Another dissertation arising from the sums of series of powers of the reciprocals of natural numbers, in which the same sums may be derived from a completely different source.

L. Euler E61 : Miscellanea Berolinonsia 7, 1743, p. 172-192

1. Not too many years ago, with the aid of the quadrature of the circle, I presented the sums of series contained in this general form

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{ etc., extended indefinitely,}$$

if *n* were an even positive number, and likewise also the sums of these series, if *n* were an odd number,

$$1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\text{ etc., extended indefinitely,}$$

and I have shown the sum always to be expressed by the same noteworthy perimeter of the circle, as the exponent *n* may indicate, this argument has been appreciated so much by the sharpest geometers [such as expressed in letters to Euler from Dan. & Joh. Bernoulli, & Goldbach at this time], as not only may they approve that especially, but truly also they may expend their enthusiasm and work towards sums requiring to be elicited by these familiar methods. I too have been occupied from that time in finding other ways by which the same may be deduced, in order that not only may I confirm the truth found more, but also so that I may extend the bounds of the analysis further in series of this kind, requiring to be treated.

2. The method, which has guided me to the summation of these same series, certainly was new and plainly not used in the customary manner ; indeed it would depend on the resolution of infinite equations, of which all the roots, of which the number was infinite, would be required to be known. For indeed I have considered this equation extending indefinitely:

$$x=s-\frac{s^3}{6}+\frac{s^5}{120}-\frac{s^7}{5040}+\frac{s^9}{362889}-\text{ etc.}$$

which maintains the relation between the arc of the circle s and its sine x, with the whole sine put $=1$. But since innumerable arcs both positive and negative may correspond to the same sine x, I have followed the previous manner with innumerable roots of this equation; and since the coefficients of each equation depends on the roots, from the comparison of these same coefficients with the roots of the equation, to arrive at the sums of the series mentioned before.

3. Indeed I have at once readily observed, this same method to be secure and cannot lead to the truth, unless it may be agreed that in an equation of infinite order I have included other roots as well besides these, which the nature of the sines has made available to me. Indeed though I might understand that other real roots, besides those designated, not to be contained in that equation, yet deservedly it will be agreed to doubt, whether or not imaginary roots may become involved; since if that may happen, the whole sums, which thence I have elicited, will be unable to agree with the truth. Again I have been confirmed in having this doubt, since in a similar manner with an elliptical arc, I have expressed its sine or corresponding applied line by a series; yet indeed equally innumerable arcs of the ellipse will be presented, which refer to the same sine, and from these no sums of the series will be allowed to be deduced, which may be agreed to be true ; the account of this inconvenience established, was far from removing the doubt that many and perhaps an infinite number of imaginary roots may enter into that equation formed from the ellipse.

4. Therefore since at that time I would have had no demonstration, from which it could be agreed to be certain to me that the equation between the arc of the circle s and its sine x to be entirely free from imaginary roots, I had began to examine the sums of the series found for the true balance of the truth and indeed in the first place, I had discovered at once this method to provide the same sum of the series

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\tfrac{1}{13}-\text{ etc.},$$

as at one time Leibnitz now had assigned, which may be indicated to be agreed on well enough, if which equation may contain imaginary roots, then the sum of these roots necessarily will be $=0$. Thereupon I have considered series of higher powers, so that I may compare with the sums found by this method, which I had elicited by approximation some time before, in which matter I have reached agreement. And on account of these reasons I was reassured that equation to be completely freed from imaginary roots, which had led me to be in an iniquitous state regarding all these sums; and thus I have not doubted these same sums as it were to be the most definitely true.

5. But another more analytical method had convinced me completely in this opinion, with the aid of which henceforth through integrations alone, I had elicited the same sum of this series

$$1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\tfrac{1}{25}+\text{ etc.},$$

and set out in a paper by Nicolas Bernoulli , by far the sharpest Geometer, in almost the same way, presented in Book X. Comment. Acad. Petropol. But, nevertheless, it will be observed, the analytical calculus to be able to treat all the same sums requiring to be elicited in this manner, yet neither I nor anyone else, have been able to provide an equivalent calculation pertaining to higher powers. Which thus almost impelled me, to the extent that I might believe there might be another way not apparent besides for the resolution of infinite equations, which might provide the sums of all the powers together.

6. The letters received by me recently from the most celebrated Daniel Bernoulli have renewed this almost completely forgotten concern, in which he raises the same objections as myself concerning the reasons about my method and likewise he has indicated the celebrated Cramer to be predisposed to the same doubts, whether or not my method may be able to be approved. Therefore these friendly admonitions thus have brought that back to me, so that I might reconsider the whole matter both for the worth of the method requiring to be demonstrated as well as I may work on another way of summing the same series requiring to be sought. Therefore for each of the nominated fields granted in this dissertation I shall resolve each situation fully. Of course initially I will show that no imaginary roots to be contained in the infinite equation mentioned above, and thence with the truth of the sums deduced to be no longer doubted. Truly in the second place a new method, and that not only especially different from the first, but also I may propose revealing a field for many others requiring to be shown, which may perform the whole operation by the rules of integration alone.

7. I have obtained the first demonstration promised from the resolution of this binomial

$$a^n + b^n$$

into its real factors. Indeed each factor of this binomial is contained in this form

$$aa - 2ab\cos \mathrm{A}\cdot\tfrac{(2k-1)\pi}{n} + bb$$

and all the factors will be obtained, if in place of $2k-1$ all the successive odd numbers smaller than the exponent n may be substituted ; and if n were an odd number, then besides these trinomial factors the simple factor $a+b$ must be added . So that if the remainder may be had

$$a^n - b^n$$

$a-b$ is its simple first factor, the remaining trinomial real factors will be contained in this form

$$aa - 2ab\cos \mathrm{A}\,\tfrac{2k\pi}{n} + bb$$

and all the factors of this kind will be obtained, if in place of $2k$ successively all the even numbers (zero excepted) may be written smaller than the exponent n ; and if n itself were an even number, the simple factor $a+b$ besides must be added. Therefore in this way all the formulas

$$a^n \pm b^n$$

with real factors will be shown entirely, the product of all of which shall return this same formula. Finally here it is required to be noted π to denote the semi circumference of the circle, of which the radius $=1$, or π to be the angle equal to two right angles.

8. Hence now from before I can assign all the roots or factors of this infinite expression

$$s-\tfrac{s^3}{1\cdot 2\cdot 3}+\tfrac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\tfrac{s^7}{1\cdot 2\cdot 3\cdots 7}+\tfrac{s^9}{1\cdot 2\cdot 3\cdots 9}-\text{etc.}$$

Indeed this expression is equivalent to :

$$\frac{e^{s\sqrt{-1}}-e^{-s\sqrt{-1}}}{2\sqrt{-1}}$$

with e denoting the number, of which the logarithm is $=1$, and since there shall be

$$e^z=\left(1+\tfrac{z}{n}\right)^n$$

with the infinite number n, the infinite expression proposed is reduced to this:

$$\frac{\left(1+\frac{s\sqrt{-1}}{n}\right)^n-\left(1-\frac{s\sqrt{-1}}{n}\right)^n}{2\sqrt{-1}},$$

of which the first factor simply is s, which indeed an inspection of the same series shows. For the remaining factors requiring to be elicited, I compare this expression with the formula a^n-b^n ; there will become

$$a=1+\tfrac{s\sqrt{-1}}{n}\quad\text{and}\quad b=1-\tfrac{s\sqrt{-1}}{n}$$

and hence

$$aa+bb=2-\tfrac{2ss}{nn}\quad\text{and}\quad 2ab=2+\tfrac{2ss}{nn}.$$

Therefore any factor will be contained in this form

$$\frac{\left(1+\frac{s\sqrt{-1}}{n}\right)^{2k}-\left(1-\frac{s\sqrt{-1}}{n}\right)^{2k}}{2\sqrt{-1}}=2-\tfrac{2ss}{nn}-2\left(1+\tfrac{ss}{nn}\right)\cos\text{A}\,\tfrac{2k\pi}{n}$$

and hence all the factors will arise, if all the successive even numbers may be substituted indefinitely for $2k$, therefore since n may denote an infinite number.

9. Moreover since n shall be an infinite number, the arc $\frac{2k\pi}{n}$ will be infinitely small, provided $2k$ also may become an infinite number, yet smaller than n. Therefore there will be

$$\cos A\,\frac{2k\pi}{n}=1-\frac{2kk\pi\pi}{nn},$$

from which the general factor will be changed into this form

$$-\frac{4ss\pi}{nn}+\frac{4kk\pi\pi}{nn},$$

from which, with the known term reduced to unity, the factor arises

$$1-\frac{ss}{kk\pi\pi},$$

which, with all the successive numbers 1, 2, 3 etc. substituted in place of k indefinitely, all the factors are produced. So that thus if k may become an infinite number, so that $2k$ to n may become a finite rational number, then on account of

$$\cos A\,\frac{2k\pi}{n}<1$$

the terms $\frac{ss}{nn}$ less than unity present will vanish and the factor $1-\cos A\,\frac{2k\pi}{n}$ will become constant and thus does not enter into the calculation, because the arc s is not present there.

10. Therefore with this agreed we have obtained all the factors of the formula proposed:

$$s-\frac{s^3}{1\cdot 2\cdot 3}+\frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{s^7}{1\cdot 2\cdot 3\cdots 7}+\text{ etc.},$$

which thus will be exactly equal to the product from these agreeing infinite factors

$$s\left(1-\frac{ss}{\pi\pi}\right)\left(1-\frac{ss}{4\pi\pi}\right)\left(1-\frac{ss}{9\pi\pi}\right)\left(1-\frac{ss}{16\pi\pi}\right)\text{ etc.},$$

from which with the coefficients of the terms of the series prepared, the sums of the series follow:

$$1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\frac{1}{5^m}+\frac{1}{6^m}+\text{etc.},$$

if m may denote some even number, therefore neither can the truth of these be doubted.

11. In a similar manner if we may consider this series

$$s-\frac{ss}{1\cdot 2}+\frac{s^4}{1\cdot 2\cdot 3\cdot 4}-\frac{s^6}{1\cdot 2\cdot 3\cdots 6}+\frac{s^8}{1\cdot 2\cdot 3\cdots 8}-\text{ etc.},$$

this may be reduced to this form:

$$\frac{\left(1+\frac{s\sqrt{-1}}{n}\right)^n+\left(1-\frac{s\sqrt{-1}}{n}\right)^n}{2}$$

with n denoting an infinite number. Therefore the divisors of the binomial

$$\left(1+\frac{s\sqrt{-1}}{n}\right)^n+\left(1-\frac{s\sqrt{-1}}{n}\right)^n$$

likewise will be divisors of the proposed series, and indeed will be of all the divisors. This form prepared with a^n+b^n will become

$$a=1+\frac{s\sqrt{-1}}{n},\ \ b=1-\frac{s\sqrt{-1}}{n},\ \ aa+bb=2-\frac{2ss}{nn}\text{ and }2ab=2+\frac{2ss}{nn};$$

therefore a divisor of each proposed formula will be contained in this expression

$$2\left(1-\frac{ss}{nn}\right)-2\left(1+\frac{ss}{nn}\right)\cos\mathrm{A}\frac{(2k-1)\pi}{n}$$

or this,

$$2\left(1-\cos\mathrm{A}\frac{(2k-1)\pi}{n}\right)-\frac{2ss}{nn}\left(1+\cos\mathrm{A}\frac{(2k-1)\pi}{n}\right).$$

But since only one unknown may be considered in the divisor, any divisor will be

$$1-\frac{ss\left(1+\cos\mathrm{A}\frac{(2k-1)\pi}{n}\right)}{nn\left(1-\cos\mathrm{A}\frac{(2k-1)\pi}{n}\right)}$$

with the known term equal to one, since in the series the first term itself is $=1$.

12. On account of the infinite number n there will become

$$1+\cos\mathrm{A}\frac{(2k-1)\pi}{n}=2\ \text{ and }\ 1-\cos\mathrm{A}\frac{(2k-1)\pi}{n}=\frac{(2k-1)^2\pi\pi}{2nn},$$

from which each individual divisor will be

$$1-\frac{4ss}{(2k-1)^2\pi\pi};$$

and if in place of $2k-1$ all the odd numbers may be substituted indefinitely, all the proposed divisors of the series will arise:

$$1-\frac{ss}{1\cdot 2}+\frac{s^4}{1\cdot 2\cdot 3\cdot 4}-\frac{s^6}{1\cdot 2\cdot 3\cdots 6}+\text{ etc.},$$

which therefore will be produced from which the number will be produced from these infinite factors

$$\left(1-\frac{4ss}{\pi\pi}\right)\left(1-\frac{4ss}{9\pi\pi}\right)\left(1-\frac{4ss}{25\pi\pi}\right)\left(1-\frac{4ss}{36\pi\pi}\right)\text{ etc.},$$

from the comparison of which with that series itself all the series of the powers are summed as before. And thus those infinite equations has been demonstrated, which I have treated at that time, no other roots to be had besides these, which I have pursued in the latter from the nature of sines and cosines.

13. With the method demonstrated, with the aid of which I have assigned previously the sums of series of this kind, I progress with pleasure to another quite different method requiring to be explained, which supplies the sums of series wonderfully well only with the help of the principles of the calculus without any other circumstances. Moreover this method depends on two theorems, the demonstration of which I have given in the above dissertation: *concerning the invention of integration, if a definite value may be attributed to the variable quantity after the integration* (E60), so that I may disclose that without demonstration.

Thus in the first place there is had:

" The differential formulas

$$\frac{x^{p-1}+x^{q-p-1}}{1+x^q}dx,$$

with the integral taken thus, so that it may vanish on putting $x=0$, if there may be put $x=1$ after the integration , will give this value

$$\frac{\pi}{q\sin \mathrm{A}\frac{p\pi}{q}},$$

with π denoting the semi circumference of the circle, of which the radius $=1$, in which circle, likewise I put the sine of the arc taken $\frac{p\pi}{q}$.

The other theorem almost the same to this thus is had :
The integral of this differential formula :

$$\frac{x^{p-1}-x^{q-p-1}}{1-x^q}dx$$

thus taken, so that it may vanish on putting $x=0$, if there may be put $x=1$ into that after the integration, will give this value:

$$\frac{\pi\cos \mathrm{A}\frac{p\pi}{q}}{q\sin \mathrm{A}\frac{p\pi}{q}} \text{ or } \frac{\pi}{q\,\mathrm{tang}\,\mathrm{A}\frac{p\pi}{q}}. \text{ "}$$

The demonstrations of these theorems plainly proceed in order ; indeed the first following the accustomed rules of these integral formulas I have investigated generally and with these found I have put unity in place of the variable x. With which done thus to be extended to a finite series of sines, which, since the arcs were appearing in an arithmetical progression, the summation was allowed and it provided these very expressions.

14. Now we may assume the first integral formula

$$\int \frac{x^{p-1}+x^{q-p-1}}{1+x^q}dx$$

which resolved into a series will give the twofold geometric progression

$$\int dx\left(x^{p-1}-x^{q+p-1}+x^{2q+p-1}-x^{3q+p-1}+\text{etc.}\right)$$
$$+\int dx\left(x^{q-p-1}-x^{2q-p-1}+x^{3q-p-1}-x^{4q-p-1}+\text{etc.}\right).$$

Therefore with the integral of this thus taken, so that it may vanish on putting $x=0$, thus will be expressed by the series

$$\frac{x^p}{p}+\frac{x^{q-p}}{q-p}-\frac{x^{q+p}}{q+p}-\frac{x^{2q-p}}{2q-p}+\frac{x^{2q+p}}{2q+p}+\frac{x^{3q-p}}{3q-p}-\text{etc.}$$

So that if now we may put $x=1$, by the theorem of this first series

$$\frac{1}{p}+\frac{1}{q-p}-\frac{1}{q+p}-\frac{1}{2q-p}+\frac{1}{2q+p}+\frac{1}{3q-p}-\frac{1}{3q+p}-\frac{1}{4q-p}+\text{etc.}$$

with the sum

$$=\frac{\pi}{q\,\mathrm{tang}\,\mathrm{A}\frac{p\pi}{q}}.$$

15. In a similar manner the other integral formula

$$\int \frac{x^{p-1}-x^{q-p-1}}{1-x^{q}}dx$$

will give with the integrated series

$$\frac{x^{p}}{p}-\frac{x^{q-p}}{q-p}+\frac{x^{q+p}}{q+p}-\frac{x^{2q-p}}{2q-p}+\frac{x^{2q+p}}{2q+p}-\frac{x^{3q-p}}{3q-p}+\text{etc.}$$

On account of which by the other theorem, if we may put $x=1$, the sum of this series

$$\frac{1}{p}-\frac{1}{q-p}+\frac{1}{q+p}-\frac{1}{2q-p}+\frac{1}{2q+p}-\frac{1}{3q-p}+\frac{1}{3q+p}-\text{etc.}$$

will become

$$=\frac{\pi\cos \mathrm{A}\frac{p\pi}{q}}{q\sin \mathrm{A}\frac{p\pi}{q}},$$

provided p and q will have been positive numbers and $q > p$, that which will be required to be assumed in the following; indeed otherwise the integral taken in this way will not vanish on putting $x=0$.

16. Let $\frac{p}{q}=s$; and with the series found multiplied by q we will have these two series reduced to a finite sum:

$$\frac{\pi}{\sin \mathrm{A}\,s\pi}=\frac{1}{s}+\frac{1}{1-s}-\frac{1}{1+s}-\frac{1}{2-s}+\frac{1}{2+s}+\frac{1}{3-s}-\frac{1}{3+s}-\text{etc.},$$

$$\frac{\pi\cos \mathrm{A}\,s\pi}{\sin \mathrm{A}\,s\pi}=\frac{1}{s}-\frac{1}{1-s}+\frac{1}{1+s}-\frac{1}{2-s}+\frac{1}{2+s}-\frac{1}{3-s}+\frac{1}{3+s}-\text{etc.},$$

and the true sums of these series, whatever number may be indicated by s, either to be rational or irrational, and thus the law of continuity is no longer infringed as before, where in place of p and q whole numbers will be required to be taken. Also, so that these sums may not depart from the truth, even if numbers greater than unity may be put in place for s. Indeed if there shall become $s=1$ or any whole number, then the series will become infinite on account of one term departing to infinity, likewise truly the sums shown on account of $\sin \mathrm{A}s\pi=0$ will increase indefinitely. Hence these sums extend so widely, that that they require no restriction arising from that.

17. Now from these general series the series for the quadrature of the circle both of Leibniz as well as Gregory and innumerable others are deduced, the precepts of which it is pleasing to show here.

Let $q=2$ and $p=1$; there will become

$$\sin \mathrm{A}\tfrac{\pi}{2}=1 \text{ and } \cos \mathrm{A}\tfrac{\pi}{2}=0$$

and hence the following series arise

$$\tfrac{\pi}{2}=1+1-\tfrac{1}{3}-\tfrac{1}{3}+\tfrac{1}{5}+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{7}+\text{etc.}$$

or

$$\tfrac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\text{ etc.}$$

and

$$\tfrac{0\pi}{4}=1-1+\tfrac{1}{3}-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{7}+\text{ etc.}$$

of which that is the Leibniz series, truly this is produced at once.
Let there be $q=3$ and $p=1$; there will become

$$\sin.\mathrm{A}\tfrac{\pi}{3}=\tfrac{\sqrt{3}}{2} \quad\text{and}\quad \cos \mathrm{A}\tfrac{\pi}{3}=\tfrac{1}{2},$$

from which the following series arise

$$\tfrac{2\pi}{3\sqrt{3}}=1+\tfrac{1}{2}-\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}+\tfrac{1}{8}-\tfrac{1}{10}-\tfrac{1}{11}+\tfrac{1}{13}+\text{etc.},$$
$$\tfrac{\pi}{3\sqrt{3}}=1-\tfrac{1}{2}+\tfrac{1}{4}-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{8}+\tfrac{1}{10}-\tfrac{1}{11}+\tfrac{1}{13}-\text{etc.}$$

Let $q=4,\ p=1$: there will become

$$\sin.\mathrm{A}\tfrac{\pi}{4}=\tfrac{1}{\sqrt{2}} \quad\text{and}\quad \cos \mathrm{A}\tfrac{\pi}{4}=\tfrac{1}{\sqrt{2}};$$

and hence the following series arise:

$$\tfrac{\pi}{2\sqrt{2}}=1+\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{11}-\tfrac{1}{13}-\text{etc.},$$
$$\tfrac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\tfrac{1}{13}-\text{etc.}$$

Let $q=6,\ \ p=1$; there will become

$$\sin \mathrm{A}\tfrac{\pi}{3}=\tfrac{1}{2} \quad\text{and}\quad \cos \mathrm{A}\tfrac{\pi}{6}=\tfrac{\sqrt{3}}{2},$$

from which the following series proceed:

$$\tfrac{\pi}{3}=1+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19}-\tfrac{1}{23}+\text{etc.},$$
$$\tfrac{\pi}{2\sqrt{3}}=1-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{17}+\tfrac{1}{19}-\tfrac{1}{23}+\text{etc.}$$

This method will supply all the former series as well.

18. Therefore since we have seen the sum of this series

$$\frac{1}{s}+\frac{1}{1-s}-\frac{1}{1+s}-\frac{1}{2-s}+\frac{1}{2+s}+\frac{1}{3-s}-\frac{1}{3+s}-\text{etc.}$$

to be

$$=\frac{\pi}{\sin \mathrm{A}\, s\pi}$$

and of this series

$$\frac{1}{s}-\frac{1}{1-s}+\frac{1}{1+s}-\frac{1}{2-s}+\frac{1}{2+s}-\frac{1}{3-s}+\text{etc.}$$

to become

$$=\frac{\pi\cos \mathrm{A}\, s\pi}{\sin \mathrm{A}\, s\pi},$$

whatever value may be attributed to the letter s, it is evident these same equalities may be used, if there may be put $s+ds$ in place of s, or, which corresponds to the same, if these series may be differentiated with the quantity s put as the variable quantity. Whereby, since there shall be

$$d\sin A\,\pi s=\pi ds\cos A\pi s \quad\text{and}\quad d\cos A\,\pi s=-\pi ds\sin A\,\pi s,$$

there will become, with the differentials taken and with the division by $-ds$ performed

$$\frac{\pi\cos \mathrm{A}\, s\pi}{(\sin \mathrm{A}\, s\pi)^2}=\frac{1}{ss}-\frac{1}{(1-s)^2}-\frac{1}{(1+s)^2}+\frac{1}{(2-s)^2}+\frac{1}{(2+s)^2}-\frac{1}{(3-s)^2}-\text{etc.}$$
$$\frac{\pi\pi}{(\sin \mathrm{A}\, s\pi)^2}=\frac{1}{ss}+\frac{1}{(1-s)^2}+\frac{1}{(1+s)^2}+\frac{1}{(2-s)^2}+\frac{1}{(2+s)^2}+\frac{1}{(3-s)^2}+\text{etc.}$$

Therefore so that if in place of s there may be restored $\frac{p}{q}$ and each may be divided by pq, the following series to be summed become:

$$\frac{\pi\pi\cos \mathrm{A}\, s\pi}{qq(\sin \mathrm{A}\, s\pi)^2}=\frac{1}{pp}-\frac{1}{(q-p)^2}-\frac{1}{(q+p)^2}+\frac{1}{(2q-p)^2}+\frac{1}{(2q+p)^2}-\text{etc.},$$
$$\frac{\pi\pi}{qq(\sin \mathrm{A}\, s\pi)^2}=\frac{1}{pp}+\frac{1}{(q-p)^2}+\frac{1}{(q+p)^2}+\frac{1}{(2q-p)^2}+\frac{1}{(2q+p)^2}+\text{etc.}$$

19. We may put $q=2$ et $p=1$; there will become $\sin \mathrm{A}\frac{\pi}{2}=1$ and $\cos \mathrm{A}\frac{\pi}{2}=0$, from which the following series will be produced:

$$0=1-1-\frac{1}{3^2}+\frac{1}{3^2}+\frac{1}{5^2}-\frac{1}{5^2}-\frac{1}{7^2}+\text{ etc.},$$
$$\frac{\pi\pi}{4}=1+1+\frac{1}{3^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{5^2}+\frac{1}{7^2}+\text{ etc.}$$

of which the first is apparent by itself, truly the latter here returns :

$$\frac{\pi\pi}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\text{etc.}$$

Let $q=3$ and $p=1$, there will become $\sin A\frac{\pi}{3}=\frac{\sqrt{3}}{2}$ and $\cos A\frac{\pi}{3}=\frac{1}{2}$, from which these two series will arise

$$\frac{2\pi\pi}{27}=1-\frac{1}{2^2}-\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}-\frac{1}{10^2}+\text{etc.}$$
$$\frac{4\pi\pi}{27}=1+\frac{1}{2^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{7^2}-\frac{1}{8^2}+\frac{1}{10^2}+\text{etc.}$$

Let $q=4$ and $p=1$; there will become $\sin A\frac{\pi}{4}=\frac{1}{\sqrt{2}}$ and $\cos A\frac{\pi}{4}=\frac{1}{\sqrt{2}}$, and hence these two series will arise :

$$\frac{2\pi\pi}{8\sqrt{2}}=1-\frac{1}{3^2}-\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}-\frac{1}{11^2}-\frac{1}{13^2}+\text{etc.},$$
$$\frac{\pi\pi}{8}=1+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{9^2}+\frac{1}{11^2}+\frac{1}{13^2}+\text{etc.}$$

Let $q=6$ and $p=1$; there will become $\sin A\frac{\pi}{6}=\frac{1}{2}$ and $\cos A\frac{\pi}{6}=\frac{\sqrt{3}}{2}$, in which case these series will be obtained :

$$\frac{2\pi\pi}{6\sqrt{3}}=1-\frac{1}{5^2}-\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}-\frac{1}{17^2}-\frac{1}{19^2}+\text{etc.},$$
$$\frac{\pi\pi}{9}=1+\frac{1}{5^2}+\frac{1}{7^2}+\frac{1}{11^2}+\frac{1}{13^2}+\frac{1}{17^2}+\frac{1}{19^2}+\text{etc.}$$

And from these series these two other principal series are derived without difficulty, which I have elicited for this kind in accordance with the aid of the preceding method, namely

$$\frac{\pi\pi}{6}=1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\frac{1}{5^2}+\frac{1}{6^2}+\text{etc.}$$
$$\frac{\pi\pi}{12}=1-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\text{etc.}$$

20. So that the sums of higher powers may be able to be found more easily by continued differentiation, we may differentiate the sums and series separately. Therefore there shall be

$$\frac{\pi}{\sin A\pi s}=P \quad\text{and}\quad \frac{\pi\cos A\pi s}{\sin A\pi s}=Q$$

and we will have the following summations expressed by the differentials of each order of P and Q :

$$+P=\frac{1}{s}+\frac{1}{1-s}-\frac{1}{1+s}-\frac{1}{2-s}+\frac{1}{2+s}+\frac{1}{3-s}-\text{etc.},$$

$$+Q=\frac{1}{s}-\frac{1}{1-s}+\frac{1}{1+s}-\frac{1}{2-s}+\frac{1}{2+s}-\frac{1}{3-s}+\text{etc.},$$

$$\frac{-dP}{1ds}=\frac{1}{ss}-\frac{1}{(1-s)^2}-\frac{1}{(1+s)^2}+\frac{1}{(2-s)^2}+\frac{1}{(2+s)^2}-\frac{1}{(3-s)^2}-\text{etc.}$$

$$\frac{-dQ}{1ds}=\frac{1}{ss}+\frac{1}{(1-s)^2}+\frac{1}{(1+s)^2}+\frac{1}{(2-s)^2}+\frac{1}{(2+s)^2}+\frac{1}{(3-s)^2}+\text{etc.}$$

$$\frac{+ddP}{1\cdot 2ds^2}=\frac{1}{s^3}+\frac{1}{(1-s)^3}-\frac{1}{(1+s)^3}-\frac{1}{(2-s)^3}+\frac{1}{(2+s)^3}+\frac{1}{(3-s)^3}-\text{etc.}$$

$$\frac{+ddQ}{1\cdot 2ds^2}=\frac{1}{s^3}-\frac{1}{(1-s)^3}+\frac{1}{(1+s)^3}-\frac{1}{(2-s)^3}+\frac{1}{(2+s)^3}-\frac{1}{(3-s)^3}+\text{etc.}$$

$$\frac{-d^3P}{1\cdot 2\cdot 3ds^3}=\frac{1}{s^4}-\frac{1}{(1-s)^4}-\frac{1}{(1+s)^4}+\frac{1}{(2-s)^4}+\frac{1}{(2+s)^4}-\frac{1}{(3-s)^4}-\text{etc.}$$

$$\frac{-d^3Q}{1\cdot 2\cdot 3ds^3}=\frac{1}{s^4}-\frac{1}{(1-s)^4}+\frac{1}{(1+s)^4}+\frac{1}{(2-s)^4}+\frac{1}{(2+s)^4}+\frac{1}{(3-s)^4}+\text{etc.}$$

Therefore generally we will have this summation :

$$\frac{\pm d^{n-1}P}{1\cdot 2\cdot 3\cdots(n-1)ds^{n-1}}=\frac{1}{s^n}\pm\frac{1}{(1-s)^n}-\frac{1}{(1+s)^n}\mp\frac{1}{(2-s)^n}+\frac{1}{(2+s)^n}\pm\frac{1}{(3-s)^n}-\text{etc.}$$

$$\frac{\pm d^{n-1}Q}{1\cdot 2\cdot 3\cdots(n-1)ds^{n-1}}=\frac{1}{s^n}\mp\frac{1}{(1-s)^n}+\frac{1}{(1+s)^n}\mp\frac{1}{(2-s)^n}+\frac{1}{(2+s)^n}\mp\frac{1}{(3-s)^n}+\text{etc.},$$

where the upper signs prevail, if n were an odd number, the lower truly, if n shall be an even number.

21. Therefore for these sums requiring to be actually determined it is necessary that we may elicit the values of the differentials of any order of each of the quantities P and Q ; so that it may be able to be put easier and more succinctly, we may put

$$\sin \mathrm{A}\pi s = x \text{ and } \cos \mathrm{A}\pi s = y$$

and there will become:

$$P=\frac{\pi}{x} \text{ and } Q=\frac{\pi y}{x}.$$

Truly again there will become

$$dx = \pi y ds \text{ and } dy = -\pi x ds,$$

from which, by the rules of differentiation, the following values are obtained :

$$\begin{aligned}
&P = \tfrac{\pi}{x},\\
&-\tfrac{dP}{ds} = \tfrac{\pi\pi}{xx}\cdot y,\\
&+\tfrac{ddP}{ds^2} = \tfrac{\pi^3}{x^3}\left(y^2+1\right),\\
&-\tfrac{d^3P}{ds^3} = \tfrac{\pi^4}{x^4}\left(y^3+5y\right),\\
&+\tfrac{d^4P}{ds^4} = \tfrac{\pi^5}{x^5}\left(y^4+18y^2+5\right),\\
&-\tfrac{d^5P}{ds^5} = \tfrac{\pi^6}{x^6}\left(y^5+58y^3+61y\right),\\
&+\tfrac{d^6P}{ds^6} = \tfrac{\pi^7}{x^7}\left(y^6+179y^4+479y^2+61\right),\\
&-\tfrac{d^7P}{ds^7} = \tfrac{\pi^8}{x^8}\left(\begin{array}{ccccccc} & +6.1 & & +4.179 & & +2.479 & \\ y^7 & & y^5 & & y^3 & & y \\ & +3.179 & & +5.479 & & +7.61 & \end{array}\right)\\
&\qquad\qquad\qquad\text{etc.},
\end{aligned}$$

from which the final expression corresponds to as similar law, with the aid of which from the differential of any order the differential of the following order can be formed. And hence the sum of this series

$$\tfrac{1}{s^n} \pm \tfrac{1}{(1-s)^n} - \tfrac{1}{(1+s)^n} \mp \tfrac{1}{(2-s)^n} + \tfrac{1}{(2+s)^n} \pm \tfrac{1}{(3-s)^n} - \text{etc.}$$

will be assigned for any value of the exponent n.

22. In a similar manner the values of the differentials of any higher order of the quantity will be found, and there will become

$$+Q=\tfrac{\pi}{x}\cdot y,\quad [\text{recall } dx=\pi y ds \text{ and } dy=-\pi x ds,]$$

$$-\tfrac{dQ}{ds}=\tfrac{\pi\pi}{xx}\cdot 1,\ [i.e.\ \tfrac{dQ}{ds}=-\tfrac{d}{ds}\tfrac{\pi}{\tan\pi s}=-\tfrac{\pi}{\tan^2\pi s}\cdot\tfrac{\pi}{\cos^2\pi s}=-\tfrac{\pi\pi}{xx}]$$

$$+\tfrac{ddQ}{ds^2}=\tfrac{\pi^3}{x^3}\cdot 2y,$$

$$-\tfrac{d^3Q}{ds^3}=\tfrac{\pi^4}{x^4}\left(4yy+2\right),$$

$$+\tfrac{d^4Q}{ds^4}=\tfrac{\pi^5}{x^5}\left(8y^3+16y\right),$$

$$-\tfrac{d^5Q}{ds^5}=\tfrac{\pi^6}{x^6}\left(16y^4+88y^3+16\right),$$

$$+\tfrac{d^6Q}{ds^6}=\tfrac{\pi^7}{x^7}\left(32y^5+416y^3+272y\right),$$

$$-\tfrac{d^7Q}{ds^7}=\tfrac{\pi^8}{x^8}\left(64y^6+1824y^4+2880y^2+272\right)$$

$$+\tfrac{d^8Q}{ds^8}=\tfrac{\pi^9}{x^9}\left(2\cdot 64y^7\ \begin{matrix}+4.1824\\ \\ +6.64\end{matrix}\,y^5\ \begin{matrix}+6.2880\\ \\ +4.1824\end{matrix}\,y^3\ \begin{matrix}+8.272\\ \\ +2.2880\end{matrix}\,y\right)$$

etc.,

Here the law of the progression is equally apparent, with the aid of which, as long as it may please, it will be allowed to progress; and hence the sum of all the series of the powers will be able to be shown, which are contained in this form

$$\tfrac{1}{s^n}\pm\tfrac{1}{(1-s)^n}+\tfrac{1}{(1+s)^n}\mp\tfrac{1}{(2-s)^n}+\tfrac{1}{(2+s)^n}\mp\tfrac{1}{(3-s)^n}+\text{etc.}$$

But in this series not only all the powers of the series will be contained, which the previous method made available, but innumerable others in addition. It may be seen there is no reason why this method may not be suitable for elucidating many other delightful matters.

DE SUMMIS SERIERUM RECIPROCARUM EX POTESTATIBUS NUMERORUM NATURALIUM ORTARUM DISSERTATIO ALTERA IN QUA EAEDEM SUMMATIONES EX FONTE MAXIME DIVERSO DERIVANTUR

L. Euler E61 : Miscellanea Berolinonsia 7, 1743, p. 172-192

1. Postquam ante complures annos summas serierum in hac forma generali contentarum

$$1+\frac{1}{2^n}+\frac{1}{3^n}+\frac{1}{4^n}+\frac{1}{5^n}+\frac{1}{6^n}+\text{ etc. in infinitum,}$$

si n fuerit numerus par positivus, simulque etiam harum serierum, si n fuerit numerus impar,

$$1-\frac{1}{3^n}+\frac{1}{5^n}-\frac{1}{7^n}+\text{ etc. in infinitum}$$

ope quadraturum circuli exhibuissem atque ostendissem summam perpetuo per eandem peripheriae circuli dignitatem, quam exponens n indicet, exprimi, argumentum hoc acutissimis Geometris tantopere placuit, ut id non solum maxime probarent, verum etiam ipsi studium atque operam ad easdem summas methodis sibi famillaribus eruendas impenderent. Ego quoque ab illo tempore multum fui occupatus in alia via investiganda, quae eodem deduceret, non tam ut veritatem inventam magis confirmarem, quam ut limites analyseos in huius generis seriebus tractandis ulterius extenderem.

2. Methodus, quae me ad istarum serierum summationem manuduxit, utique erat nova et in eiusmodi instituto plane non usitata; nitebatur enim in resolutione aequationis infinitae, cuius omnes radices, quarum numerus erat infinitus, nosse oportebat. Contemplatus sum namque hanc aequationem in infinitum excurrentem

$$x=s-\frac{s^3}{6}+\frac{s^5}{120}-\frac{s^7}{5040}+\frac{s^9}{362889}-\text{ etc.}$$

quae continet relationem inter arcum circuli s et eius sinum x sinu toto posito $=1$. Cum autem eidem sinui x innumerabiles arcus tam affirmativi quam negativi respondeant, hoc modo istius aequationis radices innumeras a posteriori eram consecutus; et quoniam coefficientes cuiusque aequationis a radicibus pendent, ex comparatione istorum coefficientium cum radicibus aequationis perveni ad summas serierum ante memoratarum.

3. Statim quidem facile perspexi istam methodum tutam esse atque ad veritatem perducere non posse, nisi constaret aequationem illam gradus infinitialias radices praeter

eas, quas natura sinuum mihi suppeditaverat, in se non complecti. Quamvis enim intelligerem alias radices reales praeter assignatas in illa aequatione non contineri, tamen merito dubitare conveniebat, an non radices imaginariae inessent; quod si evenisset, cunctae summationes, quas inde elicui, cum veritate consistere non potuissent. In hoc porro dubio confirmabar, cum simili modo ex arcu elliptico eius sinum seu applicatam respondentem per seriem expressissem; quanquam enim pariter innumerabiles arcus elliptici praesto erant, qui ad eundem sinum referantur, tamen ex iis nullas serierum summas, quae veritati essent consentaneae, deducere licuerat; cuius incommodi ratio procul dubio in pluribus ac fortasse infinitis radicibus imaginariis erat posita, quae in illam aequationem ex ellipsi formatam ingrediuntur.

4. Quoniam igitur illo tempore nullam habueram demonstrationem, qua mihi certo constaret aequationem inter arcum circuli s eiusque sinum x radicibus imaginariis omnino carere, summas serierum inventas ad trutinam veritatis examinare coepi; ac primo quidem statim deprehendi hanc methodum eandem summam seriei

$$1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\tfrac{1}{13}-\text{ etc.}$$

praebere, quam iam pridem LEIBNITIUS assignaverat, quae convenientia satis declarabat, si quas aequatio radices imaginarias contineret, earum summum necessario esse $=0$. Deinde series altiorum potestatum ita examinavi, ut summas hac methode inventas cum summis, quas paulo ante per approximationem eruerum, compararem, in quo negotio denuo consensum deprehendi. Atque ob has rationes penitus confirmabar nullis omnino radicibus imaginariis aequationem illam, quae me ad omnes illas summas perduxerat, esse inquinatam; sicque non dubitavi istas summas tanquam verissimas producere.

5. In hac opinione autem prorsus me corroborabat alia methodus mere analytica, cuius ope deinceps per solas integrationes eandem summam huius seriei

$$1+\tfrac{1}{4}+\tfrac{1}{9}+\tfrac{1}{16}+\tfrac{1}{25}+\text{ etc.}$$

elicueram, similique fere modo hanc eandem summam demonstratam dedit Vir longe acutissimus Nicolaus Bernoulli in schediasmate Tomo X. Comment. Acad. Petropol. inserto. Quanquam autem hoc modo calculus analyticus ad easdem summas omnes eruendas traduci posse videbatur, tamen neque ego neque quisquam alius pari calculo ad potestates altiores pertingere potuimus. Quod me fere eo impulit, ut crederem aliam viam praeter resolutionem aequationis infinitae non patere, quae omnium potestatum summas iunctim praeberet.

6. Curam hanc fere penitus oblitam nuper in me renovaverunt litterae a Viro Celeberrimo Daniele Bernoulli acceptae, in quibus mihi easdem dubitandi rationes circa meam

methodum obiicit simulque significavit Celeberrimum Cramerum iisdem dubiis praepediri, quominus methodum meam probare queat. Istae igitur admonitiones amicissimae eo me redegerunt, ut universum negotium de novo ruminarer atque cum in methodi bonitate demonstranda tum in alia via easdem series summandi exquirenda laborarem. In utroque igitur instituto voti plene compos factus duplex negotium in hac dissertatione absolvam. Primum scilicet demonstrabo in aequatione infinita supra commemorata radices imaginarias nullas contineri hincque de summarum inde ductarum veritate amplius dubitari non licere. Secundo vero loco novam ac non solum a prima maxime diversam methodum, sed etiam ad plurima alia lustranda campum aperientem proponam, quae per solas integrandi regulas totum negotium conficiat.

7. Demonstrationem primum promissam impetravi ex resolutione huius binomii

$$a^n + b^n$$

in factores suos reales. Unusquisque enim huius binomii factor continetur in hac forma

$$aa - 2ab\cos \mathrm{A}\frac{(2k-1)\pi}{n} + bb$$

atque omnes factores obtinentur, si loco $2k-1$ successive omnes numeri impares minores exponente n substituantur; et si n fuerit numerus impar, tum praeter hos factores trinomiales adiungi debet factor simplex $a+b$. Quodsi habeatur residuum

$$a^n - b^n$$

eius primo factor est simplex $a-b$, factores reliqui trinomiales reales continentur in hac forma

$$aa - 2ab\cos \mathrm{A}\frac{2k\pi}{n} + bb$$

cunctique huius generis factores obtinentur, si loco $2k$ successive omnes numeri pares (cyphra excepta) minores exponente n scribantur; et si n ipse fuerit numerus par, praeterea factor adiici debet simplex $a+b$. Hoc igitur modo omnes omnino formulae

$$a^n \pm b^n$$

factores reales exbiberi poterunt, quorum omnium productum hanc ipsam formulam reddat. Ceterum hic notandum est π denotare semicircumferentiam circuli, cuius radius $=1$, seu π esse angulum duobus rectis aequalem.

8. Hinc iam a priori assignare possum omnes radices seo factores huius expressionis infinitae

$$s-\frac{s^3}{1\cdot 2\cdot 3}+\frac{s^5}{1\cdot 2\cdot 3\cdot 4\cdot 5}-\frac{s^7}{1\cdot 2\cdot 3\cdots 7}+\frac{s^9}{1\cdot 2\cdot 3\cdots 9}-\text{etc.}$$

Haec enim expressio aequivalet isti

$$\frac{e^{s\sqrt{-1}}-e^{-s\sqrt{-1}}}{2\sqrt{-1}}$$

denotant *e* numerum, cuius logarithmus est $=1$, et cum sit

$$e^z=\left(1+\tfrac{z}{n}\right)^n$$

existente *n* numero infinito, reducetur expressio infinita proposita ad hanc

$$\frac{\left(1+\frac{s\sqrt{-1}}{n}\right)^n-\left(1-\frac{s\sqrt{-1}}{n}\right)^n}{2\sqrt{-1}},$$

cuius primum factor simple est *s*, quem quidem ipsa seriei inspectio monstrat. Ad reliquos factores eruendos comparo hanc expressionem cum forma a^n-b^n ; erit

$$a=1+\tfrac{s\sqrt{-1}}{n} \quad \text{et} \quad b=1-\tfrac{s\sqrt{-1}}{n}$$

hincque

$$aa+bb=2-\tfrac{2ss}{nn} \quad \text{et} \quad 2ab=2+\tfrac{2ss}{nn}.$$

Quilibet ergo factor continebitur in hac forma

$$2-\tfrac{2ss}{nn}-2\left(1+\tfrac{ss}{nn}\right)\cos\mathrm{A}\,\tfrac{2k\pi}{n}$$

hincque omnes omnino factores emergent, si pro *2k* successive omnes numeri pares in infinitum substituantur, propterea quod *n* denotat numerum infinitum.

9. Cum autem *n* sit numerus infinitus, erit arcus $\frac{2k\pi}{n}$ infinite parvus, quoad 2*k* fiat quoque infinitus numerus, minor tamen quam *n.* Erit ergo

$$\cos\mathrm{A}\,\tfrac{2k\pi}{n}=1-\tfrac{2kk\pi\pi}{nn},$$

unde factor generalis transit in hanc formam

$$-\tfrac{4ss\pi}{nn}+\tfrac{4kk\pi\pi}{nn},$$

ex quo, termino cognito ad unitatem reducto, nascitur factor

$$1-\frac{ss}{kk\pi\pi},$$

qui, loco k substitutis successive omnibus numeris 1, 2, 3 etc. in infinitum, praebet omnes factores. Quodsi autem k ita in infinitum abeat, ut $2k$ ad n acquirat rationem finitam, tum ob

$$\cos A\frac{2k\pi}{n}<1$$

termini $\frac{ss}{nn}$ prae unitate non destructa evanescent fietque factor $1-\cos A\frac{2k\pi}{n}$ constans ideoque in computum non venit, quod in eo arcus s non continetur.

10. Hoc ergo pacto nacti sumus omnes factores formulae propositae

$$s-\frac{s^3}{1\cdot2\cdot3}+\frac{s^5}{1\cdot2\cdot3\cdot4\cdot5}-\frac{s^7}{1\cdot2\cdot3\cdots7}+\text{ etc.},$$

quae adeo exacte aequalis erit producto ex his infinitis factoribus constanti

$$s\left(1-\frac{ss}{\pi\pi}\right)\left(1-\frac{ss}{4\pi\pi}\right)\left(1-\frac{ss}{9\pi\pi}\right)\left(1-\frac{ss}{16\pi\pi}\right)\text{ etc.},$$

ex quorum cum coefficientibus terminorum seriei comparatione statim sequuntur summae serierum

$$1+\frac{1}{2^m}+\frac{1}{3^m}+\frac{1}{4^m}+\frac{1}{5^m}+\frac{1}{6^m}+\text{etc.},$$

si m denotat numerum quemcunque parem, neque igitur amplius de earum veritate dubitati potest.

11. Simili modo si consideremus hanc seriem

$$s-\frac{ss}{1\cdot2}+\frac{s^4}{1\cdot2\cdot3\cdot4}-\frac{s^6}{1\cdot2\cdot3\cdots6}+\frac{s^8}{1\cdot2\cdot3\cdots8}-\text{ etc.},$$

ea reducetur ad hanc formam

$$\frac{\left(1+\frac{s\sqrt{-1}}{n}\right)^n+\left(1-\frac{s\sqrt{-1}}{n}\right)^n}{2}$$

denotante n numerum infinitum. Divisores ergo binomii

$$\left(1+\tfrac{s\sqrt{-1}}{n}\right)^n+\left(1-\tfrac{s\sqrt{-1}}{n}\right)^n$$

simul erunt divisores seriei propositae, et quidem omnes. Comparata hac forma cum a^n+b^n erit

$$a=1+\tfrac{s\sqrt{-1}}{n},\quad b=1-\tfrac{s\sqrt{-1}}{n},\quad aa+bb=2-\tfrac{2ss}{nn}\ \text{ et }\ 2ab=2+\tfrac{2ss}{nn};$$

unusquisque ergo formulae propositae divisor continetur in hae expressione

$$2\left(1-\tfrac{ss}{nn}\right)-2\left(1+\tfrac{ss}{nn}\right)\cos A\,\tfrac{(2k-1)\pi}{n}$$

vel hac

$$2\left(1-\cos A\,\tfrac{(2k-1)\pi}{n}\right)-\tfrac{2ss}{nn}\left(1+\cos A\,\tfrac{(2k-1)\pi}{n}\right).$$

Cum autem in divisore tantum ad incognitam respiciatur, erit divisor quilibet

$$1-\frac{ss\left(1+\cos A\frac{(2k-1)\pi}{n}\right)}{nn\left(1-\cos A\frac{(2k-1)\pi}{n}\right)}$$

facto termino cognito unitati aequali, quia in serie ipsa primus terminus est $=1$.

12. Ob n autem numerum infinitum erit

$$1+\cos A\,\tfrac{(2k-1)\pi}{n}=2\ \text{ et }\ 1-\cos A\,\tfrac{(2k-1)\pi}{n}=\tfrac{(2k-1)^2\pi\pi}{2nn},$$

ex quo unusquisque divisor erit

$$1-\frac{4ss}{(2k-1)^2\pi\pi};$$

et si loco $2k-1$ omnes numeri impares in infinitum usque substituantur, orientur omnes divisores seriei propositae

$$1-\tfrac{ss}{1\cdot 2}+\tfrac{s^4}{1\cdot 2\cdot 3\cdot 4}-\tfrac{s^6}{1\cdot 2\cdot 3\cdots 6}+\text{ etc.},$$

quae ergo erit productum ex his factoribus numero infinitis

$$\left(1-\tfrac{4ss}{\pi\pi}\right)\left(1-\tfrac{4ss}{9\pi\pi}\right)\left(1-\tfrac{4ss}{25\pi\pi}\right)\left(1-\tfrac{4ss}{36\pi\pi}\right)\text{ etc.},$$

ex quorum comparatione cum ipsa serie omnes series potestatum summantur ut ante. Atque sic demonstratum est aequationes illas infinitas, quas illo tempere tractavi, alias radices non habere praeter eas, quas ex natura sinuum et cosinuum a posteriore sum assecutus.

13. Demonstrata methodi, cuius ope antehac istiusmodi serierum summas assignavi, bonitate progredior ad alteram methodum ab hac prorsus diversam explicandam, quae ex solis calculi integralis principiis petita sine ullis ambagibus earundem serierum summas mira facilitate suppeditat. Nititur autem ista methodus duobus theorematis, quorum demonstrationem in dissertatione superiori *de inventione integralium, si quantitati variabili post integrationem definitus valor tribuatur* dedi, unde ea sine demonstratione depromam.
Primum ita se habet:

Formulae differentialis

$$\frac{x^{p-1}+x^{q-p-1}}{1+x^{q}}dx,$$

integrale ita sumtum, ut evanescat posito $x=0$, si post integrationem ponatur $x=1$, dabit hunc valorem

$$\frac{\pi}{q\sin\mathrm{A}\frac{p\pi}{q}},$$

denotante π semicircumferentiam circuli, cuius radius $=1$, in quo circulo simul sinum arcus $\frac{p\pi}{q}$ capi pono.

Alterum theorema huic fere simile ita se habet:
Formulae differentialis huius

$$\frac{x^{p-1}-x^{q-p-1}}{1-x^{q}}dx$$

integrale ita sumtum, ut evanescat posito $x=0$, si in eo post integrationem ponatur $x=1$, dabit hunc valorom

$$\frac{\pi\cos\mathrm{A}\frac{p\pi}{q}}{q\sin\mathrm{A}\frac{p\pi}{q}} \quad \text{seu} \quad \frac{\pi}{q\,\mathrm{tang}\,\mathrm{A}\frac{p\pi}{q}}.$$

Demonstrationes horum theorematum ordine plano procedunt; primum enim secundum regulas consuetas istarum formularum integralia generatim investigavi iisque inventis loco variabilis *oe* unitatem posui. Quo facto ad seriem sinuum finitam adeo pertigi, quae, quia arcus in progressione arithmetica progrediebantur, summationem admittebat hasque ipsas expressiones praebebat.

14. Sumamus iam priorem formulam integralem

$$\int \frac{x^{p-1}+x^{q-p-1}}{1+x^{q}}dx$$

quae in seriem resoluta dabit duplicem progressionem geometricam

$$\int dx\left(x^{p-1}-x^{q+p-1}+x^{2q+p-1}-x^{3q+p-1}+\text{etc.}\right)$$
$$+\int dx\left(x^{q-p-1}-x^{2q-p-1}+x^{3q-p-1}-x^{4q-p-1}+\text{etc.}\right).$$

Huius igitur integrale ita acceptum, ut evanescat posito $x=0$, ita per seriem exprimetur

$$\frac{x^{p}}{p}+\frac{x^{q-p}}{q-p}-\frac{x^{q+p}}{q+p}-\frac{x^{2q-p}}{2q-p}+\frac{x^{2q+p}}{2q+p}+\frac{x^{3q-p}}{3q-p}-\text{etc.}$$

Quodsi nunc ponamus $x=1$, erit per theorema prius huius seriei

$$\frac{1}{p}+\frac{1}{q-p}-\frac{1}{q+p}-\frac{1}{2q-p}+\frac{1}{2q+p}+\frac{1}{3q-p}-\frac{1}{3q+p}-\frac{1}{4q-p}+\text{etc.}$$

summa

$$=\frac{\pi}{q\operatorname{tang}\mathrm{A}\frac{p\pi}{q}}.$$

15. Simili modo altera formula integralis

$$\int \frac{x^{p-1}-x^{q-p-1}}{1-x^{q}}dx$$

per seriem integrata dabit

$$\frac{x^{p}}{p}-\frac{x^{q-p}}{q-p}+\frac{x^{q+p}}{q+p}-\frac{x^{2q-p}}{2q-p}+\frac{x^{2q+p}}{2q+p}-\frac{x^{3q-p}}{3q-p}+\text{etc.}$$

Quamobrem per alterum theorema, si ponamus $x=1$, erit huius seriei

$$\frac{1}{p}-\frac{1}{q-p}+\frac{1}{q+p}-\frac{1}{2q-p}+\frac{1}{2q+p}-\frac{1}{3q-p}+\frac{1}{3q+p}-\text{etc.}$$

summa

$$=\frac{\pi\cos\mathrm{A}\frac{p\pi}{q}}{q\sin\mathrm{A}\frac{p\pi}{q}},$$

dummodo fuerint p et q numeri affirmativi atque $q > p$, id quod in sequentibus semper assumi oportet; alioquin enim integrale hoc modo sumtum non evanesceret posito $x=0$.

16. Sit $\frac{p}{q}=s$; atque seriebus inventis per q multiplicatis habebimus has binas series ad summam finitam reductas

$$\frac{\pi}{\sin \mathrm{A}\, s\pi}=\frac{1}{s}+\frac{1}{1-s}-\frac{1}{1+s}-\frac{1}{2-s}+\frac{1}{2+s}+\frac{1}{3-s}-\frac{1}{3+s}-\text{etc.},$$
$$\frac{\pi\cos \mathrm{A}\, s\pi}{\sin \mathrm{A}\, s\pi}=\frac{1}{s}-\frac{1}{1-s}+\frac{1}{1+s}-\frac{1}{2-s}+\frac{1}{2+s}-\frac{1}{3-s}+\frac{1}{3+s}-\text{etc.},$$

harumque serierum summae verae erunt, quicunque numerus per s indicetur, sive rationalis sive irrationalis, sicque lex continuitatis non amplius infringitur ut ante, ubi loco p et q numeros integros accipi oportebat. Quin etiam hae summae a veritate non recedunt, etiamsi pro s numeri unitate maiores ponantur. Si enim sit $s=1$ vel quilibet numerus integer, tum series fiunt infinitae ob unum terminum in infinitum abeuntem, simul vero summae exhibitae ob $\sin \mathrm{A} s\pi=0$ in infinitum excrescunt. Hinc istae summae tam late patent, ut nullam prorsus restrictionem requirant.

17. Ex his seriebus generalibus iam deducuntur series pro quadratura circuli cum Leibnizii tum Gregorii aliaeque innumerabiles, quarum praecipuas hic exhibere lubet.

Sit $q=2$ et $p=1$; erit

$$\sin \mathrm{A}\,\frac{\pi}{2}=1 \quad\text{et}\quad \cos \mathrm{A}\,\frac{\pi}{2}=0$$

hincque sequentes series nascuntur

$$\frac{\pi}{2}=1+1-\frac{1}{3}-\frac{1}{3}+\frac{1}{5}+\frac{1}{5}-\frac{1}{7}-\frac{1}{7}+\text{etc.}$$

sive

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\frac{1}{11}+\text{ etc.}$$

atque

$$\frac{0\pi}{4}=1-1+\frac{1}{3}-\frac{1}{3}+\frac{1}{5}-\frac{1}{5}+\frac{1}{7}-\frac{1}{7}+\text{ etc.}$$

quarum illa est series LEIBNIZIANA, haec vero se sponte prodit.

Sit $q=3$ et $p=1$; erit

$$\sin.\mathrm{A}\,\frac{\pi}{3}=\frac{\sqrt{3}}{2} \quad\text{et}\quad \cos \mathrm{A}\,\frac{\pi}{3}=\frac{1}{2},$$

unde sequentes series nascuntur

$$\frac{2\pi}{3\sqrt{3}}=1+\frac{1}{2}-\frac{1}{4}-\frac{1}{5}+\frac{1}{7}+\frac{1}{8}-\frac{1}{10}-\frac{1}{11}+\frac{1}{13}+\text{etc.},$$
$$\frac{\pi}{3\sqrt{3}}=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\frac{1}{10}-\frac{1}{11}+\frac{1}{13}-\text{etc.}$$

Sit $q=4,\ p=1$: erit

$$\sin.\mathrm{A}\tfrac{\pi}{4}=\tfrac{1}{\sqrt{2}} \quad \text{et} \quad \cos\mathrm{A}\tfrac{\pi}{4}=\tfrac{1}{\sqrt{2}};$$

hincque sequentes oriuntur series

$$\frac{\pi}{2\sqrt{2}}=1+\tfrac{1}{3}-\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}+\tfrac{1}{11}-\tfrac{1}{13}-\text{etc.},$$
$$\tfrac{\pi}{4}=1-\tfrac{1}{3}+\tfrac{1}{5}-\tfrac{1}{7}+\tfrac{1}{9}-\tfrac{1}{11}+\tfrac{1}{13}-\text{etc.}$$

Sit $q=6,\ \ p=1$; erit

$$\sin\mathrm{A}\tfrac{\pi}{3}=\tfrac{1}{2} \quad \text{et} \quad \cos\mathrm{A}\tfrac{\pi}{6}=\tfrac{\sqrt{3}}{2},$$

unde sequentes proficiscuntur series

$$\tfrac{\pi}{3}=1+\tfrac{1}{5}-\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}+\tfrac{1}{17}-\tfrac{1}{19}-\tfrac{1}{23}+\text{etc.},$$
$$\frac{\pi}{2\sqrt{3}}=1-\tfrac{1}{5}+\tfrac{1}{7}-\tfrac{1}{11}+\tfrac{1}{13}-\tfrac{1}{17}+\tfrac{1}{19}-\tfrac{1}{23}+\text{etc.}$$

Hasque series omnes prior quoque methodus suppeditaverat.

18. Quoniam igitur vidimus summam huius seriei

$$\tfrac{1}{s}+\tfrac{1}{1-s}-\tfrac{1}{1+s}-\tfrac{1}{2-s}+\tfrac{1}{2+s}+\tfrac{1}{3-s}-\tfrac{1}{3+s}-\text{etc.}$$

esse

$$=\frac{\pi}{\sin\mathrm{A}\,s\pi}$$

et huius

$$\tfrac{1}{s}-\tfrac{1}{1-s}+\tfrac{1}{1+s}-\tfrac{1}{2-s}+\tfrac{1}{2+s}-\tfrac{1}{3-s}+\text{etc.}$$

esse

$$=\frac{\pi\cos\mathrm{A}\,s\pi}{\sin\mathrm{A}\,s\pi},$$

quicunque valor litterae s tribuatur, manifestmn est easdem aequalitates locum habere, si loco s ponatur $s+ds$, vel, quod eodem redit, si, illae series cum suis summis diiferentientur posita quantite s variabili. Quaro, cum sit

$$d\sin A\,\pi s=\pi ds\cos A\pi s \quad \text{et} \quad d\cos A\,\pi s=-\pi ds\sin A\,\pi s,$$

erit sumtis differentialibus atque ubique per $-ds$ divisione facta

$$\frac{\pi\cos A\, s\pi}{(\sin A\, s\pi)^2} = \frac{1}{ss} - \frac{1}{(1-s)^2} - \frac{1}{(1+s)^2} + \frac{1}{(2-s)^2} + \frac{1}{(2+s)^2} - \frac{1}{(3-s)^2} - \text{etc.}$$
$$\frac{\pi\pi}{(\sin A\, s\pi)^2} = \frac{1}{ss} + \frac{1}{(1-s)^2} + \frac{1}{(1+s)^2} + \frac{1}{(2-s)^2} + \frac{1}{(2+s)^2} + \frac{1}{(3-s)^2} + \text{etc.}$$

Quodsi ergo loco s restituatur $\frac{p}{q}$ et utrinque per pq dividatur, prodibunt sequentes series summatae

$$\frac{\pi\pi\cos A\, s\pi}{qq(\sin A\, s\pi)^2} = \frac{1}{pp} - \frac{1}{(q-p)^2} - \frac{1}{(q+p)^2} + \frac{1}{(2q-p)^2} + \frac{1}{(2q+p)^2} - \text{etc.},$$
$$\frac{\pi\pi}{qq(\sin A\, s\pi)^2} = \frac{1}{pp} + \frac{1}{(q-p)^2} + \frac{1}{(q+p)^2} + \frac{1}{(2q-p)^2} + \frac{1}{(2q+p)^2} + \text{etc.}$$

19. Ponamus esse $q = 2$ et $p = 1$; erit $\sin A\frac{\pi}{2} = 1$ et $\cos A\frac{\pi}{2} = 0$, unde sequentes prodibunt series

$$0 = 1 - 1 - \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{5^2} - \frac{1}{5^2} - \frac{1}{7^2} + \text{etc.},$$
$$\frac{\pi\pi}{4} = 1 + 1 + \frac{1}{3^2} + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{5^2} + \frac{1}{7^2} + \text{etc.}$$

quarum prioris veritas per se patet, posterior vero huc redit

$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \text{etc.}$$

Sit $q = 3$ et $p = 1$, erit $\sin A\frac{\pi}{3} = \frac{\sqrt{3}}{2}$ et $\cos A\frac{\pi}{3} = \frac{1}{2}$, unde nascentur istae binae series

$$\frac{2\pi\pi}{27} = 1 - \frac{1}{2^2} - \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} - \frac{1}{10^2} + \text{etc.}$$
$$\frac{4\pi\pi}{27} = 1 + \frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{7^2} - \frac{1}{8^2} + \frac{1}{10^2} + \text{etc.}$$

Sit $q = 4$ et $p = 1$; erit $\sin A\frac{\pi}{4} = \frac{1}{\sqrt{2}}$ et $\cos A\frac{\pi}{4} = \frac{1}{\sqrt{2}}$, atque hinc orientur hae duae series

$$\frac{2\pi\pi}{8\sqrt{2}} = 1 - \frac{1}{3^2} - \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} - \frac{1}{11^2} - \frac{1}{13^2} + \text{etc.},$$
$$\frac{\pi\pi}{8} = 1 + \frac{1}{3^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{9^2} + \frac{1}{11^2} + \frac{1}{13^2} + \text{etc.}$$

Sit $q = 6$ et $p = 1$; erit $\sin A\frac{\pi}{6} = \frac{1}{2}$ et $\cos A\frac{\pi}{6} = \frac{\sqrt{3}}{2}$, quo casu obtinebuntur istae series

$$\frac{2\pi\pi}{6\sqrt{3}} = 1 - \frac{1}{5^2} - \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} - \frac{1}{17^2} - \frac{1}{19^2} + \text{etc.},$$
$$\frac{\pi\pi}{9} = 1 + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \frac{1}{19^2} + \text{etc.}$$

Atque ex his seriebus non difficulter derivantur binae illae principales, quas praecedentis methodi ope in hoc genere elicui, nempe

$$\frac{\pi\pi}{6} = 1 + \frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \frac{1}{5^2} + \frac{1}{6^2} + \text{etc.}$$
$$\frac{\pi\pi}{12} = 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \frac{1}{5^2} - \frac{1}{6^2} + \text{etc.}$$

20. Quo facilius summae altiorum potestatum per continuatam differentiationem inveniri queant, summas atque series seorsim differentiemus. Sit igitur

$$\frac{\pi}{\sin A\pi s} = P \quad \text{et} \quad \frac{\pi \cos A\pi s}{\sin A\pi s} = Q$$

habebimusque sequentes summationes per differentialia cuiusque gradus ipsorum P et Q expressas

$$+P = \frac{1}{s} + \frac{1}{1-s} - \frac{1}{1+s} - \frac{1}{2-s} + \frac{1}{2+s} + \frac{1}{3-s} - \text{etc.},$$
$$+Q = \frac{1}{s} - \frac{1}{1-s} + \frac{1}{1+s} - \frac{1}{2-s} + \frac{1}{2+s} - \frac{1}{3-s} + \text{etc.},$$
$$\frac{-dP}{1ds} = \frac{1}{ss} - \frac{1}{(1-s)^2} - \frac{1}{(1+s)^2} + \frac{1}{(2-s)^2} + \frac{1}{(2+s)^2} - \frac{1}{(3-s)^2} - \text{etc.}$$
$$\frac{-dP}{1ds} = \frac{1}{ss} - \frac{1}{(1-s)^2} - \frac{1}{(1+s)^2} + \frac{1}{(2-s)^2} + \frac{1}{(2+s)^2} - \frac{1}{(3-s)^2} - \text{etc.}$$
$$\frac{-dQ}{1ds} = \frac{1}{ss} + \frac{1}{(1-s)^2} + \frac{1}{(1+s)^2} + \frac{1}{(2-s)^2} + \frac{1}{(2+s)^2} + \frac{1}{(3-s)^2} + \text{etc.}$$
$$\frac{+ddP}{1\cdot 2ds^2} = \frac{1}{s^3} + \frac{1}{(1-s)^3} - \frac{1}{(1+s)^3} - \frac{1}{(2-s)^3} + \frac{1}{(2+s)^3} + \frac{1}{(3-s)^3} - \text{etc.}$$
$$\frac{+ddQ}{1\cdot 2ds^2} = \frac{1}{s^3} - \frac{1}{(1-s)^3} + \frac{1}{(1+s)^3} - \frac{1}{(2-s)^3} + \frac{1}{(2+s)^3} - \frac{1}{(3-s)^3} + \text{etc.}$$

$$\frac{-d^3P}{1\cdot 2\cdot 3ds^3} = \frac{1}{s^4} - \frac{1}{(1-s)^4} - \frac{1}{(1+s)^4} + \frac{1}{(2-s)^4} + \frac{1}{(2+s)^4} - \frac{1}{(3-s)^4} - \text{etc.}$$
$$\frac{-d^3Q}{1\cdot 2\cdot 3ds^3} = \frac{1}{s^4} - \frac{1}{(1-s)^4} + \frac{1}{(1+s)^4} + \frac{1}{(2-s)^4} + \frac{1}{(2+s)^4} + \frac{1}{(3-s)^4} + \text{etc.}$$

Generaliter ergo habebitur ista summatio

$$\frac{\pm d^{n-1}P}{1\cdot 2\cdot 3\cdots(n-1)ds^{n-1}} = \frac{1}{s^n} \pm \frac{1}{(1-s)^n} - \frac{1}{(1+s)^n} \mp \frac{1}{(2-s)^n} + \frac{1}{(2+s)^n} \pm \frac{1}{(3-s)^n} - \text{etc.}$$
$$\frac{\pm d^{n-1}Q}{1\cdot 2\cdot 3\cdots(n-1)ds^{n-1}} = \frac{1}{s^n} \mp \frac{1}{(1-s)^n} + \frac{1}{(1+s)^n} \mp \frac{1}{(2-s)^n} + \frac{1}{(2+s)^n} \mp \frac{1}{(3-s)^n} + \text{etc.},$$

ubi signa superiora valent, si n fuerit numerus impar, inferiora vero, si n sit numerus par.

21. Ad smmuas igitur has actu determinandas necesse est, ut valores differentialimn cuiuscunque ordinis quantitatum P et Q eruamus; quod ut facilius et succinctius fieri possit, ponamus

$$\sin \mathrm{A}\pi s = x \text{ et } \cos \mathrm{A}\pi s = y$$

eritque

$$P = \tfrac{\pi}{x} \text{ et } Q = \tfrac{\pi y}{x}.$$

Porro vero erit

$$dx = \pi y ds \text{ et } dy = -\pi x ds,$$

unde per regulas differentiationis sequentes obtinentur valores

$$\begin{aligned}
&P = \tfrac{\pi}{x},\\
&-\tfrac{dP}{ds} = \tfrac{\pi\pi}{xx}\cdot y,\\
&+\tfrac{ddP}{ds^2} = \tfrac{\pi^3}{x^3}\left(y^2+1\right),\\
&-\tfrac{d^3P}{ds^3} = \tfrac{\pi^4}{x^4}\left(y^3+5y\right),\\
&+\tfrac{d^4P}{ds^4} = \tfrac{\pi^5}{x^5}\left(y^4+18y^2+5\right),\\
&-\tfrac{d^5P}{ds^5} = \tfrac{\pi^6}{x^6}\left(y^5+58y^3+61y\right),\\
&+\tfrac{d^6P}{ds^6} = \tfrac{\pi^7}{x^7}\left(y^6+179y^4+479y^2+61\right),\\
&-\tfrac{d^7P}{ds^7} = \tfrac{\pi^8}{x^8}\left(\begin{array}{llll} & +6.1 & +4.179 & +2.479 \\ y^7 & y^5 & y^3 & y \\ & +3.179 & +5.479 & +7.61 \end{array}\right)\\
&\qquad\qquad\qquad\text{etc.},
\end{aligned}$$

ex quarum ultima expressione simul lex constat, cuius ope ex differentiali cuiusvis ordinis differentiale sequentis ordinis formari potest. Hincque istius seriei

$$\tfrac{1}{s^n} \pm \tfrac{1}{(1-s)^n} - \tfrac{1}{(1+s)^n} \mp \tfrac{1}{(2-s)^n} + \tfrac{1}{(2+s)^n} \pm \tfrac{1}{(3-s)^n} - \text{etc.}$$

summa pro quovis exponentis n valore assignabitur.

22. Simili modo reperientur valores differentialium cuiuscunque ordinis alterius quantitatis Q eritque

$$+Q=\frac{\pi}{x}\cdot y,$$

$$-\frac{dQ}{ds}=\frac{\pi\pi}{xx}\cdot 1,$$

$$+\frac{ddQ}{ds^2}=\frac{\pi^3}{x^3}\cdot 2y,$$

$$-\frac{d^3Q}{ds^3}=\frac{\pi^4}{x^4}\left(4yy+2\right),$$

$$+\frac{d^4Q}{ds^4}=\frac{\pi^5}{x^5}\left(8y^3+16y\right),$$

$$-\frac{d^5Q}{ds^5}=\frac{\pi^6}{x^6}\left(16y^4+88y^3+16\right),$$

$$+\frac{d^6Q}{ds^6}=\frac{\pi^7}{x^7}\left(32y^5+416y^3+272y\right),$$

$$-\frac{d^7Q}{ds^7}=\frac{\pi^8}{x^8}\left(64y^6+1824y^4+2880y^2+272\right)$$

$$+\frac{d^8Q}{ds^8}=\frac{\pi^9}{x^9}\left(\begin{array}{llll} & +4.1824 & +6.2880 & +8.272 \\ 2\cdot 64y^7 & \quad y^5 & \quad y^3 & \quad y \\ & +6.64 & +4.1824 & +2.2880 \end{array}\right)$$

etc.,

Lex progressionis hic pariter patet, cuius ope, quousque libuerit, progredi licet; atque hinc exhiberi poterit summa omnis seriei potestatum, quae in hac forma continetur

$$\frac{1}{s^n}\pm\frac{1}{(1-s)^n}+\frac{1}{(1+s)^n}\mp\frac{1}{(2-s)^n}+\frac{1}{(2+s)^n}\mp\frac{1}{(3-s)^n}+\text{etc.}$$

In his autem seriebus non solum continentur omnes potestatum series, quas praecedens methodus suppeditaverat, sed insuper innumerabiles aliae. Quin etiam ipsa haec methodus ad plura alia praeclara eruenda idonea videtur.