E71: DISSERTATION ON CONTINUED FRACTIONS

1. Various kinds of quantities have been established in analysis which otherwise would be able to be assigned with difficulty. Evidently logarithms, circular arcs and the quadrature of other curves are irrational numbers and transcendental quantities of this kind are accustomed to be shown by infinite series which, since they shall depend on known terms, the values of these quantities are indicated well enough. But there are two kinds of these series, to which the first of these series belong, the terms of which have been joined by addition or subtraction; but the latter can refer to these, the terms of which may be linked together by multiplication. Thus in this other way the area of the circle, whose diameter $=1$, is accustomed to be expressed; nevertheless by the first way the area of the circle without doubt is said to be equal to

$$1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\text{ etc. to infinity,}$$

truly by the latter way the same area is equal to this expression

$$\frac{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10}{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdot11}\text{etc. to infinity.}$$

Of which series these will deserve to be preferred which may converge the most with the least terms taken, and provide the approximate value of the quantity sought.

2. A third series is seen now to be required to be added to these two kinds of series, not without merit, the terms of which are connected together by continued division, which series therefore will be agreed to be called *continued fractions*. Indeed there is less usage of this kind of series than for the remaining two ; just as equally distinctly the value of the quantity which it expresses and presented to the eye, truly also is extremely useful for finding an approximate value of that quantity to be found. But even now so very little of this kind of series has been developed, that besides one or other series of this kind now known, not even the method may be evident, or the true values of this kind of series found, or of the transcending quantities required to be changed into such expressions. Therefore, since for some time now I have labored over the examination of these continued fractions, and more so I have observed both from the use of these as well as a discovery of no small moment, that I may put into effect a different way for treating the same more plainly than from the other ways. Although indeed not yet in whatever way have I been able to extend the theory of this investigation to become a complete doctrine, yet these matters which I have elicited with great labour, I trust from the significant help they have brought forth, that the same theory may be brought more towards being perfected.

3. Therefore , by whatever way I may understand the name of continued fractions, in order that it may be understood more clearly, so that before everything else, I will show an example of these most fully :

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{etc.}}}}}}$$

from which, by the manner written out, any meaning of this expression will become known easily. Clearly this quantity is consistent with these two members, the whole number *a* and with a fraction, of which the numerator is α, indeed in turn the denominator is composed from two members, without doubt with a whole number *b* and a fraction, of which the numerator is β and the denominator truly in turn depends on two terms, clearly with the whole number *c* and with a fraction γ as before ; and thus so on again indefinitely. Here twofold quantities occur, which also I have distinguished by letters taken from the Latin and greek alphabets. Of these quantities these, which also I have denoted by greek letters, I will call the *numerators*, since the fractions will actually constitute the numerators of the following fractions ; truly we will call all the remaining quantities the *denominators,* expressed for distinction by Latin letters; indeed besides all except the first are parts of the denominators.
[Thus, the first division gives rise to a whole number and a remainder, less than the original divisor, this remainder is inverted to give a second whole number equal to or greater than zero and a second remainder, the original remainder is now divided by this new whole number plus its remainder, *i.e.* the expression found is inverted or put as the first remainder divided by the second division plus its new remainder, and so it is iterated either finitely or infinitely.]

4. The first person, who advanced a continued fraction of this kind, was Viscount Brouncker, as far as for me it is generally agreed, who after a communication from Wallis thus changed the same expression of the quadrature of the circle, so that he could assert the area of the circle itself to be had as the square of the diameter as 1 to

$$1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cfrac{81}{2+\text{etc.}}}}}}$$

where the numerators are the numerators of the odd numbers, the denominators truly are 2. But it is not evident by which way Brouncker came upon this expression, and deservedly it would be a source of pain, if his method should have perished, since there shall be no doubting, why by the same method more outstanding results may not be able to be shown by this manner. Indeed Wallis, while examining this fraction, was himself known to have prepared this demonstration, but which is less genuine and may be considered to be completely different from the method of the author. But Wallis may have derived this whole invention from the following theorem, which shall be :

$$a^2=(a-1)+\cfrac{1}{2(a-1)+\cfrac{9}{2(a-1)+\cfrac{25}{2(a-1)+\text{etc.}}}}\times(a+1)+\cfrac{1}{2(a+1)+\cfrac{9}{2(a+1)+\cfrac{25}{2(a+1)+\text{etc.}}}}$$

the truth of which may be confirmed well enough by induction, but, whatever the source, an analysis is not advanced, by which this theorem will be come upon.

5. Moreover from a given continued fraction of this kind, its true value can be conveniently and easily determined approximately, and why not may the bounds be defined, within which the true value may be contained, so that, if certain quadratures or other transcending quantities were expressed in this manner, by an easy business those may be able to be themselves assigned. I may show this from the form of the general continued fraction

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\text{etc.}}}}}$$

in which I put all the quantities entering to be positive. Moreover truly it is apparent a close approximation to be obtained, if the continued fraction may be interrupted somewhere, and there the closer the value going to be found, where the fraction will be continued further. Thus by taking *a* only, indeed a smaller value will be had, since the adjoined fraction is ignored completely. But by taking

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c}}$$

truly a greater value is obtained, since in the fraction the denominator *b* is just smaller than *a*. But if there may be taken

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c}}$$

again there will be had a value just smaller than the fraction $\frac{\beta}{c}$ and thence the denominator $b+\frac{\beta}{c}$ exceedingly great. And in this manner the continued fraction by being successively interrupted will produce alternatively just greater and just smaller values; from which however it will be allowed to approach to the true value of the continued fraction.

6. Therefore we will have the following series of expressions:

$$a \quad a+\frac{\alpha}{b} \quad \frac{\alpha}{b+\frac{\beta}{c}} \quad a+\frac{\alpha}{b+\frac{\beta}{c+\frac{\gamma}{d}}} \quad \text{etc.};$$

of which the odd terms are in order, so that the first, third, fifth, etc., shall be smaller than the true value of the continued fraction; but the even terms will be greater than the same. Whereby since the greater term shall be the first, the fifth greater than the third and thus henceforth, the odd terms by increasing finally will reach the true value of the continued fraction; indeed the even terms, which decrease continually, by decreasing finally descend to the true value of the continued fraction. But if these expressions may be changed into simple fractions, the following will produce the series of the same expressions

$$\frac{a}{1},\ \frac{ab+\alpha}{b},\ \frac{abc+\alpha c+\beta a}{bc+\beta},\ \frac{abcd+\alpha cd+\beta ad+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b}\ \text{etc.};$$

which if it may be examined more carefully, the law may be deduced readily, by which these terms are progressing and it is allowed for each of these composite fractions to be reduced without being continued with the aid of these troublesome fractions, as far as it pleases. Indeed these fractions at once become exceedingly large ; but in the examples in which these letters are expressed by numbers, by which such a series may be continued conveniently.

7. Moreover the law of the progression of these fractions may be understood clearly :

$$\begin{array}{cccccc} a & b & c & d & e & \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+\alpha}{b}, & \frac{abc+\alpha c+\beta a}{bc+\beta}, & \frac{abcd+\alpha cd+\beta ad+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b} & \text{etc.} \\ \alpha & \beta & \gamma & \delta & \varepsilon & \end{array}$$

Namely the denominators of the continued fractions are from these fractions written above, truly just as the numerators are from the indices written below ; but to these fractions the fraction $\frac{1}{0}$ is prefixed [*i.e.* the numerator at present is 1, but nothing has been added to the denominator to start the continued fraction] , certainly which will relate to the soon to be declared law itself. Now the law of the progression consists of this, in order that the numerator of each continued fraction multiplied by the above index together with the numerator of the preceding fraction multiplied by its own index written below shall produce the numerator of the following fraction, and in the same manner, the denominator of each fraction multiplied by its own index placed above together with the denominator of the preceding fraction multiplied by its own index written below may produce the denominator of the following fraction. Indeed this law is easily observed from the observation of these continued fractions, if they may be continued further; but also the same continued fraction can be deduced from the nature of the continued fractions ; but to place which demonstration here I judge to be superfluous.

8. If the difference of the fractions may be taken by subtracting each from the preceding, the following series will arise

$$\frac{1}{0},\quad -\frac{\alpha}{1\cdot b},\ +\frac{\alpha\beta}{b(bc+\beta)},\quad -\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}\text{ etc.},$$

of which the progression of the numerators by themselves is clear, truly the denominators are formed from the two preceding denominators. Therefore since the final term of the above series, which shows the true value of the continued fraction, may be composed from the first we may take to be *a* on rejecting $\frac{1}{0}$, and from all the differences, the true value of all the continued fractions proposed will become

$$\alpha+\frac{\alpha}{1\cdot b}-\frac{\alpha\beta}{b(bc+\beta)}+\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}-\frac{\alpha\beta\gamma\delta}{(bcd+\beta d+\gamma b)(bcde+\cdots)}+\text{etc.}$$

Thus we have an infinite series of the first kind, the terms of which are connected together by addition and subtraction, equal to the value of the continued fraction proposed; and this series definitely converges and is incredibly suitable for converging to that approximate value. If two terms may be joined together in order to avoid the reason for the opposite signs, the same continued fraction will be found to be equal to the following series :

$$a+\frac{\alpha c}{1(bc+\beta)}+\frac{\alpha\beta\gamma e}{(bc+\beta)(bcde+\beta de+\gamma be+\delta bc+\beta\delta)}+\text{ etc.},$$

of which the rule of the numerators and denominators arises at once from the above. Moreover this series is strongly convergent, and with its aid truly the approximate sum can be found.

9. Where this final series found converges more, there also the continued fraction itself is required to be considered to converge, since the given number of terms of the series corresponds to the given number of the continued fraction. Therefore it is evident the continued fraction thus to converge more rapidly, where its numerators α, β, γ, etc. shall be smaller and its denominators a, b, c etc. greater. But it is permitted to put all these numbers, both the numerators as well as the denominators, to be whole numbers ; for if they were fractions, they will be able to be transformed into integers by the known reduction of fractions, evidently the numerators and denominators of the individual fractions require to be multiplied by the same number. Therefore with all the numbers replaced with integers, both α, β, γ, etc. as well as a, b, c etc., the continued fraction converges maximally, if all the numbers α, β, γ, etc. shall be equal to unity; thence truly the convergence will be greater there, where the denominators a, b, c, d etc. were greater. Evidently numerators cannot be less than unity; if indeed the numerator were $=0$ somewhere, at that same place the continued fraction is terminated to become a finite fraction. Likewise also it may happen, if some of the denominators may become $=\infty$; at that place the continued fraction is terminated and will be changed into a finite fraction.

10. Therefore if the following continued fraction may be proposed, all the numerators of which shall be one,

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\text{etc.}}}}}}$$

the fractions of the following series will approach to its value :

$$\begin{array}{cccccc} & a & b & c & d & e \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+1}{b}, & \frac{abc+c+a}{bc+1}, & \frac{abcd+cd+ad+ab+1}{bcd+d+b} & \text{etc.,}\end{array}$$

which series may be continued with the aid of the unit index *a, b, c, d* etc. of the progression. Evidently both the numerator as well as the denominator of each fraction multiplied by the index and with the numerator and the denominator of the preceding fraction respectively increased will give the numerator and the denominator of the following fraction. Then the value of this continued fraction will be equal to the sum of the following series :

$$a+\frac{1}{1\cdot b}-\frac{1}{b(bc+1)}+\frac{1}{(bc+1)(bcd+d+b)}-\frac{1}{(bcd+d+b)(bcde+\cdots)}+\text{ etc.}$$

or to the sum of this, into which the same may be changed,

$$a+\frac{c}{bc+1}+\frac{e}{(bc+1)(bcde+de+be+bc+1)}\ +\ \text{etc.,}$$

the denominators of which series are formed from the other denominators of the above series of fractions and thus may be continued easily.

11. If in such a continued fraction, of which all the numerators are unity, the denominators were fractional numbers, it will arrange such a continued fraction to be changed into another fraction, in which both the numerators and denominators shall be whole numbers. Thus if a continued fraction of this kind shall be proposed

$$a+\cfrac{1}{\frac{b}{B}+\cfrac{1}{\frac{c}{C}+\cfrac{1}{\frac{d}{D}+\cfrac{1}{\frac{e}{E}+\text{etc.}}}}}$$

this will be transformed into the following form, with the particular fractions removed :

$$a+\cfrac{B}{b+\cfrac{BC}{c+\cfrac{CD}{d+\cfrac{DE}{e+\text{etc.}}}}}$$

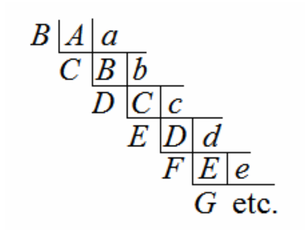
In turn in a similar manner some continued fraction can be changed into another, all the numerators of which are unity, truly the denominators are fractional numbers ; evidently there will become :

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{etc.}}}}}} = a+\cfrac{1}{\frac{b}{\alpha}+\cfrac{1}{\frac{\alpha c}{\beta}+\cfrac{1}{\frac{\beta d}{\alpha\gamma}+\cfrac{1}{\frac{\alpha\gamma e}{\beta\delta}+\cfrac{1}{\frac{\beta\delta f}{\alpha\gamma\varepsilon}+\text{etc.}}}}}}$$

which latter form may be formed easily from the former.

11[a]. Therefore since for a given continued fraction either the value of that will be true, if indeed the fraction may be terminated, or truly may be able to be shown approximately by an ordinary fraction, also in turn any ordinary fraction can be transformed into a continued fraction. First I shall show which transformation shall be required to put in place so that these may be changed into continued fractions, of which the numerators shall all be unity, and indeed the denominators to be whole numbers. Moreover every finite fraction, of which the numerator and denominator are finite whole numbers, may be transformed into a continued fraction of this kind, which is terminated somewhere; but a fraction, of which kind both the numerator and the denominator are infinitely great, are given for true continued fractions for irrational and transcending quantities, and will be the true continued fractions and will go on indefinitely. For such a continued fraction requiring to be found it will suffice for the denominators only to be assigned, since we may put all the numerators to be unity. Truly these conditions between the numerator and denominator of the proposed fraction will be found by putting in place that same operation, which is accustomed to be used for finding the greatest common divisor of these fractions. Evidently the numerator may be divided by the denominator and the denominator itself by the remainder, and thus always again the divisor by the preceding remainder. Truly the denominators of the continued fractions will arise from this continued division as many times as sought.

12. Thus if this fraction proposed $\frac{A}{B}$ is required to be transformed into a continued fraction, all its numerators shall be unity, on dividing *A* by *B* and thus the quotient *a* and the remainder *C;* the preceding divisor *B* is divided by this remainder *C* and the quotient shall be *b* and by which the quotient *C* may be divided by the remainder *D*, and thus again, finally for the remainder $=0$ the quotient may arise infinitely large. Moreover this operation may be represented in the following manner :



Therefore by this operation the quotients *a, b, c, d, e* etc. are found, from which it is understood that :

$$\frac{A}{B}=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}$$

Indeed if the remainder shall be $G=0$, there will be

$$e=\frac{E}{F} \text{ and } \frac{1}{e}=\frac{F}{E}$$

and hence again

$$d+\frac{1}{e}=d+\frac{F}{E}=\frac{D}{E} \text{ ac } \frac{1}{d+\frac{1}{e}}=\frac{E}{D},$$

$$c+\frac{1}{d+\frac{1}{e}}=c+\frac{E}{D}=\frac{C}{D}.$$

And in this manner by ascending as far as to the beginning, the continued fraction $\frac{A}{B}$ will be found.

13. If there were $A<B$ in the fraction $\frac{A}{B}$, then the first quotient *a* will be $=0$ and the first remainder $=A$, thus so that then *B* must be divided by *A* . Therefore in this case there will be

$$\frac{A}{B}=\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}$$

But in the case where $A<B$, a single term will be produced in the continued fraction, if the ratio were an integer multiple between *A* and *B* ; but the continued fraction will consist of two denominators, if the ratio $A:B$ may belong to the class of fractional ratios [*i.e.* where *A* and *B* are relatively prime] ; truly several denominators will be present, if *A* : *B* may refer to a ratio involving several fractions. Actually moreover the continued fraction may depart to infinity, if the ratio *A* to *B* were not as of one whole number to another, but were either irrational or transcendental. But it will be required to be changing expressions of this kind into continued fractions, so that they shall be set out by rational numbers, perhaps indeed approximately, just as this is accustomed to be done by decimal

fractions. Therefore such expressions may be had and continued fractions will be formed in the prescribed manner.

14. But when a fraction or other expression were converted into a continued fraction of this kind, then the approximate value of its expression can be assigned in the manner set out in §10. Just as if this expression were found :

$$\frac{A}{B} = a + \cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}$$

and the following series of fractions will be formed from the denominators *a, b, c, d* etc. :

$$\begin{array}{cccccc} & a & b & c & d & e \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+1}{b}, & \frac{abc+c+a}{bc+1}, & \frac{abcd+cd+ad+ab+1}{bcd+d+b} & \text{etc.,} \end{array}$$

these fractions will be approximately equal to the expression $\frac{A}{B}$ and they will be closer to that when they will be further from the first term. Thus moreover any of these fractions will agree together, so that others shall be unable to be shown by greater numbers, which may approach closer to the value $\frac{A}{B}$. And thus in this manner the following problem will be solved conveniently:

To change a given fraction in agreement with large numbers into a simpler one, which will approach closer than can be done with smaller numbers.

Wallis worked on this problem with great enthusiasm, but the solution he gave was exceedingly troublesome and difficult.

15. The fraction $\frac{355}{113}$ shall be proposed requiring to be adapted according to our method for the solution of this problem, which following Metius, expresses approximately the ratio of the periphery to the diameter ; therefore we may seek fractions with smaller numbers agreeing with that both differing so very little, as can be done. Therefore I divide 355 by 113 and I find:

$$\frac{355}{113} = 3+\left[\frac{16}{113}=\frac{1}{\frac{113}{16}}\right] = 3+\frac{1}{7+\frac{1}{16}} = \frac{1}{7+\frac{1}{16}}$$

from which I form the following fractions

$$\begin{matrix} & 3 & 7 & 16 \\ \frac{1}{0}, & \frac{3}{1}, & \frac{22}{7}, & \frac{355}{113}; \end{matrix}$$

therefore the fractions $\frac{3}{1}$ and $\frac{22}{7}$ approach the fraction $\frac{355}{113}$ closer than any other composed from not greater numbers; but the one $\frac{22}{7}$ will be greater, the other $\frac{3}{1}$ smaller than that proposed, as we have observed now in general above. These may be permitted to be called the *principal ratios*, for besides this other equally satisfying subsidiary or *nearby ratios* can be assigned to be sought ; clearly as the fraction $\frac{22}{7}$ to be used has been formed with the index 7 from the preceding, thus subsidiary ratios will be formed in the same way with smaller numbers being substituted in place of 7.

16. But if the ratio of the periphery to the diameter may be taken more exactly and continued division, as has been established already, may be put in place, the following series of quotients will be produced

$$3,\ 7,\ 15,\ 1,\ 292,\ 1\ ,\ 1\ ,\ 1\ ,\ 2,\ 1\ ,\ 3,\ 1\ ,\ 14 \text{ etc.},$$

from which only the following simpler fractions will be elicited

3	7	15	1	292	1	
$\frac{1}{0}$,	$\frac{3}{1}$,	$\frac{22}{7}$,	$\frac{333}{106}$,	$\frac{355}{113}$,	$\frac{103993}{33102}$	*principal ratios*
	$\frac{2}{1}$,	$\frac{19}{6}$,	$\frac{311}{99}$,		$\frac{103638}{32989}$	*nearby ratios*
	$\frac{1}{1}$,	$\frac{16}{5}$,	$\frac{289}{92}$,		$\frac{103283}{32876}$	
		$\frac{13}{4}$,	$\frac{267}{85}$,		$\frac{102928}{32763}$	
		$\frac{10}{3}$,			etc.	
		$\frac{7}{2}$,				
		$\frac{4}{1}$,				
		etc.				

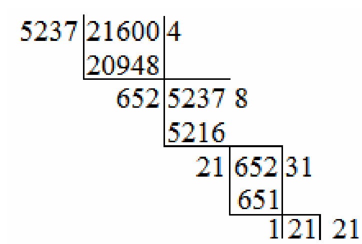
Therefore with this agreed on we have produced two kinds of fractions, of which some are exceedingly large, others exceedingly small; evidently they are exceedingly large which are contained under the indices 3, 15, 292 etc., the rest are exceedingly small. And hence it will be allowed readily to assemble the whole Wallisean table, which includes the ratios approaching closer to the true ratio of the periphery to the diameter, than can be done with smaller numbers.

17. Also by this method it will be possible to define the reasoning of the arrangement of leap years, so that the beginnings of the years may fall at the same time always. This determination depends on the magnitude of the tropical year, which the most accurate observations I may put just short of $365^{d}\,5^{h}\,49'\,8''$.

The excess therefore over 365 days will be $5^{\text{h}}\,49'\,8''$, which if it may be to the fourth part of a day, with care always may constitute the fourth part of any leap year ; but since this same excess shall be less than 6 hours, a smaller number of leap years must be taken; which shall be known from the ratio 24^{h} to $5^{\text{h}}\,49'8''$, or from the fraction

$$\frac{21600}{5237}$$

from which it follows in the interval of 21600 years only 5237 leap years will be required to be put in place. But since this period shall be exceedingly great, we will obtain smaller periods by investigating fractions with smaller numbers, which shall be approximately equal to the fraction $\frac{21600}{5237}$. Towards this end I shall establish the following division :



Now from the quotients found 4, 8, 31, 21, which will be the denominators of the continued fraction, the following fractions will be formed:

$$\begin{array}{ccccc} & 4 & 8 & 31 & 21 \\ \frac{1}{0}, & \frac{4}{1}, & \frac{33}{8}, & \frac{1027}{249}, & \frac{21600}{5237}. \end{array}$$

The second of these fractions, $\frac{4}{1}$, at once gives the account of the Julian calendar, where every four years is called a leap year. Therefore the goal will be achieved closer from the third fraction, if only 8 leap years may be put in place for 33 years. But since also it may be arranged for the period of the number of years to be had equally, we may take the fractions corresponding to the non-major ratio of four years, which will have numerators divisible by 4 ; which will be

$$\frac{136}{33},\ \frac{268}{65},\ \frac{400}{97},\ \frac{532}{129},\ \frac{664}{161}\ \text{etc.,}$$

the third of which $\frac{400}{97}$ is most convenient for the computation of the calendar.

[Note that these nearby ratios may be formed by the simple artifice of adding together the numerators and the corresponding denominators of two nearby fractions supplied, and adjusting slightly so that the numerator is divisible by 4, and the denominator is an odd relative prime.]

Moreover it is evident from this interval of 400 years only 97 leap years must be put in place, the remaining 3 [for the 100, 200, & 300 years intervals] otherwise are required to be changed here into the common 4 year intervals, which would be leap years in the Julian calendar, which also puts in place the Gregorian precept. From which it is understood it is not allowed to use a correction interval with a smaller number of years. But the most accurate is accustomed to be acquired , if the interval of 21600 years may acquire a single extra year to be changed generally every 400 years, which must be established following the Gregorian leap year constitution.

[In the Gregorian calendar, every year that is exactly divisible by four is a leap year, except for years that are exactly divisible by 100, but these centurial years are leap years if they are exactly divisible by 400. For example, the years 1700, 1800, and 1900 are not leap years, but the years 1600 and 2000 are leap years.]

18. Now we will investigate the fractions , which approach so closely to $\sqrt{2}$, that no other smaller fraction shall be able to approach closer. Truly there is :

$$\sqrt{2}=1,41421356=\tfrac{141421356}{100000000},$$

which fraction, if it may be treated by continued division just as in the prescribed manner, will give these quotients

$$1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2 \text{ etc.},$$

from which the following fractions sought will be formed, satisfying both the major as well as the minor ratios sought:

$$\begin{array}{cccccccc} 1, & 2, & 2, & 2, & 2, & 2, & 2, & 2 \text{ etc.},\\ \frac{1}{0}, & \frac{1}{1}, & \frac{3}{2}, & \frac{7}{5}, & \frac{17}{12}, & \frac{41}{29}, & \frac{99}{70}, & \frac{239}{169} \text{ etc.}\\ & & \frac{2}{1}, & \frac{4}{3}, & \frac{10}{7}, & \frac{24}{17}, & \frac{58}{41}, & \frac{140}{99}\\ > & < & > & < & > & < & > & < \end{array}$$

of which the alternate fractions with the sign > are observed greater than $\sqrt{2}$, truly the rest having the sign < are observed to be less than $\sqrt{2}$.

[It should be observed that the above and following continued fractions for $\sqrt{2}$ may be found by setting $\left(\sqrt{2}-1\right)\left(\sqrt{2}+1\right)=1,\quad \sqrt{2}=1+\frac{1}{1+\sqrt{2}}=1+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\sqrt{2}}}}$, etc.]

19. This property of $\sqrt{2}$ is noteworthy, because all the quotients besides the first are equal to two, thus so that there shall be

$$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\text{etc.}}}}}$$

Truly also in a similar manner, if $\sqrt{3}$ may be set out, the quotients are found :

$$1,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1 \text{ etc.},$$

thus so that there shall be

$$\sqrt{3}=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\text{etc.}}}}}}}$$

Although indeed it may not be agreed from the division itself, whether the quotients may be progressing further by this law, yet that may be seen to be not only probable, but also it can be shown in the following manner, where the values of each kind of continued fractions, in which the denominators shall either all be equal or alternating between two values, or three values etc., we will show how to investigate later.

19 [a]. Therefore the following continued fraction shall be proposed:

$$a+\cfrac{1}{b+\cfrac{1}{b+\cfrac{1}{b+\cfrac{1}{b+\text{etc.}}}}}$$

which may be put $=x$; there will become:

$$x-a=\cfrac{1}{b+\cfrac{1}{b+\cfrac{1}{b+\cfrac{1}{b+\text{etc.}}}}}\quad=\frac{1}{b+x-a};$$

hence there will be

$$x^2-2ax+bx+a^2-ab=1$$

and

$$x=a-\tfrac{b}{2}+\sqrt{\left(1+\tfrac{bb}{4}\right)}.$$

Whereby if there were $b=2$ and $a=1$, there will become

$$x=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\text{etc.}}}}}=\sqrt{2};$$

therefore if there may be put $b=2a$, there will become

$$\sqrt{\left(a^2+1\right)}=a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\text{etc.}}}}}$$

from which, from all the numbers, which exceed the square by one, the approximate square root can be extracted expeditely; as on putting $a=2$ the following fractions will serve for finding $\sqrt{5}$ approximately :

2	4	4	4	4	4	4
$\frac{1}{0}$,	$\frac{2}{1}$,	$\frac{9}{4}$,	$\frac{38}{17}$,	$\frac{161}{72}$,	$\frac{682}{305}$,	$\frac{2889}{1292}$ etc.
	$\frac{1}{1}$,	$\frac{7}{3}$,	$\frac{29}{13}$,	$\frac{123}{55}$,	$\frac{521}{233}$,	$\frac{2207}{987}$
		$\frac{5}{2}$,	$\frac{20}{9}$,	$\frac{85}{38}$,	$\frac{360}{161}$,	$\frac{1525}{682}$
		$\frac{3}{1}$,	$\frac{11}{5}$,	$\frac{47}{21}$,	$\frac{199}{89}$	$\frac{843}{477}$

20. Now the following continued fraction shall be proposed :

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{c+\text{etc.}}}}}}}$$

which may be put $=x$; and the value of x will be found in the following manner:

$$x-a=\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{c+\text{etc.}}}}}}}=\cfrac{1}{b+\cfrac{1}{c+x-a}}$$

hence therefore there will be $x-a=\frac{x+c-a}{bx+bc-ab+1}$ or

$$bxx+bcx-2abx=abc-a^2b+c;$$

therefore if there were $c=2a$, there will become

$$bxx = aab + 2a \text{ and } x = \sqrt{\left(a^2 + \tfrac{2a}{b}\right)}.$$

In a similar manner if there may be put:

$$x = a + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + \text{etc.}}}}}}}$$

there will become

$$x - a = \cfrac{1}{b + \cfrac{1}{c + \cfrac{1}{d + x - a}}}$$

from which it follows:

$$(bc+1)x^2 + (bcd + b + d - c - 2abc - 2a)x - abcd + a^2bc - ab - ad$$
$$+ aa - cd + ac - 1 = 0.$$

And in this way all the continued fractions, of which it is allowed to take the denominators in repetition either of all the following alternate, third, or fourth, etc. repetition to be equal to each other. Moreover always the sum or the value *x* is the root taken from the quadratic equation.

21. Before we may progress to other continued fractions, in which the denominators constitute arithmetical progressions requiring to be summed, we shall establish certain transcending quantities, which conversely in continued fractions may give the denominators advancing in arithmetic progression, where from these a plainer way may emerge of summing continued fractions of this kind. Therefore here with logarithms and with other transcending expressions attempting to grasp continued fractions of this kind, if the number, of which the hyperbolic logarithm is unity, and some of its powers may be considered. Therefore by putting this number $= e$ there will be

$$e = 2,71828182845904,$$

from which expression changed into a continued fraction there will become

$$e = 2 + \cfrac{1}{1 + \cfrac{1}{2 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{4 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{6 + \cfrac{1}{1 + \text{etc.}}}}}}}}}}$$

the third denominators of which constitute the arithmetical progression 2, 4, 6, 8 etc., the remainder are all unity. Which law, even if taken from a single observation, yet that will probably be seen to be valid indefinitely, which indeed certainly will be confirmed below. In as similar manner

$$\sqrt{e} = 1,6487212707$$

may be changed into a continued fraction, there will become :

$$\sqrt{e} = 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{5 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{9 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{13 + \text{etc.}}}}}}}}}}$$

the law of which is similar to the preceding. And it will be allowed to be observed likewise in other continued fractions, in which the powers of *e* may be changed.

22. In a similar manner I have considered the cube root taken from the number *e*, the hyperbolic logarithm of which is 1, and I have found

$$\frac{\sqrt[3]{e}-1}{2} = 0,1978062125 = \cfrac{1}{5 + \cfrac{1}{18 + \cfrac{1}{30 + \cfrac{1}{42 + \cfrac{1}{54 + \text{etc.}}}}}}$$

in the denominators of which continued fraction I have observed the first arithmetic progression.

Similarly it happens, if integer powers of the exponent of *e* may be considered. Thus we may consider the square to be found:

$$\frac{e^2-1}{2} = 3,19452804946532 = 3 + \cfrac{1}{5 + \cfrac{1}{7 + \cfrac{1}{9 + \cfrac{1}{11 + \cfrac{1}{13 + \cfrac{1}{15 + \text{etc.}}}}}}}$$

Finally also from the number *e* itself, from which formed the interrupted continued fraction has had an arithmetical progression of the denominators, I have observed with few changes a continued fraction of this kind can be formed free from breaks. For it will produce

$$\frac{e-1}{e+1}=2+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{26+\text{etc.}}}}}}}$$

In which according to the rules present, the arithmetic progression proceed with a common difference of 4.

23. Therefore since I may have observed so great an agreement between continued fractions, in which the denominators may constitute an arithmetic progression, sometimes interrupted, sometimes not interrupted ; in that case I have happened to consider, whether perhaps for a continued fraction, in which the arithmetical progression of the denominators shall be interrupted, may be able to be transformed in another arithmetical progression without the interruption. Therefore I have considered some progression *a, b, c, d, e* etc. and I have interpolated *m, n,* between any two neighbouring numbers, so that the following continued fraction may be produced

$$a+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{b+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{c+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{d+\text{etc.}}}}}}}}}}$$

and I have found that to be equal to the following continued fraction, in which the denominators may be progressing without interruption,

$$\frac{1}{mn+1}\left((mn+1)a+n+\cfrac{1}{(mn+1)b+m+n+\cfrac{1}{(mn+1)c+m+n+\cfrac{1}{(mn+1)d+m+n+\text{etc.}}}}\right).$$

The demonstration of this arrangement consists in this, that ordinary fractions, for which the value of each is agreed, shall agree amongst themselves, as will become apparent on being tested.

24. If the interpolated quantities *m, n* may be inverted in order, the difference in the latter continued fraction will be apparent only in the first term; from which the following elegant theorem is prepared well enough, so that there will become

$$a+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{b+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{c+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{d+\text{etc.}}}}}}}}}} - \left(a+\cfrac{1}{n+\cfrac{1}{m+\cfrac{1}{b+\cfrac{1}{n+\cfrac{1}{m+\cfrac{1}{c+\cfrac{1}{n+\cfrac{1}{m+\cfrac{1}{d+\text{etc.}}}}}}}}}}\right) = \frac{n-m}{mn+1}.$$

Therefore whatever numbers *a*, *b*, *c*, *d* etc. may be substituted, the difference between these two continued fractions always will be known and constant, evidently $=\frac{n-m}{mn+1}$.

25. From the same equality found between continued fractions, evidently for both interrupted as well as non-interrupted, the following equality follows on dividing one by each and adding the same quantity *A* to each :

$$A+\cfrac{1}{a+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{b+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{b+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{c+\text{etc.}}}}}}}}}}} = A+\cfrac{mn+1}{(mn+1)a+n+\cfrac{1}{(mn+1)b+m+n+\cfrac{1}{(mn+1)c+m+n+\text{etc.}}}}$$

Therefore with the help of this equation any continued fraction being interrupted by the quantities *m* and *n* will be able to be converted into another in which the denominators may be progressing without interruption. If therefore, as we had in the above fractions, there may be put $m=n=1$, the following equation will be produced :

$$A+\cfrac{1}{a+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{b+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{b+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{c+\text{etc.}}}}}}}}}}} = A+\cfrac{2}{2a+1+\cfrac{1}{2b+2+\cfrac{1}{2c+2+\text{etc.}}}}$$

Therefore from § 21, there shall be

$$\frac{1}{e-2}=1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\text{etc.}}}}}$$

there will become, on putting $A=1$, $a=2$, $b=4$ etc., as follows,

$$\frac{1}{e-2}=1+\cfrac{2}{5+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\text{etc.}}}}}}$$

and hence there will become on dividing unity by each side of the above equation,

$$e=2+\cfrac{1}{1+\cfrac{2}{5+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{26+\text{etc.}}}}}}}}$$

In a similar manner, from the same paragraph there will be found :

$$\sqrt{e}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{9+\text{ etc.}}}}}}}}=1+\cfrac{2}{3+\cfrac{1}{12+\cfrac{1}{20+\cfrac{1}{28+\text{etc.}}}}}$$

And these continued fractions now converge so much, so that it will be an easy matter to find the values of e and $\sqrt{e}$ themselves, however close they may be required to be found.

26. Truly hence also in turn a continued fraction, in which the denominators progress in order without interruption, may be changed into another, in which the denominators shall be interrupted by the two constant numbers *m* and *n;* thus I have found to become

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}=a-n+\cfrac{mn+1}{m+\cfrac{1}{n+\cfrac{1}{\frac{b-m-n}{mn+1}+\cfrac{1}{\frac{c-m-n}{mn+1}+\cfrac{1}{m+\text{etc.}}}}}}$$

or by taking the fractions in these denominators, if the need were seen, there will become t

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{etc.}}}} = a-n+\cfrac{mn+1}{m+\cfrac{1}{n+\cfrac{mn+1}{m+\cfrac{1}{\frac{mn+1}{c-m-n}+\cfrac{mn+1}{m+\cfrac{1}{n+\text{etc.}}}}}}}$$

Therefore if there may be put $m=n=1$, there will be had:

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{etc.}}}} = a-1+\cfrac{2}{1+\cfrac{1}{1+\cfrac{1}{\frac{1}{2}b-1+\cfrac{1}{1+\frac{1}{\frac{1}{2}c-1}+\cfrac{1}{\frac{1}{2}d-1+\text{etc.}}}}}}$$

27. Just as here continued fractions, of which the denominators thus interrupted are progressing in order, so that between any two consecutive terms two constant quantities shall be interposed, thus we have considered this same reduction can be extended to four, six, etc. interpolated constant quantities. But an odd number of constant quantities cannot be interpolated. Thus if between the quantities *a*, *b*, *c*, *d* etc. some contiguous quantities may be interpolated between any two these four m, n, p, q may be put, and for the sake of brevity

$$mnpq+mn+mq+pq+1=P$$

and

$$mnp+npq+m+n+p+q=Q,$$

there will become:

$$a+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{m+\cfrac{1}{m+\text{etc.}}}}}}}}} = \frac{1}{P}\left(Pa+npq+n+q+\cfrac{1}{Pb+Q+\cfrac{1}{Pc+Q+\cfrac{1}{Pd+Q+\text{etc.}}}}\right)$$

And if there were $m=p=q=1$, there will be had:

$$a+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{b+\cfrac{1}{1+\cfrac{1}{1+\text{etc.}}}}}}}} = \frac{1}{5}\left(5a+3+\cfrac{1}{5b+6+\cfrac{1}{5b+6+\cfrac{1}{5d+6+\text{etc.}}}}\right)$$

From which in turn a new version of the continued fraction arises.

28. But since in the preceding, where the number *e*, of which the logarithm is $=1$, and I have changed its powers into continued fractions, I will have observed only the arithmetical progression of the denominators nor besides have I been able to confirm the probability of this progression being continuing indefinitely, that I have brooded over mainly, so that I shall enquire into the need of this progression, in order that I may show that more firmly. And this also I have pursued in a particular way, where the integration of this equation

$$ady + y^2dx = x^{\frac{-4n}{2n+1}}dx\,,$$

I have reduced to the integration of this equation

$$adq + q^2dp = dp.$$

For on putting

$$p = (2n+1)\,x^{\frac{1}{2n+1}}\,,$$

I have found to be

$$q = \frac{a}{p} + \cfrac{1}{\frac{3a}{p} + \cfrac{1}{\frac{5a}{p} + \cfrac{1}{\frac{7a}{p} + \cdots + \cfrac{1}{\frac{(2n-1)a}{p} + \cfrac{1}{x^{\frac{2n}{2n+1}}y}}}}}$$

From which, since q may be able to be given by p and there shall be $p = (2n+1)\,x^{\frac{1}{2n+1}}$, a finite equation can be formed between *x* and *y*, which will be the integral of the equation $ady + y^2dx = x^{\frac{-4n}{2n+1}}dx$, provided *n* is a positive integer.

29. Therefore if n may be put to be an infinite number, the expression found will be a continued fraction departing to infinity, the denominators of which constitute an arithmetical progression. On account of which the following equation will be had:

$$q = \frac{a}{p} + \cfrac{1}{\frac{3a}{p} + \cfrac{1}{\frac{5a}{p} + \cfrac{1}{\frac{7a}{p} + \cfrac{1}{\frac{9a}{p} + \text{etc.}}}}}$$

and *q*, or the value of its continuous fraction will be defined from this equation :

$$adq + q^2 dq = dp.$$

Therefore there will become

$$\frac{adq}{1-qq} = dp$$

which constant must be determined from there, so that on putting $p = 0$ there may become $q = \infty$. On account of which there will become

$$\frac{a}{2} l \frac{q+1}{q-1} q = p \quad \text{and} \quad \frac{q+1}{q-1} = e^{\frac{2p}{a}}$$

from which there becomes

$$q = \frac{e^{\frac{2p}{a}}+1}{e^{\frac{2p}{a}}-1},$$

which is the value found of the continuous fraction. Then truly since there shall be

$$e^{\frac{2p}{a}} = 1 + \frac{2}{q-1},$$

there will be had

$$e^{\frac{2p}{a}} = 1 + \cfrac{2}{\frac{a-p}{p} + \cfrac{1}{\frac{3a}{p} + \cfrac{1}{\frac{5a}{p} + \cfrac{1}{\frac{7a}{p} + \text{etc.}}}}}$$

30. If there may be put $\frac{a}{2p} = s$ or $a = 2ps$, there will become

$$e^{\frac{1}{s}} = 1 + \cfrac{2}{2s - 1 + \cfrac{1}{\frac{1}{6s} + \cfrac{1}{\frac{1}{10s} + \cfrac{1}{\frac{1}{14s} + \text{etc.}}}}}$$

And from the first equation found there will become

$$\frac{e^{\frac{1}{s}}+1}{e^{\frac{1}{s}}-1} = 2s + \cfrac{1}{6s + \cfrac{1}{10s + \cfrac{1}{14s + \cfrac{1}{18s + \text{etc.}}}}}$$

If the denominators of this may be interpolated with two units, there will be had,

$$\frac{e^{\frac{1}{s}}+1}{e^{\frac{1}{s}}-1}=2s-1+\cfrac{2}{1+\cfrac{1}{3s-1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{+\cfrac{1}{5s-1+\cfrac{1}{1+\text{etc.}}}}}}}}$$

From which the following continued fraction arises

$$e^{\frac{1}{s}}=1+\cfrac{2}{s-1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3s-1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5s-1+\cfrac{1}{1+\cfrac{1}{1+\text{etc.}}}}}}}}}}$$

Truly from these formulas everything found above arises, from which we express certain powers of *e* by continued fractions, from which the necessity of the progression just observed before is understood.

31. Therefore we have obtained now the continued fraction, the denominators of which constitute an arithmetical progression and will be allowed to show its true value. But since this progression shall be only a kind of arithmetical progression, I have considered the general arithmetical progression and the continued fraction, which shall establish its denominators, I have recalled to the sum in the following way. Clearly there shall be the following continued fraction, of which the value, that I seek, I put $=s$, thus so that there shall become

$$s=a+\cfrac{1}{(1+n)a+\cfrac{1}{(1+2n)a+\cfrac{1}{(1+3n)a+\cfrac{1}{(1+4n)a+\text{etc.}}}}}$$

from which, so that I may elicit the value of s, I begin from the approximation for that. And thus there will be, by the method treated above,

$$\begin{array}{cccc} a & (1+n)a & (1+2n)a & (1+3n)a \\ \frac{1}{0}, & \frac{a}{1}, & \frac{(1+n)a^2+1}{(1+n)a}, & \frac{(1+n)(1+2n)a^3+(2+2n)a}{(1+n)(1+2n)a^2+1} \end{array} \text{ etc.,}$$

which fractions continually approach closer to the true value of s ; and the most distant limit will give the true value of *s*.

32. If these fractions may be continued further, the law will be readily observed, by which they are formed, and from that the infinite fraction may be concluded after the division of the numerator and the denominator by the first denominator to become :

$$\frac{a+\frac{1}{1\cdot na}+\frac{1}{1\cdot 2\cdot 1(1+n)n^3a^3}+\frac{1}{1\cdot 2\cdot 3\cdot 1(1+n)(1+2n)n^5a^5}+\text{etc.}}{1+\frac{1}{1(1+n)na^2}+\frac{1}{1\cdot 2(1+n)(1+2n)n^3a^4}+\frac{1}{1\cdot 2\cdot 3(1+n)(1+2n)(1+3n)n^5a^6}+\text{etc.}},$$

to which thus *s* shall be equal. Therefore on putting $a=\frac{1}{\sqrt{nz}}$ there will become

$$s=\frac{1}{\sqrt{nz}}\cdot\frac{1+\frac{z}{1\cdot 1}+\frac{z^2}{1\cdot 2\cdot 1(1+n)}+\frac{z^3}{1\cdot 2\cdot 3\cdot 1(1+n)(1+2n)}+\text{etc.}}{1+\frac{z}{1(1+n)}+\frac{z^2}{1\cdot 2(1+n)(1+2n)}+\frac{z^3}{1\cdot 2\cdot 3(1+n)(1+2n)(1+3n)}+\text{etc.}}$$

so that which value may be obtained, there may be put

$$t=1+\frac{z}{1\cdot 1}+\frac{z^2}{1\cdot 2\cdot 1(1+n)}+\frac{z^3}{1\cdot 2\cdot 3\cdot 1(1+n)(1+2n)}+\text{etc.}$$

and

$$u=1+\frac{z}{1(1+n)}+\frac{z^2}{1\cdot 2(1+n)(1+2n)}+\frac{z^3}{1\cdot 2\cdot 3(1+n)(1+2n)(1+3n)}+\text{etc.},$$

thus so that there will become :

$$s=\frac{t}{u\sqrt{nz}}.$$

Moreover, from the inspection of these two series there understood to become :

$$dt=udz;$$

and in a similar manner there will be taken to be :

$$udz+nzdu=tdz.$$

There may be put $t=vu$, from which there shall become

$$s = \frac{v}{\sqrt{nz}}.$$

there will be

$$vdu + udv = udz$$

and

$$udz + nzdu = uvdz;$$

from which it follows :

$$\frac{du}{u} = \frac{dz - dv}{v} = \frac{vdz - dz}{nz}.$$

and hence the following equation consisting between *z* and *v* only:

$$nzdv - vdz + v^2 dz = n - zdz,$$

which on substituting

$$v = z^n q \text{ and } z = r^n.$$

will be changed into this:

$$dq = q^2 dr = nr^{n-2} dr.$$

And from the equation, if *q* may be determined by *r* and there may be put

$$r = n^{-\frac{1}{n}} a^{-\frac{2}{n}},$$

the value sought will become

$$s = arq.$$

33. Therefore with the assigned value of the proposed continued fraction, which I have put *s*, with there proving to be

$$s = a + \cfrac{1}{(1+n)a + \cfrac{1}{(1+2n)a + \cfrac{1}{(1+3n)a + \cfrac{1}{(1+4n)a + \text{etc.}}}}}$$

has led to the resolution of this equation

$$dq + q^2 dr = nr^{n-2} dr,$$

thus moreover there must taken with the integral of this equation, so that on making $a = \infty$ there may become $s = \infty$, or on taking $a = 0$ there may become $s = 1$. From which the following rule arises in the integration for the constant being introduced, so that in the case, where $n > 2$ is not satisfied, there shall become $q = \infty$ on putting $r = 0$. Moreover we may put *n* to be a positive number, where a continued fraction may arise, such as we have considered up to this stage, have positive denominators.

34. Moreover the equation found

$$dq + qqdr = nr^{n-2}dr$$

agrees with the equation found proposed at one time by Ct. Riccati and from these therefore only the cases to be integrable, for which n is a number of this form $\frac{2}{2m+1}$ with m whole and positive, so that we may obtain a positive number for n. On this account therefore the case of the continued fraction

$$a+\cfrac{1}{\cfrac{(2m+3)a}{2m+1}+\cfrac{1}{\cfrac{(2m+5)a}{2m+1}+\cfrac{1}{\cfrac{(2m+7)a}{2m+1}+\text{etc.}}}}$$

the value will be able to be shown always by a finite expression. Which indeed itself may be evident ; for on making $m=0$ we will have this continued fraction

$$a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}}$$

the value of which we have now found above. For this can be reduced from that kind; for on putting $a=(2m+1)b$ there will be had

$$(2m+1)b+\cfrac{1}{(2m+3)b+\cfrac{1}{(2m+5)b+\text{etc.}}}$$

which now will be considered as known just as often as m were a positive whole number.

35. Therefore it is apparent the integration of the equation

$$dq + q^2dr = nr^{n-2}dr$$

by the resolution of this same continued fraction, leads to the integration of this equation

$$dq + q^2dr = 2dr,$$

if indeed n were $=\frac{2}{2m+1}$, with m denoting a positive integer. Which same reduction I have established now above in § 28 in the same manner, which can be done from this source. Moreover so that it may be understood, how by this account the true value of this kind of continued fraction may be found, I will consider the case $n=2$ or $m=0$, where there will become

$$s = a + \cfrac{1}{3a + \cfrac{1}{5a + \cfrac{1}{7a + \text{etc.}}}}$$

Truly s will be found from this equation

$$dq + q^2 dr = 2dr$$

which integrated in the due manner give

$$r = \frac{1}{2\sqrt{2}} l \frac{q+\sqrt{2}}{q-\sqrt{2}},$$

from which there is produced

$$q = \frac{\left(e^{2r\sqrt{2}}+1\right)\sqrt{2}}{e^{2r\sqrt{2}}-1}.$$

Truly there is

$$r = \frac{1}{a\sqrt{2}} \quad \text{and} \quad s = arq = \frac{q}{\sqrt{2}},$$

from which there will arise this value itself

$$s = \frac{e^{\frac{2}{a}}+1}{e^{\frac{2}{a}}-1},$$

just as now we have shown above (§ 29).

DE FRACTIONIBUS CONTINUIS DISSERTATIO

Commentatio 71 indicis ENESTROEMIANI

Commentarii academiae scientiarum Petropolitanae 9 (1737), 1744, p. 98-137

1. Varii in Analysin recepti sunt modi quantitates, quae alias difficulter assignari queant, commode exprimendi. Quantitates scilicet irrationales et transcendantes, cuiusmodi sunt logarithmi, arcus circulares aliarumque curvarum quadraturae, per series infinitas exhiberi solent, quae, cum terminis constent cognitis, valores illarum quantitatum satis distincte indicant. Series autem istae duplicis sunt generis, ad quorum prius pertinent illae series, quarum termini additione subtractioneve sunt connexi; ad posterius vero referri possunt eae, quarum termini multiplicatione coniunguntur. Sic utroque modo area circuli, cuius diameter est $=1$, exprimi solet; priore nimirum area circuli aequalis dicitur

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\text{ etc.in infinitum,}$$

posteriore vero modo eadem area aequatur huic expressioni

$$\tfrac{2\cdot4\cdot4\cdot6\cdot6\cdot8\cdot8\cdot10\cdot10}{3\cdot3\cdot5\cdot5\cdot7\cdot7\cdot9\cdot9\cdot11}\text{etc. in infinitum.}$$

Quarum serierum illae reliquis merito praeferuntur, quae maxime convergant et paucissimis sumendis terminis, valorem quantitatis quaesitae proxima praebeant.

2. His duobus serierum generibus non immerito superaddendum videtur tertium, cuius termini continua divisione inter se connectuntur, quas series propterea *fractiones continuas* appellare conveniet. Minus quidem usitatum est hoc serierum genus duobus reliquis; sed non solum aeque distincte valorem quantitatis, quam exprimit, ob oculos ponit, verum etiam perquam est aptum ad valorem illum proxime inveniendum. Tam parum autem hoc serierum genus etiamnum est excultum, ut praeter unam vel alteram huius generis seriem iam cognitam nequidem methodus habeatur vel huiusmodi serierum veros valores inveniendi vel datas quantitates transcendantes in tales expressiones convertendi. Cum igitur iam pridem in his fractionibus continuis examinandis laboraverim atque plura cum ad earum usum tum inventionem pertinentia non parvi momenti observaverim, ea hic exponere constitui, quo aliis viam easdem tractandi planiorem efficerem. Quamvis enim nondum ad completam huius doctrinae theoriam pertigerim, tamen haec, quae magno labore elicui, insigne adiumentum allatura esse confido ad istam doctrinam magis perficiendam.

3. Quo igitur, quid nomine fractionum continuarum intelligam, clarius percipiatur, amplissimum earum exemplum ante omnia exhibeo

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{etc.}}}}}}$$

ex quo scribendi modo quilibet significationem huius expressionis facile cognoscet. Quantitas scilicet haec constat duobus membris, numero integro α et fractione, cuius numerator est *a*, denominator vero iterum ex duobus compositus est membris, integro nimirum *b* et fractione, cuius numerator est β denominator vero rursum duobus consistit membris, integro videlicet *c* et fractione ut ante; sicque porro in infinitum. Duplices hic occurrunt quantitatas, quas etiam litteris ex latino et graeco alphabeto desumtis distinxi. Harum quantitatum eas, quas etiam graecis litteris denotavi, *numeratores* appellabo, quia fractionum sequentium numeratores revera constituunt; reliquas vero quantitates latinis litteris expressas ad distinctionem omnes *denominatores* vocabimus; omnes enim praeter primam revera sunt partes denominatorum.

4. Primus, qui, quantum mihi constat, huiusmodi fractionem continuam protulit, erat Vicecomes Brouncker, qui post communicatam secum Wallisii quadraturam circuli eandem expressionem ita commutavit, ut asseveraret aream circuli se habere ad quadratum diametri uti 1 ad

$$1+\cfrac{1}{2+\cfrac{9}{2+\cfrac{25}{2+\cfrac{49}{2+\cfrac{81}{2+\text{etc.}}}}}}$$

ubi numeratores sunt quadrata numerorum imparium, denominatores vero 2. Qua autem via Brounckerus in hanc expressionem inciderit, non constat, atque merito foret dolendum, si eius methodus periisset, cum non sit dubitandum, quin eadem methodo plura praeclara in hoc genere exhiberi possent. Wallisius quidem, dum hanc fractionem recenset, ipse demonstrationem concinnare est conatus, quae autem minus est genuina atque penitus ab auctoris methodo diversa esse videtur. Wallisius autem hanc totam inventionem derivat ex sequente theoremate, quod sit

$$a^2=(a-1)+\cfrac{1}{2(a-1)+\cfrac{9}{2(a-1)+\cfrac{25}{2(a-1)+\text{etc.}}}}\times(a+1)+\cfrac{1}{2(a+1)+\cfrac{9}{2(a+1)+\cfrac{25}{2(a+1)+\text{etc.}}}}$$

cuius veritatem per inductionem satis confirmat, sed, quod caput est, analysin non affert, qua ad hoc theorema sit perventum.

5. Commode autem atque facile ex data huiusmodi fractione continua valor eius vero proximus potest determinari, quin et limites definire licet, intra quos verus valor contineatur, ut, si quadratura quaepiam vel alia quantitas transcendens hoc modo fuerit. expressa, facili negotio ea ipsa proxime assignari queat. Ostendam hoc ex generali fractionum continuarum forma

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\text{etc.}}}}}$$

in qua omnes quantitates ingredientes affirmativas pono. Apparet autem valorem vero propinquum obtineri, si fractio continua alicubi abrumpatur, atque eo propiorem valorem inventum iri, quo longius fractio continuetur. Ita sumendo tantum a habebitur valor minor vero, cum annexa fractio tota negligatur. Sumendo autem

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c}}$$

valor habebitur maior vero, quia in fractione denominator b est iusto minor. Sin autem sumatur

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c}}$$

habebitur iterum valor iusto minor ob fractionem $\frac{\beta}{c}$ indeque denominatorem $b+\frac{\beta}{c}$ nimis magnum. Atque hoc modo fractionem continuam successive abrumpendo alternative valores iusto maiores et minores prodibunt; unde quamtumvis prope ad verum fractionis continuae valorem accedera licebit.

6. Sequens igitur habebitur expressionum series

$$a \quad a+\frac{\alpha}{b} \quad \cfrac{\alpha}{b+\cfrac{\beta}{c}} \quad a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d}}} \quad \text{etc.;}$$

quarum quae sunt ordine impares, ut prima, tertia, quinta etc., minores sunt vero fractionis continuae valore; pares autem erunt maiores eodem. Quare cum terminus tertius maior sit primo, quintus maior tertio et ita porro, termini impares crescendo tandem verum fractionis continuae valorem attingent; termini pares vero, qui continuo decrescunt, decrescendo tandem ad verum fractionis continuae valorem descendent. Si autem hae expressiones in fractiones simplices transmutentur, sequens prodibit earundem expressionum series

$$\frac{a}{1},\ \frac{ab+\alpha}{b},\ \frac{abc+\alpha c+\beta a}{bc+\beta},\ \frac{abcd+\alpha cd+\beta ad+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b} \ \text{etc.;}$$

quae si attentius inspiciatur, facile colligetur lex, qua isti termini progrediuntur cuiusque ope sine operosa fractionum illarum compositarum reductione has fractiones, quousque libuerit, continuare licet. Nimis quidem hae fractiones statim fiunt prolixae; sed in exemplis, quibus hae litterae numeris exprimuntur, perquam commode haec series continuatur.

7. Lex autem progressionis harum fractionum ex sequente schemate clare percipietur:

$$\begin{array}{cccccc} a & b & c & d & e & \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+\alpha}{b}, & \frac{abc+\alpha c+\beta a}{bc+\beta}, & \frac{abcd+\alpha cd+\beta ad+\gamma ab+\alpha\gamma}{bcd+\beta d+\gamma b} & \text{etc.} \\ \alpha & \beta & \gamma & \delta & \varepsilon & \end{array}$$

Scilicet his fractionibus supra scripti sunt denominatores fractionis continuae, infra vero numeratores tanquam indices; ipsis autem fractionibus praefixa est fractio $\frac{1}{0}$, quippe quae ex ipsa lege mox declaranda in hunc locum pertinet. Lex iam progressionis in hoc consistit ut cuiusque fractionis numerator per indicem supra scriptum multiplicatus una cum numeratore praecedentis fractionis per suum infra scriptum indicem multiplicato praebeat numeratorem sequentis fractionis, atque eodem modo cuiusque fractionis denominator per indicem suum supra positum multiplicatus una cum denominatore praecedentis fractionis per indicem suum infra scriptum multiplicato praebeat denominatorem fractionis sequentis. Lex quidem haec ex ipsa inspectione harum fractionum, si ulterius continentur, facile observatur; sed eadem etiam ex ipsa fractionum continuarum natura deduci potest; quam demonstrationem autem hic apponere superfluum iudico.

8. Si istarum fractionum differentiae capiantur subtrahendo quamque a praecedente, sequens orietur series

$$\frac{1}{0}, \quad -\frac{\alpha}{1\cdot b}, \quad +\frac{\alpha\beta}{b(bc+\beta)}, \quad -\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)} \text{ etc.,}$$

cuius numeratorum progressio per se est manifesta, denominatores vero ex binis denominatoribus praecedentibus formantur. Cum igitur superioris seriei ultimus terminus, qui verum fractionis continuae valorem exhibet, componatur ex primo, quem reiceto $\frac{1}{0}$ sumamus *a*, et omnibus differentiis, erit verus fractionis continuae propositae valor

$$\alpha+\frac{\alpha}{1\cdot b}-\frac{\alpha\beta}{b(bc+\beta)}+\frac{\alpha\beta\gamma}{(bc+\beta)(bcd+\beta d+\gamma b)}-\frac{\alpha\beta\gamma\delta}{(bcd+\beta d+\gamma b)(bcde+\cdots)}\text{etc.}$$

.

Habemus adeo seriem infinitam primi generis, cuius termini additione et subtractione inter se coniunguntur, valori fractionis continuae propositae aequalem; haecque series valde convergit atque ad valorem illum proxime inveniendum admodum est apta. Si bini termini coniungantur alternationis signorum evitandae causa, reperietur eadem fractio continua aequalis sequenti seriei

$$a+\frac{\alpha c}{1(bc+\beta)}+\frac{\alpha\beta\gamma e}{(bc+\beta)(bcde+\beta de+\gamma be+\delta bc+\beta\delta)}+\text{ etc.,}$$

cuius numeratorum et denominatorum lex ex superiore sponte se prodit. Vehementer autem haec series convergit, atque eius ope citissime vero proxima summa inveniri potest.

9. Quo magis igitur haec series ultima inventa convergit, eo magis etiam ipsa fractio continua convergere censenda est, quia datus terminorum seriei numerus dato fractionum numero fractionis continuae respondet. Perspicuum ergo est fractionem continuam eo magis convergere, quo minores sint eius numeratores $\alpha, \beta, \gamma,$ etc. maioresque denominatores a, b, c etc. Omnes autem hos numeros, tam numeratores quam denominatores, integros ponere licet; nam si essent fracti, per notam fractionum reductionem in integras transmutari possent, singularum scilicet fractionum numeratores et denominatores per eundem numerum multiplicando. Positis ergo omnibus numeris, tam $\alpha, \beta, \gamma,$ etc. quam a, b, c etc., integris fractio continua maxime converget, si omnes numeratores $\alpha, \beta, \gamma,$ etc. aequentur unitati; deinde vero convergentia eo erit maior, quo maiores fuerint denominatores a, b, c, d etc. Unitate scilicet numeratores minores esse nequeunt; si enim alicubi numerator esset $=0$, ibidem fractio continua abrumperetur foretque fractio finita. Idem quoque accidit, si denominatorum aliquis fiat $=\infty$; ibidem enim pariter fractio continua abrumpetur atque in fractionem finitam transibit.

10. Si igitur sequens proposita sit fractio continua, cuius omnes numeratores sint unitates,

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\cfrac{1}{f+\text{etc.}}}}}}$$

ad eius valorem appropinquabunt fractiones sequentis seriei

$$\begin{array}{ccccc} a & b & c & d & e \\ \frac{1}{0}, & \frac{a}{1}, & \frac{ab+1}{b}, & \frac{abc+c+a}{bc+1}, & \frac{abcd+cd+ad+ab+1}{bcd+d+b} \end{array} \text{ etc.,}$$

quae series ope unicae indicum a, b, c, d etc. progressionis continuatur. Scilicet cuiusque fractionis tam numerator quam denominator per indicem multiplicatus et praecedentis fractionis numeratore et denominatore respective auctus dabit numeratorem et denominatorem sequentis fractionis. Valor deinde huius fractionis continuae aequabitur summae sequentis seriei

$$a+\frac{1}{1\cdot b}-\frac{1}{b(bc+1)}+\frac{1}{(bc+1)(bcd+d+b)}-\frac{1}{(bcd+d+b)(bcde+\cdots)}+\text{ etc.}$$

vel summae huius, in quam ista transmutatur,

$$a+\frac{c}{bc+1}+\frac{e}{(bc+1)(bcde+de+be+bc+1)}+\text{ etc.},$$

cuius seriei denominatores formantur ex alternis denominatoribus seriei fractionum superioris ideoque facile continuantur.

11. Si in tali fractione continua, cuius numeratores omnes sunt unitates, denominatores fuerint numeri fracti, expediet talem fractionem continuam in aliam transformare, in qua tam numeratores quam denominatores sint numeri integri. Ita si huiusmodi proposita esset fractio continua

$$a+\cfrac{1}{\frac{b}{B}+\cfrac{1}{\frac{c}{C}+\cfrac{1}{\frac{d}{D}+\cfrac{1}{\frac{e}{E}+\text{etc.}}}}}$$

haec tollendis fractionibus particularibus transmutabitur in sequentem formam

$$a+\cfrac{B}{b+\cfrac{BC}{c+\cfrac{CD}{d+\cfrac{DE}{e+\text{etc.}}}}}$$

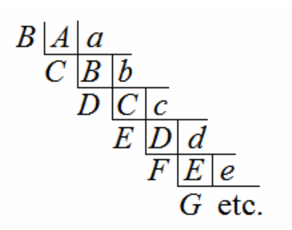
Simili modo vicissim quaevis fractio continua in aliam transmutari potest, cuius omnes numeratores sint unitates, denominatores vero numeri fracti; erit scilicet

$$a+\cfrac{\alpha}{b+\cfrac{\beta}{c+\cfrac{\gamma}{d+\cfrac{\delta}{e+\cfrac{\varepsilon}{f+\text{etc.}}}}}}=a+\cfrac{1}{\frac{b}{\alpha}+\cfrac{1}{\frac{\alpha c}{\beta}+\cfrac{1}{\frac{\beta d}{\alpha\gamma}+\cfrac{1}{\frac{\alpha\gamma e}{\beta\delta}+\cfrac{1}{\frac{\beta\delta f}{\alpha\gamma\varepsilon}+\text{etc.}}}}}}$$

quae posterior forma ex priore facile formatur.

11[a]. Cum igitur data fractione continua eius valor vel verus ipse, si quidem fractio abrumpatur, vel vero proximus per fractionem ordinariam exhiberi queat, vicissim quoque fractio ordinaria in fractionem continuam transformari poterit. Quae transmutatio quomodo sit instituenda in fractionibus continuis, quarum numeratores omnes sint unitates, denominatores vero numeri integri, primum ostendam. Omnis autem fractio finita, cuius numerator et denominator sunt numeri integri finiti, in huiusmodi fractionem continuam transformatur, quae alicubi abrumpitur; fractio autem, cuius numerator et denominator sunt numeri infinite magni, cuiusmodi dantur pro quantitatibus irrationalibus et transcendentibus, in fractionem vere continuam et in infinitum excurrentem transibit. Ad talem fractionem continuam inveniendam sufficiet denominatores tantum assignasse, cum numeratores omnes unitates esse ponamus. Hi vero invenientur inter numeratorem et denominatorem fractionis propositae eandem operationem instituendo, quae ad maximum earum communem divisorem investigandum instituі solet. Numerator scilicet per denominatorem dividatur et per residuum ipse denominator et ita porro semper per residuum praecedens divisor. Quoti vero ex hac continuata divisione orti erunt denominatores fractionis continuae quaesiti.

12. Sic si haec proposita sit fractio $\frac{A}{B}$ in fractionem continuam transmutanda, cuius omnes numeratores sint unitates, divido *A* per *B* sitque quotus *a* et residuum *C;* per hoc residuum *C* dividatur praecedens divisor *B* sitque quotus *b* residuumque *D*, per quod *C* dividatur, et ita porro, donec ad residuum $=0$ quotumque infinite magnum perveniatur. Operatio autem haec sequenti modo repraesentatur



Hac igitur operatione inveniuntur quoti *a, b, c, d, e* etc., quibus cognitis erit

$$\frac{A}{B}=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}$$

Si enim sit residuum $G=0$, erit

$$e=\frac{E}{F} \quad \text{atque} \quad \frac{1}{e}=\frac{F}{E}$$

hincque porro

$$d+\frac{1}{e}=d+\frac{E}{F}=\frac{D}{E} \quad \text{ac} \quad \frac{1}{d+\frac{1}{e}}=\frac{E}{D},$$

$$c+\cfrac{1}{d+\cfrac{1}{e}}=c+\frac{E}{D}=\frac{C}{D}.$$

Hocque modo usque ad initium ascendendo fractio continua reperietur $\frac{A}{B}$.

13. Si in fractione $\frac{A}{B}$ fuerit $A<B$, tum primus quotus *a* erit $=0$ residuumque primum $=A$, ita ut tum *B* per *A* dividi debeat. Hoc ergo casu erit

$$\frac{A}{B}=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}$$

Casu autem, quo $A<B$, unicus in fractione continua prodibit terminus, si ratio inter *A* et *B* fuerit multipla; duobus autem consistet fractio continua denominatoribus, si ratio $A:B$

pertineat ad genus rationum superparticularium ; plures vero aderunt denominatores, si ratio $A : B$ ad genus superpartientium referatur. Revera autem fractio continua in infinitum excurret, si ratio A ad B non fuerit ut numeri ad numerum, sed vel irrationalis vel transcendens. Ad huiusmodi autem expressiones in fractiones continuas transmutandas oportet, ut numeris rationalibus sint expositae, saltem vero proxime, quemadmodum hoc fieri solet per fractiones decimales. Tales igitur expressiones si habeantur, modo praescripto fractiones continuae formabuntur.

14. Cum autem fractio vel alia expressio in huiusmodi fractionem continuam fuerit conversa, tum eius expressionis valor proximus modo § 10 exposito poterit assignari. Uti si inventa fuerit haec expressio

$$\frac{A}{B}=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}$$

atque ex denominatoribus *a, b, c, d* etc. formetur sequens fractionum series

$$\begin{matrix} & a & b & c & d & e & \\ & \frac{1}{0}, & \frac{a}{1}, & \frac{ab+1}{b}, & \frac{abc+c+a}{bc+1}, & \frac{abcd+cd+ad+ab+1}{bcd+d+b} & \text{etc.,}\end{matrix}$$

hae fractiones proxime aequales erunt expressioni $\frac{A}{B}$ eoque minus distabunt, quo remotiores fuerint a prima. Ita autem quaelibet harum fractionum erit comparata, ut alia per numeros non maiores exhiberi nequeat, quae propius ad valorem $\frac{A}{B}$ accederet. Hoc itaque modo sequens problema commode solvetur:

Datam fractionem ex magnis numeris constantem in simpliciorem convertere, quae ad illam propius accedat, quam fieri potest numeris non maiorous.

Problema hoc WALLISIUS magno studio pertractavit, solutionem vero dedit vehementer operosam atque difficilem.

15. Ad methodum nostram ad solutionem huius problematis accommodandam sit proposita fractio

$$\frac{355}{113},$$

quae secundum Metium rationem peripheriae ad diametrum proxime exprimit; quaeramus igitur fractiones minoribus numeris constantes ab ista fractione tam parum discrepantes, quam fieri potest. Divido ergo 355 per 113 atque invenio

$$\frac{355}{113}=3+\frac{1}{7+\frac{1}{16}}$$

unde formo sequentes fractiones

$$\begin{array}{cccc} & 3 & 7 & 16 \\ \frac{1}{0}, & \frac{3}{1}, & \frac{22}{7}, & \frac{355}{113}; \end{array}$$

fractiones ergo $\frac{3}{1}$ et $\frac{22}{7}$ propius ad fractionem $\frac{355}{113}$ accedunt quam ullae aliae numeris non maioribus compositae; erit autem altera $\frac{22}{7}$ maior, altera $\frac{3}{1}$ minor quam proposita, uti iam supra in genere annotavimus. Has fractiones *principales* appellare liceat, nam praeter has assignari possunt aliae *minus principales* quaesito aeque satisfacientes; scilicet uti fractio $\frac{22}{7}$ ex praecedentibus cum indice 7 est formata, ita minus principales eodem modo formabuntur loco 7 minores nu_meros singulos substituendo.

16. Si autem ratio peripheriae ad diametrum exactior accipiatur divisioque continua, uti est praeceptum, instituatur, sequens quotorum series prodibit

3, 7, 15, 1, 292, 1 , 1 , 1 , 2, 1 , 3, 1 , 14 etc.,

ex quibus sequenti modo fractiones simpliciores eruentur

	3	7	15	1	292	1	
$\frac{1}{0}$,	$\frac{3}{1}$,	$\frac{22}{7}$,	$\frac{333}{106}$,	$\frac{355}{113}$,	$\frac{103993}{33102}$		*principales*
	$\frac{2}{1}$,	$\frac{19}{6}$,	$\frac{311}{99}$,		$\frac{103638}{32989}$		*minus principales*
	$\frac{1}{1}$,	$\frac{16}{5}$,	$\frac{289}{92}$,		$\frac{103283}{32876}$		
		$\frac{13}{4}$,	$\frac{267}{85}$,		$\frac{102928}{32763}$		
		$\frac{10}{3}$,			etc.		
		$\frac{7}{2}$,					
		$\frac{4}{1}$,					
		etc.					

Hoc igitur pacto duplices fractiones nacti sumus, quarum aliae nimis sunt magnae, aliae nimis parvae; nimis magnae scilicet sunt, quae sub indicibus 3, 15, 292 etc. continentur, reliquae nimis sunt parvae. Atque hinc facile integram tabulam Wallisianam condere licet, quae omnes complectitur rationes ad veram peripheriae ad diametrum rationem propius accedentes, quam fieri potest numeris non maioribus.

17. Hac etiam methodo definire licebit rationem constitutionis annorum bissextilium, quo annorum initia perpetuo in eandem tempestatem incidant. Pendet haec determinatio a quantitate anni tropici, quam iuxta accuratissimas observationes ponam

$$365^{d}\,5^{h}\,49'\,8''$$

Excessus ergo supra 365 dies erit $5^{h}\,49'\,8''$, qui si aequaretur quartae diei parti, tuto semper quartus quisque annus bissextilis constitueretur; sed cum iste excessus minor sit 6 horis, numerus annorum bissextilium minor debet accipi; quod cognoscetur ex ratione 24^{h} ad $5^{h}\,49'8''$ seu ex fractione

$$\frac{21600}{5237}$$

ex qua sequitur in intervallo 21600 annorum tantum 5237 annos bissextiles constitui oportere. Cum autem haec periodus nimis sit magna, minores obtinebimus periodos fractiones minoribus numeris constantes investigando, quae proxime fractioni $\frac{21600}{5237}$ sint aequales. In hunc finem sequentem divisionem instituo

```
5237 | 21600 | 4
     | 20948 |
         652 | 5237 | 8
             | 5216 |
                 21 | 652 | 31
                    | 651 |
                        1 | 21 | 21
```

Iam ex quotis inventis 4, 8, 31, 21, qui erunt denominatores fractionis continuae, sequentes formentur fractiones

$$\begin{array}{ccccc} & 4 & 8 & 31 & 21 \\ \frac{1}{0}, & \frac{4}{1}, & \frac{33}{8}, & \frac{1027}{249}, & \frac{21600}{5237}. \end{array}$$

Harum fractionum secunda, $\frac{4}{1}$, statim dat rationem calendarii Iuliani, quo quartus quisque annus ponitur bissextilis. Propius ergo scopus attingeretur, si annis 33 tantum 8 anni bissextiles collocarentur, ex fractione tertia. Cum autem expediat pro annorum periodo numerum pariter parem habere, sumamus fractiones minus principales quartae respondentes, quae habeant numeratores per 4 divisibiles; quae erunt

$$\frac{136}{33},\quad \frac{268}{65},\quad \frac{400}{97},\quad \frac{532}{129},\quad \frac{664}{161}\ \text{etc.},$$

quarum tertia $\frac{400}{97}$ ad computum calendarii est commodissima. Apparet autem ex ea intervallo annorum 400 tantum 97 annos bissextiles constitui debere seu tres annos hoc intervallo, qui in calendario Juliano bissextiles essent, in communes esse transmutandos, id quod etiam constitutio Gregoria praecipit. Ex quo intelligitur minore annorum intervallo accuratiorem correctionem adhiberi non passe. Accuratissime autem cum sole calendarium conciliabitur, si intervallo 21600 annorum denuo unus annus, qui secundum constitutionem Gregorianam bissextilis esse deberet, in communem transmutetur.

18. Quaeramus iam fractiones, quae ad $\sqrt{2}$ tam prope accedant, ut aliae minoribus numeris constantes propius accedere nequeant. Est vero

$$\sqrt{2}=1,41421356=\tfrac{141421356}{100000000},$$

quae fractio, si divisione continua iuxta modum praescriptum tractetur, dabit hos quotos

$$1,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2,\ 2 \text{ etc.},$$

ex quibus sequentes formabuntur fractiones quaesito satisfacientes tam principales quam minus principales

$$\begin{array}{cccccccccccccccc} 1, & & 2, & & 2, & & 2, & & 2, & & 2, & & 2, & & 2 & \text{etc.,} \\ \frac{1}{0}, & & \frac{1}{1}, & & \frac{3}{2}, & & \frac{7}{5}, & & \frac{17}{12}, & & \frac{41}{29}, & & \frac{99}{70}, & & \frac{239}{169} & \text{etc.} \\ & & & & \frac{2}{1}, & & \frac{4}{3}, & & \frac{10}{7}, & & \frac{24}{17}, & & \frac{58}{41}, & & \frac{140}{99} & \\ > & & < & & > & & < & & > & & < & & > & & < & \end{array}$$

quarum fractionum alternae signo > notatae maiores sunt quam $\sqrt{2}$, reliquae vero signum < habentes minores quam $\sqrt{2}$.

19. Notatu digna est haec proprietas ipsius $\sqrt{2}$, quod omnes quotos praeter primum habeat aequales binario, ita ut sit

$$\sqrt{2}=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\text{etc.}}}}}$$

Simili modo vero etiam, si $\sqrt{3}$ evolvatur, reperiuntur quoti

$$1,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1,\ 2,\ 1 \text{ etc.},$$

ita ut sit

$$\sqrt{3}=1+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{2+\text{etc.}}}}}}}$$

Quamvis enim non constet ex ipsa divisione, utrum quoti hac lege ulterius progrediantur, tamen id non solum probabile videtur, sed etiam sequenti modo demonstrari potest, quo valores huiusmodi fractionum continuarum, in quibus denominatores vel sunt omnes aequales vel alterni vel terni etc., a posteriori investigare docebimus.

19 [a]. Sit igitur proposita sequens fractio continua

$$a+\cfrac{1}{b+\cfrac{1}{b+\cfrac{1}{b+\cfrac{1}{b+\text{etc.}}}}}$$

quae ponatur $=x$; erit

$$x-a=\cfrac{1}{b+\cfrac{1}{b+\cfrac{1}{b+\cfrac{1}{b+\text{etc.}}}}}=\frac{1}{b+x-a};$$

hinc erit

$$x^2-2ax+bx+a^2-ab=1$$

atque

$$x=a-\tfrac{b}{2}+\sqrt{\left(1+\tfrac{bb}{4}\right)}.$$

Quare si fuerit $b=2$ et $a=1$, erit

$$x=1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\text{etc.}}}}}=\sqrt{2};$$

si ergo ponatur $b=2a$, erit

$$\sqrt{\left(a^2+1\right)}=a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\cfrac{1}{a+\text{etc.}}}}}$$

unde ex omnibus numeris, qui unitate quadratum excedunt, expedite per approximationem radix quadrata extrahi potest; uti posito $a=2$ sequentes fractiones ad $\sqrt{5}$ proxime inveniendam inservient:

2	4	4	4	4	4	4
$\frac{1}{0}$,	$\frac{2}{1}$,	$\frac{9}{4}$,	$\frac{38}{17}$,	$\frac{161}{72}$,	$\frac{682}{305}$,	$\frac{2889}{1292}$ etc.
	$\frac{1}{1}$,	$\frac{7}{3}$,	$\frac{29}{13}$,	$\frac{123}{55}$,	$\frac{521}{233}$,	$\frac{2207}{987}$
		$\frac{5}{2}$,	$\frac{20}{9}$,	$\frac{85}{38}$,	$\frac{360}{161}$,	$\frac{1525}{682}$
		$\frac{3}{1}$,	$\frac{11}{5}$,	$\frac{47}{21}$,	$\frac{199}{89}$	$\frac{843}{477}$

20. Sit nunc proposita sequens fractio continua

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{c+\text{etc.}}}}}}}$$

quae ponatur $=x$*;* atque valor ipsius *x* reperietur sequenti modo

$$x-a=\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{b+\cfrac{1}{c+\text{etc.}}}}}}}=\cfrac{1}{b+\cfrac{1}{c+x-a}}$$

hinc ergo erit $x-a=\frac{x+c-a}{bx+bc-ab+1}$ seu

$$bxx+bcx-2abx=abc-a^2b+c;$$

si ergo fuerit $c=2a$, erit

$$bxx=aab+2a \text{ atque } x=\sqrt{\left(a^2+\tfrac{2a}{b}\right)}.$$

Simili modo si ponatur

$$x=a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{etc.}}}}}}}$$

erit

$$x-a=\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+x-a}}}$$

unde sequitur

$$(bc+1)x^2+(bcd+b+d-c-2abc-2a)x-abcd+a^2bc-ab-ad$$
$$+aa-cd+ac-1=0.$$

Atque hoc modo omnes huiusmodi fractiones continuas, quarum denominatores vel omnes vel alterni vel terni vel quaterni etc. sunt inter se aequales, summare licet. Semper autem summa seu valor *x* est radix ex aequatione quadrata.

21. Antequam ad alias fractiones continuas, in quibus denominatores progressiones arithmeticas constituunt, summandas progrediamur, quantitates quasdam transcendantes evolvamus, quae in fractiones continuas conversae dent denominatores in progressione

arithmetica progredientes, quo ex his via evadat planior eiusmodi fractiones continuas summandi. Hoc igitur logarithmis aliisque expressionibus transcendentibus tentans deprehendi in eiusmodi fractiones continuas deduci, si numerus, cuius logarithmus hyperbolicus est unitas, eiusque potestates quaeque considerentur. Posito igitur hoc numero $=e$ erit

$$e=2,71828182845904,$$

qua expressione in fractionem continuam conversa erit

$$e=2+\cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{6+\cfrac{1}{1+\text{etc.}}}}}}}}}}$$

cuius denominatores terni constituunt progressionem arithmeticam 2, 4, 6, 8 etc., reliqui sunt unitates. Quae lex etsi ex sola observatione est deprehensa, tamen probabile videtur eam in infinitum valere, quod quidem infra certo confirmabitur. Simili modo si

$$\sqrt{e}=1,6487212707$$

in fractionem continuam convertatur, erit

$$\sqrt{e}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{9+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{13+\text{etc.}}}}}}}}}}}$$

cuius progressionis lex similis est praecedentis. Similiaque observare licet in aliis fractionibus continuis, in quas potestates ipsius *e* transmutantur.

22. Simili modo consideravi radicem cubicam ex numero *e*, cuius logarithmus hyperbolicus est 1, invenique

$$\frac{\sqrt[3]{e}-1}{2}=0,1978062125=\cfrac{1}{5+\cfrac{1}{18+\cfrac{1}{30+\cfrac{1}{42+\cfrac{1}{54+\text{etc.}}}}}}$$

in cuius fractionis continuae denominatoribus praeter primum progressio arithmetica observatur.

Simile accidit, si potestates exponentium integrorum ipsius *e* considerentur et in fractiones continuas transformentur. Sic considerans quadratum reperi

$$\frac{e^2-1}{2}=3,19452804946532=3+\cfrac{1}{5+\cfrac{1}{7+\cfrac{1}{9+\cfrac{1}{11+\cfrac{1}{13+\text{etc.}}}}}}$$

Deinde etiam ex ipso numero *e*, ex quo formata fractio continua interruptam habuit progressionem arithmeticam denominatorum, observavi paucis mutandis huiusmodi fractionem continuam ab interruptione liberam formari posse. Prodiit enim

$$\frac{e-1}{e+1}=2+\cfrac{1}{6+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{26+\text{etc.}}}}}}}$$

In qua regularis inest progressio arithmetica differentia 4 progrediens.

23. Cum igitur observassem tantam convenientiam inter fractiones continuas, in quibus denominatores modo interruptam modo non interruptam constituant progressionem arithmeticam, in eam incidi cogitationem, num forte fractio continua, in qua interrupta sit denominatorum progressio, in aliam non interruptam transformari possit. Consideravi igitur progressionem quamcunque *a, b, c, d, e* etc. interque binos contiguos ubique hos duos numeros *m, n* interpolavi, ut prodiret sequens fractio continua

$$a+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{b+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{c+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{d+\text{etc.}}}}}}}}}}$$

hancque inveni aequalem sequenti fractioni continuae, in qua denominatores sine interruptione progrediantur,

$$\frac{1}{mn+1}\left((mn+1)a+n+\cfrac{1}{(mn+1)b+m+n+\cfrac{1}{(mn+1)c+m+n+\cfrac{1}{(mn+1)d+m+n+\text{etc.}}}} \right).$$

Demonstratio huius convenientiae in hoc consistit, quod fractiones ordinariae, quibus ad valorem utriusque acceditur, inter se conveniant, prout tentanti patebit.

24. Si quantitates interpolatae *m, n* invertantur ordine, fractio continua posterior discrimen tantum in primo termino patietur; ex quo sequens satis elegans theorema conficitur, quo erit

$$a+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{b+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{c+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{d+\text{etc.}}}}}}}}}} - \left(a+\cfrac{1}{n+\cfrac{1}{m+\cfrac{1}{b+\cfrac{1}{n+\cfrac{1}{m+\cfrac{1}{c+\cfrac{1}{n+\cfrac{1}{m+\cfrac{1}{d+\text{etc.}}}}}}}}}} \right) = \frac{n-m}{mn+1}.$$

Quicunque ergo numeri loco *a, b, c, d* etc. substituantur, differentia inter duas fractiones continuas semper erit cognita atque constans, scilicet $=\frac{n-m}{mn+1}$.

25. Ex eadem inventa aequalitate inter fractiones continuas superiores, interruptam scilicet et non interruptam, sequens consequitur aequalitas dividendo unitatem per utramque et addendo utrinque eandem quantitatem *A*

$$A+\cfrac{1}{a+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{b+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{b+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{c+\text{etc.}}}}}}}}}} = A+\cfrac{mn+1}{(mn+1)a+n+\cfrac{1}{(mn+1)b+m+n+\cfrac{1}{(mn+1)c+m+n+\text{etc.}}}}$$

Huius ergo aequationis ope quamvis fractionem continuam interruptam habentem progressionem denominatorum binis quantitatibus *m* et *n* convertere licebit in aliam, in qua denominatores sine interruptione progrediantur. Si ergo, ut in fractionibus superioribus habuimus, ponatur $m = n = 1$, sequens prodibit aequatio

$$A+\cfrac{1}{a+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{b+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{b+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{c+\text{etc.}}}}}}}}}}} = A+\cfrac{2}{2a+1+\cfrac{1}{2b+2+\cfrac{1}{2c+2+\text{etc.}}}}$$

Cum igitur ex § 21 sit

$$\frac{1}{e-2} = 1+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{4+\text{etc.}}}}}$$

erit ponendo $A = 1,\ a = 2,\ b = 4$ etc., ut sequitur,

$$\frac{1}{e-2} = 1+\cfrac{2}{5+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\text{etc.}}}}}}$$

hincque erit unitatem per utrumque dividendo

$$e = 2+\cfrac{1}{1+\cfrac{2}{5+\cfrac{1}{10+\cfrac{1}{14+\cfrac{1}{18+\cfrac{1}{22+\cfrac{1}{26+\text{etc.}}}}}}}}$$

Simili modo ex eodem paragrapho reperietur

$$\sqrt{e}=1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{9+\text{etc.}}}}}}}}=1+\cfrac{2}{3+\cfrac{1}{12+\cfrac{1}{20+\cfrac{1}{28+\text{etc.}}}}}$$

Haeque fractiones continuae nunc inventae tantopere convergunt, ut facili negotio valores ipsorum e et $\sqrt{e}$ quantumvis prope reperiri queant.

26. Vicissim vero etiam hinc fractio continua, in qua denominatores ordine non interrupto progrediuntur, transmutari poterit in aliam, in qua denominatores interrupti sint duobus constantibus numeris m et n; ita inveni fore

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\cfrac{1}{e+\text{etc.}}}}}=a-n+\cfrac{mn+1}{m+\cfrac{1}{n+\cfrac{1}{\frac{b-m-n}{mn+1}+\cfrac{1}{\frac{c-m-n}{mn+1}+\cfrac{1}{m+\text{etc.}}}}}}$$

vel tollendo fractiones in his denominatoribus, si opus visum fuerit, erit

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{etc.}}}}=a-n+\cfrac{mn+1}{m+\cfrac{1}{n+\cfrac{mn+1}{m+\cfrac{1}{\frac{mn+1}{c-m-n}+\cfrac{mn+1}{m+\cfrac{1}{n+\text{etc.}}}}}}}$$

Si ergo ponatur $m=n=1$, habebitur

$$a+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{d+\text{etc.}}}}=a-1+\cfrac{2}{1+\cfrac{1}{1+\cfrac{1}{\frac{1}{2}b-1+\cfrac{1}{1+\cfrac{1}{\frac{1}{2}c-1}+\cfrac{1}{\frac{1}{2}d-1+\text{etc.}}}}}}$$

27. Quemadmodum hic fractiones continuas, quarum denominatores ita interrupto ordine progrediuntur, ut inter binos quosque contiguos duae interpositae sint quantitates constantes, consideravimus, ita eadem reductio extendi potest ad quatuor vel sex vel octo etc. quantitates constantes interpolatas. Numerus autem impar quantitatum constantium

interpolari nequit. Sic si inter quantitatum *a, b, c, d* etc. binas quasque contiguas interpolentur hae quatuor *m, n, p, q* ponaturque brevitatis gratia

$$mnpq+mn+mq+pq+1=P$$

et

$$mnp+npq+m+n+p+q=Q,$$

erit

$$Pb+Q+$$

$$a+\cfrac{1}{m+\cfrac{1}{n+\cfrac{1}{p+\cfrac{1}{q+\cfrac{1}{b+\cfrac{1}{c+\cfrac{1}{m+\cfrac{1}{m+\text{etc.}}}}}}}}}=\frac{1}{P}\left(Pa+npq+n+q+\cfrac{1}{Pb+Q+\cfrac{1}{Pc+Q+\cfrac{1}{Pd+Q+\text{etc.}}}}\right)$$

Atque si fuerit $m == p = q = 1$, habebitur

$$a+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{b+\cfrac{1}{1+\cfrac{1}{1+\text{etc.}}}}}}}}=\frac{1}{5}\left(5a+3+\cfrac{1}{5b+6+\cfrac{1}{5b+6+\cfrac{1}{5d+6+\text{etc.}}}}\right)$$

Ex quibus nova fractionum continuarum conversio nascitur.

28. Cum autem in praecedentibus, ubi numerum *e*, cuius logarithmus est $=1$, eiusque potestates in fractiones continuas converti, progressionem arithmeticam denominatorum tantum observaverim neque praeter probabilitatem de huius progressionis continuatione in infinitum quicquam affirmare valuerim, in id potissimum incubui, ut in huius progressionis necessitatem inquirerem eamque firmiter demonstrarem. Hocque etiam feliciter sum consecutus ex peculiari modo, quo integrationem huius aequationis

$$ady+y^2dx=x^{\frac{-4n}{2n+1}}dx$$

reduxi ad integrationem huius

$$adq+q^2dp=dp.$$

Posito enim

$$p=(2n+1)x^{\frac{1}{2n+1}}$$

inveni esse

$$q=\frac{a}{p}+\cfrac{1}{\frac{3a}{p}+\cfrac{1}{\frac{5a}{p}+\cfrac{1}{\frac{7a}{p}+\vdots+\cfrac{1}{\frac{(2n-1)a}{p}+\cfrac{1}{x^{\frac{2n}{2n+1}}y}}}}}$$

Unde, cum *q* per *p* dari queat sitque $p=(2n+1)x^{\frac{1}{2n+1}}$, formari potest aequatio finita inter *x* et *y*, quae integralis erit aequationis $ady+y^2dx=x^{\frac{-4n}{2n+1}}dx$, quoties *n* est numerus integer affirmativus.

29. Si ergo *n* ponatur numerus infinitus, expressio inventa erit fractio continua in infinitum excurrens, cuius denominatores constituent progressionem arithmeticam. Quamobrem habebitur sequens aequatio

$$q=\frac{a}{p}+\cfrac{1}{\frac{3a}{p}+\cfrac{1}{\frac{5a}{p}+\cfrac{1}{\frac{7a}{p}+\cfrac{1}{\frac{9a}{p}+\text{etc.}}}}}$$

atque *q* seu valor huius fractionis continuae ex ista aequatione

$$adq+q^2dq=dp$$

definietur. Erit vero

$$\frac{adq}{1-qq}=dp$$

quae constans ex eo debet determinari, quod posito $p=0$ fiat $q=\infty$. Quamobrem erit

$$\frac{a}{2}l\frac{q+1}{q-1}q=p \quad\text{atque}\quad \frac{q+1}{q-1}=e^{\frac{2p}{a}}$$

unde fiat

$$q=\frac{e^{\frac{2p}{a}}+1}{e^{\frac{2p}{a}}-1},$$

qui est valor fractionis continuae inventae. Deinde vero cum sit

$$e^{\frac{2p}{a}} = 1 + \frac{2}{q-1},$$

habebitur

$$e^{\frac{2p}{a}} = 1 + \cfrac{2}{\frac{a-p}{p} + \cfrac{1}{\frac{3a}{p} + \cfrac{1}{\frac{5a}{p} + \cfrac{1}{\frac{7a}{p} + \text{etc.}}}}}$$

30. Si ponatur $\frac{a}{2p} = s$ seu $a = 2ps$, erit

$$e^{\frac{1}{s}} = 1 + \cfrac{2}{2s - 1 + \cfrac{1}{\frac{1}{6s} + \cfrac{1}{\frac{1}{10s} + \cfrac{1}{\frac{1}{14s} + \text{etc.}}}}}$$

Atque ex priore inventa aequatione erit

$$\frac{e^{\frac{1}{s}}+1}{e^{\frac{1}{s}}-1} = 2s + \cfrac{1}{6s + \cfrac{1}{10s + \cfrac{1}{14s + \cfrac{1}{18s + \text{etc.}}}}}$$

Si denominatores huius interpolentur binis unitatibus, habebitur

$$\frac{e^{\frac{1}{s}}+1}{e^{\frac{1}{s}}-1} = 2s - 1 + \cfrac{2}{1 + \cfrac{1}{3s - 1 + \cfrac{1}{1 + \cfrac{1}{1 + \cfrac{1}{+ \cfrac{1}{5s - 1 + \cfrac{1}{1 + \text{etc.}}}}}}}}$$

Ex qua oritur sequens fractio continua

$$e^{\frac{1}{s}}=1+\cfrac{2}{s-1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{3s-1+\cfrac{1}{1+\cfrac{1}{1+\cfrac{1}{5s-1+\cfrac{1}{1+\cfrac{1}{1+\text{etc.}}}}}}}}}}$$

Ex his vero formulis fluunt omnes supra inventae, quibus potestates quasdam ipsius *e* per fractiones continuas expressimus, ex quo necessitas progressionis ante tantum observatae intelligitur.

31. Iam ergo nacti sumus fractionem continuam, cuius denominatores progressionem arithmeticam constituunt cuiusque valorem exhibere licuit. Cum autem haec progressio sit species tantum arithmeticae, generalem contemplatus sum progressionem arithmeticam atque fractionem continuam, cuius denominatores eam progressionem constituant, sequenti modo ad summam revocavi. Sit scilicet sequens fractio continua, cuius valorem, quem quaero, pono $=s$, ita ut sit

$$s=a+\cfrac{1}{(1+n)a+\cfrac{1}{(1+2n)a+\cfrac{1}{(1+3n)a+\cfrac{1}{(1+4n)a+\text{etc.}}}}}$$

ex qua, quo valorem ipsius *s* eruam, ab approximatione ad eum ordior. Erit itaque per methodum supra traditam

$$\begin{array}{ccccc} a & (1+n)a & (1+2n)a & (1+3n)a & \\ \frac{1}{0}, & \frac{a}{1}, & \frac{(1+n)a^2+1}{(1+n)a}, & \frac{(1+n)(1+2n)a^3+(2+2n)a}{(1+n)(1+2n)a^2+1} & \text{etc.,} \end{array}$$

quae fractiones continuo magis ad valorem verum ipsius *s* accedunt; atque fractio infinitesima verum ipsius *s* valorem dabit.

32. Si hae fractiones ulterius continuentur, facile observabitur lex, qua formatae sunt, ex eaque concludetur fractionem infinitesimam post numeratoris et denominatoris divisionem per primum denominatoris terminum fore

$$\frac{a+\frac{1}{1\cdot na}+\frac{1}{1\cdot2\cdot1(1+n)n^3a^3}+\frac{1}{1\cdot2\cdot3\cdot1(1+n)(1+2n)n^5a^5}+\text{etc.}}{1+\frac{1}{1(1+n)na^2}+\frac{1}{1\cdot2(1+n)(1+2n)n^3a^4}+\frac{1}{1\cdot2\cdot3(1+n)(1+2n)(1+3n)n^5a^6}+\text{etc.}},$$

cui adeo s aequatur. Posito ergo $a=\frac{1}{\sqrt{nz}}$ erit

$$s=\frac{1}{\sqrt{nz}}\cdot\frac{1+\frac{z}{1\cdot 1}+\frac{z^2}{1\cdot 2\cdot 1(1+n)}+\frac{z^3}{1\cdot 2\cdot 3\cdot 1(1+n)(1+2n)}+\text{etc.}}{1+\frac{z}{1(1+n)}+\frac{z^2}{1\cdot 2(1+n)(1+2n)}+\frac{z^3}{1\cdot 2\cdot 3(1+n)(1+2n)(1+3n)}+\text{etc.}}$$

qui valor quo obtineatur, ponatur

$$t=1+\frac{z}{1\cdot 1}+\frac{z^2}{1\cdot 2\cdot 1(1+n)}+\frac{z^3}{1\cdot 2\cdot 3\cdot 1(1+n)(1+2n)}+\text{etc.}$$

et

$$u=1+\frac{z}{1(1+n)}+\frac{z^2}{1\cdot 2(1+n)(1+2n)}+\frac{z^3}{1\cdot 2\cdot 3(1+n)(1+2n)(1+3n)}+\text{etc.},$$

ita ut futurum sit

$$s=\frac{t}{u\sqrt{nz}}.$$

Ex inspectione autem harum duarum serierum intelligitur fore

$$dt=udz;$$

atque simili modo deprehendetur esse

$$udz+nzdu=tdz.$$

Ponatur $t=vu$, quo sit

$$s=\frac{v}{\sqrt{nz}}.$$

erit

$$vdu+udv=udz$$

atque

$$udz+nzdu=uvdz;$$

ex quibus sequitur

$$\frac{du}{u}=\frac{dz-dv}{v}=\frac{vdz-dz}{nz}.$$

hincque sequens aequatio inter z et v tantum consistens

$$nzdv-vdz+v^2dz=n-zdz,$$

quae substituto

$$v=z^nq \text{ et } z=r^n.$$

abibit in hanc

$$dq=q^2dr=nr^{n-2}dr.$$

Ex qua aequatione si q determinetur per r ponaturque

$$r = n^{-\frac{1}{n}} a^{-\frac{2}{n}},$$

erit valor quaesitus

$$s = arq.$$

33. Assignatio ergo valoris fractionis continuae propositae, quem posui s existante

$$s = a + \cfrac{1}{(1+n)a + \cfrac{1}{(1+2n)a + \cfrac{1}{(1+3n)a + \cfrac{1}{(1+4n)a + \text{etc.}}}}}$$

perducta est ad resolutionem huius aequationis

$$dq + q^2 dr = nr^{n-2} dr,$$

ita autem huius aequationis integrale accipi debet, ut facto $a = \infty$ fiat $s = \infty$ vel posito $a = 0$ fiat $s = 1$. Unde sequens pro introducenda constante in integrando regula nascitur, ut casu, quo non est $n > 2$, fiat $q = \infty$ posito $r = 0$. Ponimus autem n esse numerum affirmativum, quo fractio continua oriatur, qualem hactenus consideravimus, denominatores affirmativos habentem.

34. Constat autem aequationem inventam

$$dq + qqdr = nr^{n-2} dr$$

congruere cum aequatione olim a Com. Riccati proposita iisque propterea tantum casibus esse integrabilem, quibus n est numerus huius formae $\frac{2}{2m+1}$ denotane m integrum eumque affirmativum, quo pro n obtineamus numeros affirmativos. Ob hos igitur casus sequentis fractionis continuae

$$a + \cfrac{1}{\frac{(2m+3)a}{2m+1} + \cfrac{1}{\frac{(2m+5)a}{2m+1} + \cfrac{1}{\frac{(2m+7)a}{2m+1} + \text{etc.}}}}$$

valor semper per expressionem finitam exhiberi poterit. Quod quidem per se facile constat; nam facto $m = 0$ habemus hanc fractionem continuam

$$a + \cfrac{1}{3a + \cfrac{1}{5a + \cfrac{1}{7a + \text{etc.}}}}$$

cuius valorem iam supra invenimus. Ad hanc vero reduci potest illa generalis; posito enim $a=(2m+1)b$ habebitur

$$(2m+1)b+\cfrac{1}{(2m+3)b+\cfrac{1}{(2m+5)b+\text{etc.}}}$$

quae in ista iam cognita toties continetur, quoties *m* fuerit numerus integer affirmativus.

35. Apparet igitur per hanc ipsam fractionum continuarum resolutionem integrationem aequationis

$$dq+q^2dr=nr^{n-2}dr$$

deduci ad integrationem huius aequationis

$$dq+q^2dr=2dr,$$

siquidem *n* fuerit $=\frac{2}{2m+1}$ denotante *m* numerum integrum affirmativum. Quam ipsam reductionem iam supra § 28 eodem modo, quo ex hoc fonte perfici potest, exposui. Quo autem intelligatur, quomodo hac ratione verus huiusmodi fractionum continuarum valor reperiatur, considerabo casum $n=2$ seu $m=0$, quo orietur

$$s=a+\cfrac{1}{3a+\cfrac{1}{5a+\cfrac{1}{7a+\text{etc.}}}}$$

Reperietur vero *s* ex hac aequatione

$$dq+q^2dr=2dr$$

quae debito modo integrata dat

$$r=\frac{1}{2\sqrt{2}}l\frac{q+\sqrt{2}}{q-\sqrt{2}},$$

ex qua prodibit

$$q=\frac{\left(e^{2r\sqrt{2}}+1\right)\sqrt{2}}{e^{2r\sqrt{2}}-1}.$$

Est vero

$$r=\frac{1}{a\sqrt{2}} \quad \text{atque} \quad s=arq=\frac{q}{\sqrt{2}},$$

unde proveniet valor ipsius

$$s=\frac{e^{\frac{2}{a}}+1}{e^{\frac{2}{a}}-1},$$

prorsus ut iam supra invenimus (§ 29).