An Addition to the Dissertation Concerning

An Infinite Number Of Curves

Of The Same Kind

[p.184]

Leonhard Euler.

§1.

In the above dissertation, in which I presented a method of finding the equations for an infinite number of curves of the same kind, I have shown how to determine the value of Q in the equation

$$dz = Pdx + Qda \,,$$

from the given equation

$$z = \int Pdx.$$

In as much as P is composed in some manner from x and a with some constants; it is clear that if $\int Pdx$ is differentiated, on putting not only x but also a to be a variable, then an equation of this form is produced:

$$dz = Pdx + Qda \,,$$

in which the value of Q depends on the known value of P. Clearly, I have shown that if the differential of P with x placed constant is Bda, then the differential of Q with a placed constant is Bdx, from which the dependence of Q on P is evident enough.

§2. Moreover with the value of Q found, the equation

$$dz = Pdx + Qda$$

symmetrically expresses the nature of the infinitude of the given curves, of which the individual members are contained separately by the equation

$$dz = Pdx \,,$$

and which now differ in turn by the diversity of the parameter or modulus a. [p.185] And on this account, the equation

$$dz = Pdx + Qda \,,$$

in which the modulus a is present as the magnitude of the variable, I have called the modular equation, following the most Cel. Hermann. [In the previous paper, a has been called the parameter, but here it seems appropriate to retain the word modulus. Such words suffer from overuse in different contexts, of course.]

§3. If Pdx is able to be integrated, or if all the given curves in order are algebraic, then the equation

$$z = \int Pdx$$

likewise is modular ; for since no differentials are present, the modulus [or parameter] a can be considered equally for both the variables x and z. But if moreover Pdx is unable to be integrated, then the equation of the modulus is also not algebraic, with the cases excepted in which

$$P = AX + BY + CZ \text{, etc.}$$

with the functions *A*, *B*, *C*, etc., of *a* present and constants, and with *X*, *Y*, *Z*, etc functions of *x* and constants only, with the modulus *a* not itself present. For even if the equation is made the differential

$$dz = Pdx ,$$

yet the equation of the algebraic counterpart of the modulus

$$z = A\int Xdx + B\int Ydx + C\int Zdx \text{ , etc}$$

is to be considered. [*i. e.* The variables *a* and *x* are separable.]

§4. Moreover, unless *P* has such a value as above, the equation of the modulus is either a first order differential equation, or it is of higher order. Indeed, the differential is of the first order if *Q* is either an algebraic quantity, or it involves the integral of *Pdx*; indeed in this case by substituting $\int Pdx$ in place of *z*, *Q* also carries the integration sign, so that thus a pure differential equation of the modulus is produced.

§5. Now in the above dissertation, *Q* is taken to have an algebraic value whenever *P* is such a function of *a* and *x*, in order that the number of the dimensions, which *a* and *x* agree upon, is everywhere the same and equal to – 1, [p.186] or whenever *Px* or *Pa* is a function of *a* and *x* of zero dimensions. Then also I have observed, whenever the letters *a* and *x* in *P* produce only a number of the same dimension everywhere, then *Q* depends on the integration of Pdx. From which, the discovery of modular equations may follow from such excellent aids, and that will aid the investigation, especially where perhaps other functions of *P* of this kind are given, which it may please to choose before others. Therefore I have decided to investigate these in the first place, since by a similar method such functions can be found.

§6. If *P* is a function of *a* and *x* of dimensions – 1, or *z* is a function of *a* and *x* of zero dimensions, it is to be shown that

$$Px + Qa = 0, \text{or } Q = -\frac{Px}{a} .$$

Therefore we may assume

$$Q = -\frac{Px}{a}$$

and we ask, what is the [corresponding] function *P* of *a* and *x*. But if

$$Q = -\frac{Px}{a} ,$$

then

$$dz = Pdx - \frac{Pxda}{a} .$$

On account of which, *P* must be such a function of *a* and *x* that it becomes integrable on multiplying it by $dx - \frac{xda}{a}$. Moreover I understand that not only can this [function *z*] be restored by integration to become an algebraic quantity, but also through some form of quadrature. If therefore we can find generally some quantity, which multiplied by $dx - \frac{xda}{a}$ makes it become integrable, then the sought value of this property of *P* is such that

$$Q = -\frac{Px}{a} .$$

[This is easily shown for a trial function of zero dimensions, such as $z = \frac{x^n}{a^n}$. Euler introduces a general algebraic integrable function *f*, using his function notation, that satisfies $\frac{\partial P}{\partial a} = \frac{dQ}{\partial x}$, etc., in the following section.]

§7. Moreover, $dx - \frac{xda}{a}$ becomes integrable if it is multiplied by $\frac{1}{a}$, for the integral is $\frac{x}{a} + c$, with *c* designating some constant quantity that does not depend on *a*. On account of which, if $f(\frac{x}{a} + c)$ denotes some function [p.187] of $\frac{x}{a} + c$, it also makes $dx - \frac{xda}{a}$ integrable, if it is multiplied by $\frac{1}{a} f(\frac{x}{a} + c)$. Which value is $P = \frac{1}{a} f(\frac{x}{a} + c)$, when it is made the most general, and $Q = -\frac{Px}{a}$. Now $f(\frac{x}{a} + c)$ is some function of *a* and *x* of zero dimensions. [Note well.] On account of which whenever *Pa* should be a function of zero dimensions of *a* and *x*, so that

$$Q = -\frac{Px}{a},$$

and thus the equation of the modulus is

$$dz = Pdx - \frac{Pxda}{a}.$$

§8. Let

$$Q = -\frac{Px}{a},$$

and let *A* be some function of *a* and constants; then let

$$dz = Pdx + Ada - \frac{Pxda}{a}, \text{or } dz - Ada = Pdx - \frac{Pxda}{a}.$$

In which equation, since $dz - Ada$ is integrable, also $Pdx - \frac{Pxda}{a}$ must be integrable. Moreover, by the preceding operation, this comes about if

$$P = \frac{1}{a} f(\frac{x}{a} + c).$$

Therefore it is then the case that

$$Q = A - \frac{x}{a^2} f(\frac{x}{a} + c).$$

By like reasoning it is understood that if

$$P = X + \frac{x}{a} f(\frac{x}{a} + c),$$

with *X* denoting some function of *x* only, to be

$$Q = A - \frac{x}{a^2} f(\frac{x}{a} + c),$$

where as before $f(\frac{x}{a} + c)$ expresses some function of *a* and *x* of zero dimensions.

§9. If

$$Q = -\frac{nPx}{a},$$

where *n* indicates some number; then

$$dz = Pdx - \frac{nPxda}{a}.$$

Hence *P* must be such a quantity, which if it is multiplied by $dx - \frac{nxda}{a}$, is returned integrable. But $dx - \frac{nxda}{a}$ becomes integrable, if it is multiplied by $\frac{1}{a^n}$, with the integral $\frac{x}{a^n}$. Whereby generally, it is the case that

$$P = \frac{1}{a^n} f(\frac{x}{a^n} + c).$$

And whenever *P* has such a value, [p.188]

$$Q = -\frac{nx}{a^{n+1}} f(\frac{x}{a^n} + c).$$

Also it is understood that if more generally

$$P = X + \frac{1}{a^n} f(\frac{x}{a^n} + c),$$

then

$$Q = A - \frac{nx}{a^{n+1}} f(\frac{x}{a^n} + c).$$

Where as before and in what follows *f* always denotes a function of any quantity that follows But *A* is some function of *a*, and *X* some function of *x* only.
[Thus, *fx* means an algebraic function of the *x* following, or with the brackets, *f(x)* means a function of several things *x*, and both notations are used ; thus, with this briefest of introductions, Euler presents one of the great advances in mathematical notation, although he does not quite mean what we now mean by *f(x)*.]

§10. Therefore where it can be discerned, or where some given value of *P* is contained in a formula found, it is necessary to put *a* equal to $b^{\frac{1}{n}}$, with which done it is evident that *Pb* becomes a function of *b* and *x* of zero dimensions, or that it produces a sum from a certain function of *x* only; for such a function, if taken, *P* has the required property, and *Q* is equal to this function taken by $-\frac{nx}{a}$ with some function of *A*.
[Thus, on making this substitution, we return to the original conditions of §6]
Moreover in general, it is observed that the quantity *P* can be augmented by a function of *x* as *X* , and similarly *Q* by a function of *a* as *A*. For if the modular equation is :

$$dz = Pdx + Qda$$

such also is the equation :

$$dz = Pdx + Xdx + Qda + Ada.$$

For on placing *du* in place of $dz - Xdx - Ada$ there is obtained :

$$du = Pdx + Qda,$$

which evidently agrees with the previous. On account of which in what follows it may be superfluous to add a function *A* of *a* to the value of the *Q* taken. Whereby we will ignore this apparent generality. [p.189]

§11. Now let

$$Q = PE$$

with *E* denoting some function of *a*. Thus

$$dz = Pdx + PEda$$

and let *P* be such a quantity, which returns $dx + Eda$ integrable. But if *P* = 1 shall be integrable with this differential, the integral is indeed $x + \int Eda$. On account of which,

$$P = f(x + \int Eda),$$

and

$$Q = Ef(x + \int Eda).$$

Or if on putting

$$\int Eda = A,$$

then

$$P = f(x + A)$$

and

$$Q = \frac{dA}{da} f(x + A).$$

Moreover, whether or not the given value of *P* is contained in this formula, has to be investigated in this way, putting

$$x = y - A,$$

and for *A* such a function of *a* and constants is sought, that makes *P* a function of y only and of constants, as the modulus *a* is no longer present.

§12. We put $Q = PY$, where *Y* is some function of *x* not involving the modulus *a*. When this put in place, the [modular] equation becomes

$$dz = Pdx + PYda,$$

and *P* is such a function that makes $dx + Yda$ integrable. Moreover on putting

$$P = \frac{1}{Y},$$

then

$$z = \int \frac{dx}{Y} + a = X + a,$$

if we put

$$\int \frac{dx}{Y} = X.$$

On account of which

$$P = \frac{1}{Y} f(X + a).$$

Hence whenever *P* has a value of this kind, then $Q = f(X + a)$ always.

§13. If now more generally we put

$$Q = PEY$$

then

$$dz = Pdx + PEYda,$$

where as before *E* denotes a function of *a*, and *Y* now of *x*. It is evident, if we put

$$P = \frac{1}{Y},$$

that the formula makes that differential integrable, and indeed it gives :

$$z = \int \frac{dx}{Y} + \int Eda,$$

or

$$z = X + A \text{, on putting } \int \frac{dx}{Y} = X.$$

On account of which,

$$P = \frac{1}{Y} f(X + A) = \frac{dX}{dx} f(X + A)$$

and in these cases we have

$$Q = \frac{dX}{da} f(X + A).$$

In these formulas logarithmic values of *A* et *X* can also be taken, as thus if $X = lT$ and $A = -lF$, then $P = \frac{dT}{Tdx} f \frac{T}{F}$ and $Q = \frac{-dF}{Fda} f \frac{T}{F}$. [p.190]

§14. Therefore all these formulas are observed to be in place, if the proposed equation is either of the form

$$dz = dXf(X + A)$$

or

$$dz = \frac{dX}{X} f \frac{X}{A}.$$

Hence whenever the proposed equation can be reduced to these forms, on substituting *X* for any function of *x* and *A* for any function of *a*, the equation of the modulus can be shown : for in the first case it is given by

$$dz = dXf(X + A) + dAf(X + A),$$

and in the second case by

$$dz = \frac{dX}{X} f \frac{X}{A} - \frac{dA}{A} f \frac{X}{A}.$$

That which is easily observed in these more general examples, is much more difficult in more specialised examples. On account of which the maximum effort is put into the reduction of special cases to these general forms, that can then be set out without difficulty, if indeed such a reduction can be made.

§15. Let $Q = PR$ be put in place, with R designating some function of *a* and *x* , then the [modular equation] becomes

$$dz = Pdx + PRda.$$

Now on finding the value of *P*, the formula *dx* + *Rda* is taken, or the equation :

$$dx - Rda = 0$$

is considered, and how the indeterminates *a* and *x* can be separated from each other is sought, or what is the same thing, by what quantity *dx* + *Rda* must be multiplied in order that it becomes integrable. Let this quantity be *S* and with *T* the integral of $Sdx + RSda$ it becomes $P = SfT$. And in these cases $Q = RSfT$. This operation appears the widest and embraces all cases, in which *Q* is known, and does not have a value depending on *z*.

§16. Moreover we can progress further and we examine those values of *P*, [p.191] in which *Q* not only depends on *P* but also depends on $\int Pdx$ or *z*. Therefore first there is placed $Q = \frac{nz}{a} - \frac{Px}{a}$, and with *n* denoting some number. Therefore the equation becomes :

$$dz = Pdx + \frac{nzda}{a} - \frac{Pxda}{a}, \text{ or } dz - \frac{nzda}{a} = Pdx - \frac{Pxda}{a}.$$

Each is multiplied by $\frac{1}{a^n}$, which produces this equation :

$$\frac{dz}{a^n} - \frac{nzda}{a^{n+1}} = P\frac{dx}{a^n} - \frac{Pxda}{a^{n+1}},$$

in which the first part is integrable. Therefore also the second part

$$P\frac{dx}{a^n} - \frac{Pxda}{a^{n+1}},$$

must be integrable, from which a suitable value is sought for the value of P. It comes about that if $P = a^{n-1}$, for which the integral is $\frac{x}{a} + c$. Whereby generally,

$$P = a^{n-1} f(\tfrac{x}{a} + c),$$

which happens if $\frac{P}{a^{n-1}}$ is a function of a and x of zero dimensions, or P a function of a and x of dimension $n - 1$. Therefore in this case,

$$nz = Px + Qa$$

as we have shown in the above dissertation [E44].

§17. Let

$$Q = \frac{nz}{a} + PEY,$$

where some E has been composed from a, and some Y from x. Thus the equation becomes :

$$dz - \frac{nzda}{a} = Pdx + PEYda,$$

and

$$\frac{dz}{a^n} - \frac{nzda}{a^{n+1}} = \frac{Pdx}{a^n} + \frac{PEYda}{a^n}$$

On account of which P must thus be adapted, so that $\frac{dx+EYda}{a^n}$ multiplied by P becomes integrable. Moreover this is done if $P = \frac{a^n}{Y}$, in which case the integral is

$$\int \frac{dx}{Y} + \int Eda \text{ or } X + A$$

on putting

$$\int \frac{dx}{Y} = X \text{ and } \int Eda = A. \text{ [p.192]}$$

Whereby it becomes :

$$P = \frac{a^n dX}{dx} f(X + A),$$

and in these cases Q becomes :

$$Q = \frac{a^n dA}{da} f(X + A) + \frac{nz}{a}.$$

If X and A depend on logarithms, then this value of P is produced:

$$\frac{a^n dX}{Xdx} f \frac{X}{A}.$$

to which there corresponds

$$Q = \frac{nz}{a} - \frac{a^n dA}{Ada} f \frac{X}{A}.$$

§18. If we put

$$Q = Fz + PEY,$$

both F and E are functions of a, Y now is a function of x. Then the modular equation becomes :

$$dz - Fzda = Pdx + PEYda.$$

On putting

$$\int Fda = lB,$$

thus in order that B is a function of a, and the above equation is divided by B, there is obtained :

$$\frac{dz}{B}-\frac{zdB}{B^2}=\frac{Pdx}{B}+\frac{PEYda}{B}.$$

Therefore since the first part is integrable, then the second part must be made integrable. This is done if $P=\frac{B}{Y}$ and then the integral is

$$\int\frac{dx}{Y}+\int Eda \text{ or } X+A.$$

On this account the value of P sought is

$$\frac{BdX}{dx}f(X+A),$$

and Q now is

$$\frac{zdB}{Bda}+\frac{BdA}{da}f(X+A).$$

It is also observed that if

$$P=\frac{BdX}{Xdx}f\frac{X}{A}$$

that Q becomes :

$$Q=\frac{zdB}{Bda}-\frac{BdA}{Ada}f\frac{X}{A}.$$

§19. It is clear that the widest solution is

$$Q=Fz+PR$$

and R is a function of a and x. For the [modular] equation is :

$$dz-Fzda=Pdx+PRda.$$

On putting

$$\int Fda=lB$$

on dividing by B, there is obtained :

$$\frac{dz}{B}-\frac{zdB}{B^2}=\frac{P}{B}(dx+Rda).$$

Now let S be the function making $dx + Rda$ integrable and let

$$\int(Sdx+SRda)=T.$$

From which it is found that

$$P=BSfT$$

to which there corresponds :

$$Q=\frac{zdB}{Bda}+BRSfT.$$

§20. Besides many values of P of this kind can be joined together, and in this way extended much wider, as if we place [p.193]

$$P=\frac{BdX}{dx}f(\mathrm{X}+\mathrm{A})+\frac{BdY}{dx}f(\mathrm{Y}+\mathrm{E})$$

then

$$Q=\frac{zdB}{Bda}+\frac{BdA}{da}f(\mathrm{X}+\mathrm{A})+\frac{BdE}{da}f(\mathrm{Y}+\mathrm{E}).$$

And in a like manner the number of terms can be increased as much as you please. Therefore in all these cases a differential equation is found of the first order of the modulus. On account of which, from these put in place, I go on to these cases to be investigated, in which a differential equation of the modulus of the first order is not given, but which yet are leading to difference of the differential modular equation. [That is, which lead to the consideration of second degree differential equations.]

§21. If therefore Q cannot be expressed algebraically either by a and x or by z, these are cases to be investigated in which the differential of Q can be produced. Moreover,

$$Q = \frac{dz-Pdx}{da},$$

hence

$$dQ = d\,\frac{dz-Pdx}{da}.$$

Whereby if the differential of Q can be expressed either with respect to a or x alone, and likewise also by z, a modular equation is obtained, which is of the second order. . Moreover it has been shown in the above dissertation [E44] that if we put

$$dP = Ldx + Mda,$$

then it is to be

$$dQ = Mdx + Nda$$

thus, so that these differentials involve the common letter M. Moreover since from the given P, M is also given, then nothing else is required, except that N should be found. On account of which in these we examine the case, in which N can be expressed either algebraically, or can be expressed in terms of Q, or by Q and z. For then the modular equation is obtained :

$$Mdx + Nda = d\,\frac{dz-Pdx}{da},$$

on putting N in place of Q with the value $\frac{dz-Pdx}{da}$.

§22. It is understood clearly enough from the preceding, that if N is determined only by a and x, [p.194] then

$$M = \frac{dX}{dx} f(\mathrm{X} + \mathrm{A}) \text{ and } N = \frac{dA}{da} f(X + A),$$

or

$$M = V + \frac{dX}{dx} f(X + A) \text{ and } N = 1 + \frac{dA}{da} f(X + A)$$

with V denoting some function of x and I some function of a. Thus from the given P M is sought, by differentiation of P with x constant, and the differential found to be divided by da . Which accomplished, or the value of M contained in the formula $V + \frac{dX}{dz} f(X + A)$ is sought, then

$$Vdx + dXf(X + A) + Ida + dAf(X + A) = d\,\frac{dz-Pdx}{da}$$

is the required equation of the modulus. It is to be observed in the following that it is possible to always put in place of $\frac{dX}{dx} f(X + A)$ some of however many of these formulas :

$$\frac{dX}{dx} f(X + A) + \frac{dY}{dx} f(Y + B) + \text{etc.}$$

But in place of $\frac{dA}{da} f(X + A)$ one must put in place $\frac{dA}{da} f(X + A) + \frac{dB}{da} f(Y + B)$ etc. Therefore with this reminder in the following we will use only a single formula $\frac{dX}{dx} f(X + A)$ and that corresponding $\frac{dA}{da} f(X + A)$.

§23. Likewise N can also depend on Q and it becomes $N = R + DQ$, where D is a function of a, and R a function of a and x to be determined from the following conditions. Therefore the equation becomes :

$$dQ - DQdq = Mdx + Rda,$$

let $Dda = \frac{dH}{H}$ and on both being divided by H there is produced:

$$\frac{dQ}{H} - \frac{QdH}{H^2} = \frac{Mdx+Rda}{H}.$$

In which equation, since the one member is integrable, from such also this expression $\frac{Mdx+Rda}{H}$ can be effected. Therefore by the preceding method, let

$$M = \frac{HdX}{dx} f(X + A) \text{ and } R = \frac{HdA}{da} f(X + A).$$

Whereby if in some example proposed, from P M is found there of such a value, then

$$N = \frac{HdA}{dx} f(X + A) + \frac{dH}{Hda^2}(dz - Pdx)$$

on putting $\frac{dH}{Hda}$ in place of Q. And thus the modular equation is made ready. [p.195]

§24. But if N does not depend on Q but on z, thus so that

$$N = R + Cz,$$

with C some function of a ; then

$$dQ - Czda = Mdx + Rda.$$

But since

$$dz - Qda = Pdx,$$

a multiple

$$Fdz - QFda = PFdx$$

of this is added, with F becoming a function of a, with which done there arises the equation

$$dQ - QFda + Fdz - Czda = (M + PF)dx + Rda.$$

Putting

$$Fda = \frac{dB}{B} \text{ and } \frac{Cda}{F} = \frac{dG}{G},$$

thus so that we make

$$F = \frac{dB}{Bda} \text{ and } C = \frac{dBdG}{BGda^2}.$$

Thus it is evident that $dQ - QFda$ is returned integrable if it is divided by B or it is multiplied by $\frac{1}{B}$, and moreover $Fdz - Czda$ is made integrable, if it is multiplied by $\frac{1}{FG}$. Whereby when the same factor returns the integrable sum of the differentials it must be $FG = B$ or $\frac{GdB}{Bda} = B$, hence this becomes $G = \frac{B^2da}{dB}$. On this account the other member divided by B can also be made integrable, clearly $\frac{(M+PF)dx+Rda}{B}$. On account of which I put

$$R = \frac{BdA}{da} f(X + A) \text{ and } M + PF = \frac{BdX}{dx} f(X + \text{A}) = M + \frac{PdB}{Bda}.$$

The proposition must be examined by an example, or in place of A, B, and X are found such functions which show the form $\frac{PdX}{dx} f(X + \text{A})$ equal to $M + \frac{PdB}{Bda}$. And from these found, it follows that

$$N = \frac{BdA}{da} f(X + A) + \frac{xdBdG}{BGda^2}$$

from which it follows that :

$$G = \frac{B^2da}{dB},$$

which value substituted in the equation

$$Mdx + Nda = d.\frac{dz-Pdx}{da}$$

gives the modular equation.

§25. Now most generally, let

$$N = R + DQ + Cz,$$

with *R, D* and *C* keeping the same values as before. Therefore the equation becomes :

$$dQ - DQda - Czda = Mdx + Rda,$$

and there is added to that the equation :

$$Fdz - FQda = PFdx,$$

from which there is obtained :

$$dQ - DQda - FQda + Fdz - Czda = (\mathrm{M} + \mathrm{PF})dx + Rda. \text{ [p.196]}$$

Moreover, on putting in place as before:

$$Dda = \frac{dH}{H}, Fda = \frac{dB}{B}, \text{and } \frac{Cda}{F} = \frac{dG}{G},$$

$dQ - DQda - FQda$ is made integrable if it is multiplied by $\frac{1}{HB}$,

and $Fdz - Czda$ becomes integrable on multiplying by $\frac{1}{FG}$. Whereby we have :

$$HB = FG = \frac{GdB}{Bda} \text{ and } G = \frac{B^2Hda}{dB}.$$

And $\frac{(M+PF)dx+Rda}{HB}$ on being returned is integrable : hence on putting

$$HB = E, R = \frac{EdA}{da} f(X + A) \text{ and } M + PF = \frac{EdX}{dx} f(X + A).$$

On account of which in the proposed case, if *A, X, E,* and *F* if possible must thus be defined so that $\frac{EdX}{dx} f(X + A)$ becomes equal to $M + PF$. And from this is found

$$\frac{EdA}{da} f(X + A) + \frac{dH}{Hda^2}(\mathrm{dz} - Pdx) + \frac{FzdG}{Gda},$$

and hence the modular equation is found.

§26. But if no particular equation of the second order of the modulus can be found, it is necessary to proceed to the third order. Therefore make

$$N = \frac{d(\frac{dz-Pdx}{da})-Mdx}{da}$$

and hence on putting

$$d\mathrm{N} = sdx + tda,$$

the equation becomes

$$sdx + tda = \mathrm{d}\left(\frac{d(\frac{dz-Pdx}{da})-Mdx}{da}\right).$$

Moreover *s* is found from *M*, since *sda* is the differential of *M* that is produced if *x* is placed constant. On account of which only *t* has to be investigated. Therefore let :

$$t = R + EN + DQ + Cz,$$

and thus

$$dN - ENda - DQda - Czda = sdx + Rda.$$

Moreover since

$$dQ - Nda = Mdx \text{ and } dz - Qda = Pdx,$$

a multiple of this is added to that equation, in order to produce this equation :

$dN - ENda - FNda + FdQ - DQda - GQda + Gdz - Czda = (s + MF + PG)dx + Rda.$
Let

$$Eda + Fda = \frac{df}{f}, \frac{Dda+Gda}{F} = \frac{dg}{g} \text{ and } \frac{Cda}{G} = \frac{dh}{h},$$

and we have :

$$f = Fg = Gh. \text{ [p.197]}$$

With which completed, the first part of the equation found divided by *f* is integrable, and on account of which being effected , $\frac{(s+MF+PG)dx+Rda}{f}$ is integrable. Therefore we are to put in place :

$$R = \frac{fdA}{da} f(X + A) \text{ and } s + MF + PG = \frac{fdX}{da} f(X + A).$$

Therefore in the proposed equation, since *s* and *M* are given from *P* , *F, G* and both *f* and *X* are to be determined from this equation. With which done, there is taken

$$g = \frac{f}{F}, h = \frac{f}{G}, C = \frac{Gdh}{hda}, D = \frac{Fdg}{gda} - G, \text{ and } E = \frac{df}{fda} - F.$$

And with these known the equation is given:
$t = R + EN + DQ + Cz,$
from which the modular equation can be easily put in place. In a similar manner from these it is also understood that for higher orders of differentials the operation must be put in place, so that the modular equations can be arrived at.

§27. In summary, what we have set out can now be reviewed up to this point, so that whatever the equation proposed, it can be reduced more easily, and in order that the progress at each order of differentiation is then seen to be more clear. Therefore, from the proposed equation $dz = Pdx$, *x* is put constant and yet from the variable *a*, there is:

$$dP = Mda,\ dM = pda, dp = rda, \text{ etc.}$$

Again let

$$Q = \frac{dz-Pdx}{da}, N = \frac{dQ-Mdx}{da}, q = \frac{dN-pdx}{da} \text{ and } s = \frac{dq-rdx}{da} \text{ etc.}$$

Where *dQ, dN, dq*, etc are differentials of *Q, N*, and *q*, which are found from the values

$$\frac{dz-Pdx}{da}, \frac{dQ-Mdx}{da}, \text{and } \frac{dq-rdx}{da}$$

with the variables *a, x* and *z* put in place. Therefore on this account, *M, p, r* etc are known from *P* alone, and now from these values we have *Q, N, q* etc. Besides, let *A, B, C, D, E, F* etc be functions of *a* and of constants, and *X, Y* etc. are functions of *x* not involving *a*. [p.198]

§28. From these premises, if *P* is such a function of *x* and *a* so that *BP* is taken in this form $\frac{dX}{dx} f(X + A)$, or a sum of several formula of this kind, then a modular equation of the first order can always be given. For it shall be :

$$PdAdx = z\frac{dBdX}{B} + QdadX$$

or

$$BPdAdx = zdBdX + BQdadX.$$

Which equation on account of the given *Q* is of the modulus corresponding to the proposed equation.

§29. Then if P is such a function of a and x in order that $BP + CM$ can be made equal to $\frac{dX}{dx} f(X + A)$, or some sum of formulas of this kind, then the modular equation rises to a differential equation of the second order. For it is :

$$BPdAdx + CMdAdx = zdBdX + BQdadX + QdCdX + CNdadX.$$

Which is the equation of the modulus sought, and it involves a second order differential equation, which that letter N introduces, which is determined by dQ and thus by ddz, ddx, and dda.

§30. But if $BP + CM + Dp$ should be equal to this formulae : $\frac{dX}{dx} f(X + A)$, or to some sum of formulas of this kind ; the modular equation is a differential equation of the third order, indeed it is produced by this equation :

$$BPdAdx + CMdAdx + DpdAdx = zdBdX + BQdadX + QdCdX + CNdadX + NdDdX + DqdadX.$$

As can be gathered from the previous treatment, only quantities depending on a can be applied to these formulas.

§31. In a similar manner the progression to higher differential can be completed. [p.199] For if $BP + CM + Dp + Er$ is equal to the formula $\frac{dX}{dx} f(X + A)$, or to a sum of several of these formulas, this modular equation arises :

$$\begin{gathered}BPdAdx + CMdAdx + DpdAdx + ErdAdx = zdBdX + BQdadX + QdCdX + \\ CNdadX + NdDdX + DqdadX + qdEdX + EsdadX\end{gathered}$$

which is a differential of the fourth order. And in this manner as far as it is wished, these operations can be easily continued, produced only from inspection.

§32. Moreover from all these examined, the maximum difficulty yet placed in many occasions is in finding the function P, or contained in the general expositions and arising somewhere. Also if indeed the general values of P, which are seen to be obtained without difficulty from assumed formulas, but yet in particular examples proposed, often can only be fitted with difficulty. The reason for this is in no manner to be attributed to the method of treatment, but to imperfect knowledge of the functions, which is thus had. On this account not only in this treatment, but in several other cases it would be of maximum usefulness if the teaching of functions were carried out better and improved.

§33. Indeed however much has been made clear to me by thinking about this matter, I have found that the best help is, if P is at once reduced to a form of this kind $\frac{dX}{dx} f(X + A)$ or a sum of forms of this kind, that is better to proceed in the following manner. First the proposed equation is not established between z and x but between z and y, thus in order that the modular equation is to be lead by $dz = Tdy$, [p.200] with T present some function of y and of the modulus a. Then such a function is taken for x of a and y, which changes T into a function of a and x held within the formula $f(X + A)$, or several similar of this, and from the multiples of these, in which X is a function of x only, and A of a. Therefore with this accomplished, the equation is produced :

$$dz = Sdx\ f(X + A)$$

where S is a quantity as simple as it is possible to produce. Whereby P is equal to $S\ f(X + A)$ and thus as with the adjoining M, p etc., so it can be compared with the more general formulas. Moreover by finding the modular equation in this way, the value of x is taken every in terms of a and y, and everywhere in place of x ; moreover in place of dx the differential of this value with the variables a and y is substituted. With which accomplished, the modular equation is obtained between a, y, and z, which was sought.

§34. Indeed examples and problems bring the most light to a fuller understanding of what we have just treated, the solution of which require the same method. But since dignity demands the proper treatment of problems, I must defer that to another time, as I may not now have the long time required.

ADDITAMENTUM AD DISSERTATIONEM DE INFINITIS CURVIS EIUSDEM GENERIS.

[p.184]

AUCTORE

Leonh. Eulero.

§1.

In superiore dissertatione, in qua methodum tradidi aequationem pro infinitis curvis eiusdem generis inveniendi, ipsius Q valorem in aequatione $dz = Pdx + Qda$ determinare docui, ex data aequatione $z = \int Pdx$. Namque si P ex x, et a cum constantibus utcunque fuerit compositum; manifestum est si $\int Pdx$ differentietur posito non solum x sed etiam a variabili, prodituram esse huius formae aequationem $dz = Pdx + Qda$, in qua valor ipsius Q necessario a quantitate P, quae est cognita, pendebit. Demonstravi scilicet, si differentiale ipsius P posito x constante fuerit Bda, fore ipsius Q differentiale posito a constante, Bdx, ex quo pendentia ipsius Q a P satis perspicitur.

§2. Cum autem inventus fuerit valor ipsius Q, aequatio $dz = Pdx + Qda$ expremet naturam infinitarum curvarum ordinatim datarum, quarum singulare seorsim continetur aequatione $dz = Pdx$, a se invicem vero differunt diversitate parametri seu moduli a. [p.185] Et hanc ob rem aequationem $dz = Pdx + Qda$ in qua modulus a tanquam quantitas variabilis inest, cum Cel. Hermanno aequationem modularem vocavi.

§3. Si Pdx integratione admittit, seu si curvae ordinatim datae omnes sunt algebraicae aequatio $z = \int Pdx$ simul erit modularis; nam quia nulla adsunt differentialia, modulus a aeque variabilis x et z poterit considerari. Sin autem Pdx integrari nequit, aequatio etiam modularis non erit algebraica, exceptis casibus quibus est $P = AX + BY + CZ$ etc. existenibus *A*, *B*, *C*, etc functionibus ipsius *a* et constantium, atque *X*, *Y*, *Z*, etc functionibus ipsius *x* et constantium tantum, modulo *a* ipsas non ingrediente. Etiamsi enim ipsa aequatio $dz = Pdx$ sit differentialis, tamen aequatio modularis $z = A\int Xdx + B\int Ydx + C\int Zdx$, etc instar algebraicae est consideranda.

§4. Nisi autem *P* talem habuerit valorem aequatio modularis vel erit differentialis gradus primi vel alterioris gradus. Differentialis quidem primi gradus erit, si *Q* vel erit quantitas algebraica, vel integrale ipsius *Pdx* involuet, hoc enim casu z loco $\int Pdx$ substitutum tollet quoque signum summatorium, ita ut aequatio modularis differentialis pura sit proditura.

§5. Deprehendi vero in superiore dissertatione, *Q* toties algebraicum habuerit valorem quoties *P* talis fuerit ipsarum *a* et *x* functio, ut numerus dimensionum, quas *a* et *x* constituunt sit ubique idem atque – 1, [p.186] seu quoties *Px* vel *Pa* fuerit functio ipsarum *a* et *x* nullius dimensionis. Deinde etiam observavi, quoties in *P* litterae *a* et *x* eundem tantum ubique constituant dimensionum numerum, toties *Q* ab integratione ipsius *Pdx* pendere. Ex quo, cum tam eximia consequantur subsidia ad aequationes modulares inveniendas, maxime iuvabit investigare, num forte aliae dentur huiusmodi functiones ipsius *P*, quae iisdem praerogativis gaudeant. Has igitur a priore investigare constitui, quo simul methodus tales functiones inveniendi aperiatur.

§6. Si *P* est functio ipsarum *a* et *x* dimensionum – 1, seu *z* functio ipsarum *a* et *x* nullius dimensionis, ostendi fore $Px + Qa = 0,$ seu $Q = -\frac{Px}{a}$. Sumamus igitur esse $Q = -\frac{Px}{a}$ et quaeramus, qualis fit *P* functio ipsarum a et x. At si $Q = -\frac{Px}{a}$ erit $dz = Pdx - \frac{Pxda}{a}$. Quamobrem *P* talis esse debebit functio ipsarum *a* et *x*, ut $dx - \frac{xda}{a}$ per eam multiplicatum evadat integrabile. Hic autem per integrabile non solum intelligo, quod integratione ad quantitatem algebraicam, sed etiam quod ad quadraturum quamcunque reducitur. Si igitur generaliter invenerimus quantitatem, in quam $dx - \frac{xda}{a}$ ductum fit integrabile, ea erit quaesitus valor ipsius P, eius proprietatis, ut sit $Q = -\frac{Px}{a}$.

§7. Fit autem $dx - \frac{xda}{a}$ integrabile si multiplicatur per $\frac{1}{a}$, integrale enim erit $\frac{x}{a} + c$, designante *c* quantitatem constantem quamcunque ab *a* non pendentem. Quocirca, si $f(\frac{x}{a} + c)$ denotat functionem [p.187] quamcunque ipsius $\frac{x}{a} + c$, fiet quoque $dx - \frac{xda}{a}$ integrabile, si multiplicetur per $\frac{1}{a} f(\frac{x}{a} + c)$. Qui valor cum sit maxime generalis, erit $P = \frac{1}{a} f(\frac{x}{a} + c)$, et $Q = -\frac{Px}{a}$. Est vero $f(\frac{x}{a} + c)$ functio quaecunque ipsarum *a* et *x* nullius dimensionis. Quamobrem quoties *Pa* fuerit functio nullius dimensionis ipsarum *a* et *x*, toties erit $Q = -\frac{Px}{a}$, ideoque aequatio modularis $dz = Pdx - \frac{Pxda}{a}$.

§8. Sit $Q=-\frac{Px}{a}$, et A functio quaecunque ipsius a et constantium; erit $dz=Pdx+Ada-\frac{Pxda}{a}$ seu $dz-Ada=Pdx-\frac{Pxda}{a}$. In qua aequatione cum $dz-Ada$ sit integrabile, debebit $Pdx-\frac{Pxda}{a}$ quoque esse integrabile. Hoc autem per praecedentem operationem evenit si $P=\frac{1}{a}f(\frac{x}{a}+c)$. Tum igitur erit $Q=A-\frac{x}{a^2}f(\frac{x}{a}+c)$. Simul ratione intelligitur si fuerit $P=X+\frac{x}{a}f(\frac{x}{a}+c)$, denotante X functionem ipsius x tantum, fore $Q=A-\frac{x}{a^2}f(\frac{x}{a}+c)$, ubi ut ante $f(\frac{x}{a}+c)$ exprimit functionem quamcunque ipsarum a et x nullius dimensionis.

§9. Si $Q=-\frac{nPx}{a}$, ubi n indicet numerum quemcunque; erit $dz=Pdx-\frac{nPxda}{a}$. Debebit ergo P talis esse quantitas, quae $dx-\frac{nxda}{a}$ si in id multiplicetur, reddat integrabile. Fit autem $dx-\frac{nxda}{a}$ integrabile, si ducatur in $\frac{1}{a^n}$, integrale enim erit $\frac{x}{a^n}$. Quare generaliter erit $P=\frac{1}{a^n}f(\frac{x}{a^n}+c)$. Atque quoties P talem habuerit valorem erit [p.188] $Q=-\frac{nx}{a^{n+1}}f(\frac{x}{a^n}+c)$. Intelligitur etiam si furerit $P=X+\frac{1}{a^n}f(\frac{x}{a^n}+c)$, fore quoque generalius $Q=A-\frac{nx}{a^{n+1}}f(\frac{x}{a^n}+c)$. Ubi ut ante et in posterum semper f denotat functionem quamcunque quantitatis sequentis. At A est functio quaecunque ipsius a, et X functio quaecunque ipsius x tantum.

§10. Quo igitur dignosci queat, an datus quispiam valor ipsius P in formula inventa contineatur, poni debebit $a=b^{\frac{1}{n}}$, quo facto videndum est , an Pb fiat functio ipsarum b et x nullius dimensionis, vel an prodeat aggregatum ex functione quadam ipsius x tantum, et tali functione. Quod si deprenendetur, habebit P proprietatem requisitam, eritque Q aequale huic ipsi functioni in $-\frac{nx}{a}$ ductae una cum functione quacunque ipsius A. In universum autem notandum est quantitatum P functione ipsius x ut X , et Q functione ipsius a ut A posse augeri. Nam si fuerit $dz=Pdx+Qda$ aequatio modularis, talis quoque erit aequatio $dz=Pdx+Xdx+Qda+Ada$. Posito enim du loco $dz-Xdx-Ada$ habebitur $du=Pdx+Qda$, quae cum priore prorsus congruit. Hancobrem superfluum foret in posterum ad valorem ipsius Q assumtum, functionem A ipsius a adiicere. Quare hanc apparentem generalitatem negligemus. [p.189]

§11. Sit nunc $Q = PE$ denotante E functionem quamcunque ipsius a. Erit itaque $dz=Pdx+PEda$ et P talis quantitas, quae reddit $dx+Eda$ integrabile. At si $P = 1$ fit integrabile hoc differentiale, integrale enim erit $x+\int Eda$. Quamobrem erit $P=f(x+\int Eda)$ et $Q=Ef(x+\int Eda)$. Sive si ponatur $\int Eda=A$, fueritque $P=f(x+A)$ erit $Q=\frac{dA}{da}f(x+A)$. Num autem datus ipsius P valor in hac formula contineatur, hoc modo est investigandum, ponatur $x=y-A$, et quaeritur, an pro A talis accipi queat functio ipsius a et constantium, ut P fiat functio solius y et constantium, quam modulus a non amplius ingrediatur.

§12. Ponamus esse $Q = PY$, ubi Y sit functio quaecunque ipsius x modulum a non involvens. Quo posito erit $dz = Pdx + PYda$, et P talis functio quae efficiat $dx + Yda$ integrabile. Posito autem $P = \frac{1}{Y}$, sit $z = \int \frac{dx}{Y} + a = X + a$, si ponatur $\int \frac{dx}{Y} = X$. Quamobrem erit $P = \frac{1}{Y} f(X + a)$. Quoties ergo P huiusmodi habuerit valorem erit semper $Q = f(X + a)$.

§13. Si nunc generalius positum $Q = PEY$ erit $dz = Pdx + PEYda$, ubi ut ante E denotat functionem ipsius a, Y vero ipsius x. Perspicuum est, si fuerit $P = \frac{1}{Y}$ formulam istam differentialem effici integrabilem, prodiret enim $z = \int \frac{dx}{Y} + \int Eda$, seu $z = X + A$ posito $\int \frac{dx}{Y} = X$. Quamobrem erit $P = \frac{1}{Y} f(X + A) = \frac{\mathrm{dX}}{\mathrm{dx}} f(X + A)$ hisque in casibus fiet $Q = \frac{\mathrm{dX}}{\mathrm{da}} f(X + A)$. Comprehenduntur in his formulis etiam logarithmici ipsarum A et X valores, ut sit fit $X = lT$ et $A = -lF$, erit $P = \frac{dT}{Tdx} f \frac{T}{F}$ et $Q = \frac{-dF}{Fda} f \frac{T}{F}$. [p.190]

§14. Perspicitur igitur omnes has formulas locum habere, si aequatio proposita fuerit vel $dz = dXf(X + A)$ vel $dz = \frac{dX}{X} f \frac{X}{A}$. Quoties ergo aequatio proposita ad has formas poterit reduci, substitutuendis X pro functione quacunque poterit reduci, substituendis X pro functione quacunque ipsius x et A pro functione quacunque ipsius a, totius aequatio modularis poterit exhiberi: erit enim priore casu
$dz = dXf(X + A) + dAf(X + A)$ in posteriore vero casu $dz = \frac{dX}{X} f \frac{X}{A} - \frac{dA}{A} f \frac{X}{A}$. Id quod quidem in his universalibus exemplis facile perspicitur, in specialioribus vero multo difficilius. Quocirca maximum positum erit subsidium in reducendis casibus particularibus ad has generales formas, id quod, si quidem talis reductio fieri potest, non difficulter praestatur.

§15. Sit ponatur $Q = PR$, designante R functionem ipsarum a et x, erit
$dz = Pdx + PRda$. Ad inveniendum nunc valorem ipsius P, sumatur formula $dx + Rda$, seu aequatio $dx - Rda = 0$ consideretur, et quaeretur quomodo indeterminatae a et x a se invicem possint separari, seu quod idem est, per qamnam quantitatem $dx + Rda$ debeat multiplicari, ut fiat integrabilis. Sit haec quantitas S et ipsius $Sdx + RSda$ integrale T erit $P = SfT$. Hisque in casibus erit $Q = RSfT$. Haec operatio latissime patet et omnes casus complectitur, quibus Q cognitum et a z non pendentem habet valorem.

§16. Progrediamur autem ulterius et in eos ipsius P valores inquiramus, [p.191] in quibus Q non solum a P sed etiam a $\int Pdx$ seu a z pendet. Ponatur igitur primo
$Q = \frac{nz}{a} - \frac{Px}{a}$, denotante n numerum quemcunq;. Erit ergo $dz = Pdx + \frac{nzda}{a} - \frac{Pxda}{a}$, seu $dz - \frac{nzda}{a} = Pdx - \frac{Pxda}{a}$. Multiplicetur utrinque per $\frac{1}{a^n}$, quo prodeat haec aequatio $\frac{dz}{a^n} - \frac{nzda}{a^{n+1}} = P\frac{dx}{a^n} - \frac{Pxda}{a^{n+1}}$, in qua prius membrum est integrabile. Debebit ergo etiam

integrabile esse alterum membrum $P\frac{dx}{a^n} - \frac{Pxda}{a^{n+1}}$, ex quo idoneus ipseus *P* valor est quaerendus. Evenit hoc si $P = a^{n-1}$, erit enim integrale $\frac{x}{a} + c$. Quare erit universaliter $P = a^{n-1} f(\frac{x}{a} + c)$, id quod contingit si $\frac{P}{a^{n-1}}$ est functio ipsarum *a* et *x* nullius dimensionis seu *P* functio ipsarum *a* et *x* dimensionum *n* – 1. Hoc igitur casu est $nz = Px + Qa$ ut in superiore dissertatione ostendimus.

§17. Sit $Q = \frac{nz}{a} + PEY$, ubi *E* ex *a*, et *Y* ex *x* utcunque est compositum. Erit itaque $dz - \frac{nzda}{a} = Pdx + PEYda$, et $\frac{dz}{a^n} - \frac{nzda}{a^{n+1}} = \frac{Pdx}{a^n} + \frac{PEYda}{a^n}$. Quamobrem *P* ita debet accommodari, ut $\frac{dx+EYda}{a^n}$ per id multiplicatum evadat integrabile. Fit hoc autem si $P = \frac{a^n}{Y}$, quo casu integrale est $\int\frac{dx}{Y} + \int Eda$ seu X + A posito $\int\frac{dx}{Y} = X$ et $\int Eda = A$. [p.192] Quare debebit esse $P = \frac{a^n dX}{dx} f(X + A)$, et in his casibus erit $Q = \frac{a^n dA}{da} f(X + A) + \frac{nz}{a}$. Si *X* et *A* a logarithmis pendeant prodibit *P* huius valoris $\frac{a^n dX}{Xdx} f\,\frac{X}{A}$, cui respondit $Q = \frac{nz}{a} - \frac{a^n dA}{Ada} f\,\frac{X}{A}$.

§18. Si ponatur $Q = Fz + PEY$, et F et E functiones sint ipsius a, Y vero ipsius x. Tum erit $dz - FzdA = Pdx + PEYda$. Ponatur $\int Fda = lB$, ita ut *B* sit functio ipsius *a*, et dividitur per *B* habebitur $\frac{dz}{B} - \frac{zdB}{B^2} = \frac{Pdx}{B} + \frac{PEYda}{B}$. Cum igitur prius membrum sit integrabile, et alterum tale effici debet. Fit hoc si $P = \frac{B}{Y}$ tumque erit integrale $\int\frac{dx}{Y} + \int Eda$ seu *X* + *A*. Quocirca erit ipsius *P* valor quaesitus $\frac{BdX}{dx} f(X + A)$, Q vero erit $\frac{zdB}{Bda} + \frac{BdA}{da} f(X + A)$. Perspicitur quoque si fuerit $P = \frac{BdX}{Xdx} f\,\frac{X}{A}$ fore $Q = \frac{zdB}{Bda} - \frac{BdA}{Ada} f\,\frac{X}{A}$.

§19. Latissime patebit solutio si ponatur $Q = Fz + PR$ et *R* fuerit functio ipsarum *a* et *x*. Erit enim $dz - Fzda = Pdx + PRda$. Posito $\int Fda = lB$ divisatur per *B* habebitur $\frac{dz}{B} - \frac{zdB}{B^2} = \frac{P}{B}(dx + Rda)$. Sit iam *S* functio efficiens *dx* + *Rda* integrabile sitque $\int(Sdx + SRda) = T$. Quo invento erit $P = BSfT$ huic respondet $Q = \frac{zdB}{Bda} + BRSfT$.

§20. Possunt praeterea plures huiusmodi valores ipsius *P* coniungi, hocque modo [p.193] multo latius extendi ut si ponatur $P = \frac{BdX}{dx} f(\mathrm{X} + \mathrm{A}) + \frac{BdY}{dx} f(\mathrm{Y} + \mathrm{E})$ erit $Q = \frac{zdB}{Bda} + \frac{BdA}{da} f(\mathrm{X} + \mathrm{A}) + \frac{BdE}{da} f(\mathrm{Y} + \mathrm{E})$. Atque simili modo numerus terminorum quantum libuerit, poterit augeri. In his igitur casibus omnibus aequatio modularis differentialis primi casus invenitur. Quamobrem his expeditis pergo ad eos casus investigandos, inquibus aequatio modularis primi gradus differentialis non datur, sed qui tamen ad aequationem modularem differentio-differentialem perducuntur.

§21. Si igitur Q neque algebraice per a et x neque per z potest exprimi, ii investigandi sunt casus quibus differentiale ipsius Q poterit exhiberi. Est autem $Q = \frac{dz-Pdx}{da}$, ergo $dQ = d\frac{dz-Pdx}{da}$. Quare si differentiale ipsius Q vel sola a et x vel per haec et Q vel etiam simul per z poterit exprimi, habebitur aequatio modularis, quae erit differentialis secundi gradus. Ostensum autem est superiore dissertatione si ponatur $dP = Ldx + Mda$ fore $dQ = Mdx + Nda$, ita ut haec differentialia commumem literum M involvant. Quia autem ex dato P etiam M datur, nil aliud requiritur, nisi ut N determinetur. Quamobrem in eos inquiremus casus, quibus N vel algebraice, vel per Q vel per Q et z exprimi potest. Tum enim habebitur aequatio modularis $Mdx + Nda = d\frac{dz-Pdx}{da}$, posito in N loco Q eius valore $\frac{dz-Pdx}{da}$.

§22. Ex praecedentibus satis intelligitur, si N per sola a et x determinatur, [p.194] fore $M = \frac{dX}{dx} f(\mathrm{X}+\mathrm{A})$ et $N = \frac{dA}{da} f(X+A)$, seu $M = V + \frac{dX}{dx} f(X+A)$ et $N = 1 + \frac{dA}{da} f(X+A)$ denotante V functionem quamcunque ipsius x et I ipsius a. Ex dato itaque P quareatur M, differentiando P posito x constante, et differentiali invento per da dividendo. Quo facto quaeratur an valor ipsius M in formula $V + \frac{dX}{dz} f(X+A)$ contineatur, erit $Vdx + dXf(X+A) + Ida + dAf(X+A) = d\frac{dz-Pdx}{da}$ aequatio modularis desiderata. Notandum est in posterum semper loco $\frac{dX}{dx} f(X+A)$ poni posse aggregatum ex quotius huiusmodi formulis $\frac{dX}{dx} f(X+A) + \frac{dY}{dx} f(Y+B) +$ etc. At loco $\frac{dA}{da} f(X+A)$ tunc poni debebit $\frac{dA}{da} f(X+A) + \frac{dB}{da} f(Y+B)$ etc. Hoc igitur monito in posterum tantum unica formula $\frac{dX}{dx} f(X+A)$ eique respondente $\frac{dA}{da} f(X+A)$.

§23. Pendeat N simul etiam a Q sitque $N = R + DQ$, ubi D sit functio ipsius a, et R functio ipsarum a et x ex conditionibus sequentibus determinanda. Erit igitur $dQ - DQdq = Mdx + Rda$, sit $Dda = \frac{dH}{H}$ et dividatur utrinque per H prodibit $\frac{dQ}{H} - \frac{QdH}{H^2} = \frac{Mdx+Rda}{H}$. In qua aequatione, cum illud membrum sit integrabile, tale quoque hoc $\frac{Mdx+Rda}{H}$ est efficiendum. Fiet igitur per praecedentem methodum $M = \frac{HdX}{dx} f(X+A)$ and $R = \frac{HdA}{da} f(X+A)$. Quare si in exemplo quopiam proposito ex P reperiatur M talis valoris, erit $N = \frac{HdA}{dx} f(X+A) + \frac{dH}{Hda^2}(dz - Pdx)$ positio $\frac{dH}{Hda}$ loco Q. Atque hinc in promtu erit aequatio modularis. [p.195]

§24. Sin N non a Q sed a z pendeat, ita ut sit $N = R + Cz$, denotante C functionem ipsius a quamcunque; erit $dQ - Czda = Mdx + Rda$. At quia est $dz - Qda = Pdx$, addatur huius multiplum $Fdz - QFda = PFdx$, existente F functione ipsius a, quo facto orietur aequatio $dQ - QFda + Fdz - Czda = (M + PF)dx + Rda$. Ponatur

$Fda = \frac{dB}{B}$ et $\frac{Cda}{F} = \frac{dG}{G}$, ita ut fit $F = \frac{dB}{Bda}$ et $C = \frac{dBdG}{BGda^2}$. Perspicuum itaque est $dQ - QFda$ integrabile reddi si dividatur per B seu multiplicetur per $\frac{1}{B}$, $Fdz - Czda$ autem fit integrabile, si multiplicatur per $\frac{1}{FG}$. Quare quo idem factor summam horum differentialium reddat integrabilem debebit esse $FG = B$ seu $\frac{GdB}{Bda} = B$, unde fiet $G = \frac{B^2 da}{dB}$. Hancobrem alterum quoque membrum per B divisum est integrabile efficiendum scilicet $\frac{(M+PF)dx+Rda}{B}$. Quocirca facio $R = \frac{BdA}{da} f(X + A)$ et $M + PF = \frac{BdX}{dx} f(X + \mathrm{A}) = M + \frac{PdB}{Bda}$. Investigari igitur debet proposito exemplo, an loco *A*, *B*, et *X* tales functiones inveniri queant, quae exhibeant formulam $\frac{PdX}{dx} f(X + \mathrm{A})$ aequalem ipsi $M + \frac{PdB}{Bda}$. Hisque inventis erit $N = \frac{BdA}{da} f(X + A) + \frac{xdBdG}{BGda^2}$ exifluente $G = \frac{B^2 da}{dB}$, qui valor in aequatione $Mdx + Nda = d.\frac{dz-Pdx}{da}$ substitutus dabit aequationem modularem.

§25. Sit nunc generalissime $N = R + DQ + Cz,$ tenentibus R, D et C iisdem quibus ante valoribus. Erit ergo $dQ - DQda - Czda = Mdx + Rda,$ addatur ad hanc aequatio $Fdz - FQda = PFdx,$ quo habeatur $dQ - DQda - FQda + Fdz - Czda = (\mathrm{M} + \mathrm{PF})dx + Rda.$ [p.196] Positis autem ut ante $Dda = \frac{dH}{H}, Fda = \frac{dB}{B}$, et $\frac{Cda}{F} = \frac{dG}{G}$, fit $dQ - DQda - FQda$ integrabile si ducatur in $\frac{1}{HB}$, et $Fdz - Czda$ integrabile fit ductum in $\frac{1}{FG}$. Quare debebit esse $HB = FG = \frac{GdB}{Bda}$ et $G = \frac{B^2 Hda}{dB}$. Atque $\frac{(M+PF)dx+Rda}{HB}$ reddendum est integrabile: fiet ergo facto $HB = E, R = \frac{EdA}{da} f(X + A)$ et $M + PF = \frac{EdX}{dx} f(X + A).$ Quocirca in casu proposito *A*, *X*, *E*, et *F* si fieri potest ita debent definire, ut $\frac{EdX}{dx} f(X + A)$ aequale fiat ipsi $M + PF$. Hocque invento erit $\frac{EdA}{da} f(X + A) + \frac{dH}{Hda^2}(dz - Pdx) + \frac{FzdG}{Gda}$, unde aequatio modularis reperitur.

§26. At si nequidem differentialis secundi gradus aequatio modularis obtineri poterit; ad differentialia tertii gradus erit procedendum. Fiet ergo $N = \frac{d(\frac{dz-Pdx}{da})-Mdx}{da}$ atque hinc posito $d\mathrm{N} = sdx + tda$, erit $sdx + tda = \mathrm{d}\left(\frac{d(\frac{dz-Pdx}{da})-Mdx}{da}\right)$. Datur autem *s* ex *M*, cum sit *sda* differentiale ipsius *M*, quod prodit, si *x* ponatur constans. Quamobrem *t* tantum debebit investigari. Sit ergo $t = R + EN + DQ + Cz,$ ideoque $dN - ENda - DQda - Czda = sdx + Rda.$ Cum sit autem $dQ - Nda = Mdx$ et $dz - Qda = Pdx,$ addantur horum multipla ad illam aequationem, ut prodeat haec aequatio

$$dN - ENda - FNda + FdQ - DQda - GQda + Gdz - Czda =$$
$$(s + MF + PG)dx + Rda.$$

Sit $Eda + Fda = \frac{df}{f}, \frac{Dda+Gda}{F} = \frac{dg}{g}$ et $\frac{Cda}{G} = \frac{dh}{h}$, fiatque $f = Fg = Gh$. [p.197] Quo facto aequationis inventae prius membrum fit integrabile divisum per *f*; hanc ob rem et $\frac{(s+MF+PG)dx+Rda}{f}$ efficiendum est integrabile. Ponendum igitur est $R = \frac{fdA}{da} f(X+A)$ et $s + MF + PG = \frac{fdX}{da} f(X+A)$. In aequatione ergo proposita, quia *s* et *M* ex *P* dantur, debent *F, G* et *f* et *X* ex hac aequatione determinari. Quo facto sumatur $g = \frac{f}{F}$ et $h = \frac{f}{G}$, et $C = \frac{Gdh}{hda}$, et $D = \frac{Fdg}{gda} - G$ et $E = \frac{df}{fda} - F$. Atque ex his cognita erit aequatio $t = R + EN + DQ + Cz$, ex qua aequatio modularis facile constatur. Simili modo ex his intelligitur quomodo pro altioribus differentialium gradibus operatio debeat institui, ut ad aequationes modulares perveniatur.

§27. In compendium nunc, quae hactenus tradidimus, redigamus tum quo facilius quaevis aequatio proposita reduci queat, tum quo processus ad cuiusque gradus differentialia clarius perspiciatur. Proposita igitur aequatione $dz = Pdx$, ponatur *x* constans et *a* tantum variabile sitque $dP = Mda, dM = pda, dp = rda$ etc. Sit porro $Q = \frac{dz-Pdx}{da}, N = \frac{dQ-Mdx}{da}, q = \frac{dN-pdx}{da}$ et $s = \frac{dq-rdx}{da}$ etc. Ubi *dQ, dN, dq*, etc sunt differentialia ipsorum *Q, N,* et *q*, quae ex valoribus $\frac{dz-Pdx}{da}, \frac{dQ-Mdx}{da}$, et $\frac{dq-rdx}{da}$ inveniuntur positis *a, x* et *z* variabilibus. Hanc igitur ob rem cognitae erunt *M, p, r* etc ex solo *P*, ex his vero habebuntur *Q, N, q* etc. Sint praeterea *A, B, C, D, E, F* etc functiones ipsius *a* et constantium, et *X, Y* etc. functiones ipsius *x* non involuentes *a*. [p.198]

§28. His praemissis si furerit *P* talis functio ipsius *x* et *a*, ut *BP* comprehendatur in hac forma $\frac{dX}{dx} f(X+A)$ seu plurium huiusmodi formularum aggregato, semper dari poterit aequatio modularis differentialis primi gradus. Namque erit $PdAdx = z\frac{dBdX}{B} + QdadX$ seu $BPdAdx = zdBdX + BQdadX$. Quae aequatio ob datum *Q* est modularis respondens aequationi propositae.

§29. Deinde si *P* talis sit functio ipsarum *a* et *x* ut *BP + CM* aequalis fieri possit $\frac{dX}{dx} f(X+A)$ seu quotcunque huiusmodi formularum aggregato, aequatio modularis ad differentialia secundi gradus ascendet. Erit enim $BPdAdx + CMdAdx = zdBdX + BQdadX + QdCdX + CNdadX$. Quae est aequatio modularis quaesita, et involuit differentialia secundi gradus, quia eam littera *N* ingreditur, quae per *dQ* ideoque per *ddz, ddx*, et *dda* determinatur.

§30. At si fuerit *BP + CM + Dp* aequalis huic formuale $\frac{dX}{dx} f(X+A)$ vel aggregato quotcunque huiusmodi formularum; aequatio modularis erit differentialis tertii gradus, prodibit enim ista aequatio
$$BPdAdx + CMdAdx + DpdAdx = zdBdX + BQdadX + QdCdX + CNdadX + NdDdX + DqdadX.$$
Quemadmodum ex ante traditis colligere licet, si modo quantitates ab *a* tantum pendentes ad has formulas accommodantur.

§31. Simili modo ad altiora differentialia progressus facile absolvitur. [p.199] Nam si $BP + CM + Dp + \mathrm{E}r$ aequetur formula $\frac{dX}{dx} f(X + A)$ vel talium plurium formularum aggregato, orietur aequatio modularis ista

$$BPdAdx + CMdAdx + DpdAdx + ErdAdx = zdBdX + BQdadX + QdCdX +$$
$$CNdadX + NdDdX + DqdadX + qdEdX + EsdadX$$

quae erit differentialis quarti gradus. Atque hoc modo quousque libuerit hae operationes facile continuantur ex sola allatarum inspectione.

§32. His autem omnibus perspectis maxima tamen difficutas saepenumero posita erit in dignoscenda functione *P*, an in his expositis generibus contineatur et in quonam genere. Etiam si enim generales ipsius *P* valores, qui ex assumtis formulis obtinentur nihil difficultatis in se habere videantur, tamen exemplis particularibus propositis accommodatio saepissime erit difficillima. Cuius rei ratio nequaquam methodo traditae est tribuenda, sed imperfectae functionum cognitioni, quae adhuc habetur. Quamobrem non solum in hoc negotio, sed in plurimis etiam aliis casibus maxime utile foret, si functionum doctrina magis perficeretur, et excoleretur.

§33. Quantum quidem mihi hac de re meditari licuit, eximium subsidium inveni, si *P* statim ad huiusmodi formam $\frac{dX}{dx} f(X + A)$ vel huiusmodi formularum aggregatum reducatur, id quod sequenti modo facillime praestatur, Prima aequatio proposita non constituatur inter *z* et *x* sed inter *z* et *y*, ita ut aequatio ad modularem perducenda sit $dz = Tdy$, [p.200] existente *T* functione ipsius y et moduli a. Tum accipiatur pro *x* talis functio ipsarum *a* et *y*, quae transmutet *T* in functionem ipsarum *a* et *x* contentam in formula $f(X + A)$, vel pluribus huic similibus, earumque multiplis, in quibus *X* est functio ipsius *x* tantum, et *A* ipsius *a*. Hoc igitur facto prodeat aequato $dz = Sdx\ f(X + A)$ ubi *S* sit quantitas tam simplex quam fieri potest. Quare P erit $S\ f(X + A)$ ideoque cum *M, p* etc. coniuncta facilius cum generalibus formulis comparatur. Inventa autem hoc modo aequatione modulari, valor ipsius *x* in *a* et *y* assumtus, ubique loco *x*, loco *dx* autem differentiale huius valoris positis *a* et *y* variabilibus substituatur. Quo facto habebitur aequatio modularis inter *a*, *y*, et *z*, quae quaerebatur.

§34. Ad pleniorem quidem methodi hactenus traditae cognitionem maximam lucem afferrent exempla et problemata, quorum solutio istam methodum requiret. Sed quia ipsorum problematum dignitas peculiarem tractationem postulat, in aliud tempus, ne hoc tempore nimis sim longus, eam differo.