\ c,\ b)$$

et

$$u = m(a,\ b,\ c,\ d,\ c,\ b) - (b, c,\ d,\ c,\ b)(b,\ c,\ d,\ c,\ b).$$

EXPOSITIO CALCULI PRO QUOLIBET NUMERO z UT FIAT

$$pp = zqq + 1$$

37. Primum igitur methodo supra exposita pro numero z ex eius radice quadrata indices investigari oportet; quam operationem autem ulterius continuari non est opus, quam donec indices ordine retrograda prodire incipiant, quo pacto semissi laboris supra explicati supersedere poterimus. Cum autem in prima periodo vel unus index medius occurrat vel bini, hi casus probe sunt distinguendi, cum, si unicus medius affuerit, inventio numerorum p et q modo in casibus II, IV, VI et VIII tradito institui debeat, sin autem bini fuerint medii, eo modo, qui in casibus I, III, V et VII est descriptus. Scilicet si prius eveniat, numeri p et q numeris x et y aequales sumuntur, sin autem posterius, uti vidimus, statui oportet $p = 2xx+1$ et $q = 2xy$, ita ut his casibus numeri p et q *caeteris* paribus multo grandiores reperiantur.

38. En igitur exempla prioris generis, quo in qualibet periodo unus datur index medius.

I. Si $z = 6$, sunt indices 2, 2, 4; hinc operatio:

$$\begin{array}{ccc} 2, & 2 & \\ \frac{1}{0}, & \frac{2}{1}, & \frac{5}{2}; \end{array}$$

hinc

$$x = 1\cdot 5 + 0\cdot 2, \qquad p = 5,$$

ergo

$$y = 1\cdot 2 + 0\cdot 1, \qquad q = 2.$$

II. Si $z = 14$, sunt indices 3, 1, 2, 1, 6:

$$\begin{array}{cccc} 3, & 1, & 2 & \\ \frac{1}{0}, & \frac{3}{1}, & \frac{4}{1} & \frac{11}{3}; \end{array}$$

hinc

$$x = 1\cdot 11 + 1\cdot 4, \qquad p = 15,$$

ergo

$$y = 1\cdot 3 + 1\cdot 1, \qquad q = 4.$$

III. Si $z = 19$, sunt indices 4, 2, 1, 3, 1, 2, 8:

$$\begin{array}{ccccc} 4, & 2, & 1, & 3 & \\ \frac{1}{0}, & \frac{4}{1}, & \frac{9}{2}, & \frac{13}{3}, & \frac{48}{11}; \end{array}$$

hinc

$$x = 3\cdot 48 + 2\cdot 13, \qquad p = 170,$$

ergo

$$y = 3\cdot 11 + 2\cdot 3, \qquad q = 39.$$

IV. Si z = 31, sunt indices 5, 1, 1, 3, 5, 3, 1, 1, 10:

$$\begin{array}{cccccc} & 5, & 1, & 1, & 3, & 5 \\ \frac{1}{0}, & \frac{5}{1}, & \frac{6}{1}, & \frac{11}{2}, & \frac{39}{7}, & \frac{206}{37}; \end{array}$$

hinc

$$x = 7\cdot 206 + 2\cdot 39, \qquad p = 1520,$$

ergo

$$y = 7\cdot 37 + 2\cdot 7, \qquad q = 273.$$

Si z = 44, sunt indices 6, 1, 1, 1, 2, 1, 1, 1, 12:

$$\begin{array}{cccccc} & 6, & 1, & 1, & 1, & 2 \\ \frac{1}{0}, & \frac{6}{1}, & \frac{7}{1}, & \frac{13}{2}, & \frac{20}{3}, & \frac{53}{8}; \end{array}$$

hinc

$$x = 3\cdot 53 + 2\cdot 20, \qquad p = 199,$$

ergo

$$y = 3\cdot 8 + 2\cdot 3, \qquad q = 30.$$

VI. Si $z = 55$, sunt indices 7, 2, 2, 2, 14:

$$\begin{array}{cccc} & 7, & 2, & 2, \\ \frac{1}{0}, & \frac{7}{1}, & \frac{15}{2}, & \frac{37}{5}; \end{array}$$

hinc

$$x = 2\cdot 37 + 1\cdot 15, \qquad p = 89,$$

ergo

$$y = 2\cdot 5 + 1\cdot 2, \qquad q = 12.$$

39. Alterius vero generis, quo bini dantur indices medii in qualibet periodo, haec adiungo exempla.

I. Si $z = 13$, sunt indices 3, 1, 1, 1, 1, 6:

$$\begin{array}{cccc} & 3, & 1, & 1, \\ \frac{1}{0}, & \frac{3}{1}, & \frac{4}{1}, & \frac{7}{2}; \end{array}$$

hinc

$$x = 2\cdot 7 + 1\cdot 4 = 18,$$
$$y = 2\cdot 2 + 1\cdot 1 = 5.$$

ergo

$$p = 2xx + 1 = 649,$$
$$q = 2xy = 180.$$

II. Si $z = 29$, sunt indices 5, 2, 1, 1, 2, 10:

$$\begin{array}{cccc} & 5, & 2, & 1, \\ \frac{1}{0}, & \frac{5}{1}, & \frac{11}{2}, & \frac{16}{3}; \end{array}$$

hinc

$$x = 3\cdot 16 + 2\cdot 11 = 70,$$
$$y = 3\cdot 3 + 2\cdot 2 = 13.$$

ergo

$$p = 2xx + 1 = 9801,$$
$$q = 2xy = 1820.$$

III. Si $z = 58$, sunt indices 7, 1, 1, 1, 1, 1, 1, 14:

$$\begin{array}{ccccc} & 7, & 1, & 1, & 1 \\ \frac{1}{0}, & \frac{7}{1}, & \frac{8}{1}, & \frac{15}{2} & \frac{23}{3}; \end{array}$$

hinc

$$x = 3\cdot 23 + 2\cdot 15 = 99,$$
$$y = 3\cdot 3 + 2\cdot 2 = 13.$$

ergo

$$p = 2xx + 1 = 19603,$$
$$q = 2xy = 2574.$$

IV. Si $z = 61$, indices sunt 7, 1, 4, 3, 1, 2, 2, 1, 3, 4, 1, 14:

$$7,\quad 1,\quad 4,\quad 3,\quad 1,\quad 2$$

$$\frac{1}{0},\ \frac{7}{1},\ \frac{8}{1},\ \frac{39}{5},\ \frac{125}{16},\ \frac{164}{21},\ \frac{453}{58};$$

hinc fit

$$x = 58\cdot 453+21\cdot 164 = 29718,$$

$$y = 58\cdot\ 58+21\cdot\ 21 = \quad 3805.$$

ergo

$$p = 2xx+1 = 1766319049,$$

$$q = 2xy \quad = 226153980.$$

40. Quodsi pro maioribus numeris *z*, quam ante sunt evoluti, quaeri debeant numeri *p* et *q*, ut sit $pp = zqq+1$, primum methodo supra exposita (§ 12) indices *v, a, b, c, d* etc. quaeri oportet, quos autem ulterius continuari non est opus, quam donec ad indicem medium vel binos medios primae periodi perveniatur; tum vero ex iis per operationes hic descriptas primo numeri *x* et *y*, tum vero ipsi quaesiti *p* et *q* determinabuntur. Id quod aliquibus exemplis illustrari conveniet.

I. Quaerantur numeri p et q, ut sit $pp = 157qq+1$

41. Cum hic sit $z = 157$, erit $v = 12$ et $\alpha = 13$, unde indicum inventio ita se habebit:

$$\begin{array}{lll}
A = 12, & \alpha = 13, & a = 1, \\
B = 1, & \beta = 12, & b = 1, \\
C = 11, & \gamma = 3, & c = 7, \\
D = 10, & \delta = 19, & d = 1, \\
E = 9, & \varepsilon = 4, & e = 5, \\
F = 11, & \zeta = 9, & f = 2, \\
G = 7, & \eta = 12, & g = 1, \\
H = 5, & \theta = 11, & \left.\begin{array}{l} h = 1 \\ i = 1 \end{array}\right\} \\
I = 6, & \iota = 11, &
\end{array}$$

Hinc ob binos medios exemplum ad genus secundum pertinet et operationes ita sunt instituendae:

$$12,\ 1,\ 1,\ 7,\ 1,\ 5,\ 2,\ 1,\ 1$$

$$\frac{1}{0},\ \frac{12}{1},\ \frac{13}{1},\ \frac{25}{2},\ \frac{188}{15},\ \frac{213}{17},\ \frac{1253}{100},\ \frac{2719}{217},\ \frac{3972}{317},\ \frac{6691}{534}.$$

Hinc erit

$$x = 534 \cdot 6691 + 317 \cdot 3972 = 4832118$$

et

$$y = 534 \cdot 534 + 317 \cdot 317 = 385645.$$

Quocirca

$$p = 2xx + 1 = 46698728731849$$

et

$$q = 2xy \quad = \quad 3726964292220$$

atque hi adeo sunt minimi numeri integri formulae $p = \sqrt{(157qq + 1)}$ satisfacientes.

II. Quaerantur numeri p et q, ut sit $pp = 367qq + 1$

42. Hic ergo est $z = 367$, $v = 19$ hincque

$$\begin{array}{lll}
A = 19, & \alpha = 6, & a = 6, \\
B = 17, & \beta = 13, & b = 2, \\
C = \ 9, & \gamma = 22, & c = 1, \\
D = 13, & \delta = 9, & d = 3, \\
E = 14, & \varepsilon = 19, & e = 1, \\
F = 5, & \zeta = 18, & f = 1, \\
G = 13, & \eta = 11, & g = 2, \\
H = 9, & \theta = 26, & h = 1, \\
I = 17, & \iota = 3, & \mathrm{i} = 12, \\
K = 19, & \chi = 2, & k = \text{medius} 19, \\
L = 19, & \lambda = 3, & l = 12.
\end{array}$$

Hoc ergo exemplum ad genus primum pertinet.

$$\begin{array}{cccccccccccc}
 & 19, & 6, & 2, & 1, & 3, & 1, & 1, & 2, & 1, & 12, & 19 \\
\frac{1}{0}, & \frac{19}{1}, & \frac{115}{6}, & \frac{249}{13}, & \frac{364}{19}, & \frac{1341}{70}, & \frac{1705}{89}, & \frac{3046}{159}, & \frac{7797}{407}, & \frac{10843}{566}, & \frac{137913}{7199}, & \frac{2631190}{137347}.
\end{array}$$

Hinc erit

$$x = 7199 \cdot 2631190 + 566 \cdot 137913$$

et

$$y = 7199 \cdot 137347 + 566 \cdot 7199,$$

ex quo minimi numeri satisfacientes sunt

$$p = 19019995568,$$
$$q = 992835687.$$

Tabula numerorum p et q,
quibus fit $pp = lqq + 1$ pro omnibus valoribus numeri l usque ad 100

l	q	p
2	3	3
3	1	2
5	4	9
6	2	5
7	3	8
8	1	3
10	6	19
11	3	10
12	2	7
13	180	649
14	4	15
15	1	4
17	8	33
18	4	17
19	39	170
20	2	9
21	12	55
22	42	197
23	5	24
24	1	5

26	10	51
27	5	26
28	24	127
29	1820	9801
30	2	11
31	273	1520
32	3	17
33	4	23
34	6	35
35	1	6

37	12	73
38	6	37
39	4	25
40	3	19
41	320	2049
42	2	13
43	531	3482
44	30	199
45	24	161
46	3588	24335
47	7	48
48	1	7
50	14	99
51	7	50
52	90	649
53	9100	66249
54	66	485
55	12	89
56	2	15
57	20	151
58	2574	19603
59	69	530
60	4	31
61	226153980	1766319049
62	8	63
63	1	8
65	16	129
66	8	65
67	5967	48842
68	4	33
69	936	7775
70	30	251
71	413	3480
72	2	17
73	267000	2281249
74	430	3699
75	3	26
76	6630	57799
77	40	351
78	6	53
79	9	80
80	1	9
82	18	163
83	9	82
84	6	55

85	30996	285769
86	1122	10405
87	3	28
88	21	197
89	53000	500001
90	2	19
91	165	1574
92	120	1151
93	1260	12151
94	221064	21432295
95	4	39
96	5	49
97	6377352	62809633
98	10	99
99	1	10

Exempla denique quaedam numerorum maiorum pro *l* assumtorum adiungam:

$$\text{Si } \ l=103, \text{ erit } \begin{cases} p = 227528, \\ q = \quad 22419; \end{cases},$$

$$\text{si } l = \ 109, \text{ erit } \begin{cases} p = 158070671986249, \\ q = 15140424455100; \end{cases}$$

$$\text{si } l = 113, \text{ erit } \begin{cases} p = 1204353, \\ q = \quad 113296; \end{cases}$$

$$\text{si } l = 157, \text{ erit } \begin{cases} p = 46698728731849 \\ q = \quad 3726964292220; \end{cases}$$

$$\text{si } l = 367, \text{ erit } \begin{cases} p = 19019995568, \\ q = \qquad 992835687. \end{cases}$$